Stochastic gradient descent

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Consider the stochastic optimization problem

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} f(\boldsymbol{x}) = \mathbb{E}[\mathbb{F}(\boldsymbol{x},\boldsymbol{\xi})], \tag{1}$$

where $\boldsymbol{\xi} \in \Omega \subset \mathbb{R}^d$ is a random vector. Suppose $f(\cdot)$ is continuous, strongly convex (with constant c) and there exists a compact level set of $f(\cdot)$, hence (1) has a unique optimal solution $\boldsymbol{x}_*$. We make the following two assumptions:

- It is possible to generate independent identically distributed samples of $\boldsymbol{\xi}$
- There exists an oracle which, for a given point $(\boldsymbol{x},\boldsymbol{\xi})$ returns a stochastic direction $D(\boldsymbol{x},\boldsymbol{\xi})$ such that $d(\boldsymbol{x}) \triangleq \mathbb{E}[D(\boldsymbol{x},\boldsymbol{\xi})]$ satisfies:

$$-(\boldsymbol{x}-\boldsymbol{x}_*)^T(f'-g(\boldsymbol{x})) \geq -\mu\left\|\boldsymbol{x}-\boldsymbol{x}_*\right\|_2^2 \qquad \text{for some } f' \in \partial f(\boldsymbol{x}), \tag{2}$$

for some $\mu \in (0,c)$.

Consider the algorithm defined by

$$\boldsymbol{x}_{j+1} = \boldsymbol{x}_j - \gamma_j D(\boldsymbol{x}_j,\boldsymbol{\xi}_j). \tag{3}$$

Each iterate $\boldsymbol{x}_j$ of such random process is a function of the history $\boldsymbol{\xi}_{[j-1]} = (\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_{j-1})$

Let $A_j \triangleq \|\boldsymbol{x}_j-\boldsymbol{x}_*\|_2^2$ and $a_j \triangleq \mathbb{E}[A_j]$. From (3) we get

$$\begin{aligned} A_{j+1} &= \frac{1}{2}\left\|\boldsymbol{x}_j-\gamma_j D(\boldsymbol{x}_j,\boldsymbol{\xi}_j)-\boldsymbol{x}_*\right\|_2^2 \\ &= A_j + \frac{1}{2}\gamma_j^2\left\|D(\boldsymbol{x}_j,\boldsymbol{\xi}_j)\right\|_2^2 - \gamma_j(\boldsymbol{x}_j-\boldsymbol{x}_*)^T D(\boldsymbol{x}_j,\boldsymbol{\xi}_j). \end{aligned} \tag{4}$$

Since $\boldsymbol{x}_j = \boldsymbol{x}_j(\boldsymbol{\xi}_{[j-1]})$ is independent of $\boldsymbol{\xi}_j$ we have

$$\begin{aligned} \mathbb{E}_{\boldsymbol{\xi}_{[j]}}[(\boldsymbol{x}_j-\boldsymbol{x}_*)^T D(\boldsymbol{x}_j,\boldsymbol{\xi}_j)] &= \mathbb{E}_{\boldsymbol{\xi}_{[j-1]}}[\mathbb{E}_{\boldsymbol{\xi}_{[j]}}[(\boldsymbol{x}_j-\boldsymbol{x}_*)^T D(\boldsymbol{x}_j,\boldsymbol{\xi}_j)]|\boldsymbol{\xi}_{[j-1]}] \\ &= \mathbb{E}_{\boldsymbol{\xi}_{[j-1]}}[(\boldsymbol{x}_j-\boldsymbol{x}_*)^T\mathbb{E}\boldsymbol{\xi}_{[j]}[D(\boldsymbol{x}_j,\boldsymbol{\xi}_j)]|\boldsymbol{\xi}_{[j-1]}] \\ &= \mathbb{E}_{\boldsymbol{\xi}_{[j-1]}}[(\boldsymbol{x}_j-\boldsymbol{x}_*)^T d(\boldsymbol{x}_j)] \end{aligned} \tag{5}$$

Let now assume that there exists $M > 0$ such that

$$\mathbb{E}[\|D(\boldsymbol{x}, \boldsymbol{\xi})\|_2^2] \leq M^2 \quad \forall \boldsymbol{x} \in \mathbb{R}^n. \tag{6}$$

Using (5) and (6) we obtain, taking expectation of both sides of (4)

$$a_{j+1} \leq a_j - \gamma_j \mathbb{E}_{\boldsymbol{\xi}_{[j-1]}}[(\boldsymbol{x}_j - \boldsymbol{x}_*)^T d(\boldsymbol{x}_j)] + \frac{1}{2}\gamma_j^2 M^2 \tag{7}$$

Since $f(\cdot)$ is strongly convex there exists $c > 0$ such that

$$(\boldsymbol{x} - \boldsymbol{y})^T (f' - g')) \geq c \|\boldsymbol{x} - \boldsymbol{y}\|_2^2, \quad \forall f' \in \partial f(\boldsymbol{x}), g' \in \partial f(\boldsymbol{y}) \tag{8}$$

By optimality of $\boldsymbol{x}_*$ we have

$$(\boldsymbol{x} - \boldsymbol{x}_*)^T f' \geq 0 \quad \forall \boldsymbol{x} \in \mathbb{R}^n, \forall f' \in \partial f(\boldsymbol{x}_*). \tag{9}$$

Inequalities (9) and (8) implies

$$(\boldsymbol{x} - \boldsymbol{x}_*)^T f' \geq c \|\boldsymbol{x} - \boldsymbol{x}_*\|_2^2 \quad \forall \boldsymbol{x} \in \mathbb{R}^n, \forall f' \in \partial f(\boldsymbol{x}). \tag{10}$$

Adding and subtracting the oracle direction $g(\boldsymbol{x})$ we get

$$(\boldsymbol{x} - \boldsymbol{x}_*)^T (f' - g(\boldsymbol{x}) + g(\boldsymbol{x})) \geq c \|\boldsymbol{x} - \boldsymbol{x}_*\|_2^2, \tag{11}$$

which can be rewritten as

$$(\boldsymbol{x} - \boldsymbol{x}_*)^T g(\boldsymbol{x}) \geq c \|\boldsymbol{x} - \boldsymbol{x}_*\|_2^2 - (\boldsymbol{x} - \boldsymbol{x}_*)^T (f' - g(\boldsymbol{x})) \tag{12}$$

From assumption (2), taking expectations of both side of (12) we obtain

$$\begin{aligned}
\mathbb{E}[(\boldsymbol{x}_j - \boldsymbol{x}_*)^T g(\boldsymbol{x}_j)] &\geq c\mathbb{E}[\|\boldsymbol{x}_j - \boldsymbol{x}_*\|_2^2)] - \mathbb{E}[(\boldsymbol{x}_j - \boldsymbol{x}_*)^T (f_j' - g(\boldsymbol{x}_j))] && (13)\\
&\geq c(1 - \frac{\mu}{c})\mathbb{E}[\|\boldsymbol{x}_j - \boldsymbol{x}_*\|_2^2)] && (14)\\
&= 2\bar{c} a_j, && (15)
\end{aligned}$$

with $\bar{c} = c(1 - \frac{\mu}{c})$ and $f_j' \in \partial f(\boldsymbol{x}_j)$. Hence from (7) follows

$$a_{j+1} \leq (1 - 2\bar{c}\gamma_j) a_j + \frac{1}{2}\gamma_j^2 M^2. \tag{16}$$

Choosing the stepsizes as $\gamma_j = \frac{\beta}{j}$ for some constant $\beta > \frac{1}{2\bar{c}}$ we get

$$a_{j+1} \leq (1 - 2\bar{c}\gamma_j) a_j + \frac{1}{2}\frac{\beta^2 M^2}{j^2}. \tag{17}$$

It follows by induction that

$$\mathbb{E}[\|\boldsymbol{x}_j - \boldsymbol{x}_*\|_2^2] = 2a_j \leq \frac{Q(\beta)}{j}, \tag{18}$$

where

$$Q(\beta) = max \left\{ \frac{\beta^2 M^2}{2\bar{c} - 1}, \|\boldsymbol{x}_1 - \boldsymbol{x}_*\|_2^2 \right\}. \tag{19}$$

When ∇f is Lipschitz continuous we also have

$$f(\boldsymbol{x}) \leq f(\boldsymbol{x}_*) + \frac{1}{2} L \|\boldsymbol{x} - \boldsymbol{x}_*\|_2^2, \quad \forall \boldsymbol{x} \in \mathbb{R}^n, \tag{20}$$

hence we can get a bound also on the function value:

$$\mathbb{E}[f(\boldsymbol{x}_j) - f(\boldsymbol{x}_*)] \leq \frac{1}{2} L \mathbb{E}[\|\boldsymbol{x}_j - \boldsymbol{x}_*\|_2^2] \leq \frac{1}{2} L Q(\beta) \tag{21}$$

Sufficient stochastic direction condition Assumption 2 can be further elaborated. Let θ be the angle between $f' \in \partial f(\boldsymbol{x})$ and $g(\boldsymbol{x})$. Write $\|g(\boldsymbol{x}_j)\| = \alpha \|\nabla f(\boldsymbol{x}_j)\|$ for some $\alpha > 0$, then

$$\|f' - g(\boldsymbol{x}_j)\|^2 = \|f'\|^2 + \|g(\boldsymbol{x}_j)\|^2 - 2\|f'\| \|g(\boldsymbol{x}_j)\| \cos\theta_j \tag{22}$$

$$= \|f'\|^2 (1 + \alpha_j^2 - 2\alpha_j \cos\theta_j). \tag{23}$$

Hence

$$(\boldsymbol{x} - \boldsymbol{x}_*)^T (\nabla f(\boldsymbol{x}) - g(\boldsymbol{x})) \leq \|\boldsymbol{x} - \boldsymbol{x}_*\| \|\nabla f(\boldsymbol{x}) - g(\boldsymbol{x})\| \tag{24}$$

$$= \|\boldsymbol{x} - \boldsymbol{x}_*\| \|f'\| (1 + \alpha_j^2 - 2\alpha_j \cos\theta_j)^{\frac{1}{2}} \tag{25}$$

A sufficient condition is thus

$$\|f'\| (1 + \alpha_j^2 - 2\alpha_j \cos\theta_j)^{\frac{1}{2}} \leq \mu \|\boldsymbol{x} - \boldsymbol{x}_*\| \tag{26}$$

In the smooth case, since $\nabla f(\boldsymbol{x}_*) = 0$, using Lipschitz continuity of ∇f (with constant L) we get

$$\|\nabla f(\boldsymbol{x}_j) - g(\boldsymbol{x})\| \leq L \|\boldsymbol{x} - \boldsymbol{x}_*\| (1 + \alpha^2 - 2\alpha\cos\theta)^{\frac{1}{2}} \tag{27}$$

Hence a sufficient condition for assumption 2 to hold is

$$1 + \alpha^2 - 2\alpha\cos\theta_j \leq (\frac{\mu}{L})^2 \tag{28}$$