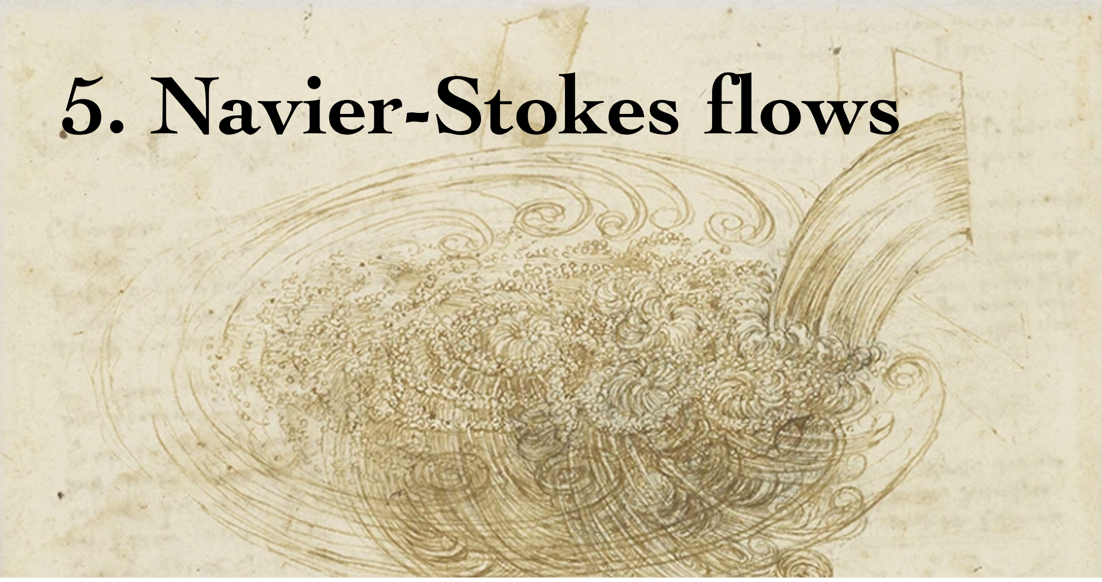
Numerical Methods for Fluid Dynamics

Emmanuel DORMY (DMA-ENS)
emmanuel.dormy@ens.fr

CFD

5. Navier-Stokes flows



Navier-Stokes flows

Claude Louis Navier
(1785-1836)

George Gabriel Stokes
(1819-1903)

Navier-Stokes equations

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\, \mathbf{u} = -\boldsymbol{\nabla} p + \frac{1}{Re}\, \Delta \mathbf{u}$$

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Numerical advection:

$$\frac{\partial u}{\partial t} = -c\,\frac{\partial u}{\partial x}$$

$$u(x,0) = u_0(x)$$

$$u(x,t) = u_0(x - ct)$$

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Three possible schemes

$$u_j^{n+1} = u_j^n - \Delta t\, c\, \delta_0 u_j^n$$

$$u_j^{n+1} = u_j^n - \Delta t\, c\, \delta_+ u_j^n$$

$$u_j^{n+1} = u_j^n - \Delta t\, c\, \delta_- u_j^n$$

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The centered scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -c\,\frac{u_{j+1}^n - u_{j-1}^n}{2\,\Delta x}$$

$$u_j^{n+1} = u_j^n - \Delta t\, c\, \delta_0 u_j^n$$

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Von Neumann
stability analysis:

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n)$$

$$\begin{aligned}\xi &= 1 - \frac{c\Delta t}{\Delta x}\left(\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2}\right)\\ &= 1 - \frac{c\Delta t}{\Delta x} i \sin(k\Delta x)\end{aligned}$$

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Modified equation

$$\frac{u^{n+1}-u^n}{\Delta t} - \frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right) =$$

$$-c\left(\frac{u_{j+1}-u_{j-1}}{2\Delta x} - \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}\right)$$

$$R_h = \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + c\frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}$$

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Modified equation:

$$R_h = \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + c\frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}$$

with $$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

At leading order:

$$\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial x} - c^2\frac{\Delta t}{2}\frac{\partial^2 u}{\partial x^2}$$

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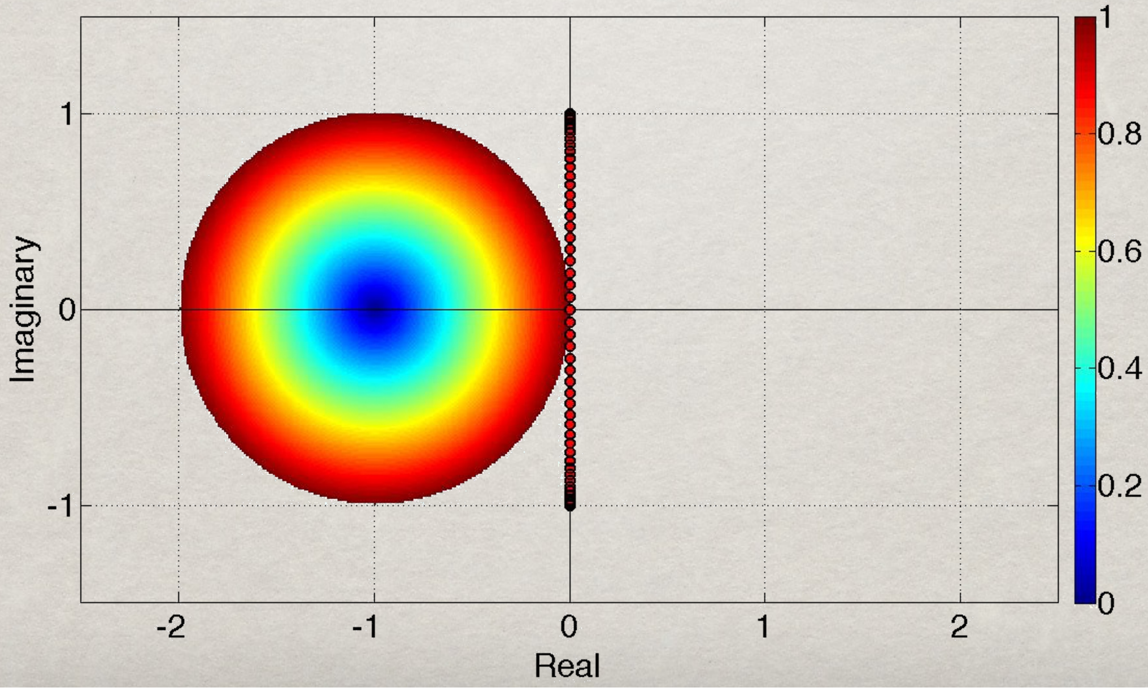
$$u_j^{n+1} = u_j^n - \Delta t \, c \, \delta_0 u_j^n$$

$$c > 0 \quad \Delta t = \Delta x / c$$



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δ_- scheme:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -c \frac{u_j^n - u_{j-1}^n}{\Delta x}$$

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x}(u_j^n - u_{j-1}^n)$$

Von Neumann stability analysis:

$$\xi = 1 - \frac{c\Delta t}{\Delta x}(1 - e^{-ik\Delta x})$$

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Von Neumann stability analysis:

$$\xi = 1 - \frac{c\Delta t}{\Delta x}(1 - e^{-ik\Delta x})$$

$$0 < \frac{c\Delta t}{\Delta x} < 1$$

for $c > 0$ Stable if $\Delta t < \Delta x / c$

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Von Neumann stability analysis:

$$\xi = 1 - \frac{c\Delta t}{\Delta x}(1 - e^{-ik\Delta x})$$

$$0 < \frac{c\Delta t}{\Delta x} < 1$$

for $c > 0$ Stable if $\Delta t < \Delta x / c$

CFL condition

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Stable if $\Delta t \ < \ \Delta x / c$

CFL condition

(1928)

Richard Courant Kurt Friedrichs Hans Lewy

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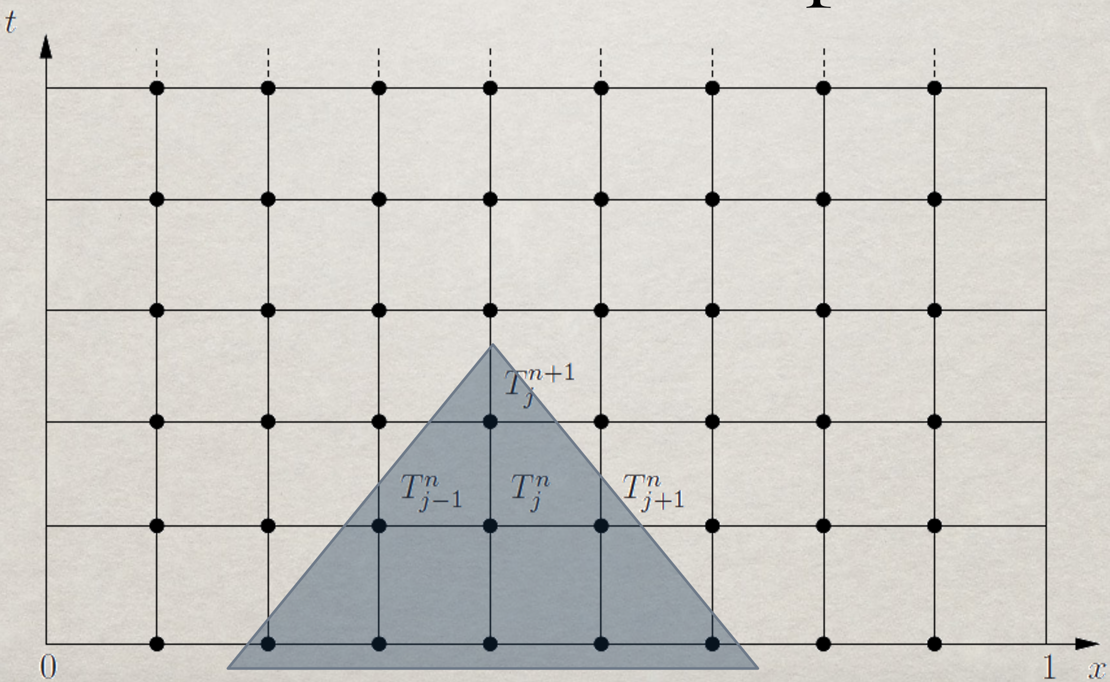
Numerical domain of dependence



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Modified equation:

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} + c\frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2} - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}\cdots$$

with $$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} + c\left(\frac{\Delta x}{2} - c\frac{\Delta t}{2}\right)\frac{\partial^2 u}{\partial x^2}\cdots$$

Numerical dissipation

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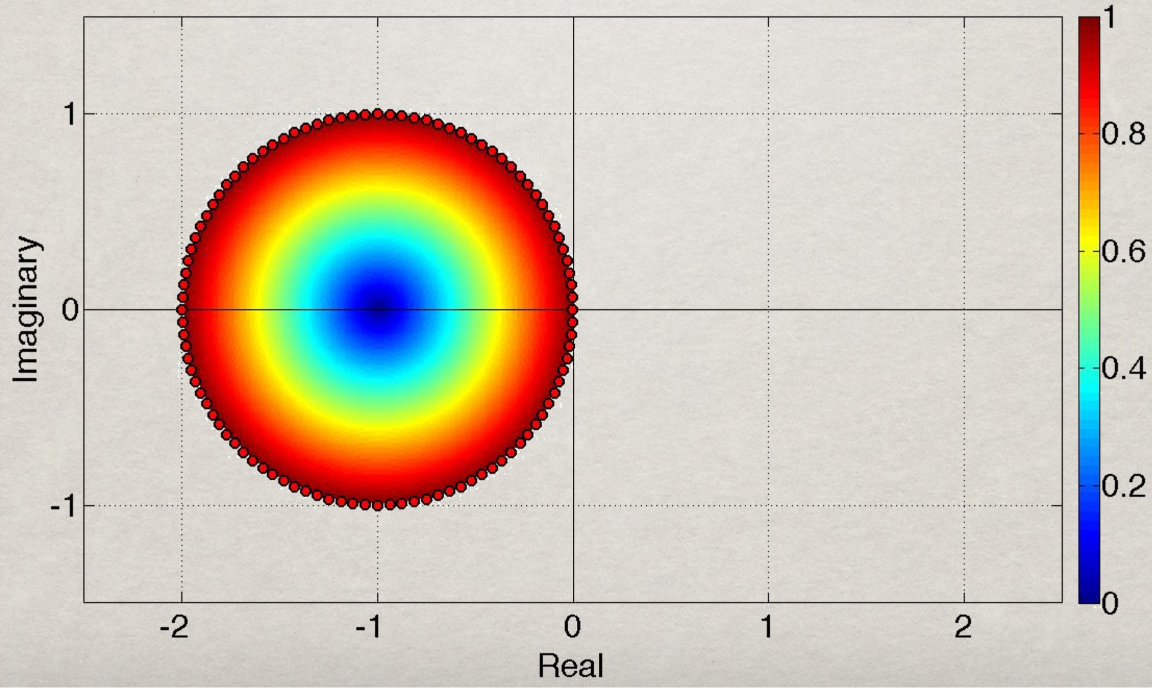
$$u_j^{n+1} = u_j^n - \Delta t\, c\, \delta_- u_j^n$$

$$c > 0 \quad \Delta t = \Delta x / c$$



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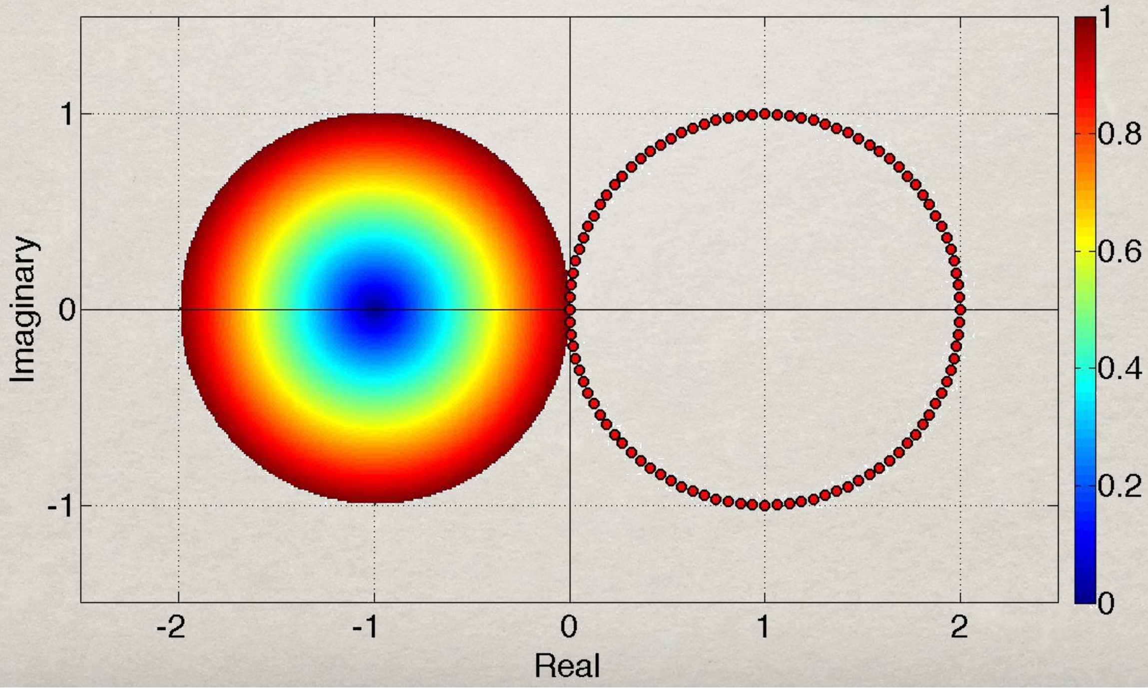
$$u_j^{n+1} = u_j^n - \Delta t\, c\, \delta_+ u_j^n$$

$$c > 0 \quad \Delta t = \Delta x / c$$



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Upwind:

$$u_j^{n+1} = u_j^n - \Delta t\, c\, \delta_- u_j^n \qquad \text{for c>0}$$

$$u_j^{n+1} = u_j^n - \Delta t\, c\, \delta_+ u_j^n \qquad \text{for c<0}$$

Lax-Friedrich:

$$u_j^{n+1} = \tfrac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right) - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

Numerical dissipation

Modified equation

$$\frac{u^{n+1} - u^n}{\Delta t} - \frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right) =$$

$$-c\left(\frac{u_{j+1} - u_{j-1}}{2\Delta x} - \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}\right)$$

At leading order:

$$\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial x} - c^2\frac{\Delta t}{2}\frac{\partial^2 u}{\partial x^2}$$

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Lax-Wendroff:

$$\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial x} - c^2\frac{\Delta t}{2}\frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{c^2\Delta t}{2}\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2}$$

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Lax-Wendroff:

$$\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial x} - c^2\frac{\Delta t}{2}\frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{c^2\Delta t}{2}\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2}$$

$$R_h = c\frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}$$

Numerical dispersion

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Semi-Lagrangian formulation

$$\frac{\partial u}{\partial t} = -c\,\frac{\partial u}{\partial x} + S \qquad\qquad \frac{\mathrm{D}u}{\mathrm{D}t} = S$$

$$\frac{\mathrm{D}u}{\mathrm{D}t} \qquad\qquad \frac{u(x_j, t^n) - u(x_j - c\Delta t, t^n)}{\Delta t}$$

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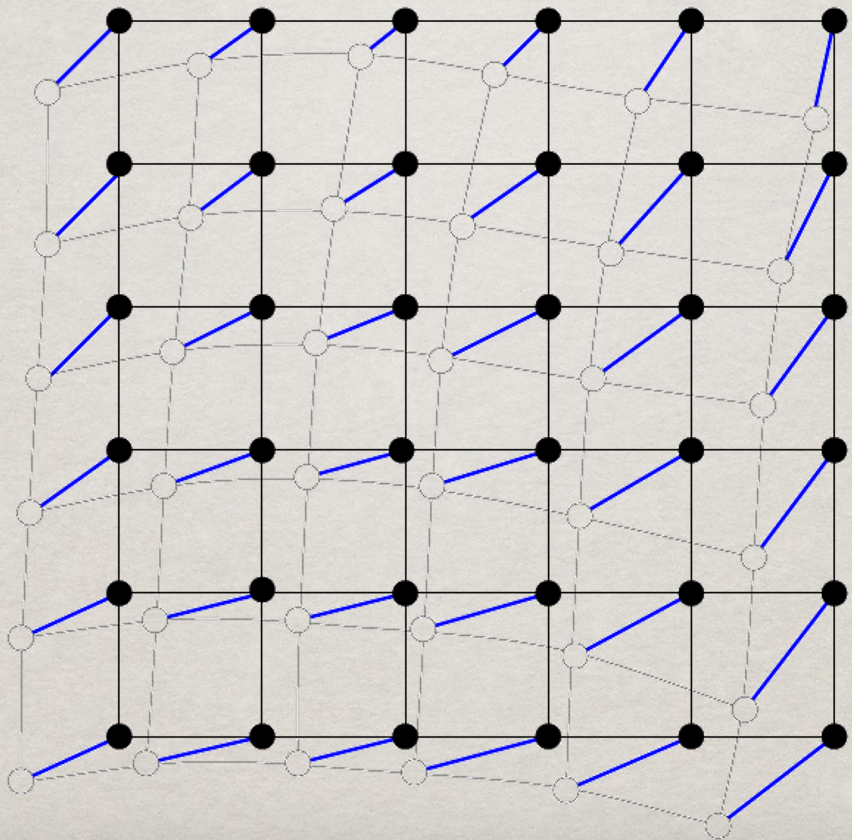

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If c>0:

$$
\begin{aligned}
u(x_j, t^{n+1}) &= u(x_j - c\Delta t, t^n) \\
&= \frac{c\Delta t}{\Delta x} u(x_{j-1}, t^n) + \frac{\Delta x - c\Delta t}{\Delta x} u(x_j, t^n) \\
&= \frac{c\Delta t}{\Delta x} u(x_{j-1}, t^n) + u(x_j, t^n) - \frac{c\Delta t}{\Delta x} u(x_j, t^n)
\end{aligned}
$$

and thus

$$
\frac{u(x_j, t^{n+1}) - u(x_j, t^n)}{\Delta t} = -c\,\frac{u(x_j, t^n) - u(x_{j-1}, t^n)}{\Delta x}
$$

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Using linear interpolation:

$$
u(x_j - c\Delta t, t^n) = \frac{c\Delta t}{\Delta x} u(x_{j-1}, t^n) + \frac{\Delta x - c\Delta t}{\Delta x} u(x_j, t^n)
$$

if c>0

$$
u(x_j - c\Delta t, t^n) = \frac{\Delta x + c\Delta t}{\Delta x} u(x_j, t^n) + \frac{-c\Delta t}{\Delta x} u(x_{j+1}, t^n)
$$

if c<0

CFD

Thus if c>0:

$$\frac{u(x_j, t^{n+1}) - u(x_j, t^n)}{\Delta t} = -c\,\frac{u(x_j, t^n) - u(x_{j-1}, t^n)}{\Delta x}$$

and if c<0:

$$\frac{u(x_j, t^{n+1}) - u(x_j, t^n)}{\Delta t} = -c\,\frac{u(x_{j+1}, t^n) - u(x_j, t^n)}{\Delta x}$$

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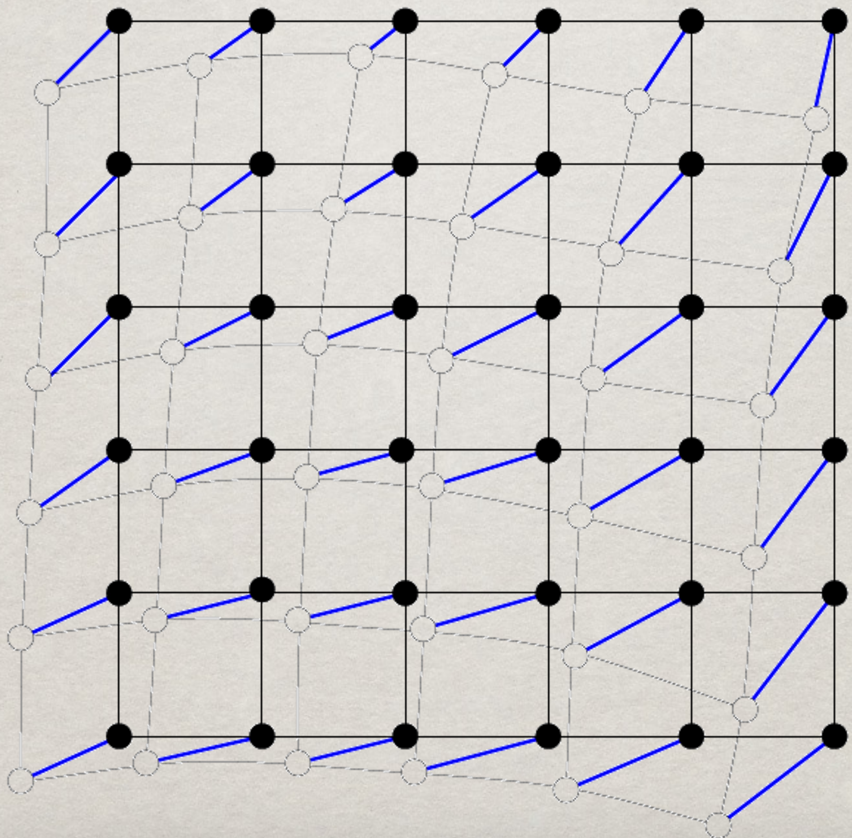

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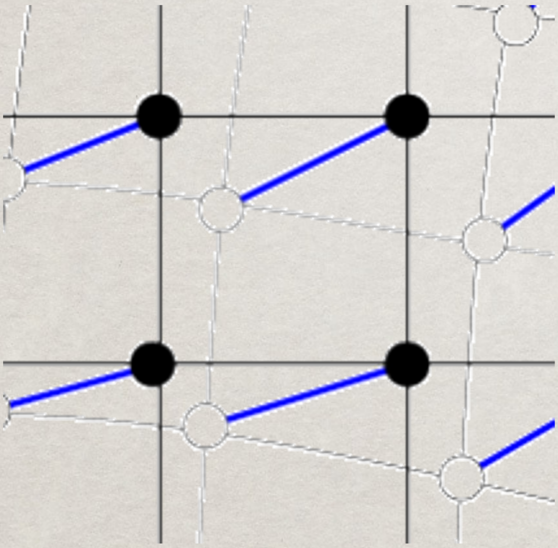

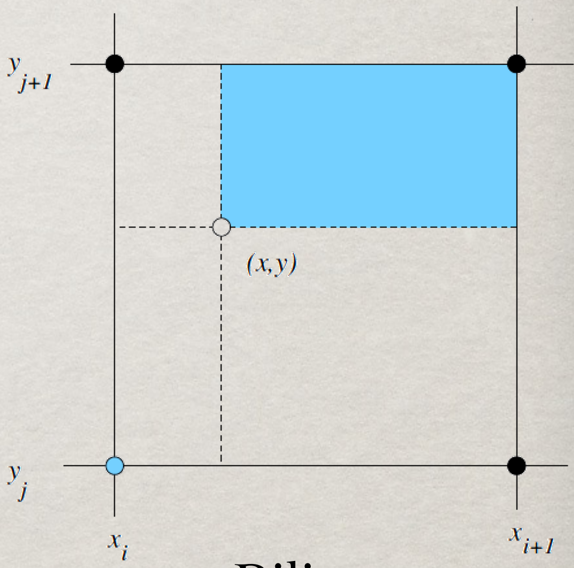

Bilinear interpolation

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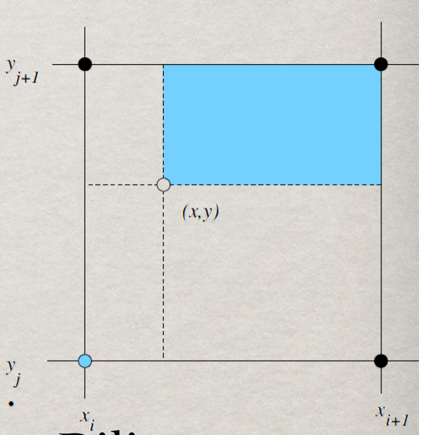
$$
\begin{aligned}
u(x,y) \quad \simeq \quad & u(x_i,y_j)\,\frac{(x_{i+1}-x)(y_{j+1}-y)}{(x_{i+1}-x_i)(y_{j+1}-y_j)} \\
+ \quad & u(x_{i+1},y_j)\,\frac{(x-x_i)(y_{j+1}-y)}{(x_{i+1}-x_i)(y_{j+1}-y_j)} \\
+ \quad & u(x_i,y_{j+1})\,\frac{(x_{i+1}-x)(y-y_j)}{(x_{i+1}-x_i)(y_{j+1}-y_j)} \\
+ \quad & u(x_{i+1},y_{j+1})\,\frac{(x-x_i)(y-y_j)}{(x_{i+1}-x_i)(y_{j+1}-y_j)}\,.
\end{aligned}
$$



Bilinear interpolation

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$$\frac{\partial \phi}{\partial t} = -\mathbf{u} \cdot \boldsymbol{\nabla} \phi$$

$$\phi^{n+1} = \texttt{SemiLag}(u, v, \phi^n)$$

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Back and Forth Error Compensation

- $\phi^\star = \texttt{SemiLag}(u, v, \phi^n)$
- $\tilde{\phi} = \texttt{SemiLag}(-u, -v, \phi^\star)$
- $\phi^\star = \phi^n + (\phi^n - \tilde{\phi})/2$
- $\phi^{n+1} = \texttt{SemiLag}(u, v, \phi^\star)$

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