MUSIC AND
ACOUSTIC ENGINEERING

NUMERICAL MODELLING AND SIMULATIONS FOR ACOUSTICS

Homework 3 - Report

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1 Finite Volume Formulation

1.1 Problem presentation

In this report, we are interested in the traffic flow equation given by :

$$\partial_t \rho(x,t) + \partial_x(\rho u) = 0, \quad \begin{cases} \forall x \in]0, L[= D_x \\ \forall t \in]0, T] = D_t \end{cases}$$

We assume that the velocity of cars $u(x,t)$ can be written as a function of the density of cars $\rho(x,t)$, such that:

$$u(\rho) = u_{\max} \cdot \left(1 - \frac{\rho}{\rho_{\max}}\right)$$

We have $\rho(x,t) \in [0, \rho_{\max}]$, and $u_{\max} > 0$ is the speed limit.
We have as initial condition:

$$\rho(x,0) = \rho_0(x), \forall x \in D_x$$

1.2 Flux models

We'll consider here 2 different models for $f(\rho)$.
The first model starts by considering $u(\rho) = u_{\max} \cdot \left(1 - \frac{\rho}{\rho_{\max}}\right)$, which leads to the flux function

$$f_1(\rho) = \rho u = u_{\max} \cdot \left(\rho - \frac{\rho^2}{\rho_{\max}}\right)$$

In this case, the characteristic speed is $c_1 = \frac{df_1}{d\rho} = u_{max}(1 - 2\frac{\rho}{\rho_{max}})$
The second model, that aims at being finer than the one above, is defined by considering a flux behavior that would mimic better the one we can find in reality. Tipically on a highway, we wish to drive at some speed u_{max} but in heavy traffic we slow down. At some point, the highway reaches its maximum capacity of cars ρ_{max} and our velocity is 0. The simplest model for this relationship between velocity and density is:

$$f_2(\rho) = \rho \log\left(\frac{\rho_{max}}{\rho}\right)$$

In this case, the characteristic speed is $c_2 = \frac{df_2}{d\rho} = \log\left(\frac{\rho_{max}}{\rho}\right) - 1$
This function has been found to provide a fairly good model for actual traffic flows.

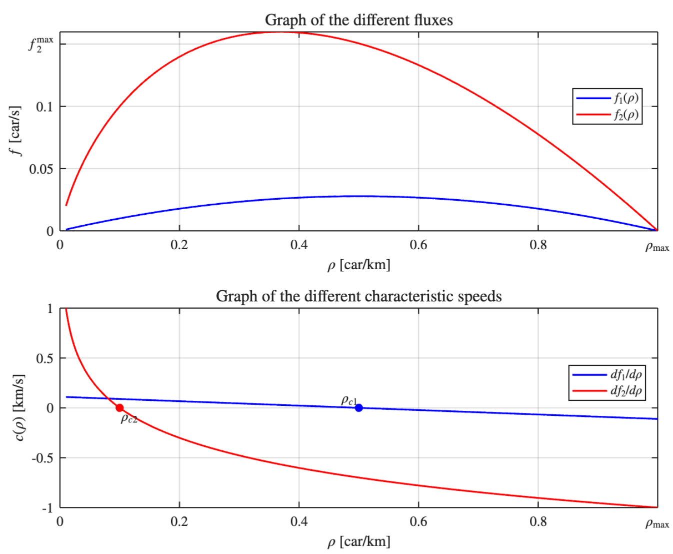


Figure 1.1: Representation of $f_{1,2}(\rho)$, and $c_{1,2}(\rho)$

Considering either f_1, or f_2, the traffic flow equation becomes:

$$\rho_t + f_x = 0, \quad \forall (x,t) \in D_x \times D_t$$

We note the critical densities, $\rho^c_{1,2}$ suchat that $\frac{df_{1,2}}{d\rho} = 0$, and we have $\begin{cases} \rho^c_1 = \rho_{max}/2 \\ \rho^c_2 = \rho_{max}/10 \end{cases}$
More precisely, we observe 3 conditions of information propagation direction:

$$\begin{cases} \rho < \rho_c & \Rightarrow c(\rho) > 0, \quad \text{information travels right} \\ \rho > \rho_c & \Rightarrow c(\rho) < 0, \quad \text{information travels left} \\ \rho = \rho_c & \Rightarrow c(\rho) = 0, \quad \text{information stands still} \end{cases}$$

1.3 Time and space meshes

The space domain $[0, L]$ discretized into $M+1$ nodes, leading us M intervals $k^x_m = [x_m, x_{m+1}]$.
The space step in between 2 consecutive nodes is $\Delta x = \frac{L}{M}$, and we have:

$$x_m = (m-1)\,\Delta x \quad \forall m \in [1, M+1]\,, \qquad x_1 = 0, \qquad x_{M+1} = L$$

The time domain $[0, T]$ discretized into $N+1$ instants, leading us N intervals $k^t_n = [t_n, t_{n+1}]$.
The time step in between 2 consecutive instants is $\Delta t = \frac{T}{N}$, and we have:

$$t_n = (n-1)\,\Delta t, \quad \forall n \in [1, N+1]\,, \qquad t_1 = 0, \qquad t_{N+1} = T$$

1.4 Problem discretization

Integrating on $\left[x_{i-1/2}, x_{i+1/2}\right]$ gives:

$$\frac{\partial}{\partial t}\int_{x_{i-1/2}}^{x_{i+1/2}} \rho\,(x,t)\,dx = -\int_{x_{i-1/2}}^{x_{i+1/2}} f_x dx = -\left[f\left(\rho\left(x_{i+1/2},t\right)\right) - f\left(\rho\left(x_{i-1/2},t\right)\right)\right]$$

We set $f\left(\rho\left(x_k,t\right)\right) = f_k\left(t\right)$ which yields:

$$\frac{\partial}{\partial t}\int_{x_{i-1/2}}^{x_{i+1/2}} \rho\,(x,t)\,dx = -\left[f_{i+1/2}\left(t\right) - f_{i-1/2}\left(t\right)\right]$$
$$\Rightarrow \frac{\partial}{\partial t}\left[\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \rho\,(x,t)\,dx\right] = -\frac{1}{\Delta x}\left[f_{i+1/2}\left(t\right) - f_{i-1/2}\left(t\right)\right]$$

We recognize the averaging operation on the left part of the equation. More precisely, we identify the average of ρ in $\left[x_{i-1/2}, x_{i+1/2}\right]$. We set :

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \rho\,(x,t)\,dx = \rho_i\left(t\right)$$

Therefore, we now consider cell averages and flux behind cells rather than nodes in the mesh, and the equation of interest is:

$$\frac{\partial}{\partial t}\rho_i\left(t\right) = -\frac{1}{\Delta x}\left[f_{i+1/2}\left(t\right) - f_{i-1/2}\left(t\right)\right]$$

Considering the discrete time instant $t^n = (n-1)\,\Delta t$, we consider the space-and-time discrete function $\rho_i\left(t^n\right) = \rho^n_i$ and we use a simple discrete derivation formula for its first order time derivative:

$$\frac{\partial}{\partial t}\rho^n_i = \frac{\rho^{n+1}_i - \rho^n_i}{\Delta t} \qquad \Rightarrow \frac{\rho^{n+1}_i - \rho^n_i}{\Delta t} = -\frac{1}{\Delta x}\left[f^n_{i+1/2} - f^n_{i-1/2}\right]$$

Where $f^n_k = f\left(\rho\left(x_k, t^n\right)\right)$
Without considerations about the specific flux we'll use, our scheme is:

$$\rho^{n+1}_i = \rho^n_i - \frac{\Delta t}{\Delta x}\left[f^n_{i+1/2} - f^n_{i-1/2}\right], \quad \forall n \in [1, N+1]\,, \forall i \in [1, M+1]$$

To solve this problem and implement our algorithm, we are interested in using the Godunov method for the flux, and consider 2 different schemes

1.5 Godunov Method - first order scheme

We consider here a constant reconstruction of the numerical solution. In our time-and-space mesh, the Godunov flux at the left $\left(f^n_{i-1/2}\right)$ and right $\left(f^n_{i+1/2}\right)$ interfaces is defined as:

$$f^n_{i+\frac{1}{2}} = \begin{cases} \min f(\rho^n_i), f(\rho^n_{i+1}) & \text{if } \rho^n_i \leq \rho^n_{i+1} \\ \max f(\rho^n_i), f(\rho^n_{i+1}) & \text{if } \rho^n_i > \rho^n_{i+1} \end{cases}, \qquad f^n_{i-\frac{1}{2}} = \begin{cases} \min f(\rho^n_{i-1}), f(\rho^n_i) & \text{if } \rho^n_{i-1} \leq \rho^n_i \\ \max f(\rho^n_{i-1}), f(\rho^n_i) & \text{if } \rho^n_{i-1} > \rho^n_i \end{cases}$$

NB: the scenarios where $\rho^n_i \leq \rho^n_{i+1}$, or $\rho^n_{i-1} \leq \rho^n_i$ implies that we have rarefaction waves, while the second on implies that we have shock waves.

1.6 Godunov Method - second order scheme

We consider here a linear reconstruction of the numerical solution and introduce linear reconstructions of ρ:

$$\begin{cases} \rho^{n-}_{i+1/2} = \rho^n_i + \frac{1}{2}\left(\rho^n_{i+1} - \rho^n_i\right) \\ \rho^{n+}_{i+1/2} = \rho^n_i - \frac{1}{2}\left(\rho^n_{i+1} - \rho^n_i\right) \end{cases}$$

To compute the limited slope σ_i, we extend the state vector by adding ghost cells at boundaries.
For each cell i, we compute the forward and backward differences: $\begin{cases} \text{forward\_diff} = \frac{\rho_{i+1}-\rho_i}{\Delta x} \\ \text{backward\_diff} = \frac{\rho_i - \rho_{i-1}}{\Delta x} \end{cases}$
The ratio of slopes is computed as:

$$r = \begin{cases} \frac{\text{forward\_diff}}{\text{backward\_diff}}, & \text{if backward\_diff} \neq 0 \\ 100 \cdot \text{sign}(\text{forward\_diff}), & \text{otherwise} \end{cases}$$

The slope limiter function $\phi(r)$ is then applied:

$$\sigma_i = \phi(r) \cdot \text{backward\_diff}$$

In our code we consider the **Monotonized Central (MC) limiter** as it is a commonly used slope limiter in high-resolution schemes for solving hyperbolic conservation laws, such as the traffic flow equation. It is designed to prevent oscillations near discontinuities while maintaining high accuracy in smooth regions. Based on the information we retrieved in literature, the MC limiter provides a balance between the minmod limiter (which is too dissipative) and the superbee limiter (which can introduce spurious oscillations).
Using the limited slopes, the left and right states at each cell interface $i + 1/2$ are reconstructed as:

$$\begin{cases} \rho^L_{i+1/2} = \rho_i + \frac{\sigma_i \Delta x}{2} \\ \rho^R_{i+1/2} = \rho_{i+1} - \frac{\sigma_{i+1}\Delta x}{2} \end{cases}$$

As we virtually extended the domain to include external cells to store extra flux information, we also perform this operation at the boundary of the domain.
The numerical flux at each interface is computed using the Godunov solver:

$$F^n_{i+1/2} = \text{Godunov\_flux}(\rho^{n,L}_{i+1/2}, \rho^{n,R}_{i+1/2}, f, \rho_c)$$
$$F^n_{i-1/2} = \text{Godunov\_flux}(\rho^{n,L}_{i-1/2}, \rho^{n,R}_{i-1/2}, f, \rho_c)$$

TMP TMP TMP Using the finite volume method formula, the density is updated:

$$\rho^{n+1}_i = \rho^n_i - \frac{\Delta t}{\Delta x}(F^n_{i+1/2} - F^n_{i-1/2})$$

We ensure physical bounds by applying:

$$\rho(:, n+1) = \max(0, \min(1, \rho(:, n+1)))$$

NB: the scenario where $\rho^{n,L}_{i\pm1/2} \leq \rho^{n,R}_{i\pm1/2}$ implies that we have rarefaction waves, while the second on implies that we have shock waves.

2 Boundary and Initial conditions

We set here $L = 1$ km, and $T = 1$ s. As we like big engines, we arbitrarily set $u_{max} = 400$ km/h $= 0.11$ km/s. We use a normalized density and set $\rho_{max} = 1$ car/km.

2.1 Traffic Jam

Here we have as initial condition: $\rho(x,0) = \begin{cases} \rho_L < \rho_{max}, & \forall x \in [0, \frac{L}{2}] \\ \rho_R = \rho_{max}, & \forall x \in]\frac{L}{2}, L] \end{cases}$

Considering the density difference at $x = L/2$, we'll have a rarefaction wave moving towards the right. The jammed area at the right of the domain will spread along the left part. Thus the density will gradually increase from the right to the left of $[0, L/2]$ from the initial value ρ_L.

Left boundary condition, $x = 0$: we assume an open boundary condition, where the density remains constant.

Right boundary condition, $x = L$: the density also remains constant here as we want to maintain the traffic jam scenario.

I.e. we have as boundary conditions, $\forall t \in [0, T]$: $\begin{cases} \rho(0,t) = \rho_L \\ \rho(L,t) = \rho_{max} \end{cases}$

2.2 Green Light

Here we have as initial condition: $\rho(x,0) = \begin{cases} \rho_L, & \forall x \in [0, \frac{L}{2}] \\ \rho_R, & \forall x \in]\frac{L}{2}, L] \end{cases}$ with $0 < \rho_R < \rho_L < \rho_{max}$.

Considering that the light turns green at $t = 0$, the vehicles at the front will start moving, creating a rarefaction wave as the density gradually decreases from ρ_{max} to ρ_L.

Since traffic is starting from a jam, it should gradually disperse as vehicles accelerate, it is appropriate to set a high density at the left boundary while allowing free outflow at the right. This allows the density wave to propagate in a physically consistent way.

Left boundary condition, $x = 0$: we assume cars to be continuously coming before the green light and that the density of cars is maximum

Right boundary condition, $x = L$: we assume that nothing stops the cars from leaving the domain, the traffic is assumed fluid and homogeneous at the right part of the domain.

I.e. we have as boundary conditions, $\forall t \in [0, T]$: $\begin{cases} \rho(0,t) = \rho_L \\ \rho(L,t) = \rho(L - \Delta x, t) \end{cases}$

2.3 Traffic flow

Here we have as initial condition: $\rho(x,0) = \begin{cases} \rho_L = \rho_{max}, & \forall x \in [0, \frac{L}{2}] \\ \rho_R = \frac{\rho_{max}}{2}, & \forall x \in]\frac{L}{2}, L] \end{cases}$

We expect to see a shock wave to propagate to the right part of the domain as the dense region collapses into a moving region with a lower density. The shock wave forms as vehicles in the jam start accelerating toward the region of lower density.

Left boundary condition, $x = 0$: we maintain $\rho_m ax$ assuming that the traffic is fully jammed at $x = 0$ and continuously pushes forward.

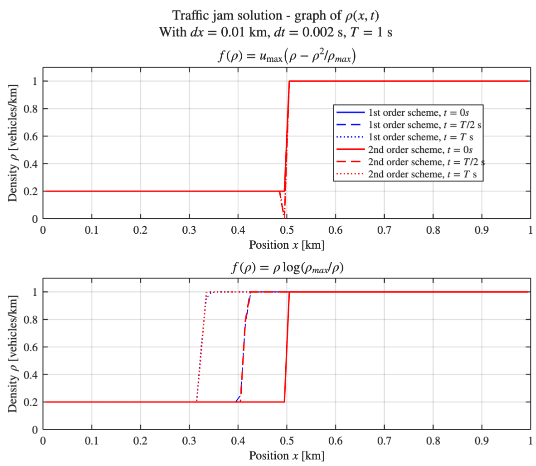
Right boundary condition, $x = L$: we assume that nothing stops the cars from leaving the domain, the traffic is assumed fluid and homogeneous at the right part of the domain.

I.e. we have as boundary conditions, $\forall t \in [0, T]$: $\begin{cases} \rho(0,t) = \rho_{max} \\ \rho(L,t) = \rho(L - \Delta x, t) \end{cases}$

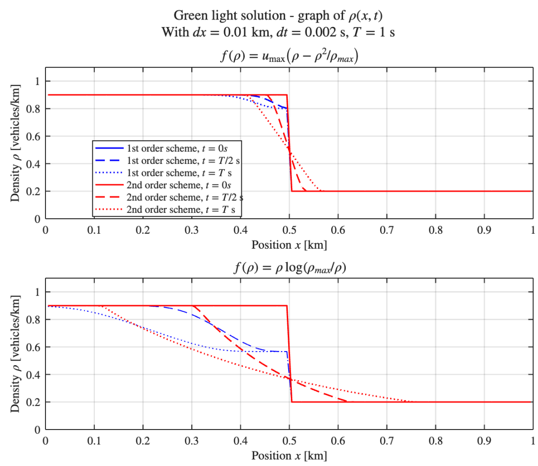
3 Numerical results

3.1 Graphs

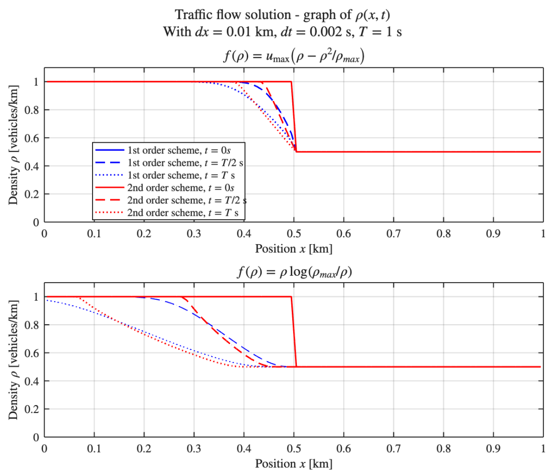
We first provide the results for both the density $\rho(x,t)$ and flux $f(\rho)$ for different instant of times: $t = (0, \frac{T}{2}, T)$.
For the density analysis, we observe that the initial discontinuity propagates accordingly to assumptions made above. We also observe a huge difference of propagation velocity in between the 2 models of flux.
We observe that the second-order scheme keeps more into account the initial discontinuity's sharpness whereas the first-order scheme is smoother. The polynomial function leading the smoothness trend around the discontinuity through time for the first order scheme is of a lower order than the one used for the 2nd order.
Since the graph of the flux is easier to analyze for phenomenon analysis, we'll expand our comments on the second set of graphs.



(a) Traffic jam scenario



(b) Green light scenario



(c) Traffic flow scenario

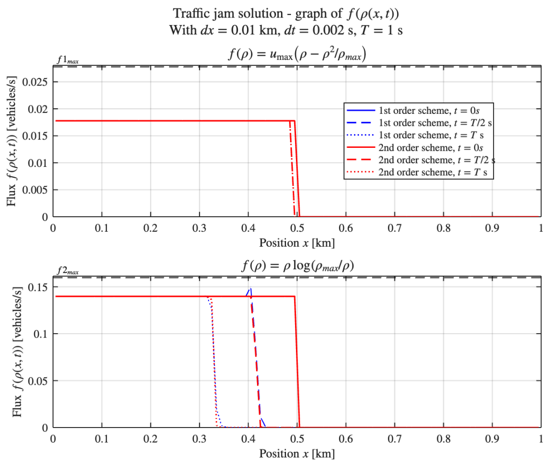
Figure 3.1: Evolution of the density for 3 discrete instants of time.

In the graphs below, we show the result of the numerical resolution of the problem of interest. We observe that for a given flux function $f_{1,2}(\rho)$, both schemes behave similarly. However, they do not provide the same trend of discontinuity propagation. The first order scheme seems to smoothen the discontinuity along its propagation while the second order scheme presents sharper and more complex profiles of the density $\rho(x,t)$ near its extremities. To have a better look at it, we can see a smoother color gradient on the first-order schemes compared to the second-order schemes.

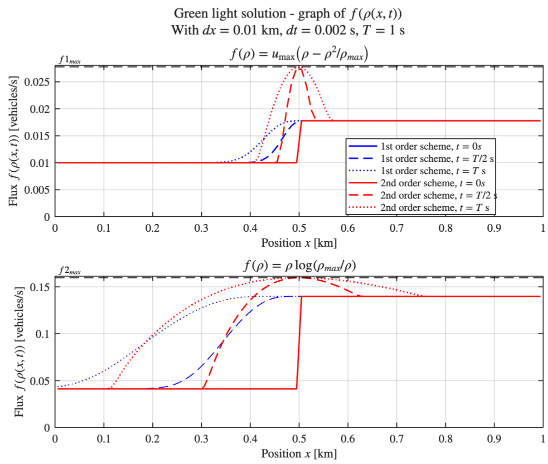
Traffic jam scenario : flux does get gradually decreased to 0 which is coherent since we expect the cars on the left part of the domain to progressively stop moving, starting from the left part of the discontinuity middle of the domain) to the left part of the domain.

Green-light scenario : flux starts increasing around the discontinuity location and is spread out in a polynomial way. This is coherent since we expect the cars to gain speed after $t = 0$. Both cars at the eft and right sides of the green light gets affected by this speed gain.

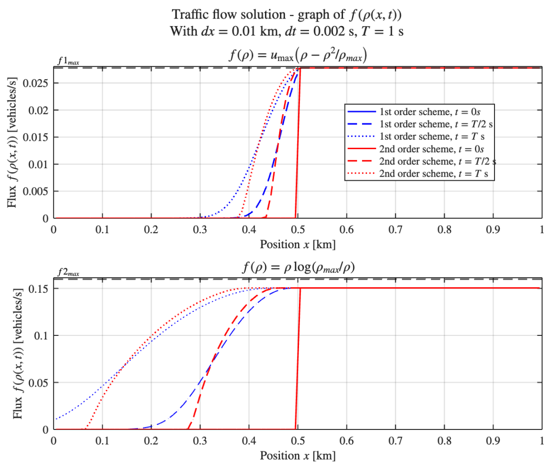
Traffic flow scenario : flux on the left part of the domain is gradually increased to match the one at the right side of the discontinuity. Indeed, in such case of traffic flow, we do expect flux to spread in the region where it is weaker.



(a) Traffic jam scenario



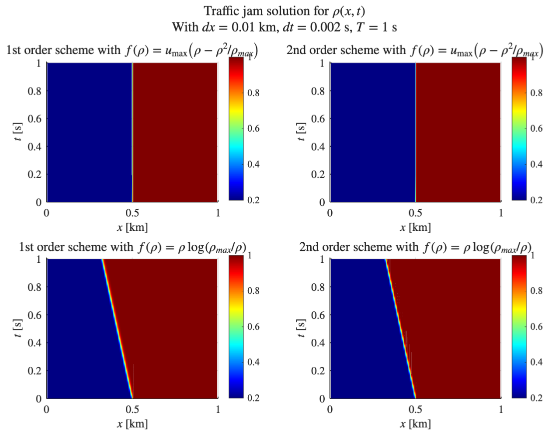
(b) Green light scenario



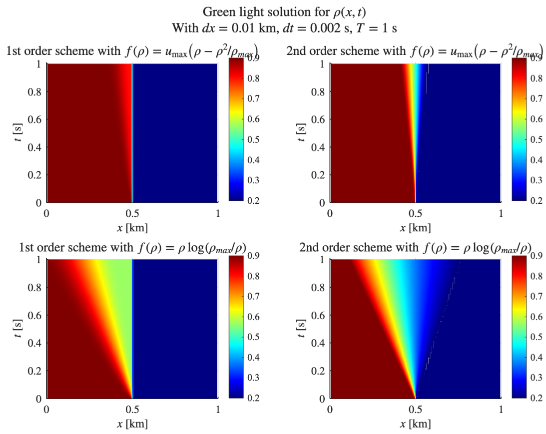
(c) Traffic flow scenario

Figure 3.2: Evolution of the flux for 3 discrete instants of time.

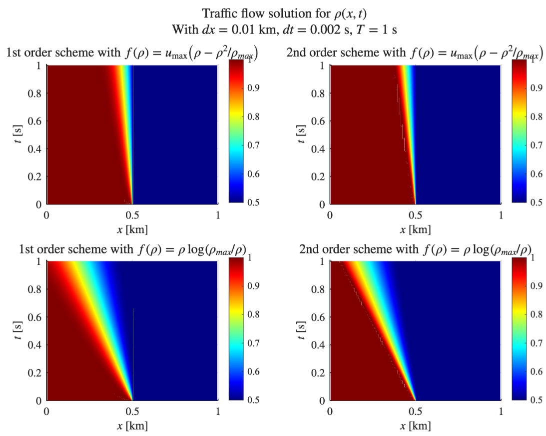
We provide below the heatmap of the density in the time-space domain to see more in details the progression it undergoes.
Once again, we do observe the sharpest transition effect of the second order scheme around the regions where the density reaches an extrema.



(a) Traffic jam scenario



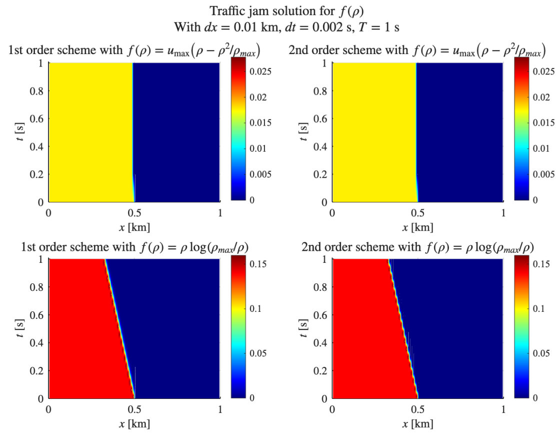
(b) Green light scenario



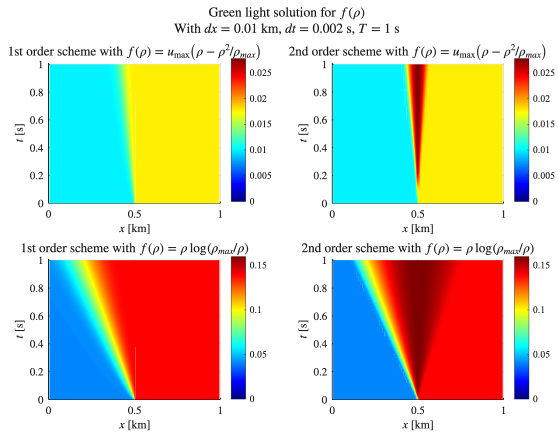
(c) Traffic flow scenario

Figure 3.3: Comparison of density evolution in different traffic scenarios.

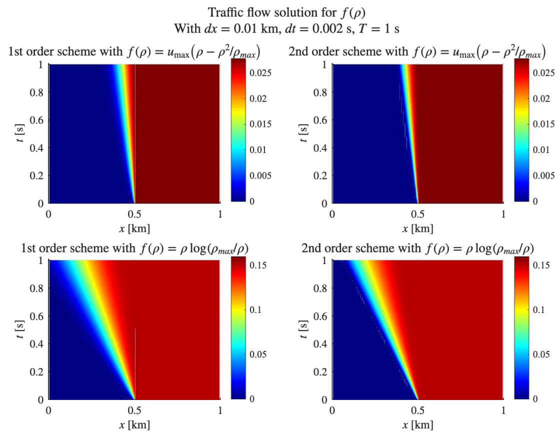
We provide below the heatmap of the flux in the time-space domain to see more in details the progression it undergoes.
The same comments made before still stand here.



(a) Traffic jam scenario



(b) Green light scenario



(c) Traffic flow scenario

Figure 3.4: Comparison of flux evolution in different traffic scenarios.

3.2 Animations

We provide below the animations of $\rho(x,t)$ obtained for $T = 5\mathrm{s}$.

We provide below the animations of $f(x,t)$ obtained for $T = 5\text{s}$.