Seminar 5 Solutions

Giulio Rossetti*

giuliorossetti94.github.io

February 13, 2025

* email: giulio.rossetti.1@wbs.ac.uk

Disclaimer

Full solutions are available on my.wbs. All exercises are examinable material, not just the ones we covered in the seminars.

Roadmap

Exercise 1

Exercise 3

Exercise 4

Exercise 6

Exercise 1

Q: The OLS estimator in a time-series setting is unbiased under the first three Gauss-Markov assumptions. true

1. linearity in params
2. no multicollinearity
3. $E[u_t | X] = 0$ strict exogeneity

$$E[\hat{\beta}] = \beta$$

Exercise 1

Q: The OLS estimator in a time-series setting is unbiased under the first three Gauss-Markov assumptions.

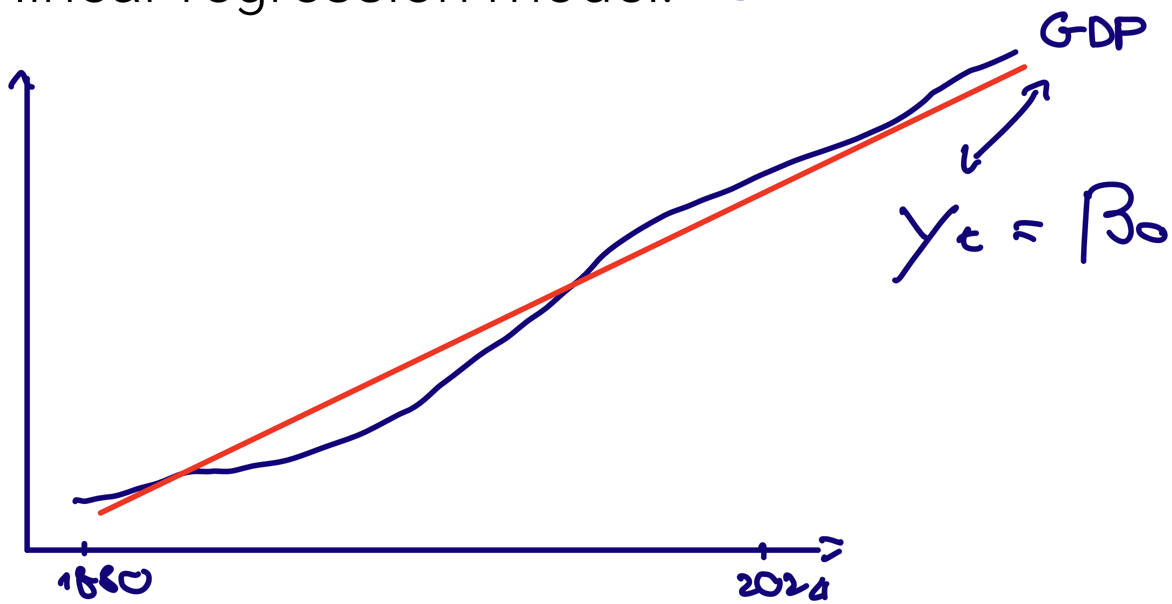
1. TS.1: Linearity in parameters
2. TS.2: No perfect collinearity
3. TS.3: Strict exogeneity/ Zero conditional mean

When we add the following two assumptions, the OLS estimator is also BLUE.

4. TS.4: Homoskedasticity $var(u_t|X) = \sigma^2$
5. TS.5: No serial correlation $corr(u_t, u_s) = 0 \quad t \neq s$

Exercise 1

Q: A trending variable cannot be used as a dependent variable in the multiple linear regression model. false



$$Y_t = \beta_0 + \beta_1 X_t + \underline{\underline{\beta_2 t}} + u_t$$

capture the trend

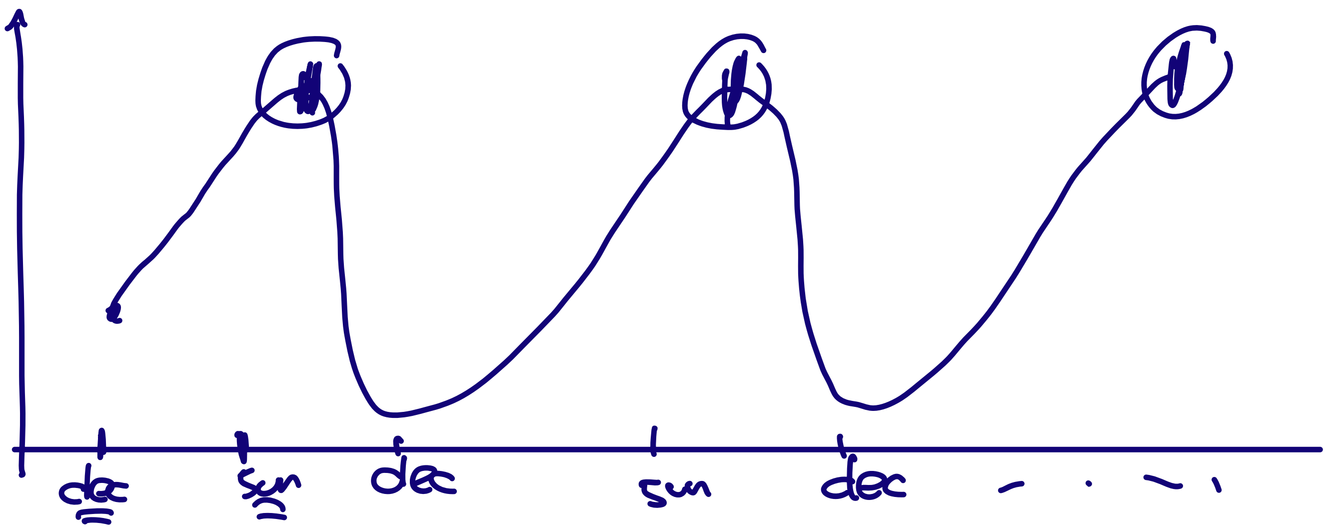
Exercise 1

Q: A trending variable cannot be used as a dependent variable in the multiple linear regression model.

- Trending variables *can* be used as dependent variables in a linear regression model.
- However, be cautious when interpreting the results:
    - spurious relationship between y_t and trending explanatory variables.
- Including a time trend in the regression is advisable when dependent and/or independent variables are trending.
- The usual R^2 measure can be misleading when the dependent variable is trending.

Exercise 1

Q: Seasonality is not an issue when using annual time-series observations. true



Exercise 1

Q: Seasonality is not an issue when using annual time-series observations.

- Each period represents a year and this is not associated with any season.

Roadmap

Exercise 1

Exercise 3

Exercise 4

Exercise 6

Exercise 3

Weakly Stationary Process: Correlation

Let $\{x_t : t = 1, 2, \ldots, T\}$ be a weakly stationary process.

Define $\gamma_h = \mathrm{Cov}(x_t, x_{t+h})$ for $h \geq 0$. Then $\gamma_0 = \mathrm{Var}(x_t)$. Show that

$$\mathrm{Corr}(x_t, x_{t+h}) = \frac{\gamma_h}{\gamma_0}.$$

Weak (or covariance) Stationarity

A stochastic process $\{x_t : t = 1, 2, \ldots\}$ is said to be weakly stationary if:

$$\mathbb{E}(x_t) = \mu, \quad \mathrm{Var}(x_t) = \sigma^2, \quad \mathrm{Cov}(x_t, x_{t+h}) = f(h).$$

$E[x_t] = E[x_{t+h}] = \mu$

A weakly stationary process is uniquely determined by its mean, variance, and autocovariance function.

Derivation: $\rho(X,Y) = \dfrac{\mathrm{cov}(X,Y)}{\mathrm{std}(X)\ \mathrm{std}(Y)}$

$$\mathrm{corr}(x_t, x_{t+h}) = \frac{\mathrm{cov}(x_t, x_{t+h})}{\sqrt{\mathrm{var}(x_t)}\sqrt{\mathrm{var}(x_{t+h})}} = \frac{\gamma_h}{\sqrt{\gamma_0}\sqrt{\gamma_0}} = \frac{\gamma_h}{\gamma_0}$$

$$\gamma_h = \mathrm{cov}(x_t, x_{t+h})$$

$h = 0$

$$\gamma_0 = \mathrm{cov}(x_t, x_{t+0}) = \mathrm{cov}(x_t, x_t) = \mathrm{var}(x_t)$$

Roadmap

Exercise 1

Exercise 3

Exercise 4

Exercise 6

Exercise 4

Suppose that a time-series process $\{x_t : t = 1, 2, \ldots, T\}$ is given by

$$x_t = z + \epsilon_t,$$

for all $t = 1, 2, \ldots, T$, where ϵ_t is an i.i.d. sequence with mean zero and variance σ_ϵ^2. The random variable z is constant over time, and it has mean zero and variance σ_z^2. Furthermore, assume that ϵ_t is uncorrelated with z.

Exercise 4

$E[X+Y] = E[X] + E[Y]$

$Var(X+Y) = var(X) + var(Y) + 2cov(X,Y)$

Q: Find the expected value and variance of x_t. Do your answers depend on t? $E[x_t]$, $var(x_t)$

$x_t = z + \varepsilon_t$ $\quad \varepsilon_t \overset{iid}{\sim} (0, \sigma^2_\varepsilon)$ $\quad z \sim (0, \sigma^2_z)$

$cov(z, \varepsilon_t) = 0$

- $E[x_t] = E[z + \varepsilon_t] = E[z] + E[\varepsilon_t] = 0$ constant

  ($E[z] = 0$, $E[\varepsilon_t] = 0$)

- $var(x_t) = var(z + \varepsilon_t) = var(z) + var(\varepsilon_t) + 2cov(z, \varepsilon_t) = \sigma^2_z + \sigma^2_\varepsilon$ constant

  ($var(z) = \sigma^2_z$, $var(\varepsilon_t) = \sigma^2_\varepsilon$, $cov(z,\varepsilon_t) = 0$)

Exercise 4

$\operatorname{cov}(X,Y) = E[XY] - E[X]E[Y]$

$x_t = z + \varepsilon_t$ $\quad \dots \quad x_{t+h} = z + \varepsilon_{t+h}$

$x_{t+1} = z + \varepsilon_{t+1}$

Q: Find $\operatorname{Cov}(x_t, x_{t+h})$ for any t and h. Is x_t a weakly stationary process?

$$\operatorname{cov}(x_t, x_{t+h}) = E[x_t x_{t+h}] - \overbrace{\underbrace{E[x_t]}_{0}\underbrace{E[x_{t+h}]}_{0}}^{0}$$

$$= E[x_t x_{t+h}]$$

$$= E[(z+\varepsilon_t)(z+\varepsilon_{t+h})]$$

$$= E[z^2 + z\varepsilon_{t+h} + \varepsilon_t z + \varepsilon_t \varepsilon_{t+h}]$$

$$= E[z^2] + \overbrace{E[z\varepsilon_{t+h}]}^{0} + \underbrace{E[\varepsilon_t z]}_{\operatorname{cov}(\varepsilon_t, z) = 0} + \overbrace{E[\varepsilon_t \varepsilon_{t+h}]}^{0}$$

$$\operatorname{cov}(\varepsilon_t, z) = E[\varepsilon_t z] - \underbrace{E[\varepsilon_t]}_{0}\underbrace{E[z]}_{0} = E[\varepsilon_t z]$$

$$= E[z^2] = \sigma_z^2$$

$$\operatorname{var}(z) = E[(z - \bar{z})^2] = E[z^2], \quad \bar{z} = 0$$

x_t is a weakly stationary process

Exercise 4

$cov(X,Y) = E[XY] - E[X]E[Y]$

$cov(z, \varepsilon_{t+h}) = E[z\varepsilon_{t+h}] - \underbrace{\underbrace{E[z]}_{0}\,\underbrace{E[\varepsilon_{t+h}]}_{0}}_{0}$

$cov(z, \varepsilon_{t+h}) = E[z\varepsilon_{t+h}]$

Q: Show that $\mathrm{Corr}(x_t, x_{t+h}) = \frac{\sigma_z^2}{\sigma_z^2+\sigma_\epsilon^2}$ for any t and h.

$$corr(x_t, x_{t+h}) = \frac{cov(x_t, x_{t+h})}{std(x_t)\,std(x_{t+h})} = \frac{cov(x_t, x_{t+h})}{var(x_t)}$$

$$= \frac{\sigma_z^2}{\sigma_z^2 + \sigma_\varepsilon^2}$$

not correlated asymptotically

$$\lim_{h\to\infty} \frac{\sigma_z^2}{\sigma_z^2+\sigma_\varepsilon^2} = \frac{\sigma_z^2}{\sigma_z^2+\sigma_\varepsilon^2} \neq 0$$

$$\lim_{h\to\infty} corr(x_t, x_{t+h}) \stackrel{?}{=} 0$$

asym uncorrelated process

$$corr(x_t, x_{t+h}) = \frac{\sigma_z^2}{\sigma_z^2+\sigma_\varepsilon^2} \times \frac{1}{h}.$$

Roadmap

Exercise 1

Exercise 3

Exercise 4

Exercise 6

Exercise 6

model $X_t = \beta_0 + \beta_1 X_{t-1} + u_t$ $|\beta_1| < 1 \Rightarrow$ stationary

AR(1)

$$\text{Return}_t = \beta_0 + \beta_1 \,\text{Return}_{t-1} + u_t, \quad u_t \sim N(0, \sigma^2).$$

Source	SS	df	MS
Model	10.6866231	1	10.6866231
Residual	3059.73817	687	4.45376735
Total	3070.42479	688	4.46282673

Number of obs =	689
F(1, 687) =	2.40
Prob > F =	0.1218
R-squared =	0.0035
Adj R-squared =	0.0020
Root MSE =	2.1104

return	Coef.	Std. Err.	t	P>|t|	[95% Conf. Interval]	
return_1	.0588984	.0380231	1.55	0.122	-.0157569	.1335538
_cons	.179634	.0807419	2.22	0.026	.0211034	.3381646

Figure: Predictive Model for Stock Returns

Q: Compute the uncoditional mean and variance of the returns.

Exercise 6

- AR(1) model: $\text{Return}_t = \beta_0 + \beta_1 \text{Return}_{t-1} + u_t$.
- uncoditional mean: $\mathbb{E}(\text{Return}_t) = \frac{\beta_0}{1-\beta_1}$.
- uncoditional variance: $\text{Var}(\text{Return}_t) = \frac{\sigma^2}{1-\beta_1^2}$.
- use $\hat{\sigma}^2$ as an estimator of σ^2

$$\hat{\sigma}^2 = \frac{\hat{u}'\hat{u}}{T-k-1}$$

$$X_t = \beta_0 + \beta_1 X_{t-1} + u_t \qquad u_t \overset{iid}{\sim} (0, \sigma^2)$$

$$E[X_t] = E[\beta_0 + \beta_1 X_{t-1} + u_t]$$

$$= E[\beta_0] + \beta_1 E[X_{t-1}] + \underbrace{E[u_t]}_{=0}$$

ws $E[X_t] = E[X_{t-1}]$

$$E[X_t] = \beta_0 + \beta_1 E[X_t]$$

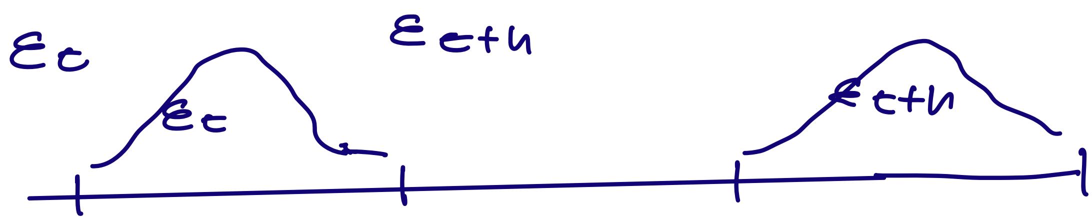
$$E[X_t] = \frac{\beta_0}{1-\beta_1} = \frac{0.17}{1-0.058} = 0.19 = E[\text{return}_t]$$

$$var(x_t) = var(\beta_0 + \beta_1 x_{t-1} + u_t) = \beta_1^2 \, var(x_{t-1}) + \sigma^2$$

$$var(x_t) = var(x_{t-1})$$

$$var(x_t) = \frac{\sigma^2}{1-\beta_1^2} = \frac{4.45}{1-0.058^2} = 4.465$$

$$\hat{\sigma}^2 = \frac{\hat{u}'\hat{u}}{T-k-1} = \frac{3059}{689-1-1} = 4.45$$



3×3

indep

	P_1 size	P_2 size	P_3 size
$P_1/_{BM}$			
$P_{2\,BM}$			
$P_{3\,BM}$			

conditional sorts

size

P_1 | P_2 | P_3

$P_{1,BM1}$

$P_{1\,BM2}$

$P_{1\,BM3}$

$P_{2\,BM1}$

⋮

P_{3BM1}