EXERCISE 01 - DEJA VU

FOR $\theta > 0$, LET $\{X_1 \dots X_m\}$ BE IID FROM THE FOLLOWING PARAMETRIC DENSITY:

$$f(x|\theta) = \frac{\theta^2}{x^3} \cdot e^{-\frac{\theta}{x}} \qquad \text{for } x \geq 0$$

HAVING MEAN $E(X) = \frac{\theta}{2}$ AND VARIANCE $Var(X) = \frac{\theta^2}{4}$.

1) FIND THE MAXIMUM LIKELIHOOD ESTIMATOR FOR θ.

THE MLE $\hat{\theta}_m$ IS THE VALUE THAT MINIMIZES $\mathcal{L}_m(\theta|X_m)$ OR, EQUIVALENTLY, $\ell(\theta|X_m)$:

$$\hat{\theta}_m = \underset{\theta \in \Theta}{\operatorname{argmax}}\ \mathcal{L}_m(\theta|X_m) = \underset{\theta \in \Theta}{\operatorname{argmax}}\ \ell_m(\theta|X_m)$$

THE MLE IS THE VALUE OF THE PARAMETER θ THAT MAXIMIZES THE JOINT PROBABILITY OF OBSERVING THE SAMPLE WE ACTUALLY COLLECTED.

THE LIKELIHOOD AND THE LOG-LIKELIHOOD FUNCTIONS ARE:

$$\mathcal{L}_m(\theta|X_m) \overset{IID}{=} \prod_{i=1}^{m} f(X_i|\theta) = \prod_{i=1}^{m} \frac{\theta^2}{X^3} \cdot e^{-\frac{\theta}{X}}$$

$$\ell_m(\theta|X_m) = \log \mathcal{L}_m(\theta|X_m) \overset{IID}{=} \sum_{i=1}^{m} \log f(X|\theta) = \sum_{i=1}^{m} \log\left(\frac{\theta^2}{X^3} \cdot e^{-\frac{\theta}{X}}\right)$$

TAKING THE DERIVATIVE AND SOLVING THE ASSOCIATED EQUATION WE GET THE MLE:

$$\frac{\partial}{\partial\theta} \ell_m(\theta|X_m) = \frac{\partial}{\partial\theta} \sum_{i=1}^{m} \log\left(\frac{\theta^2}{X^3} e^{-\frac{\theta}{X}}\right) = \sum_{i=1}^{m} \frac{\partial}{\partial\theta} \log\left(\frac{\theta^2}{X^3} \cdot e^{-\frac{\theta}{X}}\right) =$$

$$= \sum_{i=1}^{3} \left\{ \frac{1}{\frac{\theta^2}{x^3} \cdot e^{-\frac{\theta}{x}}} \left[\frac{2\theta}{x^3} \cdot e^{-\frac{\theta}{x}} + \frac{\theta^2}{x^3} \cdot e^{-\frac{\theta}{x}} \left(-\frac{1}{x} \right) \right] \right\} =$$

$$= \sum_{i=1}^{3} \left\{ \frac{1}{\frac{\theta^{\cancel{2}}}{\cancel{x^3}} \cdot \cancel{e^{-\frac{\theta}{x}}}} \left[\frac{\cancel{\theta}}{\cancel{x^3}} \cdot \cancel{e^{-\frac{\theta}{x}}} \left(2 + \theta \left(-\frac{1}{x} \right) \right) \right] \right\} =$$

$$= \sum_{i=1}^{3} \left[\frac{1}{\theta} \left(2 - \frac{\theta}{x} \right) \right] = \sum_{i=1}^{3} \left(\frac{2}{\theta} - \frac{1}{x} \right) =$$

$$= \sum_{i=1}^{3} \frac{2}{\theta} - \sum_{i=1}^{3} \frac{1}{x}$$

$$\sum_{i=1}^{3} \frac{2}{\theta} - \sum_{i=1}^{3} \frac{1}{x} = 0 \Rightarrow m \frac{2}{\theta} - \sum_{i=1}^{3} \frac{1}{x} = 0 \Rightarrow$$

$$\Rightarrow m \frac{2}{\theta} = \sum_{i=1}^{3} \frac{1}{x} \Rightarrow \theta = \frac{2m}{\sum_{i=1}^{3} \frac{1}{x_i}}$$

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2.1) WE WANT TO PROVE THAT w^* IS THE VALUE THAT MINIMIZES THE VARIANCE.

TO DO SO WE COMPUTE THE MINIMUM OF THE VARIANCE AND WE SHOW THAT THE OBTAINED EQUATION IS EQUIVALENT TO THE w^*'S ONE (THAT WE TAKE FROM EXERCISE'S SPECIFICATION)

- STEP 1 = WE WRITE THE VARIANCE FUNCTION AS:

$$Var(w\cdot X + (1-w)\cdot Y) = w^2 Var(X) + (1-w)^2 Var(Y) + 2w(1-w)Cov(X,Y)$$

- STEP 2 = COMPUTE THE DERIVATIVE

$$\frac{\partial Var(wX+(1-w)Y)}{\partial w} = 2w\,Var(X) - 2Var(Y) + 2w\,Var(Y) + 2Cov(X,Y) - 4w\,Cov(X,Y) =$$

$$= 2w\left[Var(X) + Var(Y) - 2Cov(X,Y)\right] + 2Cov(X,Y) - 2Var(Y)$$

- STEP 3 = EQUAL THE DERIVATIVE TO ZERO

$$\frac{\partial Var(wX+(1-w)Y)}{\partial w} = 0 \iff 2w\left[Var(X)+Var(Y)-2Cov(X,Y)\right] = 2Var(Y) - 2Cov(X,Y)$$

$$\iff w = \frac{Var(Y) - Cov(X,Y)}{Var(X) + Var(Y) - 2Cov(X,Y)}$$

$$\iff w = \frac{\hat{\sigma}^2_Y - \hat{\sigma}_{X,Y}}{\hat{\sigma}^2_X + \hat{\sigma}^2_Y - 2\hat{\sigma}_{XY}}$$

Note: WE SUBSTITUTE
$Var(Y) = \hat{\sigma}^2_Y$
$Var(X) = \hat{\sigma}^2_X$
$Cov(X,Y) = \hat{\sigma}_{X,Y}$

CONCLUSION = THE EQUATION WE OBTAINED IS EQUAL TO w^*'S ONE, HENCE w^* MINIMIZES THE VARIANCE