Assignment
Numerical theory

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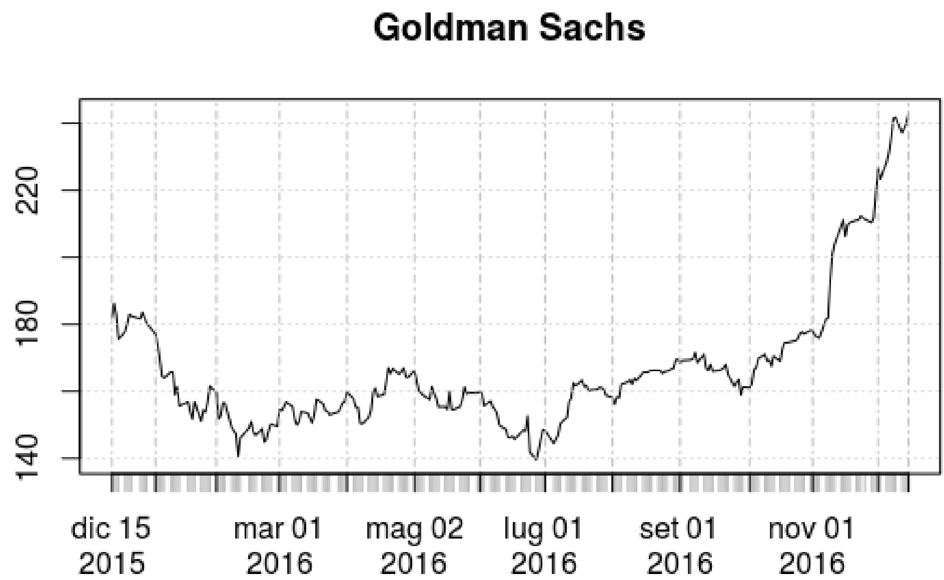
Abstract

In this paper, I want to estimate the theoretical prices of the Europian options of Goldman Sachs company, and confront them with the real prices, available in the market. In order to do so, I will start considering the Black&Scholes model, and I will test the assumptions which this model implies. Even if the assumptions should not be fulfilled, however, I will estimate the theoretical price, using the B&S framework, and I will use these as a benchmark, for evaluating others models. Considering the market price, I want to know which is the volatility implied in a B&S framework, and confront it, with the historical volatility.
I'll move further, tring to estimate a stochastic process which fits better the dynamics of our data, and I will try to estimate the theoretical options price, using it. I will end with the confront of the estimated prices using B&S framework, the estimated stochastic proces, and the prices available in the market.
I expect that the estimates made using B&S model don't match the real prices.

1 The Data

I downloaded the time series of the Goldman Sachs' prices stock (from 17/12/2015 to 15/12/2016), so the time reference is the year, and S_i represents daily data. The data are downloaded from Yahoo Finance.



2 My Hypothesis

If S_i is a solution to the differential equation:

(1)
$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Where μ represents the drift of the process and σ^2 is the volatility, S_t is a Geometric Brownian motion. Therefore

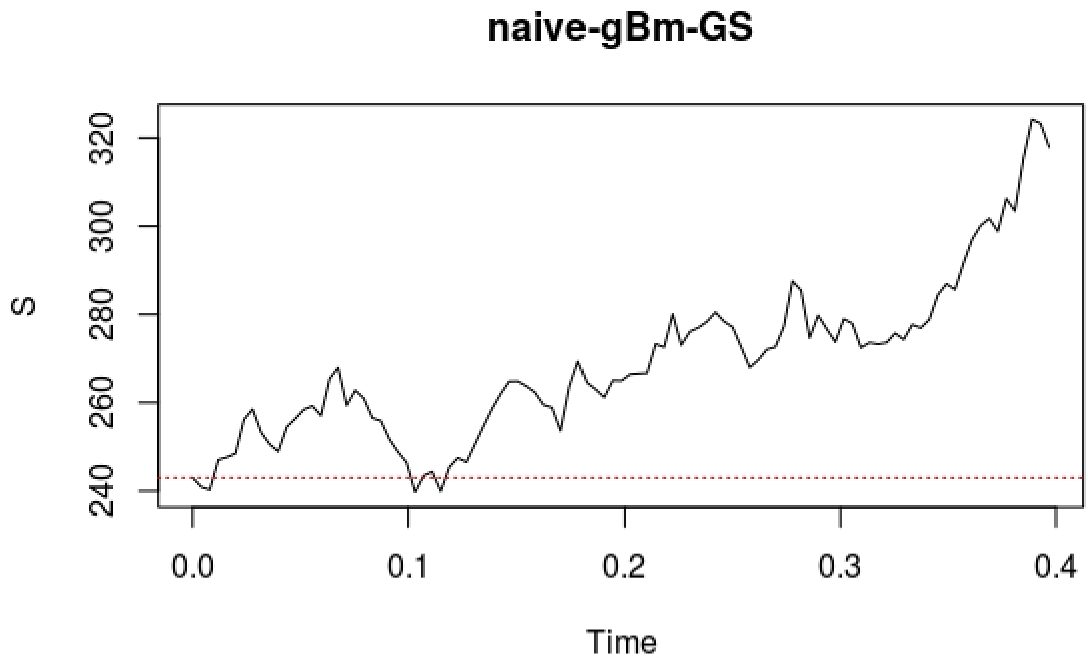
(2)
$$X_i = log(\frac{S(t_i)}{S(t_{i-1})})$$

(3)
$$X_i \sim N(\alpha \Delta t, \sigma^2 \Delta t)$$

Being Δt a constant we can obtain an unbiased estimates of the parameters μ and σ from the sample of i.i.d X_i. So, $\alpha = 0.286847$, $\sigma = 0.272263$ and $\mu = 0.3239106$. theoretically we have that, the stochastic differential equation is:

(4)
$$dS_t = (0.3239106)S_t dt + (0.272263)S_t dW_t$$

We can obtain a simulation of the path of this process,using, SDE, with *S(0)=243*, which is:



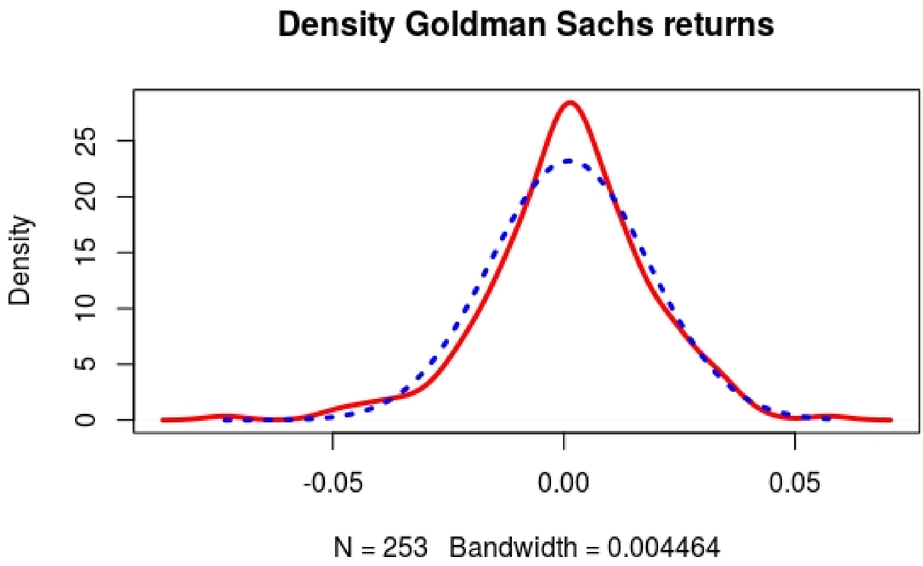
In order to say that this is the correct stochastic process, we have to check, if the difference between the log returns, X_i, have the following properties:

- **Normality**
- **Indipendence**
- **Stationarity**

Normality

Wev are going to check, if

(5)
$$X_t - X_s \sim N(0, t-s)$$

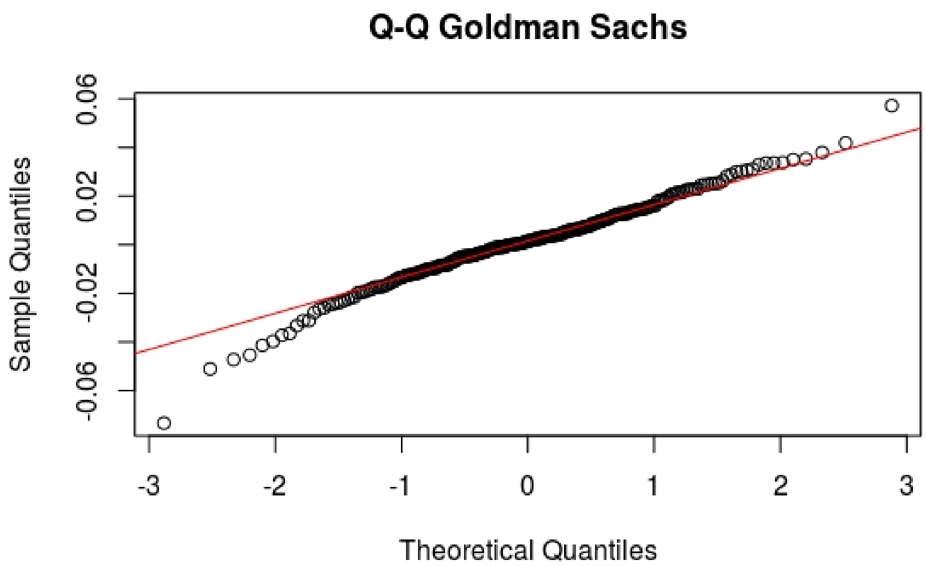


Comparing our distribution with the theoretical distribution, we can see that it is not perfectly symmetric, and with not negligible probability density on the tails. In order to confirm our intuition, we run the main tests for normality:

- **Shapiro Test** where H_0 = Normally distributed
- **Jarque- Bera** where H_0 = Skewness is zero and the excess kurtosis is zero
- **Kolmogorov- Smirnov** where H_0 = Normally distributed

Test	Statistic	P-value
Shapiro Test	0.97975	0.001134
Jarque- Bera	34.7746	$< 2.811e^{-08}$
Kolmogorov- Smirnov	0.47934	$< 2.2e^{-16}$

In all three tests the p-value is lesser than $\alpha = 0.05$, our significance level, so we can reject the hypothesis that the differences between the log returns are normally distributed.
As we could expect, the Q-Q plot below, confirms the result of the test.



Indipendence

The indipendence implies that:

$$X_t - X_s \quad and \quad X_u - X_v \quad are \quad indipendent \quad for \quad (s,t) \cap (v,u) = \varnothing$$

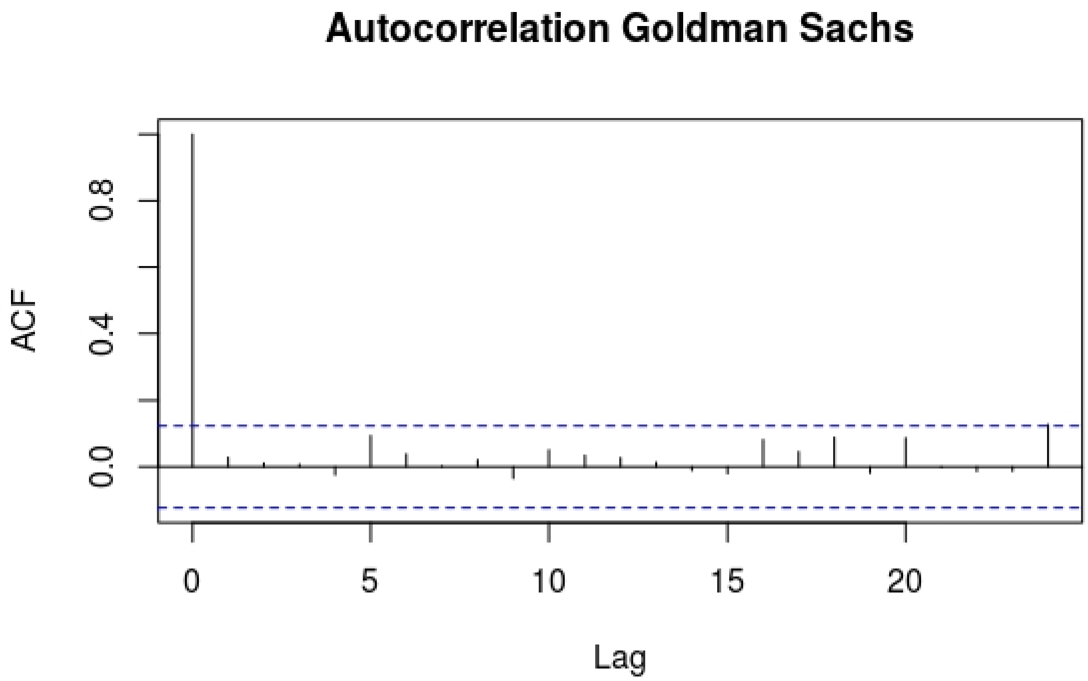
(6)

- **Box-Pierce test** where H_0 = Indipendently distributed
- **Box-Ljung test** where H_0 = Indipendently distributed

In order to see, if they are indipendent, we run the main tests of indipendence, considering the difference between the log returns.

Test	lag	Statistic	P-value
Box-Pierce	1	0.13595	0.7123
	5	2.4836	0.779
Box-Ljung	1	2.5491	0.7691
	5	2.4836	0.779

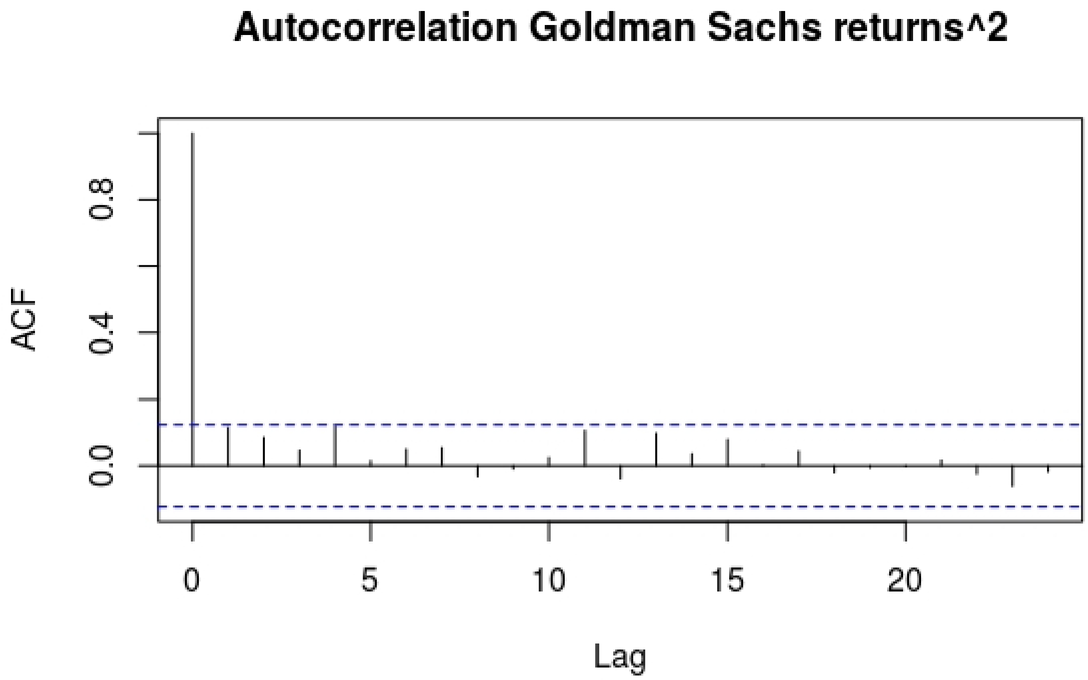
As we can see the Null Hypothesis is verified, they are uncorrelated, both considering one lag and 5 lags.



Considering the square of the difference between the log returns

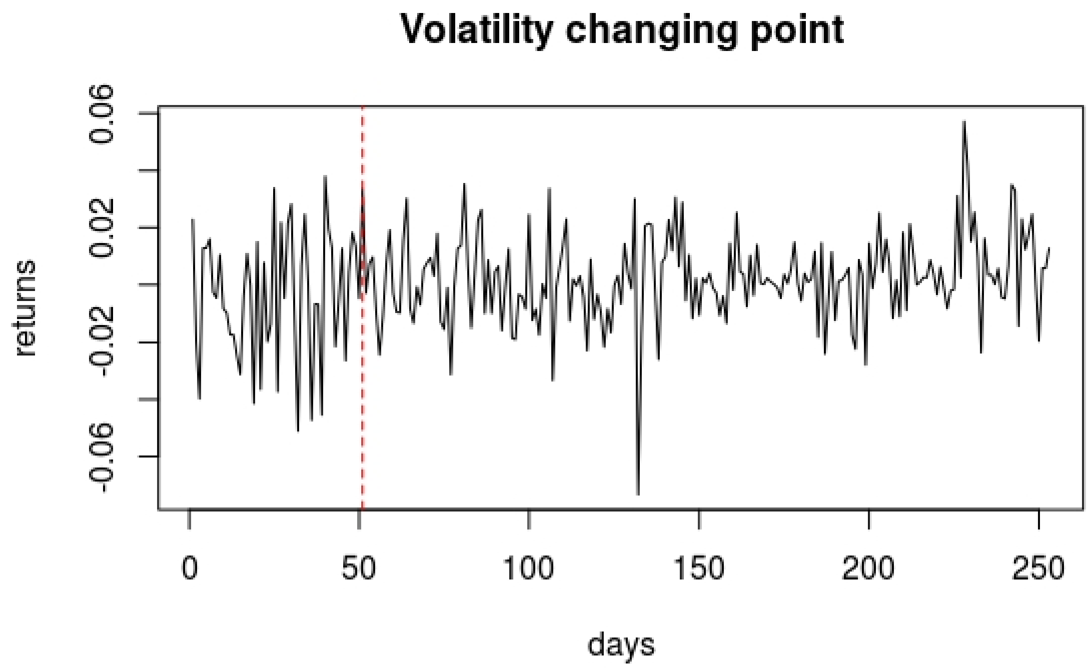
Test	lag	Statistic	P-value
Box-Pierce	1	9.077	0.106
	5	11.164	0.3449
Box-Ljung	1	3.2573	0.0711
	5	9.2428	0.09976

As in the case of the simple difference of the log returns, with a significance level of 5 %, we can say that they are uncorrelated.



Stationarity

$X_t - X_s$ for $t > s \geq 0$ depends only on $t - s$ and not by t and/or s separately. Then plot below show that, in our data the volatility changes. I used the changing point function to obtain this result.



These results show that our hypotesis are not fulfilled, so the Geometric Brownian motion is not the best way to model the asset price dynamics. But it is still a good reference point for evaluating the goodness of other model.

3 Call and Put prices

Now, I want to estimate the price of a call option and of a put option. In order to do it, I will estimate these prices using the Black&Scholes formulas. Starting from the basic equation of the Black&Scholes model:

$$P(t) = S(t)\Phi(d_1) - e^{-r(T-t)}K\Phi(d_2) \tag{7}$$

with

$$d_1 = d_2 + \sigma\sqrt{T} \tag{8}$$

$$d_2 = \frac{ln\frac{S(0)}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \tag{9}$$

Using this framework we can solve the above equation with respect to σ, for fixed value of K_i and $P(t)_i$.

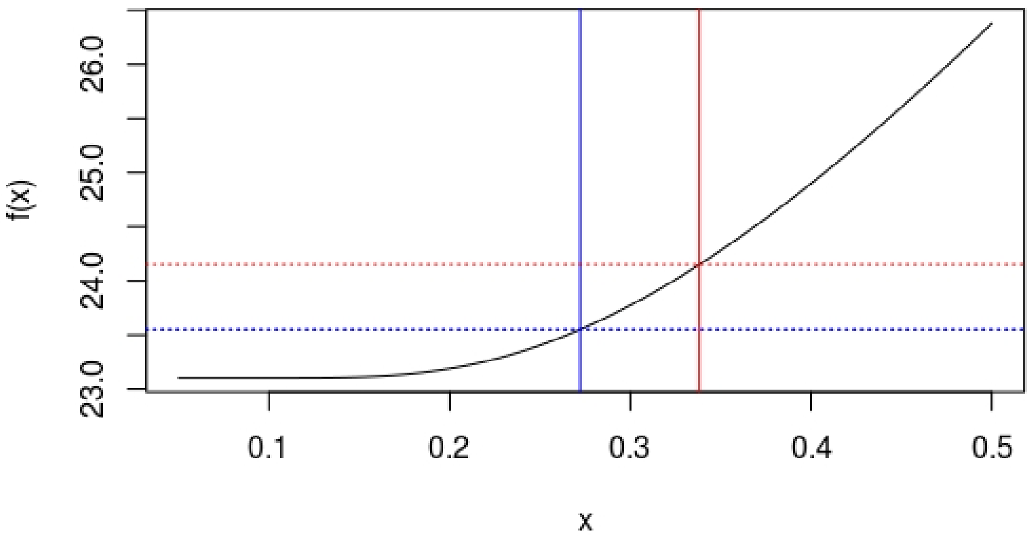
3.1 Call Prices

I collect the prices, date 15/12/2016, of a call option with expiring date (30/12/2016).

Strike price	180	185	190	195	200	205	210	215	220	222.5	225
Call Price	33.45	39.65	55.55	50.6	43.45	39	34	30.3	24.15	22.9	20.1
Strike price	227.5	230	232.5	235	237.5	240	245	250	255	260	265
Call Price	17	15.5	12.6	10.6	8.7	7	4.3	2.45	1.3	0.7	0.4

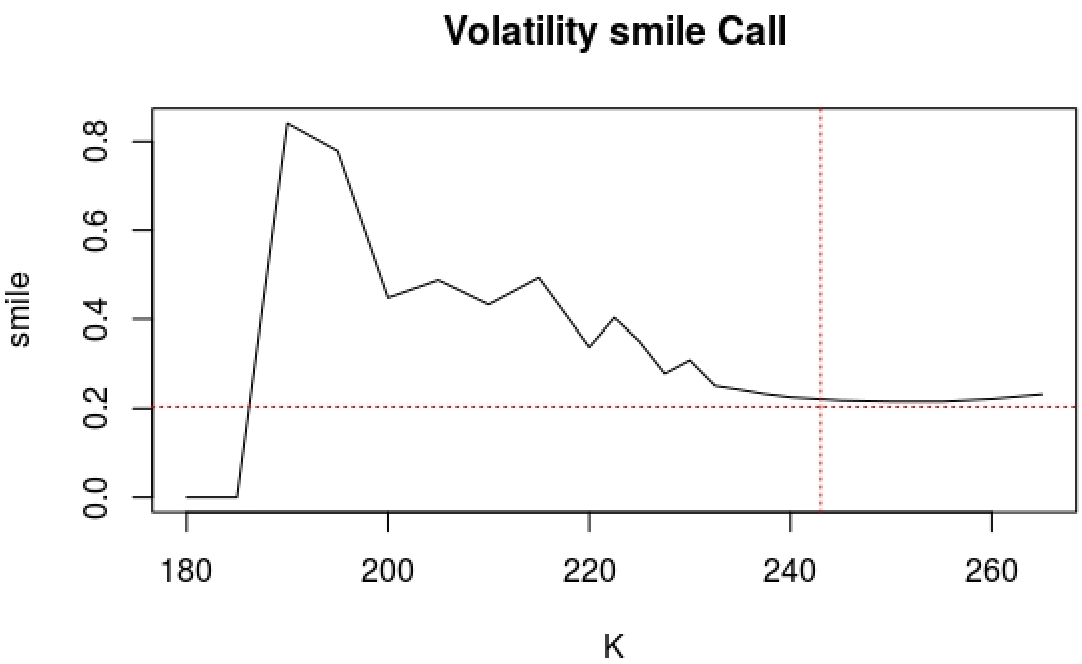
For evaluating the theoretical call price in the Black& Scholes model, I will use as yearly interest rate, the federal fund rate, which is equal to 0.81%. The actual share price is 243 (15/12/2016). The Historical Volatility is $\sigma = 0.272263$ Using GBSoptions, I get that the theoretical price for a call option with strike price $K = 220$, and expiration date (30/12/2016) is

	Put Price	Volatility
Market	24.15	0.3380374
Theoretical	23.5516	0.272263



The intersection of the blue lines represents the theoretical price estimated using the historical volatility, in GBS option. The intersection of the red lines, represents the market price of the call option and the implied volatility. As we can see, the volatility of the market is higher than the historical volatility, and even the market price is higher than the theoretical price. We deduct that, the market expects an higher probability of exercising this contract. Because the implied volatility incorporates the expectations of market participants on the options and the underlying assets.

Now consider the same effects, on the volatility, for all strike prices K_i, and relative $P_i(t)$.



Plotted as a function of the strike prices K_i, the implied volatility designs a curve. In this case, we can say that the market expects that the stock price will not rise over the actual level.

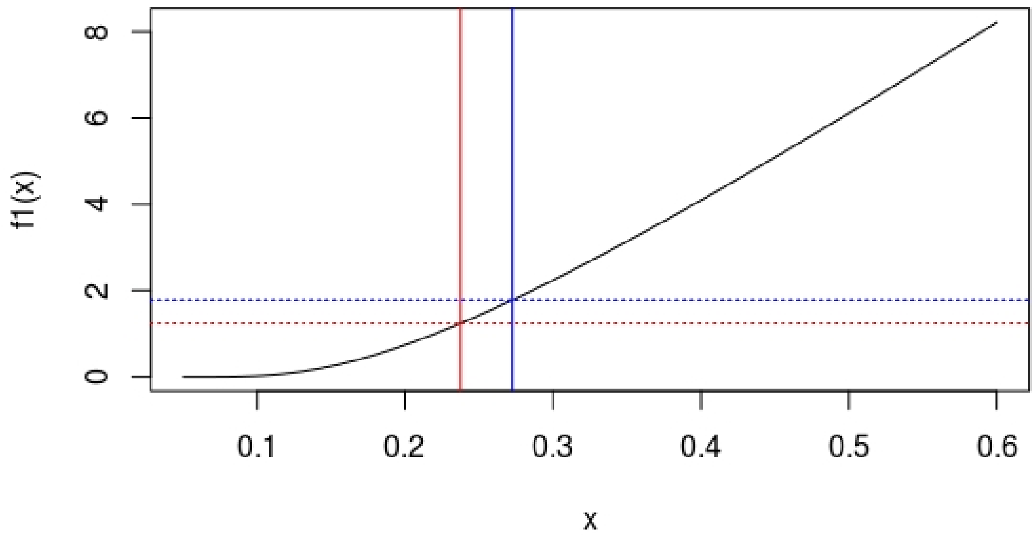
3.2 Put prices

Put options Goldman Sachs expiring date (30/12/2016)

Strike Price	192.5	195	197.5	200	202.5	205	207.5	210	212.5	215	217.5
Put Price	0.39	0.40	0.31	0.13	0.15	0.15	0.17	0.19	0.22	0.27	0.33
Strike Price	220	222.5	225	227.5	230	232.5	235	237.5	240	245	250
Put Price	0.41	0.54	0.71	0.86	1.24	1.65	2.2	2.91	3.8	6.25	9.60

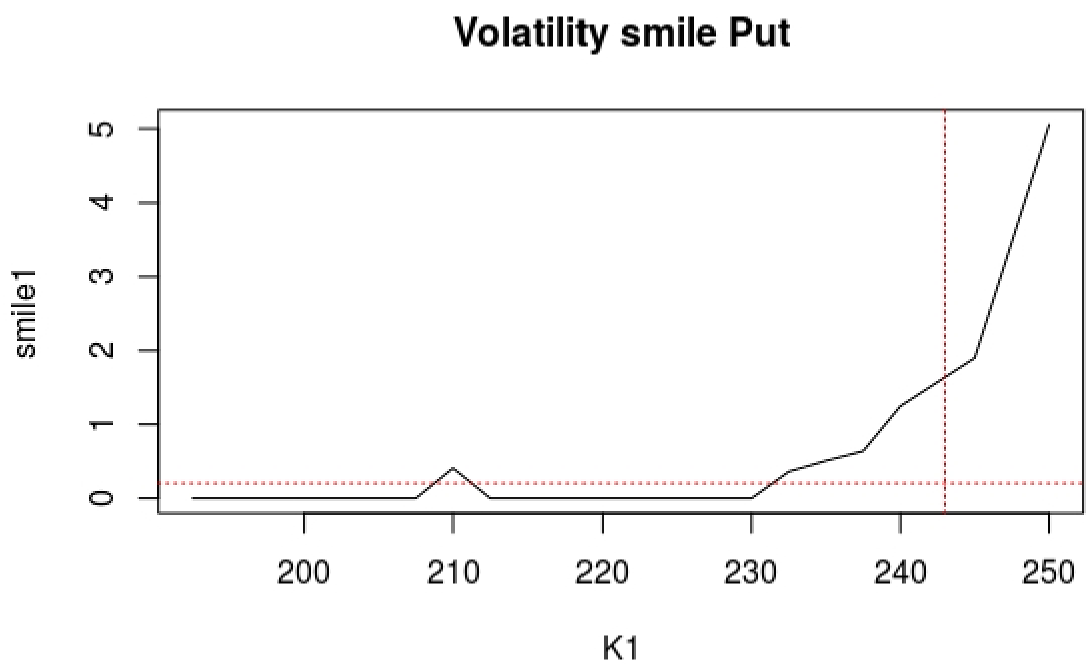
Using GBSoptions, I get that the theoretical price for a put option with strike price $K = 230$, and expiration date (30/12/2016) is

	Put Price	Volatility
Market	1.24	0.237722
Theoretical	1.7730	0.272263



As in the case of the call option, the intersection of the blue lines represents the theoretical price estimated using the historical volatility, in GBSoption. The intersection of the red lines, represents the market price of the call option and the implied volatility. In this case the volatility implied is lower than the historical volatility, and the market price, also, is lower than the theoretical one. This is interpreted again to mean that the market participants expect low probability of exercising the contract.

Now consider the same effects, on the volatility, for all strike prices K_i, and relative $P_i(t)$.



The implied volatility changes for given maturity T and different values of the strike price K_i, but in a non-linear way.
As we can see the probability of exercising this contract is very low, the market expects that the price of the stock will not go below 230.

4 Model

Now we proceed, checking for other models that can fit the dynamics of our stock prices,better than the Geometric Brownian motion. We will choose our model usig AIC information criteria. AIC is a tool for model selection, it is not a statistical test, indeed, the AIC statistic says nothing about the quality of the model in an absolute sense, i.e., if all the candidate models fit poorly AIC will not give any warning of that. AIC deals with the trade-off between the goodness of fit of the model and the complexity of the model.
Given a set of candidate models for the data, the preferred model is the one with the minimum AIC value. Hence AIC not only rewards goodness of fit, but also includes a penalty that is an increasing function of the number of estimated parameters. I will consider the following diffusion processes:

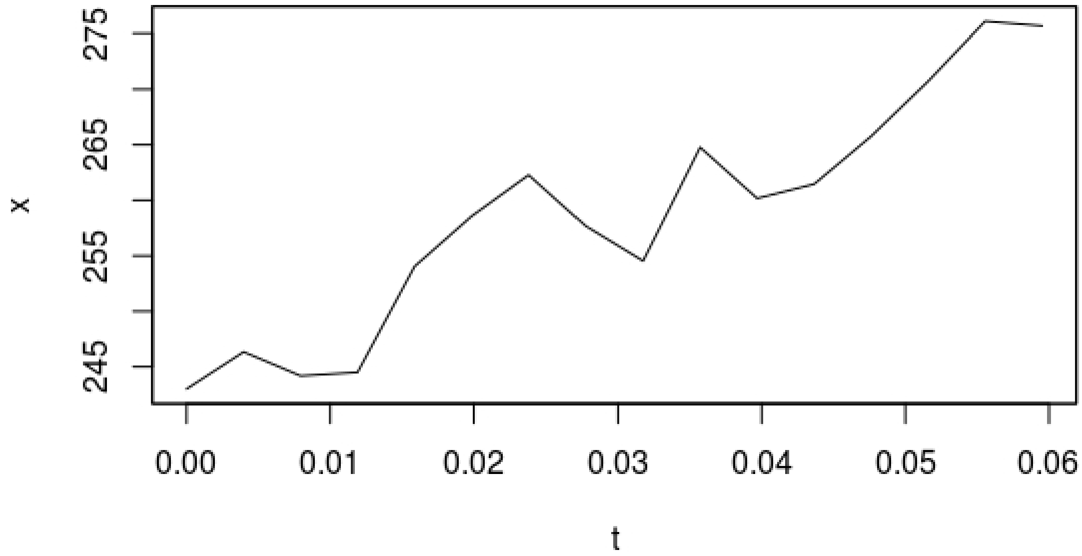
- **GBM**
$$\left(\ dx\ \right) = \left(\ \mu \cdot x\ \right) dt + \left[\ \sigma \cdot x\ \right]' \left(\ dW1\ \right)$$
$$\left(\ x(0) = 0\ \right)$$
- **CIR**
$$\left(\ dx\ \right) = \left(\ \alpha \cdot (\mu - x)\ \right) dt + \left[\ \sigma \cdot \sqrt(x)\ \right]' \left(\ dW1\ \right)$$
$$\left(\ x(0) = 0\ \right)$$
- **CIR2**
$$\left(\ dx\ \right) = \left(\ \mu \cdot x\ \right) dt + \left[\ \sigma \cdot \sqrt(x)\ \right]' \left(\ dW1\ \right)$$
$$\left(\ x(0) = 0\ \right)$$
- **CKLS**
$$\left(\ dx\ \right) = \left(\ \mu \cdot x\ \right) dt + \left[\ \sigma \cdot x^{\gamma}\ \right]' \left(\ dW1\ \right)$$
$$\left(\ x(0) = 0\ \right)$$
- **VAS**
$$\left(\ dx\ \right) = \left(\ \alpha \cdot (\mu - x)\ \right) dt + \left[\ \sigma\ \right]' \left(\ dW1\ \right)$$
$$\left(\ x(0) = 0\ \right)$$

These are the estimated value, through quasi maximum likeliood.

	Estimates									
	μ	std.Er	σ	std.Er	α	std.Er	γ	std. Er	L	AIC
GBM	0.321	0.2706	0.2722	0.012	0	0	0	0	1233.58	1237.58
CIR	149.88	15.77	3.5	0.158	-3.64	2.46	0	0	1231.45	1237.45
CIR2	0.3625	0.2728	3.52	0.16	0	0	0	0	1233.301	1237.301
CKLS	0.3543	0.2727	2.22	NaN	0	0	0.5898	NaN	1233.098	1239.098
VAS	152.6	12.5	44.98	2.024	-4.36	2.22	0	0	1233.362	1239.362

Base on the lowest AIC, I choose the CIR2, stochastic process.

We simulate the path of this process with initial value $x(0) = 243$, via MonteCarlo simulation,



Now we will try to estimate the options prices, using the process derived.
In order to do so, we will use the Monte Carlo approach.
Monte Carlo approach is an iterative procedure which simulates many path of our stochastich process, generating a large number of future prices of the underlying asset, calculate the payoff of the option for each of the potential underlying price paths, average them, and then discount the average to determine the price.

Call

Using the model selected "CIR2", The call payoff function:

$$f(K) = max(S_t - K, 0) \tag{10}$$

I will estimate the price of a call option with expiring date (30/12/2016) and strike price K=220. I will confront the result with the previous estimates, so the price under the Black& Scholes framework.

	Call Price
Market	24.15
Theoretical	23.5516
Theoretical "CIR2"	29.38

Put

Using the model selected "CIR2", I will estimate the price of a put option with expiring date (30/12/2016) and strike price K=230. The payoff function of a Put option:

$$f(K) = max(K - S_t, 0) \tag{11}$$

I will confront the result with the previous estimates, so the price under the Black& Scholes framework.

	Put Price
Market	1.24
Theoretical	1.7730
Theoretical "CIR2"	0.371

Conclusion

As we could expect the estimated prices via Black & Scholes model, didn't match the prices of the market, because the assumptions of the model are not fulfilled enrirely. But in a wider prospect, we obtained a framework for evaluating other models.
Because, we end up with, the Black & Scholes model on one side, with all its assumptions, and the real prices on the other side. This allows us to understand where we are going. Because,if the estimates using other models will up this intervall, this means that we are going in the right direction, or at least not in the wrong, otherwise we are not taking into consideration something that is important, or we are misjudging something.
Also the Black& Scholes assumptions, allow us to build new models with modificated version of them. For example taking into account heteroskedasticity, or correlation. In our case the estimated values using CIR2 are not in this interval, mainly this means that modificating the diffusion process is not enough, we should build a model which takes in consideration heteroskedasticity, and poisson process, as in the Lèvy processes. Now that we have a framework we can try to work on fitting a better model.