NUMERICAL MODELING AND SIMULATION FOR ACOUSTICS
A.A. 2024/2025

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HOMEWORK 2

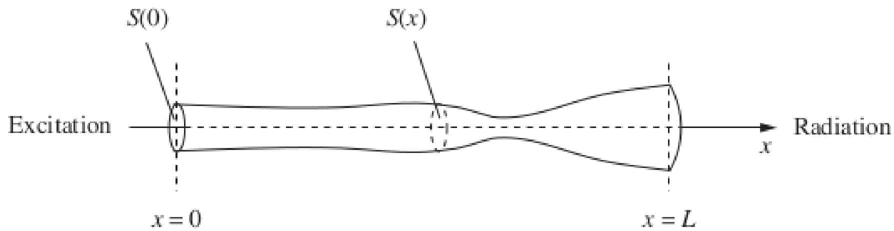

Figure 1: Example of a 1D acoustic tube, with excitation at the left end, and radiated sound produced at the right end.

Consider the following wave propagation problem (Webster's equation) in a longitudinal domain having variable cross section $S(x)$, cf. Figure 1, and lenght $L > 0$:

$$\begin{cases} S\varphi_{tt} = \gamma^2 (S\varphi_x)_x + f & (x,t) \in (0,1) \times (0,T], \\ \varphi_x(0,t) = 0 & t \in (0,T], \\ \varphi_t(1,t) = 0 & t \in (0,T], \\ \varphi(x,0) = \varphi_0(x) & x \in (0,1), \\ \varphi_t(x,0) = \varphi_1(x) & x \in (0,1), \end{cases} \tag{1}$$

where $\gamma = c/L$ and f, φ_0 and φ_1 are given functions.

1. Consider the following finite difference scheme for the solution to (1)

$$\begin{aligned} \varphi_k^{n+1} = \lambda^2 \frac{(S_{k+1} + S_k)}{2[S]_k} \varphi_{k+1}^n + \lambda^2 \frac{(S_k + S_{k-1})}{2[S]_k} \varphi_{k-1}^n \\ + \left(2 - \lambda^2 \frac{(S_{k+1} + 2S_k + S_{k-1})}{2[S]_k}\right) \varphi_k^n - \varphi_k^{n-1} + f_k^n, \end{aligned} \tag{2}$$

where $\lambda = \gamma \dfrac{\Delta t}{\Delta x}$, being Δt the time step and Δx the mesh size, and where $[S]_k$ is a second order approximation $y = a_0 + a_1 x + a_2 x^2$ of the profile section S obtained as

$$\begin{cases} a_0 + a_1 x_{k-1} + a_2 x_{k-1}^2 = S_{k-1}, \\ a_0 + a_1 x_k + a_2 x_k^2 = S_k, \\ a_0 + a_1 x_{k+1} + a_2 x_{k+1}^2 = S_{k+1}, \end{cases} \tag{3}$$

for any grid node x_k. Verify that for a constant section profile S, scheme (2) reduces to the leap-frog scheme

$$\varphi_k^{n+1} - 2\varphi_k^n + \varphi_k^{n-1} = \lambda^2 \left(\varphi_{k+1}^n - 2\varphi_k^n + \varphi_{k-1}^n\right) + f_k^n.$$

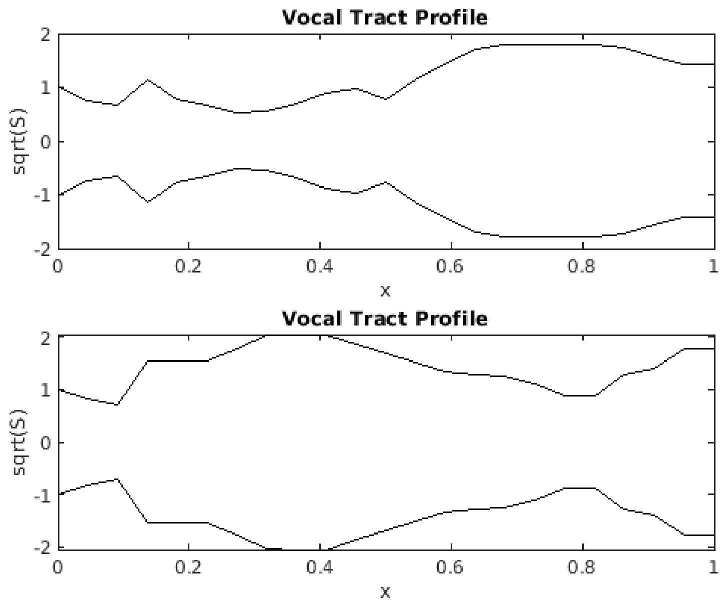

Figure 2: Vocal tract profiles exercise 4. You can obtain the plots above as well as the definition of sections S by using the conde `Sections.m`.

2. Write two Matlab codes that implement the schemes at the previous step. Verify your implementation by considering the following data:

   – $L = 1$, $c = 1$, $S(x) = 1$,
   – $L = 1$, $c = 1$, $S(x) = (1 + 2x)^2$.

   Consider as exact solution to problem (1) the function $\varphi_{ex}(x,t) = \cos(\pi(\frac{x}{2} + 1))\cos(3\pi t)$ and compute the remaining data accordingly. Report the plot of the computed solution obatined as well ase the computed discretization errors in a table. Consider both space and time grid refinements. Comment on the results.

4. Solve problem (1) with the following data:

   – $L = 1$, $c = 2$, $f = \varphi_0 = \varphi_1 = 0$,

   and

$$\varphi_t(1,t) = 0, \quad \varphi_x(0,t) = u_{in} = -\frac{1}{2S_0}\left(\sin(\pi(t-0.1)/0.05) + |\sin(\pi(t-0.1)/0.05)|\right), \quad (4)$$

   by considering the staggered scheme at point 2. Consider $S(x) = 1$ and the sections S reported in the file `Sections.m`, cf. also Figure 2. Report the results and comment on them.