# CFD Course Report - Week 12

Your Name

October 22, 2024

**Abstract**

This report summarizes the computational fluid dynamics (CFD) simulations and analyses conducted during week 12 of the course.

# 1 Task 1: Elementary Vector Calculus

## 1.1 Gradient of a scalar $\phi$:

$$grad(\phi) = \nabla\phi = \frac{\partial \phi}{\partial x_i} = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) \tag{1}$$

## 1.2 Divergence of the velocity vector $\vec{u}$:

$$div(\vec{u}) = \nabla \cdot \vec{u} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{2}$$

## 1.3 Curl of the velocity vector (vorticity):

$$rot(\vec{u}) = \nabla \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{3}$$

## 1.4 Material derivative of a scalar:

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + (\vec{u} \cdot \nabla)\phi = \frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + w\frac{\partial \phi}{\partial z} \tag{4}$$

## 1.5 Material derivative of the velocity vector:

$$\begin{aligned} \frac{D\vec{u}}{Dt} &= \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} \\ &= \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t}\right) + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)(u, v, z) \\ &= \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t}\right) + \frac{\partial u_i}{\partial x_i}(u, v, z) \end{aligned} \tag{5}$$

## 1.6 Rate-of-strain-tensor:

$$S = \frac{1}{2}\left(\nabla\vec{u} + (\nabla\vec{u})^T\right) = \frac{1}{2}\begin{pmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} & \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} & 2\frac{\partial w}{\partial z} \end{pmatrix} \tag{6}$$

## 1.7 Divergence of the rate-of-strain tensor:

$$\begin{aligned} \nabla \cdot \mathbf{S} = \frac{\partial S}{\partial x_i} &= \frac{1}{2}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\begin{pmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} & \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} & 2\frac{\partial w}{\partial z} \end{pmatrix} \\ &= \frac{1}{2}\Bigg(2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 w}{\partial x \partial z} + \frac{\partial^2 u}{\partial z^2}, \\ &\quad \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} + 2\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial z}, \\ &\quad \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 w}{\partial y^2} + 2\frac{\partial^2 w}{\partial z^2}\Bigg) \end{aligned} \tag{7}$$

# 2 Task 2: Global versus local rotation

## 2.1 Sketch of the flows:

**Line vortex flow:** $\vec{u} = [u_r, u_\theta, u_z] = \left[0, -\frac{\alpha}{r}, 0\right]$

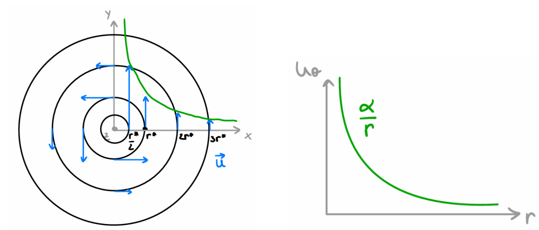


Figure 1: Line vortex flow

**Plane shear flow:** $\vec{u} = [u, v, w] = [\beta y, -0, 0]$

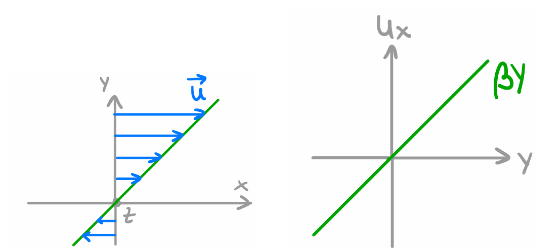


Figure 2: Plane shear flow

**Flow between rotating cylinders:** $\vec{u} = [u_r, u_\theta, u_z] = \left[0, Ar + \frac{B}{r}, 0\right]$

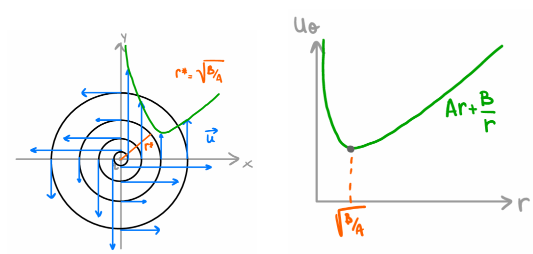


Figure 3: Flow between rotating cylinders

## 2.2 Local and global rotations:

To verify wheter a flow features a local/global rotation we need to compute its vorticity, which is a vector quantity defined as $\vec{\omega} = \nabla \times \mathbf{u}$.

**Line vortex flow:**

For this specific flow we have that the vorticity vector is $\vec{\omega} = [0, 0, 0]$, so this flow does not show a local nor a global rotation since the vorticity is the null vector.

**Plane shear flow:**

Making usage of the previously delined formula for the vorticity, we find that $\vec{\omega} = [0, 0, -\beta]$ which indicates that we have a shear-induced rotation around the $z$-axis. Also this flow does not feature a global rotation because it is characterized by a velocity gradient in the $y$-direction (shear) rather than a circular motion and this causes fluid elements to rotate locally due to the velocity differences at different $y$-positions, but there is no coordinated or structured rotation of the entire flow field. The flow is linear, not circular, so it doesn't exhibit global rotation.

**Flow between rotating cylinders:**

As in the former cases we compute the vorticity: $\vec{\omega} = [0, 0, 2A]$. Also the third flow exibits a local rotation due to the non-zero vorticiy, but unlike the previous cases the rotating cylinders create a structured and circular flow with a consistent rotational effect through the domain.

In all three cases, we could have reached similar conclusions by observing the graphs shown above.

# 3 Task 5: Simplifications of Navier-Stokes Equations