

On equiangular descent directions

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We are interested in finding a direction \mathbf{d} which makes equal angles with all the the vectors $\{\mathbf{g}_i\}_{i=1}^N$, which we suppose to have unary norm. This problem can be formulated as

$$\begin{aligned} \min_{c, \mathbf{d}} \quad & -c \\ \text{subject to} \quad & \mathbf{g}_i^T \mathbf{d} = c \quad i = 1, \dots, N \\ & \mathbf{d}^T \mathbf{d} = 1. \end{aligned} \tag{1}$$

Note that c represents the cosine between the direction and the \mathbf{g}_i 's The Lagrangian of the problem above is

$$\mathcal{L}(c, \mathbf{d}, \boldsymbol{\lambda}, \mu) = -c + \boldsymbol{\lambda}^T (G\mathbf{d} - \mathbf{1}c) + \mu(\mathbf{d}^T \mathbf{d} - 1), \tag{2}$$

where

$$G \triangleq \begin{bmatrix} \mathbf{g}_1^T \\ \mathbf{g}_2^T \\ \vdots \\ \mathbf{g}_N^T \end{bmatrix} \tag{3}$$

We find the solution of 1 imposing

$$\nabla \mathcal{L}_c = -1 - \boldsymbol{\lambda}^T \mathbf{1} = 0 \tag{4}$$

$$\nabla \mathcal{L}_d = G^T \boldsymbol{\lambda} + 2\mu \mathbf{d} = 0 \tag{5}$$

$$\nabla \mathcal{L}_\lambda = G\mathbf{d} - \mathbf{1}c = 0 \tag{6}$$

$$\nabla \mathcal{L}_\mu = \mathbf{d}^T \mathbf{d} - 1 = 0. \tag{7}$$

From (5) and 6 we get

$$GG^T \boldsymbol{\lambda} = -2\mu c \mathbf{1}, \tag{8}$$

hence, supposing \mathbf{g}_i 's linearly independent we write

$$\boldsymbol{\lambda} = (-2\mu c)(GG^T)^{-1} \mathbf{1} \tag{9}$$

From 5 we have

$$\boldsymbol{\lambda}^T GG^T \boldsymbol{\lambda} = 4\mu^2, \tag{10}$$

which, combined with (8) and (4) yields

$$c = 2\mu \quad (11)$$

Again from (4) and (9) we get

$$\mathbf{1}^T \lambda = -2uc \mathbf{1} (GG^T)^{-1} \mathbf{1}, \quad (12)$$

hence

$$\mu \pm \frac{1}{2\sqrt{\mathbf{1}(GG^T)^{-1}\mathbf{1}}}. \quad (13)$$

Since we are interested in a positive value for c the solution is

$$c = \frac{1}{\sqrt{\mathbf{1}(GG^T)^{-1}\mathbf{1}}} \quad (14)$$

$$\lambda = (-c^2)(GG^T)^{-1} \mathbf{1} \quad (15)$$

$$d = -\frac{G^T \lambda}{c} \quad (16)$$

$$(17)$$

Solving using QR factorization It is sufficient to compute $\mathbf{b} = (GG^T)^{-1} \mathbf{1}$ to easily find all the other quantities of interest. We can find \mathbf{b} decomposing G^T with the QR factorization

$$G^T = QR, \quad (18)$$

which leads to

$$R^T Q^T Q R b = R^T R b = \mathbf{1}. \quad (19)$$

Hence, we first solve $R^T \mathbf{x} = \mathbf{1}$ w.r.t. \mathbf{x} and then $Rb = \mathbf{x}$ w.r.t. \mathbf{b}

Thus, solving problem 1 amounts to the QR factorization of G and to solving two triangular linear system.