NUMERICAL MODELING AND SIMULATION FOR ACOUSTICS
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Homework 3

Traffic flow

Let $\rho(x,t)$ denote the density of cars (vehicle/km) and $u(x,t)$ the velocity of cars. Since cars are conserved

$$\rho_t(x,t) + (\rho u)_x = 0 \quad (x,t) \in (0,L) \times (0,T]. \tag{1}$$

Assume that u is a function of ρ

$$u(\rho) = u_{max}\left(1 - \frac{\rho}{\rho_{max}}\right),$$

where $0 \leq \rho \leq \rho_{max}$ and u_{max} is the speed limit, and the following initial condition

$$\rho(x,0) = \rho_0(x) \quad x \in (0,L).$$

Consider a cartesian mesh for the domain $[0,L] \times [0,T]$ by dividing the interval $[0,T]$ into N steps $0 = t_0 \leq t_1 \leq ... \leq t_N = T$ with constant time step $\Delta t = T/N$ and $t_n = n\Delta t$ for $n = 1,2,...,N$ and the space interval $[0,L]$ into M nodes $0 \leq x_0 \leq x_1 \leq ... \leq x_M = L$ with constant spacing step $h = ih$ for $i = 1,2,...,M$.

1. Write the finite volume approximation of (1) by considering the Gudunov method with either constant (first order scheme) and linear riconstruction (second order scheme) of the numerical solution. For the latter consider the following **Lax-Wendroff** scheme

$$\begin{cases} U^-_{i+\frac{1}{2}} = U_i + \frac{1}{2}(U_{i+1} - U_i), \\ U^+_{i+\frac{1}{2}} = U_i - \frac{1}{2}(U_{i+1} - U_i). \end{cases}$$

2. Set $L = 1, T = 1$ and the following initial and boundary conditions

   1. **Traffic Jam**: $\rho(x,0) = \rho_R = \rho_{max}$ for $x \geq \frac{1}{2}$ and $\rho(x,0) = \rho_L < \rho_{max}$ otherwise.
   2. **Green Light**: $\rho(x,0) = \rho_R$ for $x \geq \frac{1}{2}$ and $\rho(x,0) = \rho_L$ otherwise, with $0 < \rho_R < \rho_L < \rho_{max}$.
   3. **Traffic flow**: $\rho(x,0) = \rho_R = \frac{1}{2}\rho_{max}$ for $x \geq \frac{1}{2}$ and $\rho(x,0) = \rho_L = \rho_{max}$ otherwise.

   Consider for all cases a proper boundary condition. Compare the solutions obtained by using methods at the previous point. Comment on the results trying to give a physical explanation to the results.

3. Tipically on a highway, we wish to drive at some speed u_{max} but in heavy traffic we slow down. At some point, the highway reaches its maximum capacity of cars $\rho_{\max}$ and our velocity is zero. The simplest model for this relationship between velocity and density is

that given below. This function has been found to provide a fairly good model for actual traffic flows. For example, a good fit to actual data was obtained using the function

$$f(\rho) = \rho \log\left(\frac{\rho_{max}}{\rho}\right).$$

Repeat the points at the previous step by considering (1) with the above flux definition and comment on the results.