On equiangular descent directions

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We are interested in finding a direction $\boldsymbol{d}$ which makes equal angles with all the the vectors $\{\boldsymbol{g}_i\}_{i=1}^{N}$, which we suppose to have unary norm. This problem can be formulated as

$$
\begin{aligned}
\min_{c,\boldsymbol{d}} \quad & -c \\
\text{subject to} \quad & \boldsymbol{g}_i^T \boldsymbol{d} = c \quad i = 1,\ldots,N \\
& \boldsymbol{d}^T \boldsymbol{d} = 1.
\end{aligned}
\tag{1}
$$

Note that c represents the cosine between the direction and the $\boldsymbol{g}_i$'s The Lagrangian of the problem above is

$$
\mathcal{L}(c,\boldsymbol{d},\boldsymbol{\lambda},\mu) = -c + \boldsymbol{\lambda}^T(G\boldsymbol{d} - \mathbb{1}c) + \mu(\boldsymbol{d}^T\boldsymbol{d} - 1),
\tag{2}
$$

where

$$
G \triangleq \begin{bmatrix} \boldsymbol{g}_1^T \\ \boldsymbol{g}_2^T \\ \vdots \\ \boldsymbol{g}_N^T \end{bmatrix}
\tag{3}
$$

We find the solution of (1) imposing

$$
\nabla \mathcal{L}_c = -1 - \boldsymbol{\lambda}^T \mathbb{1} = 0
\tag{4}
$$

$$
\nabla \mathcal{L}_d = G^T \lambda + 2\mu \boldsymbol{d} = 0
\tag{5}
$$

$$
\nabla \mathcal{L}_\lambda = G\boldsymbol{d} - \mathbb{1}c = 0
\tag{6}
$$

$$
\nabla \mathcal{L}_\mu = \boldsymbol{d}^T \boldsymbol{d} - 1 = 0.
\tag{7}
$$

From (5) and (6) we get

$$
GG^T \boldsymbol{\lambda} = -2\mu c \, \mathbb{1},
\tag{8}
$$

hence, supposing $\boldsymbol{g}_i$'s linearly independent we write

$$
\lambda = (-2uc)(GG^T)^{-1} \mathbb{1}
\tag{9}
$$

From (5) we have

$$
\boldsymbol{\lambda}^T GG^T \boldsymbol{\lambda} = 4\mu^2,
\tag{10}
$$

which, combined with (8) and (4) yields

$$c = 2\mu \tag{11}$$

Again from (4) and (9) we get

$$\mathbb{1}^T\lambda = -2uc\,\mathbb{1}^T(GG^T)^{-1}\mathbb{1}, \tag{12}$$

hence

$$\mu \pm \frac{1}{2\sqrt{\mathbb{1}^T(GG^T)^{-1}\mathbb{1}}}. \tag{13}$$

Since we are interested in a positive value for c the solution is

$$c = \frac{1}{\sqrt{\mathbb{1}^T(GG^T)^{-1}\mathbb{1}}} \tag{14}$$

$$\boldsymbol{\lambda} = (-c^2)(GG^T)^{-1}\mathbb{1} \tag{15}$$

$$\boldsymbol{d} = -\frac{G^T\boldsymbol{\lambda}}{c} \tag{16}$$

$$\tag{17}$$

Solving using QR factorization   It is sufficient to compute $\boldsymbol{b} = (GG^T)^{-1}\mathbb{1}$ to easily find all the other quantities of interest. We can find $\boldsymbol{b}$ decomposing G^T with the QR factorization

$$G^T = QR, \tag{18}$$

which leads to

$$R^TQ^TQRb = R^TR\boldsymbol{b} = \mathbb{1}. \tag{19}$$

Hence, we first solve $R^T\boldsymbol{x} = \mathbb{1}$ w.r.t. $\boldsymbol{x}$ and then $R\boldsymbol{b} = \boldsymbol{x}$ w.r.t. $\boldsymbol{b}$

Thus, solving problem (1) amounts to the QR factorization of G^T and to solving two triangular linear systems.