Exercise 1

November 6, 2024

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1 Fundamentals of Differential Equations

1.a Difference between ordinary derivative, partial derivative, and material (total) derivative

- **Ordinary derivative** $\left(\frac{d}{dt}\right)$: Describes the rate of change of a function with respect to one variable. It is used for functions depending on a single variable, such as $f(t)$.
- **Partial derivative** $\left(\frac{\partial}{\partial t}\right)$: Describes the rate of change of a multivariable function with respect to one of its variables, while holding other variables constant. This is often used in multivariable functions such as $f(x,t)$, where we can find $\frac{\partial f}{\partial t}$ while x remains fixed.
- **Material (total) derivative** $\left(\frac{D}{Dt}\right)$: is a measure of the rate of change of a physical quantity (like velocity or temperature) experienced by an observer moving with the fluid. It combines both local and convective rates of change as, for example, in a function $f(x,t)$, $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$ for some velocity field u.

1.b Ordinary and partial differential equations

- **Ordinary Differential Equations (ODEs)**: These involve derivatives with respect to a single variable. For example, $\frac{dy}{dt} = y$ is an ODE.
- **Partial Differential Equations (PDEs)**: These involve partial derivatives with respect to multiple variables. For instance, the heat equation $\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$ is a PDE.

1.c Order of a differential equation

The order of a differential equation is the highest order of derivative present in the equation.

- **First-order ODE**: $\frac{dy}{dt} = ky$.
- **Second-order PDE**: The wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$.
- **Third-order ODE**: $\frac{d^3 y}{dt^3} + \frac{d^2 y}{dt^2} + y = 0$.

1.d Linear and non-linear differential equations

- **Linear Differential Equations**: These have terms that are linear in the unknown function and its derivatives. For example, $\frac{dy}{dt} + 3y = 0$ is linear.
- **Non-linear Differential Equations**: These have terms that are non-linear in the unknown function or its derivatives. For instance, the Navier-Stokes equation $\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{f}$ is non-linear. This non-linearity arises due to the convective term $\mathbf{u}\cdot\nabla\mathbf{u}$, which represents the interaction of the velocity field with itself. Specifically, $\mathbf{u}\cdot\nabla\mathbf{u}$ is non-linear because it involves the product of the velocity field $\mathbf{u}$ with its own gradient $\nabla\mathbf{u}$.

1.e Initial value problem (IVP) and boundary value problem (BVP)

- **Initial Value Problem (IVP)**: A problem that requires solving a differential equation with specified initial conditions, such as $y(0) = y_0$, in time.
- **Boundary Value Problem (BVP)**: A problem where the solution to a differential equation is sought within a specified range, with conditions, usually Dirichlet or Neumann, given at the boundaries of the range, like $u(0) = 0$ and $u(1) = 1$.

1.f Parabolic and elliptic PDE examples and their conditions

The difference between parabolic and elliptic PDEs can be defined through the computation of a discriminant $\Delta = b^2 - 4ac$, where a, b, and c are coefficients from the second-order PDE of the form $a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x\partial y} + c\frac{\partial^2 u}{\partial y^2} + \ldots = 0$. If $\Delta = 0$, the PDE is parabolic, and if $\Delta < 0$, the PDE is elliptic.

- **Parabolic PDE**: The heat equation $\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$ is parabolic and typically requires both initial and boundary conditions.
- **Elliptic PDE**: Laplace's equation $\nabla^2 u = 0$ is elliptic and usually requires boundary conditions but not initial conditions, since it does not depend on time.

References

[1] *CFD Repository*,
Available at: `https://github.com/GiuseppePisante/CFD.git`

[2] *GitHub Copilot*,
GitHub. Available at: `https://github.com/features/copilot`