Assignment. Numerical simulation of von Kármán vortex streets

In fluid dynamics, a Kármán vortex street is a repeating pattern of swirling vortices, caused by a process known as vortex shedding, which is responsible for the unsteady separation of flow of a fluid around blunt bodies (see for instance this photograph). It forms only at a certain range of flow velocities. The aim of this assignment is to show evidences of this phenomenon through numerical simulations.

To do so, we choose to work with the simplified framework of a two-dimensional rectangular box in which a viscous fluid flows from one side to the opposite one, encountering an obstacle in its path. The motion of the fluid is governed the Navier-Stokes equation for an incompressible flow

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)-\mu\Delta\boldsymbol{u}+\nabla p=\boldsymbol{0},$$

$$\nabla\cdot\boldsymbol{u}=0,$$

where ρ is the mass density (assumed constant), $\boldsymbol{u}$ is the fluid velocity, μ is the dynamic viscosity of the fluid and p is the pressure. These equations hold in the domain Ω which is made of the box minus the obstacle contained in it. There are completed by some initial and boundary conditions. These are:

- at the initial time, the fluid is at rest and both the velocity and pressure are null in the domin,
- on the left side of the domain, the flow is incoming with a velocity equal to $U\boldsymbol{e_1}$ and the pressure satisfies $\frac{\partial p}{\partial x_1}=0$,
- on the right side of the domain, the flow is free so that $\frac{\partial u_1}{\partial x_1}=\frac{\partial u_2}{\partial x_1}=0$ and $p=0$,
- on the horizontal sides, the walls are impenetrable so that $u_2=0$ and a slip condition is imposed so that $\frac{\partial u_1}{\partial x_2}=0$ and $\frac{\partial p}{\partial x_2}=0$.

The next step is to adimensionalise the problem so that a dimensionless parameter characterising the flow appears. Assuming that L denote the characteristic length of the obstacle (the length of the obstacle in the transverse direction to the flow) and setting

$$\tilde{\boldsymbol{u}}=\frac{\boldsymbol{u}}{U},\ \tilde{p}=\frac{p}{\rho U^2},\ \tilde{\boldsymbol{x}}=\frac{\boldsymbol{x}}{L}\text{ and }\tilde{t}=\frac{U}{L}t$$

we get (dropping the tildes for the sake of readibility) the following partial differential equations

$$\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\frac{1}{\mathrm{Re}}\Delta\boldsymbol{u}+\nabla p=\boldsymbol{0},$$

$$\nabla\cdot\boldsymbol{u}=0,$$

where Re is the Reynolds number of the flow, defined by

$$\mathrm{Re}=\frac{\rho U L}{\mu},$$

which measures the ratio of inertial to viscous forces in the flow around the obstacle.

Using the space discretisation method of your choice, simulate the flow around the obstable and characterise a limiting value of the Reynolds number for which there is a transition from a laminar flow to a turbulent flow, using the apparition of vortices as a reference. The proposed numerical scheme may combine two techniques largely used for numerical computations in fluid dynamics: the *Chorin projection method* and the *semi-Lagrangian scheme*. The following reference may be of help [Boy01].

References

[Boy01] J. P. Boyd. *Chebyshev and Fourier Spectral Methods*. Dover, second revised edition, 2001.