Controllability, observability and Noninteracting control of the Cubli

Corso di LM in Ingegneria Robotica ed Automazione
Controllo dei Robot

Students:
Nicola Piga
Giulio Romualdi

Supervisors:
Prof. Antonio Bicchi
Ing. Manuel Bonilla
Ing. Manolo Garabini

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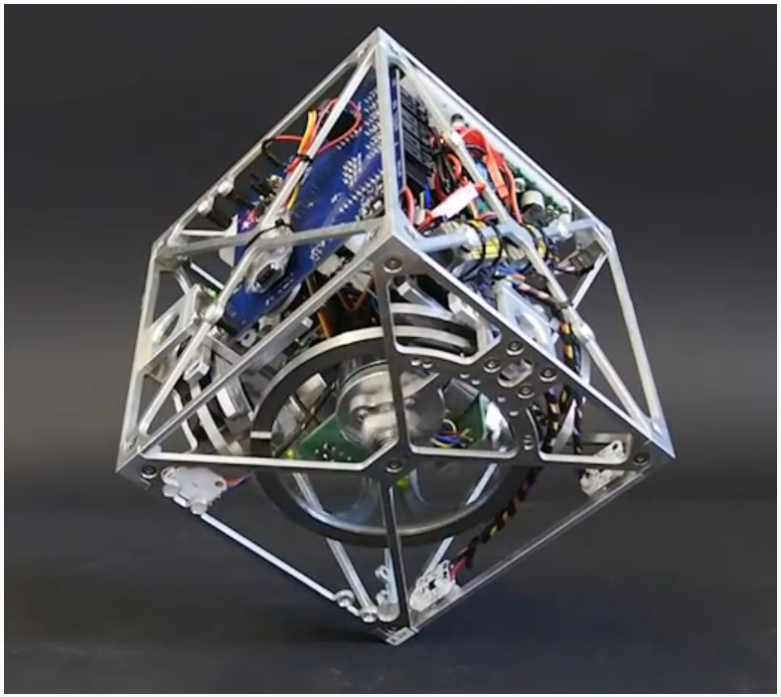
Università di Pisa

Introduction

Brief description of the system



The Cubli is a reaction wheel based 3D inverted pendulum consisting of a 15 cm sided cube with flywheels mounted on three faces.

Contents

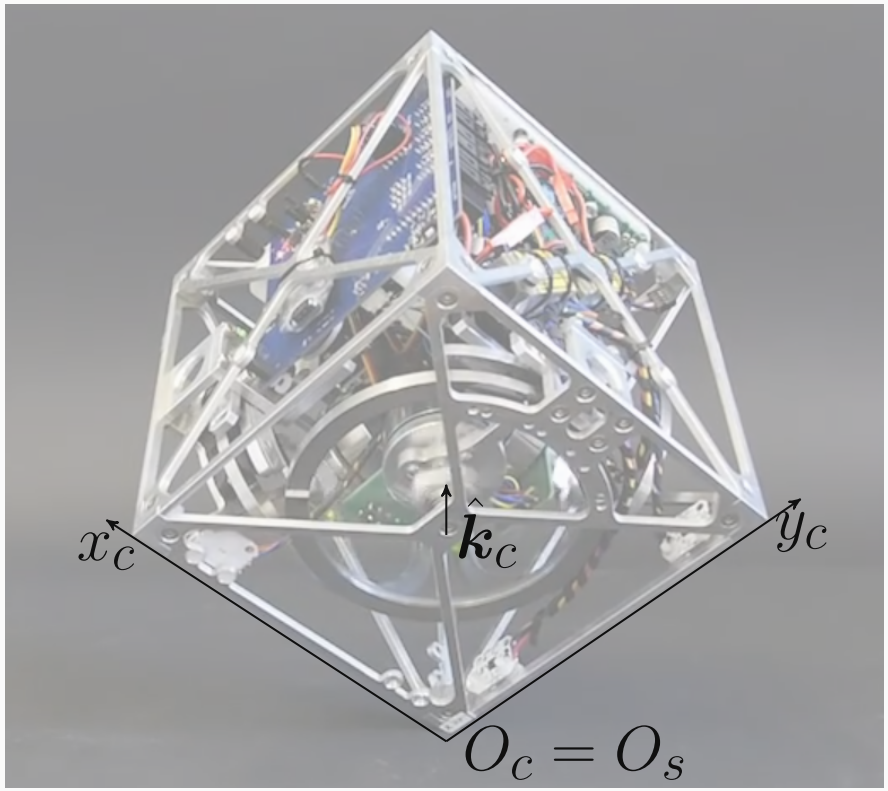
The contents presented are

- equation of motions of the system in standard manipulator form and control-affine form
- controllabilty of the nonlinear system
- observability of the nonlinear system
- noninteracting control and results of simulations

Equations of motion

Configuration vector

$$\boldsymbol{q} = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & q_x & q_y & q_z \end{bmatrix}^{\mathrm{T}}$$



Attitude

- Body fixed reference frame $\{C\} = (O_c; \hat{\boldsymbol{i}}_c, \hat{\boldsymbol{j}}_c, \hat{\boldsymbol{k}}_c)$
- Inertial reference frame $\{S\} = (O_s; \hat{\boldsymbol{i}}_s, \hat{\boldsymbol{j}}_s, \hat{\boldsymbol{k}}_s)$
- $R_{SC}(\boldsymbol{\theta}) = R_z(\theta_1)R_y(\theta_2)R_x(\theta_3)$

Flywheels

q_x, q_y and q_z are the flywheel angular positions

Notation [1]

In the follwing this notation will be used

Velocities

- ${}^{c}\boldsymbol{v}_{G_c}$: cubic frame CoM linear velocity
- ${}^{c}\boldsymbol{v}_{G_i}$: i-th flywheel CoM linear velocity
- ${}^{c}\boldsymbol{\omega}_{c}$: cubic frame angular velocity
- ${}^{c}\boldsymbol{\omega}_{i}$: i-th flywheel angular velocity

Inertial quantities

- m_c: cubic frame mass
- m_i: i-th flywheel mass
- ${}^{c}\boldsymbol{G}_{c}$: cubic frame CoM
- ${}^{c}\boldsymbol{G}_{i}$: i-th flywheel CoM
- ${}^{c}\mathfrak{J}_{G_c}$: cubic frame inertia matrix
- ${}^{c}\mathfrak{J}_{G_i}$: i-th flywheel inertia matrix

[1] All quantities expressed in $\{C\}$

Inertia matrices and parameters

Inertia matrices

$$
{}^{c}\mathfrak{J}_{G_x} = \begin{bmatrix} \frac{mr^2}{2} & 0 & 0 \\ 0 & \frac{m\left(h^2+3r^2\right)}{12} & 0 \\ 0 & 0 & \frac{m\left(h^2+3r^2\right)}{12} \end{bmatrix}
$$

$$
{}^{c}\mathfrak{J}_{G_y} = \begin{bmatrix} \frac{m\left(h^2+3r^2\right)}{12} & 0 & 0 \\ 0 & \frac{mr^2}{2} & 0 \\ 0 & 0 & \frac{m\left(h^2+3r^2\right)}{12} \end{bmatrix}
$$

$$
{}^{c}\mathfrak{J}_{G_z} = \begin{bmatrix} \frac{m\left(h^2+3r^2\right)}{12} & 0 & 0 \\ 0 & \frac{m\left(h^2+3r^2\right)}{12} & 0 \\ 0 & 0 & \frac{mr^2}{2} \end{bmatrix}
$$

$$
{}^{c}\mathfrak{J}_{G_c} = \begin{bmatrix} \frac{m_c a^2}{6} & 0 & 0 \\ 0 & \frac{m_c a^2}{6} & 0 \\ 0 & 0 & \frac{m_c a^2}{6} \end{bmatrix}
$$

Parameters

- $M = 2.5\,\mathrm{kg}$ (cubic frame mass)
- $a = 0.15\,\mathrm{m}$ (cube side)
- $m = 0.204\,\mathrm{kg}$ (flywheel mass)
- $r = 0.05\,\mathrm{m}$ (flywheel radius)
- $h = 0.005\,\mathrm{m}$ (flywheel height)

Kinematics

$$
\begin{aligned}
{}^c\boldsymbol{\omega}_c &= \begin{bmatrix} {}^c\tilde{J}_\omega & 0_{3\times 3} \end{bmatrix} \dot{\boldsymbol{q}} = {}^c J_\omega \dot{\boldsymbol{q}} \\
{}^c\boldsymbol{\omega}_x &= \begin{bmatrix} {}^c\tilde{J}_\omega & \boldsymbol{e}_1 & \mathbf{0} & \mathbf{0} \end{bmatrix} \dot{\boldsymbol{q}} = {}^c J_{\omega_x} \dot{\boldsymbol{q}} \\
{}^c\boldsymbol{\omega}_y &= \begin{bmatrix} {}^c\tilde{J}_\omega & \mathbf{0} & \boldsymbol{e}_2 & \mathbf{0} \end{bmatrix} \dot{\boldsymbol{q}} = {}^c J_{\omega_y} \dot{\boldsymbol{q}} \\
{}^c\boldsymbol{\omega}_z &= \begin{bmatrix} {}^c\tilde{J}_\omega & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_3 \end{bmatrix} \dot{\boldsymbol{q}} = {}^c J_{\omega_z} \dot{\boldsymbol{q}}
\end{aligned}
$$

$$
{}^c\tilde{J}_\omega(\boldsymbol{q}) = \begin{bmatrix} -s_2 & 0 & 1 \\ c_2 s_3 & c_3 & 0 \\ c_2 c_3 & -s_3 & 0 \end{bmatrix}
$$

$$
{}^c\boldsymbol{v}_{G_i} = \cancelto{0}{{}^c\boldsymbol{v}_{O_c}}{}^{2} + {}^c\hat{\boldsymbol{\omega}}_c\, {}^c\boldsymbol{G}_* = -\,{}^c\hat{\boldsymbol{G}}_*\, {}^c\boldsymbol{\omega}_c = -\,{}^c\hat{\boldsymbol{G}}_*\, {}^c J_\omega \dot{\boldsymbol{\theta}} = -\,{}^c\hat{\boldsymbol{G}}_*\, {}^c\tilde{J}_\omega \dot{\boldsymbol{q}}
$$

$$
{}^c\boldsymbol{G}_c = \begin{bmatrix} a/2 \\ a/2 \\ a/2 \end{bmatrix} \quad {}^c\boldsymbol{G}_x = \begin{bmatrix} 0 \\ a/2 \\ a/2 \end{bmatrix} \quad {}^c\boldsymbol{G}_y = \begin{bmatrix} a/2 \\ 0 \\ a/2 \end{bmatrix} \quad {}^c\boldsymbol{G}_z = \begin{bmatrix} a/2 \\ a/2 \\ 0 \end{bmatrix}
$$

[2] Spherical joint in $O_c = O_s$

Equation in standard manipulator form

$$B(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{Q}$$

Color code

Cubic frame Fly wheel x Fly wheel y Fly wheel z

Joint space inertia matrix $B(\boldsymbol{q})$

$$\begin{aligned} B(\boldsymbol{q}) &= m_c\, {}^c J_\omega^{\mathrm{T}}\, {}^c\hat{\boldsymbol{G}}_c^{\mathrm{T}}\, {}^c\hat{\boldsymbol{G}}_c\, {}^c J_\omega + {}^c J_\omega^{\mathrm{T}}\, {}^c\tilde{\mathfrak{J}}_{G_c}\, {}^c J_\omega + m_x\, {}^c J_\omega^{\mathrm{T}}\, {}^c\hat{\boldsymbol{G}}_x^{\mathrm{T}}\, {}^c\hat{\boldsymbol{G}}_x\, {}^c J_\omega + {}^c J_{\omega_x}^{\mathrm{T}}\, {}^c\tilde{\mathfrak{J}}_{G_x}\, {}^c J_{\omega_x} \\ &\quad + m_y\, {}^c J_\omega^{\mathrm{T}}\, {}^c\hat{\boldsymbol{G}}_y^{\mathrm{T}}\, {}^c\hat{\boldsymbol{G}}_y\, {}^c J_\omega + {}^c J_{\omega_y}^{\mathrm{T}}\, {}^c\tilde{\mathfrak{J}}_{G_y}\, {}^c J_{\omega_y} + m_z\, {}^c J_\omega^{\mathrm{T}}\, {}^c\hat{\boldsymbol{G}}_z^{\mathrm{T}}\, {}^c\hat{\boldsymbol{G}}_z\, {}^c J_\omega + {}^c J_{\omega_z}^{\mathrm{T}}\, {}^c\tilde{\mathfrak{J}}_{G_z}\, {}^c J_{\omega_z} \\ &= B(\theta_2, \theta_3) \end{aligned}$$

Coriolis $C(\boldsymbol{q}, \dot{\boldsymbol{q}})$

$$C_{ij} = \sum_{k=1}^{6} \Gamma_{jk}^i \dot{q}_k$$

$$\Gamma_{jk}^i = \frac{1}{2}\left(\frac{\partial B_{ij}}{\partial q_k} + \frac{\partial B_{ik}}{\partial q_j} - \frac{\partial B_{jk}}{\partial q_i}\right)$$

Gravitational forces $G(\boldsymbol{q})$

$$\boldsymbol{G} = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^{\mathrm{T}}$$

$$\begin{aligned} U(\boldsymbol{q}) = -\, {}^c\boldsymbol{g}^{\mathrm{T}}(&m_c\, {}^c\boldsymbol{G}_c + m_x\, {}^c\boldsymbol{G}_x \\ &+ m_y\, {}^c\boldsymbol{G}_y + m_z\, {}^c\boldsymbol{G}_z) \end{aligned}$$

$${}^c\boldsymbol{g} = R_{SC}^{\mathrm{T}}\, {}^s\boldsymbol{g} = R_{SC}^{\mathrm{T}} \begin{bmatrix} 0 & 0 & -g \end{bmatrix}^{\mathrm{T}}$$

Derivation of the generalized forces Q

$\boldsymbol{\tau} = \begin{bmatrix} \tau_x & \tau_y & \tau_z \end{bmatrix}^{\mathrm{T}}$ are the torques applied to the flywheels

$$
\begin{aligned}
Q_i = & \tau_x \hat{\boldsymbol{i}}_c^T \frac{\partial({}^c\boldsymbol{\omega}_x)}{\partial \dot{q}_i} + \tau_y \hat{\boldsymbol{j}}_c^T \frac{\partial({}^c\boldsymbol{\omega}_y)}{\partial \dot{q}_i} + \tau_z \hat{\boldsymbol{k}}_c^T \frac{\partial({}^c\boldsymbol{\omega}_z)}{\partial \dot{q}_i} - \\
& \tau_x \hat{\boldsymbol{i}}_c^T \frac{\partial({}^c\boldsymbol{\omega}_c)}{\partial \dot{q}_i} - \tau_y \hat{\boldsymbol{j}}_c^T \frac{\partial({}^c\boldsymbol{\omega}_c)}{\partial \dot{q}_i} - \tau_z \hat{\boldsymbol{k}}_c^T \frac{\partial({}^c\boldsymbol{\omega}_c)}{\partial \dot{q}_i}
\end{aligned}
$$

$$
\boldsymbol{Q} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \boldsymbol{\tau} \end{bmatrix} = F\boldsymbol{\tau} = \begin{bmatrix} 0_{3\times 3} \\ I_{3\times 3} \end{bmatrix} \boldsymbol{\tau}
$$

$\mathrm{rank}(F) = 3 < 6 \quad \forall \boldsymbol{q} \implies$ underactuated system

Equations in control affine form

State space vector

$$\tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{\boldsymbol{x}}_1^{\mathrm{T}} & \tilde{\boldsymbol{x}}_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

$$\tilde{\boldsymbol{x}}_1 = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & q_x & q_y & q_z \end{bmatrix} \quad \tilde{\boldsymbol{x}}_2 = \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 & \dot{q}_x & \dot{q}_y & \dot{q}_z \end{bmatrix}$$

$$\begin{aligned} \dot{\tilde{\boldsymbol{x}}} &= \begin{bmatrix} \tilde{\boldsymbol{x}}_2 \\ -B(\tilde{\boldsymbol{x}}_1)^{-1}(C(\tilde{\boldsymbol{x}}_1, \tilde{\boldsymbol{x}}_2)\tilde{\boldsymbol{x}}_2 + \boldsymbol{G}(\tilde{\boldsymbol{x}}_1)) \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ B(\tilde{\boldsymbol{x}}_1)^{-1}\boldsymbol{Q} \end{bmatrix} \\ &= \begin{bmatrix} \tilde{\boldsymbol{x}}_2 \\ -B^{-1}(C\tilde{\boldsymbol{x}}_2 + \boldsymbol{G}) \end{bmatrix} + \begin{bmatrix} 0_{6\times 3} \\ \left(B(\tilde{\boldsymbol{x}}_1)^{-1}\right)_{(:,4:6)} \end{bmatrix} \boldsymbol{\tau} \\ &= \tilde{\boldsymbol{f}}(\tilde{\boldsymbol{x}}) + \tilde{g}(\tilde{\boldsymbol{x}})\boldsymbol{\tau} = \tilde{\boldsymbol{f}}(\tilde{\boldsymbol{x}}) + \tilde{g}(\tilde{\boldsymbol{x}})\boldsymbol{u} \end{aligned}$$

Equations in control affine form (continued)

B, C and $\boldsymbol{G}$ do not depend on q_x, q_y and q_z and their dynamics can be neglected

$$\boldsymbol{x} = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 & \dot{q}_x & \dot{q}_y & \dot{q}_z \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{\theta} & \dot{\boldsymbol{\theta}} & \dot{\boldsymbol{q}}_{wheels} \end{bmatrix}^{\mathrm{T}}$$

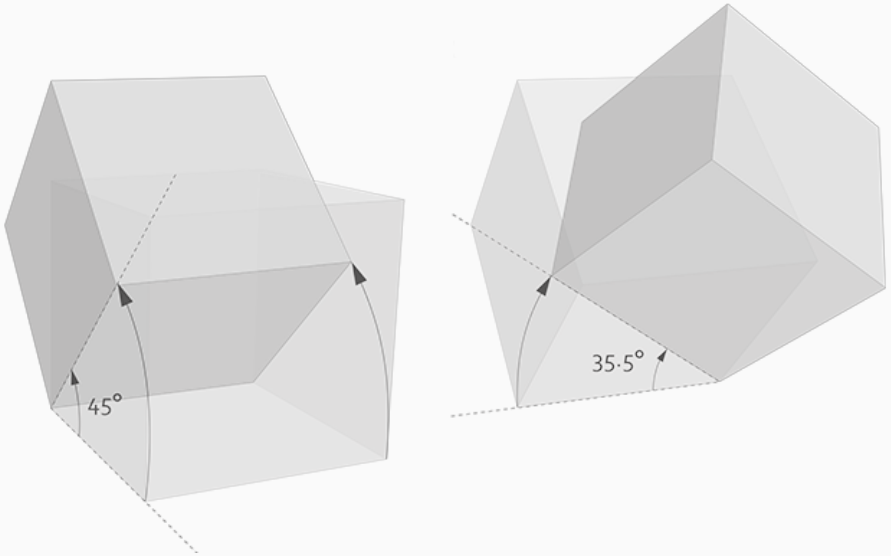
$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^{\mathrm{T}} \end{cases}$$

where

$$g(\boldsymbol{x}) = \left(B(\theta_2, \theta_3)^{-1}\right)_{(:,4:6)}$$

and $\boldsymbol{f}(\boldsymbol{x})$ do not depend on θ_1

Equlibrium points



Upright equilibria (unstable)

$$\mathcal{E}_1 = \left\{ \boldsymbol{x} \mid \theta_1 \in [-\pi, \pi),\ \ \theta_2 = -\mathrm{atan}\frac{\sqrt{2}}{2},\ \ \theta_3 = \frac{\pi}{4},\ \ \dot{\boldsymbol{\theta}} = \mathbf{0},\ \ \dot{\boldsymbol{q}}_{wheels} = \mathbf{0},\ \ \boldsymbol{u} = \mathbf{0} \right\}$$

Hanging equilibria (stable)

$$\mathcal{E}_2 = \left\{ \boldsymbol{x} \mid \theta_1 \in [-\pi, \pi),\ \ \theta_2 = \mathrm{atan}\frac{\sqrt{2}}{2},\ \ \theta_3 = -\frac{3}{4}\pi,\ \ \dot{\boldsymbol{\theta}} = \mathbf{0},\ \ \dot{\boldsymbol{q}}_{wheels} = \mathbf{0},\ \ \boldsymbol{u} = \mathbf{0} \right\}$$

Controllability

Small time local controllability

Given a point $\boldsymbol{x}_0$ and an arbitrarily small neighbourhood $V(\boldsymbol{x}_0)$ define the set

$$R_T^V(\boldsymbol{x}_0) = \{\boldsymbol{x}(\boldsymbol{x_0}, T, \bar{\boldsymbol{u}}(t)) \mid \boldsymbol{x}(\boldsymbol{x_0}, \tau, \bar{\boldsymbol{u}}(t)) \in V(\boldsymbol{x}_0) \ \ \forall \tau \in [0, T]\}$$

with $\bar{\boldsymbol{u}}$ appropriate inputs

A nonlinear system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ is said locally-locally controllable (l.l.c.) from $\boldsymbol{x}_0$ if

$$\forall \ V(\boldsymbol{x}_0) \ \exists \ T \ \ s.t. \ R_T^V(\boldsymbol{x}_0) \ \supseteq \ B(\boldsymbol{x}_0)$$

with $B(\boldsymbol{x}_0)$ an open neighbourhood

If the time T can be arbitrarily small the system is said small time locally controllable (s.t.l.c.)

Theorem

Consider a nonlinear system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ *and an equilibrium point* $\boldsymbol{x}_{eq}$ *such that* $\boldsymbol{f}(\boldsymbol{x}_{eq}, \boldsymbol{0}) = \boldsymbol{0}$. *If the linear approximation of the system around the point* $\boldsymbol{x}_{eq}$ *is completely controllable then the system is small-time locally controllable in* $\boldsymbol{x}_{eq}$.

Controllability of the linear approximation of the Cubli

$$\dot{\boldsymbol{\delta x}} = \frac{\partial \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})}{\partial \boldsymbol{x}}\Big|_{\mathcal{E}_1} \boldsymbol{\delta x} + \frac{\partial \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})}{\partial \boldsymbol{u}}\Big|_{\mathcal{E}_1} \boldsymbol{\delta u}$$
$$= A\boldsymbol{\delta x} + B\boldsymbol{\delta u}$$

where $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})$ does not depend on θ_1

The pair (A, B) is not completely controllable. Indeed

$$rank\left(\begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}\right) = 8 < n = 9$$

It is not possible to conclude on the small-time local controllability of the system in those points belonging to $\mathcal{E}_1$

If $R_T^V(\boldsymbol{x}_0)$ only contains an open set of $\boldsymbol{x}_0$ the small time local controllability becomes small-time local accessibility (s.t.l.a).

$$\Rightarrow$$

s.t.l.c s.t.l.a

$$\not\Leftarrow$$

Small time local accessibility (continued)

Consider a control affine system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \sum_{j=1}^{m} \boldsymbol{g}_j(\boldsymbol{x}) u_j$$

and the distributions

$$\Delta_0 = \mathrm{span}\{\boldsymbol{g}_1, \ \ldots \ , \boldsymbol{g}_m\} \qquad \Delta = \mathrm{span}\{\boldsymbol{f}, \ \boldsymbol{g}_1, \ \ldots \ , \boldsymbol{g}_m\}$$

Theorem

Consider the smallest Δ-invariant distribution containing Δ_0 called accessibility distribution $< \Delta | \Delta_0 >$.

a. *If the dimension of $< \Delta | \Delta_0 >= n$ in $\boldsymbol{x}_0$ then system is s.t.l.a in $\boldsymbol{x}_0$;*
b. *If $\dim(< \Delta | \Delta_0 >) = r < n$ in a neighbourhood of $\boldsymbol{x}_0$ then the set $R_T^V(\boldsymbol{x}_0)$ is contained in a submanifold of dimension r of the n-dimensional state space, and contains an open set in that submanifold.*

Accessibility distribution

In order to evaluate $< \Delta | \Delta_0 >$ the following filtration of distributions

$$\begin{cases} \Delta_1 & = \Delta_0 + [\Delta_0, \Delta] \\ & \vdots \\ \Delta_k & = \Delta_{k-1} + [\Delta_{k-1}, \Delta] \end{cases}$$

is performed until an integer k is found s.t.

$$\dim(\Delta_k(\boldsymbol{x}_0)) = \dim(\Delta_{k+1}(\boldsymbol{x}_0))$$

Accessibility distribution of the Cubli

Using the Symbolic Math Toolbox from MATLAB it turns out that for all $\boldsymbol{x}_0 \in \mathcal{E}_1$

$$k = 3$$

$$\dim(< \Delta | \Delta_0 >) = 8 < n = 9$$

Hence $R_T^V(\mathcal{E}_1)$ is containted in a submanifold of dimension 8

Weak local accessibility

Although the system is not s.t.l.a in the points of interest it could be local accessible in a weaker sense, i.e., without the requirement that the time T is arbitrarily small.

Theorem

A necessary condition for weak local accessibility is

$$\dim(< \Delta | \Delta >) = n$$

Weak local accessibility of the Cubli

Using the Symbolic Math Toolbox from MATLAB it turns out that

$$\dim < \Delta | \Delta >= 8 < n = 9 \quad \forall \boldsymbol{x}_0 \in \mathcal{E}_1$$

The system is not locally accessible in any sense hence it can't be locally controllable

Observability

Local observability and indistinguishability

Theory

The concept of local observability can be explained in terms of the indistinguishability between two "near" *initial* states $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{x}} + \boldsymbol{\delta x}$.

The initial states $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{x}} + \boldsymbol{\delta x}$ are said to be indistinguishable in the interval $[0, T]$ if

$$\boldsymbol{y}(\bar{\boldsymbol{x}}, \boldsymbol{u}, t) = \boldsymbol{y}(\bar{\boldsymbol{x}} + \boldsymbol{\delta x}, \boldsymbol{u}, t) \ \ \forall \boldsymbol{u} \in U \ \ \forall t \in [0, T]$$

where $U \subseteq \{\boldsymbol{u}(\cdot) : [0, T] \to \mathbb{R}^m\}$

$\bar{\boldsymbol{x}}_1 I_T^U \bar{\boldsymbol{x}}_2$ is the notation used to denote two indistinguishable initial states $\bar{\boldsymbol{x}}_1$ and $\bar{\boldsymbol{x}}_2$ in the interval $[0, T]$

Theorem

Consider a nonlinear system

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}) \end{cases}$$

and an equilibrium point $\boldsymbol{x}_{eq}$ *such that* $\boldsymbol{f}(\boldsymbol{x}_{eq}, \boldsymbol{0}) = \boldsymbol{0}$. *If the linear approximation of the system around the point* $\boldsymbol{x}_{eq}$ *is completely observable then there are no indistinguishable points from* $\boldsymbol{x}_{eq}$ *in a small enough neighbourhood of* $\boldsymbol{x}_{eq}$.

Observability of the linear approximation of the Cubli

$$\begin{cases} \dot{\boldsymbol{\delta x}} = \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})}{\partial \boldsymbol{x}}\right|_{\mathcal{E}_1} \boldsymbol{\delta x} + \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})}{\partial \boldsymbol{u}}\right|_{\mathcal{E}_1} \boldsymbol{\delta u} = A\boldsymbol{\delta x} + B\boldsymbol{\delta u} \\ \boldsymbol{\delta y} = \left.\frac{\partial \boldsymbol{h}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\mathcal{E}_1} \boldsymbol{\delta x} = C\boldsymbol{\delta x} \end{cases}$$

where $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ does not depend on θ_1

The pair (A, C) is not completely observable. Indeed

$$rank\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = 6 < n = 9$$

It is not possible to conclude on the local observability of the system in those points belonging to $\mathcal{E}_1$

Nonlinear local observability

Theory

Consider a control affine system, its outputs

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \sum_{j=1}^{m} \boldsymbol{g}_j(\boldsymbol{x}) u_j \quad \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x})$$

the codistribution and the distribution

$$\Omega_0 = \mathrm{span}\{d\boldsymbol{h}\} \quad \Delta = \mathrm{span}\{\boldsymbol{f}, \ \boldsymbol{g}_1, \ \ldots \ , \boldsymbol{g}_m\}$$

Theorem

Consider the smallest Δ-invariant codistribution containing Ω_0 called observability codistribution $< \Delta | d\boldsymbol{h} >$.

a. *If the $\dim(< \Delta | d\boldsymbol{h} >) = n$ in $\bar{\boldsymbol{x}}$ then there are no initial states "near" $\bar{\boldsymbol{x}}$ that are indistinguishable from it and the system is said locally observable in $\bar{\boldsymbol{x}}$;*

b. *If $\dim(< \Delta | \Omega_0 >) = d < n$ in a neighbourhood of $\bar{\boldsymbol{x}}$ then the $(n-d)$-dimensional distribution $< \Delta | \Omega_0 >^{\perp}$ that annihilates the observability codistribution is involutive.*

Observability codistribution

Theory

In order to evaluate $< \Delta | d\boldsymbol{h} >$ the following filtration of codistributions

$$\begin{cases} \Omega_1 & = \Omega_0 + L_\Delta \Omega_0 \\ & \vdots \\ \Omega_k & = \Omega_{k-1} + L_\Delta \Omega_{k-1} \end{cases}$$

is performed until an integer k is found such that

$$\dim(\Omega_k(\bar{\boldsymbol{x}})) = \dim(\Omega_{k+1}(\bar{\boldsymbol{x}}))$$

Observability codistribution for the Cubli

Any given initial state in which the cubic frame is standing in the upright configuration with non zero angular rates $\dot{\boldsymbol{\theta}}$ gives

$$\dim(< \Delta | d\boldsymbol{h} >) = n$$

hence the system is locally observable in those points.

Example 1

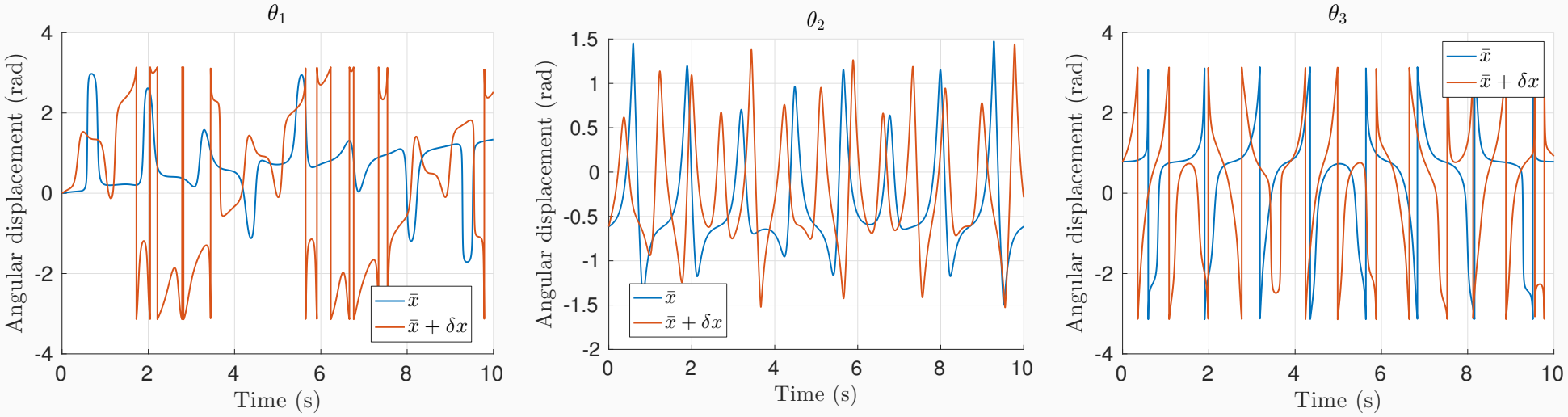
Consider the initial state

$$\bar{\boldsymbol{x}} = \begin{bmatrix} * & -\mathrm{atan}(\sqrt{2}/2) & \pi/4 & \bar{\dot{\boldsymbol{\theta}}} & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$$

It is physically reasonable that an initial state $\bar{\boldsymbol{x}} + \boldsymbol{\delta x}$ where

$$\boldsymbol{\delta x} = \begin{bmatrix} 0 & \dots & \boldsymbol{\delta\dot{\theta}} & \dots & 0 \end{bmatrix}^{\mathrm{T}}$$

produces a different output.



Example 2

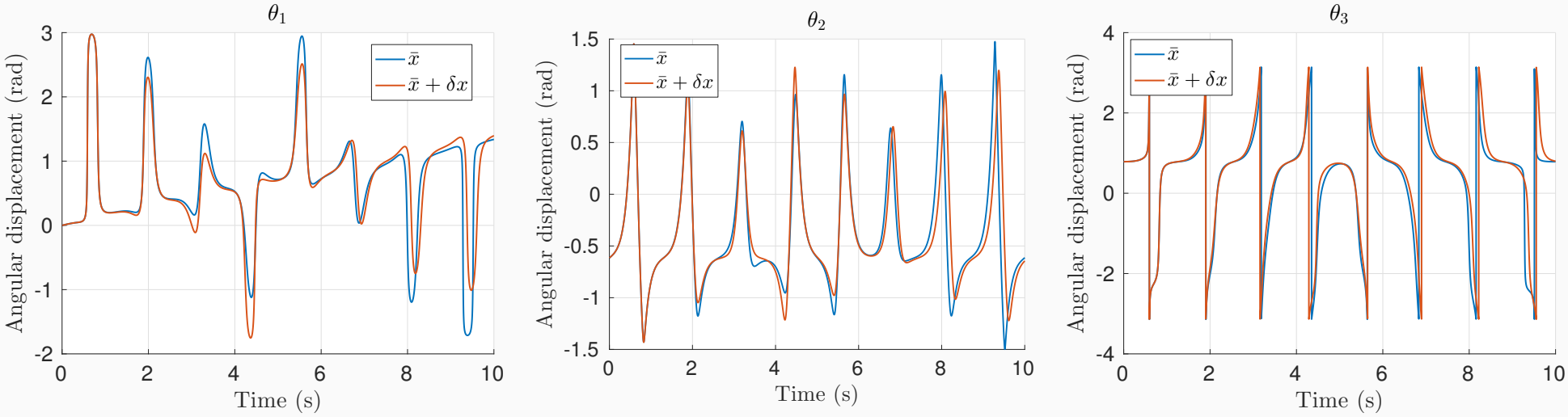
Consider the initial state

$$\bar{\boldsymbol{x}} = \begin{bmatrix} * & -\mathrm{atan}(\sqrt{2}/2) & \pi/4 & \bar{\dot{\boldsymbol{\theta}}} & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$$

An initial state $\bar{\boldsymbol{x}} + \boldsymbol{\delta x}$ where

$$\boldsymbol{\delta x} = \begin{bmatrix} 0 & \dots & \boldsymbol{\delta \dot{q}}_{wheels} \end{bmatrix}^{\mathrm{T}}$$

produces a different output.



Observability codistribution for the Cubli $\bar{\boldsymbol{x}} \in \mathcal{E}_1$

Using the Symbolic Math Toolbox from MATLAB it turns out that for all $\bar{\boldsymbol{x}} \in \mathcal{E}_1$

$$\dim(< \Delta | d\boldsymbol{h} >) = 6 < n = 9$$

but the codistribution is not regular then the theorem cannot be applied.

Noninteracting controller

Noniteracting control

Theory

Consider a nonlinear control affine system with m inputs and m outputs

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \sum_{i=1}^{m} \boldsymbol{g}_i(\boldsymbol{x}) u_i$$

$$y_1 = h_1(\boldsymbol{x})$$

$$\dots$$

$$y_m = h_m(\boldsymbol{x})$$

and an initial point $\boldsymbol{x}^0$

The problem of noninteracting control is stated as finding a regular static state feedback

$$\boldsymbol{u}(\boldsymbol{x}) = \alpha(\boldsymbol{x}) + \beta(\boldsymbol{x})\boldsymbol{v} \quad \boldsymbol{x} \in B_\delta(\boldsymbol{x}^0)$$

such that in the resulting closed loop system each output y_i is affected only by the corresponding input v_i and not by the others.

Noniteracting control (continued)

Suppose that the system has a vector relative degree $\{r_1, \ldots, r_m\}$ at $\boldsymbol{x}^0$ hence it admits a normal form locally around $\boldsymbol{x}^0$

$$\dot{\xi}_1^i = \xi_2^i$$

$$\ldots$$

$$\dot{\xi}_{r_i-1}^i = \xi_{r_i}^i$$

$$1 \leq i \leq m$$

$$\begin{bmatrix} \dot{\xi}_{r_1}^1 \\ \vdots \\ \dot{\xi}_{r_m}^m \end{bmatrix} = \boldsymbol{b}(\boldsymbol{\xi}, \boldsymbol{\eta}) + A(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{u}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{q}(\boldsymbol{\xi}, \boldsymbol{\eta}) + \boldsymbol{p}(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{u}$$

$$\begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{bmatrix} = \Phi(\boldsymbol{x})$$

with $A(\boldsymbol{x})$ nonsingular at $\boldsymbol{x}^0$

Noniteracting control (continued)

Proposition (Isidori)

Suppose

$$L_{\boldsymbol{g}_j} L_{\boldsymbol{f}}^k h_i(\boldsymbol{x}) = 0 \;\; \forall \boldsymbol{x} \in B_\delta(\boldsymbol{x}^0) \tag{1}$$

$$1 \leq j \leq m, \;\; 1 \leq i \leq m, \;\; k < r_i - 1$$

and

$$\begin{bmatrix} L_{\boldsymbol{g}_1} L_{\boldsymbol{f}}^{r_i - 1} h_i(\boldsymbol{x}^0) & \dots & L_{\boldsymbol{g}_m} L_{\boldsymbol{f}}^{r_i - 1} h_i(\boldsymbol{x}^0) \end{bmatrix} \neq \begin{bmatrix} 0 & \dots & 0 \end{bmatrix} \;\; 1 \leq i \leq m \tag{2}$$

Then the noninteracting control problem is solvable iff the matrix

$$A(\boldsymbol{x}^0) \text{ is nonsingular} \tag{3}$$

i.e. if the system has a vector relative degree $\{r_1, \dots, r_m\}$ at $\boldsymbol{x}^0$

It should be noted that (3) $\implies$ (2)

Noniteracting control (continued)

Indeed the regular static feedback

$$\boldsymbol{u} = -A^{-1}(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{b}(\boldsymbol{\xi}, \boldsymbol{\eta}) + A^{-1}(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{v}$$

results in the closed loop system

$$\dot{\xi}_1^i = \xi_2^i \qquad\qquad \dot{\boldsymbol{\eta}} = \hat{\boldsymbol{q}}(\boldsymbol{\xi}, \boldsymbol{\eta}) + \hat{\boldsymbol{p}}(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{v}$$

$$\dots$$

$$\dot{\xi}_{r_i-1}^i = \xi_{r_i}^i$$

$$\dot{\xi}_{r_i}^i = v_i$$

$$y_i = \xi_1^i$$

$$1 \leq i \leq m$$

For the cubli system

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} \\ \quad = \boldsymbol{f}(\boldsymbol{x}) + \begin{bmatrix} 0_{3\times 3} \\ \left(B(\theta_2, \theta_3)^{-1}\right)_{(:,4:6)} \end{bmatrix} \boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^{\mathrm{T}} \end{cases}$$

it can be found that

$$L_{\boldsymbol{g}_j} L_{\boldsymbol{f}}^0 h_i = L_{\boldsymbol{g}_j} h_i = \begin{bmatrix} \boldsymbol{e}_i & 0_{6\times 1} \end{bmatrix} \boldsymbol{g}_j = g_j^i = 0 \qquad 1 \leq j \leq 3, \;\; 1 \leq i \leq 3$$

This verifies the condition (1) of the proposition from Isidori

Also

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{y} = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} = \boldsymbol{b}(\boldsymbol{x}) + A(\boldsymbol{x})\boldsymbol{u}$$

where

$$\boldsymbol{b}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})_{(4:6)}$$

and

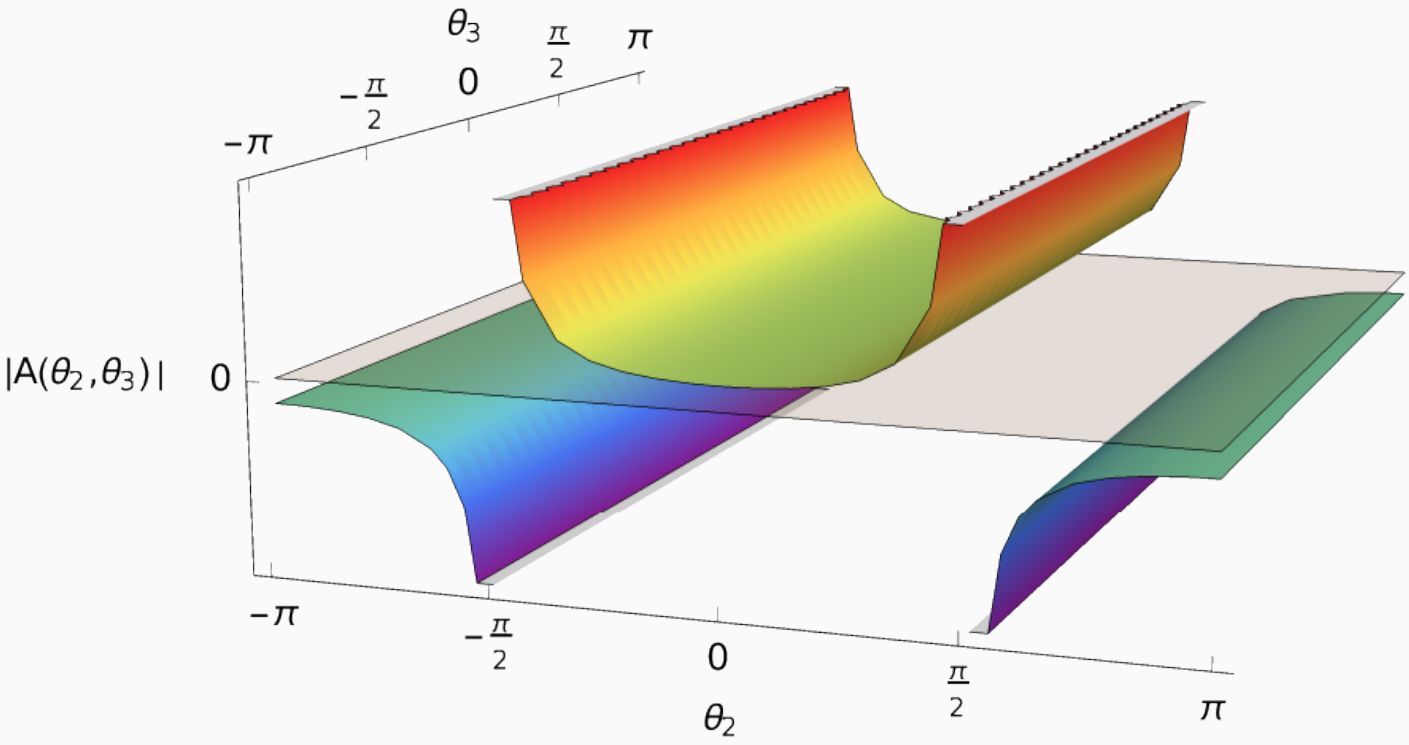
$$A(\boldsymbol{x}) = \left(B(\theta_2, \theta_3)^{-1}\right)_{(1:3,4:6)}$$

is nonsingular everywhere

This verifies the condition (3) of the proposition from Isidori

Existence of a vector relative degree

Application



$$|A| \neq 0 \;\; \forall \theta_2, \theta_3 \implies \{r_1, r_2, r_3\} = \{2, 2, 2\} \;\; \forall \boldsymbol{x}$$

The resulting standard noninteracting feedback is

$$\boldsymbol{u}_{NIC} = -\left(B(\theta_2, \theta_3)^{-1}\right)^{-1}_{(1:3,4:6)} \left(\boldsymbol{f}_{(4:6)}(\boldsymbol{x}) - \boldsymbol{v}\right)$$

Unobservable part of the system

The unobservable part of the system is obtained by completely specifying the coordinate transformation Φ

$$\boldsymbol{y} = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^{\mathrm{T}} \implies \Phi(\boldsymbol{x})_{1:6} = \begin{bmatrix} \theta_1 & \dot{\theta}_1 & \theta_2 & \dot{\theta}_2 & \theta_3 & \dot{\theta}_3 \end{bmatrix}^{\mathrm{T}}$$

The remaining part is chosen as

$$\Phi(\boldsymbol{x})_{1:6} = \begin{bmatrix} \dot{q}_x & \dot{q}_y & \dot{q}_z \end{bmatrix}^{\mathrm{T}}$$

i.e. the unobservable part of the closed loop system is the closed loop dynamics of the flywheels

$$\begin{aligned}
\dot{\boldsymbol{\eta}} = \begin{bmatrix} \ddot{q}_x \\ \ddot{q}_y \\ \ddot{q}_z \end{bmatrix} &= \boldsymbol{f}_{(7:9)}(\boldsymbol{x}) + \left(B(\theta_2, \theta_3)^{-1}\right)_{(4:6,4:6)} \boldsymbol{u}_{NIC} \\
&= \boldsymbol{f}_{(7:9)}(\boldsymbol{x}) - \left(B(\theta_2, \theta_3)^{-1}\right)_{(4:6,4:6)} \left(B(\theta_2, \theta_3)^{-1}\right)^{-1}_{(1:3,4:6)} \left(\boldsymbol{f}_{(4:6)}(\boldsymbol{x}) - \boldsymbol{v}\right) \\
&= \hat{\boldsymbol{q}}(\boldsymbol{x}) + \hat{\boldsymbol{p}}(\theta_2, \theta_3)\boldsymbol{v}
\end{aligned}$$

With reference to the normal form in coordinates $[\boldsymbol{\xi}^{\mathrm{T}}, \boldsymbol{\eta}^{\mathrm{T}}]$ the input

$$\boldsymbol{u} = -A(\mathbf{0}, \boldsymbol{\eta}(t))^{-1}\boldsymbol{b}(\mathbf{0}, \boldsymbol{\eta}(t))$$

determines the zero dynamics of the system

$$\begin{cases} \dot{\boldsymbol{\eta}}(t) = \boldsymbol{q}_0(\mathbf{0}, \boldsymbol{\eta}(t)) \\ \boldsymbol{\eta}(0) = \boldsymbol{\eta}_0 \end{cases}$$

where

$$\boldsymbol{q}_0(\boldsymbol{\xi}, \boldsymbol{\eta}) = \boldsymbol{q}(\boldsymbol{\xi}, \boldsymbol{\eta}) - \boldsymbol{p}(\boldsymbol{\xi}, \boldsymbol{\eta})A(\mathbf{0}, \boldsymbol{\eta}(t))^{-1}\boldsymbol{b}(\mathbf{0}, \boldsymbol{\eta}(t))$$

For the system under examination

$$\begin{aligned}\dot{\boldsymbol{\eta}}(t) &= \boldsymbol{f}(\boldsymbol{x}_0(t))_{(7:9)} - \boldsymbol{p}(\boldsymbol{x}_0(t))A(\boldsymbol{x}_0(t))^{-1}\boldsymbol{f}(\boldsymbol{x}_0(t))_{(4:6)} \\ &\equiv \boldsymbol{0} - \boldsymbol{p}(\boldsymbol{x}_0(t))A(\boldsymbol{x}_0(t))^{-1}\boldsymbol{0} \\ &\equiv \boldsymbol{0}\end{aligned}$$

where $\boldsymbol{x}_0(t) = \begin{bmatrix} 0 & -\mathrm{atan}\frac{\sqrt{2}}{2} & \frac{\pi}{4} & 0 & 0 & 0 & \boldsymbol{\eta}(t) \end{bmatrix}^{\mathrm{T}}$

Hence when the system is still in the upright position the initial flywheel velocites are conserved.

The same results apply if $\boldsymbol{x}_0(t)_{(1)} \in [-\pi, \pi[$ due to symmetry of the system.

Control objectives

Perform a rotation about the z axis of the inertial reference frame $\{S\}$ while the balance on one of the corner is maintained

Control using different Euler parametrization

The attitude controller was developed using two different parametrization of the R_{SC} matrix

- Euler ZYX:
  - the desired trajectory is easily specified since the yaw motion only depends on θ_1
- Euler XYZ:
  - the desired trajectory is more complex sice the yaw motion depends on all three angles
  - it allows to develop some useful considerations about the zero dynamics of the system

Control law

XYZ

A suitable control law $\boldsymbol{v}$ for the XYZ case is

$$v_i(t) = \ddot{\theta}_i^{des}(t) + k_d(\dot{\theta}_i^{des}(t) - \dot{\theta}_i(t)) + k_p(\theta_i^{des}(t) - \theta_i(t))$$

where $\theta_i^{des}(t)$ is evaluated using a Spherical linear interpolation between the initial and final poses of the cubic frame.

The unobservable part of the closed loop system (the flywheels velocities) subjected to the input $\boldsymbol{v}$

$$\dot{\boldsymbol{\eta}} = \hat{\boldsymbol{q}}(\boldsymbol{x}) + \hat{\boldsymbol{p}}(\theta_2, \theta_3)\boldsymbol{v}$$

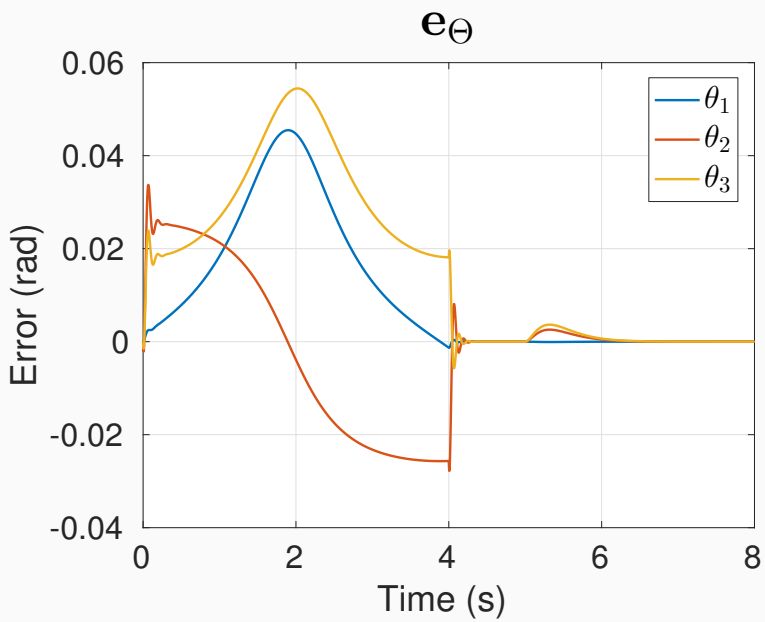
is not zero at the end of rotation and is conserved as dictated by the zero dynamics of the system. A possible solution is to switch to a LQR based control to slow down the flywheels.

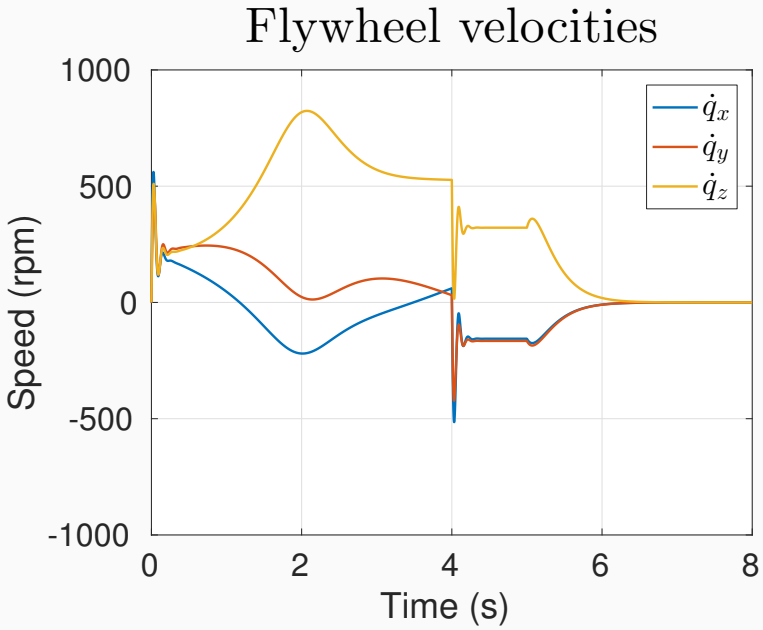
This approach works if the flywheels velocities before the switch are "not too high".

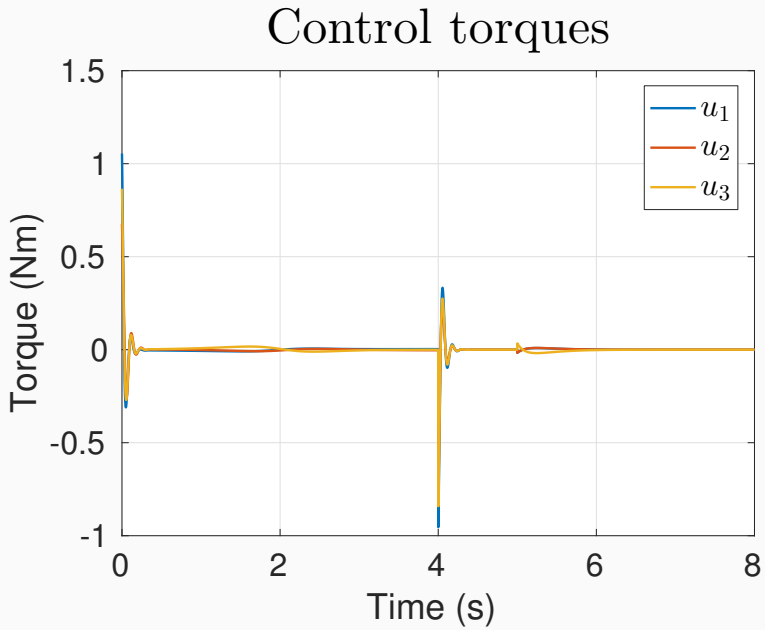
Results

XYZ

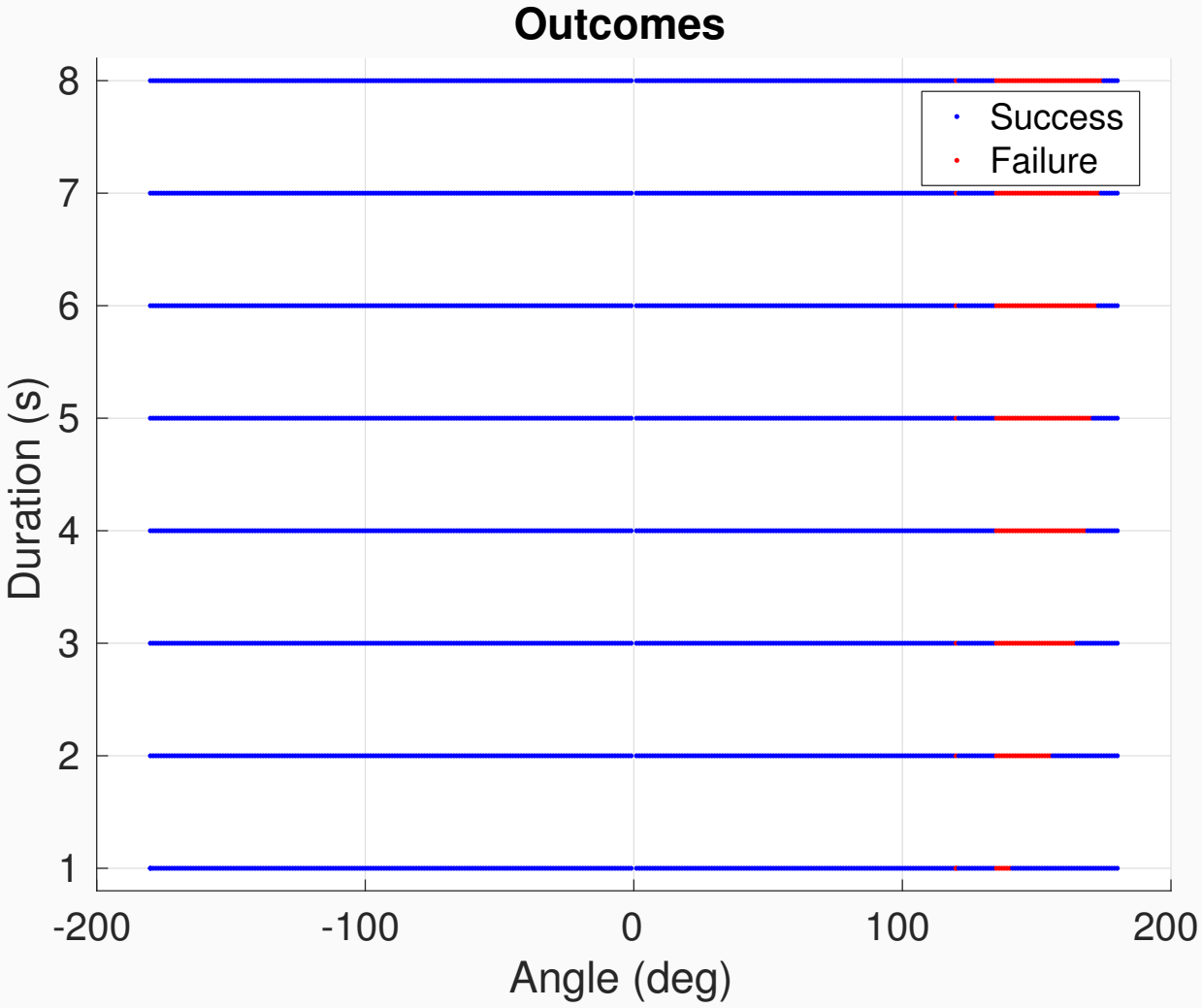
- $k_p = 2500,\ k_d = 50$
- rotation of 180° in 4 s
- LQR control actived at $t = 5$ s



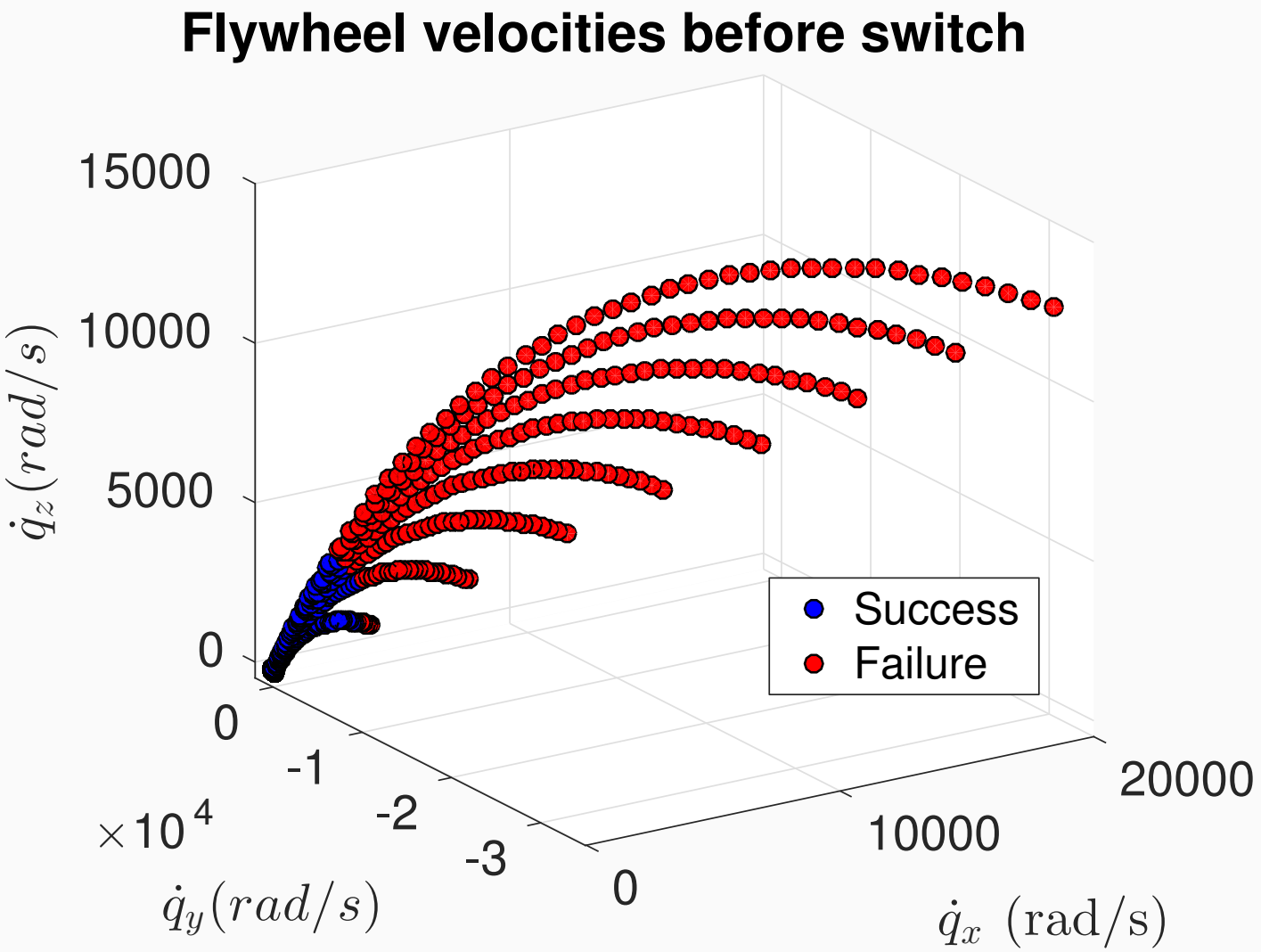




The outcome of the switch to the LQR controller is shown for several angles of rotation and several durations of the movement



The failures happen when the flywheel velocities are too high as expected



Control law

ZYX

A suitable control law $\boldsymbol{v}$ for the ZYX case is

$$v_1(t) = \ddot{\theta}_1^{des}(t) + k_d(\dot{\theta}_1^{des}(t) - \dot{\theta}_1(t)) + k_p(\theta_1^{des}(t) - \theta_1(t))$$

$$v_2(t) = -k_d\dot{\theta}_2(t) + k_p\left(-\mathrm{atan}\frac{\sqrt{2}}{2} - \theta_2(t)\right)$$

$$v_3(t) = -k_d\dot{\theta}_3(t) + k_p\left(\frac{\pi}{4} - \theta_3(t)\right)$$

where

$$\theta_1^{des}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

with boundary conditions

$$\theta_1^{des}(0) \in [-\pi, \pi) \quad \dot{\theta}_1^{des}(0) = 0 \quad \ddot{\theta}_1^{des}(0) = 0$$

$$\theta_1^{des}(t_f) \in [-\pi, \pi) \quad \dot{\theta}_1^{des}(t_f) = 0 \quad \ddot{\theta}_1^{des}(t_f) = 0$$

Results ($\theta_1^{des} = 0^\circ \rightarrow 100^\circ \rightarrow 0^\circ$) $k_p = 10$ $k_d = 0.1$ ZYX

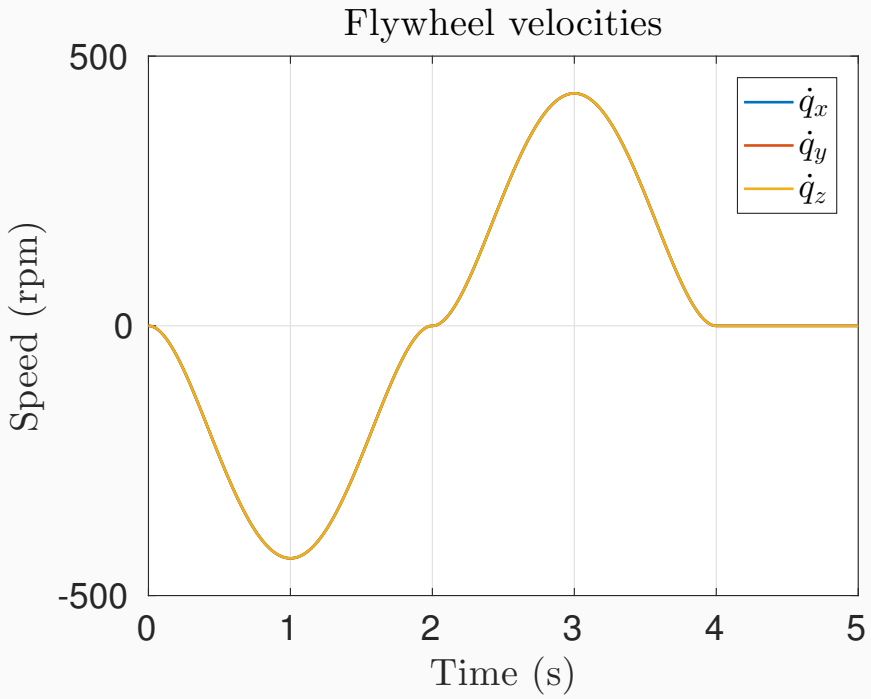
Results (continued)

ZYX

The flywheel velocities $\boldsymbol{\eta}(t)$ which are solution to the differential equation

$$\begin{cases} \dot{\boldsymbol{\eta}} = \hat{\boldsymbol{q}}(\boldsymbol{x}) + \hat{\boldsymbol{p}}\left(-\mathrm{atan}\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)\boldsymbol{v}_{PD} \\ \boldsymbol{\eta}(0) = \boldsymbol{0} \end{cases}$$

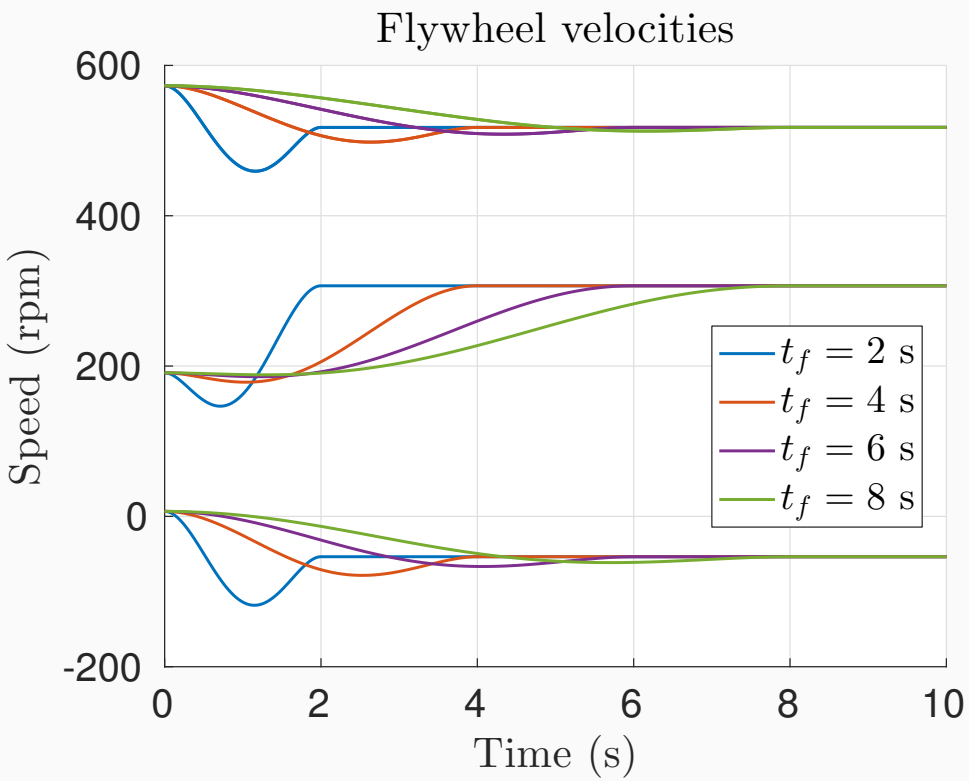
are bounded and tends to zero



Results (continued)

ZYX

Non-zero initial conditions $\eta(0)$ produce, in general, non-zero flywheel velocities at the time t_f in which the rotation is completed.



Since the output is "zeroed" at $t = t_f$ the zero dynamics of the system dictate the conservation of the flywheels velocities.

Conclusions

In conclusion the behavior of the zero dynamics suggests that, at least for the yaw motion of interest, it is better to choose an outer loop control $\boldsymbol{v}$ such that the flywheels velocites are zero at the end of the motion as happens in the ZYX case.

If it is not the case an additional controller is required to slow down the flywheel velocities as much as possible compatibly with their initial conditions.