VON KÁRMÁN VORTEX STREETS

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Introduction

von Kármán vortex street

The *von Kármán vortex street* phenomenon is a classic example of pattern formation in flows behind bodies, characterized by alternating vortices.

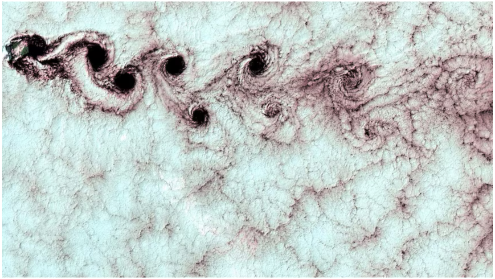


Figure 1: Atmospheric von Kármán vortex street, showcasing swirling vortices caused by airflow around a mountain (taken from (Wikipedia contributors n.d.))

Importance and Objectives

Why It Matters:

- Signigicance in cloud formation, turbulence and vibration induction.
- Key study subject in aerodynamic and fluiddynamic pattern analysis, essential for shape optimization and surface coating development.

Project Goals:

- Simulating von Kármán vortex street in 2D Navier-Stokes equation using Chorin's method.
- Analyzing flow patterns, optimizing for various shapes and configurations.

Methods

Posing the Problem

2D incompressible Navier-Stokes Eq.

Adimensionalizing the problem

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{\mathrm{Re}} \Delta \mathbf{u} + \nabla p = 0$$
$$\nabla \cdot \mathbf{u} = 0$$

with $\mathbf{u} = (u, v)$ the velocity field and $\mathrm{Re} := \frac{\rho U L}{\mu}$ is the *Reynolds number*.

Boundary Conditions:
Incoming flow from the left side, between two walls and free on the right.

Left:	$u = 1,$	$v = 0,$	$\partial_{\mathbf{n}} p = 0$
Right:	$\partial_{\mathbf{n}} u = 0,$	$\partial_{\mathbf{n}} v = 0,$	$p = 0$
Top and Bottom (Slip):	$\partial_{\mathbf{n}} u = 0,$	$v = 0,$	$\partial_{\mathbf{n}} p = 0$

Discretization and Boundary Conditions

Border of the Domain:

- Domain discretization: Red for the domain, black dots for inner cells, blue for ghost cells.
- Boundary condition treatment: e.g., $v_{0,j} = -v_{1,j}$ in ghost cells to set $v = 0$ on the left boundary.

Body:

- The fluid is set to be at rest, simulating a no-slip boundary condition on the body's surface.

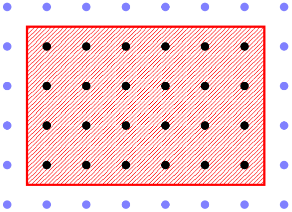


Figure 2: Grid representation with ghost cells.

Chorin's Splitting Method

Overview

Chorin's method simplifies the Navier-Stokes equations into sequential steps, enhancing numerical stability and computational efficiency (Chorin 1967; Boyd 2001).

Steps:

1. Advection: $\dfrac{\mathbf{u}^a - \mathbf{u}^n}{\Delta t} + \mathbf{u}^n \cdot \nabla \mathbf{u}^n = 0$
2. Diffusion: $\dfrac{\mathbf{u}^* - \mathbf{u}^a}{\Delta t} = \dfrac{1}{\mathrm{Re}} \Delta \mathbf{u}^n$
3. Pressure: $\Delta p^{n+1} = \dfrac{1}{\Delta t} \nabla \cdot \mathbf{u}^*$
4. Velocity correction: $\mathbf{u}^{n+1} = \mathbf{u}^* - \Delta t \nabla p^{n+1}$

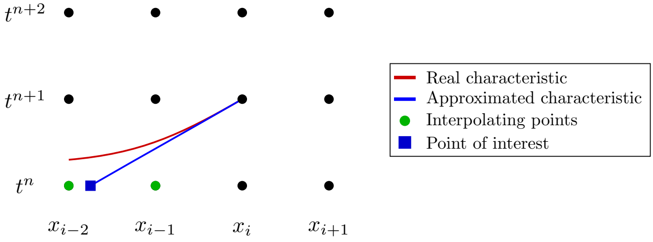
1st Step: Semi-Lagrangian method

Solving the advection equation with velocity $\mathbf{u}$:

$$\frac{\mathrm{D}\psi}{\mathrm{D}t} = \frac{\partial \psi}{\partial t} + \mathbf{u} \cdot \nabla \psi = 0$$

Discretization of the material derivative:

$$\frac{\psi(\mathbf{x}_{i,j}, t^{n+1}) - \psi(\mathbf{x}_{i,j} - \Delta t \mathbf{u}(\mathbf{x}_{i,j}, t^n), t^n)}{\Delta t} = 0$$



3rd Step: Solving the Poisson Equation for Pressure

Approach:

- Laplacian approximation: Employ a 5-point stencil with finite difference method.
- Matrix representation: Formulate as $\mathbf{Ap} = \mathbf{f}$.

Solution via Cholesky decomposition:

- Transform $\mathbf{A}$ for positive definiteness: $-\mathbf{Ap} = -\mathbf{f}$.
- Cholesky decomposition: Efficiently solves for $\mathbf{p}$.
- Stability and speed in solving $\mathbf{Ap} = \mathbf{f}$, critical for fluid dynamics simulations.

Numerical Results

Observing von Kármán Vortex Street

Setup:

- Flow with $\mathrm{Re} = 500$ ($\Leftrightarrow$ $U = 0.03012$).
- Simulation domain: $\mathrm{d}x = \mathrm{d}y = 0.01$ with a 500×100 grid.
- Circular obstacle with radius $r = 0.125$ initiates flow.

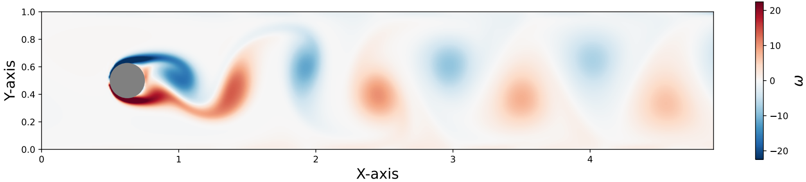


Figure 3: Vorticity $\omega = \nabla \times \mathbf{u}$ highlighting alternating vortices

Obervations:

- Laminar front with $u = 1$
- Von Kármán vortex street and periodic pattern formation

Influence of Reynolds Number on Flow Dynamics

- The Reynolds number dictates flow behavior in fluid dynamics.
- At a critical $\mathrm{Re} \approx 175$, flow transitions from laminar to unstable, eventually forming a von Kármán vortex street.

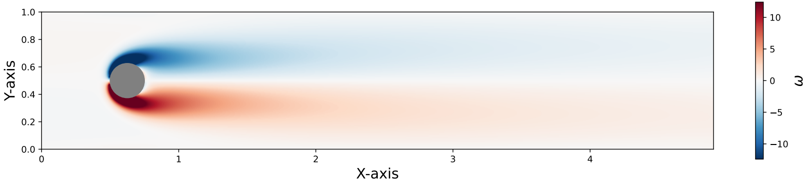


Figure 4: Laminar flow at $\mathrm{Re} = 100$ ($\Leftrightarrow\ U = 0.006$)

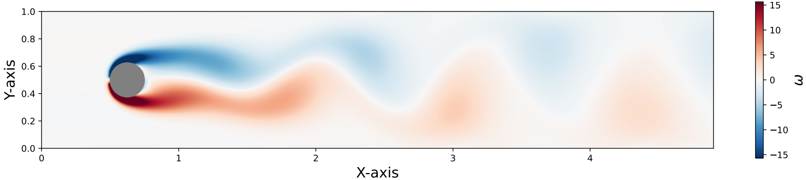


Figure 5: Vortex street at critical $\mathrm{Re} = 190$ ($\Leftrightarrow\ U = 0.0114$)

Influence of the shape on the flow

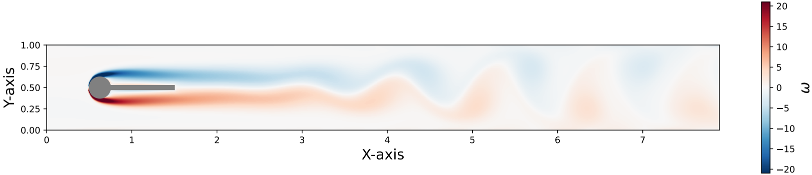


Figure 6: Flow around circle with fin at $\mathrm{Re} = 500$ ($\Leftrightarrow$ $U = 0.03012$)

Observation:

- Fin alters flow, delaying the separation.
- Flow transitions at $\mathrm{Re} \approx 500$, enhancing mixing and vortex complexity.

Optimized shape: Airfoil

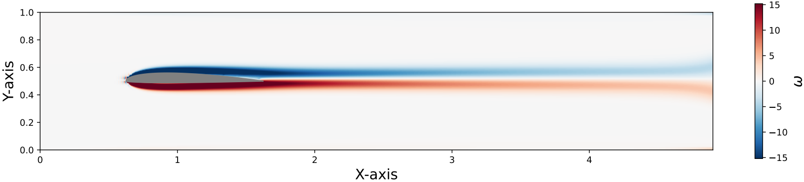


Figure 7: Flow with airfoil at $\mathrm{Re} = 500$ ($\Leftrightarrow$ $U = 0.1076$)

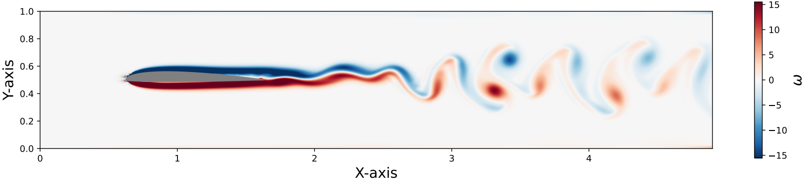


Figure 8: Flow with airfoil at $\mathrm{Re} = 5500$ ($\Leftrightarrow$ $U = 1.1833$)

Key Insight: Airfoil shape minimizes vortex shedding. Streamlined design reduces flow separation and turbulence at higher speeds.

Conclusion

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Summary:

- Utilized Chorin's method for numerical analysis of von Kármán vortex street.
- Adopted second-order schemes for advection, diffusion, and pressure.
- Implemented no-slip conditions for shape-specific flow behavior.

Implications:

- Demonstrated the influence of shape and Reynolds number on flow dynamics.
- Identified critical transitions in flow patterns, emphasizing the importance of aerodynamic design.
- Showcased the potential for optimizing performance and efficiency in engineering applications.

References

References I

Boyd, John P. (2001). *Chebyshev and Fourier Spectral Methods*. 2nd edition. Dover Publications.

Chorin, A. J. (1967). "The numerical solution of the Navier-Stokes equations for an incompressible fluid". In: *Bull. Am. Math. Soc.* 73.6, pp. 928–931. doi: 10.1090/S0002-9904-1967-11853-6.

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Thank you for your attention!