Advanced topics on Algorithms

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Approximation algorithms
Episode II

minimum Steiner Tree problem

minimum Steiner Tree problem

Input:

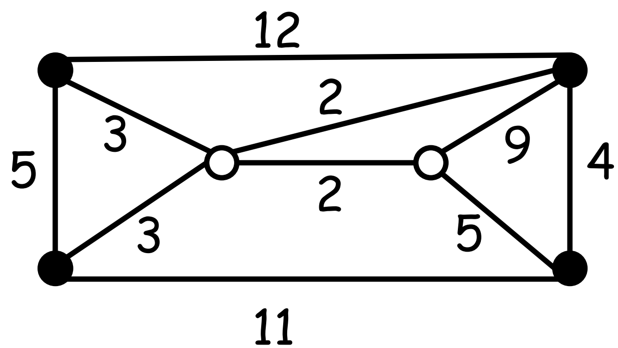
- undirected graph $G=(V,E)$ with non-negative edge costs
- subset of *required* vertices $R \subseteq V$; $V-R$ are called *Steiner* vertices

Feasible solution:

a tree T containing all the required vertices and any subset of Steiner ones

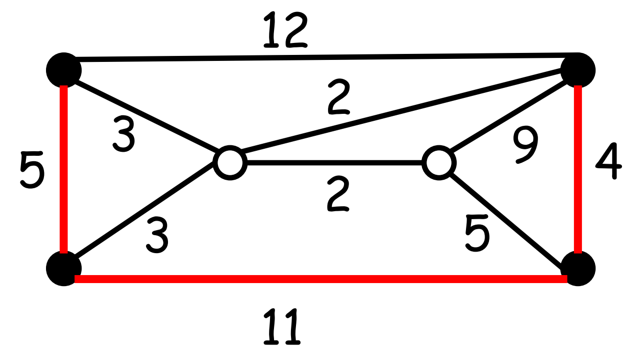
measure (min):

cost of T : $\sum_{e \in E(T)} c(e)$



● : required vertices

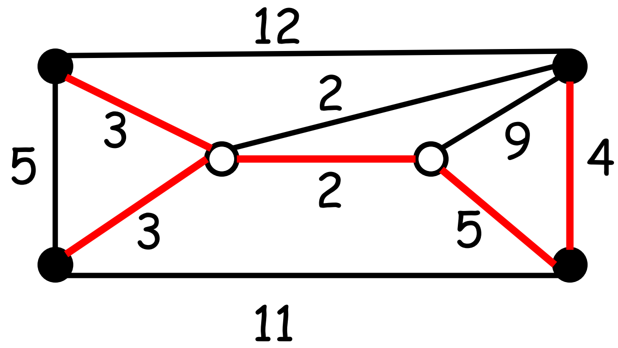
○ : Steiner vertices



a Steiner tree of cost 20

● : required vertices

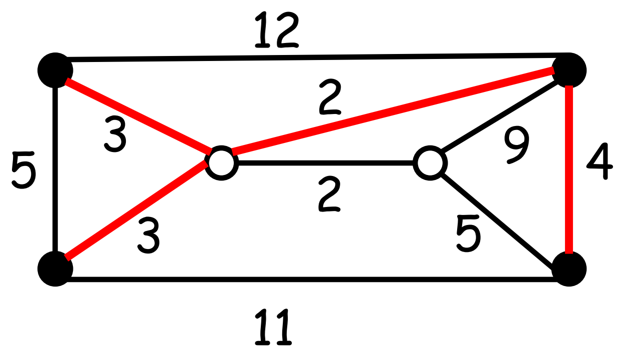
○ : Steiner vertices



a better Steiner tree of cost 17

● : required vertices

○ : Steiner vertices



a better Steiner tree of cost 12

special case: R=V

- Minimum Spanning Tree (MST) problem
- poly-time solvable

● : required vertices

○ : Steiner vertices

minimum Steiner Tree problem

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metric Steiner tree problem:

- G is complete, and
- edge costs satisfy the *triangle inequality*
  for every u,v,w : $c(u,v) \leq c(u,w)+c(w,v)$

Theorem

There is an approximation factor preserving reduction from the Steiner tree problem to the metric Steiner tree problem.

proof

let I be an instance of the ST problem consisting of graph $G=(V,E)$ and required vertices R.

in poly-time

instance I' of metric ST problem:

- $G'=(V,E')$ complete; $c'(u,v)$ in G'= cost of any u-v shortest path in G
- $R'=R$

since for every $(u,v)\in E$, $c'(u,v)\leq c(u,v)$, $OPT(I')\leq OPT(I)$.

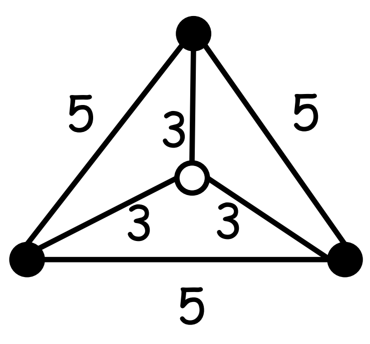
any steiner tree T' of I' can be converted in poly-time into a steiner tree T of I of at most the same cost:

- replace each edge (u,v) of T' with the shortest path in G
- pick any spanning tree T of the obtained subgraph of G

$\text{cost}(T)\leq \text{cost}(T')$

Algorithm

output a Minimum Spanning Tree (MST) of the subgraph of G induced by R



● : required vertices

○ : Steiner vertices

Algorithm

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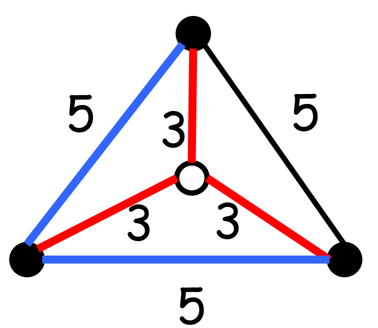


OPT=9

returned tree T has cost: 10

● : required vertices

○ : Steiner vertices

Theorem

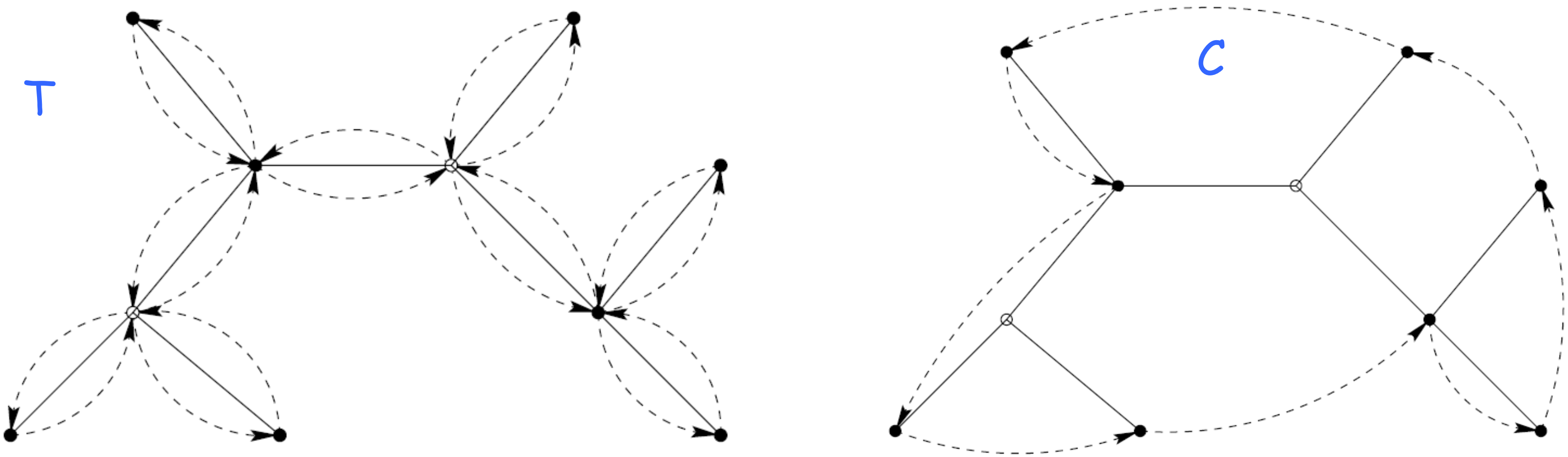
The algorithm is a 2-approximation algorithm for metric ST problem.

proof

let T be an optimal Steiner tree of cost OPT, and M the MST on R.

double the edges of T obtaining an Eulerian graph of cost 2 OPT

consider an Eulerian tour of cost 2 OPT



obtain a Hamiltonian cycle C on R by traversing the Eulerian tour and "shortcutting" Stainer vertices and previously visited vertices of R

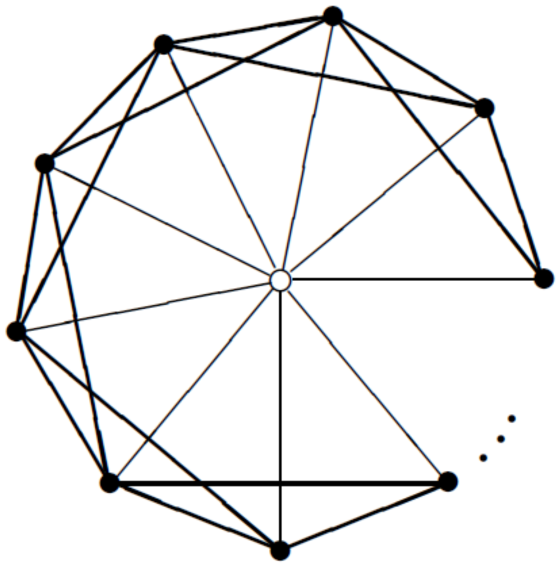
by triangle inequality: cost(C)≤2 OPT

Since C is a spanning subgraph of G[R]: cost(M)≤ cost(C)

■

tight example

n+1
vertices



- edges incident to the Steiner vertex have cost 1
- all the other edges have cost 2

returned solution has cost $2(n-1)$

OPT=n

Steiner Tree: state of the art

2	[Takahashi & Matsuyama, J.of Math. Jap, 1980]
11/6= 1.834	[Zelikovsky, Algorithmica 93]
1.746	[Berman & Ramaiyer, SODA 92]
$1+\ln 2+\varepsilon=1.693$	[Zelikovsky, Tech. Rep. 96]
$5/3+\varepsilon = 1.667$	[Promel & Steger, STACS 96]
1.644	[Karpinski & Zelikovsky, JOCO 97]
1.598	[Hougardy &. Promel, SODA 99]
$1+(\ln 3)/2+\varepsilon=1.55$	[Robins & Zelikovsky, SODA 2000]
$\ln 4 +\varepsilon =1.39$	[Byrka et al., STOC 2010]

Traveling Salesman Problem (TSP)

Traveling Salesman Problem

Input:

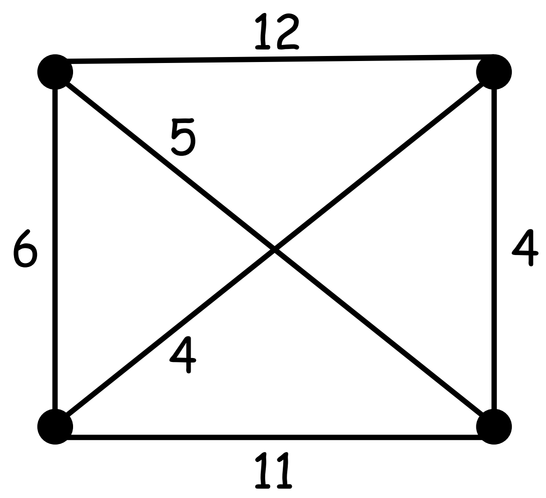
undirected complete graph $G=(V,E)$ with non-negative edge costs

Feasible solution:

a cycle C visiting every vertex exactly once

measure (min):

cost of C : $\sum_{e \in E(C)} c(e)$



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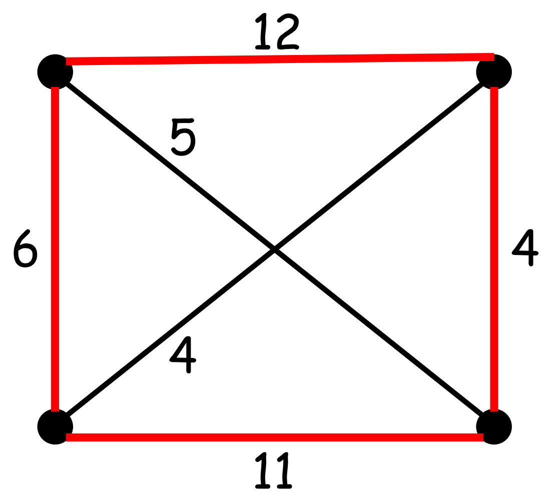

a tour of cost 33

Traveling Salesman Problem

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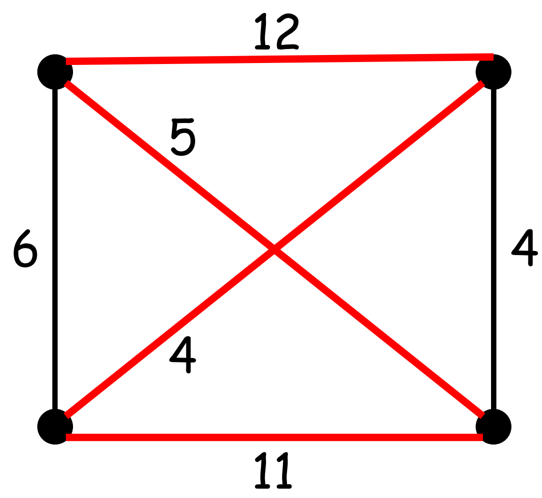

a better tour of cost 32

Traveling Salesman Problem

Input:

undirected complete graph $G=(V,E)$ with non-negative edge costs

Feasible solution:

a cycle C visiting every vertex exactly once

measure (min):

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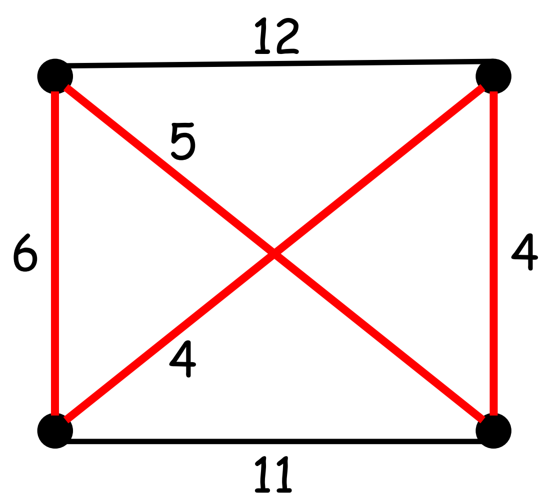

a better tour of cost 19

Traveling Salesman Problem

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Feasible solution:

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metric TSP:

edge costs satisfy the *triangle inequality*

for every u,v,w : $c(u,v) \leq c(u,w)+c(w,v)$

Theorem

For any polynomial time computable function $\alpha(n)$, TSP cannot be approximated within a factor of $\alpha(n)$, unless P=NP.

proof

by contradiction: let A be a $\alpha(n)$-apx algorithm.

We use A to decide Hamiltonian cycle.

Let G be an instance of the Hamiltonian cycle. Define G':

- $G'=(V,E')$ complete;
- $c(u,v)=1$ if $(u,v)\in E(G)$; $c(u,v)=n\alpha(n)$ otherwise

Clearly:

- if G has a Hamiltonian cycle, then optimal TSP tour in G' costs n
- if G does not have a Hamiltonian cycle, then optimal TSP tour is of cost $> n\alpha(n)$

$\Rightarrow$

G has an Hamiltonian cycle iff A returns a tour of cost n

■

Algorithm (metric TSP – factor 2)

1. Find an MST T of G
2. Double every edge of T to obtain an Eulerian graph
3. Find an Eulerian tour τ on this graph
4. Output the tour that visits vertices of G in the order of their first appearance in τ. Let C be this tour.

Theorem

The above algorithm is a 2-approximation algorithm for metric TSP.

proof

removing an edge from an optimal TSP tour gives us a spanning tree of G

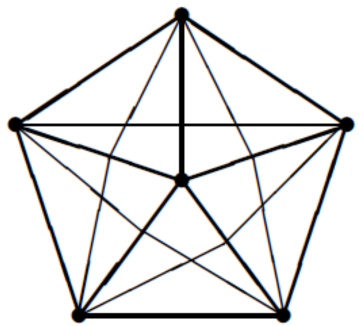
Thus: $\text{cost}(T) \leq OPT$

We have:

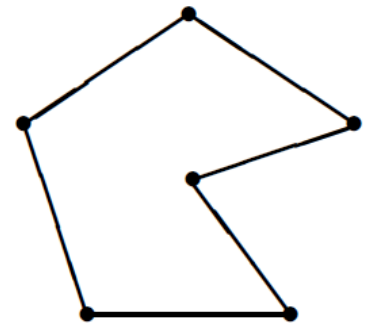
$$\text{cost}(C) \leq \text{cost}(\tau) = 2\text{cost}(T) \leq 2\ OPT$$

■

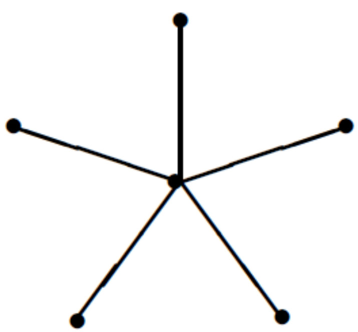
tight example



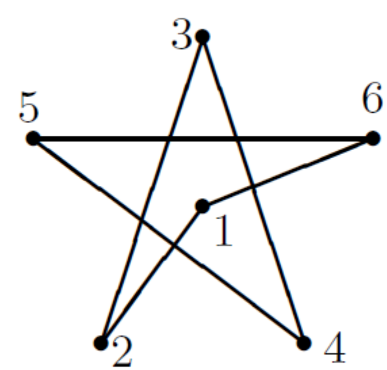
- n vertices
- think edges have cost 1 (star+(n-1)-cycle)
- all the other edges have cost 2



optimal tour of cost OPT=n



feasible MST



returned tour of cost 2n-2
(for the feasible specified order)

idea: find a cheaper Eulerian subgraph/tour

recall:

- a graph is Eulerian iff all vertices have even degree
- in every undirected graph, the number of odd-degree vertices is even

Christofides, 1976

Algorithm (metric TSP – factor 3/2)

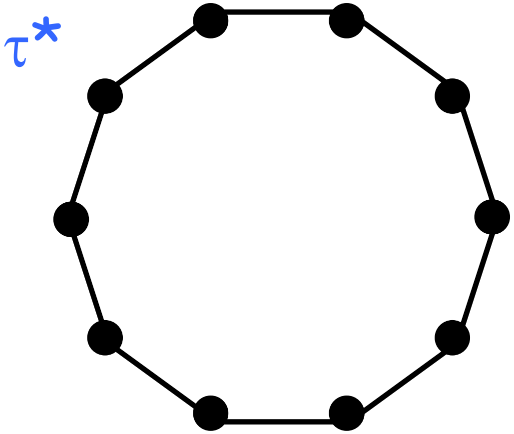
1. Find an MST *T* of *G*
2. Compute a minimum cost perfect matching , *M*, on the set V' of odd-degree vertices of *T*. Add *M* to *T* and obtain an Eulerian graph
3. Find an Eulerian tour τ on this graph
4. Output the tour that visits vertices of *G* in the order of their first appearance in τ. Let *C* be this tour.

Lemma

Let $V' \subseteq V$, such that $|V'|$ is even, and let M be a minimum cost perfect matching on V'. Then, cost(M)$\leq$OPT/2.

proof

let τ^* be an optimal TSP of cost OPT.

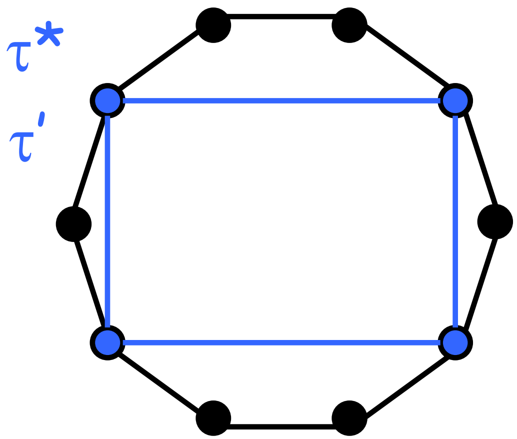


Lemma

Let $V' \subseteq V$, such that $|V'|$ is even, and let M be a minimum cost perfect matching on V'. Then, $\text{cost}(M) \leq OPT/2$.

proof

let τ^* be an optimal TSP of cost OPT.



$\text{cost}(\tau') \leq \text{cost}(\tau^*)$

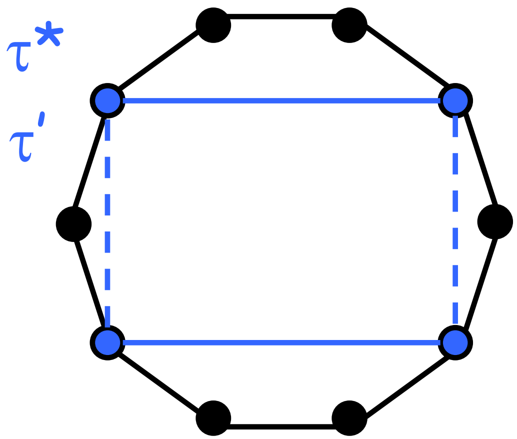
let τ' be the tour on V' obtained by shortcutting τ^*.

Lemma

Let $V' \subseteq V$, such that $|V'|$ is even, and let M be a minimum cost perfect matching on V'. Then, $cost(M) \leq OPT/2$.

proof

let τ^* be an optimal TSP of cost OPT.



$cost(\tau') \leq cost(\tau^*)$

let τ' be the tour on V' obtained by shortcutting τ^*.

τ' is the union of 2 perfect matching on V', say M_1 and M_2.

$$cost(M) \leq \min\{cost(M_1), cost(M_2)\} \leq \tfrac{1}{2}\, cost(\tau') \leq \tfrac{1}{2}\, OPT$$

■

Theorem
Christofides's algorithm is a 3/2-approximation algorithm for metric TSP.

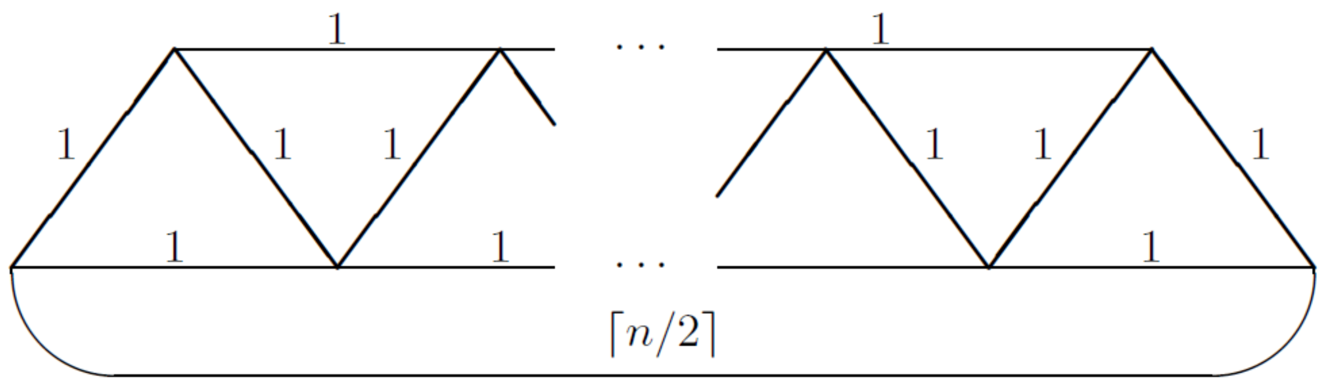
proof

We have:

$$cost(C) \leq cost(\tau) = cost(T) + cost(M) \leq OPT + \tfrac{1}{2}\, OPT \leq 3/2\ OPT$$

■

tight example



- n vertices with n odd
- feasible MST: a path of n-1 edges
- matching: a single edge of cost $\lceil n/2 \rceil$

OPT=n

returned tour of cost n-1+$\lceil n/2 \rceil$)

TSP: state of the art

3/2 [Christofides, 1976]

STOC 2021

1 / 93 100%

A (Slightly) Improved Approximation Algorithm for Metric TSP

Anna R. Karlin[*], Nathan Klein[†] and Shayan Oveis Gharan[‡]

University of Washington

March 16, 2022

Abstract

For some $\epsilon > 10^{-36}$ we give a randomized $3/2 - \epsilon$ approximation algorithm for metric TSP.