Exercise 4

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1 Stability of the unsteady advection-diffusion equation

1.a Discretization of of the unsteady one-dimensional advection-diffusion equation

We start with the one-dimensional advection-diffusion equation:

$$\rho\left(\frac{\partial\Phi}{\partial t}+U\frac{\partial\Phi}{\partial x}\right)=\alpha\frac{\partial^2\Phi}{\partial x^2}.$$

We use central finite differencing in space and explicit Euler in time. The terms are discretized as follows:

- $\frac{\partial\Phi}{\partial t}\approx\frac{\Phi_i^{n+1}-\Phi_i^n}{\Delta t}$
- $\frac{\partial\Phi}{\partial x}\approx\frac{\Phi_{i+1}^n-\Phi_{i-1}^n}{2\Delta x}$
- $\frac{\partial^2\Phi}{\partial x^2}\approx\frac{\Phi_{i+1}^n-2\Phi_i^n+\Phi_{i-1}^n}{\Delta x^2}$

Substituting these approximations into the original equation, we get:

$$\rho\left(\frac{\Phi_i^{n+1}-\Phi_i^n}{\Delta t}\right)+\rho U\left(\frac{\Phi_{i+1}^n-\Phi_{i-1}^n}{2\Delta x}\right)=\alpha\left(\frac{\Phi_{i+1}^n-2\Phi_i^n+\Phi_{i-1}^n}{\Delta x^2}\right).$$

Multiplying through by Δt:

$$\rho\Phi_i^{n+1}-\rho\Phi_i^n+\rho U\Delta t\left(\frac{\Phi_{i+1}^n-\Phi_{i-1}^n}{2\Delta x}\right)=\alpha\Delta t\left(\frac{\Phi_{i+1}^n-2\Phi_i^n+\Phi_{i-1}^n}{\Delta x^2}\right).$$

Rearranging to isolate Φ_i^{n+1} on the left-hand side:

$$\rho\Phi_i^{n+1}=\rho\Phi_i^n-\rho U\Delta t\left(\frac{\Phi_{i+1}^n-\Phi_{i-1}^n}{2\Delta x}\right)+\alpha\Delta t\left(\frac{\Phi_{i+1}^n-2\Phi_i^n+\Phi_{i-1}^n}{\Delta x^2}\right).$$

Dividing through by ρ:

$$\Phi_i^{n+1}=\Phi_i^n-c\left(\frac{\Phi_{i+1}^n-\Phi_{i-1}^n}{2}\right)+d\left(\Phi_{i+1}^n-2\Phi_i^n+\Phi_{i-1}^n\right),$$

where:

$$c=\frac{U\Delta t}{\Delta x},\quad d=\frac{\alpha\Delta t}{\rho(\Delta x)^2}.$$

The parameter c represents the Courant number and quantifies the influence of advection in the simulation. For stability and accuracy in explicit schemes, c should typically be small to satisfy the Courant-Friedrichs-Lewy condition that requires $c\leq 1$. If c is too large, the simulation may become unstable or inaccurate because the flow travels across more than one grid cell per time step. The parameter d measures the contribution of diffusion to the simulation and it should typically be small to ensure numerical stability. The diffusion stability condition generally requires $d\leq 1/2$ to ensure that diffusion does not dominate excessively or destabilize the solution.

References

[1] *CFD Repository*,
Available at: `https://github.com/GiuseppePisante/CFD.git`

[2] *GitHub Copilot*,
GitHub. Available at: `https://github.com/features/copilot`