Seminar talk 8.

Definition and examples of the DeConcini-Procesi wonderful model of the complement of a hyperplane arrangement

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15/06/2023

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1 Introduction

The main goal of this talk, is to give the definition of the De Concini-Procesi wonderful model of the complement of a hyperplane arrangement and to give some illustrative examples which highlight the main properties of the model. The most important references are [Fei05] and [DCP95]. This is what we are going to do today and in the future talks. In this talk we are going to introduce some smooth variety called the wonderful model associated to a hyperplane arrangement. Then, in the next talks (Talks 9 and 10) we are going to study a projective version of this model (same model but with the arrangement seen in a projective space) and we are going to find out that the cohomology of this projective version will only depend on the underlying matroid. This will suggest a definition of the Chow ring of a matroid. Finally, we are going to use this to prove the Rota-Heron-Welsh log-concavity conjecture from Talk 7 (Talks 11 and 12).

We start by recalling in which setting we are working. Let $\mathcal{A} := \{U_1, \dots, U_n\}$ be a complex hyperplane arrangement in a complex vector space V, and consider the topological space $\mathcal{M}(\mathcal{A}) := V \setminus \bigcup \mathcal{A}$. While having arrangement in a real vector space, is not interesting from a topological point of view, in the complex case we have already pretty interesting example in $\mathbb{C}^1$, where removing the origin, gives us a space homotopic to S^1. Another interesting example is $\mathbb{C}^2$ without the two coordinate hyperplanes, that is homotopy equivalent to $S^1 \times S^1$.

We have a combinatorial data associated with an arrangement, the intersection lattice $\mathcal{L} = \mathcal{L}(\mathcal{A})$. As with any poset, we can consider the order complex $\Delta(\overline{\mathcal{L}})$ of the proper part, $\overline{\mathcal{L}} := \mathcal{L} \setminus \{\hat{0}, \hat{1}\}$[1], of the intersection lattice, i.e., the abstract simplicial complex formed by the linearly ordered subsets in $\overline{\mathcal{L}}$,

$$\Delta(\overline{\mathcal{L}}) = \{X_{i_1} < \cdots < X_{i_k} \mid X_{i_j} \in \overline{\mathcal{L}}\}$$

[1] Here, we use the usual notation from the theory of partially ordered sets. We denote the unique minimum in $\mathcal{L}$ corresponding to the empty intersection, (i.e. the ambient space V) by $\hat{0}$ and the unique maximum of $\mathcal{L}$ (the overall intersection of subspaces in $\mathcal{A}$) by $\hat{1}$.

Besides $\Delta(\overline{\mathcal{L}})$, we will often refer to **the cone over** $\Delta(\overline{\mathcal{L}})$ obtained by extending the linearly ordered sets in $\mathcal{L}$ by the maximal element $\hat{1}$ in $\mathcal{L}$. We will denote this complex by $\Delta(\mathcal{L} \setminus \hat{0})$ or $\Delta(\mathcal{L}_{>0})$.

Example 1.1. Easy example of lattices and Δ.

Remark 1.2. For hyperplane arrangements, the homotopy type of $\Delta(\overline{\mathcal{L}})$ is well known: the complex is homotopy equivalent to a wedge of spheres of dimension equal to the codimension of the total intersection of $\mathcal{A}$ minus 2. The number of spheres can as well be read from the intersection lattice since it is the absolute value of its Mobius function.[2]

In order to have a standard example at hand, we briefly discuss braid arrangements. This class of arrangements has figured prominently in many places and has helped develop lots of arrangement theory over the last decades.

Example 1.3 (Braid arrangements). From now on, when we write K, we mean $\mathbb{R}$ or $\mathbb{C}$. Consider the complex arrangement $\mathcal{A}_{n-1}^{K}$ in an n-dimensional space over K, given by the hyperplanes

$$H_{i,j} : x_i = x_j, \text{ for } 1 \leq i < j \leq n.$$

We observe that the diagonal $d := \{x \in K^n \mid x_1 = \cdots = x_n\}$ is the intersection of all the hyperplanes in the arrangement. So we can consider, without loss of information, $\mathcal{A}_{n-1}^{K}$ to be the subset of an $n-1$-dimensional vector space $V/d \cong \{x \in K^n \mid \sum x_i = 0\}$ (this explains why the index is $n-1$ instead of n).

We have that $\mathcal{M}(\mathcal{A}_{n-1}^{\mathbb{C}})$ is the space usually denote

$$F(n, \mathbb{C}) := \{x \in \mathbb{C}^n \mid x_i \neq x_j \text{ if } i \neq j\},$$

and this is the classifying space of the braid group (and this explains the name of the arrangements).

We can identify the lattice of this hyperplane arrangement, with the *partition lattice* Π_n[3], given by the partitions of n ordered by reverse refinement. The correspondence can be easily described: the blocks of a partition correspond to sets of coordinates with identical entries, thus to the set of points in the corresponding intersection of hyperplanes.

Now, some picture. Picture of the arrangement for $n = 2$ and $K = \mathbb{R}$, picture of the poset and bijection with Π_3. Picture of $\Delta(\overline{\mathcal{L}})$. Check that mobius function is 2 and codimension of the total intersection minus 2 is 2.

Example 1.4. Other example. Boo_3.

2 The definition of the model

We are going to give 2 definition of the model. The first one is usually usefull for technical purposes, the second one is (or should be) more intuitive:

Definition 2.1. Let $\mathcal{A}$ be an arrengement of real or complex linear subspace in V a vector space. Consider the map

$$\Psi : \mathcal{M}(\mathcal{A}) \to V \times \prod_{X \in \mathcal{L}_{>0}} \mathbb{P}(V/X),$$

$$x \to (x, ([x + X])_{x \in \mathcal{L}_{>0}}).$$

[2]Reminder on what is the Mobius function: the Mobius function $\mu_{:} L \to \mathbb{Z}$ (with L a poset) is defined as $\mu(x) = 1$ if $x = \hat{0}$ and $\mu(x) = -\sum_{y<x>} \mu(y)$ otherwise.

[3]Notice that we can identify $\mathcal{A}_{n-1}^{\mathbb{R}}$ with the lattice of flats of $\mathcal{M}(K_n)$.

This encodes the relative position of each point in the arrangement complement $\mathcal{M}(\mathcal{A})$ with respect to the intersection of hyperplane in $\mathcal{A}$. The map Ψ is an open embedding; the closure of its image is called the (maximal) **De Concini–Procesi wonderful model** for $\mathcal{A}$ and is denoted by $Y_{\mathcal{A}}$.

Definition 2.2. Let $\mathcal{A}$ be an arrangement of real or complex linear subspace in V. We extend the opposite order $\mathcal{L}_{>0}^{op}$ on $\mathcal{L}_{>0}$ to a total order $X_1 < \cdots < X_t$. The (maximal) **De Concini–Procesi wonderful model** for $\mathcal{A}$ is the result $Y_{\mathcal{A}}$ of successively blowing up subspaces $X_1, \ldots, X_t$, respectively their proper transforms.

Remark 2.3. We notice that what happens in codimension one, is never relevant in both definition.

References

[DCP95] Corrado De Concini and Claudio Procesi. Wonderful models of subspace arrangements. *Selecta Mathematica*, 1:459–494, 1995.

[Fei05] Eva Maria Feichtner. De concini-procesi wonderful arrangement. *Combinatorial and computational geometry*, 52:333, 2005.