Contents

Introduction

In this report the theory of nonlinear systems is applied to a mechanical system known in the literature as the Cubli [1]. It is a reaction wheel based 3D inverted pendulum consisting of a 15 cm sided cube with flywheels mounted on the three faces.

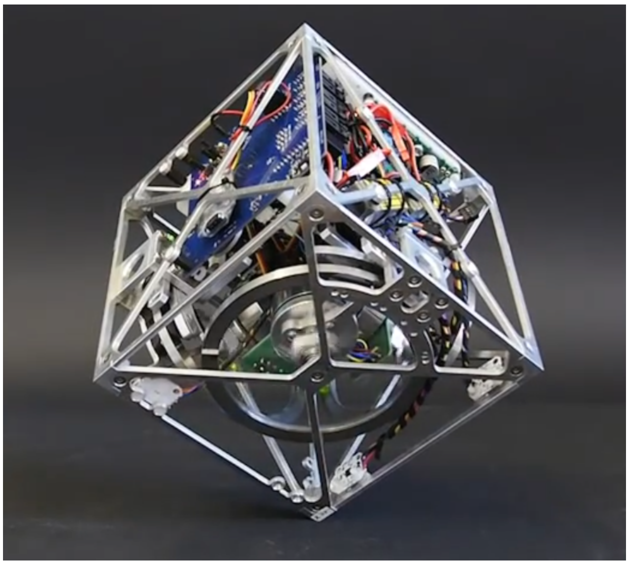


Figure 1: The Cubli

In the section one the equations of motion of the system are derived in standard manipulator form and are rewritten in control affine form which is required to apply the theory of nonlinear systems. Also the equilibria of the system are described.

In the section two and three the theory regarding nonlinear controllability and observability is briefly recalled and then is applied to the system.

In the last section a nonlinear controller is developed using the Noninteracting approach, from Isidori, which is introduced preliminarly. Then the results of the simulation in which the cube perform a pure yaw motion while balancing on its corner are presented.

1 Equations of motion

In this section the equations of motion for the Cubli are derived in the standard manipulator equation form and then rewritten in a control affine form.

1.1 Standard manipulator equation form

In order to write the equations in the standard manipulator equation form

$$B(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{Q} \tag{1}$$

the inertia matrix B, the Coriolis matrix C, the gravity term $\boldsymbol{G}$ and the generalized forces $\boldsymbol{Q}$ are derived.

1.1.1 Definition of a configuration vector

A set of independent generalized coordinates that completely specify the configuration of the system is given by

$$\boldsymbol{q} = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & q_x & q_y & q_z \end{bmatrix}^T$$

where the angles θ_i describe the attitude of the system and the angles q_i describe the orientation of the flywheels with respect to the frame of the Cubli.

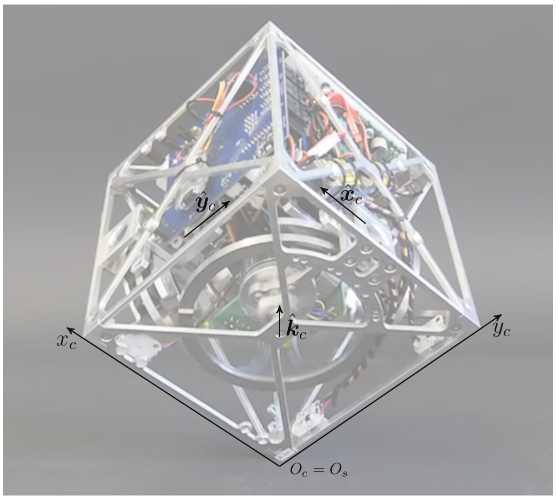


Figure 2: Reference frames

In order to represent the attitude of the system two reference systems are defined (see fig. 2), a body fixed reference frame $\{C\} = (O_c; \hat{\boldsymbol{\imath}}_c, \hat{\boldsymbol{\jmath}}_c, \hat{\boldsymbol{k}}_c)$ and an inertial reference frame $\{S\} = (O_s; \hat{\boldsymbol{\imath}}_s, \hat{\boldsymbol{\jmath}}_s, \hat{\boldsymbol{k}}_s)$ The attitude of the system is then defined to be the orientation of the frame $\{C\}$ with respect to the frame $\{S\}$ described by the rotation matrix R_{SC}

$$R_{SC}(\boldsymbol{\theta}) = R_z(\theta_1)R_y(\theta_2)R_x(\theta_3) = \begin{bmatrix} c_1c_2 & -c_3s_1 + c_1s_2s_3 & c_1c_3s_2 + s_1s_3 \\ c_2s_1 & c_1c_3 + s_1s_2s_3 & c_3s_1s_2 - c_1s_3 \\ -s_2 & c_2s_3 & c_2c_3 \end{bmatrix}$$

where $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$.

1.1.2 Derivation of the inertia matrix B

The inertia matrix B can be derived from the total kinetic energy of the system T

$$T(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}} B(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{2}$$

In the case of the Cubli the kinetic energy T evaluates to

$$\begin{aligned} T(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= \frac{1}{2}(m_c\, {}^c\boldsymbol{v}_{G_c}^{\mathrm{T}}\, {}^c\boldsymbol{v}_{G_c} + {}^c\boldsymbol{\omega}_c^{\mathrm{T}}\, {}^c\mathfrak{I}_{G_c}\, {}^c\boldsymbol{\omega}_c) \\ &+ \frac{1}{2}(m_x\, {}^c\boldsymbol{v}_{G_x}^{\mathrm{T}}\, {}^c\boldsymbol{v}_{G_x} + {}^c\boldsymbol{\omega}_x^{\mathrm{T}}\, {}^c\mathfrak{I}_{G_x}\, {}^c\boldsymbol{\omega}_x) \\ &+ \frac{1}{2}(m_y\, {}^c\boldsymbol{v}_{G_y}^{\mathrm{T}}\, {}^c\boldsymbol{v}_{G_y} + {}^c\boldsymbol{\omega}_y^{\mathrm{T}}\, {}^c\mathfrak{I}_{G_y}\, {}^c\boldsymbol{\omega}_y) \\ &+ \frac{1}{2}(m_z\, {}^c\boldsymbol{v}_{G_z}^{\mathrm{T}}\, {}^c\boldsymbol{v}_{G_z} + {}^c\boldsymbol{\omega}_z^{\mathrm{T}}\, {}^c\mathfrak{I}_{G_z}\, {}^c\boldsymbol{\omega}_z) \end{aligned} \tag{3}$$

where ${}^c\boldsymbol{v}_{G_c}$ and ${}^c\boldsymbol{v}_{G_i}$ are the linear velocities of the centre of mass of the cubic frame and of the flywheels expressed in $\{C\}$, ${}^c\boldsymbol{\omega}_c$ and ${}^c\boldsymbol{\omega}_i$ are the angular velocites of the cubic frame and of the flywheels expressed in $\{C\}$, m_c and m_i are the mass of the cubic frame and of the flywheels, ${}^c\mathfrak{I}_{G_c}$ and ${}^c\mathfrak{I}_{G_i}$ are the inertia matrices of the cubic frame and of the flywheels with respect to their CoMs and expressed in $\{C\}$.

The angular velocity ${}^c\boldsymbol{\omega}_c$ can be expressed as a function of the configuration vector $\boldsymbol{q}$ and its derivative $\dot{\boldsymbol{q}}$

$${}^c\boldsymbol{\omega}_c = {}^cJ_\omega(\boldsymbol{\theta})\dot{\boldsymbol{q}} = {}^cJ_\omega(\boldsymbol{q})\dot{\boldsymbol{q}}$$

where ${}^cJ_\omega(\boldsymbol{q})$ is an analytical Jacobian of the form

$${}^cJ_\omega(\boldsymbol{q}) = \begin{bmatrix} {}^c\tilde{J}_\omega & 0_{3\times3} \end{bmatrix} = \begin{bmatrix} -s_2 & 0 & 1 & \\ c_2s_3 & c_3 & 0 & 0_{3\times3} \\ c_2c_3 & -s_3 & 0 & \end{bmatrix} \tag{4}$$

As regards the angular velocites of the flywheels ${}^{c}\boldsymbol{\omega}_x$, ${}^{c}\boldsymbol{\omega}_y$ and ${}^{c}\boldsymbol{\omega}_z$ the addition theorem for angular velocites can be used resulting in

$$
\begin{aligned}
{}^{c}\boldsymbol{\omega}_x &= \begin{bmatrix} {}^{c}\tilde{J}_\omega & \boldsymbol{e}_1 & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \dot{\boldsymbol{q}} = {}^{c}J_{\omega_x}\dot{\boldsymbol{q}} \\
{}^{c}\boldsymbol{\omega}_y &= \begin{bmatrix} {}^{c}\tilde{J}_\omega & \boldsymbol{0} & \boldsymbol{e}_2 & \boldsymbol{0} \end{bmatrix} \dot{\boldsymbol{q}} = {}^{c}J_{\omega_y}\dot{\boldsymbol{q}} \\
{}^{c}\boldsymbol{\omega}_z &= \begin{bmatrix} {}^{c}\tilde{J}_\omega & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{e}_3 \end{bmatrix} \dot{\boldsymbol{q}} = {}^{c}J_{\omega_z}\dot{\boldsymbol{q}}
\end{aligned}
\tag{5}
$$

The linear velocities ${}^{c}\boldsymbol{v}_{\boldsymbol{G}_i}$ can be evaluated using the well known formula for the velocity of two points fixed on a rigid body

$$
{}^{c}\boldsymbol{v}_{\boldsymbol{G}_i} = \cancelto{0}{{}^{c}\boldsymbol{v}_{O_c}} + {}^{c}\hat{\boldsymbol{\omega}}_c\,{}^{c}\boldsymbol{G}_* = -\,{}^{c}\hat{\boldsymbol{G}}_*\,{}^{c}\boldsymbol{\omega}_c = -\,{}^{c}\hat{\boldsymbol{G}}_*\,{}^{c}J_\omega\dot{\boldsymbol{\theta}} = -\,{}^{c}\hat{\boldsymbol{G}}_*\,{}^{c}\tilde{J}_\omega\dot{\boldsymbol{q}}
$$

where the position $\boldsymbol{G}$ of the CoMs with respect to origin of $\{C\}$ is evaluated with the hypothesis that the masses of the rigid bodies are distributed uniformly

$$
{}^{c}\boldsymbol{G}_c = \begin{bmatrix} a/2 \\ a/2 \\ a/2 \end{bmatrix} \quad
{}^{c}\boldsymbol{G}_x = \begin{bmatrix} 0 \\ a/2 \\ a/2 \end{bmatrix} \quad
{}^{c}\boldsymbol{G}_y = \begin{bmatrix} a/2 \\ 0 \\ a/2 \end{bmatrix} \quad
{}^{c}\boldsymbol{G}_z = \begin{bmatrix} a/2 \\ a/2 \\ 0 \end{bmatrix}
$$

and the velocity of the point O_c evaluates to zero because in the model considered a spherical joint constraints one of the vertex of the cubic frame from moving.

Finally the inertia matrices of the cubic frame and of the flywheels evaluate to

$$
{}^{c}\mathfrak{I}_{G_x} = \begin{bmatrix} \frac{m_x r^2}{2} & 0 & 0 \\ 0 & \frac{m_x\left(h^2+3r^2\right)}{12} & 0 \\ 0 & 0 & \frac{m_x\left(h^2+3r^2\right)}{12} \end{bmatrix} \quad
{}^{c}\mathfrak{I}_{G_y} = \begin{bmatrix} \frac{m_y\left(h^2+3r^2\right)}{12} & 0 & 0 \\ 0 & \frac{m_y r^2}{2} & 0 \\ 0 & 0 & \frac{m_y\left(h^2+3r^2\right)}{12} \end{bmatrix}
$$

$$
{}^{c}\mathfrak{I}_{G_z} = \begin{bmatrix} \frac{m_z\left(h^2+3r^2\right)}{12} & 0 & 0 \\ 0 & \frac{m_z\left(h^2+3r^2\right)}{12} & 0 \\ 0 & 0 & \frac{m_z r^2}{2} \end{bmatrix} \quad
{}^{c}\mathfrak{I}_{G_c} = \begin{bmatrix} \frac{m_c a^2}{6} & 0 & 0 \\ 0 & \frac{m_c a^2}{6} & 0 \\ 0 & 0 & \frac{m_c a^2}{6} \end{bmatrix}
$$

where h and r are the height and the radius of flywheels respectively.

By inserting all the quantities derived up to now in the equation (3) and using the more general form of T given in equation (2) the inertia matrix B is obtained

$$
\begin{aligned}
B(\boldsymbol{q}) = {} & m_c\,{}^{c}J_\omega^{\mathrm{T}}\,{}^{c}\hat{\boldsymbol{G}}_c^{\mathrm{T}}\,{}^{c}\hat{\boldsymbol{G}}_c\,{}^{c}J_\omega + {}^{c}J_\omega^{\mathrm{T}}\,{}^{c}\mathfrak{I}_{G_c}\,{}^{c}J_\omega \\
& + m_x\,{}^{c}J_\omega^{\mathrm{T}}\,{}^{c}\hat{\boldsymbol{G}}_x^{\mathrm{T}}\,{}^{c}\hat{\boldsymbol{G}}_x\,{}^{c}J_\omega + {}^{c}J_{\omega_x}^{\mathrm{T}}\,{}^{c}\mathfrak{I}_{G_x}\,{}^{c}J_{\omega_x} \\
& + m_y\,{}^{c}J_\omega^{\mathrm{T}}\,{}^{c}\hat{\boldsymbol{G}}_y^{\mathrm{T}}\,{}^{c}\hat{\boldsymbol{G}}_y\,{}^{c}J_\omega + {}^{c}J_{\omega_y}^{\mathrm{T}}\,{}^{c}\mathfrak{I}_{G_y}\,{}^{c}J_{\omega_y} \\
& + m_z\,{}^{c}J_\omega^{\mathrm{T}}\,{}^{c}\hat{\boldsymbol{G}}_z^{\mathrm{T}}\,{}^{c}\hat{\boldsymbol{G}}_z\,{}^{c}J_\omega + {}^{c}J_{\omega_z}^{\mathrm{T}}\,{}^{c}\mathfrak{I}_{G_z}\,{}^{c}J_{\omega_z}
\end{aligned}
$$

Since the jacobians ${}^{c}J_{\omega}$, ${}^{c}J_{\omega_x}$, ${}^{c}J_{\omega_y}$, ${}^{c}J_{\omega_z}$ only depends on the angles θ_2 and θ_3 and the other quantities appearing in the inertia matrix are constant then

$$B(\boldsymbol{q}) = B(\theta_2, \theta_3)$$

This fact is a consequence of the form of the analytical jacobian ${}^{c}\tilde{J}_{\omega}$ and does not depend on the particular Euler parametrization chosen.

1.1.3 Derivation of the Coriolis matrix C

The Coriolis matrix C can be written as

$$C_{ij} = \sum_{k=1}^{6} \mathit{\Gamma}^{i}_{jk} \dot{q}_k$$

where $\mathit{\Gamma}^{i}_{jk}$ are the Christoffel Symbol of the First Kind

$$\mathit{\Gamma}^{i}_{jk} = \frac{1}{2}\left(\frac{\partial B_{ij}}{\partial q_k} + \frac{\partial B_{ik}}{\partial q_j} - \frac{\partial B_{jk}}{\partial q_i}\right)$$

1.1.4 Derivation of the gravity term $\boldsymbol{G}$

The gravity term $\boldsymbol{G}$ can be evaluated as

$$\boldsymbol{G} = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^{\mathrm{T}}$$

where U is the total potential energy of the system given by

$$U(\boldsymbol{q}) = -\,{}^{c}\boldsymbol{g}^{\mathrm{T}}(m_c\,{}^{c}\boldsymbol{G}_c + m_x\,{}^{c}\boldsymbol{G}_x + m_y\,{}^{c}\boldsymbol{G}_y + m_z\,{}^{c}\boldsymbol{G}_z)$$

and the gravitational acceleration vector ${}^{c}\boldsymbol{g}$ can be obtained as

$${}^{c}\boldsymbol{g} = R_{SC}^{\mathrm{T}}\,{}^{s}\boldsymbol{g} = R_{SC}^{\mathrm{T}}\begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

1.1.5 Derivation of the generalized forces $\boldsymbol{Q}$

The i-th generalized force Q_i for the Cubli evaluates to

$$\begin{aligned} Q_i = &\tau_x \hat{\boldsymbol{i}}_c^T \frac{\partial({}^{c}\boldsymbol{\omega}_x)}{\partial \dot{q}_i} + \tau_y \hat{\boldsymbol{j}}_c^T \frac{\partial({}^{c}\boldsymbol{\omega}_y)}{\partial \dot{q}_i} + \tau_z \hat{\boldsymbol{k}}_c^T \frac{\partial({}^{c}\boldsymbol{\omega}_z)}{\partial \dot{q}_i} - \\ &\left(\tau_x \hat{\boldsymbol{i}}_c^T \frac{\partial({}^{c}\boldsymbol{\omega}_c)}{\partial \dot{q}_i} + \tau_y \hat{\boldsymbol{j}}_c^T \frac{\partial({}^{c}\boldsymbol{\omega}_c)}{\partial \dot{q}_i} + \tau_z \hat{\boldsymbol{k}}_c^T \frac{\partial({}^{c}\boldsymbol{\omega}_c)}{\partial \dot{q}_i}\right) \end{aligned}$$

where τ_x, τ_y and τ_z are the torques applied to the flywheels and $-\tau_x$, $-\tau_y$ and $-\tau_z$ are the torques applied to the cubic frame.

By using the Jacobians defined in equations (4) and (5) the expression becomes

$$\begin{aligned}Q_i =&\tau_x \hat{\boldsymbol{i}}_c^T \, {}^c J_{\omega_x}(:,i) + \tau_y \hat{\boldsymbol{j}}_c^T \, {}^c J_{\omega_y}(:,i) + \tau_z \hat{\boldsymbol{k}}_c^T \, {}^c J_{\omega_z}(:,i) - \\ &(\tau_x \hat{\boldsymbol{i}}_c + \tau_y \hat{\boldsymbol{j}}_c + \tau_z \hat{\boldsymbol{k}}_c)^T \, {}^c J_\omega(:,i)\end{aligned}$$

Since the columns of the Jacobians are the same

$${}^c J_{\omega_x}(:,i) = {}^c J_{\omega_y}(:,i) = {}^c J_{\omega_z}(:,i) = {}^c J_\omega(:,i) = {}^c \tilde{J}_\omega(:,i)$$

for $i = 1, 2, 3$ then

$$Q_i = (\tau_x \hat{\boldsymbol{i}}_c + \tau_y \hat{\boldsymbol{j}}_c + \tau_z \hat{\boldsymbol{k}}_c - \tau_x \hat{\boldsymbol{i}}_c - \tau_y \hat{\boldsymbol{j}}_c - \tau_z \hat{\boldsymbol{k}}_c)^T \, {}^c \tilde{J}_\omega(:,i) = 0$$

As regards the other indexes

$${}^c J_{\omega_x}(:,i+3) = {}^c J_{\omega_y}(:,i+3) = {}^c J_{\omega_z}(:,i+3) = \boldsymbol{e}_i \quad {}^c J_\omega(:,i+3) = \boldsymbol{0}$$

for $i = 1, 2, 3$ resulting in

$$Q_4 = \tau_x \hat{\boldsymbol{i}}_c^T \boldsymbol{e}_1 + \tau_y \hat{\boldsymbol{j}}_c^T \boldsymbol{e}_1 + \tau_z \hat{\boldsymbol{k}}_c^T \boldsymbol{e}_1 - (\tau_x \hat{\boldsymbol{i}}_c + \tau_y \hat{\boldsymbol{j}}_c + \tau_z \hat{\boldsymbol{k}}_c)^T \boldsymbol{0} = \tau_x \boldsymbol{e}_1^T \boldsymbol{e}_1 = \tau_x$$

$$Q_5 = \tau_x \hat{\boldsymbol{i}}_c^T \boldsymbol{e}_2 + \tau_y \hat{\boldsymbol{j}}_c^T \boldsymbol{e}_2 + \tau_z \hat{\boldsymbol{k}}_c^T \boldsymbol{e}_2 - (\tau_x \hat{\boldsymbol{i}}_c + \tau_y \hat{\boldsymbol{j}}_c + \tau_z \hat{\boldsymbol{k}}_c)^T \boldsymbol{0} = \tau_y \boldsymbol{e}_2^T \boldsymbol{e}_2 = \tau_y$$

$$Q_6 = \tau_x \hat{\boldsymbol{i}}_c^T \boldsymbol{e}_3 + \tau_y \hat{\boldsymbol{j}}_c^T \boldsymbol{e}_3 + \tau_z \hat{\boldsymbol{k}}_c^T \boldsymbol{e}_3 - (\tau_x \hat{\boldsymbol{i}}_c + \tau_y \hat{\boldsymbol{j}}_c + \tau_z \hat{\boldsymbol{k}}_c)^T \boldsymbol{0} = \tau_z \boldsymbol{e}_3^T \boldsymbol{e}_3 = \tau_z$$

The resulting vector of generalized forces is

$$\boldsymbol{Q} = \begin{bmatrix} 0 & 0 & 0 & \tau_x & \tau_y & \tau_z \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & \boldsymbol{\tau} \end{bmatrix}^T = F\boldsymbol{\tau}$$

where

$$F = \begin{bmatrix} 0_{3\times3} \\ I_{3\times3} \end{bmatrix}$$

has rank $3 < 6$ for every configuration resulting in a underactuated system.

1.2 Equations in control affine form

The equation (1) in standard manipulator form can be written in a control affine form for control purposes with the following procedure.

First a new state $\tilde{x}$ for the system is defined

$$\tilde{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix}$$

The time rate of change of the state can then be written using (1)

$$
\begin{aligned}
\dot{\tilde{\boldsymbol{x}}} = \begin{bmatrix} \dot{\tilde{\boldsymbol{x}}}_1 \\ \dot{\tilde{\boldsymbol{x}}}_2 \end{bmatrix} &= \begin{bmatrix} \tilde{\boldsymbol{x}}_2 \\ -B(\tilde{\boldsymbol{x}}_1)^{-1}(C(\tilde{\boldsymbol{x}}_1, \tilde{\boldsymbol{x}}_2)\tilde{\boldsymbol{x}}_2 + \boldsymbol{G}(\tilde{\boldsymbol{x}}_1)) \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ B(\tilde{\boldsymbol{x}}_1)^{-1}\boldsymbol{Q} \end{bmatrix} \\
&= \begin{bmatrix} \tilde{\boldsymbol{x}}_2 \\ -B^{-1}(C\tilde{\boldsymbol{x}}_2 + \boldsymbol{G}) \end{bmatrix} + \begin{bmatrix} 0_{6\times 3} \\ B(\tilde{\boldsymbol{x}}_1)^{-1}F \end{bmatrix} \boldsymbol{\tau} \\
&= \begin{bmatrix} \tilde{\boldsymbol{x}}_2 \\ -B^{-1}(C\tilde{\boldsymbol{x}}_2 + \boldsymbol{G}) \end{bmatrix} + \begin{bmatrix} 0_{6\times 3} \\ (B(\tilde{\boldsymbol{x}}_1)^{-1})_{(:,4:6)} \end{bmatrix} \boldsymbol{\tau} \\
&= \tilde{\boldsymbol{f}}(\tilde{\boldsymbol{x}}) + \tilde{g}(\tilde{\boldsymbol{x}})\boldsymbol{\tau} \\
&= \tilde{\boldsymbol{f}}(\tilde{\boldsymbol{x}}) + \tilde{g}(\tilde{\boldsymbol{x}})\boldsymbol{u}
\end{aligned} \tag{6}
$$

thus obtaining a control affine form of the equations.

It can be verified that B, C and $\boldsymbol{G}$ does not depend on the angular position of the flywheels. Since the position of the flywheels is of no interest for control purposes the 4-th, 5-th and 6-th rows of (6) can be neglected and the state can be reduced to

$$
\begin{aligned}
\boldsymbol{x} &= \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 & \dot{q}_x & \dot{q}_y & \dot{q}_z \end{bmatrix}^{\mathrm{T}} \\
&= \begin{bmatrix} \boldsymbol{\theta} & \dot{\boldsymbol{\theta}} & \dot{\boldsymbol{q}}_{wheels} \end{bmatrix}^{\mathrm{T}}
\end{aligned}
$$

The time rate of change of the state updates to

$$
\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} \tag{7}
$$

where $\boldsymbol{f}$ and g are obtained from $\tilde{\boldsymbol{f}}$ and $\tilde{g}$ by removing the 4-th, 5-th and 6-th rows.

The outputs of the system are chosen as

$$
\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^T \tag{8}
$$

1.3 Equlibrium points

Among the many equilibria of the system (7) the sets

$$
\mathcal{E}_1 = \left\{ \boldsymbol{x} \mid \theta_1 \in [-\pi, \pi),\ \ \theta_2 = -\mathrm{atan}\frac{\sqrt{2}}{2},\ \ \theta_3 = \frac{\pi}{4},\ \ \dot{\boldsymbol{\theta}} = \boldsymbol{0},\ \ \dot{\boldsymbol{q}}_{wheels} = \boldsymbol{0},\ \ \boldsymbol{u} = \boldsymbol{0} \right\}
$$

$$
\mathcal{E}_2 = \left\{ \boldsymbol{x} \mid \theta_1 \in [-\pi, \pi),\ \ \theta_2 = \mathrm{atan}\frac{\sqrt{2}}{2},\ \ \theta_3 = -\frac{3}{4}\pi,\ \ \dot{\boldsymbol{\theta}} = \boldsymbol{0},\ \ \dot{\boldsymbol{q}}_{wheels} = \boldsymbol{0},\ \ \boldsymbol{u} = \boldsymbol{0} \right\}
$$

are of interest. The set $\mathcal{E}_1$ is that of the upright equilibria which is unstable and the set $\mathcal{E}_2$ is of that of the hanging equilibria which is stable in the sense of Laypunov.

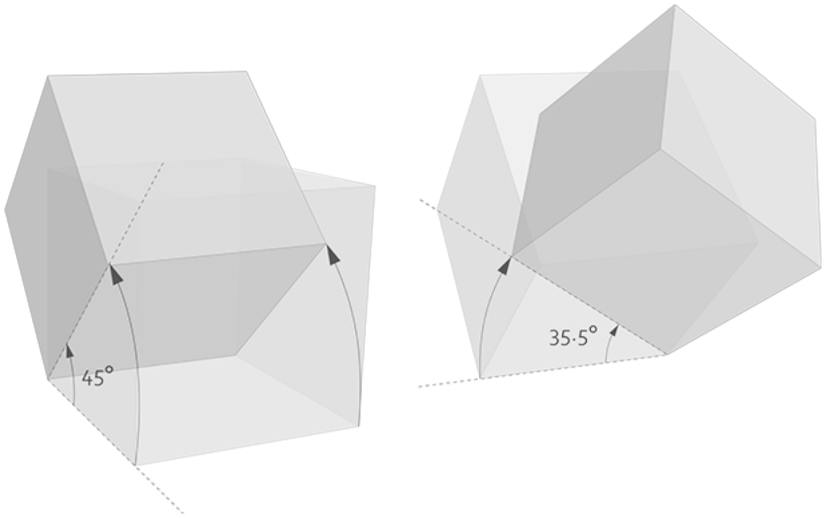


Figure 3: Upright equilibium configuration of the Cubli

The values of the angles θ_1, θ_2 and θ_3 can be easily found from the geometry of the cubic frame (see fig 3). In particular the angle θ_1 clearly represents the yaw of the cubic frame in the upright or hanging position.

It should be noted that other equilibria exist in which the system is at rest in the upright or hanging position while the flywheels rotates at a non zero constant angular velocity.

1.4 Parameters of the system

In the rest of this report the following values are used for the parameters of the system

- $M = 2.5\,\mathrm{kg}$;
- $m = 0.204\,\mathrm{kg}$;
- $a = 0.15\,\mathrm{m}$;
- $r = 0.05\,\mathrm{m}$;
- $h = 0.005\,\mathrm{m}$;

2 Controllability

In this section the nonlinear *local* controllability of the system is discussed.

A nonlinear system is locally controllable in a point $\boldsymbol{x}_0$ if there exists a neighbourhood $B_\epsilon(\boldsymbol{x}_0)$ such that for every point $\boldsymbol{x}_f$ in that neighbourhood a control law $\boldsymbol{u}(t)$ exists such that $\boldsymbol{x}(T, \boldsymbol{x}_0, \boldsymbol{u}) = \boldsymbol{x}_f$. If T can be arbitrarily small then the system is defined as locally controllable in small-time from $\boldsymbol{x}_0$.

Local controllability can be assessed using tools taken from differential geometry of manifolds and Lie algebras. However LTI theory can also be used in some special cases that are described in the following.

2.1 Linear controllability

In order to study the local controllability of nonlinear systems using the theory of LTI systems the following theorem can be used.

Theorem 1 *Consider a nonlinear system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ and an equilibrium point $\boldsymbol{x}_{eq}$ such that $\boldsymbol{f}(\boldsymbol{x}_{eq}, \boldsymbol{0}) = \boldsymbol{0}$. If the linear approximation of the system around the point $\boldsymbol{x}_{eq}$ is completely controllable then the system is small-time locally controllable in $\boldsymbol{x}_{eq}$.*

It is recalled that a linear system

$$\dot{\boldsymbol{\delta x}} = A\boldsymbol{\delta x} + B\boldsymbol{\delta u}$$

where

$$\boldsymbol{\delta x}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{eq} \quad \boldsymbol{\delta u}(t) = \boldsymbol{u}(t) - \boldsymbol{u}_{eq}$$

is completely controllable if the rank of the controllability matrix R_c

$$R_c = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} \tag{9}$$

is equal to the dimension of the state n.

For the system of equations (7) it can be found that the linear approximation around each equilibrium point belonging to $\mathcal{E}_1$ is not completely controllable. Indeed the controllability matrix has rank $8 < n = 9$ for every $\boldsymbol{x}_{eq} \in \mathcal{E}_1$ hence it is not possible to conclude on the small-time local controllability of the system in those points.

As an aside is interesting to study the stabilizability of the linear approximation. Using the Control System Toolbox from MATLAB a change of coordinates can be found such that

$$\dot{\boldsymbol{\delta z}} = \bar{A}\boldsymbol{\delta z} + \bar{B}\boldsymbol{\delta u}$$

where

$$\bar{A} = TAT^{\mathrm{T}} = \begin{bmatrix} A_{uc} & 0 \\ A_{21} & A_c \end{bmatrix}$$

$$\bar{B} = TB = \begin{bmatrix} 0 \\ B_c \end{bmatrix}$$

and T is an unitary matrix. For the system under investigation $A_{uc} = 0$, i.e., the uncontrollable eigenvalue is 0 with algebraic multiplicity one hence the system is stabilizable with a control law of the form $\boldsymbol{\delta u} = K\boldsymbol{\delta x}$ for some appropriate constant matrix K. Using arguments based on the theory of Lyapunov stability it can be also conluded that the nonlinear system of equations (7) subjected to the same control law has a *stable* point of equilibrium at $\boldsymbol{x}_{eq} \in \mathcal{E}_1$.

2.2 Nonlinear local accessibility

Local controllability of nonlinear systems is a quite difficult property to be verified. For this reason a different definition of local controllability is used in which, in addition, is required that the trajectory from the initial point $\boldsymbol{x}_0$ to the final point $\boldsymbol{x}(T)$ never goes outside of a small neighbourhood $V(\boldsymbol{x}_0)$. To be more precise a set $R_T^V(\boldsymbol{x}_0)$ is defined

$$R_T^V(\boldsymbol{x}_0) = \{\boldsymbol{x}(\boldsymbol{x_0}, T, \bar{\boldsymbol{u}}(t)) \mid \boldsymbol{x}(\boldsymbol{x_0}, \tau, \bar{\boldsymbol{u}}(t)) \in V(\boldsymbol{x}_0) \;\; \forall \tau \in [0, T]\}$$

Then a nonlinear system is said locally-locally controllable (l.l.c.) from $\boldsymbol{x}_0$ if, for any arbitrarily small neighbourhood $V(\boldsymbol{x}_0)$ there exists T such that $R_T^V(\boldsymbol{x}_0)$ contains an open neighbourhood of $\boldsymbol{x}_0$. Again if the time T can be arbitrarily small the system is said small time locally controllable (s.t.l.c.).

In some cases however $R_T^V(\boldsymbol{x}_0)$ only contains an open set of $\boldsymbol{x}_0$ and the definition of small-time local controllability becomes small-time local accessibility (s.t.l.a). It should be noted that s.t.l.c implies s.t.l.a but the converse does not hold.

In order to study the small-time locally accessibility of a system written in control affine form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} \tag{10}$$

the distributions

$$\Delta_0 = \mathrm{span}\{\boldsymbol{g}_1, \;\; \dots \;\; , \boldsymbol{g}_m\}$$

$$\Delta = \mathrm{span}\{\boldsymbol{f}, \;\; \boldsymbol{g}_1, \;\; \dots \;\; , \boldsymbol{g}_m\}$$

have to be considered. The following theorem from Chow holds

Theorem 2 *Consider the smallest Δ-invariant distribution containing Δ_0 called accessibility distribution $< \Delta|\Delta_0 >$.*

a. If the dimension of $< \Delta|\Delta_0 >= n$ in $\boldsymbol{x}_0$, then system (10) is s.t.l.a in $\boldsymbol{x}_0$;

b. If $\dim(< \Delta|\Delta_0 >) = r < n$ in a neighbourhood of $\boldsymbol{x}_0$ then the set $R_T^V(\boldsymbol{x}_0)$ is contained in a submanifold of dimension r of the n-dimensional state space, and contains an open set in that submanifold.

In order to evaluate the accessibility distribution for the system under examination (equation 7) the following filtration of distributions

$$\begin{cases} \Delta_1 & = \Delta_0 + [\Delta_0, \Delta] \\ & \vdots \\ \Delta_k & = \Delta_{k-1} + [\Delta_{k-1}, \Delta] \end{cases}$$

was performed until an integer k was found such that

$$\dim(\Delta_k(\boldsymbol{x}_0)) = \dim(\Delta_{k+1}(\boldsymbol{x}_0))$$

It turns out that for all $\boldsymbol{x}_0 \in \mathcal{E}_1$

$$\dim(< \Delta|\Delta_0 >) = 8 < n = 9$$

Hence $R_T^V(\mathcal{E}_1)$ is containted in a submanifold of dimension 8.

Although the system is not s.t.l.a in the points of interest it could be local accessible in a weaker sense, i.e., without the requirement that the time T is arbitrarily small. A *necessary* condition for weak local accessibility is that the dimension of the smallest Δ-invariant distribution containing Δ is equal to n. For the system under examination it can be found that

$$\dim < \Delta, \Delta >= 8 < n = 9$$

for every $\boldsymbol{x}_0 \in \mathcal{E}_1$.

In conclusion, at least for the equilibria that belong to $\mathcal{E}_1$, the system is not locally accessible in any sense hence it can't be locally controllable.

3 Observability

In this section the nonlinear *local* observability of the system is discussed. The concept of local observability can be explained in terms of the indistinguishability between a given *initial* state $\bar{\boldsymbol{x}}$ and another initial state "near" $\bar{\boldsymbol{x}}$ namely $\bar{\boldsymbol{x}} + \boldsymbol{\delta x}$.

The initial states $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{x}} + \boldsymbol{\delta x}$ are said to be indistinguishable in the interval $[0, T]$ if for every input $\boldsymbol{u}$ the outputs $\boldsymbol{y(\bar{x}, u, t)} = \boldsymbol{y(\bar{x} + \delta x, u, t)}$ for every $t \in [0, T]$.

Local indistinguishability can be assessed using appropriate tools from the theory of nonlinear systems and in some cases using the theory of LTI systems as explained in the following.

3.1 Linear Observability

In order to study the local indistinguishability between two initial states using the theory of LTI systems the following theorem can be used.

Theorem 3 *Consider a nonlinear system*

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}) \end{cases}$$

and an equilibrium point $\boldsymbol{x}_{eq}$ *such that* $\boldsymbol{f}(\boldsymbol{x}_{eq}, \boldsymbol{0}) = \boldsymbol{0}$. *If the linear approximation of the system around the point* $\boldsymbol{x}_{eq}$ *is completely observable then there are no indistinguishable points from* $\boldsymbol{x}_{eq}$ *in a small enough neighbourhood of* $\boldsymbol{x}_{eq}$.

It is recalled that a linear system

$$\dot{\boldsymbol{\delta x}} = A\boldsymbol{\delta x} + B\boldsymbol{\delta u}$$

$$\dot{\boldsymbol{\delta y}} = C\boldsymbol{\delta x}$$

where

$$\boldsymbol{\delta x}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{eq} \quad \boldsymbol{\delta u}(t) = \boldsymbol{u}(t) - \boldsymbol{u}_{eq} \quad \boldsymbol{\delta y}(t) = \boldsymbol{y}(t) - \boldsymbol{h}(\boldsymbol{x}_{eq}, \boldsymbol{u}_{eq})$$

is observable if the rank of the observability matrix R_o

$$R_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{11}$$

is equal to the dimension of the state n.

For the system of equations (7) and (8) it can be found that the linear approximation around each equilibrium point belonging to $\mathcal{E}_1$ is not completely observable and the matrix R_o has rank $6 < n = 9$. As a consequence it is not possible to conclude on the local indistinguishability relative to those points.

3.2 Nonlinear Observability

As done for the nonlinear local controllability the definition of local indistinguishability should be refined with the requirement that the inputs $\boldsymbol{u}$ are such that the state trajectory never goes outside of a neighbourhood of the initial states for every $t \in [0, T]$. The notation used to denote two indistinguishable initial states $\bar{\boldsymbol{x}}_1$ and $\bar{\boldsymbol{x}}_2$ in the interval $[0, T]$ is

$$\bar{\boldsymbol{x}}_1 I_T^U \bar{\boldsymbol{x}}_2$$

where $U \subseteq \{\boldsymbol{u}(\cdot) : [0, T] \to \mathbb{R}^m\}$ contains input functions which satisfy the hypothesis given above.

In order to study the nonlinear locally indistinguishability of a system written in control affine form

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}) \end{cases} \tag{12}$$

the codistribution

$$\Omega_0 = \mathrm{span}\{d\boldsymbol{h}\}$$

and the distribution

$$\Delta = \mathrm{span}\{\boldsymbol{f},\ \boldsymbol{g}_1,\ \dots\ , \boldsymbol{g}_m\}$$

have to be considered. The following theorem holds

Theorem 4 *Consider the smallest Δ-invariant codistribution containing Ω_0 called observability codistribution $< \Delta | d\boldsymbol{h} >$.*

a. *If the dimension of $< \Delta | d\boldsymbol{h} >$ is equal to n in $\bar{\boldsymbol{x}}$ then there are no initial states "near" $\bar{\boldsymbol{x}}$ that are indistinguishable from it and the system (12) is said locally observable in $\bar{\boldsymbol{x}}$;*

b. *If $dim(< \varDelta | \varOmega_0 >) = d < n$ in a neighbourhood of $\bar{\boldsymbol{x}}$ then the $(n-d)$-dimensional distribution $< \varDelta | \varOmega_0 >^{\perp}$ that annihilates the observability codistribution is involutive. Clearly such distribution evaluated at $\bar{\boldsymbol{x}}$ identifies the displacements $\boldsymbol{\delta x}$ such that $\bar{\boldsymbol{x}} I_T^U (\bar{\boldsymbol{x}} + \boldsymbol{\delta x})$*

In order to evaluate the observability codistribution for the system under examination (equations 7 and 8) the following filtration of codistributions

$$\begin{cases} \Omega_1 & = \Omega_0 + L_\Delta \Omega_0 \\ & \vdots \\ \Omega_k & = \Omega_{k-1} + L_\Delta \Omega_{k-1} \end{cases}$$

was performed until an integer k was found such that

$$\dim(\Delta_k(\bar{\boldsymbol{x}})) = \dim(\Delta_{k+1}(\bar{\boldsymbol{x}}))$$

It turns out that for all $\bar{\boldsymbol{x}} \in \mathcal{E}_1$, i.e.,the cubic frame is standing still in the upright configuration with zero flywheel velocities,

$$\dim(< \Delta, d\boldsymbol{h} >) = 6 < n = 9$$

but the codistribution is not regular, i.e., the dimension is not constant in a neighbourhood of a given $\bar{\boldsymbol{x}} \in \mathcal{E}_1$ hence the theorem (4) cannot be applied. Conversely any given initial state in which the cubic frame is standing still in the upright configuration with *non zero angular rates* gives

$$\dim(< \Delta, d\boldsymbol{h} >) = n$$

hence the system is locally observable in those points.

4 Control of the attitude of the system

In this section a *noninteracting* approach to the control of the attitude of the system is introduced. The results of the simulations executed with MATLAB Simulink are then presented.

4.1 Noninteracting Control

Noninteracting Control is defined as the problem of finding a feedback law in order to reduce a multivariable system, from an input-output point of view, to a series of independent single-input single-output channels. The following presentation is taken from Isidori [2].

The starting point is that of a nonlinear system in control affine form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \sum_{i=1}^{m} \boldsymbol{g}_i(\boldsymbol{x}) u_i$$

$$y_1 = h_1(\boldsymbol{x})$$

$$\dots$$

$$y_m = h_m(\boldsymbol{x})$$

and an initial point $\boldsymbol{x}^0$ of the vector field $\boldsymbol{f}(x)$. Then the problem of noninteracting control is stated as the problem of finding a static state feedback [1] of the form

$$u_i = \alpha_i(\boldsymbol{x}) + \sum_{j=1}^{m} \beta_{ij}(\boldsymbol{x}) v_j \tag{13}$$

defined in a neighborhood U of $\boldsymbol{x}^0$ such that in the resulting closed loop system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \sum_{i=1}^{m} \alpha_i(\boldsymbol{x}) + \sum_{j=1}^{m} \left(\sum_{i=1}^{m} \boldsymbol{g}_i(\boldsymbol{x}) \beta_{ij}(\boldsymbol{x}) \right) v_j$$

$$y_1 = h_1(\boldsymbol{x})$$

$$\dots$$

$$y_m = h_m(\boldsymbol{x})$$

each output y_i is affected only by the corresponding input v_i and not by others.

The main result [2] about the noninteracting control problem is the following.

[1] To be more precise the static state feedback should be *regular*, i.e., the matrix $\beta(\boldsymbol{x})$ is nonsingular for all x.

Proposition 1 *Consider a multivariable nonlinear system with m inputs and m outputs*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \sum_{i=1}^{m} \boldsymbol{g}_i(\boldsymbol{x}) u_i$$
$$y_1 = h_1(\boldsymbol{x})$$
$$\dots$$
$$y_m = h_m(\boldsymbol{x})$$

Suppose

$$L\boldsymbol{g}_j L_{\boldsymbol{f}}^k h_i(\boldsymbol{x}) = 0 \quad 1 \leq j \leq m, \;\; 1 \leq i \leq m, \;\; k < r_i - 1 \tag{14}$$

for all $\boldsymbol{x}$ in a neighborhood of $\boldsymbol{x}^0$ and

$$\left[L_{\boldsymbol{g}_1} L_{\boldsymbol{f}}^{r_i-1} h_i(\boldsymbol{x}^0) \;\; \dots \;\; L_{\boldsymbol{g}_m} L_{\boldsymbol{f}}^{r_i-1} h_i(\boldsymbol{x}^0) \right] \neq \left[0 \;\; \dots \;\; 0 \right] \;\; 1 \leq i \leq m$$

Then the noninteracting control problem is solvable if and only if the matrix

$$[A(\boldsymbol{x})]_{ij} = a_{ij}(\boldsymbol{x}) = L_{\boldsymbol{g}_j} L_{\boldsymbol{f}}^{r_i-1} h_i(\boldsymbol{x}) \tag{15}$$

evaluated at the point $\boldsymbol{x}^0$ is nonsingular.

The condition (14) and the non singularity of the matrix A also correspond to the definition of a multivariable nonlinear system with a *vector relative degree* $\{r_1, \dots, r_m\}$ at the point $\boldsymbol{x}^0$ and imply that *i-th* output of the system has to be differentiated r_i times in order to have at least one component of the input vector $\boldsymbol{u}(t_0)$ explicitly appearing at the time t_0 in which the state assumes the value $\boldsymbol{x}^0$.

In order to find the actual form of the controls (13) the sufficiency of the proposition (1) is explained.

First of all the system is written in a special form called *normal form* obtained using the following coordinates transformation

$$\Phi(x) = [\phi_1^1(\boldsymbol{x}), \dots, \phi_{r_1}^1(\boldsymbol{x}), \dots, \phi_1^m(\boldsymbol{x}), \dots, \phi_{r_m}^m(\boldsymbol{x}), \phi_{r+1}(\boldsymbol{x}), \dots, \phi_n(\boldsymbol{x})]$$

$$\phi_1^i(\boldsymbol{x}) = h_i(\boldsymbol{x})$$
$$\phi_2^i(\boldsymbol{x}) = L_{\boldsymbol{f}} h_i(\boldsymbol{x})$$
$$\dots$$
$$\phi_{r_i}^i(\boldsymbol{x}) = L_{\boldsymbol{f}}^{r_i-1} h_i(\boldsymbol{x})$$

where n is the dimension of the state $\boldsymbol{x}$ and $r = r_1 + \ldots + r_m$. The fact that the system has a vector relative degree at $\boldsymbol{x}^0$ assures that if $r < n$ it is always possible to find $n - r$ functions $\phi_{r+1}, \ldots, \phi_n$ such that $\Phi(\boldsymbol{x})$ has a jacobian matrix which is nonsingular at $\boldsymbol{x}^0$ hence it can be used as a local coordinates transformation in a neighborhood of $\boldsymbol{x}^0$. The new state is now defined as

$$\boldsymbol{z} = \Phi(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{bmatrix}$$

where

$$\boldsymbol{\xi} = [(\boldsymbol{\xi}^1)^T, \ldots, (\boldsymbol{\xi}^m)^T]^T$$

$$\boldsymbol{\xi}^i = \begin{bmatrix} \xi_1^i \\ \xi_2^i \\ \vdots \\ \xi_{r_i}^i \end{bmatrix} = \begin{bmatrix} h_i(\boldsymbol{x}) \\ L_{\boldsymbol{f}} h_i(\boldsymbol{x}) \\ \vdots \\ L_{\boldsymbol{f}}^{r_i - 1} h_i(\boldsymbol{x}) \end{bmatrix}$$

and

$$\boldsymbol{\eta} = \begin{bmatrix} \phi_{r+1}(\boldsymbol{x}) \\ \phi_{r+2}(\boldsymbol{x}) \\ \vdots \\ \phi_n(\boldsymbol{x}) \end{bmatrix}$$

Using the definition of vector relative degree the time rate of change of $\boldsymbol{\xi}^i$ evaluates to

$$\dot{\xi}_1^i = \xi_2^i$$

$$\ldots$$

$$\dot{\xi}_{r_i - 1}^i = \xi_{r_i}^i$$

$$\dot{\xi}_{r_i}^i = b_i(\boldsymbol{\xi}, \boldsymbol{\eta}) + \sum_{j=1}^{m} a_{ij}(\boldsymbol{\xi}, \boldsymbol{\eta}) u_j$$

for $1 \leq i \leq m$ where

$$b_i(\boldsymbol{\xi}, \boldsymbol{\eta}) = L_{\boldsymbol{f}}^{r_i} h_i(\Phi^{-1}(\boldsymbol{\xi}, \boldsymbol{\eta})) \;\; 1 \leq i \leq m$$

and

$$a_{ij}(\boldsymbol{\xi}, \boldsymbol{\eta}) = L_{\boldsymbol{g}_j} L_{\boldsymbol{f}}^{r_i - 1} h_i(\Phi^{-1}(\boldsymbol{\xi}, \boldsymbol{\eta})) \;\; 1 \leq i, j \leq m$$

is the same as defined in (15) but evaluated in $\boldsymbol{x} = \Phi^{-1}(\boldsymbol{\xi}, \boldsymbol{\eta})$. As regards the remaining part of the new coordinates $\boldsymbol{\eta}$ only a generic equation of the form

$$\dot{\boldsymbol{\eta}} = \boldsymbol{q}(\boldsymbol{\xi}, \boldsymbol{\eta}) + \boldsymbol{p}(\boldsymbol{\xi}, \boldsymbol{\eta}) \boldsymbol{u}$$

can be stated for some vector valued functions $\boldsymbol{q}$ and $\boldsymbol{p}$.

Collecting the coefficients $b_i(\boldsymbol{\xi}, \boldsymbol{\eta})$ in a vector $\boldsymbol{b}(\boldsymbol{x})$ evaluated at $\boldsymbol{x} = \Phi^{-1}(\boldsymbol{\xi}, \boldsymbol{\eta})$ then the vector containing the higher order derivatives of each output can be written as

$$\begin{bmatrix} \dot{\xi}^1_{r_1} \\ \vdots \\ \dot{\xi}^m_{r_m} \end{bmatrix} = \boldsymbol{b}(\boldsymbol{\xi}, \boldsymbol{\eta}) + A(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{u}$$

If now a feedback law of the form

$$\boldsymbol{u} = -A^{-1}(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{b}(\boldsymbol{\xi}, \boldsymbol{\eta}) + A^{-1}(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{v}$$

is selected one obtains

$$\begin{aligned} \begin{bmatrix} \dot{\xi}^1_{r_1} \\ \vdots \\ \dot{\xi}^m_{r_m} \end{bmatrix} =& \boldsymbol{b}(\boldsymbol{\xi}, \boldsymbol{\eta}) + A(\boldsymbol{\xi}, \boldsymbol{\eta})(-A^{-1}(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{b}(\boldsymbol{\xi}, \boldsymbol{\eta}) + A^{-1}(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{v}) \\ =& \boldsymbol{b}(\boldsymbol{\xi}, \boldsymbol{\eta}) - \boldsymbol{b}(\boldsymbol{\xi}, \boldsymbol{\eta}) + \boldsymbol{v} = \boldsymbol{v} \end{aligned}$$

In summary the closed loop system is characterized by the m sets of equations

$$\begin{aligned} \dot{\xi}^i_1 &= \xi^i_2 \\ &\dots \\ \dot{\xi}^i_{r_i-1} &= \xi^i_{r_i} \\ \dot{\xi}^i_{r_i} &= v_i \\ y_i &= \xi^i_1 \end{aligned}$$

for $1 \leq i \leq m$, together with

$$\begin{aligned} \dot{\boldsymbol{\eta}} =& \boldsymbol{q}(\boldsymbol{\xi}, \boldsymbol{\eta}) - \boldsymbol{p}(\boldsymbol{\xi}, \boldsymbol{\eta})A^{-1}(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{b}(\boldsymbol{\xi}, \boldsymbol{\eta}) + \boldsymbol{p}(\boldsymbol{\xi}, \boldsymbol{\eta})A^{-1}(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{v} \\ =& \hat{\boldsymbol{q}}(\boldsymbol{\xi}, \boldsymbol{\eta}) + \hat{\boldsymbol{p}}(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{v} \end{aligned}$$

As shown in the block diagram in Figure 4 each input v_i controls only the output y_i through a chain of r_i integrators as required by the noninteractive control problem. If $r = r_1 + \ldots + r_m < n$ the closed loop system also has an unobservable part which is affected by all inputs and all states but does not affect the outputs.

Reverting back to the original set of coordinates of the system the feedback law results

$$\boldsymbol{u} = \boldsymbol{\alpha}(\boldsymbol{x}) + \beta(\boldsymbol{x})\boldsymbol{v}$$

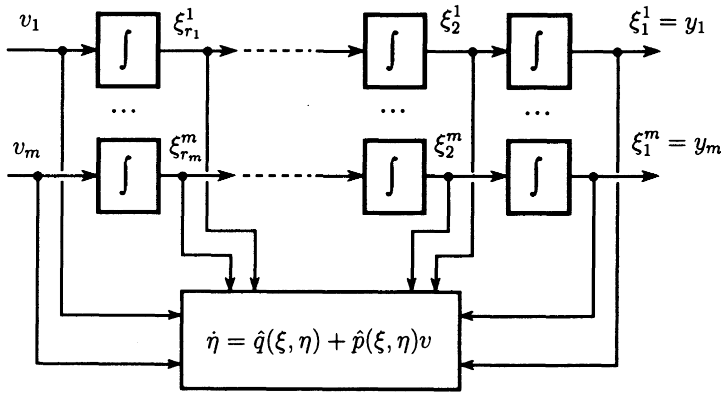


Figure 4: Noninteractive control block diagram

where

$$\boldsymbol{\alpha}(\boldsymbol{x}) = -A^{-1}\boldsymbol{b}(\boldsymbol{x}) \qquad \beta(\boldsymbol{x}) = A^{-1}(\boldsymbol{x})$$

This form of feedback is also called *Standard Noninteractive Feedback* and in general is defined only *locally* in the state space, i.e., for all $\boldsymbol{x}$ near $\boldsymbol{x}^0$ at which the matrix $A(\boldsymbol{x})$ is nonsingular.

4.2 Application of the noninteractive feedback

Consider the system (7)

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} \\ &= \boldsymbol{f}(\boldsymbol{x}) + \begin{bmatrix} 0_{3\times 3} \\ (B(\theta_2,\theta_3)^{-1})_{(:,4:6)} \end{bmatrix} \boldsymbol{u}\end{aligned}$$

with outputs given in eq. (8)

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) = \begin{bmatrix}\theta_1 & \theta_2 & \theta_3\end{bmatrix}^T$$

The purpose of this section is to assess whether, by means of a feedback of the form (13), a closed loop system in which each angle θ_i is controlled independently of others can be obtained. If this is possible than the final aim is to obtain a second order dynamics of the form

$$\ddot{\theta}_i = \ddot{\theta}_i^{des} + k_d(\dot{\theta}_i^{des} - \dot{\theta}_i) + k_p(\theta_i^{des} - \theta_i) \tag{16}$$

where $\ddot{\theta}_i^{des}$, $\dot{\theta}_i^{des}$ and θ_i^{des} are the desired trajectories.

In order to apply the noninteracting control approach to the system it is required to verify if the system has a vector relative degree at some point $\boldsymbol{x}^0$ in the state space. Simple calculations

$$L_{\boldsymbol{g}_j} L_{\boldsymbol{f}}^0 h_i = L_{\boldsymbol{g}_j} h_i = \begin{bmatrix} \boldsymbol{e}_i & 0_{6x1} \end{bmatrix} \boldsymbol{g}_j = g_j^i = 0 \qquad 1 \leq j \leq 3,\ 1 \leq i \leq 3$$

show that the condition (14) is verified for every $\boldsymbol{x}$ in the state space. Also the second derivative of the output with respect to time can be written as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{y} = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} = \boldsymbol{b}(\boldsymbol{x}) + A(\boldsymbol{x})\boldsymbol{u}$$

where

$$\boldsymbol{b}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})_{(4:6)}$$

and

$$A(\boldsymbol{x}) = \left(B(\theta_2, \theta_3)^{-1}\right)_{(1:3,4:6)}$$

As shown in figure 5 the determinant of the matrix A is nonzero for every

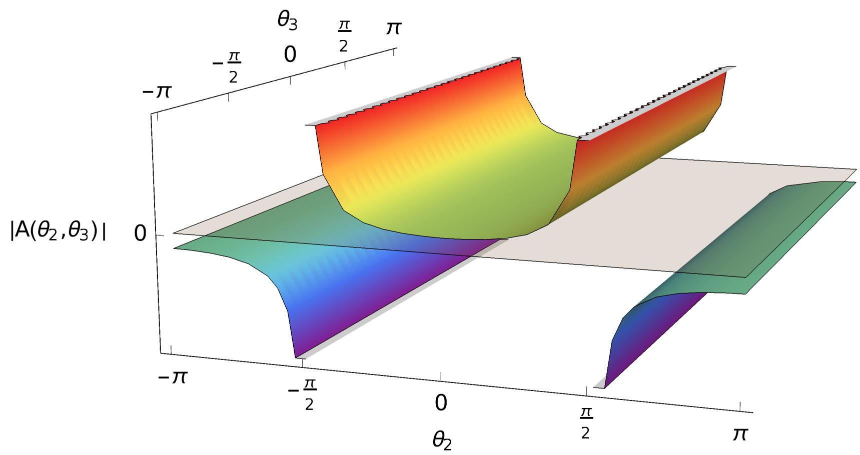


Figure 5: Determinant of the decoupling matrix A

θ_2 and θ_3 hence the system has a vector relative degree $\{2, 2, 2\}$ everywhere. The resulting standard noninteractive feedback is

$$\boldsymbol{u}_{NIC} = -\left(B(\theta_2, \theta_3)^{-1}\right)^{-1}_{(1:3,4:6)} \left(\boldsymbol{f}_{(4:6)}(\boldsymbol{x}) - \boldsymbol{v}\right)$$

where $\boldsymbol{v}$ can be chosen as

$$v_i(\theta_i, \dot{\theta}_i) = \ddot{\theta}_i^{des} + k_d(\dot{\theta}_i^{des} - \dot{\theta}_i) + k_p(\theta_i^{des} - \theta_i) \tag{17}$$

in order to obtain the desired closed loop dynamics (eq. 16).

4.2.1 Unobservable part of the system

In order to write the equations of the unobservable part of the closed loop system the coordinate transformation Φ must be specified entirely. The choice of the outputs

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) = \begin{bmatrix}\theta_1 & \theta_2 & \theta_3\end{bmatrix}^T$$

implies

$$\begin{aligned} \phi_1(\boldsymbol{x}) = \theta_1 \quad \phi_3(\boldsymbol{x}) = \theta_2 \quad \phi_5(\boldsymbol{x}) = \theta_3 \\ \phi_2(\boldsymbol{x}) = \dot{\theta}_1 \quad \phi_4(\boldsymbol{x}) = \dot{\theta}_2 \quad \phi_6(\boldsymbol{x}) = \dot{\theta}_3 \end{aligned}$$

The remaining 3 functions $\phi_7(\boldsymbol{x})$, $\phi_8(\boldsymbol{x})$ and $\phi_9(\boldsymbol{x})$ are chosen to be the last three components of the state

$$\phi_7(\boldsymbol{x}) = x_7 = \dot{q}_x \quad \phi_8(\boldsymbol{x}) = x_8 = \dot{q}_y \quad \phi_9(\boldsymbol{x}) = x_9 = \dot{q}_z$$

The resulting mapping $\Phi(\boldsymbol{x})$ has a jacobian

$$\frac{\partial \Phi(\boldsymbol{x})}{\partial \boldsymbol{x}} = \begin{bmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_3 & \boldsymbol{e}_5 & \boldsymbol{e}_2 & \boldsymbol{e}_4 & \boldsymbol{e}_6 & \boldsymbol{e}_7 & \boldsymbol{e}_8 & \boldsymbol{e}_9\end{bmatrix}$$

which is nonsingular and hence can be used as coordinate transformation everywhere.

The dynamics of the unobservable part of the closed loop system, i.e., of the flywheels, can be written as

$$\begin{aligned} \dot{\boldsymbol{\eta}} = \begin{bmatrix}\ddot{q}_x \\ \ddot{q}_y \\ \ddot{q}_z\end{bmatrix} &= \boldsymbol{f}_{(7:9)}(\boldsymbol{x}) + \left(B(\theta_2,\theta_3)^{-1}\right)_{(4:6,4:6)} \boldsymbol{u}_{NIC} \\ &= \boldsymbol{f}_{(7:9)}(\boldsymbol{x}) - \left(B(\theta_2,\theta_3)^{-1}\right)_{(4:6,4:6)} \left(B(\theta_2,\theta_3)^{-1}\right)^{-1}_{(1:3,4:6)} \left(\boldsymbol{f}_{(4:6)}(\boldsymbol{x}) - \boldsymbol{v}\right) \\ &= \hat{\boldsymbol{q}}(\boldsymbol{x}) + \hat{\boldsymbol{p}}(\theta_2,\theta_3)\boldsymbol{v} \end{aligned}$$

4.3 Results of simulation

In the following figures the results of the simulation done in MATLAB Simulink are presented. The desired trajectory, corresponding to a rotation about the z axis of the inertial reference frame $\{S\}$ while the balance on one of the corner is maintained, was chosen as

$$\theta_1^{des}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

with boundary conditions

$$\begin{aligned} \theta_1^{des}(0) = 0 \quad \dot{\theta}_1^{des}(0) = 0 \quad \ddot{\theta}_1^{des}(0) = 0 \\ \theta_1^{des}(t_f) = \pi \quad \dot{\theta}_1^{des}(t_f) = 0 \quad \ddot{\theta}_1^{des}(t_f) = 0 \end{aligned}$$

and

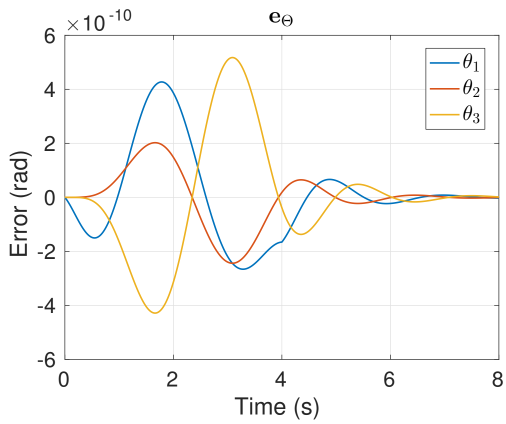
$$\theta_2^{des}(t) \equiv -\mathrm{atan}\frac{\sqrt{2}}{2}$$
$$\theta_3^{des}(t) \equiv \frac{\pi}{4}$$

The resulting control law is

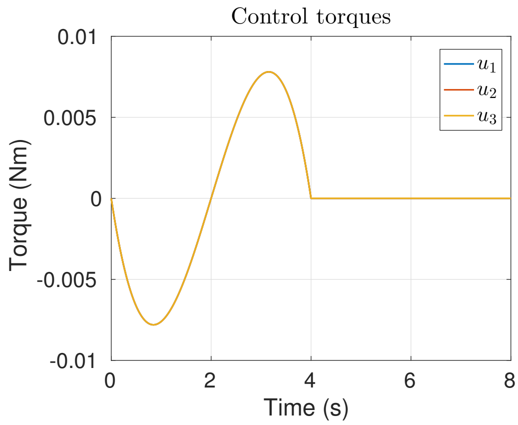
$$\begin{aligned} v_1(t) &= \ddot{\theta}_1^{des}(t) + k_d(\dot{\theta}_1^{des}(t) - \dot{\theta}_1(t)) + k_p(\theta_1^{des}(t) - \theta_1(t)) \\ v_2(t) &= -k_d\dot{\theta}_2(t) + k_p\left(-\mathrm{atan}\frac{\sqrt{2}}{2} - \theta_2(t)\right) \\ v_3(t) &= -k_d\dot{\theta}_3(t) + k_p\left(\frac{\pi}{4} - \theta_3(t)\right) \end{aligned} \tag{18}$$

It should be noted that the initial state $\boldsymbol{x}(0)$ and the final state $\boldsymbol{x}(t_f)$ corresponding to the desired trajectory belong to the set of equilibria $\mathcal{E}_1$ provided that the velocities of the flywheels are zero in $t = 0$ and $t = t_f$.

In figure 1a the orientation error is shown while in figure 1b the required torques are shown.



(a) Orientation error



(b) Control torques

Table 1: Results of simulation ($k_p = 10$, $k_d = 0.1$)

4.3.1 Steady state behavior of the unobservable subsystem

As shown in the figure 6 the flywheel velocities $\boldsymbol{\eta}(t)$ which are solution to the differential equation

$$\begin{cases} \dot{\boldsymbol{\eta}} = \hat{\boldsymbol{q}}(\boldsymbol{x}) + \hat{\boldsymbol{p}}\left(-\mathrm{atan}\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)\boldsymbol{v} \\ \boldsymbol{\eta}(0) = \boldsymbol{0} \end{cases}$$

with $\boldsymbol{v}$ given in (18) are bounded and tends to zero.

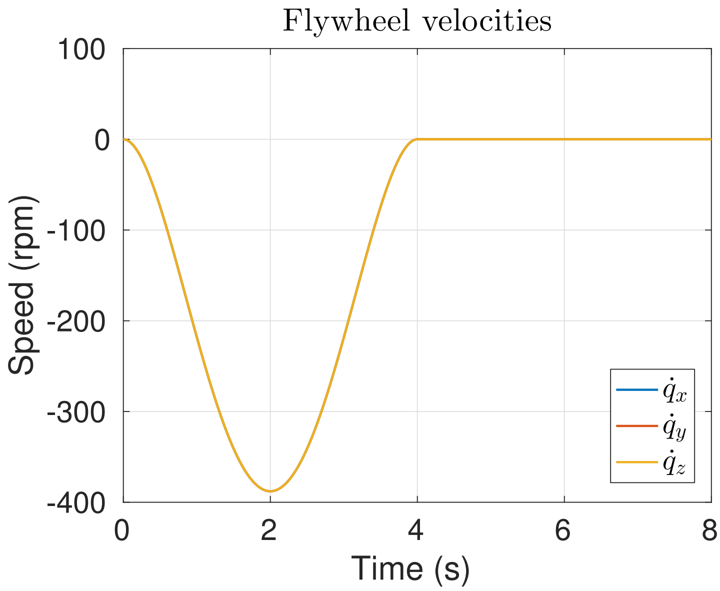


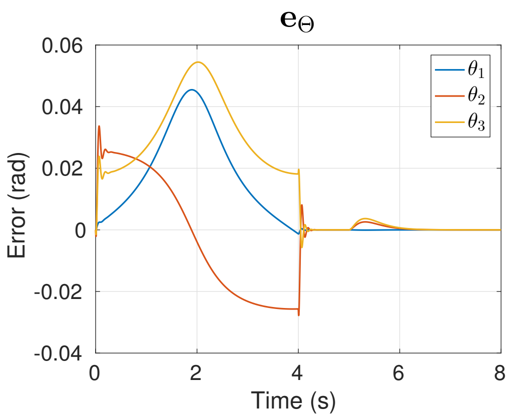
Figure 6: Flywheel velocities ($k_p = 10$, $k_d = 0.1$)

4.3.2 Use of an alternative Euler parametrization

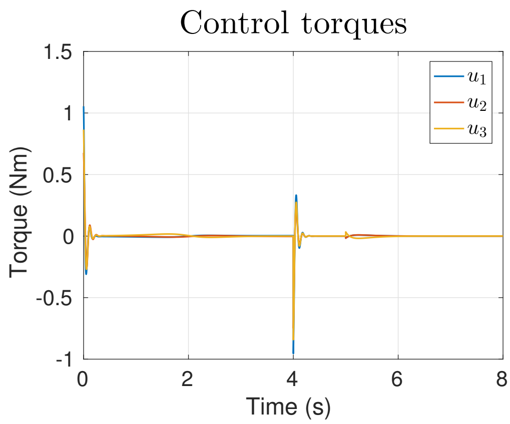
In this section the results of the same simulation obtained using a different parametrization of the rotation matrix R_{SC}, i.e. Euler XYZ instead of Euler ZYX, are presented. The theory described up to now can be still applied with the exception that the rotation of the cubic frame around the z axis of the intertial reference frame $\{S\}$ cannot be mapped to a single angle. Instead all three angles have to change and the intermediate poses of the frame, which determine the desired trajectory, can be found using a spherical linear interpolation approach (Slerp). Using this set-up however the simulations reveal that the velocites of the flywheels reamain non-zero but constant at the end of the rotation. In other words the state $\boldsymbol{\eta}$ of the unobservable part of the closed loop system subjected to the new slerp-based input does not tend to zero. Altough impractical in a real implementation this behavior is compatible with the existence of equilibria in which the system is at rest in the upright position while the flywheels rotates at a non zero constant angular velocity.

A possible solution to slow down the flywheel velocities is to switch to a LQR based control after the rotation is completed. Such a controller should be synthesized using the linear approximation of the system around an equilibrium point where the system is at rest in the upright position with zero flywheel velocities. Extensive simulations show that the linear approximation around such equilibrium points is always stabilizable. However there are cases in which the flywheel velocities before the switching are too high and the state of the system is out of the region of attraction of the LQR controller.

In the figures below an example of the described approach is presented. The desired trajectory and the NIC controller allow to rotate the cubic frame by an angle of π in 4 s with zero steady state error. As can be seen from figure 7 the flywheel velocities reamain non-zero but constant after the rotation. At time $t = 5$ s the LQR controller is actived and the flywheel velocities tends to zero asymptotically.



(a) Orientation error



(b) Control torques

Table 2: Results of simulation ($k_p = 2500$, $k_d = 50$)

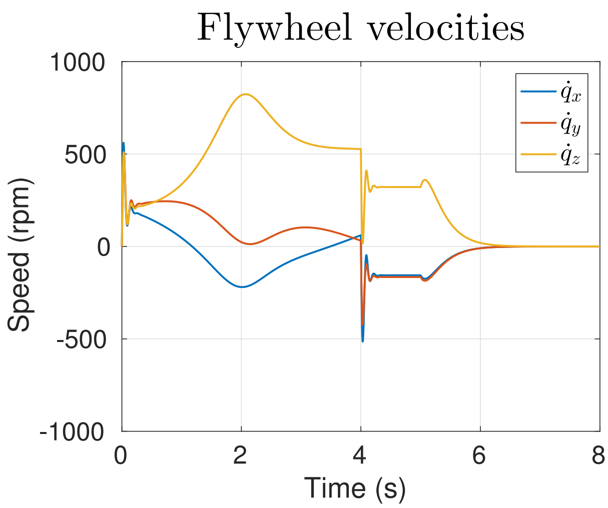


Figure 7: Flywheel velocities ($k_p = 2500$, $k_d = 50$)

In the figure 8 the outcome of the switch to the LQR controller is shown for several angles of rotation and several durations of the movement. In the

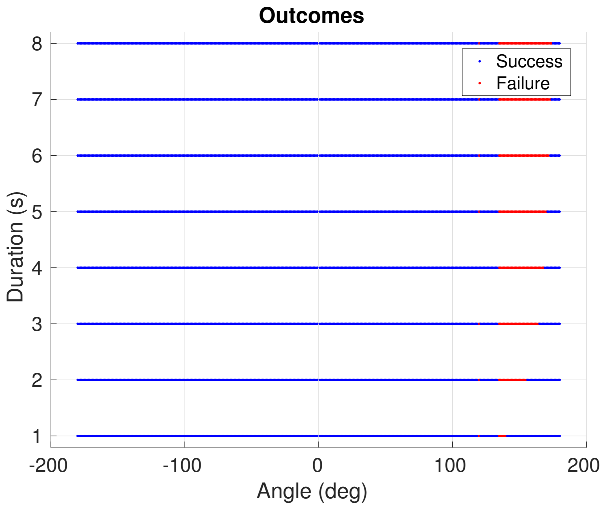


Figure 8: Outcome of the switch to LQR

figure 9 it is shown that the failures happen when the flywheel velocities are too high.

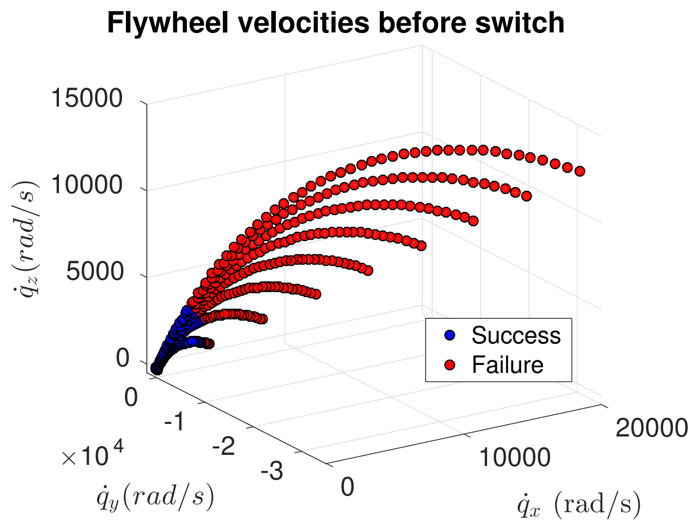


Figure 9: Flywheel velocities before the switch to LQR

References

[1] M. Gajamohan, M. Muehlebach, T. Widmer, and R. D'Andrea, "The Cubli: A Reaction Wheel Based 3D Inverted Pendulum," *Proceedings of the European Control Conference (ECC)*, pp. 268–274, 2013.

[2] A. Isidori, "Nonlinear Control Systems, Second Edition," 1999.