PHYSICS

CHAPTER 3 – KINEMATICS IN TWO DIMENSION, VECTORS

3.1 Vectors and Scalars

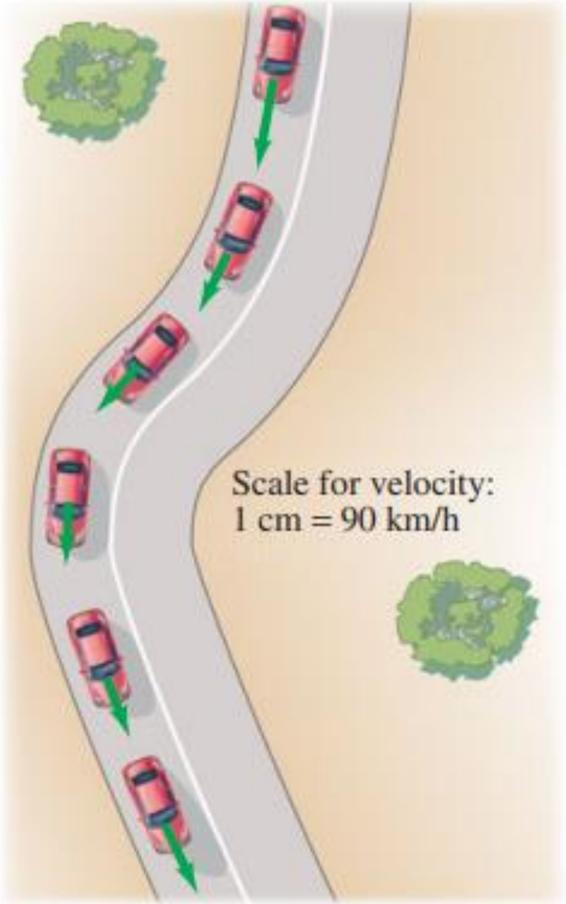


FIGURE 3–1 Car traveling on a road, slowing down to round the curve. The green arrows represent the velocity vector at each position.

A vector quantity has both **magnitude** and **direction**.
Some vector quantities: displacement, velocity, force, momentum.
A scalar quantity has only **magnitude**.
Some scalar quantities: mass, time, temperature.

When deciding what to wear before going outside, you look at the temperature. '… **degrees**' is the information of **TEMPERATURE**, which is a scalar quantity sufficient for us.

While swimming in the sea, if you want to move toward the shore, you must consider the current. To do that, you need to know both the **speed** and **direction** of the current. Hence, **velocity** is a **vector quantity**.

As seen in the figure beside, while a car moves along the road and takes a turn, the green arrows represent the velocity vector of the car at each position. Therefore, the concept of ***"velocity"*** not only indicates how fast an object moves but also specifies the ***direction*** of its motion.

Scalar Quantities:

- A SCALAR is a quantity of physics that has MAGNITUDE only, however, direction is not associated with it.
- Magnitude – A numerical value with units.

Scalar Example	Magnitude
Speed	20 m/s
Distance	10 m
Age	15 years
Heat	1000 calories

Vector Quantities:

- A **VECTOR** is a quantity which has both MAGNITUDE and DIRECTION.
- Examples: force, displacement, velocity….

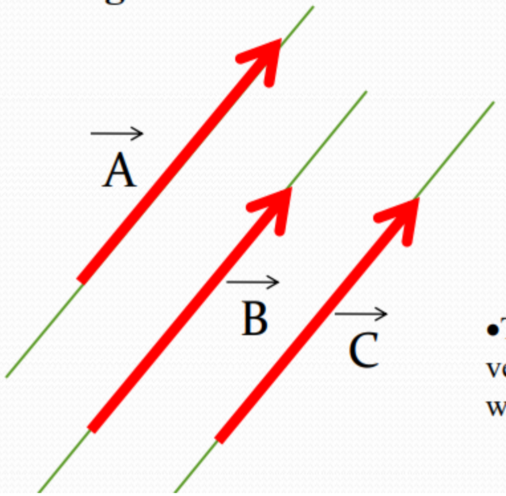
Vector	Magnitude & Direction
Velocity	20 m/s, North
Acceleration	10 m/s/s, East
Force	5 N, West

Vectors are typically illustrated by drawing an ARROW above the symbol. The arrow is used to convey direction and magnitude.

Vectors can be represented in **boldface**, such as $\mathbf{V}$, or with an **arrow** above the letter, such as $\vec{\boldsymbol{V}}$.

Bold font	$\boldsymbol{A}$
Arrow on top	$\vec{A}$

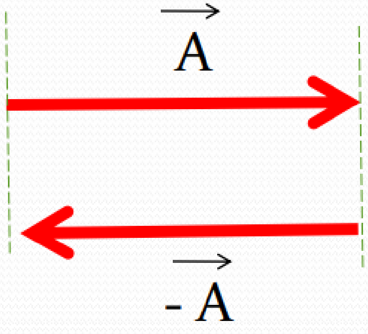
- **Equality of Two Vectors**
  - Two vectors are **equal** if they have **the same magnitude and the same direction.**



$\vec{A} = \vec{B} = \vec{C}$

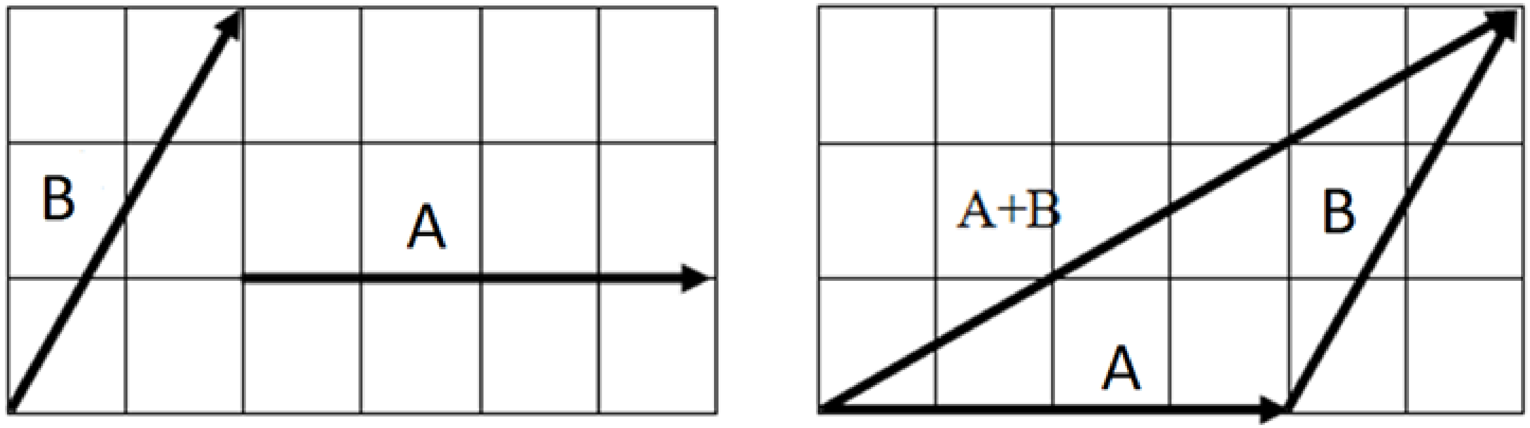
- This property allows us to translate a vector parallel to itself in a diagram without affecting the vector.

- **Negative Vectors**
  - Two vectors are **negative** if they have the same magnitude but are 180° apart (opposite directions)



Adding vectors using tail-to-tip (Polygon) method

On a diagram, draw one of the vectors, call it **A**. Next draw the second vector, call it **B**, and place its tail at the tip of the first vector and being sure its direction is correct. The arrow drawn from the tail of the first vector to the tip of the second vector represents the sum, or resultant, of the two vectors.



Tail-to-Tip Method for Multiple Vectors

The tail-to-tip method can be applied to **three or more vectors**.
Resultant vector: drawn from the **tail of the first vector** to the **tip of the last vector**.

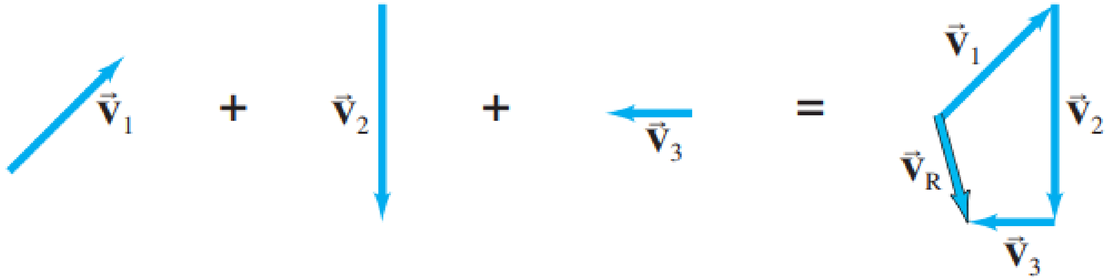


FIGURE 3–5 The resultant of three vectors: $\vec{V}_R = \vec{V}_1 + \vec{V}_2 + \vec{V}_3$.

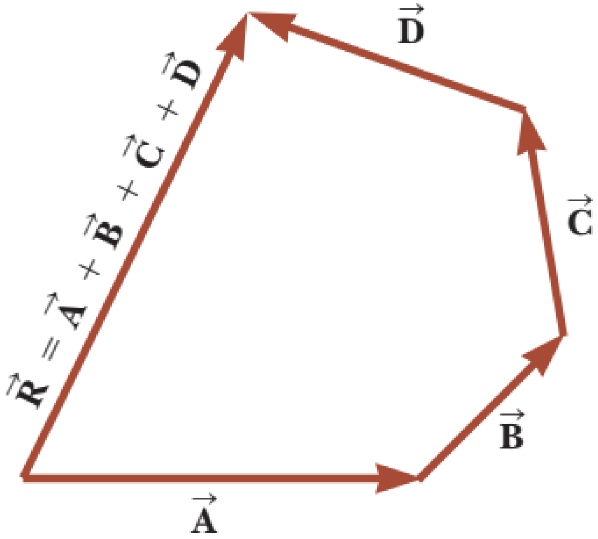
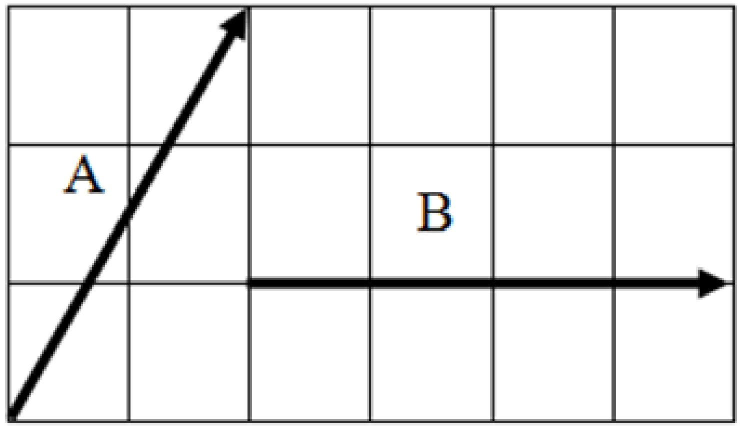
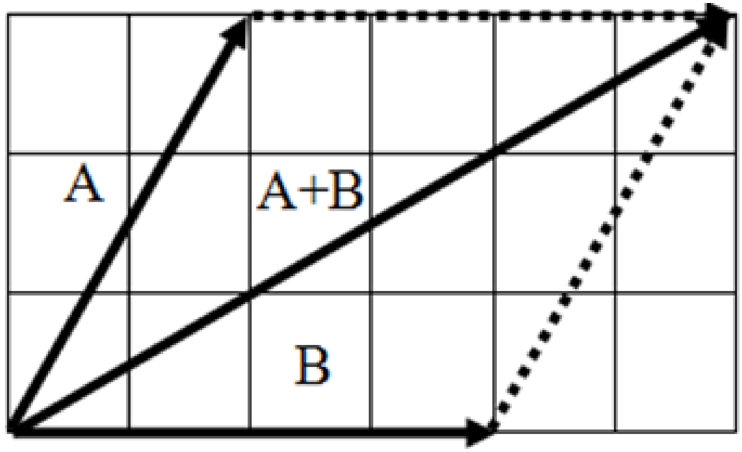


Figure 3.7 Geometric construction for summing four vectors. The resultant vector $\vec{R}$ is by definition the one that completes the polygon.

Parallelogram Method

A second way to add two vectors is the parallelogram method. It is fully equivalent to the tail-to-tip method. Draw the vectors so that their initial points coincide. Then draw lines to form a complete parallelogram. The diagonal from the initial point to the opposite vertex of the parallelogram is the resultant

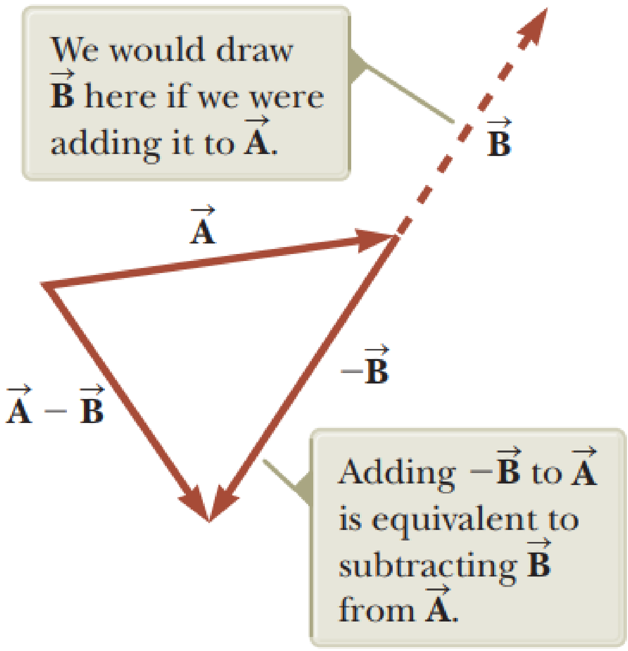




Subtracting vector

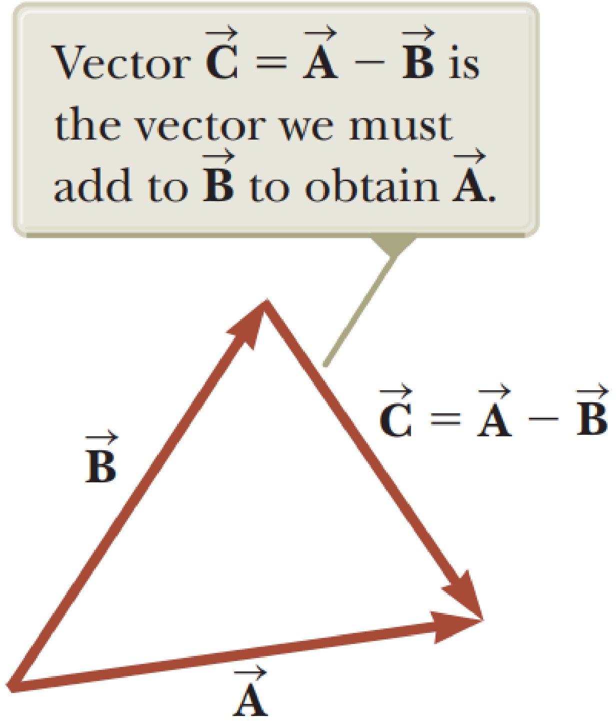
The operation of vector subtraction makes use of the definition of the negative of a vector. We define the operation $\vec{A} - \vec{B}$ as vector $-\vec{B}$ to vector $\vec{A}$.

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$



Subtracting vector

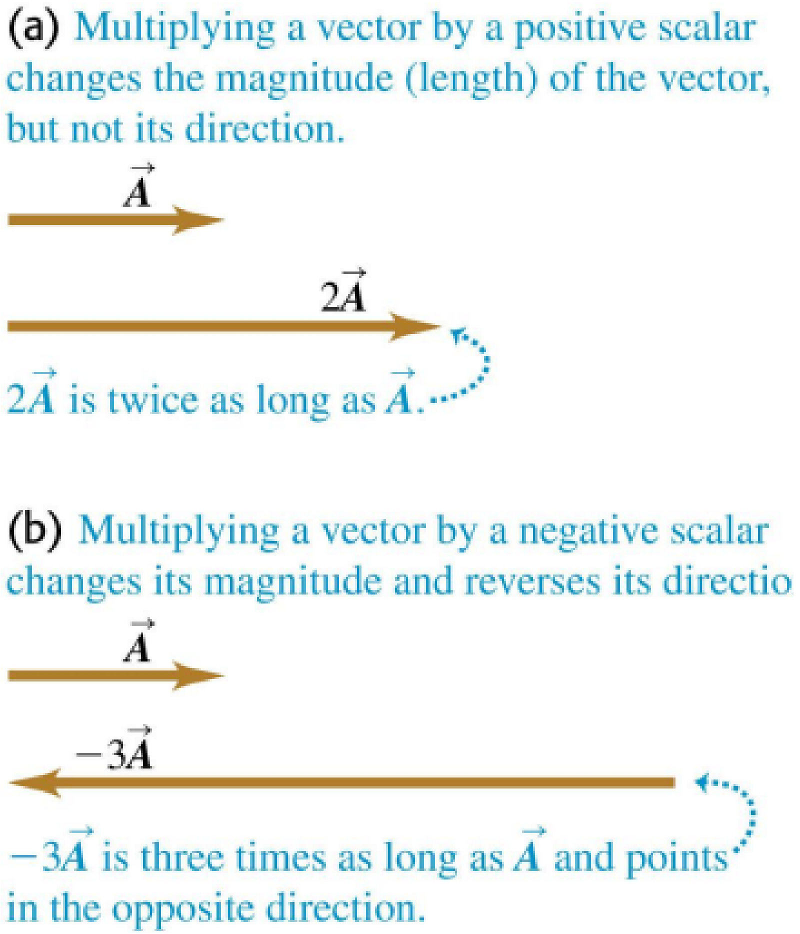
Another way of looking at vector substraction is to notice that the difference $\vec{A} - \vec{B}$ between to vectors $\vec{A}$ and $\vec{B}$ is what you have to add to the second vector to obtain the first. In this case, as Figure below shows, the vector $\vec{A} - \vec{B}$ points from the second vector to the tip of the first.



Multiplying a Vector by a Scalar

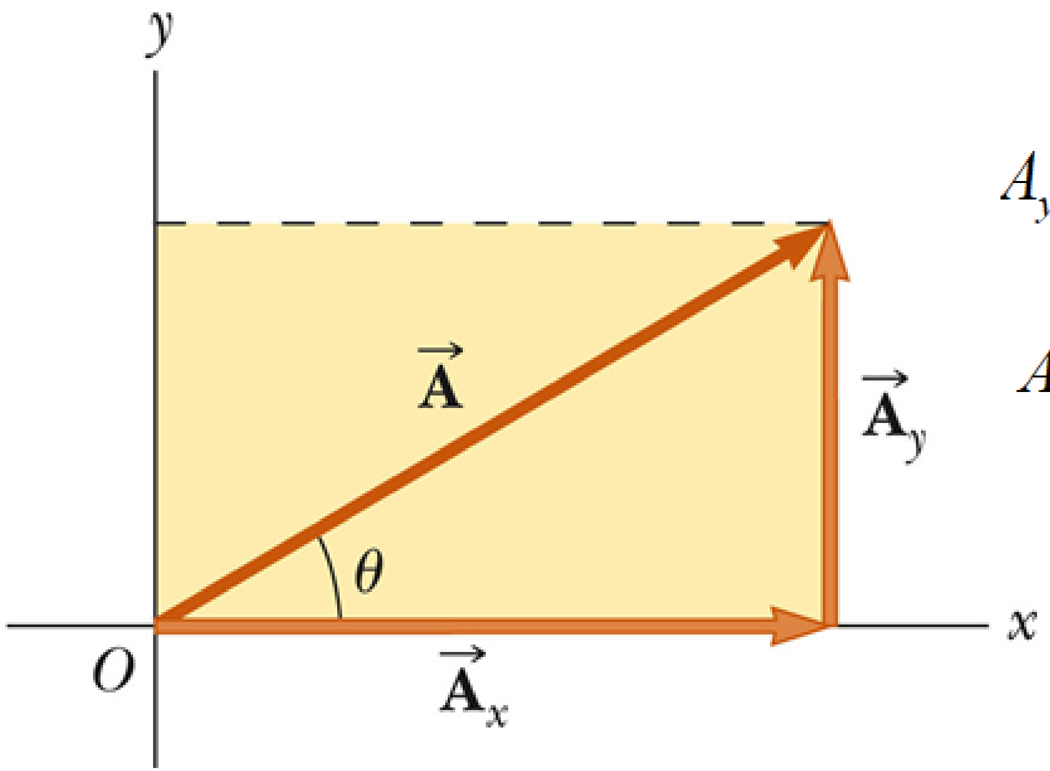
- If c is a scalar, the product $c\vec{A}$ has magnitude $|c|A$.

- Multiplication of a vector by a positive scalar and a negative scalar.



3-4 Adding Vectors by components

- Assume you are given a vector $\vec{\mathbf{A}}$
- It can be expressed in terms of two other vectors, $\vec{\mathbf{A}}_x$ and $\vec{\mathbf{A}}_y$
- These three vectors form a right triangle
- $\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y$



$$A_y = A\sin\theta$$

$$A_x = A\cos\theta$$

$$A = \sqrt{A_x^2 + A_y^2}$$

$$\theta = \tan^{-1}\frac{A_y}{A_x}$$

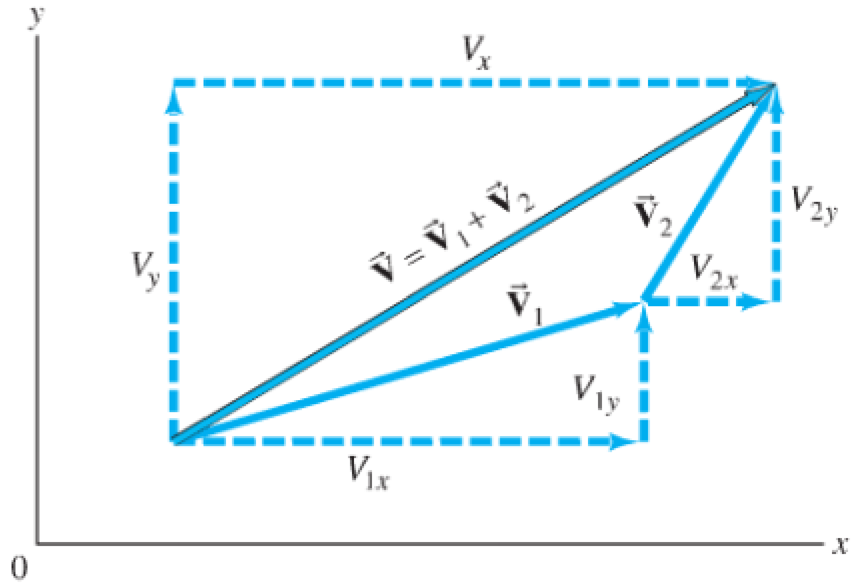
We can now discuss how to add vectors using components. The first step is to resolve each vector into its components. Next we can see, using Fig. 3-12, that the addition of any two $\overrightarrow{\boldsymbol{V_1}}$ and $\overrightarrow{\boldsymbol{V_2}}$ to give a resultant, $\vec{\boldsymbol{V}} = \overrightarrow{\boldsymbol{V_1}} + \overrightarrow{\boldsymbol{V_2}}$, implies that

$$V_x = V_{1x} + V_{2x}$$

$$V_y = V_{1y} + V_{2y}$$

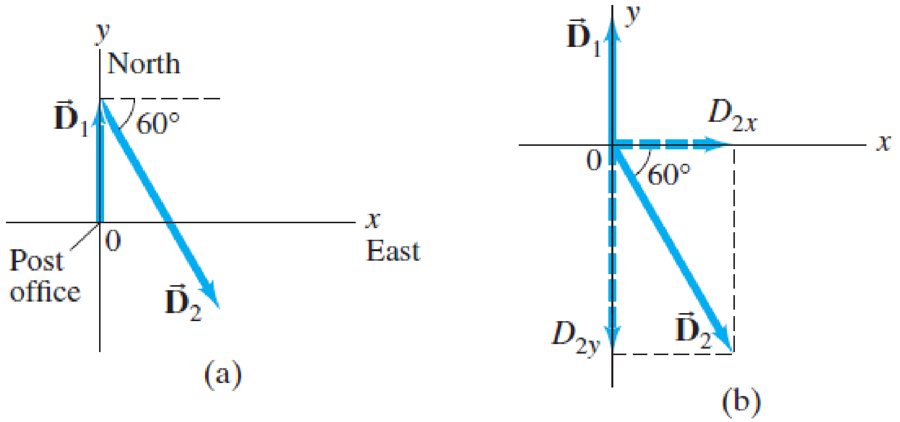
FIGURE 3–12 The components of $\vec{\mathbf{V}} = \vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2$ are

$$V_x = V_{1x} + V_{2x}$$
$$V_y = V_{1y} + V_{2y}.$$



3-4 Adding Vectors by components

EXAMPLE 3–2 **Mail carrier's displacement.** A rural mail carrier leaves the post office and drives 22.0 km in a northerly direction. She then drives in a direction 60.0° south of east for 47.0 km (Fig. 3–15a). What is her displacement from the post office?



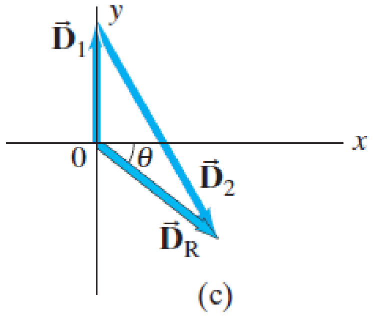


FIGURE 3–15 Example 3–2. (a) The two displacement vectors, $\vec{\mathbf{D}}_1$ and $\vec{\mathbf{D}}_2$. (b) $\vec{\mathbf{D}}_2$ is resolved into its components. (c) $\vec{\mathbf{D}}_1$ and $\vec{\mathbf{D}}_2$ are added to obtain the resultant $\vec{\mathbf{D}}_R$. The component method of adding the vectors is explained in the Example.

SOLUTION Resolve each displacement vector into its components, as shown in Fig. 3–15b. Since $\vec{\mathbf{D}}_1$ has magnitude 22.0 km and points north, it has only a y component:

$$D_{1x} = 0, \qquad D_{1y} = 22.0\,\text{km}.$$

$\vec{\mathbf{D}}_2$ has both x and y components:

$$D_{2x} = +(47.0\,\text{km})(\cos 60°) = +(47.0\,\text{km})(0.500) = +23.5\,\text{km}$$
$$D_{2y} = -(47.0\,\text{km})(\sin 60°) = -(47.0\,\text{km})(0.866) = -40.7\,\text{km}.$$

Notice that D_{2y} is negative because this vector component points along the negative y axis. The resultant vector, $\vec{\mathbf{D}}_R$, has components:

$$D_{Rx} = D_{1x} + D_{2x} = 0\,\text{km} + 23.5\,\text{km} = +23.5\,\text{km}$$
$$D_{Ry} = D_{1y} + D_{2y} = 22.0\,\text{km} + (-40.7\,\text{km}) = -18.7\,\text{km}.$$

This specifies the resultant vector completely:

$$D_{Rx} = 23.5\,\text{km}, \qquad D_{Ry} = -18.7\,\text{km}.$$

We can also specify the resultant vector by giving its magnitude and angle using Eqs. 3–4:

$$D_R = \sqrt{D_{Rx}^2 + D_{Ry}^2} = \sqrt{(23.5\,\text{km})^2 + (-18.7\,\text{km})^2} = 30.0\,\text{km}$$

$$\tan\theta = \frac{D_{Ry}}{D_{Rx}} = \frac{-18.7\,\text{km}}{23.5\,\text{km}} = -0.796.$$

A calculator with a key labeled INV TAN, or ARC TAN, or TAN^{-1} gives $\theta = \tan^{-1}(-0.796) = -38.5°$. The negative sign means $\theta = 38.5°$ below the x axis, Fig. 3–15c. So, the resultant displacement is 30.0 km directed at 38.5° in a southeasterly direction.

3-4 Adding Vectors by components

PROBLEM SOLVING

Adding Vectors

Here is a brief summary of how to add two or more vectors using components:

1. **Draw a diagram**, adding the vectors graphically by either the parallelogram or tail-to-tip method.
2. **Choose** x and y **axes**. Choose them in a way, if possible, that will make your work easier. (For example, choose one axis along the direction of one of the vectors, which then will have only one component.)
3. **Resolve** each vector into its x and y **components**, showing each component along its appropriate (x or y) axis as a (dashed) arrow.
4. **Calculate each component** (when not given) using sines and cosines. If θ_1 is the angle that vector $\vec{\mathbf{V}}_1$ makes with the positive x axis, then:
$$V_{1x} = V_1 \cos\theta_1, \qquad V_{1y} = V_1 \sin\theta_1 .$$
Pay careful attention to **signs**: any component that points along the negative x or y axis gets a minus sign.
5. **Add** the x **components** together to get the x component of the resultant. Similarly for y:
$$V_{Rx} = V_{1x} + V_{2x} + \text{any others}$$
$$V_{Ry} = V_{1y} + V_{2y} + \text{any others}.$$
This is the answer: the components of the resultant vector. Check signs to see if they fit the quadrant shown in your diagram (point 1 above).
6. If you want to know the **magnitude and direction** of the resultant vector, use Eqs. 3–4:
$$V_R = \sqrt{V_{Rx}^2 + V_{Ry}^2}, \qquad \tan\theta = \frac{V_{Ry}}{V_{Rx}}.$$
The vector diagram you already drew helps to obtain the correct position (quadrant) of the angle θ.

EXAMPLE 3–3 **Three short trips.** An airplane trip involves three legs, with two stopovers, as shown in Fig. 3–16a. The first leg is due east for 620 km; the second leg is southeast (45°) for 440 km; and the third leg is at 53° south of west, for 550 km, as shown. What is the plane's total displacement?

APPROACH We follow the steps in the Problem Solving Strategy above.

SOLUTION

1. **Draw a diagram** such as Fig. 3–16a, where $\vec{\mathbf{D}}_1$, $\vec{\mathbf{D}}_2$, and $\vec{\mathbf{D}}_3$ represent the three legs of the trip, and $\vec{\mathbf{D}}_R$ is the plane's total displacement.
2. **Choose axes:** Axes are also shown in Fig. 3–16a: x is east, y north.
3. **Resolve components:** It is imperative to draw a good diagram. The components are drawn in Fig. 3–16b. Instead of drawing all the vectors starting from a common origin, as we did in Fig. 3–15b, here we draw them "tail-to-tip" style, which is just as valid and may make it easier to see.
4. **Calculate the components:**

$$\begin{aligned}
\vec{\mathbf{D}}_1: D_{1x} &= +D_1 \cos 0^\circ &&= D_1 &&= 620\text{ km} \\
D_{1y} &= +D_1 \sin 0^\circ &&= 0\text{ km} \\
\vec{\mathbf{D}}_2: D_{2x} &= +D_2 \cos 45^\circ &&= +(440\text{ km})(0.707) &&= +311\text{ km} \\
D_{2y} &= -D_2 \sin 45^\circ &&= -(440\text{ km})(0.707) &&= -311\text{ km} \\
\vec{\mathbf{D}}_3: D_{3x} &= -D_3 \cos 53^\circ &&= -(550\text{ km})(0.602) &&= -331\text{ km} \\
D_{3y} &= -D_3 \sin 53^\circ &&= -(550\text{ km})(0.799) &&= -439\text{ km.}
\end{aligned}$$

We have given a minus sign to each component that in Fig. 3–16b points in the $-x$ or $-y$ direction. The components are shown in the Table in the margin.

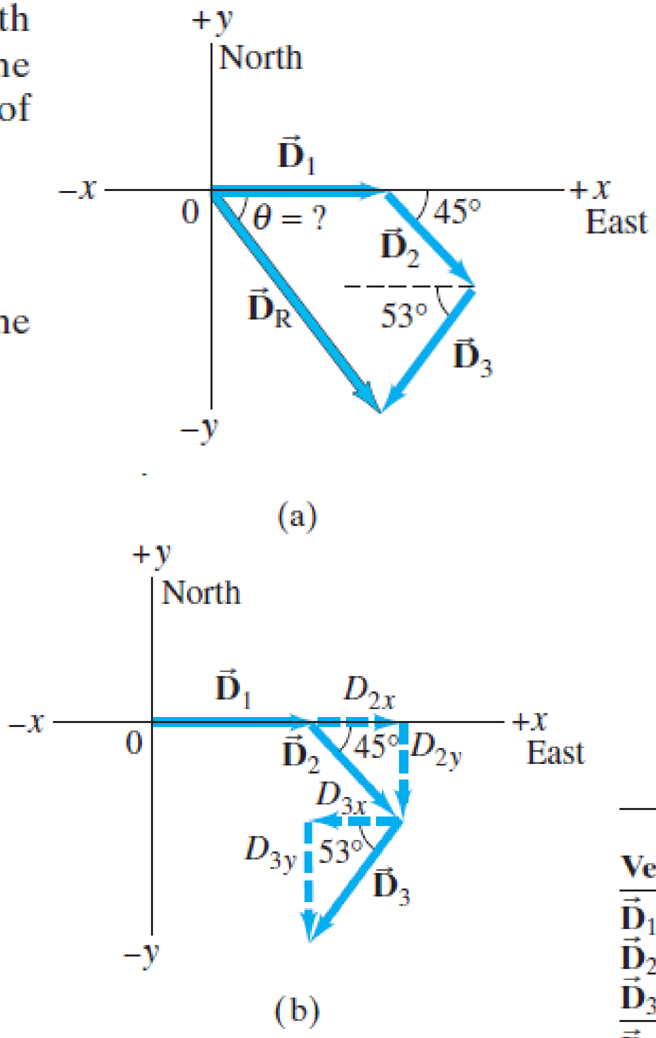


FIGURE 3–16 Example 3–3.

Vector	Components x (km)	y (km)
$\vec{\mathbf{D}}_1$	620	0
$\vec{\mathbf{D}}_2$	311	−311
$\vec{\mathbf{D}}_3$	−331	−439
$\vec{\mathbf{D}}_R$	600	−750

3-4 Adding Vectors by components

5. **Add the components**: We add the x components together, and we add the y components together to obtain the x and y components of the resultant:

$$D_{Rx} = D_{1x} + D_{2x} + D_{3x} = 620\text{ km} + 311\text{ km} - 331\text{ km} = 600\text{ km}$$

$$D_{Ry} = D_{1y} + D_{2y} + D_{3y} = 0\text{ km} - 311\text{ km} - 439\text{ km} = -750\text{ km}.$$

The x and y components of the resultant are 600 km and −750 km, and point respectively to the east and south. This is one way to give the answer.

6. **Magnitude and direction**: We can also give the answer as

$$D_R = \sqrt{D_{Rx}^2 + D_{Ry}^2} = \sqrt{(600)^2 + (-750)^2}\text{ km} = 960\text{ km}$$

$$\tan\theta = \frac{D_{Ry}}{D_{Rx}} = \frac{-750\text{ km}}{600\text{ km}} = -1.25, \qquad \text{so } \theta = -51^\circ.$$

Thus, the total displacement has magnitude 960 km and points 51° below the x axis (south of east), as was shown in our original sketch, Fig. 3–16a.

3-5 Unit Vectors

A ***unit vector*** is a dimensionless vector with a magnitude of exactly 1. Unit vectors are used to specify a direction and have no other physical significance.

$$\left|\hat{\mathbf{i}}\right| = \left|\hat{\mathbf{j}}\right| = \left|\hat{\mathbf{k}}\right| = 1$$

Using unit vectors, any vector can be written in terms of its components:

$$\vec{\mathbf{V}} = V_x\hat{\mathbf{i}} + V_y\hat{\mathbf{j}} + V_z\hat{\mathbf{k}}.$$

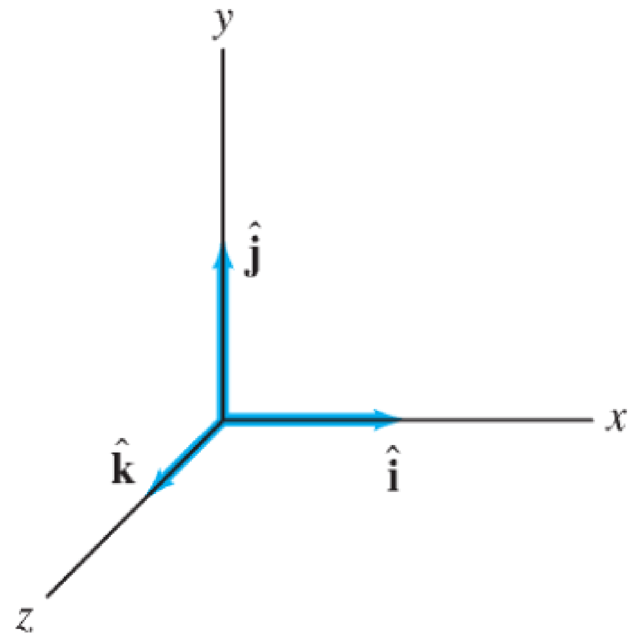


FIGURE 3–15 Unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ along the x, y, and z axes.

Adding Vectors Using Unit Vectors

- Using $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$
- Then $\vec{\mathbf{R}} = \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}\right) + \left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}}\right)$

$$\vec{\mathbf{R}} = \left(A_x + B_x\right)\hat{\mathbf{i}} + \left(A_y + B_y\right)\hat{\mathbf{j}}$$

$$\vec{\mathbf{R}} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}}$$

- and so $R_x = A_x + B_x$ and $R_y = A_y + B_y$

$$R = \sqrt{R_x^2 + R_y^2} \qquad \theta = \tan^{-1}\frac{R_y}{R_x}$$

Three-Dimensional Extension

- Using $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$
- Then $\vec{\mathbf{R}} = \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right) + \left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}\right)$

$$\vec{\mathbf{R}} = \left(A_x + B_x\right)\hat{\mathbf{i}} + \left(A_y + B_y\right)\hat{\mathbf{j}} + \left(A_z + B_z\right)\hat{\mathbf{k}}$$

$$\vec{\mathbf{R}} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}} + R_z\hat{\mathbf{k}}$$

- and so $R_x = A_x + B_x$, $R_y = A_y + B_y$, and $R_z = A_z + B_z$

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} \qquad \theta = \cos^{-1}\frac{R_x}{R}, etc.$$

$$\vec{A} = 3\hat{i} + 4\hat{j} + 12\hat{k}, \quad |\vec{A}| = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{169} = 13$$

$$\vec{B} = 2\hat{i} + 6\hat{j} + 3\hat{k}, \quad |\vec{B}| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{49} = 7$$

$$\vec{A} + \vec{B} = (3+2)\hat{i} + (4+6)\hat{j} + (12+3)\hat{k} = 5\hat{i} + 10\hat{j} + 15\hat{k}$$

$$|\vec{A} + \vec{B}| = \sqrt{5^2 + 10^2 + 15^2} = \sqrt{25 + 100 + 225} = \sqrt{350} \approx 18.708$$

Scalar Product

Two vectors can be multiplied together to give the **scalar product**, also known as the **dot product**.

The result of this multiplication is a *scalar* quantity, hence the name.

The scalar product of two vectors **a** and **b** is defined as:

$$\boxed{\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta}$$

where θ is the angle between **a** and **b** when they are placed tail to tail.

Note that θ is always taken to be between $0°$ and $180°$.

$\boxed{\text{If a.b = 0 then either a = 0, b = 0 or } \mathbf{a} \text{ and } \mathbf{b} \text{ are perpendicular.}}$

In particular, for the unit base vectors **i**, **j** and **k**:

$$\mathbf{i}.\mathbf{j} = \mathbf{i}.\mathbf{k} = \mathbf{j}.\mathbf{i} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = \mathbf{k}.\mathbf{j} = 0$$

Also, if two vectors are parallel the angle between them is taken as 0°.

Since cos 0° = 1 we can conclude from the scalar product that:

$\boxed{\text{If two vectors } \mathbf{a} \text{ and } \mathbf{b} \text{ are parallel, } \mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|.}$

In particular, for the unit base vectors **i**, **j** and **k**:

$$\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$$

Scalar Product

Find the scalar product of **a** = **i** + 2**j** + **k** and **b** = –**i** + 4**j** + 3**k**.

a.b = (**i** + 2**j** + **k**).(–**i** + 4**j** + 3**k**)

= **i**.(–**i** + 4**j** + 3**k**) + 2**j**.(–**i** + 4**j** + 3**k**) + **k**.(–**i** + 4**j** + 3**k**)

= –**i**.**i** + 4**i**.**j** + 3**i**.**k** – 2**j**.**i** + 8**j**.**j** + 6**j**.**k** – **k**.**i** + 4**k**.**j** + 3**k**.**k**

Since **i**.**j** = **i**.**k** = **j**.**i** = **j**.**k** = **k**.**i** = **k**.**j** = 0, this reduces to:

a.b = –**i**.**i** + 8**j**.**j** + 3**k**.**k**

And since **i**.**i** = **j**.**j** = **k**.**k** = 1,

a.b = –1 + 8 + 3 = 10

In general, if **a** = a_1**i** + a_2**j** + a_3**k** and **b** = b_1**i** + b_2**j** + b_3**k** then **a.b** = $a_1b_1 + a_2b_2 + a_3b_3$.

Prove that the vectors **a** = –3**i** + **j** + 2**k** and **b** = 2**i** + 8**j** – **k** are perpendicular.

$$\mathbf{a}.\mathbf{b} = (-3\mathbf{i} + \mathbf{j} + 2\mathbf{k}).(2\mathbf{i} + 8\mathbf{j} - \mathbf{k})$$
$$= (-3 \times 2) + (1 \times 8) + (2 \times -1)$$
$$= -6 + 8 - 2$$
$$= 0$$

Since **a** and **b** are both non-zero vectors, and their scalar product is 0, they must be perpendicular.

Finding the angle between two vectors

A useful application of the scalar product is in finding the angle between two vectors.

To do this we can write the scalar product in the form:

$$\boxed{\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}}$$

Finding the Angle Between Two 3D Vectors Using the Dot Product

General Formula

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta$$

where

- $\mathbf{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$
- $\mathbf{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$
- θ is the **angle between** the two vectors

Step 1: Dot Product in Component Form

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

Step 2: Magnitudes of the Vectors

$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$|\mathbf{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

Step 3: Calculate the Angle

$$\boxed{\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\,\sqrt{B_x^2 + B_y^2 + B_z^2}}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\,\sqrt{B_x^2 + B_y^2 + B_z^2}}\right)}$$

Example:

Let

$$\mathbf{A} = 2\hat{i} + 3\hat{j} + \hat{k}, \quad \mathbf{B} = \hat{i} + 2\hat{j} + 2\hat{k}$$

1. **Dot product:**

$$\mathbf{A} \cdot \mathbf{B} = (2)(1) + (3)(2) + (1)(2) = 2 + 6 + 2 = 10$$

2. **Magnitudes:**

$$|\mathbf{A}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

$$|\mathbf{B}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$$

3. **Find the angle:**

$$\cos\theta = \frac{10}{(3)(\sqrt{14})} = \frac{10}{3\sqrt{14}}$$

$$\theta = \cos^{-1}\left(\frac{10}{3\sqrt{14}}\right)$$

4. **Numerical value:**

$$\frac{10}{3\sqrt{14}} = \frac{10}{11.22} = 0.891$$

$$\boxed{\theta = 26.9^\circ}$$

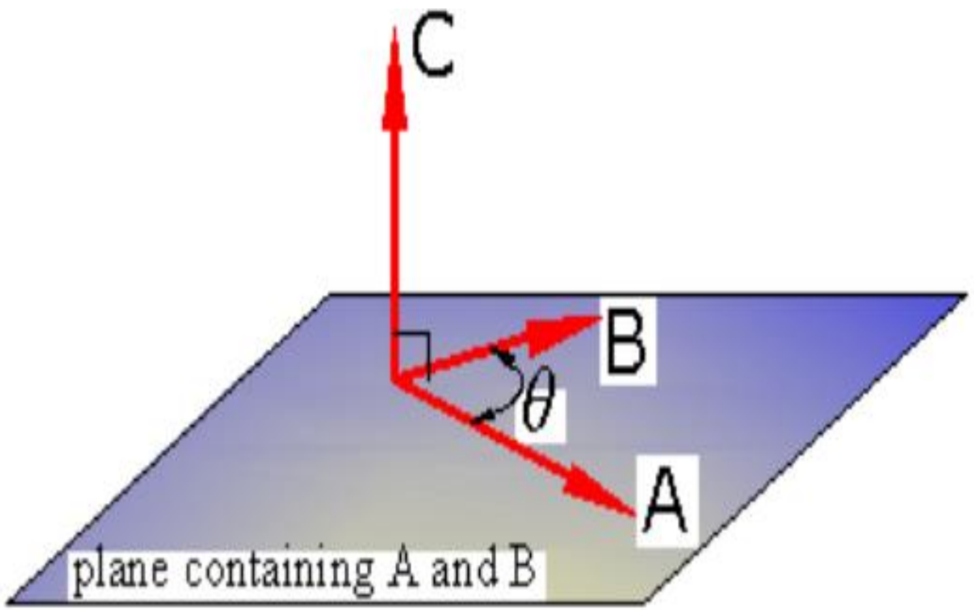
Vector Product: The vector product (also called cross product) of two vectors $\vec{A}$ and $\vec{B}$ is a vector that is perpendicular to both $\vec{A}$ and $\vec{B}$.

$$\vec{A} \times \vec{B} = \vec{C}$$

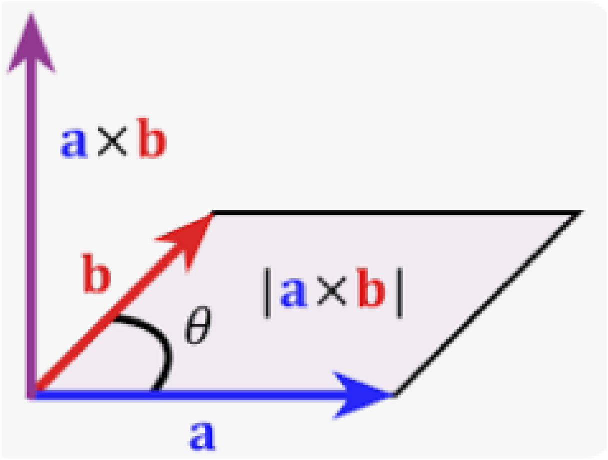
Magnitude: $|\vec{A} \times \vec{B}| = |\vec{C}| = |\vec{A}| \cdot |\vec{B}| \cdot \sin\theta$

θ= Angle between $\vec{A}$ and $\vec{B}$ $(0° \leq \theta \leq 180°)$

The vector product (vector $\vec{C}$) must be perpendicular to the plane that contains $\vec{A}$ and $\vec{B}$



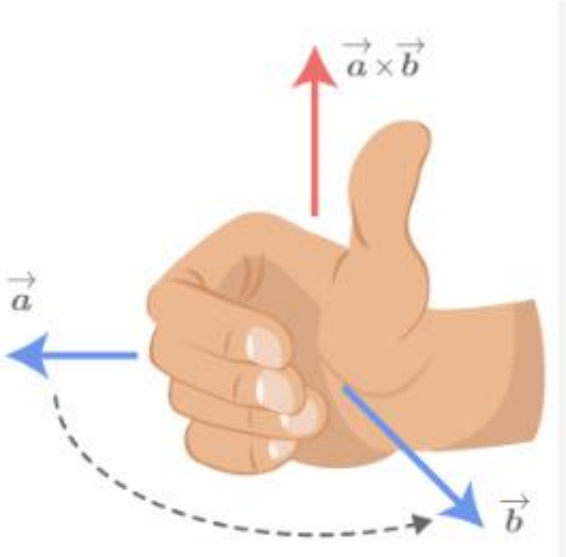
Vector Product



Geometrical Meaning:

The magnitude of $\vec{a} \times \vec{b}$ equals the **area of the parallelogram** formed by $\vec{a}$ and $\vec{b}$.

$$\text{Area} = |\vec{a} \times \vec{b}|$$



Direction (Right-Hand Rule):

- Point fingers of your right hand along $\vec{a}$
- Rotate towards $\vec{b}$
- Thumb shows the direction of $\vec{a} \times \vec{b}$

All Combinations of Unit Vector Cross Products

1. Cross Products with Themselves (Zero Result)

$$\hat{i} \times \hat{i} = \vec{0}$$

$$\hat{j} \times \hat{j} = \vec{0}$$

$$\hat{k} \times \hat{k} = \vec{0}$$

2. Cross Products in Positive (Right-Hand) Order

$$\hat{i} \times \hat{j} = \hat{k}$$

$$\hat{j} \times \hat{k} = \hat{i}$$

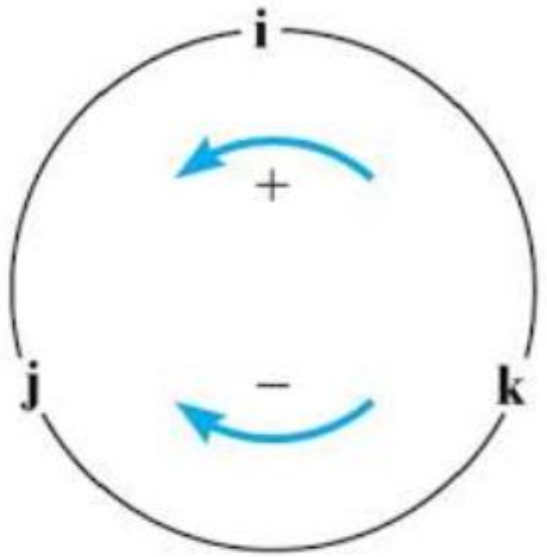
$$\hat{k} \times \hat{i} = \hat{j}$$

3. Cross Products in Reverse (Negative) Order

$$\hat{j} \times \hat{i} = -\hat{k}$$

$$\hat{k} \times \hat{j} = -\hat{i}$$

$$\hat{i} \times \hat{k} = -\hat{j}$$



If

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}, \quad \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

then

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\hat{i} - (A_xB_z - A_zB_x)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

Vector Product

Example: Cross Product of Two 3D Vectors

Let

$$\vec{A} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$$

$$\vec{B} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$$

Step 1: Write Determinant Form

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 4 \\ 1 & -2 & 1 \end{vmatrix}$$

Step 2: Expand the Determinant

$$\vec{A} \times \vec{B} = \hat{\imath}(3 \cdot 1 - 4 \cdot (-2)) - \hat{\jmath}(2 \cdot 1 - 4 \cdot 1) + \hat{k}(2 \cdot (-2) - 3 \cdot 1)$$

Step 3: Simplify Each Term

$$\vec{A} \times \vec{B} = \hat{\imath}(3 + 8) - \hat{\jmath}(2 - 4) + \hat{k}(-4 - 3)$$

$$\vec{A} \times \vec{B} = 11\hat{\imath} - (-2)\hat{\jmath} - 7\hat{k}$$

Step 4: Final Result

$$\boxed{\vec{A} \times \vec{B} = 11\hat{\imath} + 2\hat{\jmath} - 7\hat{k}}$$

Vector Product

Type	Physics Quantity / Formula	Vector Expression	Remarks
Dot Product	Work done by a force	$W = \vec{F} \cdot \vec{d} = Fd\cos\theta$	Scalar quantity; measures energy transfer along displacement
Dot Product	Power delivered by a force	$P = \vec{F} \cdot \vec{v} = Fv\cos\theta$	Scalar; rate of doing work
Dot Product	Projection of a vector along a direction	$A_{\parallel} = \vec{A} \cdot \hat{n}$	Scalar component along unit vector $\hat{n}$
Cross Product	Torque	$\vec{\tau} = \vec{r} \times \vec{F} = rF\sin\theta\,\hat{n}$	Vector; direction perpendicular to plane of $\vec{r}$ and $\vec{F}$
Cross Product	Angular momentum	$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$	Vector; measures rotational effect of motion

- Consider a particle moving in the **xy-plane**.
- At time t_1, particle is at point P_1 with **position vector** $\vec{r}_1$.
- At time t_2, particle is at point P_2 with **position vector** $\vec{r}_2$.

Displacement Vector:

- Generalization of 1D displacement.
- Defined as the **vector change in position**:

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$

The position vector of a point P_1 in unit vector form is written as:

$$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

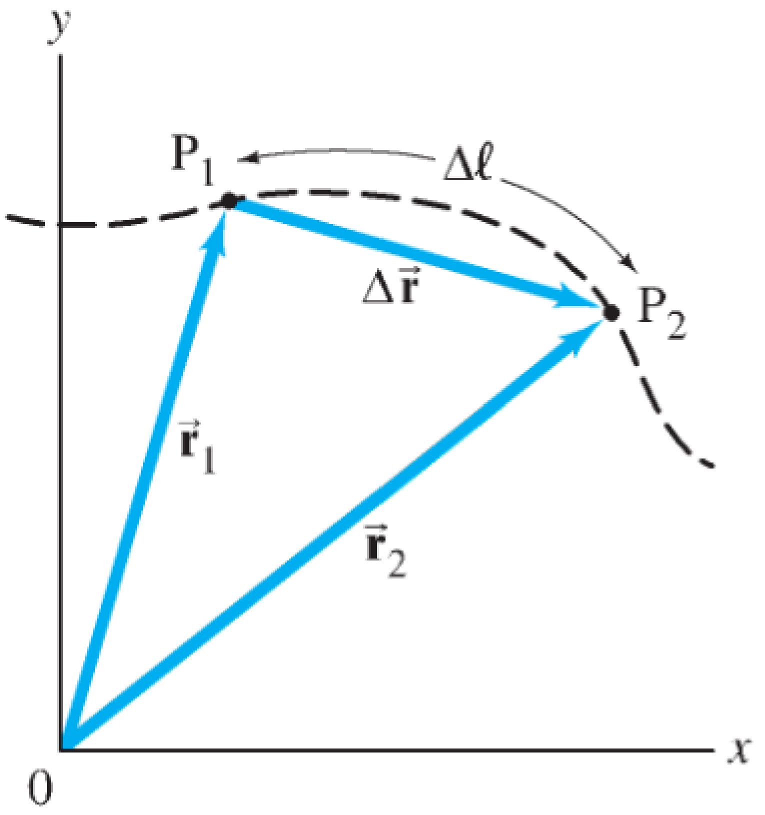
where x_1, y_1, z_1 are the coordinates of P_1. Similarly, for point P_2:

$$\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

The displacement vector is then:

$$\Delta\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

If motion occurs along the x-axis only, $y_2 - y_1 = 0$ and $z_2 - z_1 = 0$, reducing the displacement magnitude to $\Delta r = x_2 - x_1$, consistent with one-dimensional motion.



Average and Instantaneous Velocity

- **Average velocity** over time interval $\Delta t = t_2 - t_1$:

$$\vec{v}_{avg} = \frac{\Delta \vec{r}}{\Delta t}$$

- **Instantaneous velocity** is the limit of average velocity as $\Delta t \to 0$:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

- The direction of $\vec{v}$ at any instant is **tangent to the path** at that point.

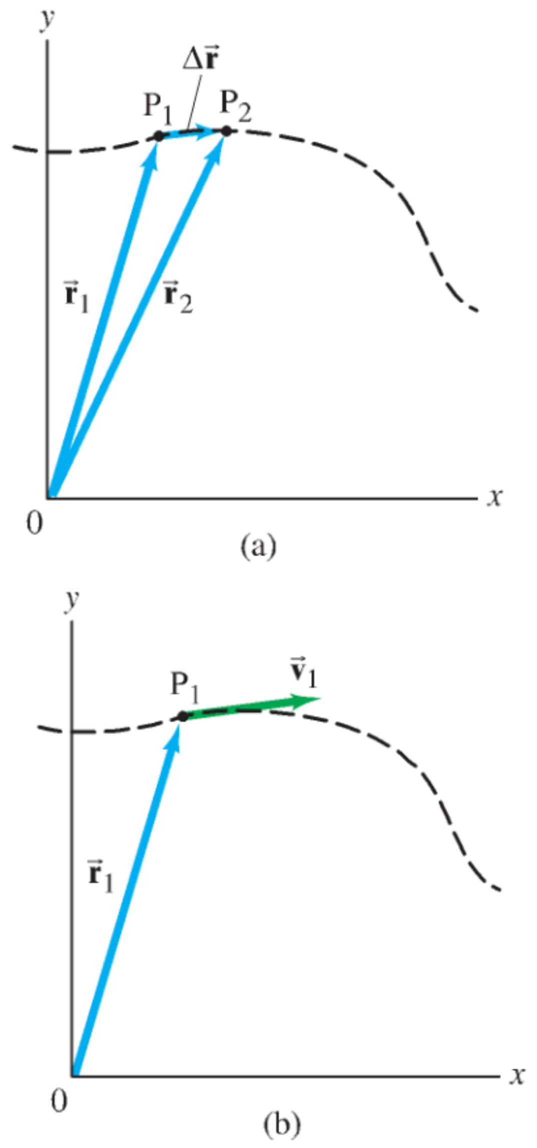


FIGURE 3–17 (a) As we take Δt and $\Delta\vec{\mathbf{r}}$ smaller and smaller [compare to Fig. 3–16] we see that the direction of $\Delta \mathbf{r}$ and of the instantaneous velocity ($\Delta\vec{\mathbf{r}}/\Delta t$, where $\Delta t \to 0$) is (b) tangent to the curve at P_1.

3-6 Vector Kinematics

Instantaneous Velocity in 3D

- Instantaneous velocity **v** is the **time derivative of the position vector**:

$$\vec{v} = \frac{d\vec{r}}{dt}$$

- In component form:

$$\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

- Components:
  - $v_x = \frac{dx}{dt}$
  - $v_y = \frac{dy}{dt}$
  - $v_z = \frac{dz}{dt}$
- Unit vectors are constant: $\frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = \frac{d\hat{k}}{dt} = 0$

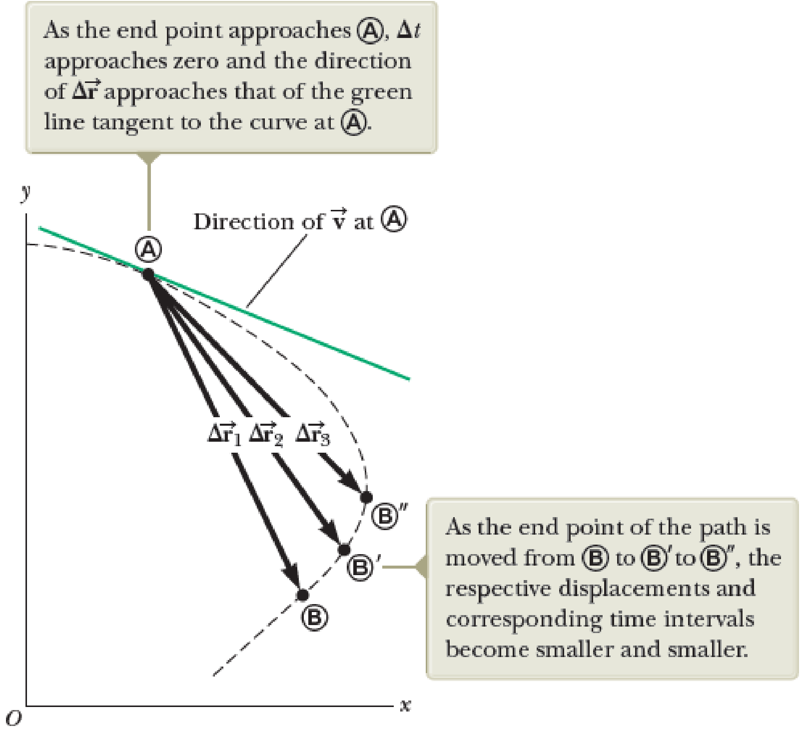


Figure 4.2 As a particle moves between two points, its average velocity is in the direction of the displacement vector $\Delta\vec{r}$. By definition, the instantaneous velocity at Ⓐ is directed along the line tangent to the curve at Ⓐ.

Acceleration in 2D and 3D

- **Average acceleration** over a time interval $\Delta t = t_2 - t_1$:

$$\vec{a}_{\text{avg}} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

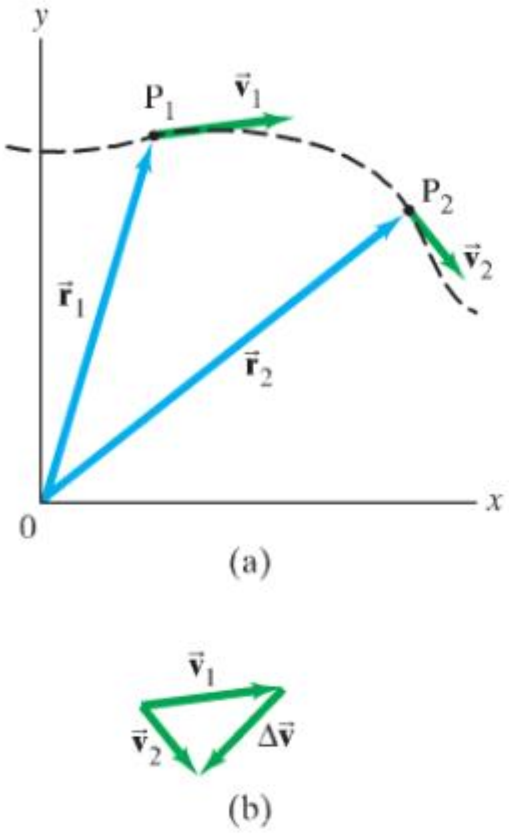
- **Key points:**
  - $\vec{v}_2$ may point in a different direction than $\vec{v}_1$.
  - Acceleration can arise from:
    - Change in **magnitude** of velocity
    - Change in **direction** of velocity
    - Change in **both** magnitude and direction

- **Instantaneous acceleration:**

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

  - It is the **time derivative of the velocity vector**.

FIGURE 3–18 (a) Velocity vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ at instants t_1 and t_2 for a particle at points P_1 and P_2, as in Fig. 3–16. (b) The direction of the average acceleration is in the direction of $\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_1$.



Instantaneous Acceleration

The **instantaneous acceleration vector** represents how the **velocity of a particle changes at a specific moment in time.**

It is defined as the **limit of the average acceleration** as the time interval (Δt) approaches zero:

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

Component Form

In three dimensions, the instantaneous acceleration vector can be expressed in terms of its Cartesian components as:

$$\vec{a} = \frac{dv_x}{dt}\,\hat{i} + \frac{dv_y}{dt}\,\hat{j} + \frac{dv_z}{dt}\,\hat{k}$$

or equivalently,

$$\vec{a} = \frac{d^2x}{dt^2}\,\hat{i} + \frac{d^2y}{dt^2}\,\hat{j} + \frac{d^2z}{dt^2}\,\hat{k}$$

Important Notes:

- Acceleration occurs whenever the **magnitude or direction of velocity changes**.
- Examples:
  - Car moving at constant speed around a curve (direction changes)
  - Child on a merry-go-round (direction changes)
- The terms **velocity** and **acceleration** usually refer to **instantaneous values**; use "average" when discussing average quantities

Acceleration is not just about speeding up or slowing down; it also includes changes in the direction of motion.

EXAMPLE 3–5 **Position given as a function of time.** The position of a particle as a function of time is given by

$$\vec{\mathbf{r}} = \left[(5.0\,\mathrm{m/s})t + (6.0\,\mathrm{m/s^2})t^2\right]\hat{\mathbf{i}} + \left[(7.0\,\mathrm{m}) - (3.0\,\mathrm{m/s^3})t^3\right]\hat{\mathbf{j}},$$

where r is in meters and t is in seconds. (*a*) What is the particle's displacement between $t_1 = 2.0\,\mathrm{s}$ and $t_2 = 3.0\,\mathrm{s}$? (*b*) Determine the particle's instantaneous velocity and acceleration as a function of time. (*c*) Evaluate $\vec{\mathbf{v}}$ and $\vec{\mathbf{a}}$ at $t = 3.0\,\mathrm{s}$.

SOLUTION (*a*) At $t_1 = 2.0\,\mathrm{s}$,

$$\begin{aligned}\vec{\mathbf{r}}_1 &= \left[(5.0\,\mathrm{m/s})(2.0\,\mathrm{s}) + (6.0\,\mathrm{m/s^2})(2.0\,\mathrm{s})^2\right]\hat{\mathbf{i}} + \left[(7.0\,\mathrm{m}) - (3.0\,\mathrm{m/s^3})(2.0\,\mathrm{s})^3\right]\hat{\mathbf{j}} \\ &= (34\,\mathrm{m})\hat{\mathbf{i}} - (17\,\mathrm{m})\hat{\mathbf{j}}.\end{aligned}$$

Similarly, at $t_2 = 3.0\,\mathrm{s}$,

$$\vec{\mathbf{r}}_2 = (15\,\mathrm{m} + 54\,\mathrm{m})\hat{\mathbf{i}} + (7.0\,\mathrm{m} - 81\,\mathrm{m})\hat{\mathbf{j}} = (69\,\mathrm{m})\hat{\mathbf{i}} - (74\,\mathrm{m})\hat{\mathbf{j}}.$$

Thus

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 = (69\,\mathrm{m} - 34\,\mathrm{m})\hat{\mathbf{i}} + (-74\,\mathrm{m} + 17\,\mathrm{m})\hat{\mathbf{j}} = (35\,\mathrm{m})\hat{\mathbf{i}} - (57\,\mathrm{m})\hat{\mathbf{j}}.$$

That is, $\Delta x = 35\,\mathrm{m}$, and $\Delta y = -57\,\mathrm{m}$.

(b) To find velocity, we take the derivative of the given $\vec{\mathbf{r}}$ with respect to time, noting (Appendix B–2) that $d(t^2)/dt = 2t$, and $d(t^3)/dt = 3t^2$:

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = [5.0\,\mathrm{m/s} + (12\,\mathrm{m/s^2})t]\hat{\mathbf{i}} + [0 - (9.0\,\mathrm{m/s^3})t^2]\hat{\mathbf{j}}.$$

The acceleration is (keeping only two significant figures):

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = (12\,\mathrm{m/s^2})\hat{\mathbf{i}} - (18\,\mathrm{m/s^3})t\,\hat{\mathbf{j}}.$$

Thus $a_x = 12\,\mathrm{m/s^2}$ is constant; but $a_y = -(18\,\mathrm{m/s^3})t$ depends linearly on time, increasing in magnitude with time in the negative y direction.

(c) We substitute $t = 3.0\,\mathrm{s}$ into the equations we just derived for $\vec{\mathbf{v}}$ and $\vec{\mathbf{a}}$:

$$\vec{\mathbf{v}} = (5.0\,\mathrm{m/s} + 36\,\mathrm{m/s})\hat{\mathbf{i}} - (81\,\mathrm{m/s})\hat{\mathbf{j}} = (41\,\mathrm{m/s})\hat{\mathbf{i}} - (81\,\mathrm{m/s})\hat{\mathbf{j}}$$

$$\vec{\mathbf{a}} = (12\,\mathrm{m/s^2})\hat{\mathbf{i}} - (54\,\mathrm{m/s^2})\hat{\mathbf{j}}.$$

Their magnitudes at $t = 3.0\,\mathrm{s}$ are $v = \sqrt{(41\,\mathrm{m/s})^2 + (81\,\mathrm{m/s})^2} = 91\,\mathrm{m/s}$, and $a = \sqrt{(12\,\mathrm{m/s^2})^2 + (54\,\mathrm{m/s^2})^2} = 55\,\mathrm{m/s^2}$.

3-6 Vector Kinematics

Constant Acceleration

When the **acceleration vector** $\vec{a}$ has a constant **magnitude and direction**, its components are constant:

$$a_x = \text{constant}, \quad a_y = \text{constant}, \quad a_z = \text{constant}$$

In this case, the **average acceleration** equals the **instantaneous acceleration** at every moment.

The **equations of motion** for constant acceleration in one dimension (from Chapter 2) apply **separately to each component** of two- or three-dimensional motion.

If the initial velocity is

$$\vec{v_0} = v_{x0}\hat{\imath} + v_{y0}\hat{\jmath}$$

then the position, velocity, and acceleration vectors can be written using the same kinematic equations for each direction:

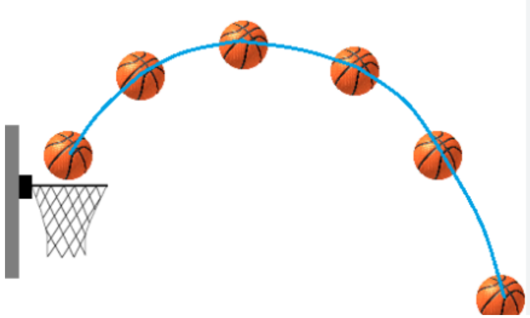
$$v_x = v_{x0} + a_x t$$
$$x = x_0 + v_{x0}t + \tfrac{1}{2}a_x t^2$$
$$v_y = v_{y0} + a_y t$$
$$y = y_0 + v_{y0}t + \tfrac{1}{2}a_y t^2$$

TABLE 3–1 Kinematic Equations for Constant Acceleration in 2 Dimensions

x Component (horizontal)		*y* Component (vertical)
$v_x = v_{x0} + a_x t$	(Eq. 2–12a)	$v_y = v_{y0} + a_y t$
$x = x_0 + v_{x0}t + \frac{1}{2}a_x t^2$	(Eq. 2–12b)	$y = y_0 + v_{y0}t + \frac{1}{2}a_y t^2$
$v_x^2 = v_{x0}^2 + 2a_x(x - x_0)$	(Eq. 2–12c)	$v_y^2 = v_{y0}^2 + 2a_y(y - y_0)$

Key Idea:

Constant acceleration in 2D or 3D behaves just like in 1D, but the equations apply **independently to each perpendicular component** of motion.



Projectile Motion

- Occurs when an object moves along a curved path and returns to the ground (e.g., a baseball).
- **Assumptions:**
  1. Acceleration due to gravity (g) is constant and downward.
  2. Air resistance is negligible.
- Under these assumptions, the trajectory is always **parabolic**.

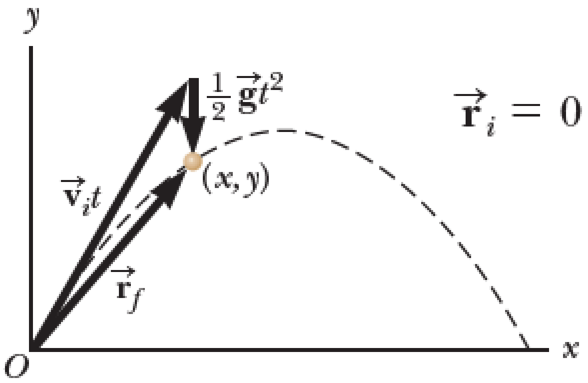


Figure 4.8 The position vector $\vec{\mathbf{r}}_f$ of a projectile launched from the origin whose initial velocity at the origin is $\vec{\mathbf{v}}_i$. The vector $\vec{\mathbf{v}}_i t$ would be the displacement of the projectile if gravity were absent, and the vector $\frac{1}{2}\vec{\mathbf{g}}t^2$ is its vertical displacement from a straight-line path due to its downward gravitational acceleration.

Position Vector:

$$\vec{r}(t) = \vec{r}_i + \vec{v}_i t + \frac{1}{2}\vec{g}t^2$$

- Initial velocity components:

$$v_{xi} = v_i \cos\theta_i, \quad v_{yi} = v_i \sin\theta_i$$

- Final position = initial position + straight-line displacement + vertical displacement due to gravity.

Projectile Motion: Two-Dimensional Analysis

- Two-dimensional motion with constant acceleration can be analyzed as **combination of two independent motions** in the x and y directions, wi accelerations a_x and a_y.
- **Projectile motion:**
  - Horizontal acceleration: $a_x = 0$ → **constant velocity**
  - Vertical acceleration: $a_y = -g$ → **constant acceleration**

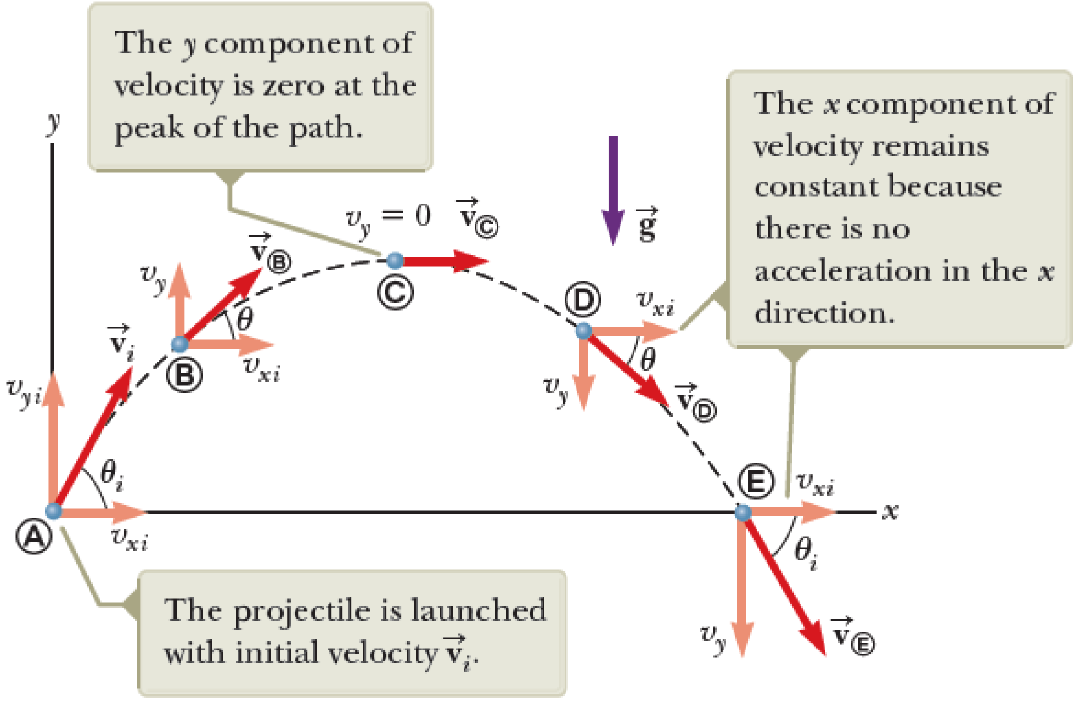
1. **Horizontal motion (constant velocity):**

$$x_f = x_i + v_{xi}t$$

2. **Vertical motion (constant acceleration):**

$$v_{yf} = v_{yi} - gt, \quad v_{y,\text{avg}} = \frac{v_{yi} + v_{yf}}{2}$$

$$y_f = y_i + v_{yi}t - \frac{1}{2}gt^2, \quad v_{yf}^2 = v_{yi}^2 - 2g(y_f - y_i)$$



$$v_{xi} = v_i \cos\theta_i, \quad v_{yi} = v_i \sin\theta_i$$

- **Key Point:** The horizontal and vertical components are **completely independent**, with **time t as the common variable** linking them.

Special Case of Projectile Motion

- Projectile is launched from the origin ($t_i = 0$) with an upward velocity and lands at the same horizontal level.
- Two key points in the trajectory:
  1. **Peak point A:** $(R/2, h)$, where h is the **maximum height**.
  2. **Point B:** $(R, 0)$, where R is the **horizontal range**.
- Maximum height h and horizontal range R can be expressed in terms of **initial speed** v_i, **launch angle** θ_i, **and gravity** g.

Maximum Height of a Projectile

- At the **peak**, the vertical velocity is zero: $v_{yA} = 0$.
- Using $v_y = v_{yi} - gt$, the time to reach the peak is:

$$t_A = \frac{v_i \sin\theta_i}{g}$$

- Substituting t_A into the vertical displacement equation:

$$y_f = y_i + v_{yi}t - \frac{1}{2}gt^2$$

- The **maximum height** is:

$$h = \frac{v_i^2 \sin^2\theta_i}{2g}$$

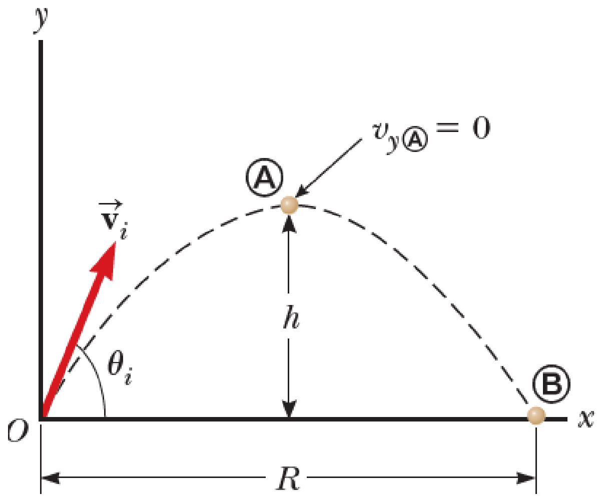


Figure 4.9 A projectile launched over a flat surface from the origin at $t_i = 0$ with an initial velocity $\vec{\mathbf{v}}_i$. The maximum height of the projectile is h, and the horizontal range is R. At Ⓐ, the peak of the trajectory, the particle has coordinates $(R/2, h)$.

Horizontal Range of a Projectile

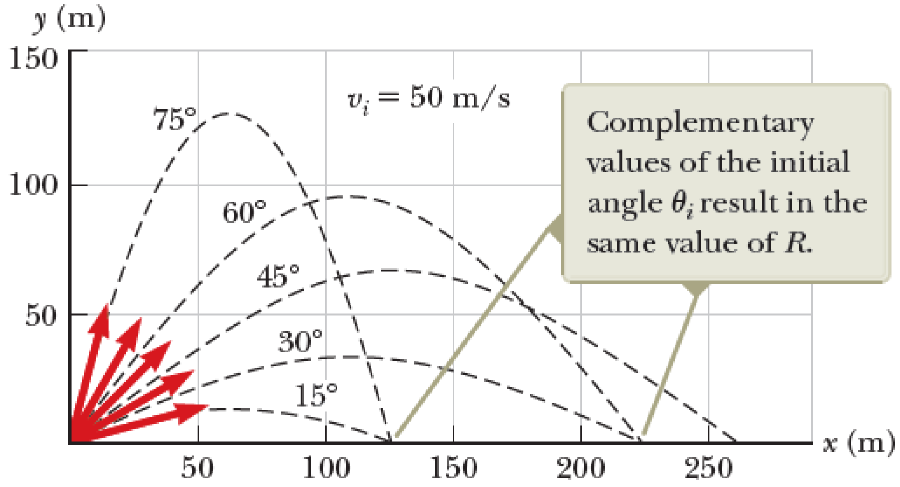
- The **range** R is the horizontal position at $t_B = 2t_A$, twice the time to reach the peak.
- Using constant horizontal velocity $v_x = v_i \cos\theta_i$:

$$R = v_x t_B = v_i \cos\theta_i \cdot 2t_A = \frac{2v_i^2 \sin\theta_i \cos\theta_i}{g}$$

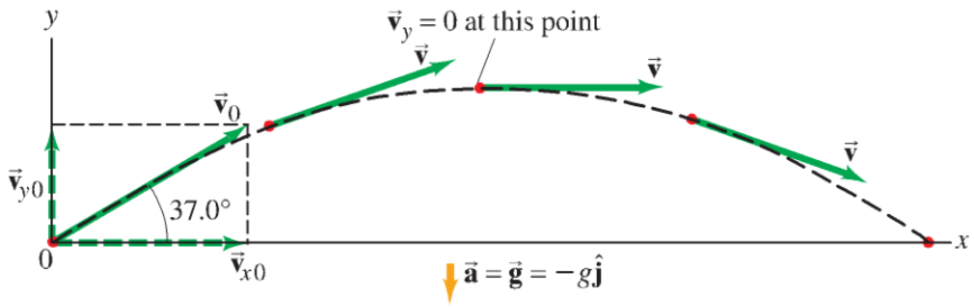
- Using the identity $\sin 2\theta = 2\sin\theta\cos\theta$:

$$R = \frac{v_i^2 \sin 2\theta_i}{g}$$

- **Maximum range:** $R_{\max} = \frac{v_i^2}{g}$, occurs at $\theta_i = 45^\circ$.
- For a given initial speed, two complementary angles (θ_i and $90^\circ - \theta_i$) can produce the same range, but the **maximum height and time of flight differ** for each angle.



EXAMPLE 3–7 **A kicked football.** A football is kicked at an angle $\theta_0 = 37.0°$ with a velocity of 20.0 m/s, as shown in Fig. 3–24. Calculate (*a*) the maximum height, (*b*) the time of travel before the football hits the ground, (*c*) how far away it hits the ground, (*d*) the velocity vector at the maximum height, and (*e*) the acceleration vector at maximum height. Assume the ball leaves the foot at ground level, and ignore air resistance and rotation of the ball.



Velocity components

$$v_{0x} = v_0 \cos\theta = 20\cos 37^\circ \approx 16.0 \text{ m/s}, \quad v_{0y} = v_0 \sin\theta = 20 \sin 37^\circ \approx 12.0 \text{ m/s}$$

(a) Maximum height

$$v_y^2 = v_{0y}^2 - 2gh_{\text{max}} \implies h_{\text{max}} = \frac{v_{0y}^2}{2g} = \frac{12.0^2}{2 \cdot 9.8} \approx 7.35 \text{ m}$$

(b) Time of flight

$$t_{\text{total}} = \frac{2v_{0y}}{g} = \frac{2 \cdot 12.0}{9.8} \approx 2.45 \text{ s}$$

(c) Horizontal range

$$x_{\text{range}} = v_x t_{\text{total}} = 16.0 \cdot 2.45 \approx 39.2 \text{ m}$$

(d) Velocity at maximum height

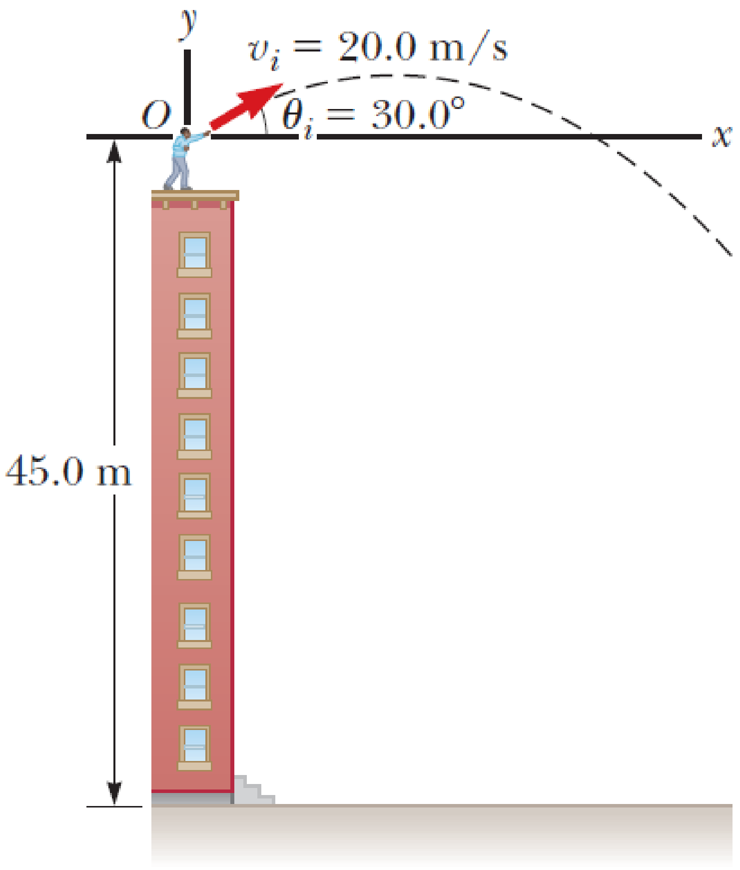
$$v_y = 0, \quad v_x = 16.0 \text{ m/s} \implies \vec{v}_{\text{max height}} = 16.0\,\hat{\imath} \text{ m/s}, \quad |\vec{v}| = 16.0 \text{ m/s}$$

(e) Acceleration at maximum height

$$\vec{a} = -g\hat{j} = -9.8\,\hat{j} \text{ m/s}^2$$

Example: A stone is thrown from the top of a building upward at an angle of 30.0° to the horizontal with an initial speed of 20.0 m/s as shown in the figure. The height from which the stone is thrown is 45.0 m above the ground.

(A) How long does it take the stone to reach the ground?

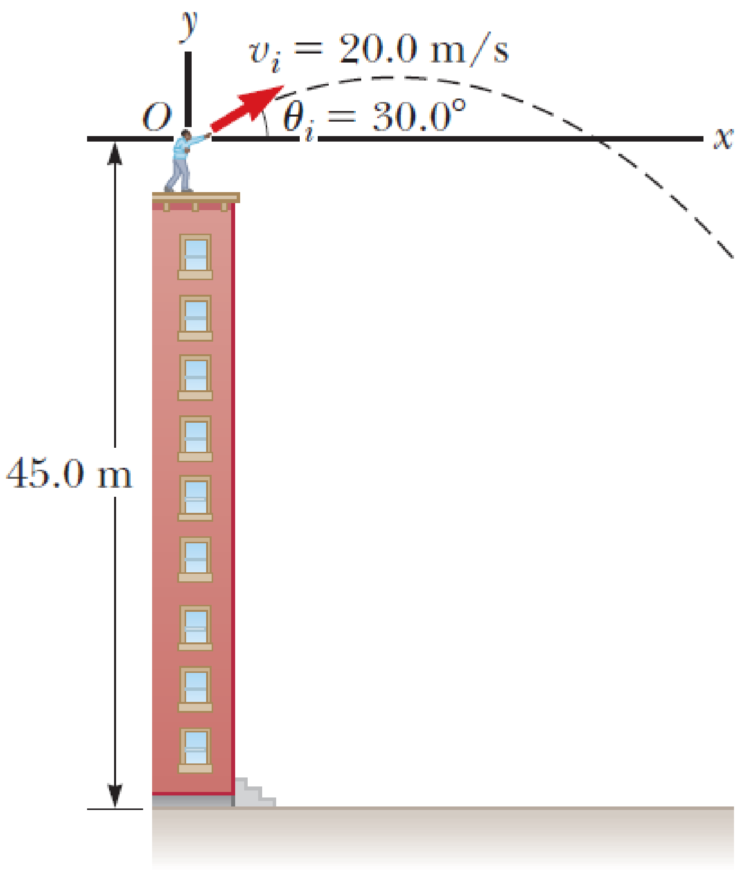


A stone is thrown from a height $y_0 = 45$ m with $v_0 = 20$ m/s at $\theta = 30°$. The vertical component is $v_{0y} = v_0 \sin\theta = 10$ m/s. Taking downward as negative ($y = -45$ m):

$$y = v_{0y}t - \frac{1}{2}gt^2 = -45 \implies -45 = 10t - 4.9t^2 \implies 4.9t^2 - 10t - 45 = 0$$

$$t = \frac{10 + \sqrt{10^2 + 4 \cdot 4.9 \cdot 45}}{9.8} = \frac{10 + \sqrt{982}}{9.8} \approx 4.22 \text{ s}$$

(B) What is the speed of the stone just before it strikes the ground?



$$v_y = v_{0y} - gt$$

Substitute numerical values ($v_{0y} = 10.0$ m/s, $g = 9.8$ m/s², $t = 4.22$ s):

$$v_y = 10.0 - 9.8 \times 4.22 \approx 10.0 - 41.3 \approx -31.3 \text{ m/s}$$

The horizontal component remains constant:

$$v_x = v_{0x} = 17.3 \text{ m/s}$$

Now, the **speed** of the stone just before hitting the ground:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(17.3)^2 + (-31.3)^2} = \sqrt{299.3 + 979.7} = \sqrt{1279} \approx 35.8 \text{ m/s}$$

Projectile Motion Is Parabolic

The path of a projectile is a **parabola** if air resistance is ignored and g is constant. By expressing vertical displacement y as a function of horizontal displacement x, we eliminate time t from the motion equations:

$$x = v_{x0}t, \quad y = v_{y0}t - \frac{1}{2}gt^2$$

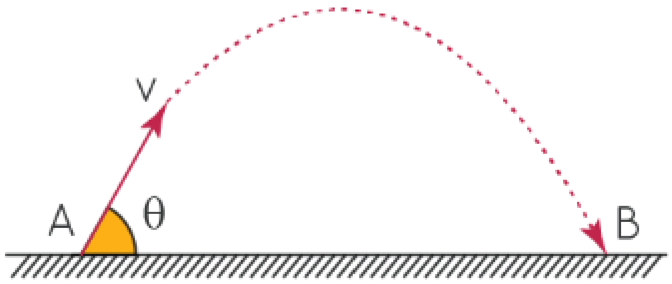
Substituting $t = x/v_{x0}$ into the vertical equation gives:

$$y = \frac{v_{y0}}{v_{x0}}x - \frac{g}{2v_{x0}^2}x^2$$

This shows y as a **quadratic function of** x:

$$y = Ax - Bx^2$$

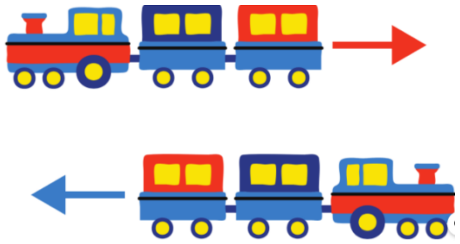
where A and B are constants, confirming that the projectile's trajectory is a **parabola**.



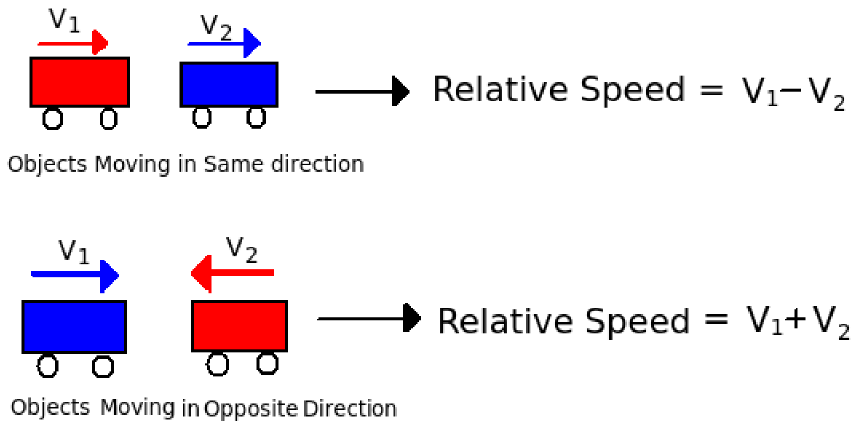
$$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta}$$

3-9 Relative Velocity

- The observed speed of an object depends on the observer's frame of reference.
- Example 1: Two trains approaching each other at 80 km/h relative to Earth.
  - Earth observer: each train moves at 80 km/h
  - Observer on one train: the other train moves at 160 km/h
- Example 2: A car at 90 km/h passes a car at 75 km/h in the same direction.
  - Relative speed = 90 km/h – 75 km/h = 15 km/h
- Velocities along the same line → simple addition or subtraction
- Velocities in different directions → use vector addition
- Always specify the **reference frame** when stating a velocity



The velocity of an object relative to one frame of reference can be found by vector addition if its velocity relative to a second frame of reference, and the relative velocity of the two reference frames, are known.



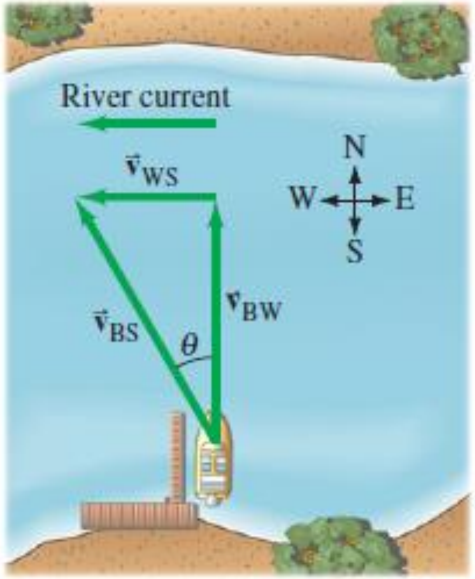


FIGURE 3–28 A boat heads north directly across a river which flows west. Velocity vectors are shown as green arrows:

$\vec{v}_{BS}$ = velocity of Boat with respect to the Shore,

$\vec{v}_{BW}$ = velocity of Boat with respect to the Water,

$\vec{v}_{WS}$ = velocity of Water with respect to the Shore (river current).

As it crosses the river, the boat is dragged downstream by the current.

Relative Velocity and Subscript Notation

- Mistakes often occur when adding or subtracting wrong velocities.
- Always **draw a diagram** and **label velocities carefully.**
- Use **two subscripts** for each velocity:
  - First subscript → object
  - Second subscript → reference frame

Example (Boat Crossing a River):

- v_{BW}: Boat relative to Water
- v_{WS}: Water relative to Shore (current)
- v_{BS}: Boat relative to Shore

Relation:

$$\vec{\mathbf{v}}_{BS} = \vec{\mathbf{v}}_{BW} + \vec{\mathbf{v}}_{WS}$$

- The **inner subscripts** (W) cancel; the **outer subscripts** (B and S) give the result.
- This subscript rule helps write correct equations for velocities in different frames.

3-9 Relative Velocity

Question:

A boat can move at **1.85 m/s** in still water.

The river flows **west** with a speed of **1.20 m/s.**

The boat wants to go **straight north** across the river.

At what angle upstream (toward the east) should the boat head so that it moves directly across the river?

Explanation:

- If the boat heads straight north, the river current will carry it west.
- To go straight across, the boat must point a little east to cancel the current's effect.
- The boat's velocity relative to the shore is the sum of:
  - Its velocity relative to the water
  - The water's velocity relative to the shore

Solution:

$$\sin\theta = \frac{v_{WS}}{v_{BW}} = \frac{1.20}{1.85} = 0.6486$$

$$\theta = 40.4^\circ$$

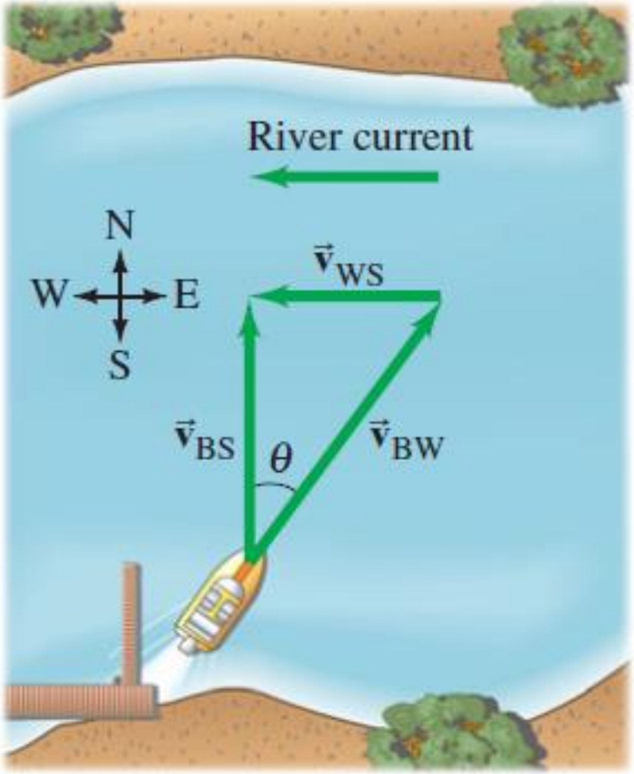


FIGURE 3–29 Example 3–10.

EXAMPLE 3–11 **Heading across the river.** The same boat ($v_{BW} = 1.85\ \mathrm{m/s}$) now heads directly across the river whose current is still $1.20\ \mathrm{m/s}$. (*a*) What is the velocity (magnitude and direction) of the boat relative to the shore? (*b*) If the river is 110 m wide, how long will it take to cross and how far downstream will the boat be then?

APPROACH The boat now heads directly across the river and is pulled downstream by the current, as shown in Fig. 3–30. The boat's velocity with respect to the shore, $\vec{\mathbf{v}}_{BS}$, is the sum of its velocity with respect to the water, $\vec{\mathbf{v}}_{BW}$, plus the velocity of the water with respect to the shore, $\vec{\mathbf{v}}_{WS}$: just as before,

$$\vec{\mathbf{v}}_{BS} = \vec{\mathbf{v}}_{BW} + \vec{\mathbf{v}}_{WS}.$$

SOLUTION (*a*) Since $\vec{\mathbf{v}}_{BW}$ is perpendicular to $\vec{\mathbf{v}}_{WS}$, we can get v_{BS} using the theorem of Pythagoras:

$$v_{BS} = \sqrt{v_{BW}^2 + v_{WS}^2} = \sqrt{(1.85\ \mathrm{m/s})^2 + (1.20\ \mathrm{m/s})^2} = 2.21\ \mathrm{m/s}.$$

We can obtain the angle (note how θ is defined in Fig. 3–30) from:

$$\tan\theta = v_{WS}/v_{BW} = (1.20\ \mathrm{m/s})/(1.85\ \mathrm{m/s}) = 0.6486.$$

A calculator with a key INV TAN or ARC TAN or TAN^{-1} gives $\theta = \tan^{-1}(0.6486) = 33.0°$. Note that this angle is not equal to the angle calculated in Example 3–10.

(*b*) The travel time for the boat is determined by the time it takes to cross the river. Given the river's width $D = 110\ \mathrm{m}$, we can use the velocity component in the direction of D, $v_{BW} = D/t$. Solving for t, we get $t = 110\ \mathrm{m}/1.85\ \mathrm{m/s} = 59.5\ \mathrm{s}$. The boat will have been carried downstream, in this time, a distance

$$d = v_{WS}t = (1.20\ \mathrm{m/s})(59.5\ \mathrm{s}) = 71.4\ \mathrm{m} \approx 71\ \mathrm{m}.$$

NOTE There is no acceleration in this Example, so the motion involves only constant velocities (of the boat or of the river).

FIGURE 3–30 Example 3–11. A boat heading directly across a river whose current moves at $1.20\ \mathrm{m/s}$.