Exercise 6

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1 Finite-difference method on a non-uniform grip

1.a Transformation from from physical space into computational space

Using the chain rule, the first derivative of Φ with respect to x is given by:

$$\frac{d\Phi}{dx} = \frac{d\Phi}{d\xi} \cdot \frac{d\xi}{dx}$$

Since we assume that ξ is the computational space where grid points are equally spaced, we express $d\xi/dx$ in terms of x:

$$\frac{d\xi}{dx} = \frac{1}{\frac{dx}{d\xi}}$$

Substituting this into the previous equation:

$$\frac{d\Phi}{dx} = \frac{d\Phi}{d\xi} \cdot \frac{1}{\frac{dx}{d\xi}}$$

Thus, we obtain the compact form:

$$\frac{d\Phi}{dx} = \frac{\frac{d\Phi}{d\xi}}{\frac{dx}{d\xi}}. \tag{1}$$

The second derivative is defined as:

$$\frac{d^2\Phi}{dx^2} = \frac{d}{dx}\left(\frac{d\Phi}{dx}\right).$$

Substituting $\frac{d\Phi}{dx} = \frac{\frac{d\Phi}{d\xi}}{\frac{dx}{d\xi}}$:

$$\frac{d^2\Phi}{dx^2} = \frac{d}{dx}\left(\frac{\frac{d\Phi}{d\xi}}{\frac{dx}{d\xi}}\right).$$

Using the chain rule:

$$\frac{d^2\Phi}{dx^2} = \frac{d}{d\xi}\left(\frac{\frac{d\Phi}{d\xi}}{\frac{dx}{d\xi}}\right) \cdot \frac{d\xi}{dx}.$$

Again, substituting $\frac{d\xi}{dx} = \frac{1}{\frac{dx}{d\xi}}$, we get:

$$\frac{d^2\Phi}{dx^2} = \frac{\frac{d}{d\xi}\left(\frac{\frac{d\Phi}{d\xi}}{\frac{dx}{d\xi}}\right)}{\frac{dx}{d\xi}}.$$

Now, applying the quotient rule to differentiate:

$$\frac{d}{d\xi}\left(\frac{\frac{d\Phi}{d\xi}}{\frac{dx}{d\xi}}\right) = \frac{\frac{d^2\Phi}{d\xi^2} \cdot \frac{dx}{d\xi} - \frac{d\Phi}{d\xi} \cdot \frac{d^2x}{d\xi^2}}{\left(\frac{dx}{d\xi}\right)^2}.$$

Substituting this result:

$$\frac{d^2\Phi}{dx^2} = \frac{\frac{\frac{d^2\Phi}{d\xi^2} \cdot \frac{dx}{d\xi} - \frac{d\Phi}{d\xi} \cdot \frac{d^2x}{d\xi^2}}{\left(\frac{dx}{d\xi}\right)^2}}{\frac{dx}{d\xi}}.$$

Multiplying by $\frac{1}{\frac{dx}{d\xi}}$, we obtain:

$$\frac{d^2\Phi}{dx^2} = \frac{\frac{d^2\Phi}{d\xi^2}}{\left(\frac{dx}{d\xi}\right)^2} - \frac{\frac{d^2x}{d\xi^2} \cdot \frac{d\Phi}{dx}}{\left(\frac{dx}{d\xi}\right)^2}. \tag{2}$$

These expressions can be used to transform differential equations from physical space into computational space.

1.b Non-uniform grid

Non-uniform grids are often used because they allow for adaptive resolution in areas where higher accuracy or detail is needed such as boundary layers, shocks, or vortices without increasing computational cost significantly across the entire domain. By refining the grid only where necessary, non-uniform grids reduce the number of total grid points, saving memory and computational time. Furthermore non-uniform grids are better suited for domains with irregular or complex geometries since the grid can better conform to the shape of the domain, improving accuracy in boundary condition enforcement.

1.c Central finite-difference approximation on an non-equispaced grid

To derive central finite-difference approximations for the first and second derivatives in the computational space ξ with second-order accuracy, we start from equation (1) and (2). For the first derivative we start computing the central finite-difference approximation for the term:

$$\frac{d\Phi}{d\xi} \approx \frac{\Phi_{i+1} - \Phi_{i-1}}{2\Delta\xi},$$

where $\Delta\xi = 1$. Here $\frac{dx}{d\xi}$ is computed based on the grid points x_i.

$$\frac{dx}{d\xi} \approx \frac{x_{i+1} - x_{i-1}}{2\Delta\xi}.$$

Substituting into the equation (1) we get:

$$\frac{d\Phi}{dx} \approx \frac{\frac{\Phi_{i+1}-\Phi_{i-1}}{2}}{\frac{x_{i+1}-x_{i-1}}{2}} = \frac{\Phi_{i+1} - \Phi_{i-1}}{x_{i+1} - x_{i-1}}. \tag{3}$$

Similary, to derive an approximation for the second derivative, we start computing the central finite-difference approximation for the term:

$$\frac{d^2\Phi}{d\xi^2} \approx \frac{\Phi_{i+1} - 2\Phi_i + \Phi_{i-1}}{\Delta\xi^2}.$$

where $\Delta\xi = 1$. Here $\frac{d^2x}{d\xi^2}$ is computed based on the grid points x_i.

$$\frac{d^2x}{d\xi^2} \approx x_{i+1} - 2x_i + x_{i-1}.$$

Substituting into the equation (2) we get:

$$\frac{d^2\Phi}{dx^2} \approx \frac{\Phi_{i+1} - 2\Phi_i + \Phi_{i-1}}{\left(\frac{x_{i+1}-x_{i-1}}{2}\right)^2} - \frac{(x_{i+1} - 2x_i + x_{i-1}) \cdot \frac{\Phi_{i+1}-\Phi_{i-1}}{x_{i+1}-x_{i-1}}}{\left(\frac{x_{i+1}-x_{i-1}}{2}\right)^2}. \tag{4}$$

1.d Discretized steady one-dimensional advection-diffusion equation

Using central finite-difference approximations for the first and second derivatives on a non-uniform grid, we want to discretize the steady one-dimensional advection-diffusion equation. Substituting equation (3) and (4) into the advection-diffusion equation leads to:

$$\frac{\Phi_{i+1} - \Phi_{i-1}}{x_{i+1} - x_{i-1}} = \frac{1}{\mathrm{Pe}} \left[\frac{\Phi_{i+1} - 2\Phi_i + \Phi_{i-1}}{\left(\frac{x_{i+1}-x_{i-1}}{2}\right)^2} - \frac{(x_{i+1} - 2x_i + x_{i-1}) \cdot \frac{\Phi_{i+1}-\Phi_{i-1}}{x_{i+1}-x_{i-1}}}{\left(\frac{x_{i+1}-x_{i-1}}{2}\right)^2} \right].$$

Rearranging the above equation, the discretized equation for an interior point i becomes:

$$a_{i-1}\Phi_{i-1} + b_i\Phi_i + c_{i+1}\Phi_{i+1} = 0,$$

where:

$$a_{i-1} = \frac{-1}{\Delta x_i} + \frac{1}{\mathrm{Pe}} \left(\frac{1}{\Delta x_i^2} - \frac{x_{i+1} - 2x_i + x_{i-1}}{\Delta x_i^2} \right),$$

$$b_i = \frac{2}{\mathrm{Pe} \cdot \Delta x_i^2},$$

$$c_{i+1} = \frac{1}{\Delta x_i} + \frac{1}{\mathrm{Pe}} \left(\frac{1}{\Delta x_i^2} + \frac{x_{i+1} - 2x_i + x_{i-1}}{\Delta x_i^2} \right).$$

Here, $\Delta x_i = x_{i+1} - x_{i-1}$. The discretized system can be expressed in matrix form as:

$$\mathbf{A\Phi} = \mathbf{b},$$

where:

- $\mathbf{A}$ is an $N \times N$ sparse coefficient matrix,
- $\mathbf{\Phi} = [\Phi_1, \Phi_2, \ldots, \Phi_N]^T$ is the vector of unknowns,
- $\mathbf{b}$ is the source term (zero for homogeneous equations).

For N grid points, the coefficient matrix $\mathbf{A}$ is tridiagonal for interior points:

$$\mathbf{A} = \begin{bmatrix} b_1 & c_2 & 0 & \cdots & 0 \\ a_2 & b_2 & c_3 & \cdots & 0 \\ 0 & a_3 & b_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & c_{N-1} \\ 0 & 0 & 0 & a_N & b_N \end{bmatrix}.$$

References

[1] *CFD Repository*,
Available at: `https://github.com/GiuseppePisante/CFD.git`

[2] *GitHub Copilot*,
GitHub. Available at: `https://github.com/features/copilot`