Advanced topics on Algorithms

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What: 3 topics, 4 lectures per topic

approximation algorithms:
- well-established field
- widely used approach for (NP-)hard problems
- cool techniques: rounding, dual-fitting, primal-dual approach

parameterized algorithms:
- multivariate analysis of algorithms
- refined notions of efficiency and hardness
- cool techniques: color coding, kernelization, treewidth

advanced data structures:
- major core of algorithmic problems
- DSs for geometric data, big data, static trees, strings
- cool techniques: fractional cascading, indirection

lecturer of this part:
Alessandro Straziota

How (to get credits)

- attend lectures
- final oral exam and/or class presentation (of uncovered material)

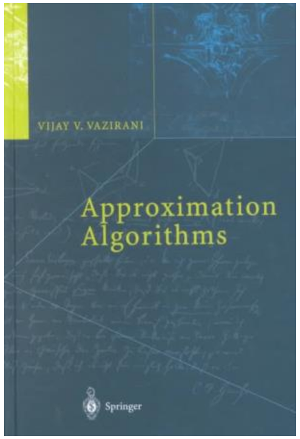
Why

- expand your background: wider view of the huge world of algorithms
- useful: be better theorists and practitioners
- fun: amazing material and techniques

any question?

Approximation algorithms:
Episode I
(pilot)

main reference:



Def.

An α-approximation algorithm for an optimization problem is a polynomial-time algorithm that for all instances of the problem produces a solution whose value is within a factor of α the value of an optimal solution.

α: approximation ratio or approximation factor

minimization problem:

- $\alpha \geq 1$
- for each instance x, the returned solution s has cost $cost(s) \leq \alpha\ OPT(x)$

maximization problem:

- $\alpha \leq 1$
- for each instance x, the returned solution s has $value(s) \geq \alpha\ OPT(x)$

minimum Vertex Cover problem

min cardinality Vertex Cover problem

Input:

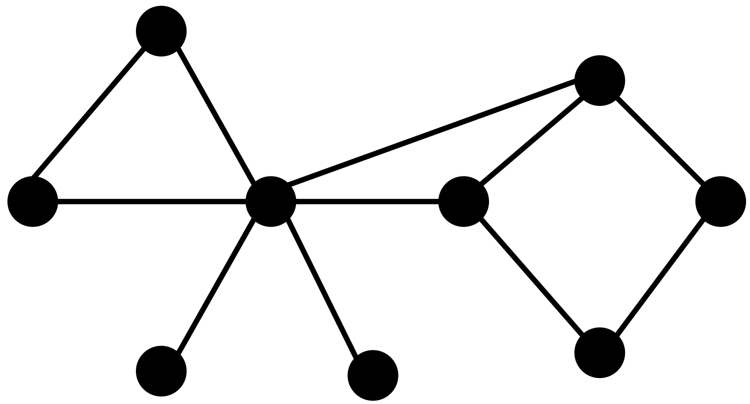
an undirected graph $G=(V,E)$

Feasible solution:

$U \subseteq V$ such that every edge $(u,v) \in E$ is covered, i.e. $u \in U$ or $v \in U$

measure (min):

cardinality of U



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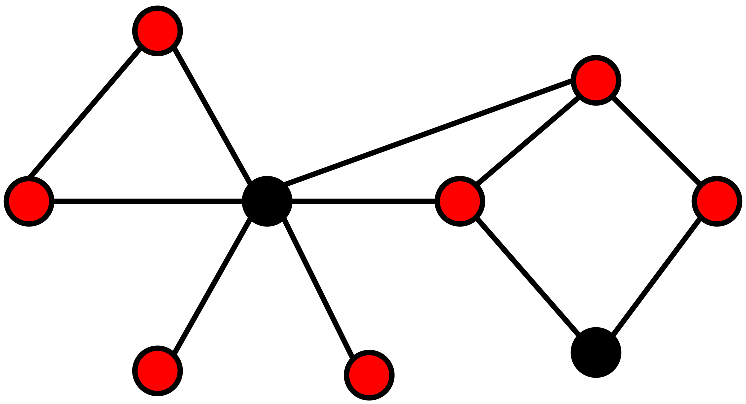

a vertex cover of size 7

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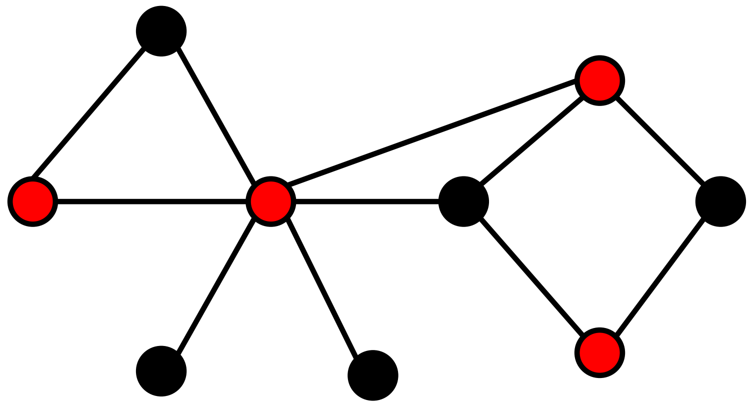
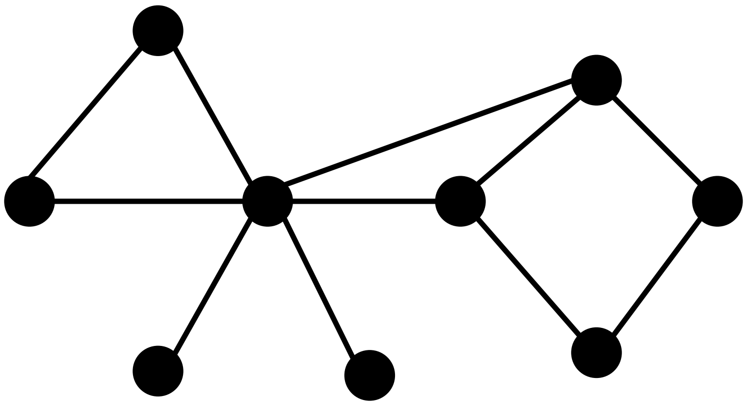


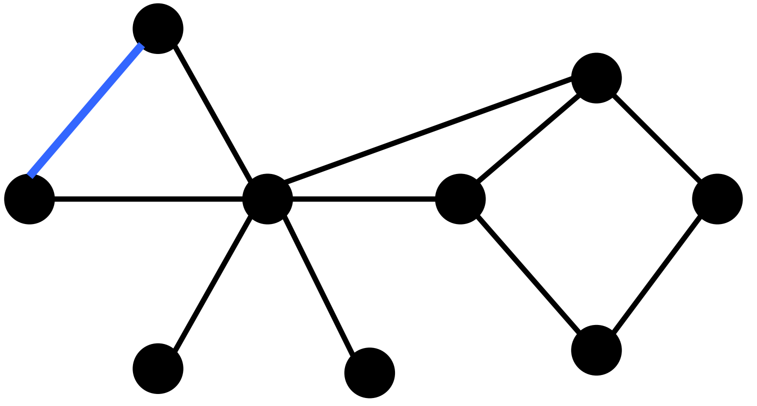
a better vertex cover of size 4

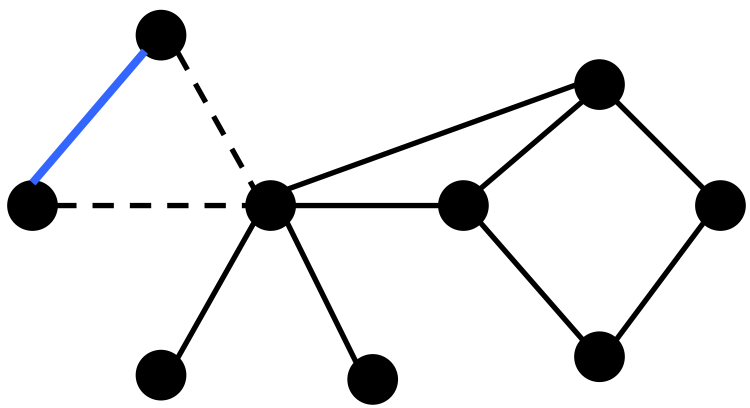
Def.
Given a graph $G=(V,E)$, a subset of edges $M \subseteq E$ is a matching if no two edges in M share an endpoint.

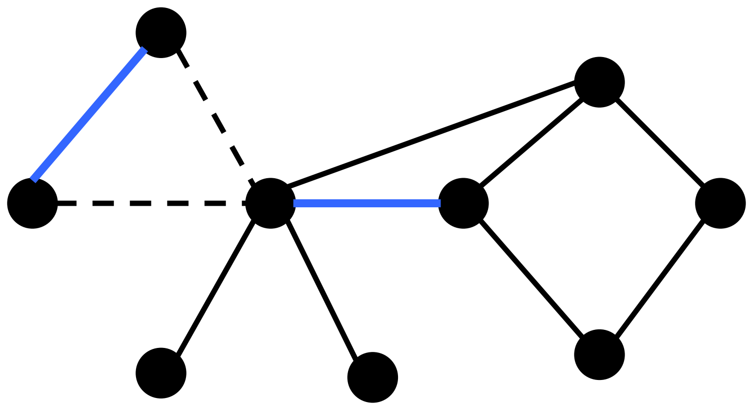
Def.
A matching $M \subseteq E$ is maximal if for every $e \in E \setminus M$, $M \cup \{e\}$ is not a matching.

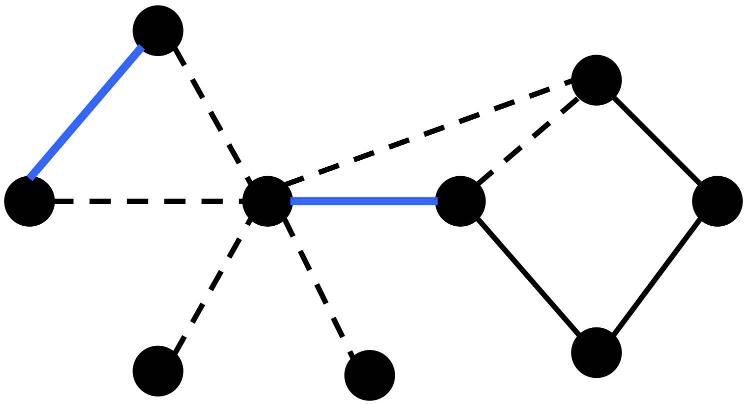
Algorithm 1.2 (Cardinality vertex cover)
Find a maximal matching in G and output the set of matched vertices.

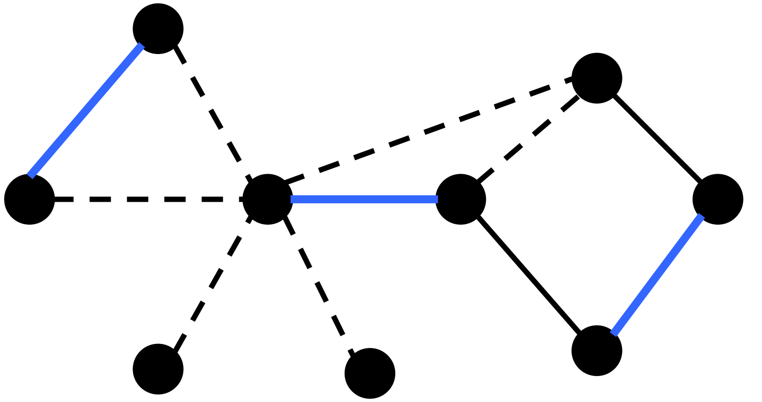


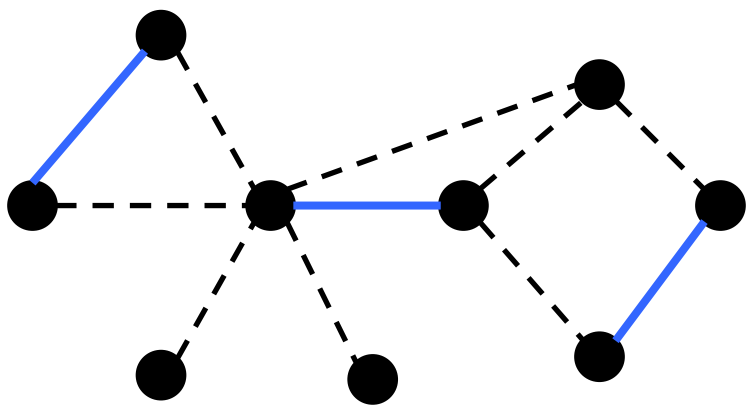


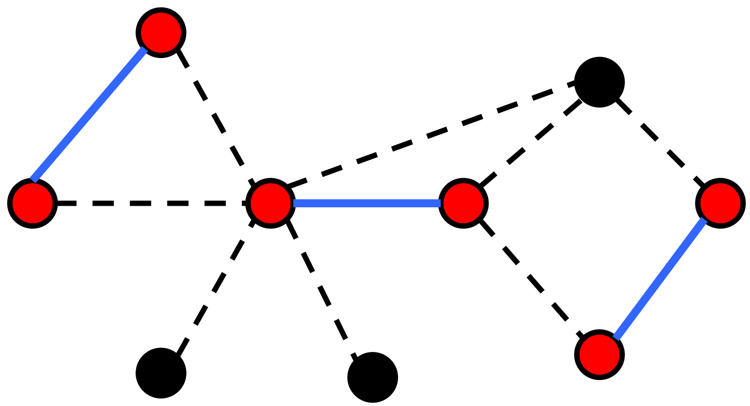












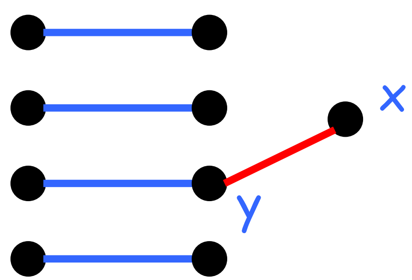
the computed vertex
cover of size 6

Lemma

The algorithm returns a feasible VC.

proof

let $M \subseteq E$ be the maximal matching computed by the algorithm.



edges in M are clearly covered

for maximality of M any other edge (x,y) shares and endpoint with some edge in M...

...and thus it is covered

■

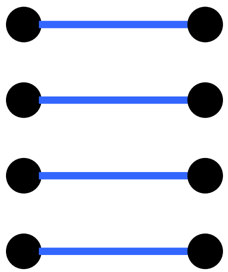
Theorem

The algorithm is a 2-approximation algorithm for the VC problem.

proof

The returned solution is a feasible VC (previous lemma)

let $M \subseteq E$ be the maximal matching computed by the algorithm, and U the corresponding VC.



any optimal solution must have size OPT at least $|M|$

Lower bounding scheme: the size of any maximal matching is a lower bound to the size of an optimal VC

thus:

$$|U|=2|M|\leq 2OPT$$

■

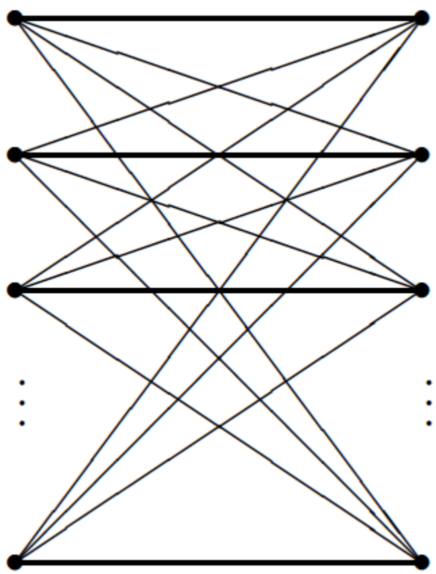
Three important questions:

1. Can the approximation ratio of Algorithm 1.2 be improved by a better analysis?

2. Can an approximation algorithm with a better apx ratio be designed using the lower bounding scheme of Algorithm 1.2, i.e. the size of a maximal matching?

3. Is there some other lower bounding scheme that can lead to a better approximation algorithm for VC?

1. Can the approximation ratio of Algorithm 1.2 be improved by a better analysis? NO



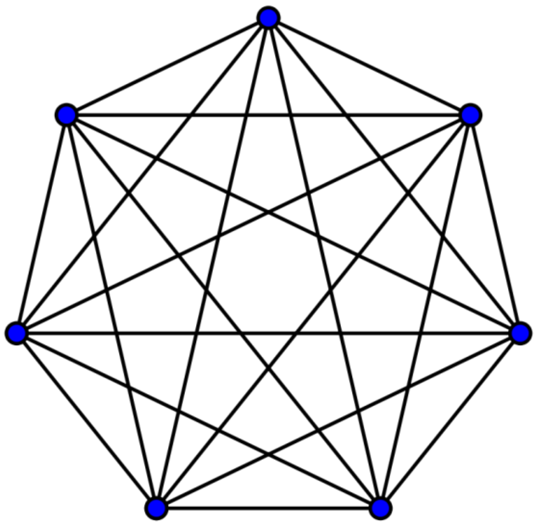
tight example

complete bipartite graph $K_{n,n}$

Algorithm 1.2 will pick all the $2n$ vertices

OPT=n (one side is an optimal VC)

2. Can an approximation algorithm with a better apx ratio be designed using the lower bounding scheme of Algorithm 1.2, i.e. the size of a maximal matching? NO



complete graph K_n where n is odd

size of any maximal matching is $(n-1)/2$

OPT=$n-1$

3. Is there some other lower bounding scheme that can lead to a better approximation algorithm for VC? OPEN

Partial answer:

Theorem

Assuming the unique games conjecture holds, if there exists an α-approximation algorithm for the VC problem with $\alpha<2$, then P=NP.

roughly: a particular problem (called unique games) is NP-hard

Minimum Set Cover problem

minimum Set Cover problem

Input:

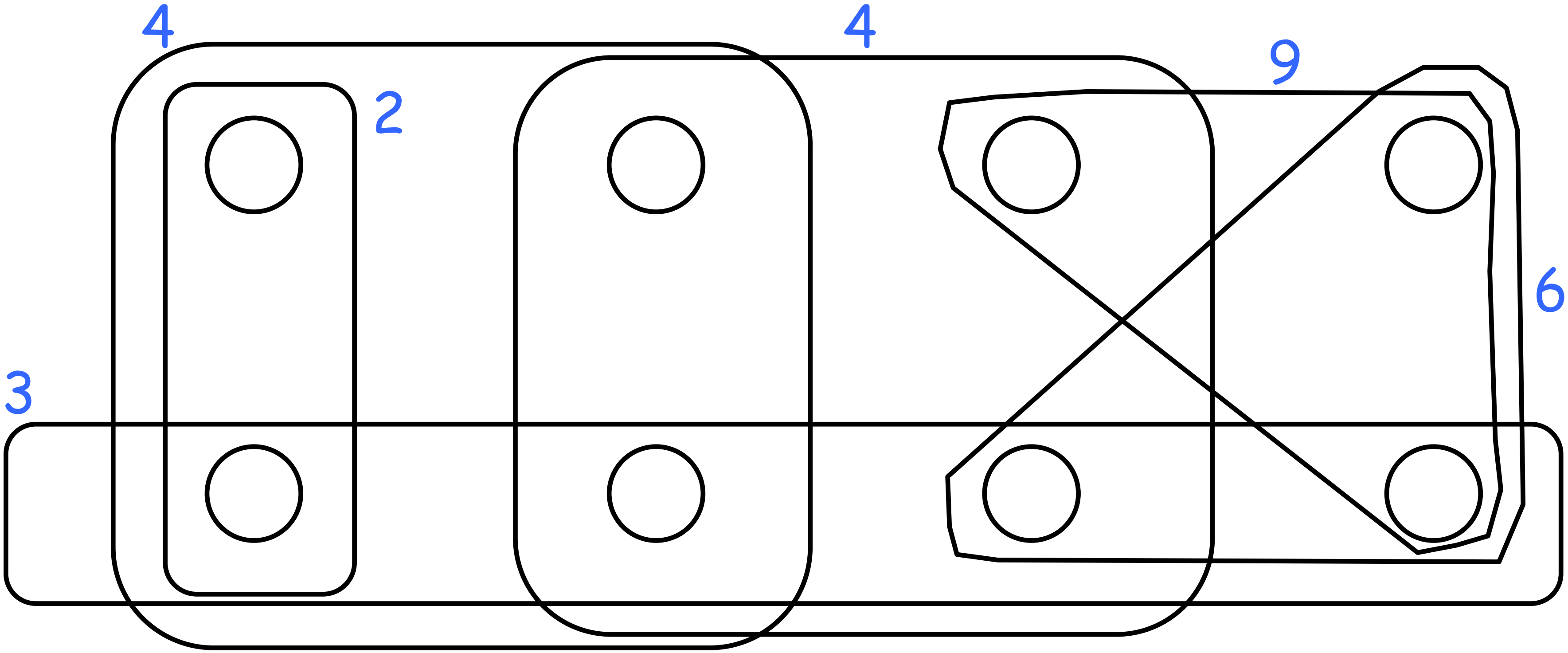
- universe U of n elements
- a collection of subsets of U, $\mathcal{S}=\{S_1,\ldots,S_k\}$
- each $S\in\mathcal{S}$ has a positive cost $c(S)$

Feasible solution:

a subcollection $\mathcal{C}\subseteq\mathcal{S}$ that covers U (whose union is U)

measure (min):

cost of $\mathcal{C}$: $\sum_{S\in\mathcal{C}} c(S)$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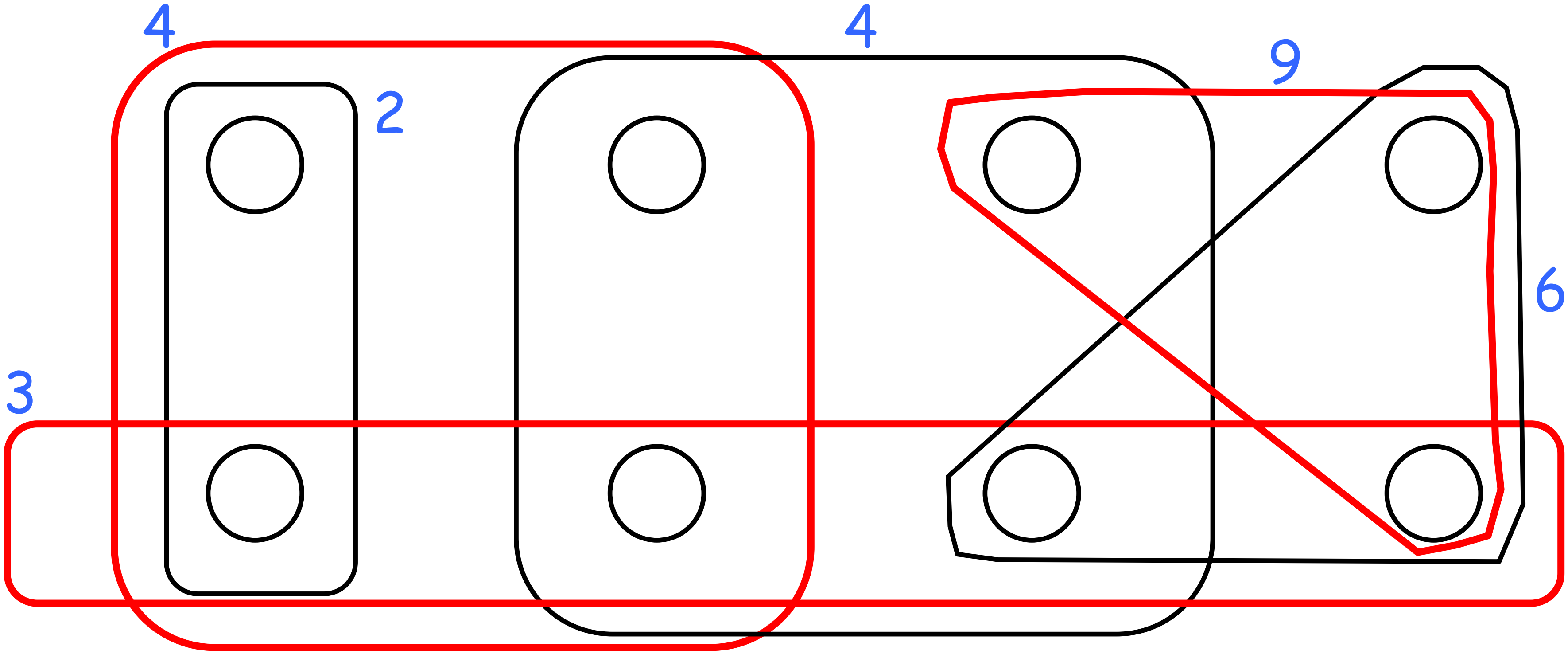
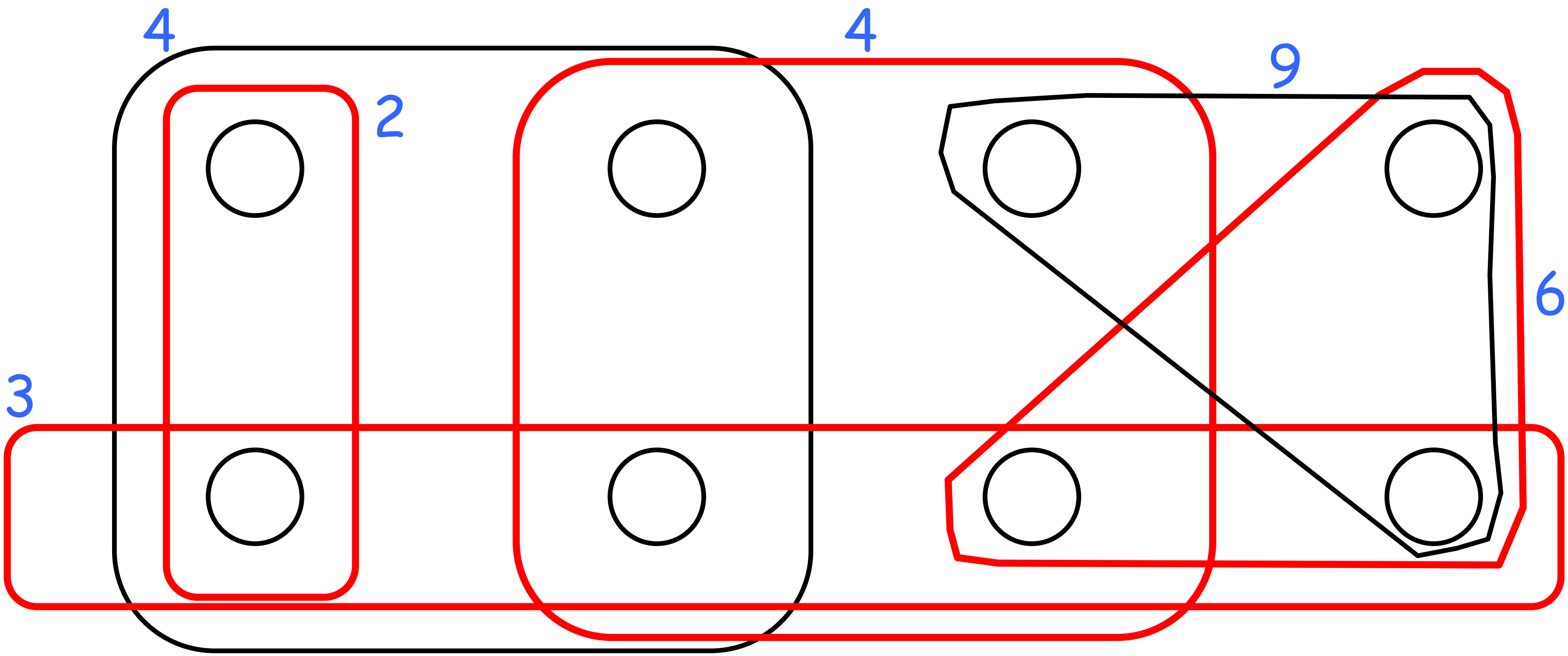
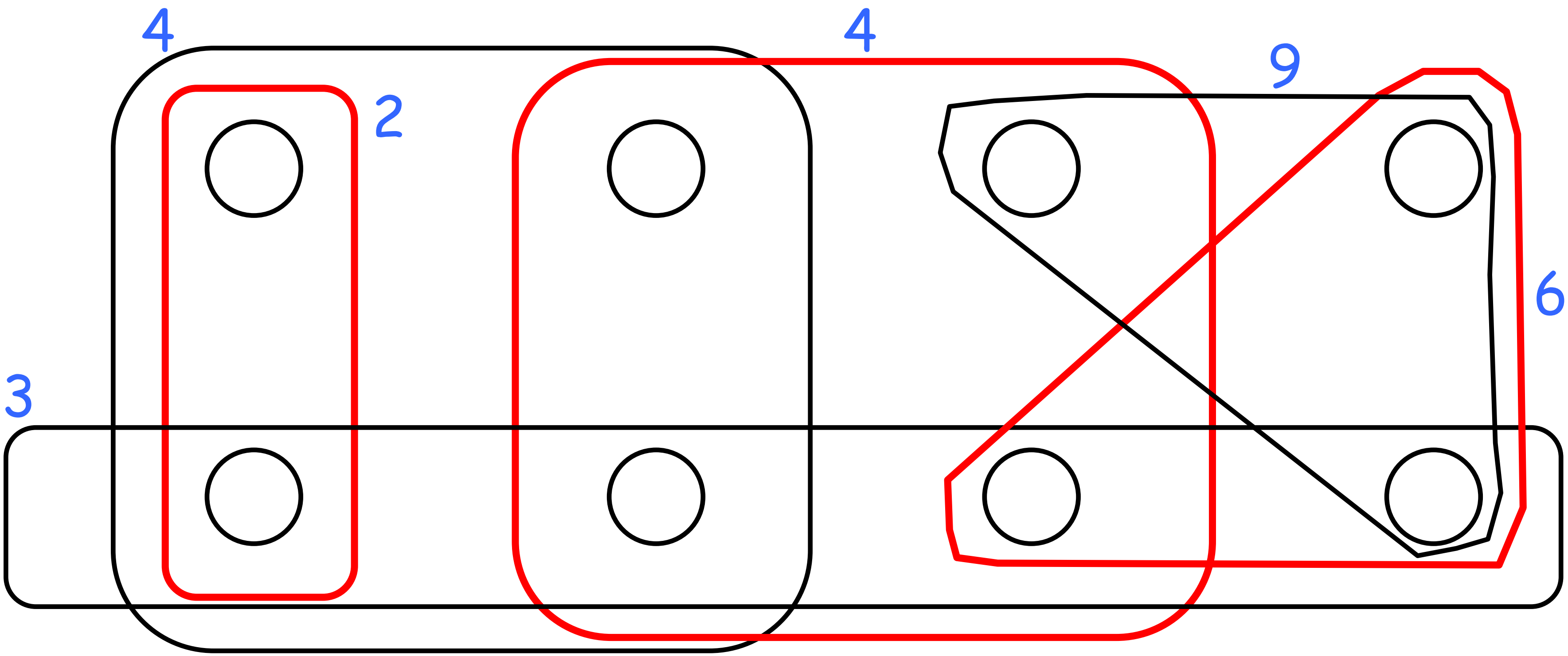


a set cover of
cost 16



a better set
cover of cost 15



a better set
cover of cost 12

greedy strategy: pick the most cost-effective set and remove the covered elements, until all elements are covered.

Let C be the set of elements already covered.

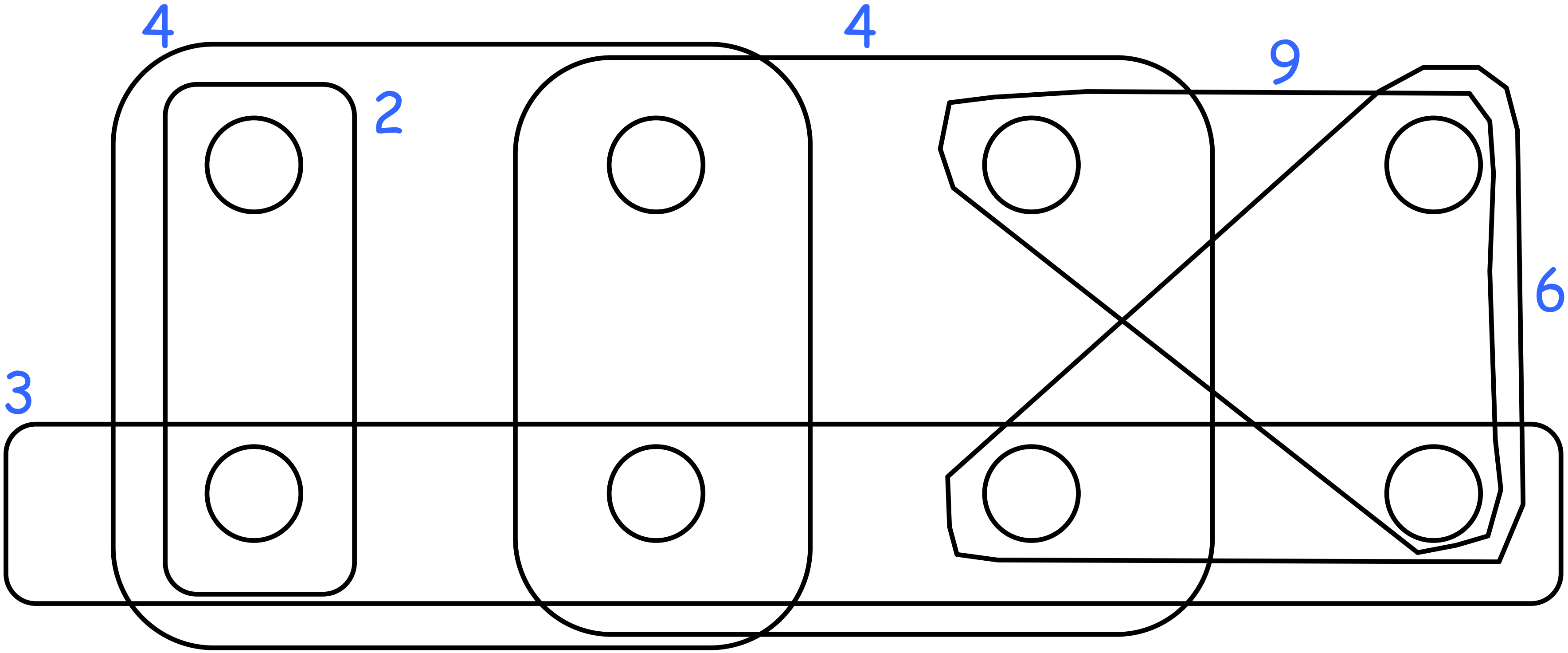
cost-effectiveness of S: c(S)/|S-C|

average cost at which S covers new elements

Algorithm 2.2 (Greedy set cover algorithm)

1. $C \leftarrow \emptyset$
2. While $C \neq U$ do
   Find the most cost-effective set in the current iteration, say S.
   Let $\alpha = \frac{\text{cost}(S)}{|S-C|}$, i.e., the cost-effectiveness of S.
   Pick S, and for each $e \in S - C$, set $\text{price}(e) = \alpha$.
   $C \leftarrow C \cup S$.
3. Output the picked sets.

average cost at which e is covered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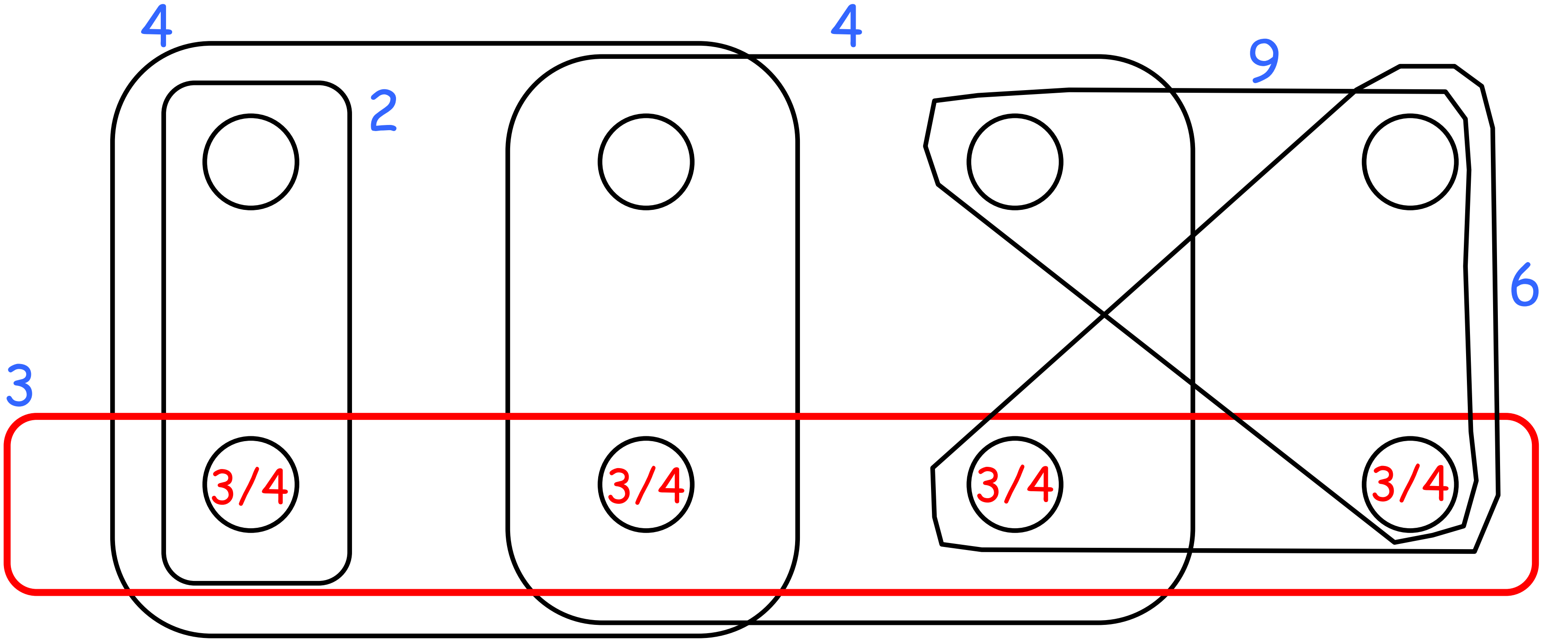
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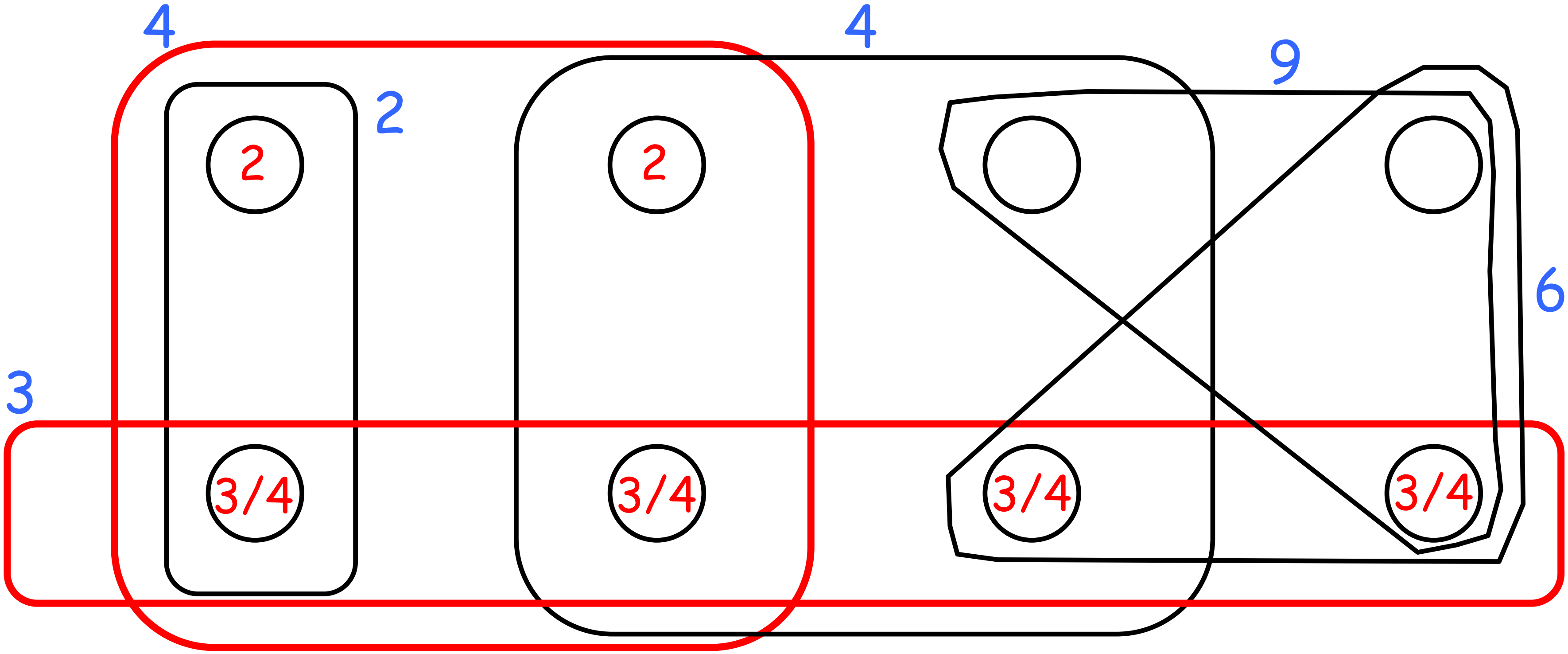
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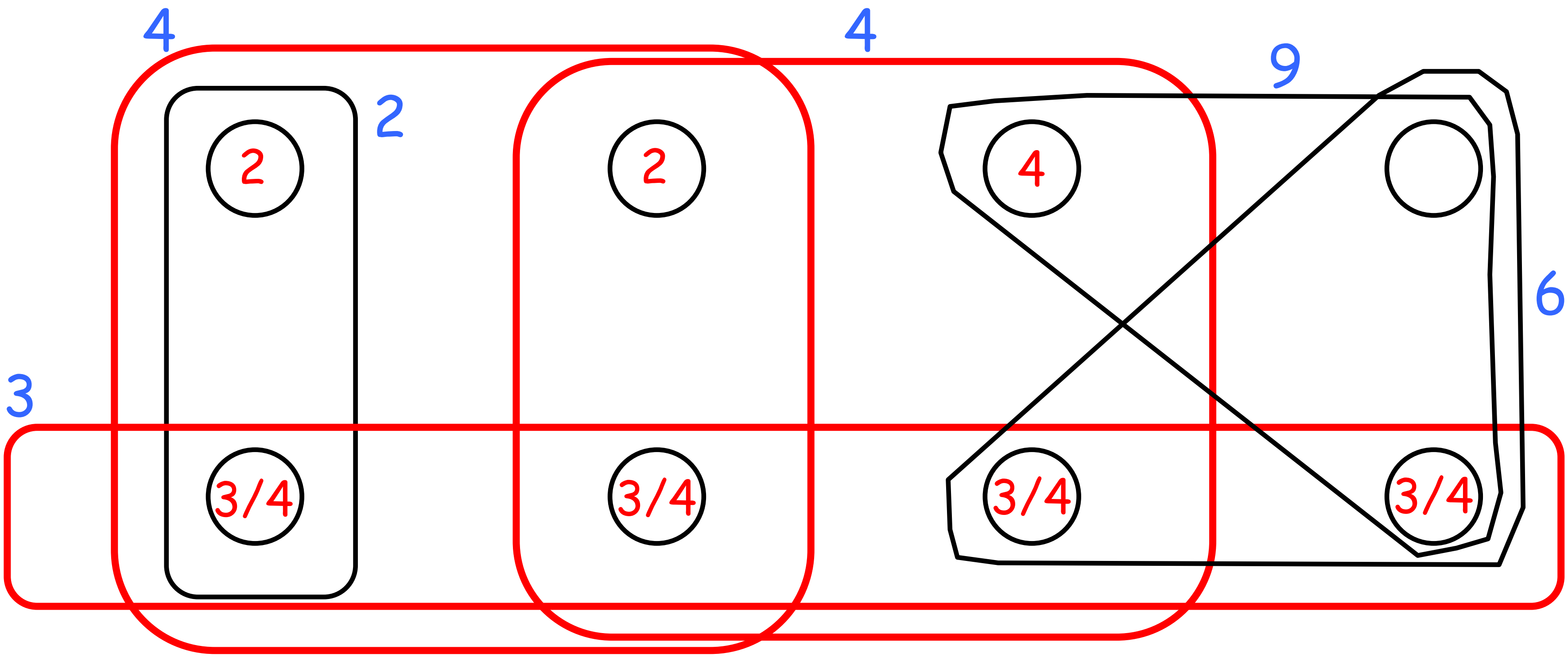

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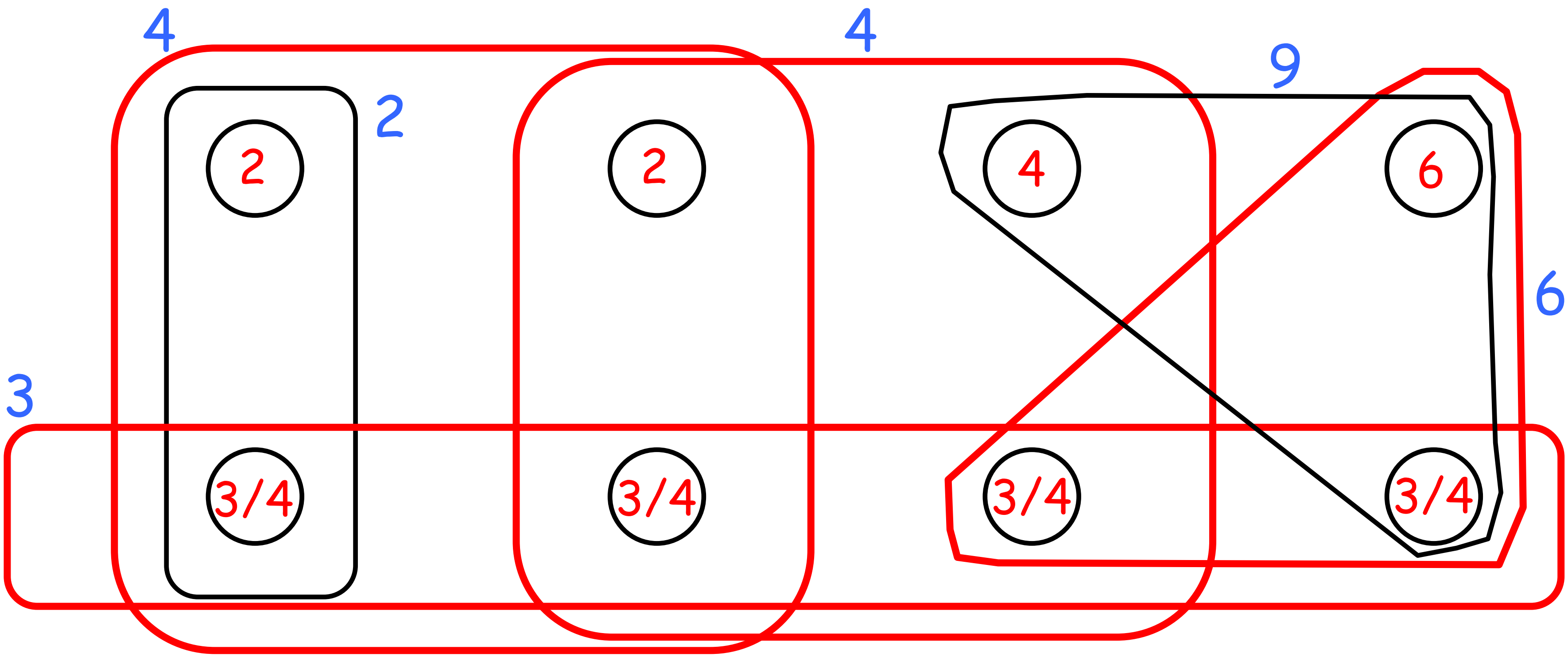


the computed set
cover of cost 17

Number the elements of U in order in which they were covered, resoving ties arbitrarily. Let $e_1,\ldots, e_n$ this numbering.

Lemma

For each $k\in\{1,\ldots,n\}$, $\text{price}(e_k) \leq OPT/(n-k+1)$

proof

at any iteration, the leftovers sets of the optimal solution can cover all the remaining elements $C'=U-C$ at cost at most OPT.

one of these leftovers sets has cost-effectiveness at most $OPT/|C'|$

at iteration in which e_k is covered, C' contains at least n-k+1 elements.

by the greedy choice:

$$\text{price}(e_k) \leq OPT/|C'| \leq OPT/(n-k+1)$$

■

Theorem

The greedy algorithm is H_n factor approximation algorithm for the minimum Set Cover problem, where $H_n = 1+1/2+\ldots+1/n$.

proof

Since the cost of each picked set is distributed among the new covered elements:

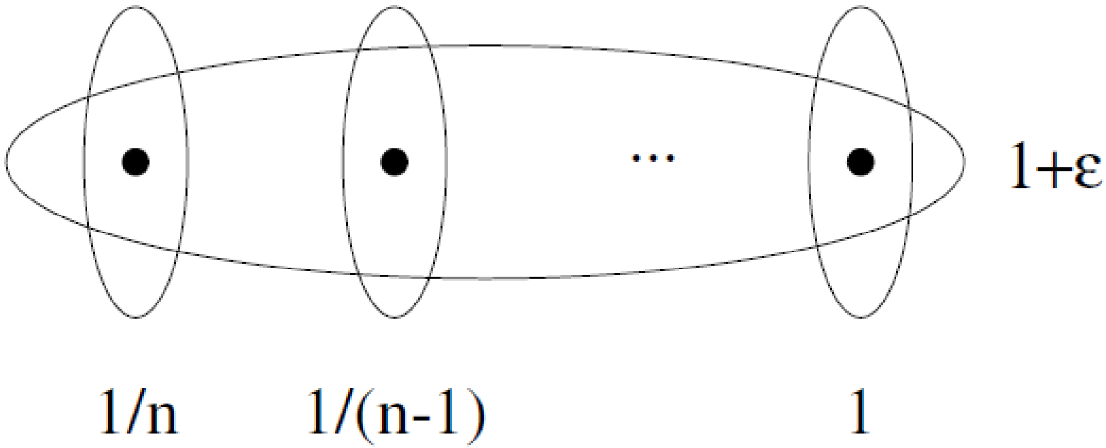
$$\text{cost of the cover} = \sum_{k=1}^{n} \text{price}(e_k) \le \sum_{k=1}^{n} OPT/(n-k+1) \le H_n OPT$$

∎

$$H_n = \sum_{k=1}^{n} 1/k \le \ln n + 1$$

n-th *harmonic number*

tight example



the greedy alg computes a cover having cost H_n

OPT=$1+\varepsilon$

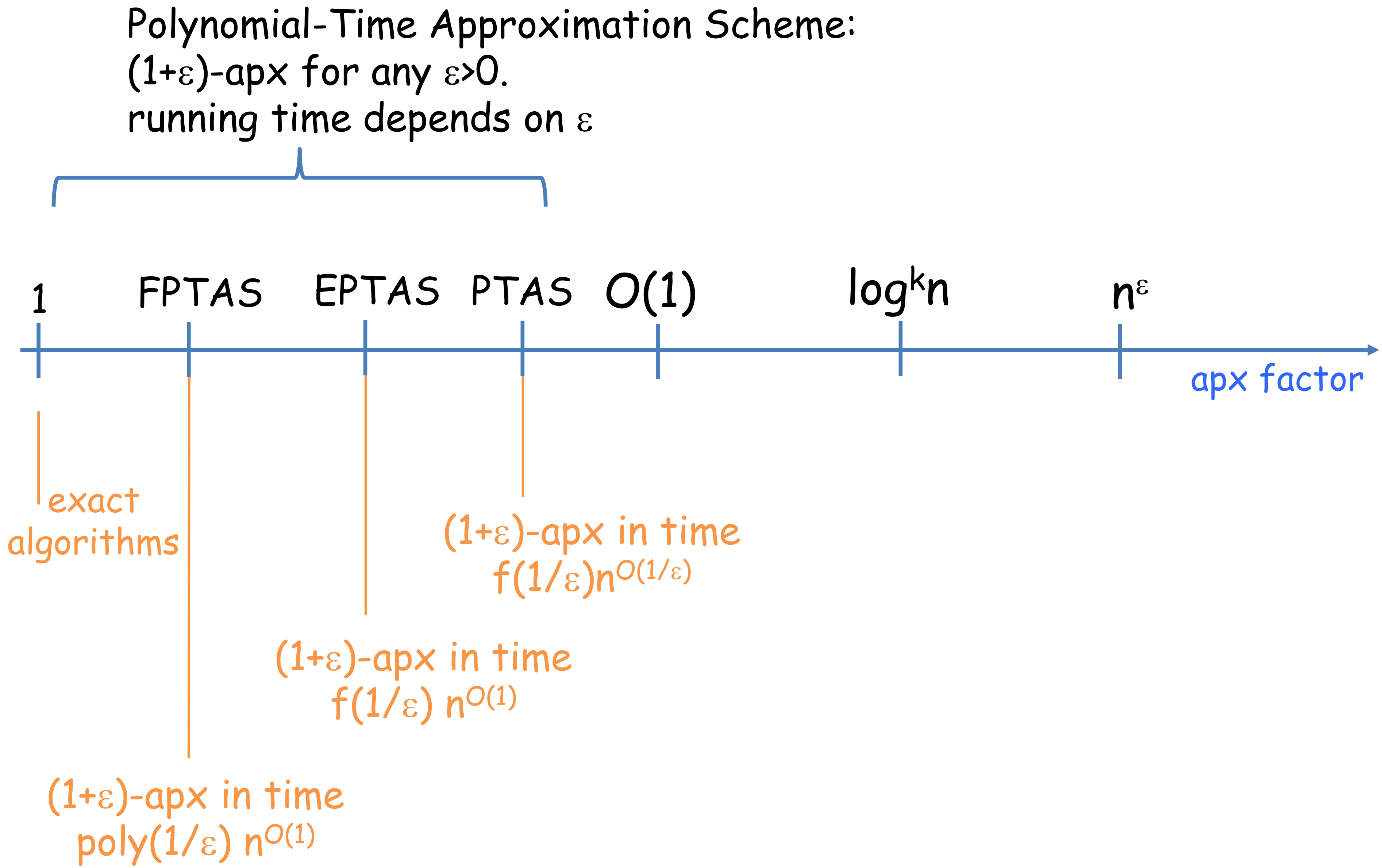
Theorem

There exists some constant $c>0$ such that if there exists a $(c \ln n)$-apx algorithm for the unweighted SC problem, then P=NP.

Theorem

If there exists a $(c \ln n)$-apx algorithm for the unweighted SC problem, for some constant $c<1$, then there is an $O(n^{O(\log \log n)})$-time alg for each NP-complete problem.

the approximation game: get better and better approximation factor



Application to shortest superstring

the shortest superstring problem

Input:

a set of n strings over a finite alphabet $S=\{s_1,\ldots,s_n\}$

Feasible solution:

a string s that contains each s_i as a substring

measure (min):

length of s

notice: w.l.o.g. we can assume no string s_i is a substring of another s_j

S={abbc, cccaab, bccc}

a solution of length 12: abbcccaabccc

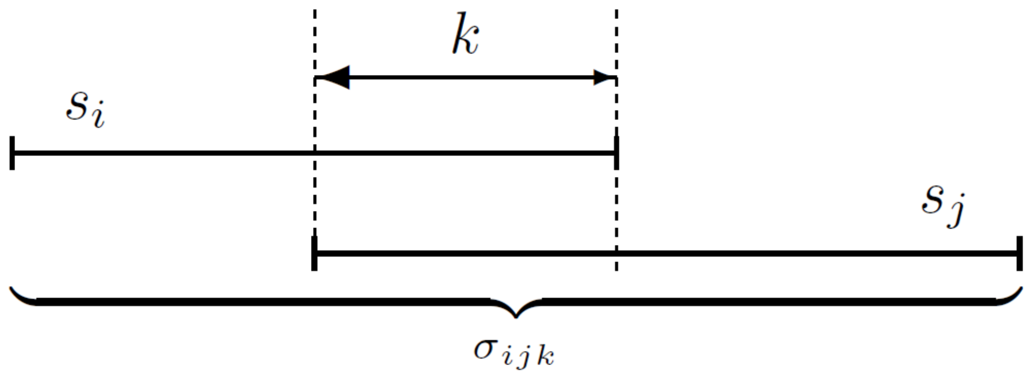
a better solution of length 9: bcccaabbc

reducing the problem to set cover

for s_i, $s_j \in S$, and $k>0$

if the last k symbols of s_i are the same as the first k symbols of s_j,

let σ_{ijk} be the string obtained by overlapping those k positions



let M be the set of the strings σ_{ijk} for all valid choices of i, j, k.

for a given string π,

let $\text{set}(\pi)=\{s\in S : s \text{ is a substring of } \pi\}$

the Set Cover instance:

- the set of objects is S
- collection of subsets: we have $\text{set}(\pi)$ for each $\pi \in S\cup M$ of cost $|\pi|$

Algorithm 2.10 (Shortest superstring via set cover)

1. Use the greedy set cover algorithm to find a cover for the instance $\mathcal{S}$. Let $\mathrm{set}(\pi_1), \ldots, \mathrm{set}(\pi_k)$ be the sets picked by this cover.
2. Concatenate the strings $\pi_1, \ldots, \pi_k$, in any order.
3. Output the resulting string, say s.

Theorem

The above algorithm is a $2H_n$-approximation algorithm for the shortest superstring problem.

proof

since we have computed a set cover, every $s \in S$ is a substring of some π_j

⇒ the computed string is a feasible superstring

OPT: the value of the optimal solution for the shortest superstring

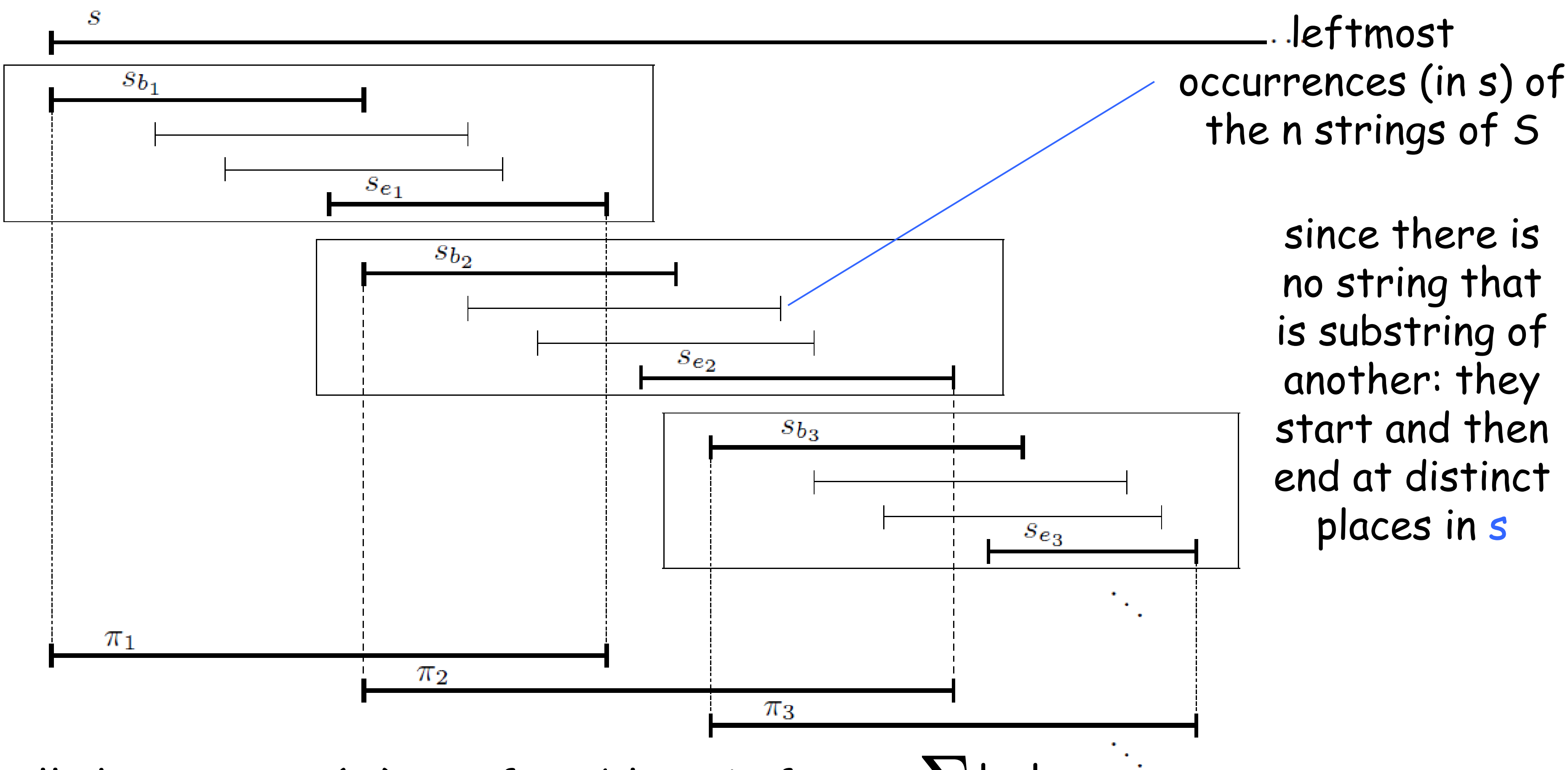
OPT_{SC}: the value of the optimal solution for the SC instance

claim: $OPT_{SC} \leq 2$ OPT.

⇒ the computed string has length $\leq H_n\, OPT_{SC} \leq 2H_n\, OPT$

Let s be the optimal superstring of length OPT

we show there is a feasible SC of cost at most 2 OPT



since there is no string that is substring of another: they start and then end at distinct places in s

all the sets set(π_i) is a feasible SC of cost $\sum_i |\pi_i|$

notice: π_i and π_{i+2} do not overlap

$$\sum_i |\pi_i| \leq 2\ |s| \leq 2\ OPT$$