On equiangular descent directions

Giulio Galvan

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We are interested in finding a direction $\boldsymbol{d}$ which makes equal angles with all the the vectors $\{\boldsymbol{g}_i\}_{i=1}^{N}$, which we suppose to have unary norm. This problem can be formulated as

$$\begin{aligned} \min_{c,\boldsymbol{d}} \quad & -c \\ \text{subject to} \quad & \boldsymbol{g}_i^T \boldsymbol{d} = c \quad i = 1, \ldots, N \\ & \boldsymbol{d}^T \boldsymbol{d} = 1. \end{aligned} \tag{1}$$

Note that c represents the cosine between the direction and the $\boldsymbol{g}_i$'s The Lagrangian of the problem above is

$$\mathcal{L}(c, \boldsymbol{d}, \boldsymbol{\lambda}, \mu) = -c + \boldsymbol{\lambda}^T (G\boldsymbol{d} - \mathbb{1}c) + \mu(\boldsymbol{d}^T \boldsymbol{d} - 1), \tag{2}$$

where

$$G \triangleq \begin{bmatrix} \boldsymbol{g}_1^T \\ \boldsymbol{g}_2^T \\ \vdots \\ \boldsymbol{g}_N^T \end{bmatrix} \tag{3}$$

We find the solution of 1 imposing

$$\nabla \mathcal{L}_c = -1 - \boldsymbol{\lambda}^T \mathbb{1} = 0 \tag{4}$$

$$\nabla \mathcal{L}_d = G^T \lambda + 2\mu \boldsymbol{d} \; = 0 \tag{5}$$

$$\nabla \mathcal{L}_\lambda = G\boldsymbol{d} - \mathbb{1}c = 0 \tag{6}$$

$$\nabla \mathcal{L}_\mu = \boldsymbol{d}^T \boldsymbol{d} - 1 = 0. \tag{7}$$

From (5) and 6 we get

$$GG^T \boldsymbol{\lambda} = -2\mu c \, \mathbb{1}, \tag{8}$$

hence, supposing $\boldsymbol{g}_i$'s linearly independent we write

$$\lambda = (-2uc)(GG^T)^{-1} \mathbb{1} \tag{9}$$

From 5 we have

$$\boldsymbol{\lambda}^T GG^T \boldsymbol{\lambda} = 4\mu^2, \tag{10}$$

which, combined with (8) and (4) yields

$$c = 2\mu \tag{11}$$

Again from (4) and (9) we get

$$\mathbb{1}^T \lambda = -2uc\, \mathbb{1}^T (GG^T)^{-1} \mathbb{1}, \tag{12}$$

hence

$$\mu \pm \frac{1}{2\sqrt{\mathbb{1}^T (GG^T)^{-1} \mathbb{1}}}. \tag{13}$$

Since we are interested in a positive value for c the solution is

$$c = \frac{1}{\sqrt{\mathbb{1}^T (GG^T)^{-1} \mathbb{1}}} \tag{14}$$

$$\boldsymbol{\lambda} = (-c^2)(GG^T)^{-1} \mathbb{1} \tag{15}$$

$$\boldsymbol{d} = -\frac{G^T \boldsymbol{\lambda}}{c} \tag{16}$$

$$\tag{17}$$

Solving using QR factorization It is sufficient to compute $\boldsymbol{b} = (GG^T)^{-1}\mathbb{1}$ to easily find all the other quantities of interest. We can find $\boldsymbol{b}$ decomposing G^T with the QR factorization

$$G^T = QR, \tag{18}$$

which leads to

$$R^T Q^T Q R b = R^T R \boldsymbol{b} = \mathbb{1}. \tag{19}$$

Hence, we first solve $R^T \boldsymbol{x} = \mathbb{1}$ w.r.t. $\boldsymbol{x}$ and then $R\boldsymbol{b} = \boldsymbol{x}$ w.r.t. $\boldsymbol{b}$

Thus, solving problem 1 amounts to the QR factorization of G and to solving two triangular linear systems.