MAP, for a single variable, using a Laplace prior: $x_i \sim \mathcal{N}(\mu, \sigma^2)$

$$\mu \sim \frac{1}{2b} e^{-|\mu|/b} = L(\mu, b)$$

We skip steps and go to:

$$\frac{\partial \mathcal{L}}{\partial \mu} = 0 \Leftrightarrow 0 = \sum_{i=1}^{N} \frac{(x_i - \mu)}{\sigma^2} + \frac{\partial}{\partial \mu} \log L$$

$$\frac{\partial}{\partial \mu} \log L = \frac{\partial}{\partial \mu} \left(\log \frac{1}{2b} - \frac{|\mu|}{b} \right)$$

$$= 0 - \frac{1}{b} \underbrace{\frac{\partial}{\partial \mu} |\mu|}_{\text{sub-gradient}}$$

sub-gradient: $\begin{cases} +1 & (\mu > 0) \\ 0 & (\mu = 0) \\ -1 & (\mu < 0) \end{cases}$

let's call this $\text{sign}(\mu)$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \mu} = 0 \Leftrightarrow 0 = \sum_{i=1}^{N} x_i - N\mu - \frac{\sigma^2}{b} \text{sign}(\mu)$$

$$\Rightarrow \quad \mu + \frac{\sigma^2}{Nb} \text{sign}(\mu) = \frac{1}{N} \sum_{i=1}^{N} x_i$$

$$\Rightarrow \quad \text{i.e. } \mu = \underbrace{\mu_{MSE}}_{\text{empirical mean}} - \frac{\sigma^2}{Nb} \text{sign}(\mu)$$

$$\Rightarrow \begin{cases} \text{if } \mu > 0 \text{ (self-consistently)}, & \mu = \mu_{MSE} - \frac{\sigma^2}{Nb} \times 1 \\ \text{if } \mu < 0 \text{ (self-consistently)}, & \mu = \mu_{MSE} + \frac{\sigma^2}{Nb} \end{cases}$$

What if $\mu_{MSE} \in \left[-\frac{\sigma^2}{Nb}, \frac{\sigma^2}{Nb} \right]$?

$\Rightarrow$ Then, we get $\mu = 0$! See sub-gradients

Note this remark:

- if $\mu > 0$, $\mu_{MSE} - \frac{\sigma^2}{Nb} \times (+1) > 0$, $\mu_{MSE} > \frac{\sigma^2}{Nb}$

- if $\mu < 0$, $\mu_{MSE} - \frac{\sigma^2}{Nb}(-1) < 0$, $\mu_{MSE} < -\frac{\sigma^2}{Nb} < 0$

$\rightarrow$ In both cases, $|\mu_{MSE}| > \frac{\sigma^2}{Nb}$

And in particular, $\text{sign}(\mu_{MSE}) = \text{sign}(\mu)$.

So we simplify to:

$$\mu = \mu_{MSE} - \frac{\sigma^2}{Nb}\,\text{sign}(\mu_{MSE})$$

which is a closed form.

- if $\mu = 0$, then $\left(\mu_{MSE} - \frac{\sigma^2}{Nb}\,\partial\text{sign}(\mu)\right) = 0$

and

$$\mu_{MSE} - \frac{\sigma^2}{Nb} \times \underbrace{[-1, 1]}_{\text{we can choose}}$$

$$\Leftrightarrow$$

$$\mu_{MSE} \in \left[-\frac{\sigma^2}{Nb}, \frac{\sigma^2}{Nb}\right]$$

Ccl°:

$$\mu = \begin{cases} 0 & \text{if } \mu_{MSE} \text{ in } \left[-\frac{\sigma^2}{Nb}, \frac{\sigma^2}{Nb}\right] \\ \mu_{MSE} - \frac{\sigma^2}{Nb}\,\text{sign}(\mu_{MSE}) & \text{(else)} \end{cases}$$