Exercise 1

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Task 2.2: Order Reduction

The governing equation of the damped oscillator is given by:

$$m\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + cy(t) = 0 \tag{1}$$

with initial conditions:

$$y(0) = s_0, \quad \frac{dy(0)}{dt} = v_0.$$

We aim to transform this second-order ODE into a system of first-order differential equations. To reduce a second-order ODE to a system of first-order ODEs we can introduce new variables to represent the derivatives of the function $y(t)$. In particular, we define:

$$y_1(t) = y(t)$$

and introduce a new variable $y_2(t)$ to represent the first derivative of $y(t)$:

$$y_2(t) = \frac{dy(t)}{dt}.$$

Since $\frac{dy_2(t)}{dt} = \frac{d^2y(t)}{dt^2}$, we can substitute this into the original equation to obtain:

$$m\frac{dy_2(t)}{dt} + by_2(t) + cy_1(t) = 0. \tag{2}$$

In this way, we can express the problem as two coupled first-order differential equations:

$$\begin{cases} \frac{dy_1(t)}{dt} & = y_2(t), \\ \frac{dy_2(t)}{dt} & = -\frac{b}{m}y_2(t) - \frac{c}{m}y_1(t). \end{cases}$$

However, we also need to rewrite the initial conditions for $y(t)$ and $\frac{dy(t)}{dt}$:

- $y_1(0) = s_0$,
- $y_2(0) = v_0$.

This approach allows us to solve the system using methods suited for first-order differential equations, enabling easier numerical or analytical analysis.

Task 2.3: Blasius Equation

Part (a): Convert the Blasius Equation to a System of First-Order ODEs

The Blasius equation is given by:

$$f''' + \frac{1}{2}ff'' = 0 \tag{3}$$

with $f' = \frac{u}{U_\infty}$. Three boundary conditions are necessary to solve this equation:

- $\eta = 0$: $f' = f = 0$ (no-slip condition)
- $\eta \to \infty$: $f' = 1$ (free outer flow)

We aim to transform this third-order ODE into a system of first-order differential equations. To reduce a third-order ODE to a system of first-order ODEs we can introduce new variables to represent the derivatives of the function $f(\eta)$. In particular, we define:

$$y_1 = f, \quad y_2 = f' = \frac{df}{d\eta}, \quad y_3 = f'' = \frac{d^2f}{d\eta^2}$$

Then, the derivatives of these variables with respect to η are:

$$\frac{dy_1}{d\eta} = y_2, \quad \frac{dy_2}{d\eta} = y_3$$

Now, we can substitute this into the original Blasius equation to obtain:

$$\frac{dy_3}{d\eta} = -\frac{1}{2}y_1y_3$$

In this way, we can express the problem as three coupled first-order differential equations:

$$\begin{cases} \frac{dy_1}{d\eta} = y_2 \\ \frac{dy_2}{d\eta} = y_3 \\ \frac{dy_3}{d\eta} = -\frac{1}{2} y_1 y_3 \end{cases}$$

with boundary conditions:

- At $\eta = 0$: $y_1 = 0$, $y_2 = 0$
- As $\eta \to \infty$: $y_2 = 1$

Part (b): Providing an Initial Condition for $f''(0)$

To solve this problem as an initial value problem, we need an initial value for $f''(0)$. However, the boundary condition $y_2(\infty) = 1$ is specified at infinity, making it impractical to impose this condition directly at a finite point. To address this, we can use an iterative approach:

1. Guess an initial value for $f''(0)$.
2. Integrate the system of equations from $\eta = 0$ to a sufficiently large value of η where $y_2(\eta)$ approaches a constant. To solve this system numerically, we use the Runge-Kutta method of fourth order (RK4) that allows to approximate solutions to ordinary differential equations.

   For a step size h, the RK4 method computes the next values y_{i+1} as follows:

$$k_1 = h \cdot f(\eta_i, y_i)$$
$$k_2 = h \cdot f\left(\eta_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right)$$
$$k_3 = h \cdot f\left(\eta_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right)$$
$$k_4 = h \cdot f(\eta_i + h, y_i + k_3)$$

   The next value of the solution is updated by:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

3. Check if $y_2(\eta)$ approaches 1 as $\eta \to \infty$. If $y_2(\eta)$ is not close to 1, adjust the initial guess for $y_3(0)$ iterate this process until the condition $y_2(\infty) = 1$ (or close to it) is satisfied within a desired tolerance.

This iterative approach allows us to find an appropriate initial condition for $y_3(0) = f''(0)$ that satisfies the boundary condition at infinity.