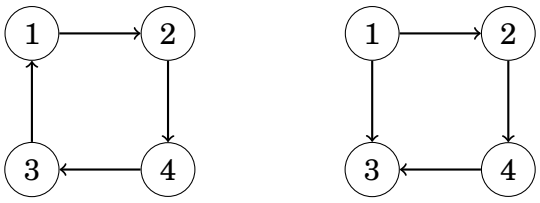
Chapter 17

Strong connectivity

In a directed graph, the edges can be traversed in one direction only, so even if the graph is connected, this does not guarantee that there would be a path from a node to another node. For this reason, it is meaningful to define a new concept that requires more than connectivity.

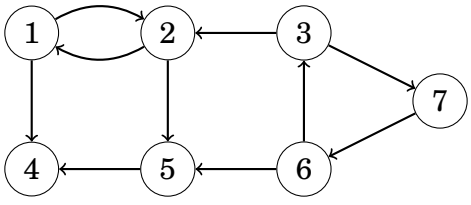
A graph is **strongly connected** if there is a path from any node to all other nodes in the graph. For example, in the following picture, the left graph is strongly connected while the right graph is not.



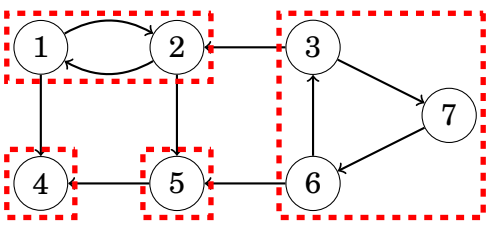
The right graph is not strongly connected because, for example, there is no path from node 2 to node 1.

The **strongly connected components** of a graph divide the graph into strongly connected parts that are as large as possible. The strongly connected components form an acyclic **component graph** that represents the deep structure of the original graph.

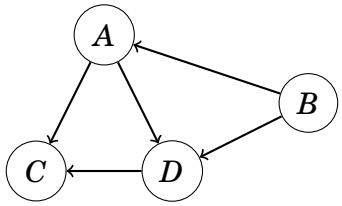
For example, for the graph



the strongly connected components are as follows:



The corresponding component graph is as follows:



The components are $A = \{1,2\}$, $B = \{3,6,7\}$, $C = \{4\}$ and $D = \{5\}$.

A component graph is an acyclic, directed graph, so it is easier to process than the original graph. Since the graph does not contain cycles, we can always construct a topological sort and use dynamic programming techniques like those presented in Chapter 16.

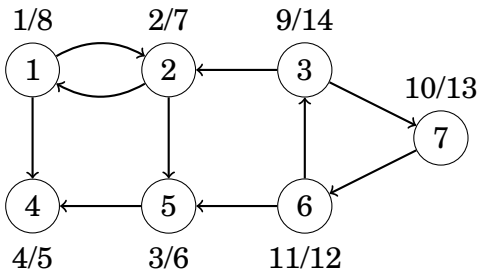
Kosaraju's algorithm

Kosaraju's algorithm[1] is an efficient method for finding the strongly connected components of a directed graph. The algorithm performs two depth-first searches: the first search constructs a list of nodes according to the structure of the graph, and the second search forms the strongly connected components.

Search 1

The first phase of Kosaraju's algorithm constructs a list of nodes in the order in which a depth-first search processes them. The algorithm goes through the nodes, and begins a depth-first search at each unprocessed node. Each node will be added to the list after it has been processed.

In the example graph, the nodes are processed in the following order:



The notation x/y means that processing the node started at time x and finished at time y. Thus, the corresponding list is as follows:

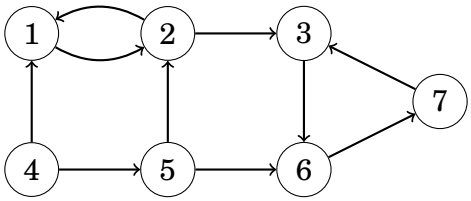
[1] According to [1], S. R. Kosaraju invented this algorithm in 1978 but did not publish it. In 1981, the same algorithm was rediscovered and published by M. Sharir [57].

node	processing time
4	5
5	6
2	7
1	8
6	12
7	13
3	14

Search 2

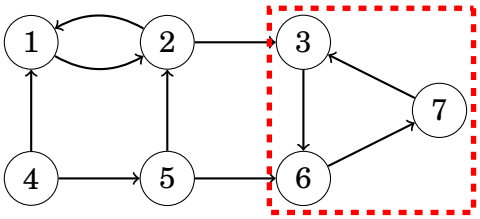
The second phase of the algorithm forms the strongly connected components of the graph. First, the algorithm reverses every edge in the graph. This guarantees that during the second search, we will always find strongly connected components that do not have extra nodes.

After reversing the edges, the example graph is as follows:



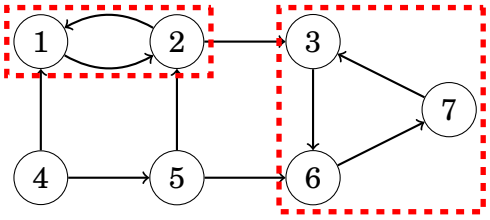
After this, the algorithm goes through the list of nodes created by the first search, in *reverse* order. If a node does not belong to a component, the algorithm creates a new component and starts a depth-first search that adds all new nodes found during the search to the new component.

In the example graph, the first component begins at node 3:

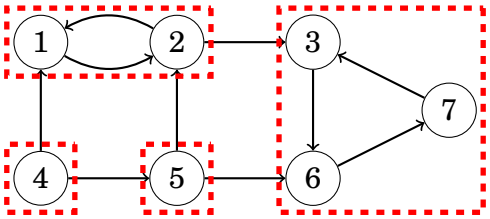


Note that since all edges are reversed, the component does not ”leak” to other parts in the graph.

The next nodes in the list are nodes 7 and 6, but they already belong to a component, so the next new component begins at node 1:



Finally, the algorithm processes nodes 5 and 4 that create the remaining strongly connected components:



The time complexity of the algorithm is $O(n+m)$, because the algorithm performs two depth-first searches.

2SAT problem

Strong connectivity is also linked with the **2SAT problem**[2]. In this problem, we are given a logical formula

$$(a_1 \vee b_1) \wedge (a_2 \vee b_2) \wedge \cdots \wedge (a_m \vee b_m),$$

where each a_i and b_i is either a logical variable ($x_1, x_2, \ldots, x_n$) or a negation of a logical variable ($\neg x_1, \neg x_2, \ldots, \neg x_n$). The symbols "$\wedge$" and "$\vee$" denote logical operators "and" and "or". Our task is to assign each variable a value so that the formula is true, or state that this is not possible.

For example, the formula

$$L_1 = (x_2 \vee \neg x_1) \wedge (\neg x_1 \vee \neg x_2) \wedge (x_1 \vee x_3) \wedge (\neg x_2 \vee \neg x_3) \wedge (x_1 \vee x_4)$$

is true when the variables are assigned as follows:

$$\begin{cases} x_1 = \text{false} \\ x_2 = \text{false} \\ x_3 = \text{true} \\ x_4 = \text{true} \end{cases}$$

[2]The algorithm presented here was introduced in [4]. There is also another well-known linear-time algorithm [19] that is based on backtracking.