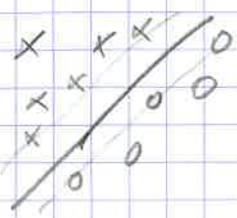
SVM₁ Separable case.



First, a remark: • $w\vec{x}+b=0$ defines a plane

• $d_{\pm} = \dfrac{\vec{w}\vec{x}+b}{\|w\|_2}$ is the distance between the point $\vec{x}$ and the plane $(\vec{w}, b)$. (signed distance)

• On one side of the plane, $d_{\pm} \geqslant 0$, on the other side, we have $d_{\pm} \leqslant 0$.

• If $\vec{x}_n$ is well classified, $\text{sign}(d_{\pm}) = \pm 1 = t_n$.

• So, for well-classified points, $d_{\pm}(\vec{x}_n) \cdot t_n = \dfrac{t_n(w\vec{x}_n+b)}{\|w\|_2} = d$ is the (positive) distance between $\vec{x}_n$ and the plane.

$$d = \frac{t_n(wx_n+b)}{\|w\|} \geqslant 0 \quad \text{is a distance (positive)}$$

• Rescaling: the plane $(\vec{w}, b)$ is the same as the plane encoded in $(c\vec{w}, cb)$, $(\forall c \in \mathbb{R}^{+*} \ (c>0))$:

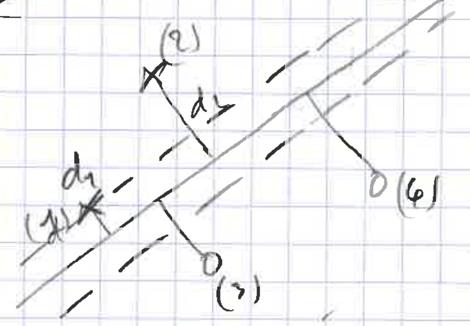
we can check this: $d' = \dfrac{t_n(\vec{w}.\cancel{c}x_n + b\cancel{c})}{\cancel{c}\|w\|} = d$.

• So, we can always choose c (rescale $\|w\|$) such that we have $t_n(\vec{w}\vec{x}_n + b) \geqslant 1$, $\forall n$ (the choice of "1" is arbitrary)

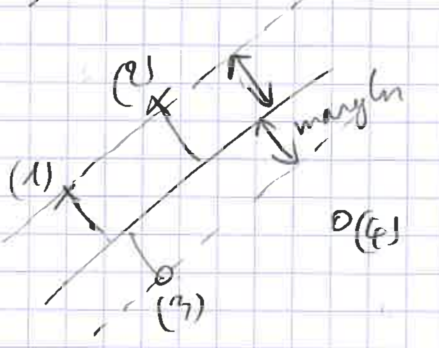
• There are many solutions to correctly classify a linearly separable set of data.

• We call "margin" the minimal distance between the points and the plane:

$$\text{margin} = \min_n (d(\vec{x}_n)) = \min_n \left(\frac{t_n(\vec{w}\vec{x}_n+b)}{\|\vec{w}\|_2} \right)$$



d_1 is minimal, so the margin is $marg = d_1 = d(\vec{x}_1)$ in this example.



here, d_3 is the smallest dist (almost equal to d_1), so

margin $= d_3 = d(\vec{x}_3)$.

This $(\vec{w}, b)$ seems better than the first one.

- The core idea of SVM (also re-named "Separators with Vast Margin") is to find the solution with the largest margin, which is expected to work better than those with small margin.

- So, we define the SVM solut° (among all the sol° to the pbm of linear separation) as:

$$\underbrace{w^*_{SVM} = \text{argmax}_{w,b} \left(\underbrace{\min_n \left(\underbrace{\frac{t_n(w \vec{x}_n + b)}{\|w\|_2}}_{\text{distance}} \right)}_{\text{margin}} \right)}_{\text{the largest margin (or its corresp. } \vec{w}\text{, rather)}}$$

From there, it's just maths (optimizat°).

$$b^*, w^*_{SVM} = \underset{w,b}{\mathrm{argmax}}\left(\min_n \left(\frac{t_n(\vec{w}\vec{x}_n + b)}{\|w\|_2} \right)\right)$$

$$= \underset{w,b}{\mathrm{argmax}}\left(\frac{1}{\|w\|_2} \min_n \left(\underbrace{\qquad\qquad}_{} \right)\right)$$

the distances d_n can be set to ≥ 1
by rescaling $\vec{w}, b$ by a constant $c > 0$.

$$= \underset{w,b \text{ such that } t_n(w\vec{x}_n + b) \geq 1, \forall n}{\mathrm{argmax}} \left(\frac{1}{\|w\|_2} \right)$$

because $\frac{1}{2x^2}$ is $\Downarrow$ for all $x > 0$, argmax becomes argmin

$$= \underline{\mathrm{argmin}} \left(\frac{1}{2}\|w\|_2^2 \right)$$

$$= \underset{w,b}{\mathrm{argmin}} \Big(\underbrace{\frac{1}{2}\|w\|_2^2 - \sum_n \overset{\geq 0}{\alpha_n} \big(\overset{\geq 0}{t_n(wx_n + b) - 1} \big)}_{\hat{=} \mathcal{L}} \Big)$$

$$= \mathrm{argmin}_{w,b} (\mathcal{L})$$

The α_n are positive, they are __Lagrange multipliers__.

We can now solve for $\vec{w}, b$:

$$\vec{\nabla}_w \mathcal{L} = \vec{0} \iff \frac{\cancel{2}}{\cancel{2}}\vec{w} - \sum_n \alpha_n t_n (\vec{x}_n) = \vec{0}$$

$$\iff \vec{w} = \sum_n \alpha_n t_n \vec{x}_n$$

$$\frac{\partial \mathcal{L}}{\partial b} = 0 \iff \sum_n \alpha_n t_n = 0$$

We now need to find the α_n, so we just moved (re-wrote) the problem. At its $\nabla \mathcal{L} = \vec{0}$ point (extremum), $\mathcal{L}$ can be written: (we replace $\vec{w}$ with its expression in α_n)

$$\mathcal{L} = \frac{1}{2}\|w\|_2^2 - \sum_n \vec{w}\, \underbrace{\alpha_n t_n x_n}_{=\vec{w}} + \sum_n \alpha_n t_n + \sum_n \alpha_n$$

$$= \underbrace{\frac{1}{2}\vec{w}\cdot\vec{w} - \vec{w}\cdot\vec{w}}_{-1/2\,\vec{w}\cdot\vec{w}} + \underbrace{\sum_n \alpha_n t_n}_{=0} + \sum_n \alpha_n$$

$$= -\frac{1}{2}\left(\sum_n \alpha_n x_n t_n\right)\left(\sum_m \alpha_m x_m t_m\right) + 0 + \sum_n \alpha_n$$

$$= \sum_n \alpha_n - \frac{1}{2}\sum_n\sum_m \left(\alpha_n \alpha_m t_n t_m \underbrace{\vec{x}_n \vec{x}_m}\right)$$

dual form.

$\vec{x}_n \vec{x}_m$ can be replaced with $K(x_n, x_m)$

This is then the kernelized SVM
Note that we needed this dual form.

. At prediction time, we would have:

$$y^{pred} = sign(\vec{w}\,\vec{x}_{test}) = sign\left(\sum_n \alpha_n t_n \underbrace{\vec{x}_n \cdot \vec{x}_{test}}\right)$$

we can insert $K(\vec{x}_n, \vec{x}_{test})$ here

. The goal is then to find the best α_n.

. The KKT conditions tell us things about the α_n that do define the $\vec{w}^*_{SVM}$, $\alpha^*_m = \underset{\alpha_1 \dots \alpha_N}{argmin}(\mathcal{L})$

KKT: $\begin{cases} \alpha_n \geq 0 \\ t_n d(x_n) - 1 \geq 0 \\ \alpha_n (t_n d(x_n) - 1) = 0 \end{cases}$ $\left(d(x_n) = \frac{\vec{w}x + b}{\|w\|}\right)$

i.e the α_n are Lagrange multipliers, $\alpha_n \geq 0$
i.e each inequality is either saturated ($t_n d(x_n) - 1 = 0$), ↳ point $\vec{x}_n$ is on the margin
or the α_n is 0 (point ignored)

There are 2 kind of points: $\begin{cases} \alpha_n = 0, & t_n d(x_n) - 1 > 0 : \text{points well. class. outside the margin} \\ \alpha_n > 0, & \text{———} = 0 : \text{on the margin} \end{cases}$

→ These are the support vectors!

Note that the sol^o is $w = \sum_n \alpha_n t_n x_n = \sum_{S.V}$ —
a linear combinat^n of the support vectors only!

b can be computed: