Sparse Temporal Spanners with Low Stretch

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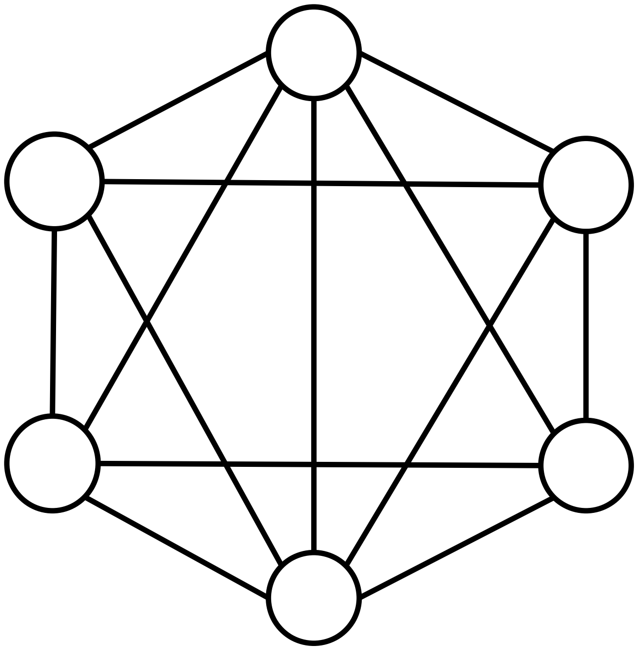
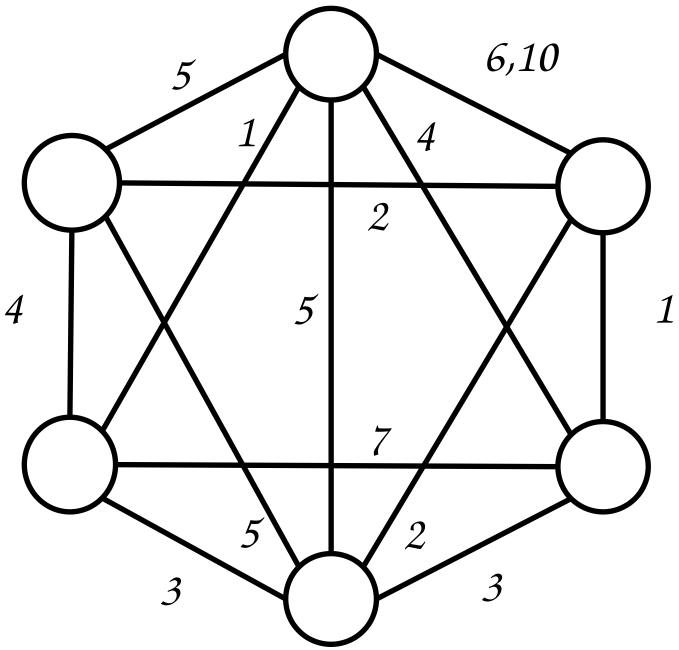
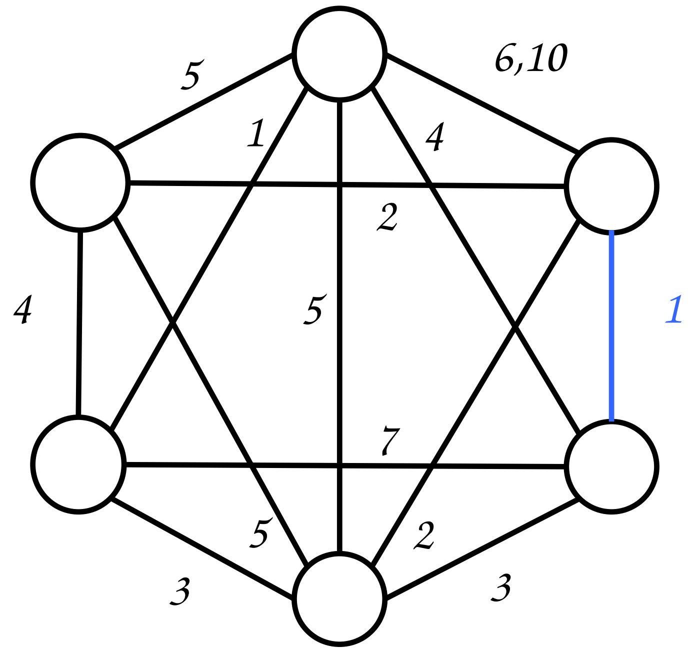
Temporal graphs



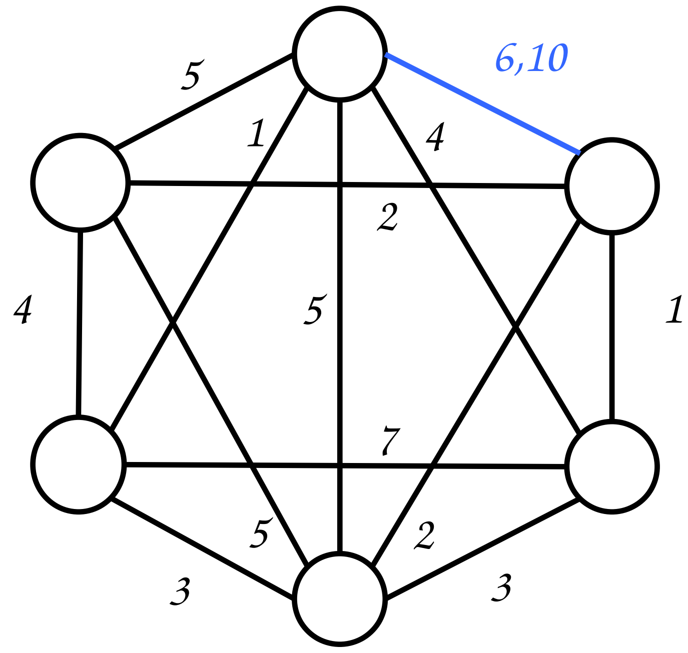
Temporal graphs



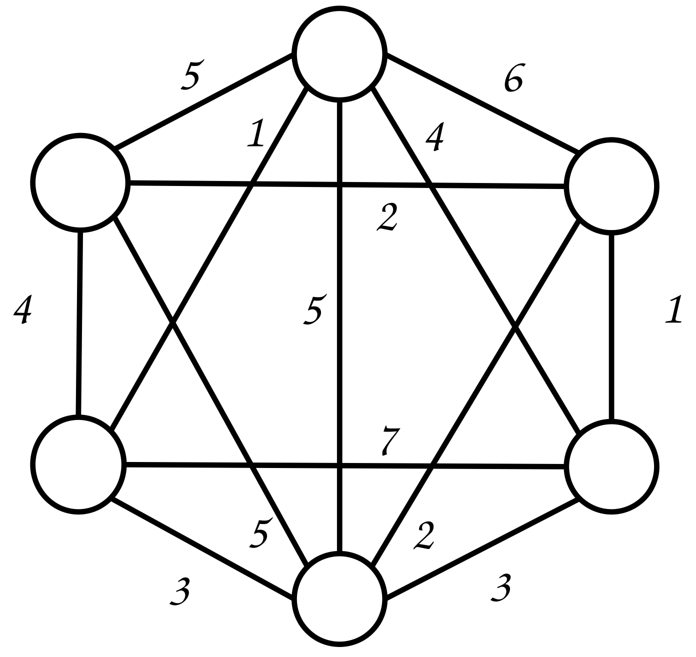
Temporal graphs



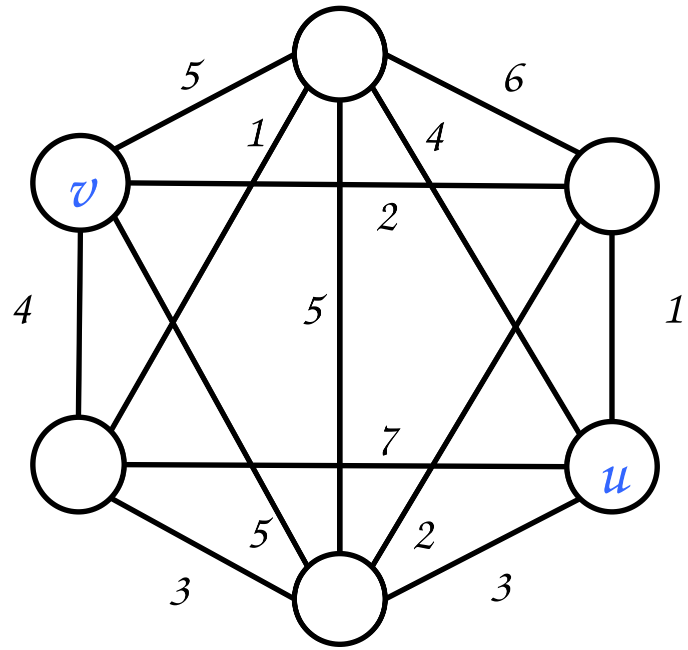
Temporal graphs



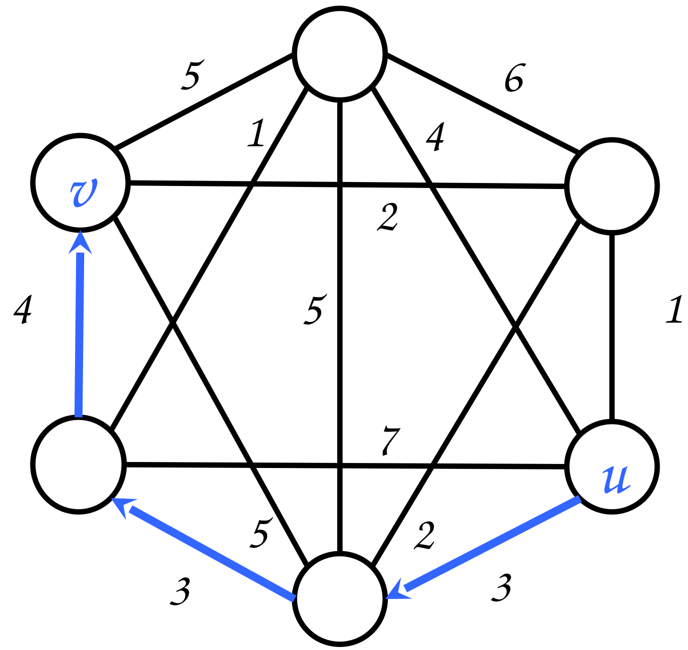
Temporal graphs



Temporal graphs



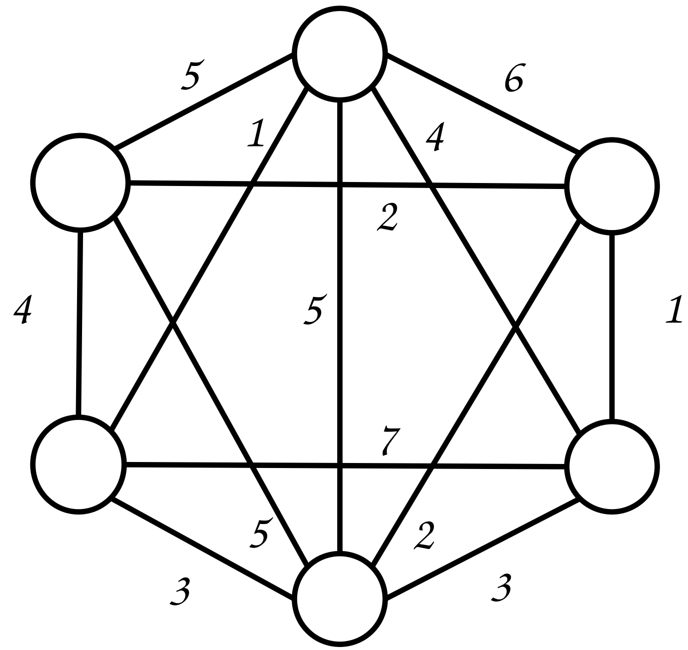
Temporal graphs



u-v temporal path: *u-v path of non-decreasing time-labels*

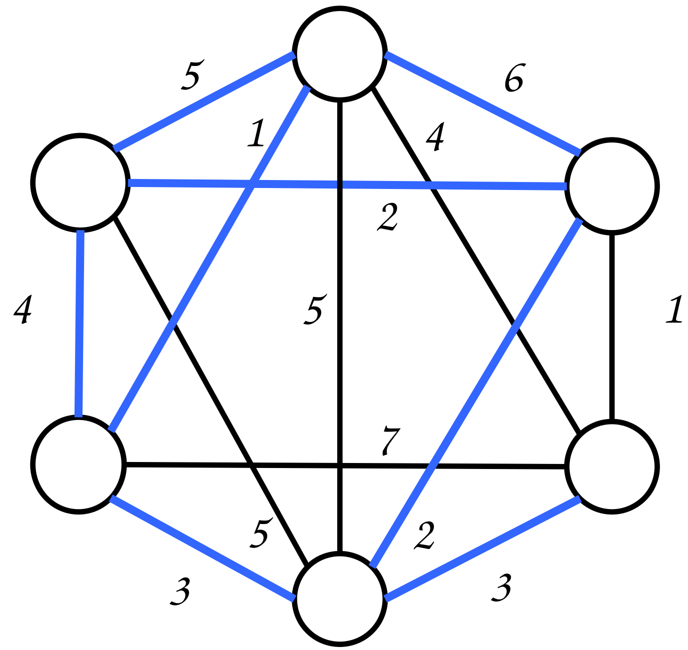
Temporal spanner:

a subgraph $\mathcal{H}$ of $\mathcal{G}$ that preserves pairwise temporal connectivity



Temporal spanner:

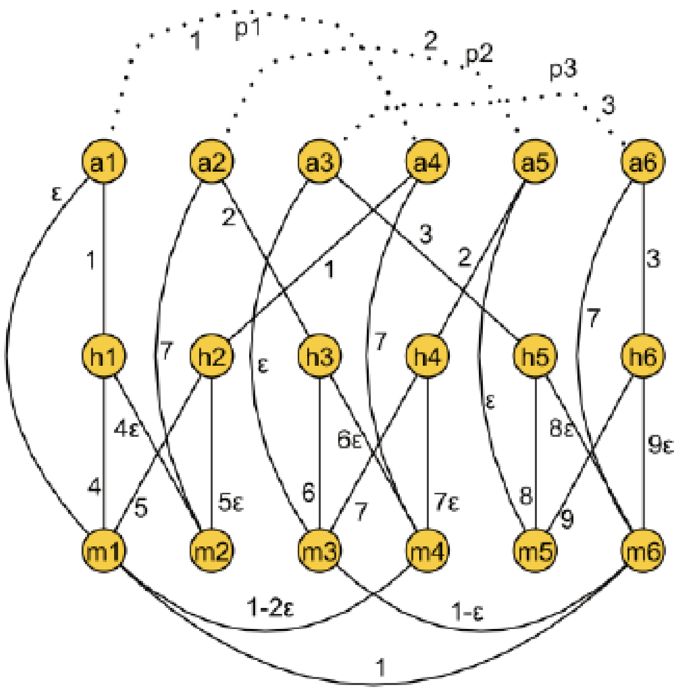
a subgraph H of G that preserves pairwise temporal connectivity



Kempe et al. [STOC'00]: find temporal spanners of small size (#of edges)

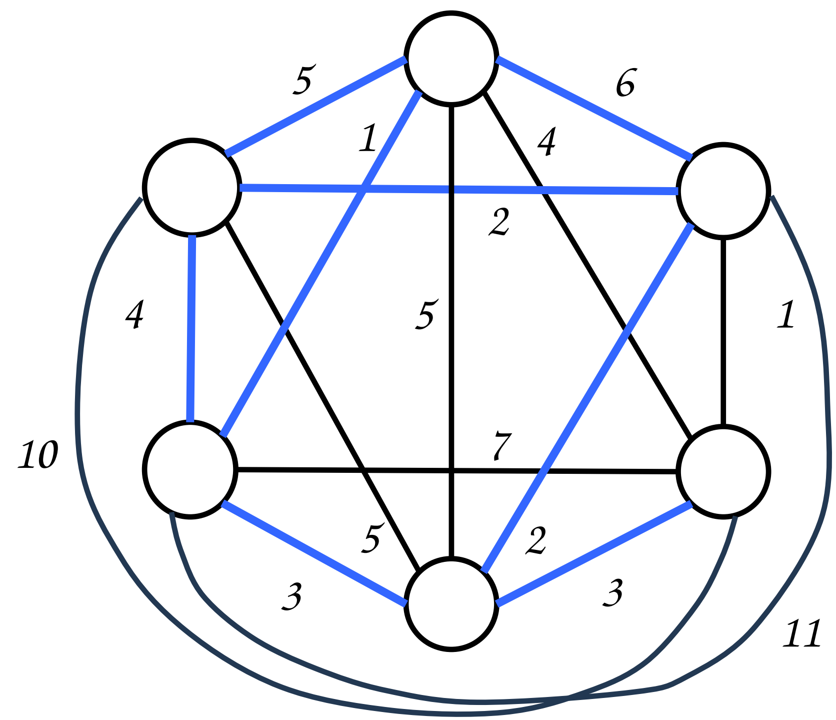
Axiotis and Fotakis
ICALP 2016

Lower Bound of $\Omega(n^2)$



Casteigts et al.
ICALP 2019

Upper Bound of $O(n \log n)$
for temporal cliques

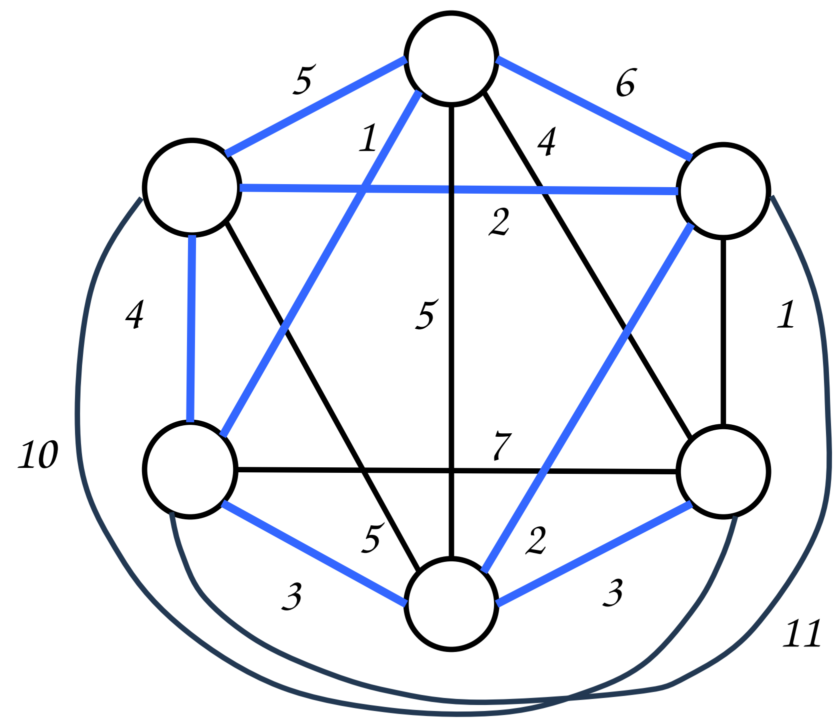


only preserves reachability

no guarantees on the distances

Casteigts et al.
ICALP 2019

Upper Bound of O(n log n)
for temporal cliques



Temporal spanner with stretch α:

a subgraph $\mathcal{H}$ of $\mathcal{G}$ such that for every pair of vertices u and v

$$dist_{\mathcal{H}}(u,v) \leq \alpha\ dist_{\mathcal{G}}(u,v)$$

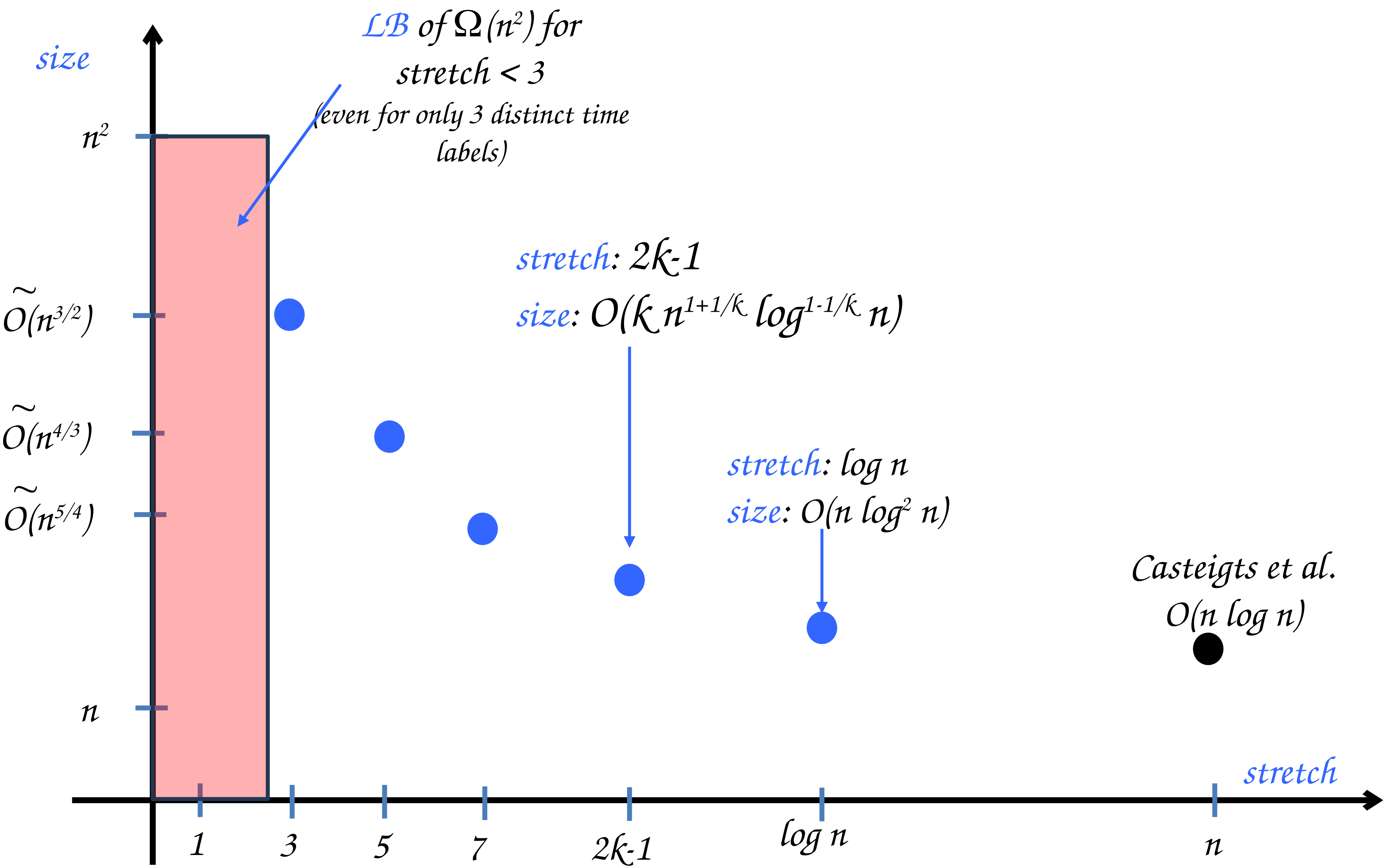
generalization of standard notion of distance for static graphs

strong lower bounds

distance:

1. *min Length (#of edges)*
2. *min Arrival Time*
3. *max Departure Time*
4. *min Travel time*

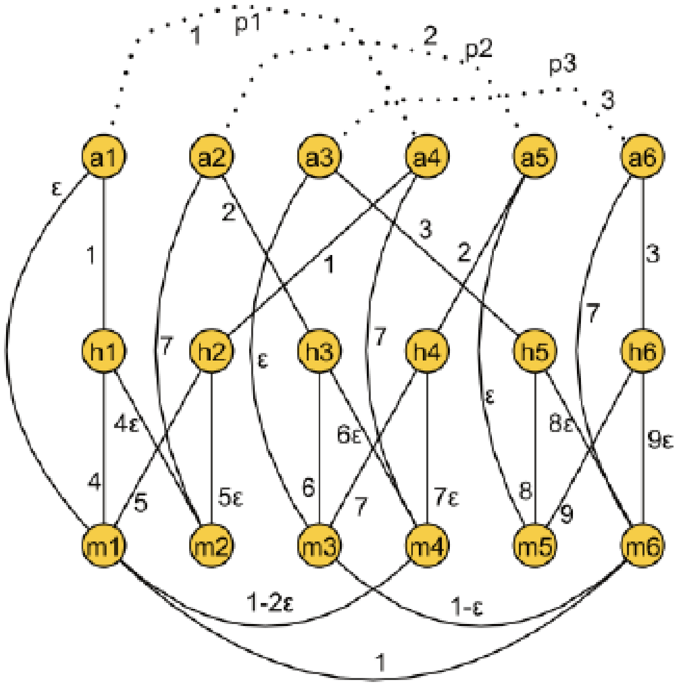
Our results I: cliques



Our results II: general graphs

Axiotis and Fotakis
ICALP 2016

Lower Bound of $\Omega(n^2)$



Our results II: single-source spanners for general graphs

a subgraph $\mathcal{H}$ *of* $\mathcal{G}$ *such that for every vertex* v

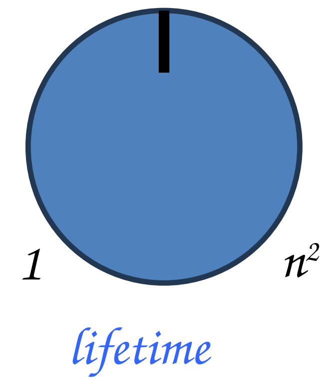
$$dist_{\mathcal{H}}(s,v) \leq \alpha \; dist_{\mathcal{G}}(s,v)$$

UB: *stretch*: $1+\varepsilon$ *size*: $O\left(n \frac{\log^4 n}{\log(1+\varepsilon)}\right)$

LB: *size* $\Omega(n^2)$ *for stretch 1*

Our results III: the role of lifetime

lifetime: number L of distinct time-labels



Our results III: the role of lifetime

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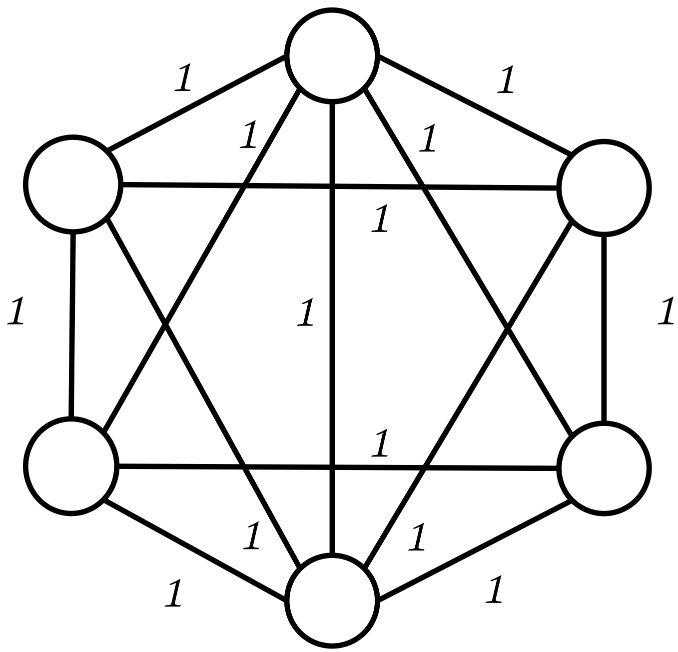

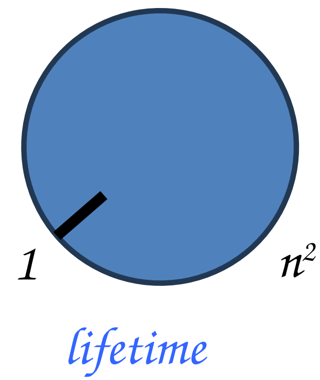

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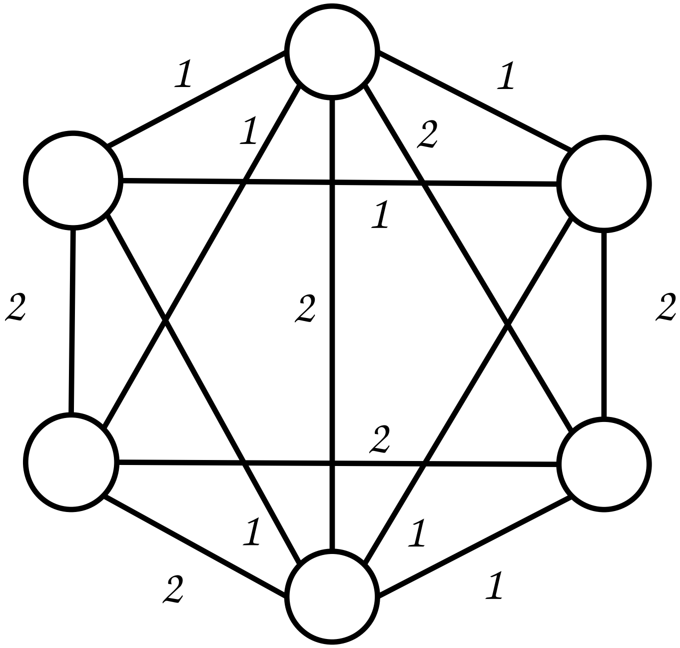

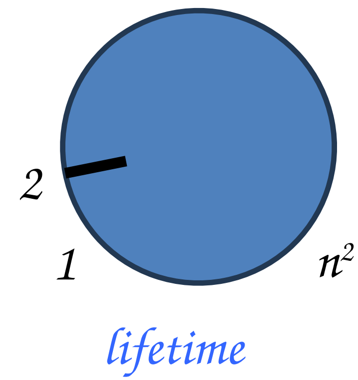

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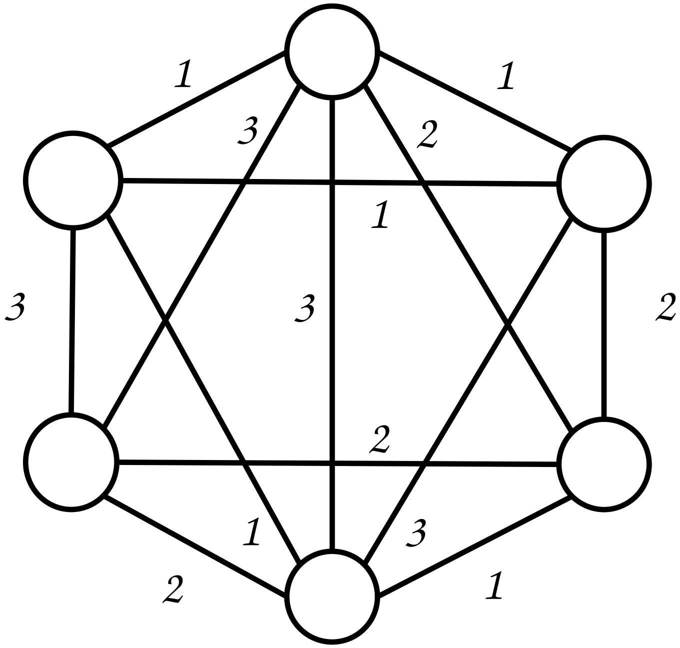
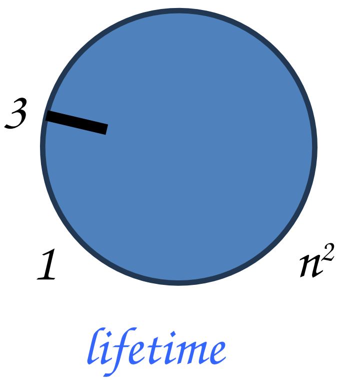
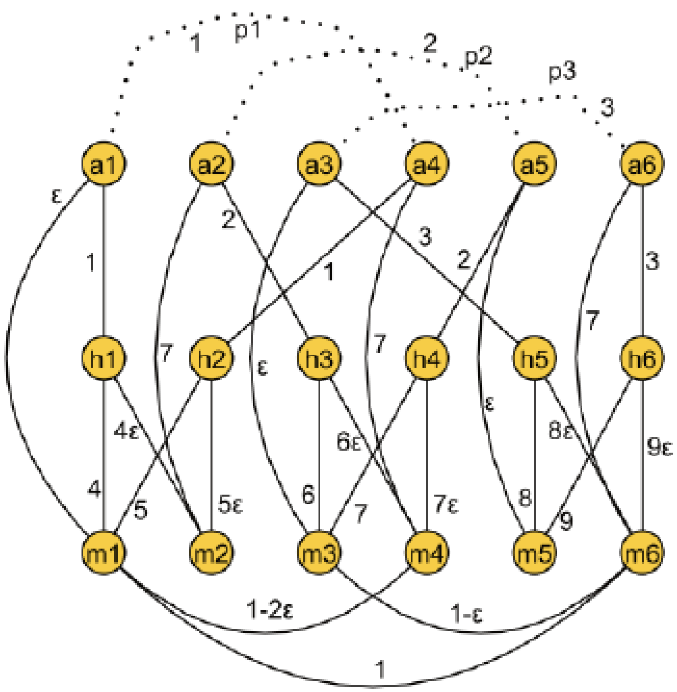


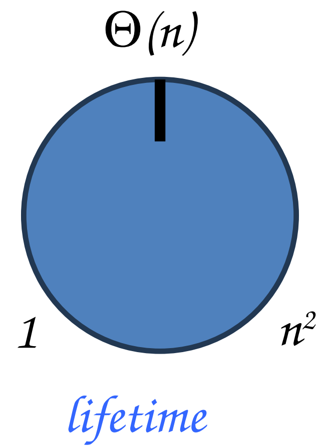


Our results III: *the role of* *lifetime*

lifetime: *number L of distinct time-labels*

Axiotis and Fotakis
ICALP 2016
Lower Bound of $\Omega(n^2)$



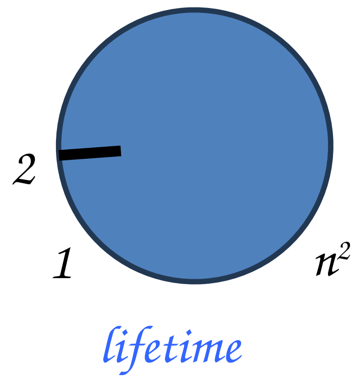


Our results III: the role of lifetime

Cliques

α=2

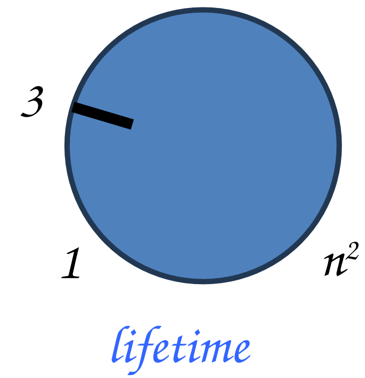
$\mathcal{L}$=2 $O(n \log n)$



Our results III: the role of lifetime

Cliques

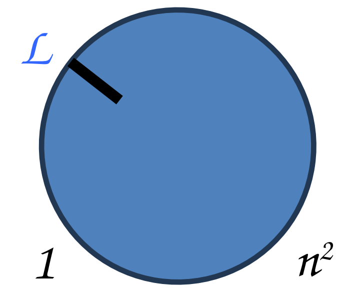
	$\alpha=2$	$\alpha=3$
$\mathcal{L}=2$	$O(n \log n)$	
$\mathcal{L}=3$	$\Omega(n^2)$	$O(n \log n)$



Our results III: the role of lifetime

Cliques

	$\alpha=2$	$\alpha=3$
$\mathcal{L}=2$	$O(n \log n)$	
$\mathcal{L}=3$	$\Omega(n^2)$	$O(n \log n)$
$\mathcal{L}$		$O(2^{\mathcal{L}} n \log n)$



lifetime

Our results III: the role of lifetime

Cliques

	$\alpha=2$	$\alpha=3$
$\mathcal{L}=2$	$O(n \log n)$	
$\mathcal{L}=3$	$\Omega(n^2)$	$O(n \log n)$
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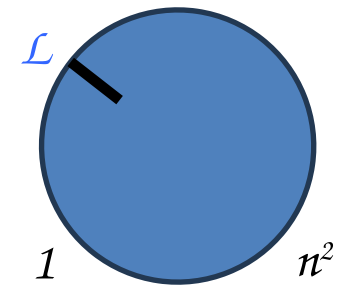


lifetime

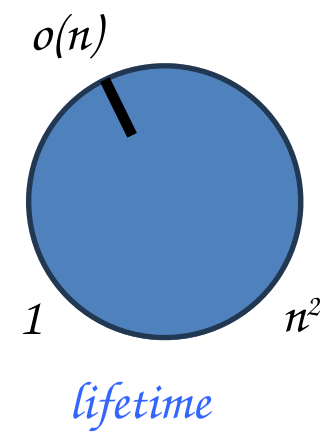
General graphs

an α-spanner of size $f(n)$ for static graphs → an α-spanner of size $O(\mathcal{L} f(n))$ for temporal graphs of lifetime $\mathcal{L}$

Our results III: the role of lifetime

Cliques

	$\alpha=2$	$\alpha=3$
$\mathcal{L}=2$	$O(n \log n)$	
$\mathcal{L}=3$	$\Omega(n^2)$	$O(n \log n)$
$\mathcal{L}$		$O(2^{\mathcal{L}} n \log n)$



General graphs

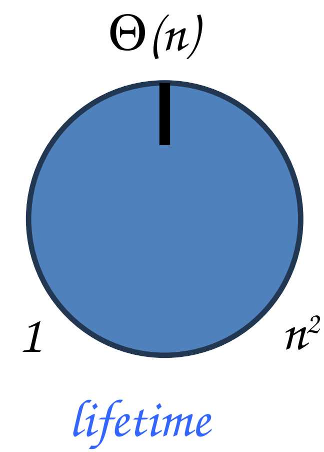
an α-spanner of size $f(n)$ for static graphs → an α-spanner of size $O(\mathcal{L} f(n))$ for temporal graphs of lifetime $\mathcal{L}$

temporal spanner of
stretch $\log n$ and size $o(n^2)$
for any temporal graph o lifetime $\mathcal{L}=o(n)$

Our results III: *the role of lifetime*

Cliques

	$\alpha=2$	$\alpha=3$
$\mathcal{L}=2$	$O(n \log n)$	
$\mathcal{L}=3$	$\Omega(n^2)$	$O(n \log n)$
$\mathcal{L}$		$O(2^{\mathcal{L}} n \log n)$



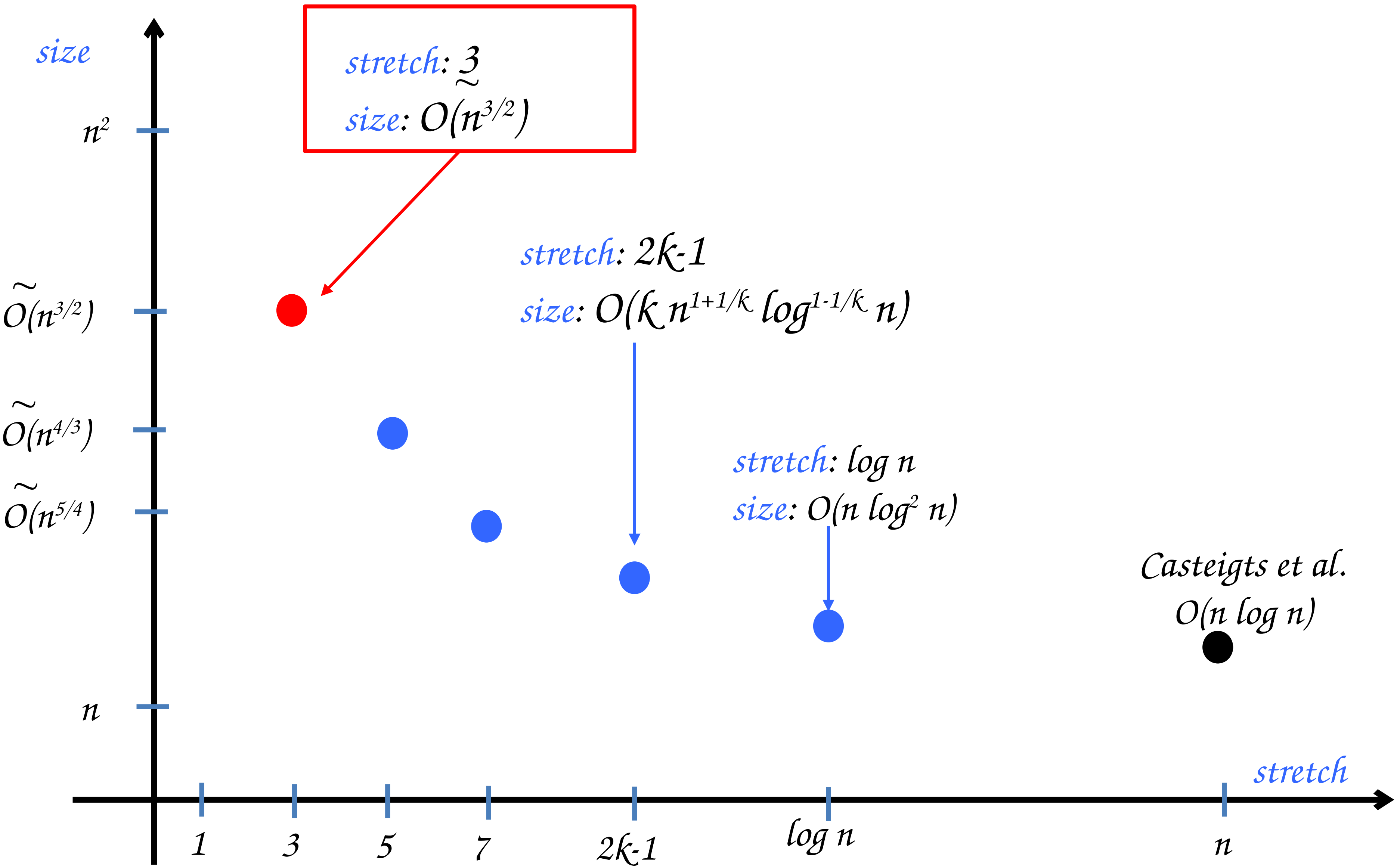
General graphs

an α-spanner of size $f(n)$ for static graphs → *an α-spanner of size $O(\mathcal{L} f(n))$ for temporal graphs of lifetime $\mathcal{L}$*

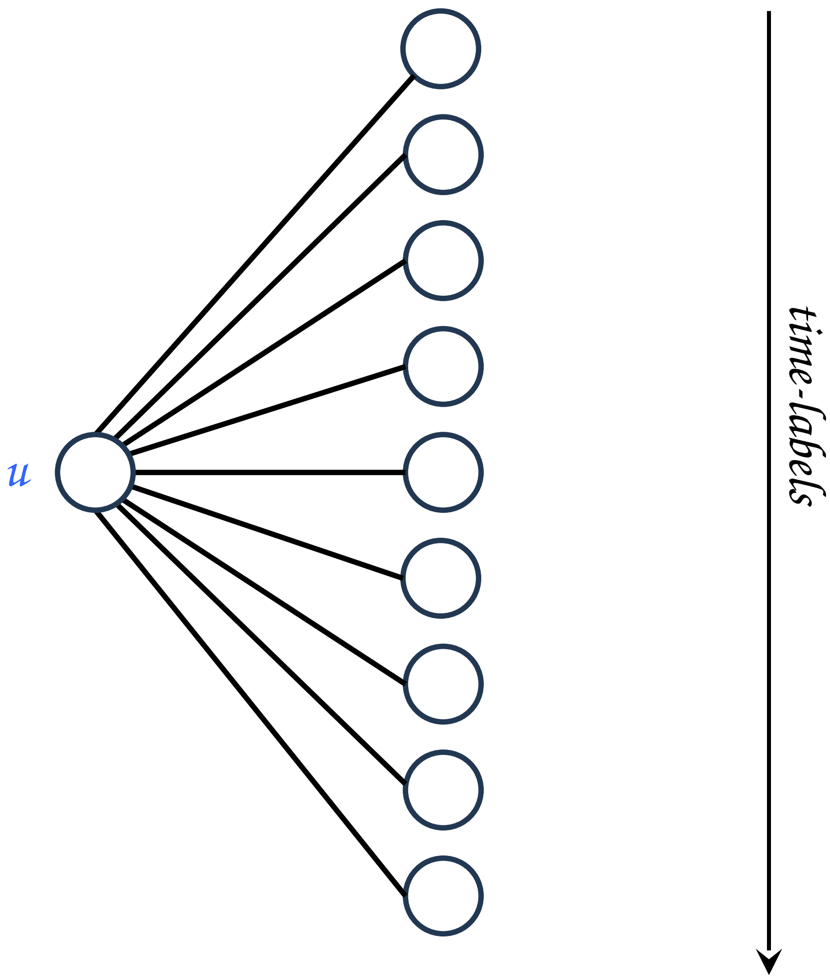
temporal spanner of
stretch $\log n$ and size $o(n^2)$
for any temporal graph o lifetime $\mathcal{L}=o(n)$

size $\Omega(n^2)$
for general graph with $\mathcal{L}=\Theta(n)$

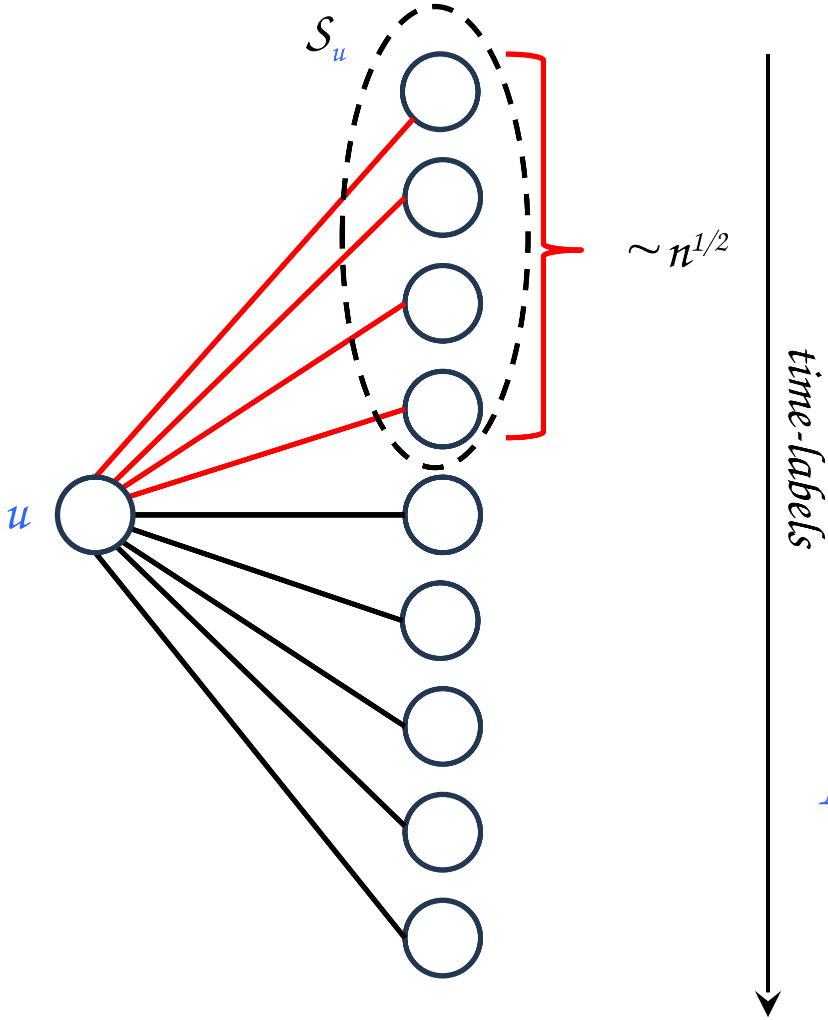
Our results I: cliques



for every $u \in \mathcal{V}$



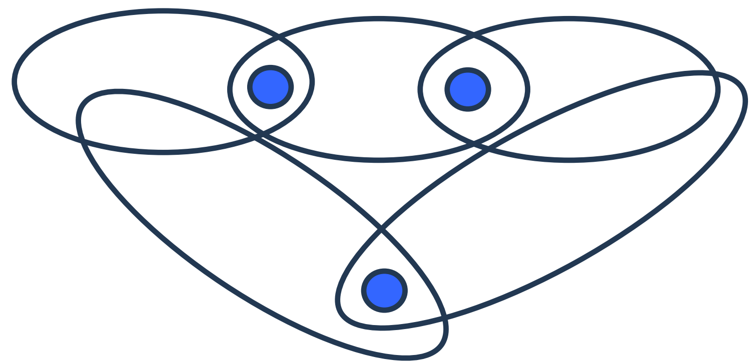
for every $u \in V$



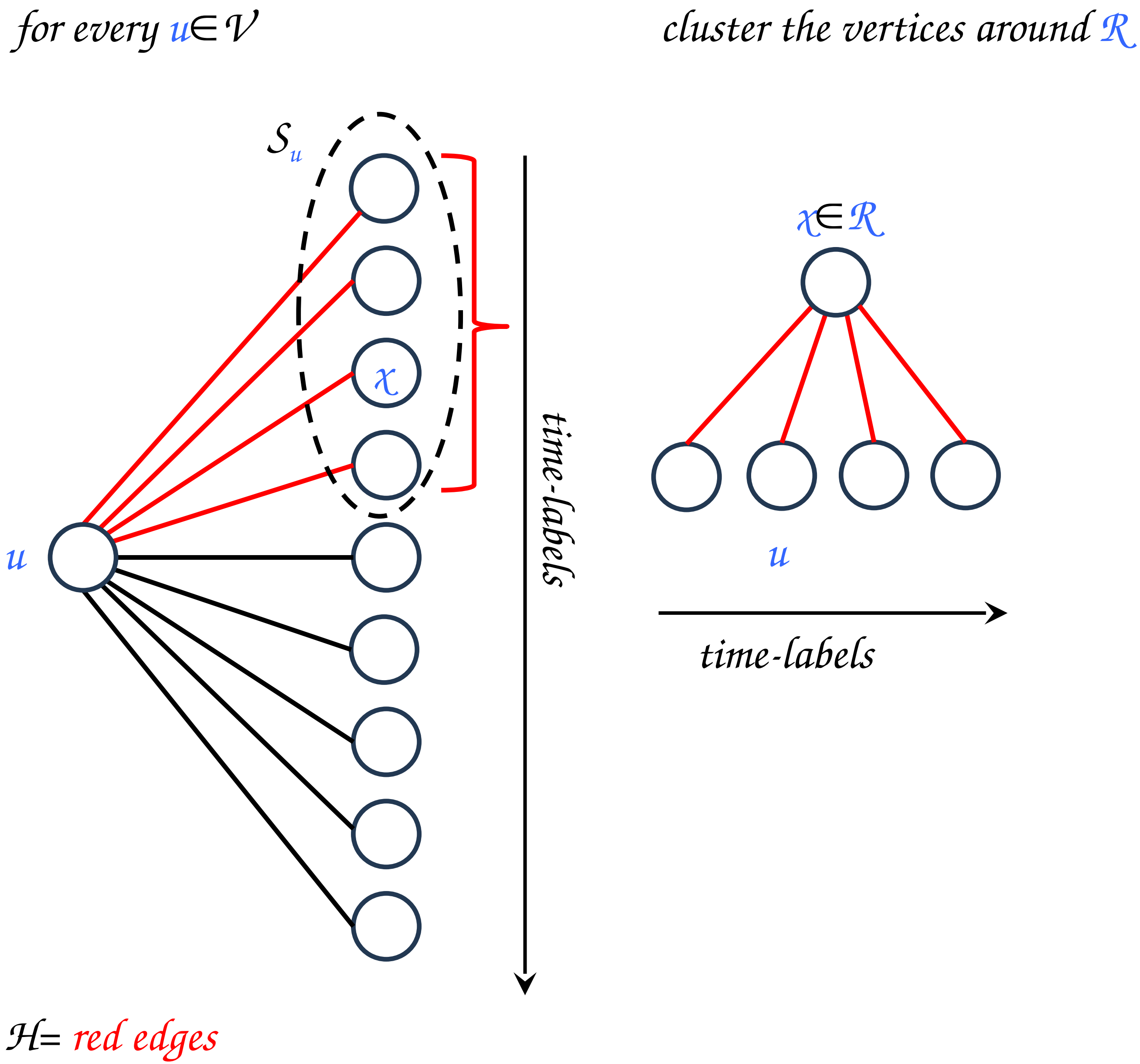
$\mathcal{H}$:= red edges

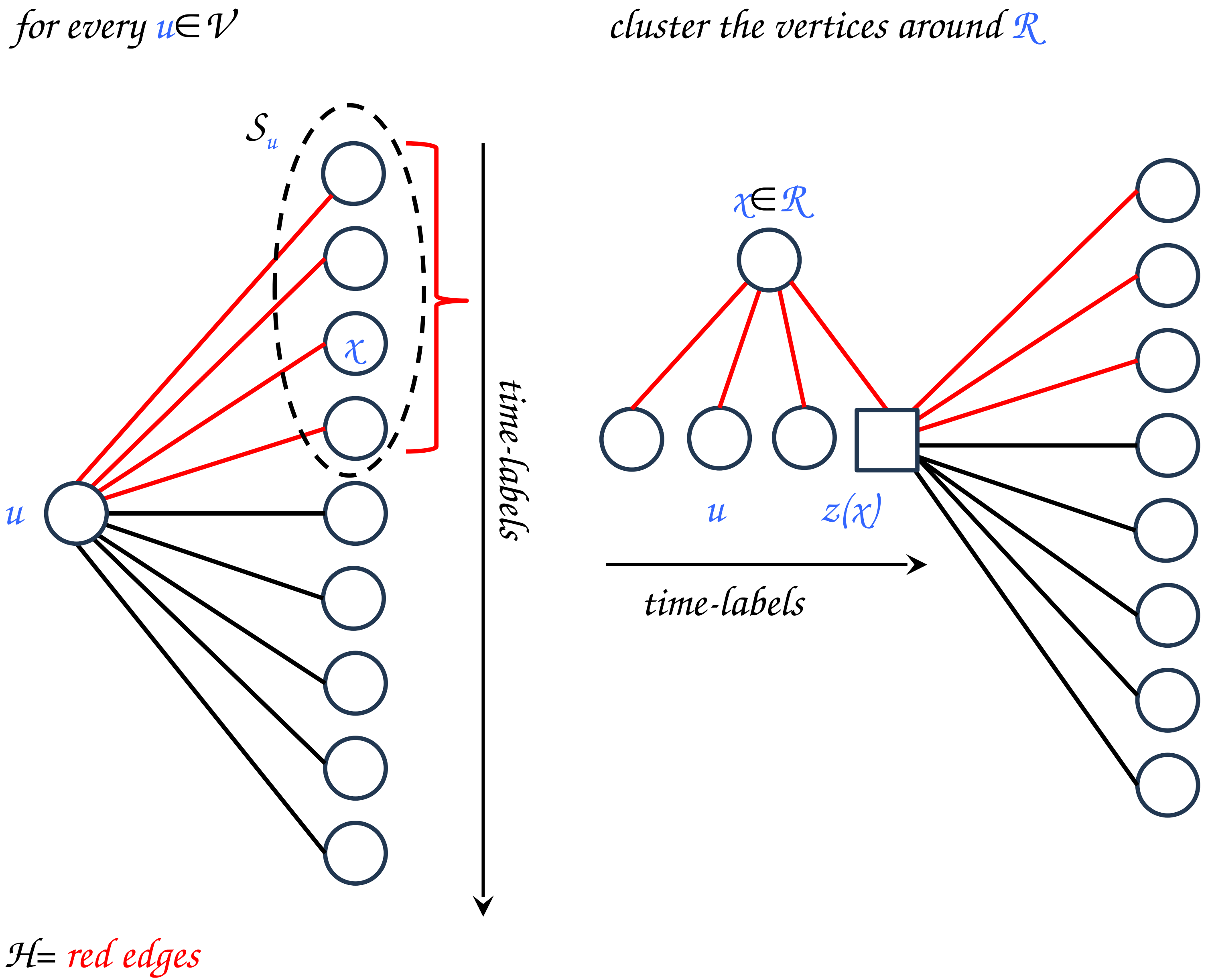
- # red edges $\tilde{O}(n^{3/2})$

compute a hitting set $\mathcal{R}$ of S_u's



Lemma: $$|\mathcal{R}| = \tilde{O}\left(\frac{n}{|S_u|}\right) = \tilde{O}\left(\frac{n}{n^{1/2}}\right) = \tilde{O}(n^{1/2})$$


for every u∈V
cluster the vertices around R
S_u
x
u
time-labels
x∈R
u
time-labels
H= red edges

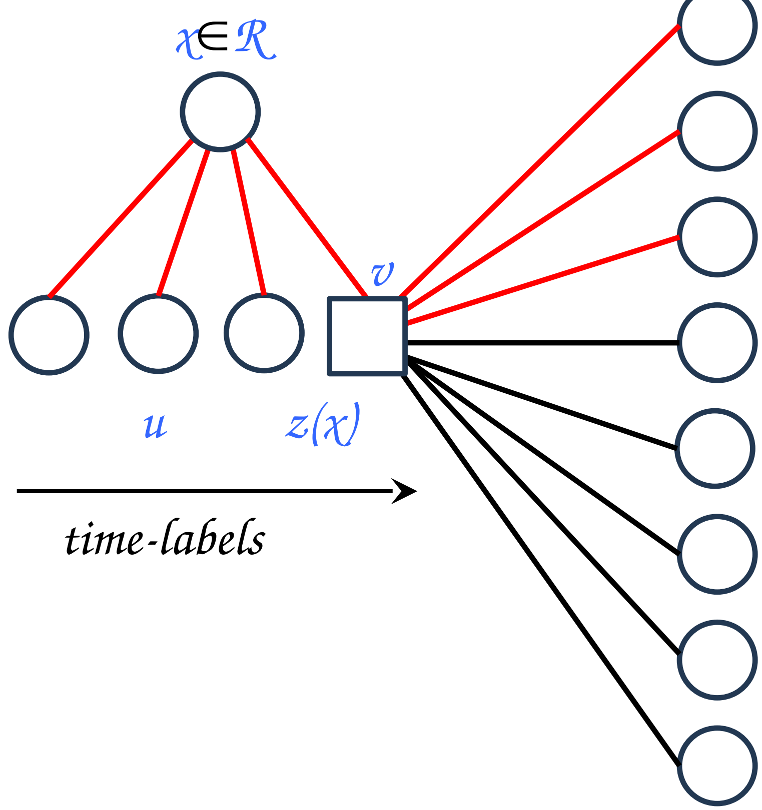

for every u∈V
cluster the vertices around R
S_u
x
u
time-labels
x∈R
u
z(x)
time-labels
H= red edges

for every $u \in \mathcal{V}$

cluster the vertices around $\mathcal{R}$

for any $v \in \mathcal{V}$

case: $v = z(x)$



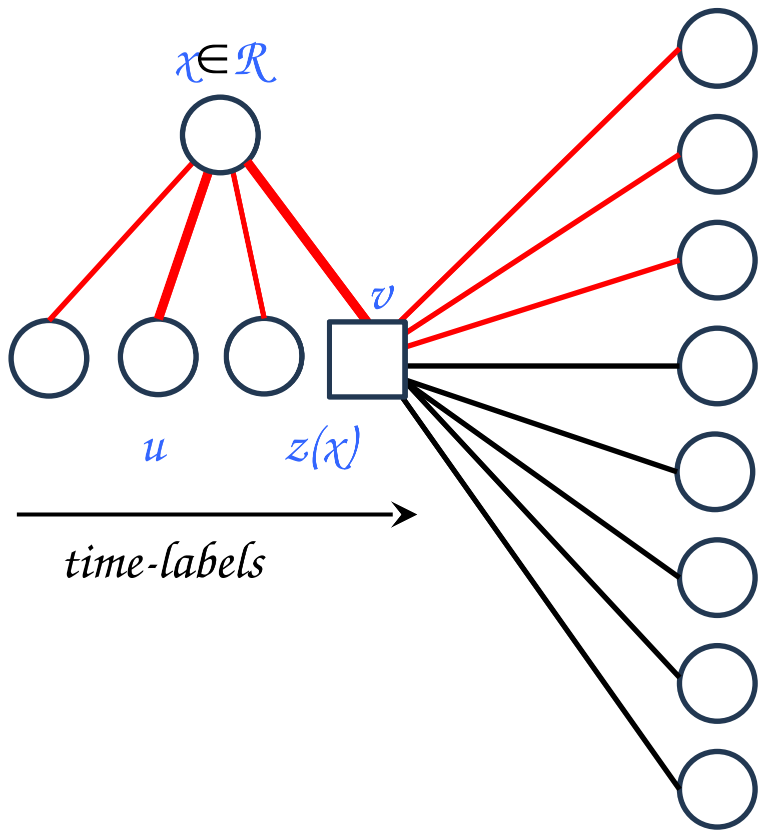
$\mathcal{H}$= *red edges*

for every $u \in \mathcal{V}$

cluster the vertices around $\mathcal{R}$

for any $v \in \mathcal{V}$

case: $v = z(x)$



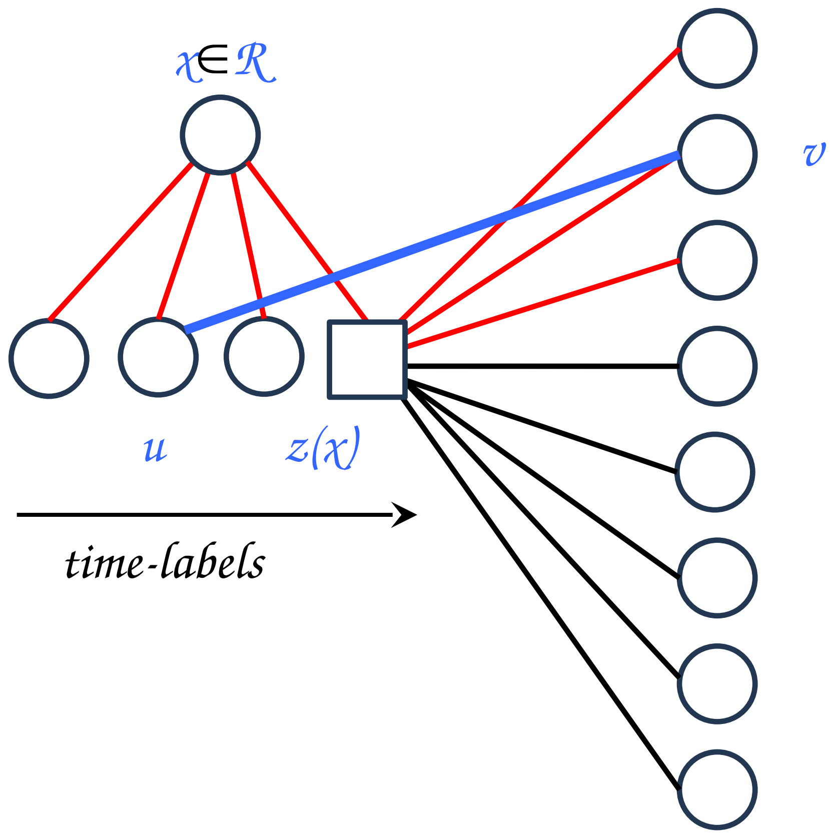
$\mathcal{H}$= red edges

for every $u \in \mathcal{V}$

cluster the vertices around $\mathcal{R}$

for any $v \in \mathcal{V}$

case: $v \in \mathcal{S}_{z(x)}$

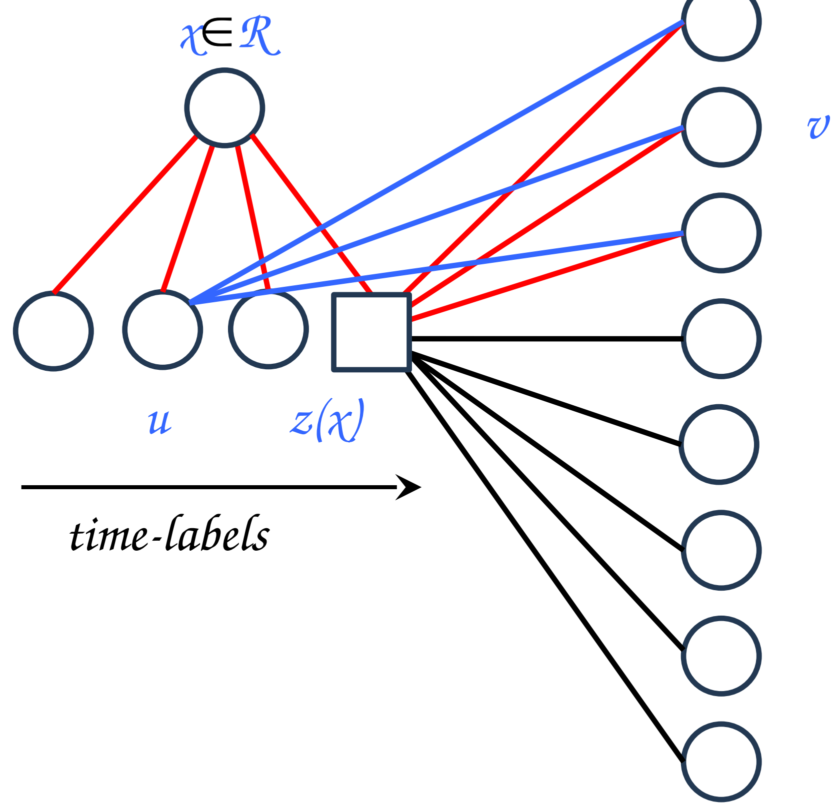


$\mathcal{H}$= red edges

for every $u \in \mathcal{V}$ *cluster the vertices around* $\mathcal{R}$

for any $v \in \mathcal{V}$

case: $v \in \mathcal{S}_{z(x)}$



$\mathcal{H}$= *red edges* + *blue edges*

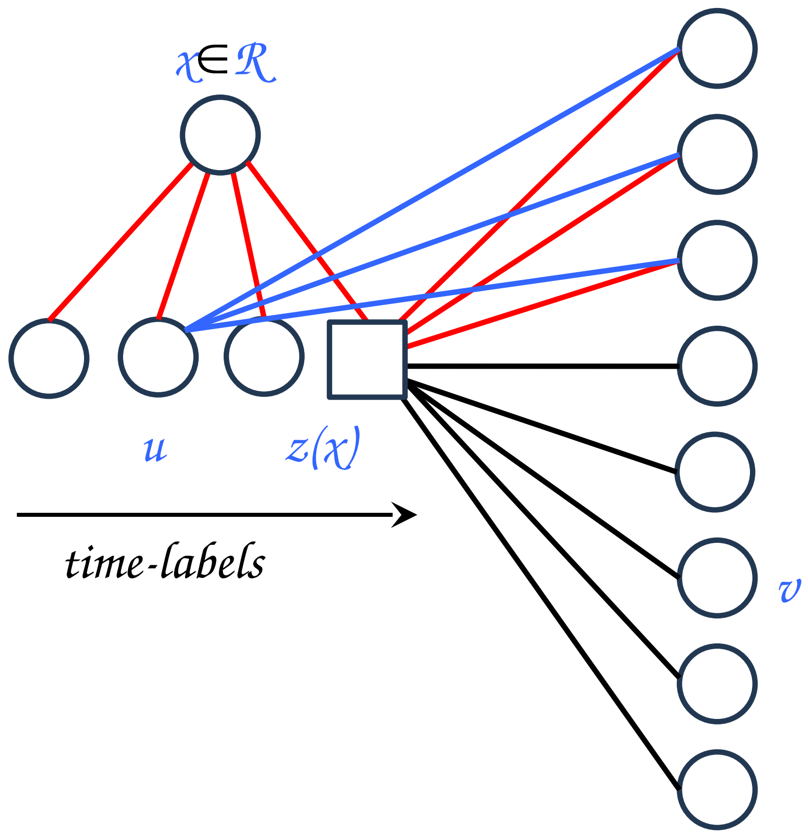
- # *blue edges* $\tilde{O}(n^{3/2})$

for every $u \in \mathcal{V}$

cluster the vertices around $\mathcal{R}$

for any $v \in \mathcal{V}$

case: $v \in \mathcal{V} \setminus \mathcal{S}_{z(x)}$



$\mathcal{H}$= red edges + blue edges

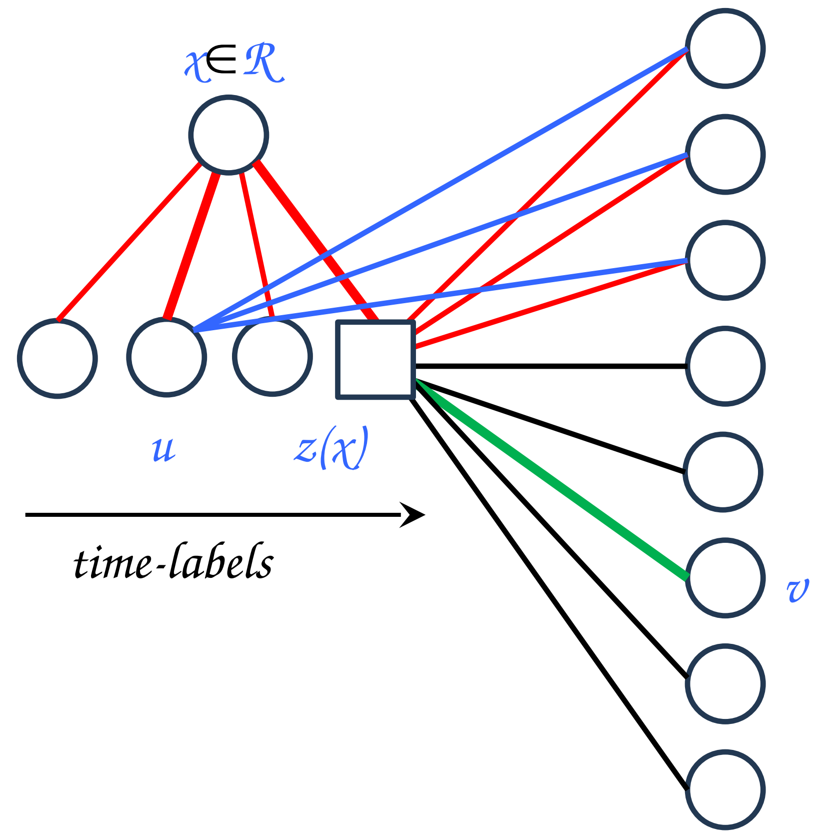
- # blue edges $\tilde{O}(n^{3/2})$

for every $u \in \mathcal{V}$

cluster the vertices around $\mathcal{R}$

for any $v \in \mathcal{V}$

case: $v \in \mathcal{V} \setminus \mathcal{S}_{z(x)}$



$\mathcal{H}$= *red edges* + *blue edges*

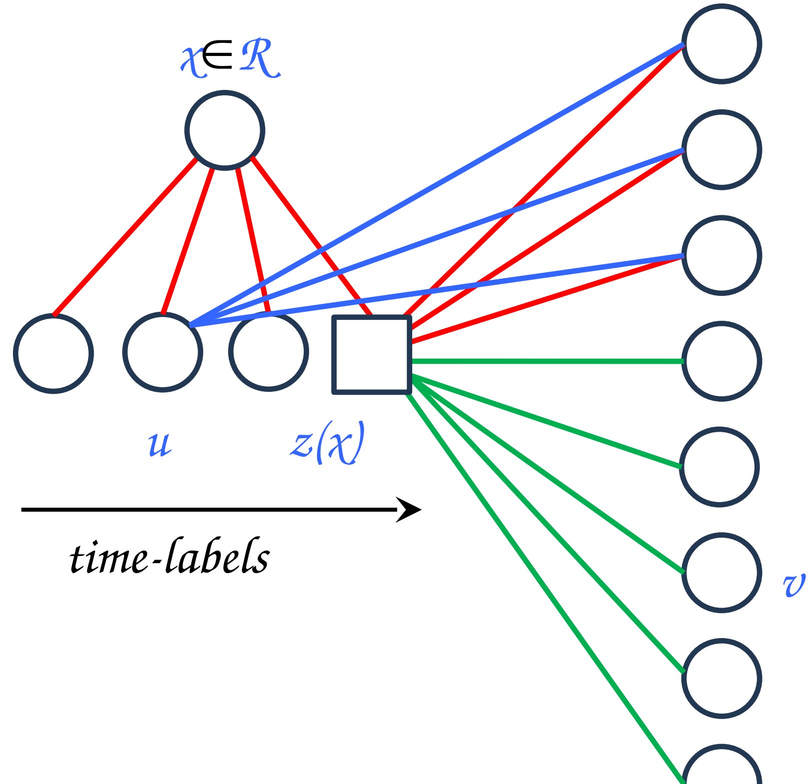
- # blue edges $\tilde{O}(n^{3/2})$

for every $u \in \mathcal{V}$

cluster the vertices around $\mathcal{R}$

for any $v \in \mathcal{V}$

case: $v \in \mathcal{V} \setminus \mathcal{S}_{z(x)}$

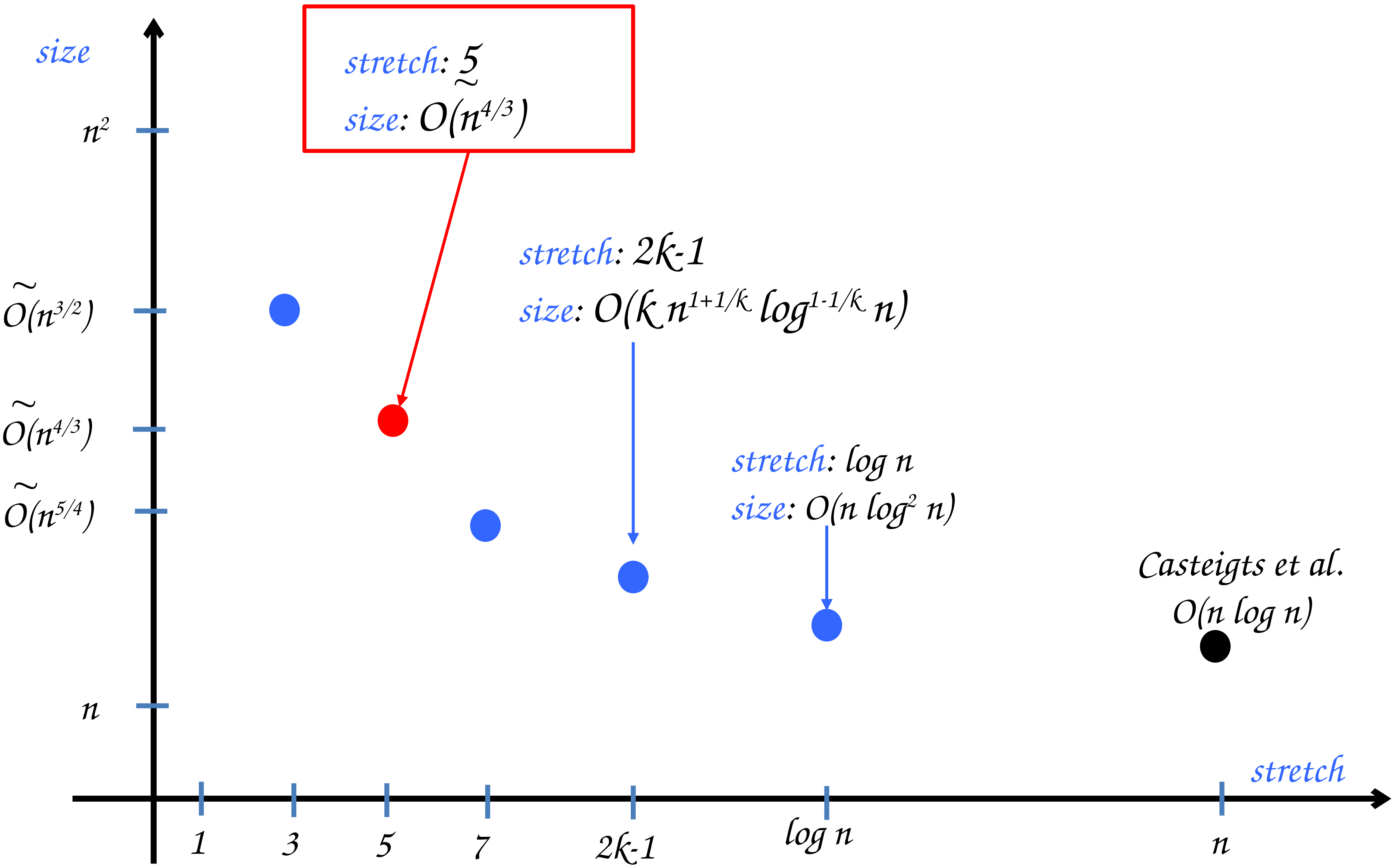


$\mathcal{H}$= red edges + blue edges + green edges

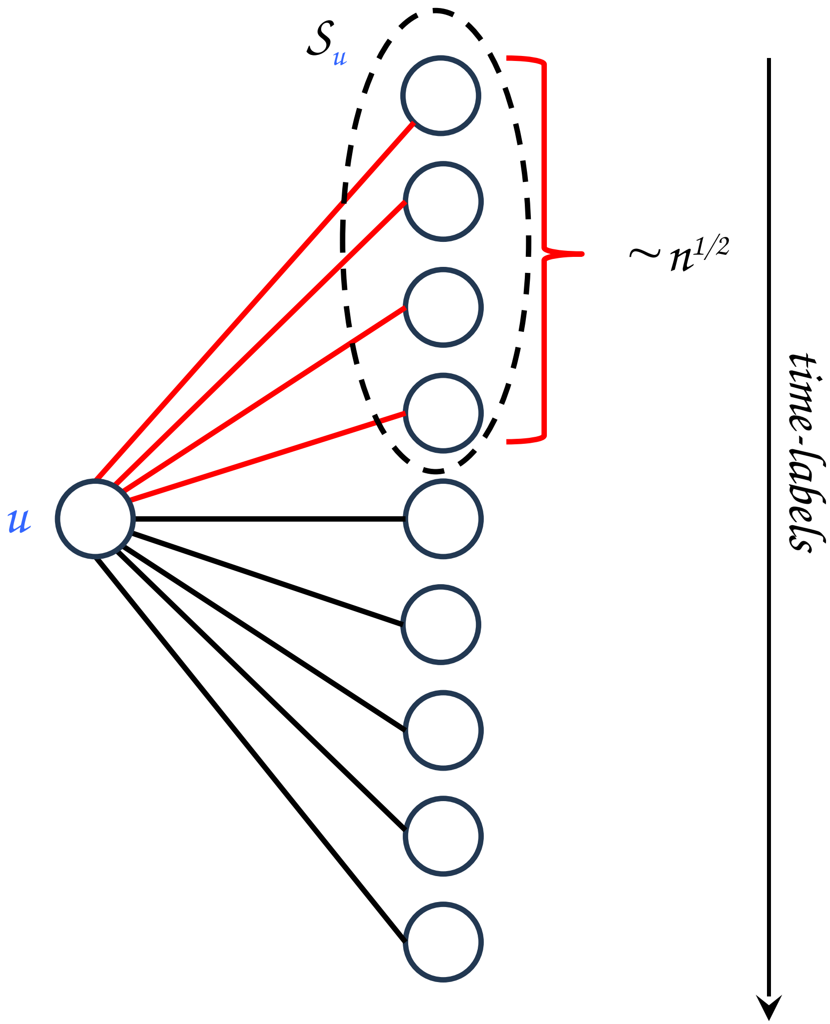
- # green edges $O(n^{3/2})$

$\mathcal{H}$ is a 3- spanner of size $O(n^{3/2})$

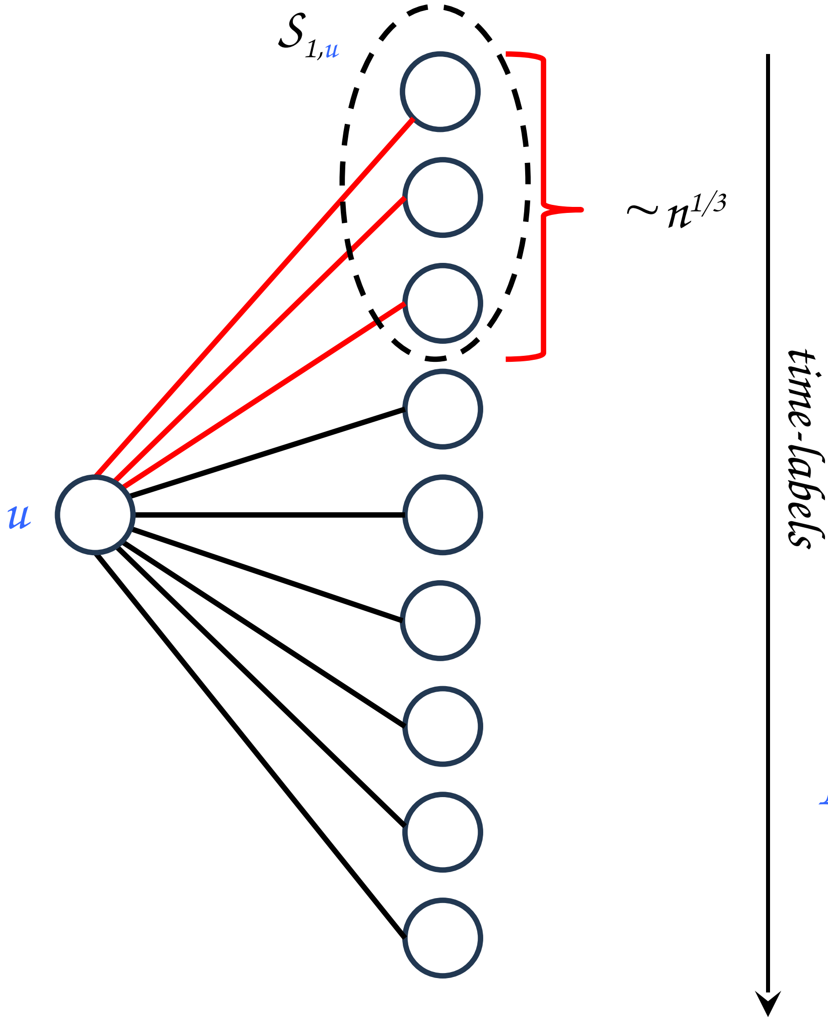
Our results I: cliques



for every $u \in V$



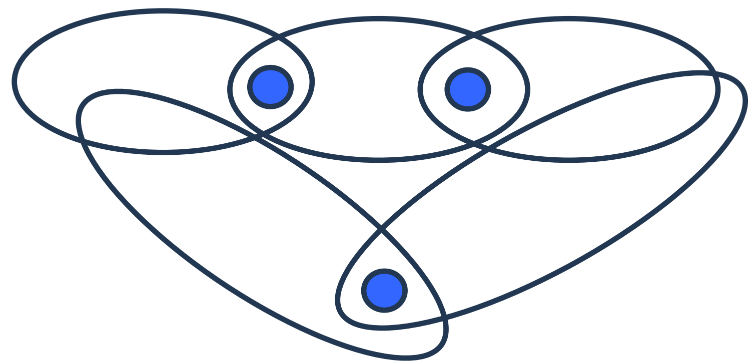
for every $u \in \mathcal{V}$



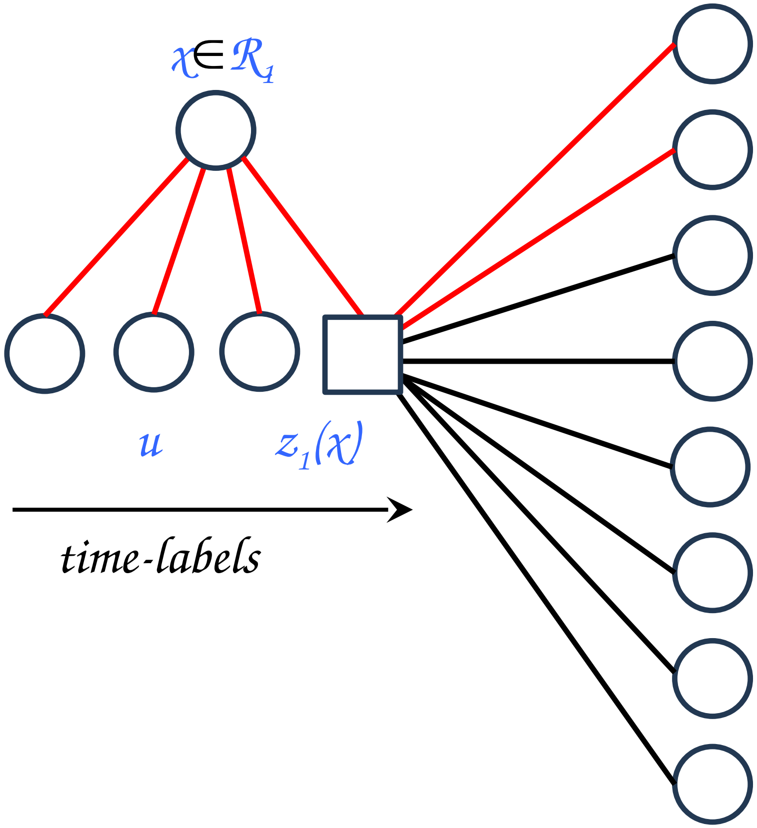
$\mathcal{H} :=$ red edges

- # red edges $\tilde{O}(n^{4/3})$

compute a hitting set $\mathcal{R}_1$ of $S_{1,u}$'s



Lemma: $|\mathcal{R}_1| = \tilde{O}\left(\frac{n}{|S_{1,u}|}\right) = \tilde{O}\left(\frac{n}{n^{1/3}}\right) = \tilde{O}(n^{2/3})$

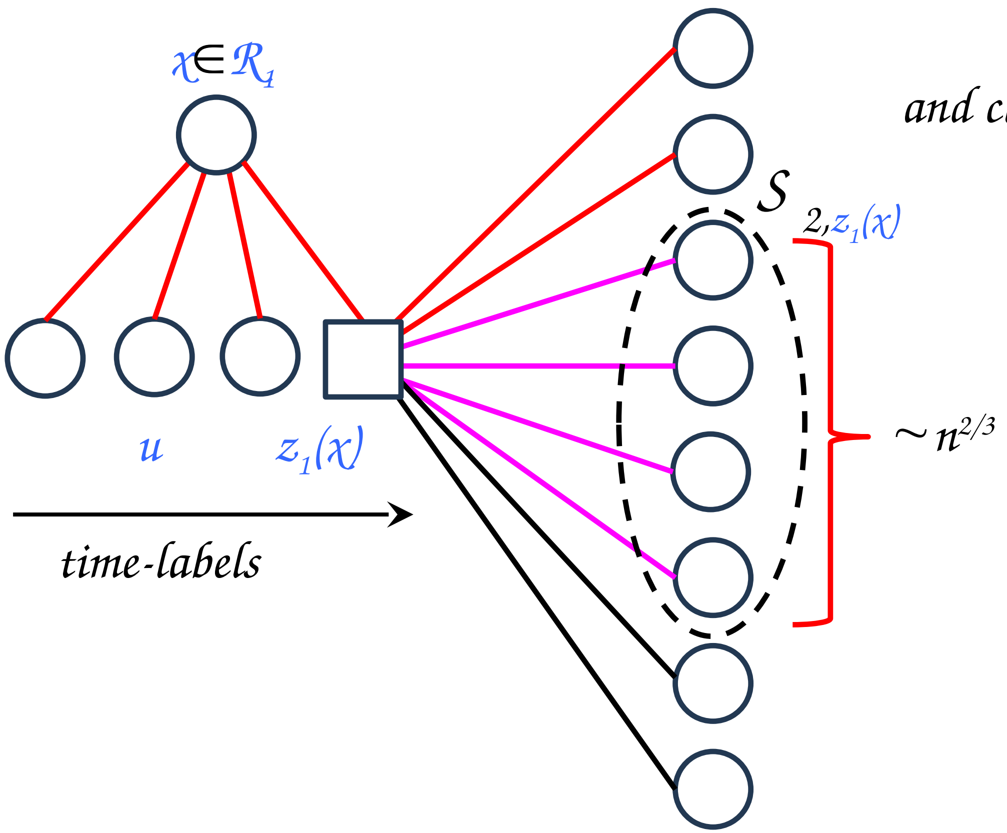


$x \in \mathcal{R}_1$
u
$z_1(x)$
time-labels

compute a second-level hitting set $\mathcal{R}_2$ of

$$\{\mathcal{S}_{2,z}\}_{z\in Z}$$

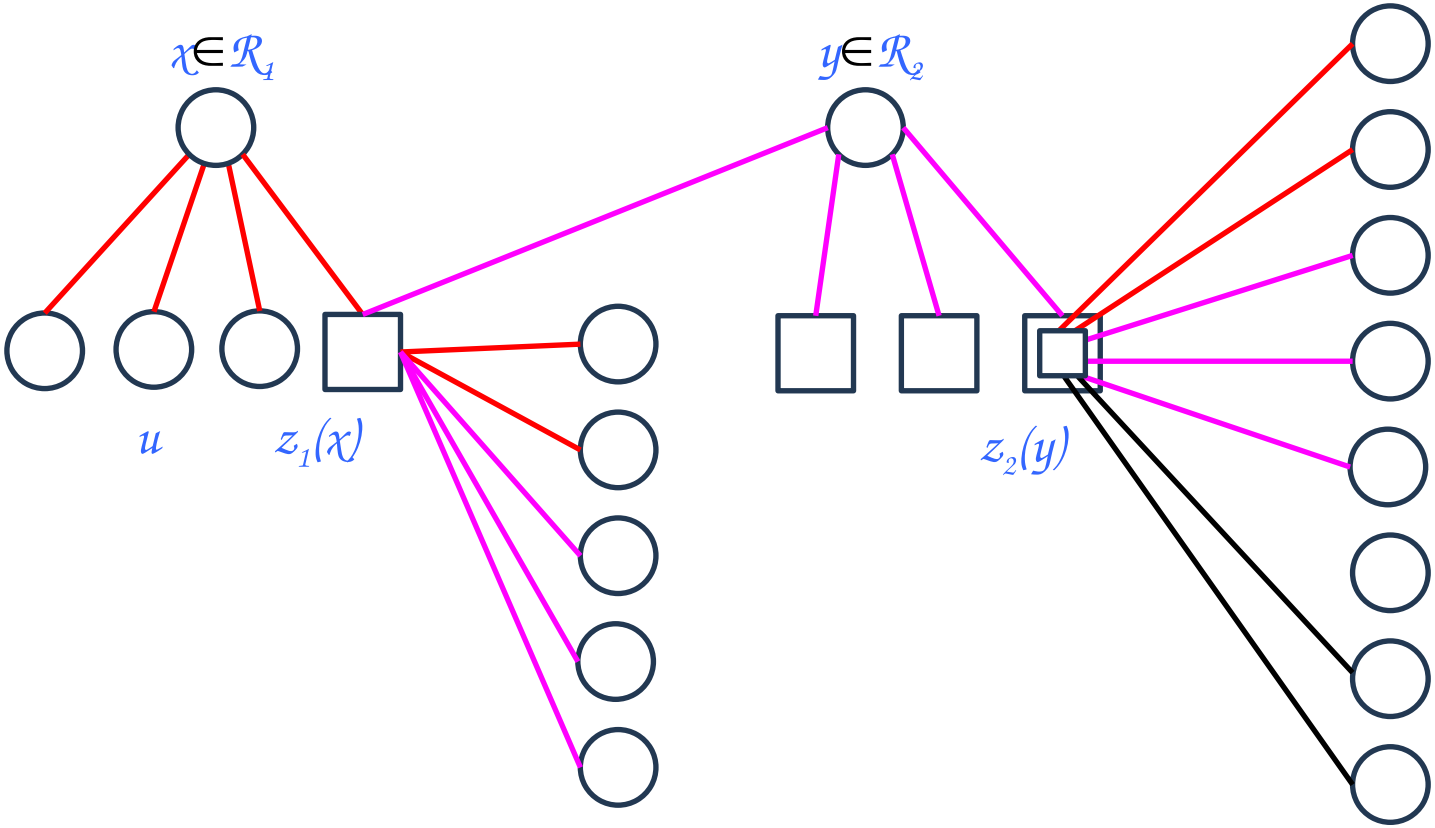
and cluster special vertices around $\mathcal{R}_2$



$\mathcal{H}$= red edges + purple edges

- # red + #purple edges $\tilde{O}(n^{4/3})$

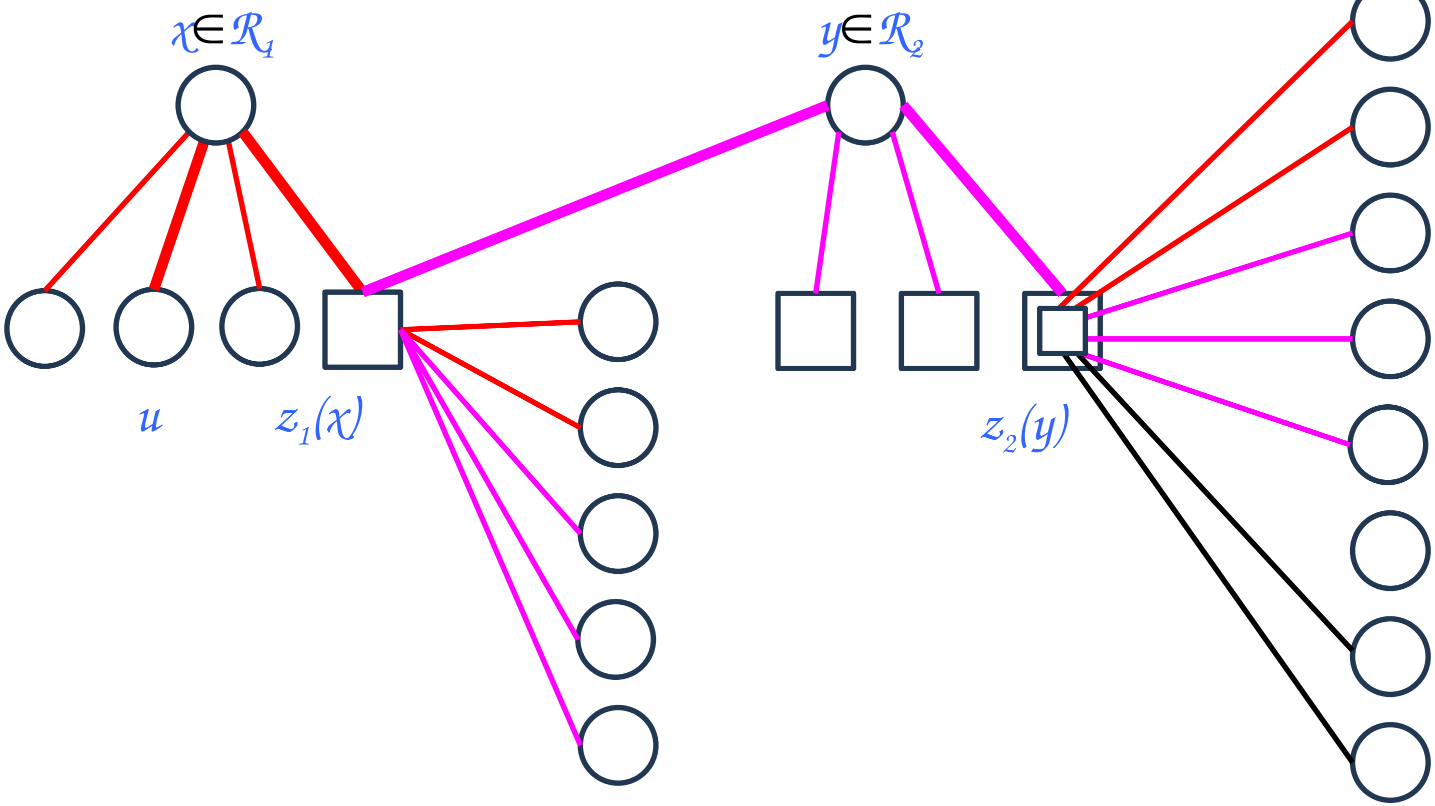
$\mathcal{H}$ = red edges + purple edges



for any $v \in \mathcal{V}$

case: $v = z_1(x)$ or $v = z_2(y)$

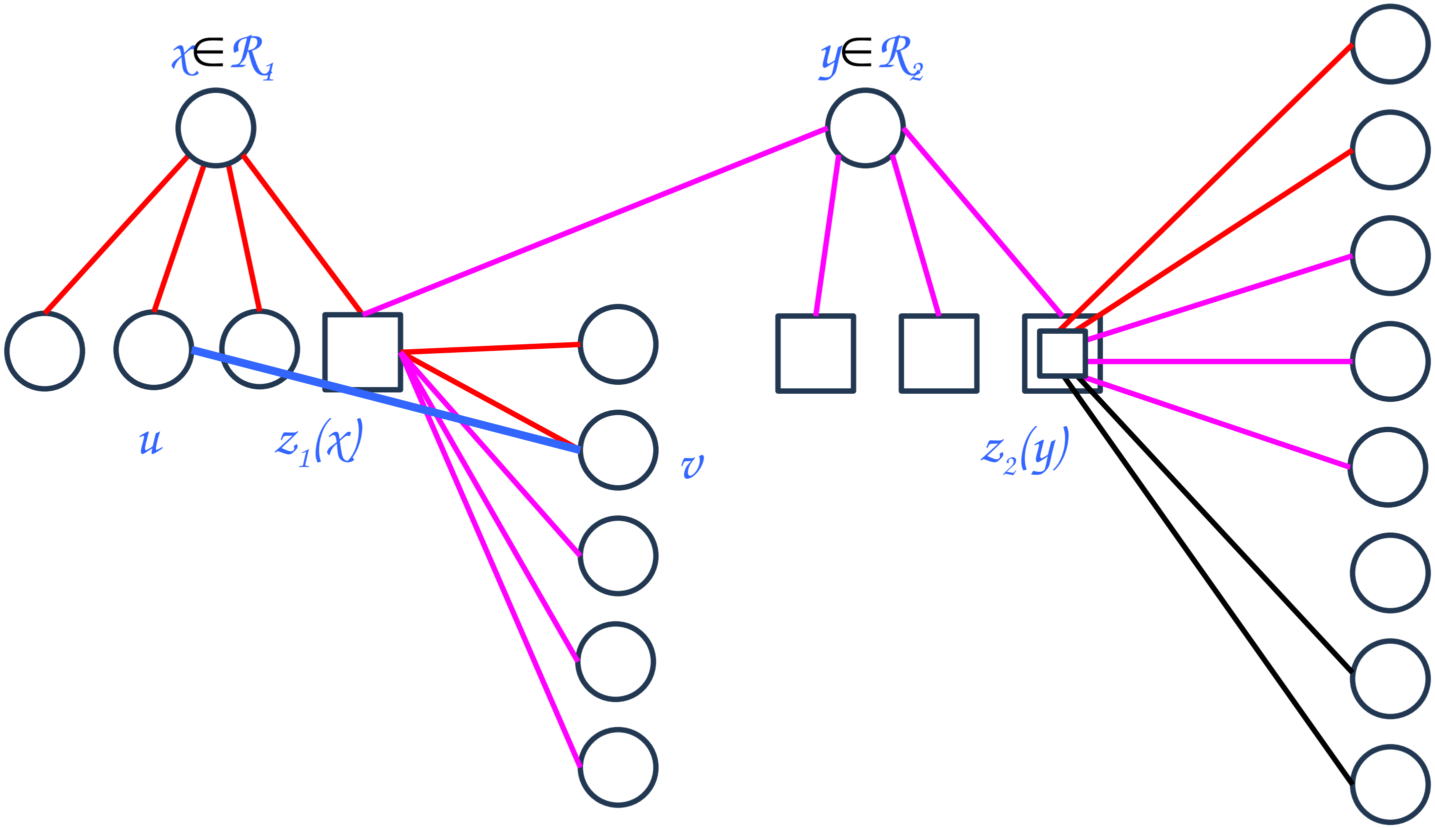
$\mathcal{H}$ = red edges + purple edges



for any $v \in \mathcal{V}$

case: $v = z_1(x)$ or $v = z_2(y)$

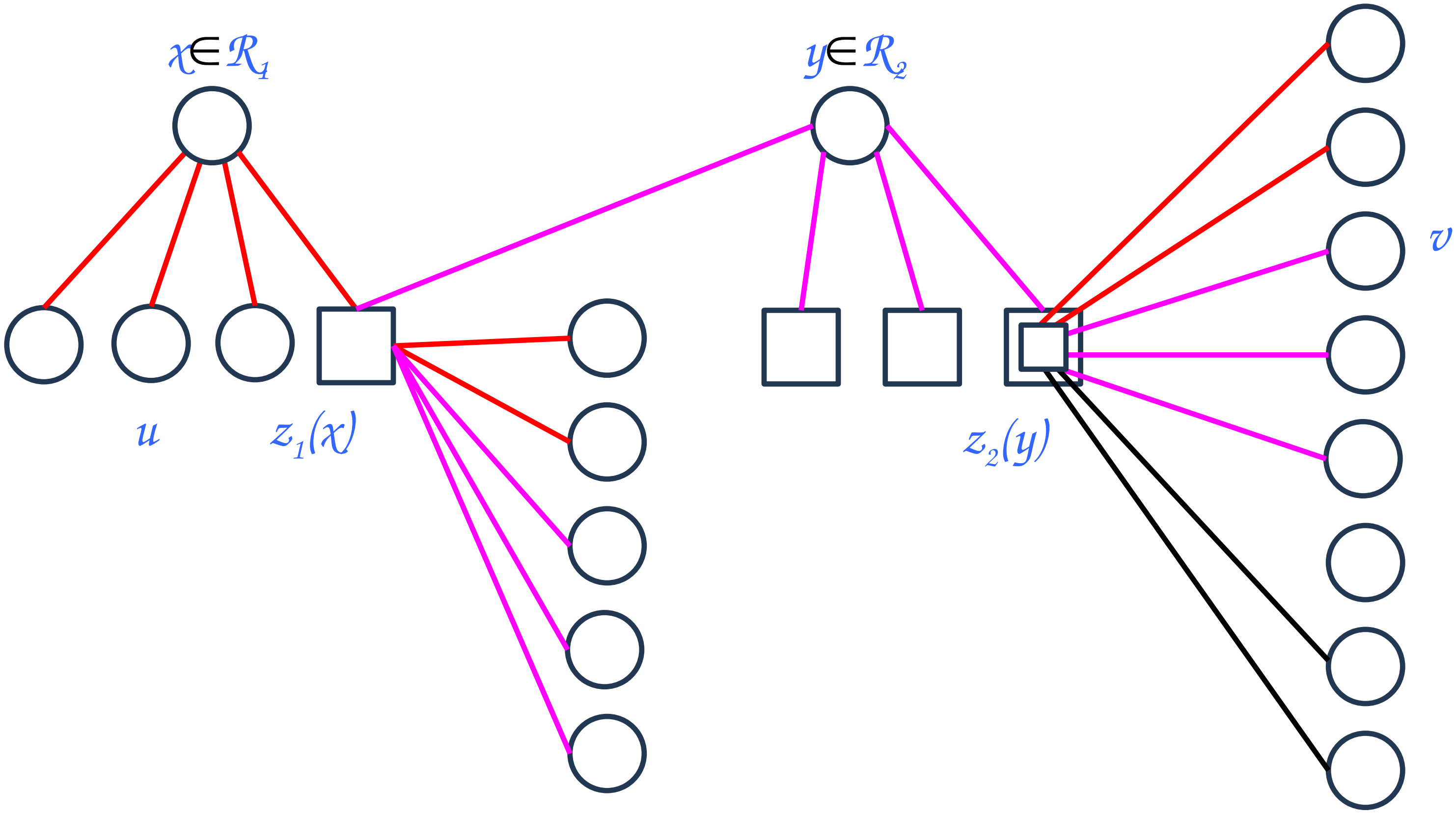
$\mathcal{H}$ = red edges + purple edges



for any $v \in \mathcal{V}$

case: $v \in \mathcal{S}_{1,z_1(x)}$

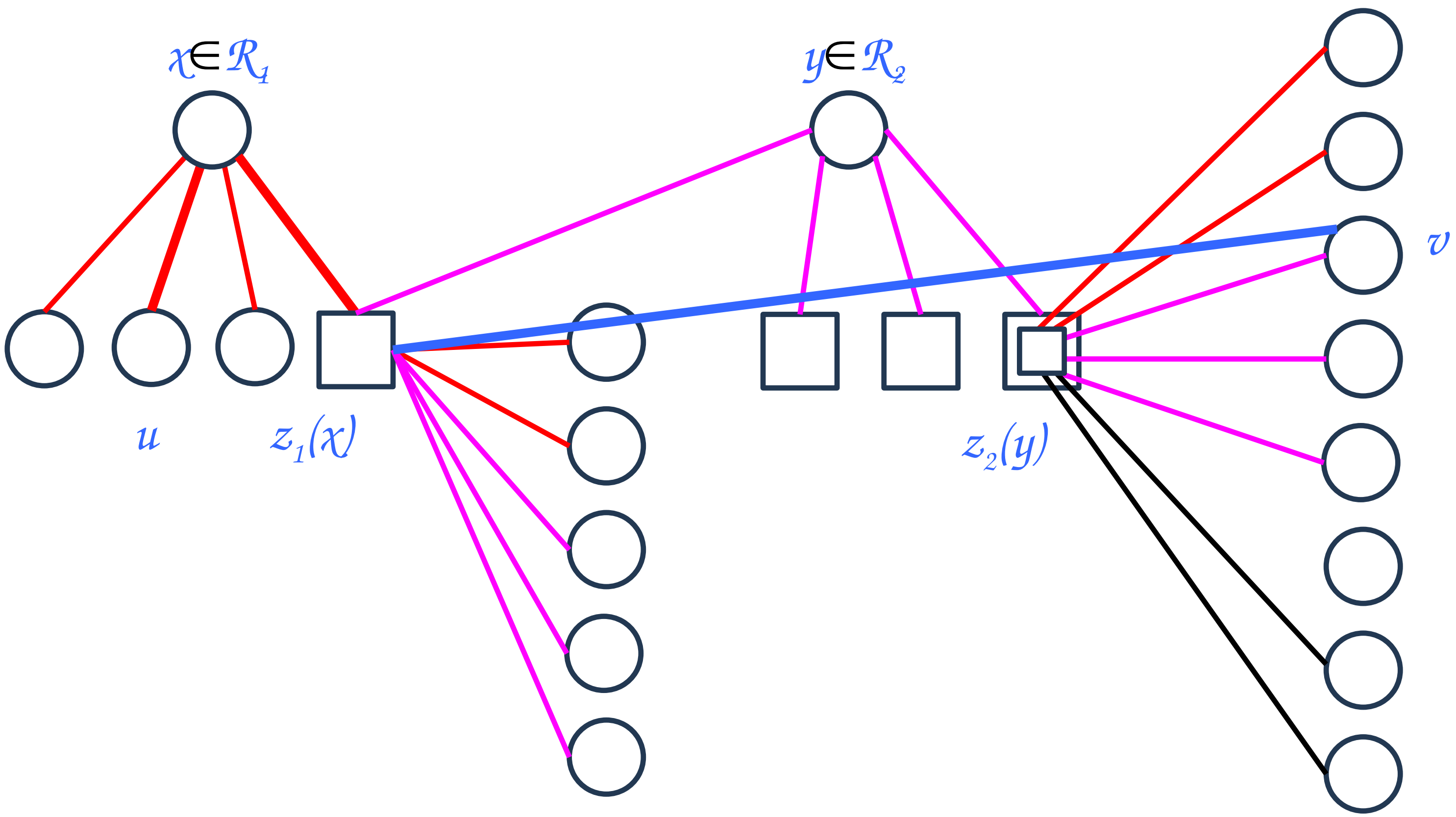
$\mathcal{H}$= red edges + purple edges



for any $v \in \mathcal{V}$

case: $v \in \mathcal{S}_{1,z_2(y)}$ *or* $v \in \mathcal{S}_{2,z_2(y)}$

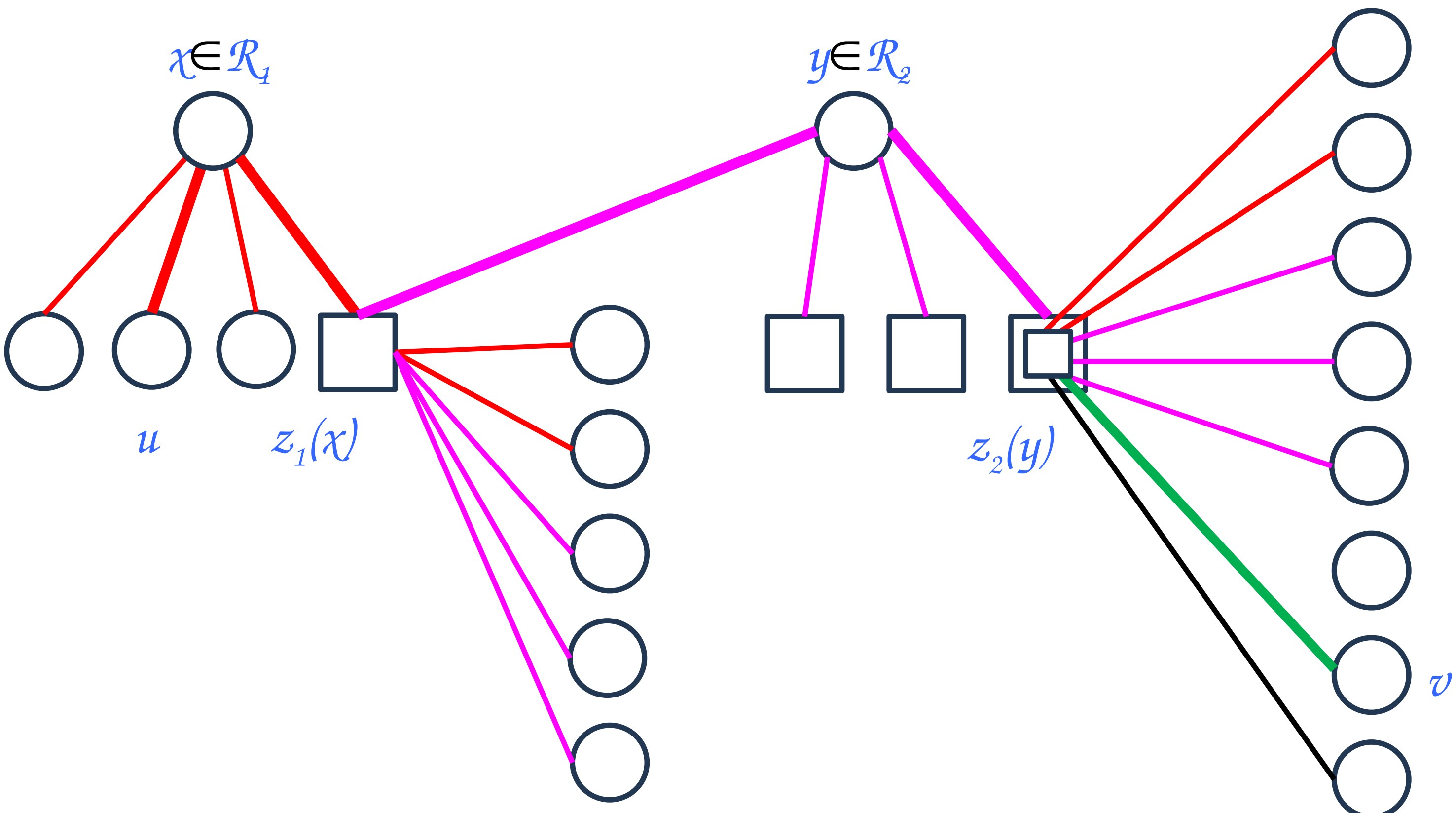
$\mathcal{H}$= red edges + purple edges + blue edges



for any $v \in \mathcal{V}$

case: $v \in \mathcal{S}_{1,z_2(y)}$ or $v \in \mathcal{S}_{2,z_2(y)}$

$\mathcal{H}$ = red edges + purple edges + blue edges + green edges



for any $v \in \mathcal{V}$

case: $v \notin \mathcal{S}_{1,z_2(y)}$ or $v \notin \mathcal{S}_{2,z_2(y)}$

$\mathcal{H}$ is a 5-spanner of size $O(n^{4/3})$

Selected open problems

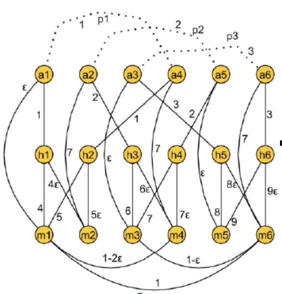
Cliques

3-spanner

$\Omega(n^{1+\varepsilon})$ *vs* $\widetilde{O}(n)$

Beyond Cliques

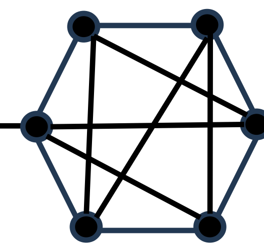
(dense)
general graphs



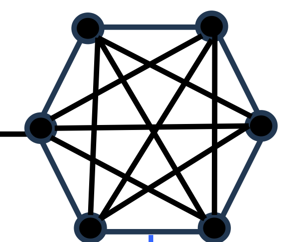
$\Omega(n^2)$ *for connectivity*

???

(n-2)-regular graph



cliques



for connectivity and logarithmic stretch $\tilde{O}(n)$

for stretch 2k-1 $\tilde{O}(n^{1+1/k})$

Thanks for your attention!