Exercise 1

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1 Fundamentals of Differential Equations

1.a Difference between ordinary derivative, partial derivative, and material (total) derivative

- **Ordinary derivative** $\left(\frac{d}{dt}\right)$: Describes the rate of change of a function with respect to one variable. It is used for functions depending on a single variable, such as $f(t)$.
- **Partial derivative** $\left(\frac{\partial}{\partial t}\right)$: Describes the rate of change of a multivariable function with respect to one of its variables, while holding other variables constant. This is often used in multivariable functions such as $f(x,t)$, where we can find $\frac{\partial f}{\partial t}$ while x remains fixed.
- **Material (total) derivative** $\left(\frac{D}{Dt}\right)$: is a measure of the rate of change of a physical quantity (like velocity or temperature) experienced by an observer moving with the fluid. It combines both local and convective rates of change as, for example, in a function $f(x,t)$, $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$ for some velocity field u.

1.b Ordinary and partial differential equations

- **Ordinary Differential Equations (ODEs)**: These involve derivatives with respect to a single variable. For example, $\frac{dy}{dt} = y$ is an ODE.
- **Partial Differential Equations (PDEs)**: These involve partial derivatives with respect to multiple variables. For instance, the heat equation $\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$ is a PDE.

1.c Order of a differential equation

The order of a differential equation is the highest order of derivative present in the equation.

- **First-order ODE**: $\frac{dy}{dt} = ky$.
- **Second-order PDE**: The wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$.
- **Third-order ODE**: $\frac{d^3 y}{dt^3} + \frac{d^2 y}{dt^2} + y = 0$.

1.d Linear and non-linear differential equations

- **Linear Differential Equations**: These have terms that are linear in the unknown function and its derivatives. For example, $\frac{dy}{dt} + 3y = 0$ is linear.
- **Non-linear Differential Equations**: These have terms that are non-linear in the unknown function or its derivatives. For instance, the Navier-Stokes equation $\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{f}$ is non-linear. This non-linearity arises due to the convective term $\mathbf{u}\cdot\nabla\mathbf{u}$, which represents the interaction of the velocity field with itself. Specifically, $\mathbf{u}\cdot\nabla\mathbf{u}$ is non-linear because it involves the product of the velocity field $\mathbf{u}$ with its own gradient $\nabla\mathbf{u}$.

1.e Initial value problem (IVP) and boundary value problem (BVP)

- **Initial Value Problem (IVP)**: A problem that requires solving a differential equation with specified initial conditions, such as $y(0) = y_0$, in time.
- **Boundary Value Problem (BVP)**: A problem where the solution to a differential equation is sought within a specified range, with conditions, usually Dirichlet or Neumann, given at the boundaries of the range, like $u(0) = 0$ and $u(1) = 1$.

1.f Parabolic and elliptic PDE examples and their conditions

The difference between parabolic and elliptic PDEs can be defined through the computation of a discriminant $\Delta = b^2 - 4ac$, where a, b, and c are coefficients from the second-order PDE of the form $a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x\partial y} + c\frac{\partial^2 u}{\partial y^2} + \ldots = 0$. If $\Delta = 0$, the PDE is parabolic, and if $\Delta < 0$, the PDE is elliptic.

- **Parabolic PDE**: The heat equation $\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$ is parabolic and typically requires both initial and boundary conditions.
- **Elliptic PDE**: Laplace's equation $\nabla^2 u = 0$ is elliptic and usually requires boundary conditions but not initial conditions, since it does not depend on time.

2 Order Reduction

The governing equation of the damped oscillator is given by:

$$m\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + cy(t) = 0 \tag{1}$$

with initial conditions:

$$y(0) = s_0, \quad \frac{dy(0)}{dt} = v_0.$$

We aim to transform this second-order ODE into a system of first-order differential equations. To reduce a second-order ODE to a system of first-order ODEs we can introduce new variables to represent the derivatives of the function $y(t)$. In particular, we define:

$$y_1(t) = y(t)$$

and introduce a new variable $y_2(t)$ to represent the first derivative of $y(t)$:

$$y_2(t) = \frac{dy(t)}{dt}.$$

Since $\frac{dy_2(t)}{dt} = \frac{d^2y(t)}{dt^2}$, we can substitute this into the original equation to obtain:

$$m\frac{dy_2(t)}{dt} + by_2(t) + cy_1(t) = 0. \tag{2}$$

In this way, we can express the problem as two coupled first-order differential equations:

$$\begin{cases} \frac{dy_1(t)}{dt} & = y_2(t), \\ \frac{dy_2(t)}{dt} & = -\frac{b}{m}y_2(t) - \frac{c}{m}y_1(t). \end{cases}$$

However, we also need to rewrite the initial conditions for $y(t)$ and $\frac{dy(t)}{dt}$:

- $y_1(0) = s_0$,
- $y_2(0) = v_0$.

This approach allows us to solve the system using methods suited for first-order differential equations, enabling easier numerical or analytical analysis.

3 Blasius Equation

Part (a): Convert the Blasius Equation to a System of First-Order ODEs

The Blasius equation is given by:

$$f''' + \frac{1}{2}ff'' = 0 \tag{3}$$

with $f' = \frac{u}{U_\infty}$. Three boundary conditions are necessary to solve this equation:

- $\eta = 0$: $f' = f = 0$ (no-slip condition)
- $\eta \to \infty$: $f' = 1$ (free outer flow)

We aim to transform this third-order ODE into a system of first-order differential equations. To reduce a third-order ODE to a system of first-order ODEs we can introduce new variables to represent the derivatives of the function $f(\eta)$. In particular, we define:

$$y_1 = f, \quad y_2 = f' = \frac{df}{d\eta}, \quad y_3 = f'' = \frac{d^2f}{d\eta^2}$$

Then, the derivatives of these variables with respect to η are:

$$\frac{dy_1}{d\eta} = y_2, \quad \frac{dy_2}{d\eta} = y_3$$

Now, we can substitute this into the original Blasius equation to obtain:

$$\frac{dy_3}{d\eta} = -\frac{1}{2}y_1y_3$$

In this way, we can express the problem as three coupled first-order differential equations:

$$\begin{cases} \frac{dy_1}{d\eta} = y_2 \\ \frac{dy_2}{d\eta} = y_3 \\ \frac{dy_3}{d\eta} = -\frac{1}{2} y_1 y_3 \end{cases}$$

with boundary conditions:

- At $\eta = 0$: $y_1 = 0$, $y_2 = 0$
- As $\eta \to \infty$: $y_2 = 1$

Part (b): Providing an Initial Condition for $f''(0)$

To solve this problem as an initial value problem, we need an initial value for $f''(0)$. However, the boundary condition $y_2(\infty) = 1$ is specified at infinity, making it impractical to impose this condition directly at a finite point. To address this, we can use an iterative approach:

1. Guess an initial value for $f''(0)$.
2. Integrate the system of equations from $\eta = 0$ to a sufficiently large value of η where $y_2(\eta)$ approaches a constant. To solve this system numerically, we use the Runge-Kutta method of fourth order (RK4) that allows to approximate solutions to ordinary differential equations.

   For a step size h, the RK4 method computes the next values y_{i+1} as follows:

$$k_1 = h \cdot f(\eta_i, y_i)$$
$$k_2 = h \cdot f\left(\eta_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right)$$
$$k_3 = h \cdot f\left(\eta_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right)$$
$$k_4 = h \cdot f(\eta_i + h, y_i + k_3)$$

   The next value of the solution is updated by:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

3. Check if $y_2(\eta)$ approaches 1 as $\eta \to \infty$. If $y_2(\eta)$ is not close to 1, adjust the initial guess for $y_3(0)$ iterate this process until the condition $y_2(\infty) = 1$ (or close to it) is satisfied within a desired tolerance.

This iterative approach allows us to find an appropriate initial condition for $y_3(0) = f''(0)$ that satisfies the boundary condition at infinity.

References

[1] *CFD Repository*,
Available at: `https://github.com/GiuseppePisante/CFD.git`

[2] *GitHub Copilot*,
GitHub. Available at: `https://github.com/features/copilot`