Exercise 5

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1 Implicit Euler scheme for diffusion equation

1.a Discretization

Recalling the general form of the implicit Euler method:

$$\phi_{i,j}^{n+1} = \phi_{i,j}^{n} + f(\phi^{n+1}, t^{n+1})\Delta t$$

we get for the diffusion equation:

$$\phi_{i,j}^{n+1} = \phi_{i,j}^{n} + \alpha(\frac{\partial^2 \phi_{i,j}^{n+1}}{\partial x^2} + \frac{\partial^2 \phi_{i,j}^{n+1}}{\partial y^2})\Delta t$$

Then we use the second order central difference scheme for the spatial derivatives:

$$\frac{\partial^2 \phi_{i,j}^{n+1}}{\partial x^2} = \frac{\phi_{i+1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i-1,j}^{n+1}}{\Delta x^2}$$

$$\frac{\partial^2 \phi_{i,j}^{n+1}}{\partial y^2} = \frac{\phi_{i,j+1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j-1}^{n+1}}{\Delta y^2}$$

Substituting the above equations into the diffusion equation, we get:

$$\phi_{i,j}^{n+1} = \phi_{i,j}^{n} + \alpha \left(\frac{\phi_{i+1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i-1,j}^{n+1}}{\Delta x^2} + \frac{\phi_{i,j+1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j-1}^{n+1}}{\Delta y^2} \right) \Delta t \tag{1}$$

where we can rearrange the terms to get the following expression:

$$\phi_{i,j}^{n+1}(1 + 2\alpha\Delta t(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2})) - \alpha\Delta t \left(\frac{\phi_{i+1,j}^{n+1} + \phi_{i-1,j}^{n+1}}{\Delta x^2} + \frac{\phi_{i,j+1}^{n+1} + \phi_{i,j-1}^{n+1}}{\Delta y^2} \right) = \phi_{i,j}^{n} \tag{2}$$

1.b Consistency proof

In order to prove consistency of the discretization we have to show that the truncation error T goes to zero as the grid spacing $(\Delta x, \Delta y)$ goes to zero and the time step (Δt) goes to zero.

We start with the exact form of the diffusion equation:

$$\frac{\partial \phi}{\partial t} = \alpha \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right). \tag{3}$$

The discretized scheme for the implicit Euler method is described by equation (1).

To analyze the truncation error, we expand $\phi_{i+1,j}^{n+1}, \phi_{i-1,j}^{n+1}, \phi_{i,j+1}^{n+1}, \phi_{i,j-1}^{n+1}$ using Taylor series around $\phi_{i,j}^{n+1}$. For example, the expansion for $\phi_{i+1,j}^{n+1}$ is given by:

$$\phi_{i+1,j}^{n+1} = \phi_{i,j}^{n+1} + \Delta x \frac{\partial \phi}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 \phi}{\partial x^2} + \frac{\Delta x^3}{6} \frac{\partial^3 \phi}{\partial x^3} + \cdots,$$

and similar expansions hold for the other terms. Then we substitute these expansions into the discretized scheme. For example, the term

$$\frac{\phi_{i+1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i-1,j}^{n+1}}{\Delta x^2}$$

becomes:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\Delta x^2}{12} \frac{\partial^4 \phi}{\partial x^4} + \cdots.$$

Substituting all the Taylor expansions into the discretized equation and rearranging terms, we obtain the residual or truncation error as:

$$T = \Delta t \frac{\partial^2 \phi}{\partial t^2} + \frac{\Delta x^2}{12} \frac{\partial^4 \phi}{\partial x^4} + \frac{\Delta y^2}{12} \frac{\partial^4 \phi}{\partial y^4} + \cdots. \tag{4}$$

As $\Delta t \to 0$, $\Delta x \to 0$, and $\Delta y \to 0$, the truncation error T approaches zero, demonstrating that the discretization is consistent.

1.c Stability criteria

To determine the stability criterion for the implicit Euler scheme using the Von Neumann method, we start with the discretized equation (1) and we introduce the parameters $r_x = \frac{\alpha \Delta t}{\Delta x^2}$ and $r_y = \frac{\alpha \Delta t}{\Delta y^2}$, then the equation becomes:

$$\phi_{i,j}^{n+1} = \phi_{i,j}^{n} + r_x \left(\phi_{i+1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i-1,j}^{n+1}\right) + r_y \left(\phi_{i,j+1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j-1}^{n+1}\right).$$

Rearranging terms, we write:

$$\phi_{i,j}^{n+1} \left(1 + 2r_x + 2r_y\right) = \phi_{i,j}^{n} + r_x \left(\phi_{i+1,j}^{n+1} + \phi_{i-1,j}^{n+1}\right) + r_y \left(\phi_{i,j+1}^{n+1} + \phi_{i,j-1}^{n+1}\right).$$

The Von Neumann method assumes a Fourier mode solution of the form:

$$\phi_{i,j}^{n} = G^n e^{I(k_x i \Delta x + k_y j \Delta y)},$$

where G is the amplification factor,k_x and k_y are the wave numbers in the x- and y-directions, and $I = \sqrt{-1}$.

We substituting this Fourier mode into the discretized equation, and the neighboring terms become:

$$\phi_{i+1,j}^{n+1} = Ge^{Ik_x \Delta x} \phi_{i,j}^{n+1}, \quad \phi_{i-1,j}^{n+1} = Ge^{-Ik_x \Delta x} \phi_{i,j}^{n+1},$$

$$\phi_{i,j+1}^{n+1} = Ge^{Ik_y \Delta y} \phi_{i,j}^{n+1}, \quad \phi_{i,j-1}^{n+1} = Ge^{-Ik_y \Delta y} \phi_{i,j}^{n+1}.$$

Using these expressions, substitute into the equation. Simplify the exponentials using $e^{I\theta} + e^{-I\theta} = 2\cos\theta$, leading to:

$$G\phi_{i,j}^{n+1} \left(1 + 2r_x + 2r_y\right) = \phi_{i,j}^{n} + 2r_x G\phi_{i,j}^{n+1} \cos(k_x \Delta x) + 2r_y G\phi_{i,j}^{n+1} \cos(k_y \Delta y).$$

Factoring G from the left-hand side:

$$G \left(1 + 2r_x + 2r_y - 2r_x \cos(k_x \Delta x) - 2r_y \cos(k_y \Delta y)\right) = 1.$$

We solve for G:

$$G = \frac{1}{1 + 2r_x + 2r_y - 2r_x \cos(k_x \Delta x) - 2r_y \cos(k_y \Delta y)}.$$

The scheme is stable if $|G| \leq 1$. For the implicit Euler method, the denominator in G is always greater than 1 for all k_x and k_y. So we have that:

$$|G| \leq 1, \tag{5}$$

which implies unconditional stability.

1.d Convergence proof

Since the scheme is both consistent and stable, for the Lax equivalence theorem, the scheme is convergent.

References

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