# Some Discrete-Time *SI*, *SIR*, and *SIS* Epidemic Models

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## ABSTRACT

Discrete-time models, or difference equations, of some well-known *SI*, *SIR*, and *SIS* epidemic models are considered. The discrete-time *SI* and *SIR* models give rise to systems of nonlinear difference equations that are similar in behavior to their continuous analogues under the natural restriction that solutions to the discrete-time models be positive. It is important that the entire system be considered since the difference equation for infectives *I* in an *SI* model has a logistic form which can exhibit period-doubling and chaos for certain parameter values. Under the restriction that *S* and *I* be positive, these parameter values are excluded. In the case of a discrete *SIS* model, positivity of solutions is not enough to guarantee asymptotic convergence to an equilibrium value (as in the case of the continuous model). The positive feedback from the infective class to the susceptible class allows for more diverse behavior in the discrete model. Period-doubling and chaotic behavior is possible for some parameter values. In addition, if births and deaths are included in the *SI* and *SIR* models (positive feedback due to births) the discrete models exhibit periodicity and chaos for some parameter values. Single-population and multi-population, discrete-time epidemic models are analyzed.

## 1. INTRODUCTION

Discrete-time models or difference equations are used to formulate some standard *SI*, *SIR*, and *SIS* epidemic models. The continuous approximations of these models are used more often in modeling situations because of their mathematical tractability. Difference equations are not as well-behaved as their continuous approximations. Simple nonlinear difference equations can exhibit chaotic behavior. The following logistic difference equations are two examples that have received much attention (e.g., [16, 19]):

$$x_{n+1} = (1+r)x_n(1-x_n) \qquad (1)$$
$$x_{n+1} = x_n \exp(r(1-x_n)). \qquad (2)$$

As the value of $r$ increases above 2, there is period doubling and eventually chaos. The difference equations for infectives in the standard discrete-time *SI* and *SIS* models studied in Sections 2 and 4 have the

form of (1) and therefore, for sufficiently large time steps the number of infectives can behave *chaotically*.

Schaffer and Kot [22] fit the logistic equation (2) to Poincaré sections of measles data and found that the estimated range of parameters was in the region where $r > 2$. Schaffer [21] and Olsen and Schaffer [17] showed that measles data closely agreed with an *SEIR* differential equation model with a periodic contact rate. The parameter values were in a range that exhibited chaotic dynamics [3,17,21]. These results indicate a deterministic component in the underlying model (with possible seasonal forcing). However, the underlying model cannot be a simple continuous *SIR*-type model (with constant parameters) because such models do not exhibit periodic behavior [10]. It is shown in the present investigation that the simple discrete-time *SI* and *SIR* models do not have periodic behavior either. They behave qualitatively similar to their continuous counterpart under the necessary restriction that solutions remain positive. However, if there is positive feedback to the susceptible class, as in discrete *SIS* models or *SIR* models with births and deaths, then periodic behavior is possible.

When the time step is sufficiently small, differential equations are good approximations to the discrete formulations. In the case of simple epidemics these continuous approximations are justified for an *SI* or an *SIR* model, since the behavior in the discrete-time model with any time step that yields positive solutions is the same qualitatively as in the continuous model (when the time step approaches zero). However, in the case of discrete *SIS* models or *SIR* models with births and deaths, the continuous approximation is only justified for certain parameter values.

In the following sections, *SI*, *SIR*, and *SIS* discrete-time models are analyzed and their solutions compared with their analogous continuous models. The analysis is straightforward, but has not been presented elsewhere for all of these basic discrete-time models. Allen et al. [1], Longini [14], and Rvachev and Longini [20] have presented some of the basic properties for general multi-population, discrete-time *SIR* models. Hethcote [8,9] gives some excellent reviews of the continuous *SI*, *SIR*, and *SIS* epidemic models and discusses many variations of these basic models.

## 2. *SI* MODEL

The discrete-time *SI* epidemic model, where $S$ represents susceptibles and $I$ represents infectives has the following form:

$$S_{n+1} = S_n \left( 1 - \frac{\alpha \Delta t}{N} I_n \right) \tag{3}$$

$$I_{n+1} = I_n \left( 1 + \frac{\alpha \Delta t}{N} S_n \right), \tag{4}$$

with positive initial conditions $S_0 > 0$ and $I_0 > 0$ satisfying $S_0 + I_0 = N$, where $\alpha$ ( $> 0$) is the contact rate, i.e., the average number of individuals with whom an infectious individual makes sufficient contact (to pass infection) during a unit time interval [1,8,20], $N$ is the total population size, and the subscript $n$ represents the time $n\Delta t$ ( $> 0$). Thus, $S_n$ is the size of the susceptible subpopulation at time $n\Delta t$. The above system is deterministic; however, $S_n$ and $I_n$ could represent the expected values of random variables from a stochastic model [15].

There are two basic assumptions in these simple epidemic models: (i) the population mixes homogeneously (each individual is equally likely to contract the disease), and (ii) the total population size remains constant. This latter assumption follows directly from the system of difference equations ($S_n + I_n = N$, $n = 1,2,\ldots$) and the assumption that solutions are positive.

To ensure solutions to (3) and (4) are positive, restrictions must be put on the parameters. A necessary and sufficient condition to ensure that $S_n$ is positive for all initial conditions (and $I_n < N$) is

$$\alpha\,\Delta t \leqslant 1 \tag{5}$$

or $\Delta t \leqslant 1/\alpha$. This latter inequality implies that the time step $\Delta t$ must be less than the average time required for a successful contact.

It is easy to establish the global behavior of this model. First note that $S_n$ decreases monotonically and $I_n$ increases monotonically. Thus, they approach an equilibrium, $(S^*, I^*)$, where $I^* > 0$. The unique equilibrium for which $I^*$ is positive is $S^* = 0$ and $I^* = N$. Therefore, the entire population eventually becomes infected.

The parameter $\alpha$ is expressed as a rate so that the continuous analogue of (3) and (4) can be obtained easily. With the approximation, $(S_{n+1} - S_n)/\Delta t \approx dS/dt$, the analogous differential system has the following form:

$$\frac{dS}{dt} = -\frac{\alpha}{N}SI$$

$$\frac{dI}{dt} = \frac{\alpha}{N}SI,$$

with positive initial conditions satisfying $S(0) + I(0) = N$.

The solution for the continuous model can be obtained exactly [4,8]. In the $SI$ differential equation model, substitution of $N - I$ for $S$ leads to a logistic differential equation for $I$ whose exact solution is $I(t) = I(0)N/[I(0) + \exp(-\alpha t)(N - I(0))]$. $I(t)$ approaches $N$ monotonically. The continuous model exhibits the same behavior as the discrete model.

An alternate way to verify the global behavior of the discrete model is to use the same type of argument that was used for the continuous

model. Substitute $x_n = \alpha \Delta t I_n / (N(1 + \alpha \Delta t))$ and $S_n = N - I_n$ into (3) and (4). This leads to the discrete logistic equation

$$x_{n+1} = (1 + \alpha \Delta t) x_n (1 - x_n). \tag{6}$$

The additional restriction that $I_n < N$ requires the inequality $x_n < x^* = \alpha \Delta t / (1 + \alpha \Delta t)$. The inequality will hold for all initial conditions $x_0$ if and only if $x^* \leqslant 0.5$. (The maximum of $y = (1 + \alpha \Delta t) x (1 - x)$ must occur to the right of the line $x = x^*$ [5,12].) Thus, it follows that $\alpha \Delta t \leqslant 1$; (5) holds. Solutions converge monotonically to the equilibrium $x^*$.

Another form for a discrete equation for the infective class can be obtained from the solution of the continuous model (logistic equation for infectives). It does not require condition (5) because solutions are positive for positive initial conditions. However, it cannot be justified biologically. The exact logistic difference equation is given by [18]:

$$I_{n+1} = \frac{N \lambda I_n}{N + (\lambda - 1) I_n}, \tag{7}$$

where $\lambda = \exp(\alpha \Delta t)$. Equation (7) and the identity, $S_n = N - I_n$ give the same solutions at $n = 0, 1, \dots$ as the continuous $SI$ model.

The discrete-time, multi-population $SI$ model has the same monotonic behavior as the single-population model. Consider the following $SI$ model with $K$ subpopulations:

$$S_{n+1}^i = S_n^i \left( 1 - \sum_{k=1}^{K} \frac{\alpha_{ik} \Delta t}{N^i} I_n^k \right)$$

$$I_{n+1}^i = I_n^i + S_n^i \sum_{k=1}^{K} \frac{\alpha_{ik} \Delta t}{N^i} I_n^k,$$

where $i = 1, \dots, K$ and with initial conditions $S_0^i > 0$, $I_0^i \geqslant 0$ ($I_0^k > 0$ for some $k$) satisfying $S_0^i + I_0^i = N^i =$ the size of the $i$th subpopulation. The parameter $\alpha_{ik}$ is the average number of contacts per unit time of an infective in group $k$ with individuals in group $i$ [9]. Again the total subpopulation size remains constant; $S_n^i + I_n^i = N^i$ and solutions to the above system are nonnegative for all initial conditions if and only if $\max_i \{ \sum_{k=1}^{K} \alpha_{ik} \Delta t N^k / N^i \} \leqslant 1$. Each $S_n^i$ is strictly monotonically decreasing for each $i$ and must approach zero since this is the only steady-state value other than $N^i$; $I_n^i$ approaches $N^i$.

The requirement that solutions to the discrete-time $SI$ system be positive guarantees that the discrete and continuous systems behave

similar qualitatively and ensures that the differential system approximates well the discrete system. However, as indicated in Figure 1, the continuous and discrete models approach the equilibrium at different rates.

## 3.  *SIR* MODEL

The discrete model for the standard *SIR* model divides the population into three subgroups: susceptibles, infectives, and removed or isolated ($R$). The difference equations have following form:

$$S_{n+1} = S_n\left(1 - \frac{\alpha \Delta t}{N} I_n\right) \tag{8}$$

$$I_{n+1} = I_n\left(1 - \gamma \Delta t + \frac{\alpha \Delta t}{N} S_n\right) \tag{9}$$

$$R_{n+1} = R_n + \gamma \Delta t I_n, \tag{10}$$

with $S_0 > 0$, $I_0 > 0$, and $R_0 \geqslant 0$ satisfying $S_0 + I_0 + R_0 = N$, where $\gamma$ ($> 0$) is the probability that one infective will be removed from the infection process during a unit time interval (relative removal rate). Unlike the *SI* model, individuals in the *SIR* model recover from the disease and become permanently immune ($R$ subgroup). It is easy to
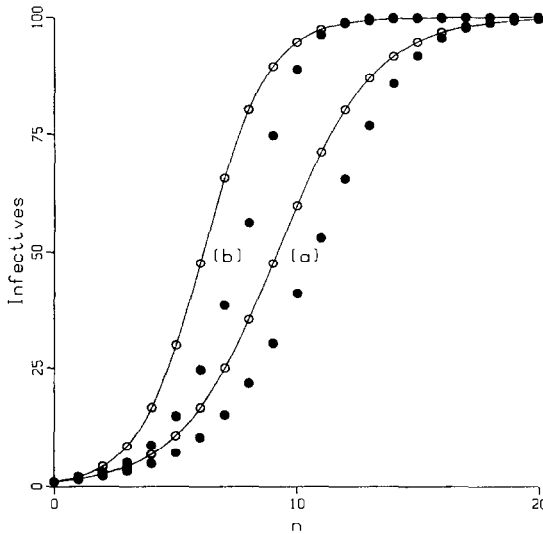


FIG. 1. Number of infectives in the discrete (●●●) and continuous (O-O-O) *SI* models, $\Delta t = 0.25$, $N = 100$, and $I_0 = 1$. (*a*) $\alpha = 2$, (*b*) $\alpha = 3$.

see that the total population size remains constant, $S_n + I_n + R_n = N$. Solutions to the discrete system are positive for $n = 1, 2, \ldots$ for all initial conditions if and only if

$$\max\{\gamma \Delta t, \alpha \Delta t\} \leqslant 1. \qquad (11)$$

Thus, $\Delta t \leqslant \min\{1/\alpha, 1/\gamma\}$; the time step must be less than the average time required for a successful contact and less than the average infectious period.

The global behavior of system (8), (9), and (10) is easy to establish. Let $\mathscr{R} = S_0 \alpha / (N\gamma)$ be the reproductive rate [2]. The value of $\mathscr{R}$ determines the global behavior of the discrete $SIR$ model. It is important to note that $S_n$ is strictly decreasing and $R_n$ is strictly increasing. Let $S_\infty = \lim_{n \to \infty} S_n \geqslant 0$, which depends on the initial conditions. If $S_0 \leqslant N\gamma/\alpha$ or $\mathscr{R} \leqslant 1$, then $I_1 \leqslant I_0$ and because $S_n$ is decreasing, $I_{n+1} \leqslant I_n$; there is no epidemic. In the other case, if $S_0 > N\gamma/\alpha$, then $I_1 > I_0$; the infective class initially increases. It must be the case that $S_\infty < N\gamma/\alpha$ (no more epidemics can occur) because otherwise $I_n$ increases to a positive equilibrium $I_\infty$ which implies $R_n$ approaches infinity as $n \to \infty$, an impossibility. Also, the infective class eventually decreases and approaches zero. In addition, it can easily be shown that $S_\infty > 0$ (see Lemma 1 in the Appendix); there always remain some susceptibles after the epidemic has ended. The behavior of the discrete $SIR$ model is illustrated in Figure 2.

The continuous version of this $SIR$ model behaves in the same manner as the discrete model [8]. The continuous $SIR$ model has the following form:

$$\frac{dS}{dt} = -\frac{\alpha}{N}SI$$

$$\frac{dI}{dt} = I\left(\frac{\alpha}{N}S - \gamma\right)$$

$$\frac{dR}{dt} = R + \gamma I,$$

where $S(0) + I(0) + R(0) = N$. The reproductive rate in the continuous case is $\mathscr{R} = S(0)\alpha/(N\gamma)$. If $\mathscr{R} \leqslant 1$, there is no epidemic, but if $\mathscr{R} > 1$, there is an epidemic [8].

The discrete-time, multi-population $SIR$ model exhibits the same characteristic behavior as the single-population model. The $SIR$ model
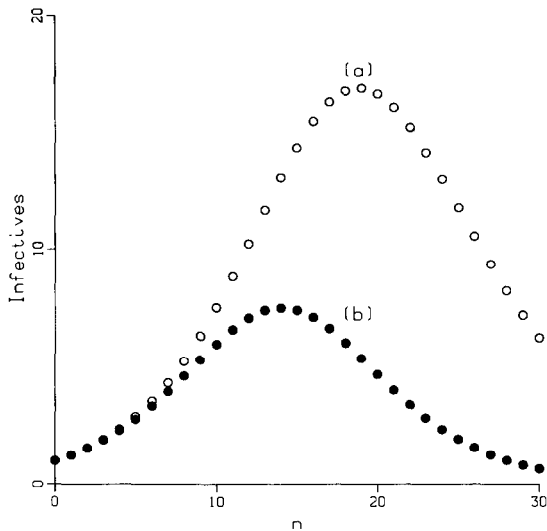
FIG. 2. Number of infectives in the discrete $SIR$ model, $\Delta t = 0.25$, $N = 100$, $S_0 = 99$, and $I_0 = 1$. (a) $\alpha = 2$, $\gamma = 1$, and $\mathscr{R} = 1.98$. (b) $\alpha = 3$, $\gamma = 2$, and $\mathscr{R} = 1.485$.

with $K$ subpopulations has the following form:

$$S_{n+1}^i = S_n^i \left( 1 - \sum_{k=1}^{K} \frac{\alpha_{ik} \Delta t}{N^i} I_n^k \right)$$

$$I_{n+1}^i = I_n^i (1 - \gamma_i \Delta t) + S_n^i \sum_{k=1}^{K} \frac{\alpha_{ik} \Delta t}{N^i} I_n^k$$

$$R_{n+1}^i = R_n^i + \gamma_i \Delta t I_n^i,$$

where $i = 1, \ldots, K$ and initial conditions $S_0^i > 0$, $I_0^i \geq 0$ ($I_0^k > 0$ for some $k$), and $R_0^i \geq 0$ satisfying $S_0^i + I_0^i + R_0^i = N^i$. Again, the total population size remains constant, $S_n^i + I_n^i + R_n^i = N^i$ for all $n$ and solutions are nonnegative for all initial conditions if and only if $\max_i \{ \sum_{k=1}^{K} \alpha_{ik} \Delta t N^k / N^i, \gamma_i \Delta t \} \leq 1$. Note that $S_n^i$ is monotonically decreasing, $R_n^i$ is monotonically increasing, and they are both bounded; therefore, they must approach a limit. It follows from the difference equation for $R_n^i$ that $I_n^i$ approaches zero as $n \to \infty$.

To determine whether an epidemic occurs within a subpopulation of the multi-population $SIR$ model is not as straightforward as in the

single-population case. If the value of $\mathscr{R}_i = S_0^i \alpha_{ii} /(\gamma_i N^i) > 1$, then the number of infectives in the $i$th subpopulation will initially increase as in Figure 3. However, $\mathscr{R}_i \leqslant 1$ is not enough to ensure that there will be no epidemic in the $i$th subpopulation; the size of the other infective subpopulations is required also as in Figure 4. Instead of considering infective subpopulations separately, the size of the entire infective population may be considered: $I_n = \sum_{i=1}^K I_n^i$. If $\max_k \{\sum_{i=1}^K S_0^i \alpha_{ik} /(\gamma_k N^i)\}$ $\leqslant 1$, then $I_n$ decreases with $n$; there is no epidemic as in Figure 5. However, if $\min_k \{\sum_{i=1}^K S_0^i \alpha_{ik} /(\gamma_k N^i)\} > 1$, then $I_n$ increases with $n$; there is an epidemic (see Figure 3). A reproductive rate cannot be simply defined because it depends on initial conditions.

## 4. SIS MODEL

The SIS epidemic model has been used to describe sexually transmitted diseases [8, 11, 13]. Individuals that are cured do not develop permanent immunity as in the SIR model, but are immediately susceptible to the disease again. The SIS model removes individuals from the infective class to the susceptible class; hence, there is no removed class. The
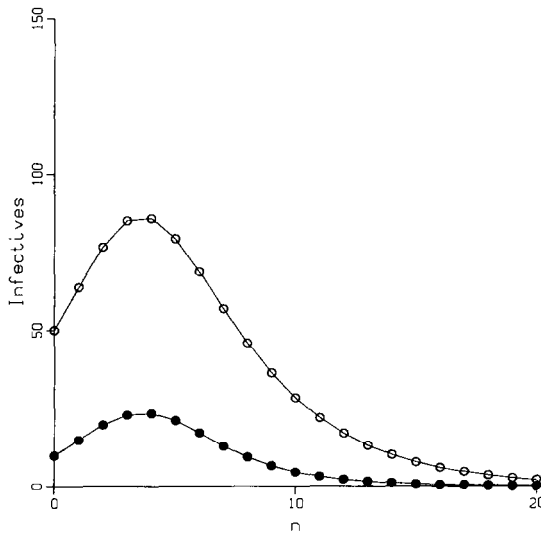


FIG. 3. Number of infectives in a two-population, discrete SIR model (population 1: ●-●-●, population 2: O-O-O), $\Delta t = 0.25$, $\alpha_{11} = 2$, $\alpha_{12} = 0.5$, $\alpha_{21} = 4$, $\alpha_{22} = 2$, $\gamma_1 = 2$, $\gamma_2 = 1$, $N^1 = 100$, and $N^2 = 200$. The initial conditions are $I_0^1 = 10$, $S_0^1 = 90$, $I_0^2 = 50$, and $S_0^2 = 150$. Note the $\mathscr{R}_1 = 0.9$, $\mathscr{R}_2 = 1.5$, and $\min_k \{\sum_{i=1}^2 S_0^i \alpha_{ik} /(\gamma_k N^i)\} = 1.95$. There is an epidemic in both populations.
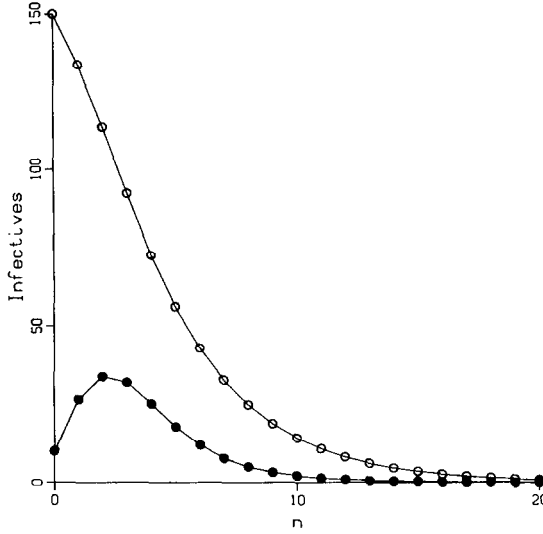
FIG. 4. Number of infectives in a two-population discrete *SIR* model with the same parameters as in Figure 3. The initial conditions are $I_0^1 = 10$, $S_0^1 = 90$, $I_0^2 = 150$, and $S_0^2 = 50$. Note that $\mathscr{R}_1 = 0.9$ and $\mathscr{R}_2 = 0.5$. There is an epidemic in the first population.

difference equations have the following form:

$$S_{n+1} = S_n \left(1 - \frac{\alpha \, \Delta t}{N} I_n\right) + \gamma \, \Delta t \, I_n \tag{12}$$

$$I_{n+1} = I_n \left(1 - \gamma \, \Delta t + \frac{\alpha \, \Delta t}{N} S_n\right), \tag{13}$$

with positive initial conditions $S_0 > 0$ and $I_0 > 0$ satisfying $S_0 + I_0 = N$. The population size remains constant and solutions are positive for all initial conditions if and only if the following inequalities hold (see Lemma 2 in the Appendix):

$$\gamma \, \Delta t \leqslant 1 \quad \text{and} \quad \alpha \, \Delta t < \left(1 + \sqrt{\gamma \, \Delta t}\right)^2. \tag{14}$$

The basic reproductive rate for this model is $\mathscr{R} = \alpha / \gamma$. If $\mathscr{R} \leqslant 1$, then it follows that $I_{n+1} < I_n$ because $0 < S_n < N$ (solutions are positive). In this case, it is easy to show that the monotonic limit is $(S^*, I^*) = (N, 0)$. Suppose $S^* < N$, then there exists $n_1$ and $\epsilon$ such that for all $n \geqslant n_1$,
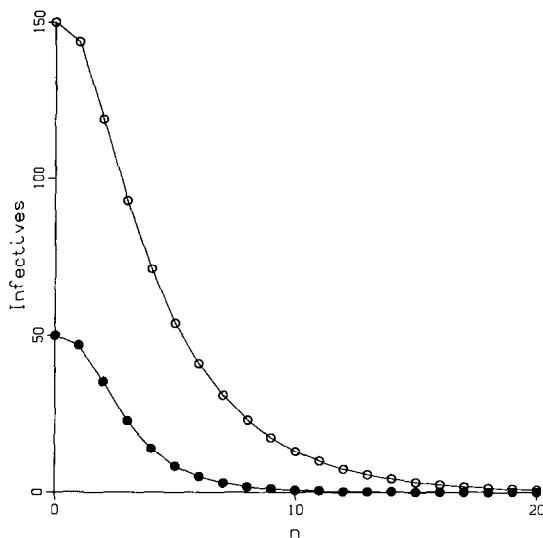
FIG. 5. Number of infectives in a two-populations discrete *SIR* model with the same parameters as in Figure 3. The initial conditions are $I_0^1 = 50$, $S_0^1 = 50$, $I_0^2 = 150$, and $S_0^2 = 50$. Note that $\mathscr{R}_1 = 0.5$, $\mathscr{R}_2 = 0.5$, and $\max_k\{\Sigma_{i=1}^2 S_0^i \alpha_{ik} /(\gamma_k N^i)\} = 1$. There is no epidemic in either population.

$S_n < S^* + \epsilon < N$, and

$$I_{n+1} \leq I_n(1 - \gamma\Delta t + \alpha\Delta t(S^* + \epsilon)/N)$$
$$= \rho I_n.$$

Because $\rho < 1$ it follows that $I^* = 0$, contradicting the fact that $S^* < N$. In the case that $\mathscr{R} > 1$, substitution of $S_n = N - I_n$ and $x_n = \alpha\Delta t I_n /[N(1 + \alpha\Delta t - \gamma\Delta t)]$ into (13) yields the normalized logistic difference equation:

$$x_{n+1} = (1 + \alpha\Delta t - \gamma\Delta t)x_n(1 - x_n).$$

For $\mathscr{R} > 1$ the restriction on the parameters necessary for positive solutions, inequality (14), is not sufficient to guarantee convergence. If, in addition to inequality (14), $\alpha$ is restricted so that $\alpha\Delta t \leq 2 + \gamma\Delta t$, then solutions will converge to a stable endemic equilibrium, $S^* = \gamma N/\alpha$, $I^* = N - S^*$ as in Figure 6. If $0.25 < \gamma\Delta t \leq 1$ and $2 + \gamma\Delta t < \alpha\Delta t \leq (1 + \sqrt{\gamma\Delta t})^2$, then monotonic convergence to an endemic equilibrium is no longer possible. It is the positive feedback to the susceptible class through the recovery parameter $\gamma$ and a sufficiently large
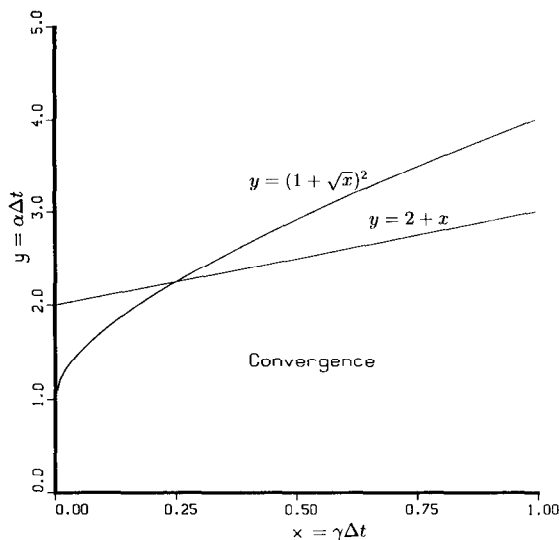
FIG. 6. Parameter space where solutions to the discrete *SIS* model are positive: $0 < \gamma \Delta t \leqslant 1$ and $0 < \alpha \Delta t \leqslant (1 + \sqrt{\gamma \Delta t})^2$. If, in addition, $\alpha \Delta t \leqslant 2 + \gamma \Delta t$, solutions converge to an equilibrium value. Note that if the horizontal axis is relabeled as $x = \gamma \Delta t + \beta \Delta t$, then the parameter space again defines regions where solutions are positive and where solutions converge to an equilibrium value for the discrete *SIS* model with birth and deaths.

contact rate $\alpha$ that allows for period-doubling and chaotic behavior, the same behavior as in (1) and in Figure 7.

The continuous *SIS* model does not exhibit periodicity. If

$$\frac{dS}{dt} = -\frac{\alpha}{N} SI + \gamma I$$

$$\frac{dI}{dt} = I \left( \frac{\alpha}{N} S - \gamma \right),$$

where $S(0) + I(0) = N$, the exact solution can be calculated. If $\mathscr{R} \leqslant 1$, then the disease dies out, and if $\mathscr{R} > 1$, then there is a globally stable endemic equilibrium [8]. Two particular cases for $\mathscr{R} > 1$ are illustrated for the discrete and continuous models in Figure 7.

There are other discrete models that behave exactly as their continuous counterpart. They can be obtained from the solution to the continuous model (infective class has a logistic solution). The particular form of
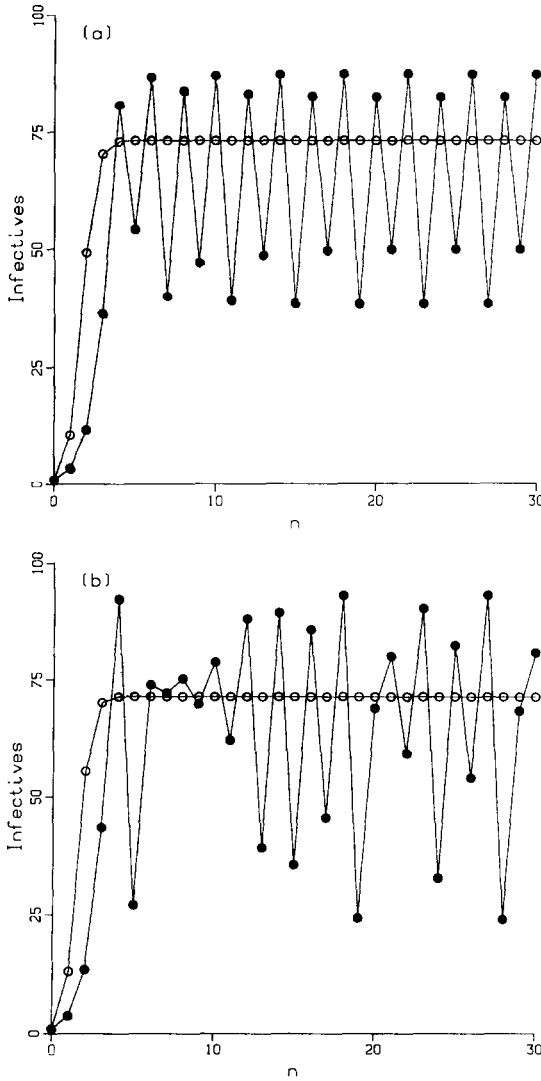
FIG. 7. Number of infectives in the discrete (●●●) and continuous (O-O-O) SIS models, $\gamma = 2.$, $\Delta t = 0.5$, $N = 100$, and $I_0 = 1$. (a) $\alpha = 7$, $I^* \approx 71.4$, and $\mathscr{R} = 3.5$, a four-point cycle in the discrete model corresponding to $r = 2.5$ in (1). (b) $\alpha = 7.5$, $I^* \approx 73.3$, and $\mathscr{R} = 3.75$; the exact period is difficult to ascertain in the discrete model. It corresponds to $r = 2.75$ in (1).

the discrete model depends on the value of $\mathscr{R}$. If $\mathscr{R} \neq 1$, then

$$I_{n+1} = \frac{I^* \lambda I_n}{I^* + I_n(\lambda - 1)} \tag{15}$$

and $S_n = N - I_n$, where $I^* = (\alpha - \gamma)N/\alpha$ and $\lambda = \exp((\alpha - \gamma)\Delta t)$. In the case that $\mathscr{R} = 1$, the infectives have the following form:

$$I_{n+1} = \frac{NI_n}{N + \alpha \Delta t I_n}. \tag{16}$$

If $\mathscr{R} \leqslant 1$, $I_n$ approaches zero ($S_n$ approaches $N$). If $\mathscr{R} > 1$, the infectives $I_n$ approach the positive equilibrium $I^*$ and $S_n$ approaches $N - I^*$. These difference equations do not exhibit period-doubling or chaos; only convergence to an equilibrium value is possible.

The discrete-time, multi-population $SIS$ model with $K$ subpopulations has the following form:

$$S_{n+1}^i = S_n^i \left( 1 - \sum_{k=1}^{K} \frac{\alpha_{ik} \Delta t}{N^i} I_n^k \right) + \gamma_i \Delta t I_n^i$$

$$I_{n+1}^i = I_n^i(1 - \gamma_i \Delta t) + S_n^i \sum_{k=1}^{K} \frac{\alpha_{ik} \Delta t}{N^i} I_n^k,$$

where $S_0^i + I_0^i = N^i$, $S_0^i > 0$ and $I_0^i \geqslant 0$ ($I_0^k > 0$ for some $k$). Solutions satisfy $S_n^i + I_n^i = N^i$ and are nonnegative if and only if

$$\max_i \{a_i, \gamma_i \Delta t\} \leqslant 1 \quad \text{and} \quad \alpha_{ii} \Delta t \leqslant \left( \sqrt{1 - a_i} + \sqrt{\gamma_i \Delta t} \right)^2,$$

for $i = 1, \ldots, K$ where $a_i = \sum_{k \neq i} \alpha_{ik} \Delta t N^k / N^i$ (see Lemma 3 in the Appendix). Nonnegativity requires a relatively small between-population contact rate ($a_i$) as well as a relatively small within-population contact rate ($\alpha_{ii}$). However, even the conditions for nonnegative solutions do not rule out periodicity or chaos.

Although global stability for the discrete-time, multi-population model is not easy to establish, local stability of the noninfection state is straightforward and similar to the continuous case [13]. Substitution of $S_n^i = N^i - I_n^i$ into the equation for $I_{n+1}^i$ and linearization of the infective equations about the origin yields $I_{n+1} \approx A I_n$, where $I_n = [I_n^i]$, $A = [a_{ij}]$, $a_{ii} = 1 - \gamma_i \Delta t + \alpha_{ii} \Delta t$, and $a_{ij} = \alpha_{ij} \Delta t$. Since the matrix $A$ is positive, by Perron's Theorem [7], there is a positive eigenvalue $\bar{\lambda}$ of

maximum modulus. If $\bar{\lambda} < 1$, the origin is locally stable and the infection dies out.

In the continuous case the maximum real part of the eigenvalues of the linearized matrix $A$ ($s(A)$) determines global behavior. Lajmanovich and Yorke [13] showed that if $s(A) \leqslant 0$, then the origin is globally stable and if $s(A) > 0$, then there is a globally stable endemic equilibrium. The latter result was verified with Lyapunov-type arguments.

In the case of two subpopulations ($K = 2$) with $\alpha_{ii} = 0$, a basic reproductive rate $\mathcal{R}$ can be simply defined in the continuous model [13]. The condition $s(A) \leqslant 0$ is equivalent to $\mathcal{R} \leqslant 1$, where $\mathcal{R} = \alpha_{12}\alpha_{21} / (\gamma_1\gamma_2)$. This same reproductive rate was found in a special case for the discrete-time model when $K = 2$ by Martin et al. [15] in a study of a sexually transmitted disease. For simplification purposes let $I^1 = x$, $I^2 = y$, $N^1 = W$, and $N^2 = M$ ($I^1$ represents females and $I^2$ represents males). The model is given below:

$$x_{n+1} = x_n(1 - \gamma_1 \Delta t) + \frac{\alpha_{12} \Delta t}{W}(W - x_n)y_n \qquad (17)$$

$$y_{n+1} = y_n(1 - \gamma_2 \Delta t) + \frac{\alpha_{21} \Delta t}{M}(M - y_n)x_n. \qquad (18)$$

In this model it was assumed there are no homosexual contacts; males do not infect other males and females do not infect other females ($\alpha_{ii} = 0$) and $\alpha_{12}M = \alpha_{21}W$. The $SIS$ model gives positive solutions if

$$\max_{i, i \neq j}\left\{\alpha_{ij} \Delta t N^j / N^i, \gamma_i \Delta t\right\} \leqslant 1.$$

Martin et al. [15] showed for system (17) and (18) (using Lyapunov-type arguments) that if the basic reproductive rate $\mathcal{R} \leqslant 1$ (and solutions are positive), then solutions tend to the zero state; there is no epidemic. In the other case, if $\mathcal{R} > 1$ and

$$\max_{i, i \neq j}\left\{2\alpha_{ji} \Delta t + \gamma_i \Delta t\right\} < 1, \qquad (19)$$

then solutions converge monotonically to an endemic equilibrium, $x^* = (\alpha_{12}\alpha_{21} - \gamma_1\gamma_2)MW/(\alpha_{21}(W\gamma_1 + M\alpha_{12}))$ and $y^* = (\alpha_{12}\alpha_{21} - \gamma_1\gamma_2)MW/(\alpha_{12}(M\gamma_2 + W\alpha_{21}))$. (Another weaker, but more complicated condition than inequality (19) was also given in reference [15].) Numerical solutions indicate that the restriction (19) is sufficient, but not necessary for global stability of an endemic equilibrium as in Figure 8. In the particular case considered by (17) and (18), where $\alpha_{ii} = 0$, it appears that solutions converge to an equilibrium and the only require-
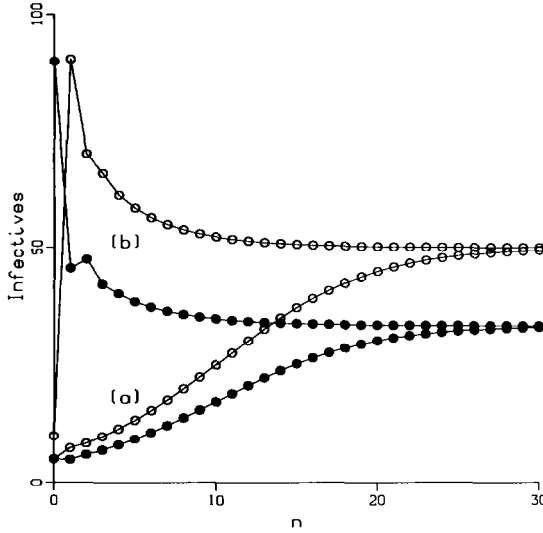
FIG. 8. Number of infectives in the two-population, discrete *SIS* model (17) and (18), females ($x_n$, ●-●-●), males ($y_n$, ○-○-○), $\Delta t = 0.25$ $\alpha_{12} = 2$, $\alpha_{21} = 4$, $\gamma_1 = 2 = \gamma_2$, $M = 200$, $W = 100$, $\mathscr{R} = 2$, $x^* = 33.3$, and $y^* = 50$. (*a*) $x_0 = 5$ and $y_0 = 5$, (*b*) $x_0 = 90$ and $y_0 = 10$. (Note that $\max_{i, i \neq j}\{\alpha_{ij} \Delta t N^j / N^i, \gamma_i \Delta t\} \leqslant 1$, but inequality (19) does not hold.)

ment is that solutions are positive. Further analysis is required to verify this conclusion. However, if $\alpha_{ii} > 0$ and sufficiently large, there are parameter values that give rise to periodicity and chaos just as in the single-population case.

## 5.  MODELS WITH BIRTHS AND DEATHS

Next we consider the basic models with births and deaths. To keep the population size constant it is assumed that birth rate ($\beta$) equals death rate. The discrete-time *SIS* model with births and deaths has the following form:

$$S_{n+1} = S_n\left(1 - \frac{\alpha \Delta t}{N} I_n\right) + \gamma \Delta t I_n + \beta \Delta t (N - S_n)$$

$$I_{n+1} = I_n\left(1 - \gamma \Delta t - \beta \Delta t + \frac{\alpha \Delta t}{N} S_n\right),$$

with positive initial conditions $S_0 > 0$ and $I_0 > 0$ satisfying $S_0 + I_0 = N$. Assume $\gamma \geqslant 0$ and $\alpha, \beta > 0$. Thus, the *SI* model is a special case of the above *SIS* model; if $\gamma = 0$ the *SI* model is obtained. Solutions of the

above system are positive and satisfy $S_n + I_n = N$ for all initial conditions if and only if

$$(\gamma + \beta)\,\Delta t \leqslant 1 \quad \text{and} \quad \alpha\,\Delta t < \left(1 + \sqrt{(\beta + \gamma)\,\Delta t}\,\right)^2.$$

(See the note following Lemma 2 in the Appendix.) The basic reproductive rate for the above system is $\mathscr{R} = \alpha/(\gamma + \beta)$. If $\mathscr{R} \leqslant 1$, it can be shown in a manner similar to the case without births or deaths ($\beta = 0$) that $\lim_{n \to \infty} I_n = 0$ and $\lim_{n \to \infty} S_n = N$. If $\mathscr{R} > 1$, the *SI* and *SIS* model with births and deaths may experience the same diverse behavior as (1). This can be seen by considering the normalized logistic difference equation for infectives. Let $x_n = \alpha\,\Delta t\, I_n/[N(1 + \alpha\,\Delta t - \gamma\,\Delta t - \beta\,\Delta t)]$, then $x_{n+1} = x_n(1 + \alpha\,\Delta t - \gamma\,\Delta t - \beta\,\Delta t)(1 - x_n)$. However, if $\alpha\,\Delta t \leqslant 2 + \gamma\,\Delta t + \beta\,\Delta t$ (see Figure 6 with the horizontal axis relabeled as $\gamma\,\Delta t + \beta\,\Delta t$), the value of $\mathscr{R}$ completely determines the behavior of the *SIS* model; if $\mathscr{R} \leqslant 1$, then solutions converge to $(N,0)$ and if $\mathscr{R} > 1$, then solutions converge to the endemic equilibrium, $S^* = (\gamma + \beta)N/\alpha$ and $I^* = N - S^*$. This latter behavior describes the behavior of the continuous-time *SIS* model with births and deaths; the behavior is determined by the value of $\mathscr{R}$ [8]. A difference equation that mimics the behavior of the continuous-time model can be formulated using the continuous-time model; (15) and (16) are obtained, where $\lambda = \exp((\alpha - \gamma - \beta)\,\Delta t)$ and $I^* = N(\alpha - \gamma - \beta)/\alpha$.

For the discrete-time *SIR* model with births and deaths the same wide array of behavior is possible as in the discrete-time *SIS* model. Consider

$$S_{n+1} = S_n\left(1 - \frac{\alpha\,\Delta t}{N} I_n\right) + \beta\,\Delta t(N - S_n)$$

$$I_{n+1} = I_n\left(1 - \gamma\,\Delta t - \beta\,\Delta t + \frac{\alpha\,\Delta t}{N} S_n\right)$$

$$R_{n+1} = R_n(1 - \beta\,\Delta t) + \gamma\,\Delta t\, I_n,$$

where $S_0$, $I_0 > 0$ and $R_0 \geqslant 0$, $S_0 + I_0 + R_0 = N$, and $\alpha, \beta, \gamma > 0$. Solutions are nonnegative for all initial conditions if and only if

$$(\gamma + \beta)\,\Delta t \leqslant 1 \quad \text{and} \quad \alpha\,\Delta t \leqslant \left(1 + \sqrt{\beta\,\Delta t}\,\right)^2.$$

(See the note following Lemma 3 in the Appendix.) If the basic reproductive rate $\mathscr{R} = \alpha/(\gamma + \beta) \leqslant 1$, then $\lim_{n \to \infty} I_n = 0$, but if $\mathscr{R} > 1$, numerical simulations show that periodic behavior is possible as in Figure 9. However, if $\alpha$ and $\beta$ are sufficiently small, then the behavior
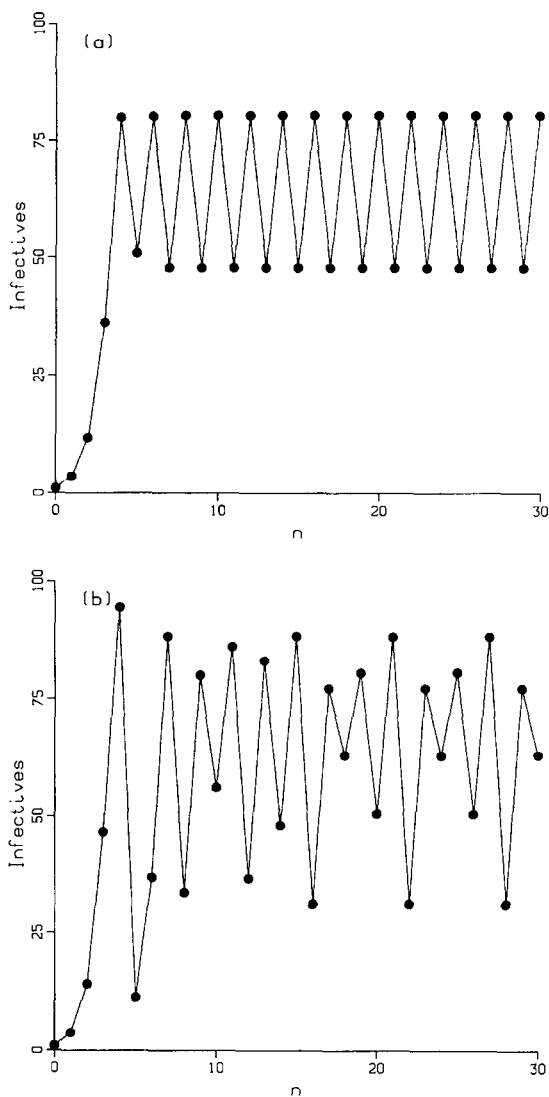
FIG. 9. Number of infectives in the discrete *SIR* model with births and deaths, $\gamma = 0.1$, $\beta = 1.9$, $\Delta t = 0.5$, $N = 100$, $S_0 = 99$, $I_0 = 1$, and $R_0 = 0$. (*a*) $\alpha = 7$, $I^* \approx 67.9$, and $\mathcal{R} = 3.5$, a two-point cycle. (*b*) $\alpha = 7.7$, $I^* \approx 70.3$, and $\mathcal{R} = 3.85$; the exact period is difficult to ascertain.

is completely determined by $\mathscr{R}$; if $\mathscr{R} \leqslant 1$, then solutions converge to $(S_\infty, 0, R_\infty)$ and if $\mathscr{R} > 1$, numerical simulations show that solutions converge to $S^* = N(\gamma + \beta)/\alpha$, $I^* = \beta N(\alpha - \gamma - \beta)/(\alpha(\gamma + \beta))$, and $R^* = \gamma N(\alpha - \gamma - \beta)/(\alpha(\gamma + \beta))$ (the same behavior as the continuous-time $SIR$ model with births and deaths [8]).

Discrete-time, multi-population $SI$, $SIS$ and $SIR$ models with births and deaths are more difficult to analyze, but as in the case of the multi-population $SIS$ model, conditions for nonnegative solutions and local stability can be established. Consider the multi-population $SIS$ model with births and deaths:

$$S_{n+1}^i = S_n^i \left( 1 - \sum_{k=1}^K \frac{\alpha_{ik} \Delta t}{N^i} I_n^k \right) + \gamma_i \Delta t I_n^i + \beta_i \Delta t (N^i - S_n^i)$$

$$I_{n+1}^i = I_n^i(1 - \gamma_i \Delta t - \beta_i \Delta t) + S_n^i \sum_{k=1}^K \frac{\alpha_{ik} \Delta t}{N^i} I_n^k,$$

where $S_0^i + I_0^i = N^i$, $S_0^i > 0$ and $I_0^i \geqslant 0$ ($I_0^k > 0$ for some $k$). Solutions satisfy $S_n^i + I_n^i = N^i$ and are nonnegative for all initial conditions if and only if

$$\gamma_i \Delta t + \beta_i \Delta t \leqslant 1 \quad \text{and} \quad \alpha_{ii} \Delta t \leqslant \left( \sqrt{1 - a_i} + \sqrt{\gamma_i \Delta t + \beta_i \Delta t} \right)^2,$$

for $i = 1, \ldots, K$. Recall that $a_i = \sum_{k \neq i} \alpha_{ij} \Delta t N^k / N^i$. The multi-population $SIR$ model with births and deaths has the form:

$$S_{n+1}^i = S_n^i \left( 1 - \sum_{k=1}^K \frac{\alpha_{ik} \Delta t}{N^i} I_n^k \right) + \beta_i \Delta t (N^i - S_n^i)$$

$$I_{n+1}^i = I_n^i(1 - \gamma_i \Delta t - \beta_i \Delta t) + S_n^i \sum_{k=1}^K \frac{\alpha_{ik} \Delta t}{N^i} I_n^k,$$

$$R_{n+1}^i = R_n^i(1 - \beta_i \Delta t) + \gamma_i \Delta t I_n^i,$$

where $S_0^i + I_0^i + R_0^i = N^i$, $S_0^i > 0$, $I_0^i \geqslant 0$ ($I_0^k > 0$ for some $k$), and $R_0^i \geqslant 0$. Solutions satisfy $S_n^i + I_n^i + R_n^i = N^i$ and are nonnegative for all initial conditions if and only if

$$\gamma_i \Delta t + \beta_i \Delta t \leqslant 1 \quad \text{and} \quad \alpha_{ii} \Delta t \leqslant \left( \sqrt{1 - a_i} + \sqrt{\beta_i \Delta t} \right)^2,$$

for $i = 1, \ldots, K$. (See the notes following Lemma 3 in the Appendix.) For both of these models there is only one noninfection state, a state where

some $I^j = 0$. The noninfection state is given by $S^i = N^i$, $I^i = 0$, and $R^i = 0$ for all of the $i$ subpopulations. The local stability conditions for this state can be obtained by checking that the Jury conditions are satisfied [6]. From the results of the single-population models with births and deaths, it is clear that periodicity and chaos are possible for sufficiently large $\alpha_{ii} \Delta t$ and $\beta_i \Delta t$.

## 6. FINAL REMARKS

The simple discrete-time $SI$ and $SIR$ epidemic models without births or deaths mimic the behavior of the continuous-time models, simple covergence to an equilibrium. However, the behavior in the discrete-time $SI$, $SIR$, and $SIS$ models with some type of positive feedback to the susceptible class (i.e., through recovery or births) differs from their continuous analogues. The same type of behavior that occurs in the well-known difference equation

$$x_{n+1} = (1 + r)x_n(1 - x_n),$$

(periodicity and chaos) is possible in these discrete-time models with positive feedback. The fact that the time step $\Delta t$ must be sufficiently large for this behavior to occur is not surprising. However, the magnitude of the time step alone is not sufficient to guarantee period-doubling or chaos. In addition, the contact rate ($\alpha$ or $\alpha_{ii}$) must be sufficiently large. Although this behavior is not possible in the continuous-time analogues, Aron and Schwartz [3] showed the significance of the contact rate in a continuous-time SEIR model with a periodic contact rate; periodicity and chaos are possible for a sufficiently large mean contact rate.

It should be noted that an alternate formulation for the discrete-time epidemic models can be obtained under the assumption that the probability a susceptible individual does not become infective in time $\Delta t$ is $\exp(- \alpha \Delta t I_n / N)$ (derived from the Poisson probability distribution) [1]. For example, in the $SI$ model, $S_{n+1} = S_n \exp(- \alpha \Delta t I_n / N)$ and $I_{n+1} = I_n + S_n(1 - \exp(- \alpha \Delta t I_n / N))$. These equations are approximated by (3) and (4).

## APPENDIX

*LEMMA 1*

In the single-population, discrete-time SIR model, $S_\infty > 0$.

First, note that

$$S_{n+1} = S_0 \prod_{k=0}^{n} \left(1 - \frac{\alpha \Delta t}{N} I_k\right). \tag{20}$$

Furthermore, since $\lim_{n\to\infty} S_n = S_\infty < N\gamma/\alpha$, there exists and $n_1$ such that for all $n \geq n_1$, $S_n < N\gamma/\alpha$. Thus, $1 - \gamma\Delta t + \alpha\Delta t S_n/N < 1$. Let $r_k = 1 - \gamma\Delta t + \alpha\Delta t S_n/N$ for $n = n_1 + k$, $k = 0, 1, 2, \ldots$. Note that $r_{k+1} < r_k$ because $S_n$ is strictly monotonically decreasing. Now choose $n_2 \geq n_1$ such that for all $n \geq n_2$, $cr_0 I_n < 1 - r_0$, where $c = \alpha\Delta t/N$. This is possible since $I_n$ approaches zero. Now $S_{n_2+1} = S_0 \prod_{k=0}^{n_2}(1 - cI_k) = S_0 K > 0$ and

$$S_{n_2+n+1} = S_0 K \prod_{k=n_2+1}^{n_2+n} \left(1 - cI_k\right)$$

$$\geq S_0 K \prod_{k=1}^{n} \left(1 - c\tilde{r}^k I_{n_2}\right)$$

$$\geq S_0 K \left(1 - I_{n_2} c \sum_{k=1}^{\infty} \tilde{r}^k\right)$$

$$= S_0 K \left(1 - I_{n_2} \frac{c\tilde{r}}{1 - \tilde{r}}\right),$$

where $\tilde{r} = 1 - \gamma\Delta t + cS_{n_2}$. The above inequalities hold because $I_{n_2+k} \leq \tilde{r}^k I_{n_2}$, and $\prod_{k=1}^{n}(1 - c\tilde{r}^k I_{n_2}) \geq 1 - I_{n_2} c \sum_{k=1}^{n} \tilde{r}^k$. The right-hand side of the last inequality is strictly positive and independent of $n$ because $n_2$ was chosen so that $c\tilde{r}I_{n_2} < 1 - \tilde{r}$. Hence, $S_\infty > 0$; there always remains some susceptibles in the population after the epidemic has ended.

An implicit expression for $S_\infty$ in terms of $R_\infty$ can be obtained in the continuous-time model by integrating $dS/S = -\alpha\, dR/(\gamma N)$ [9]. This is not possible in the discrete-time model; however, from expression (20) $S_\infty = S_0 \prod_{k=0}^{\infty}(1 - (\alpha\Delta t/N)I_k)$, which clearly shows the dependence of $S_\infty$ on the initial conditions.

*LEMMA 2*

Solutions to the single-population, discrete-time SIS model are positive for all initial conditions if and only if $\gamma\Delta t \leq 1$ and $\alpha\Delta t < (1 + \sqrt{\gamma\Delta t})^2$.

Let $I_0 = \epsilon$ and $S_0 = N - \epsilon$, where $0 < \epsilon < N$. We show that $0 < I_1 < N$ if and only if the above conditions hold. It follows from (13) that

$$I_1 = \epsilon\left(1 - \gamma\Delta t + \alpha\Delta t \frac{(N - \epsilon)}{N}\right)$$

$$= -\frac{\alpha\Delta t\, \epsilon^2}{N} + \epsilon(1 - \gamma\Delta t + \alpha\Delta t)$$

$$= p(\epsilon).$$

Thus, we need to show that the parabola $p(\epsilon)$ satisfies $0 < p(\epsilon) < N$ for $0 < \epsilon < N$. Note that $p(0) = 0$, $p(N) = N(1 - \gamma \Delta t)$, and the vertex of $p$ is $(\epsilon^*, p^*)$, where $\epsilon^* = N(1 - \gamma \Delta t + \alpha \Delta t)/(2\alpha \Delta t)$ and $p^* = N(1 - \gamma \Delta t + \alpha \Delta t)^2/(4\alpha \Delta t)$. Thus, $0 < p(\epsilon) < N$ for $0 < \epsilon < N$ if and only if

    (i)   $\gamma \Delta t \leqslant 1$ and either
    (ii)  $\epsilon^* \geqslant N$ or
    (iii) $\epsilon^* < N$ and $p^* < N$.

Condition (ii) is equivalent to $\alpha \Delta t \leqslant 1 - \gamma \Delta t$. Condition (iii) requires $\alpha \Delta t > 1 - \gamma \Delta t$ and $(1 - \gamma \Delta t + \alpha \Delta t)^2 < 4\alpha \Delta t$. The latter two inequalities hold if and only if $1 - \gamma \Delta t < \alpha \Delta t < (1 + \sqrt{\gamma \Delta t})^2$. Thus, the lemma follows.

Note that Lemma 2 can be applied to the discrete-time *SIS* model with births and deaths. If, in the proof of Lemma 2, $\gamma$ is replaced by $\gamma + \beta$, then solutions to the discrete-time *SIS* model are positive for all initial conditions if and only if $(\gamma + \beta)\Delta t \leqslant 1$ and $\alpha \Delta t < (1 + \sqrt{(\gamma + \beta)\Delta t})^2$.

*LEMMA 3*

*Solutions to the multi-population, discrete-time SIS model are nonnegative for all initial conditions if and only if $\max_i\{a_i, \gamma_i \Delta t\} \leqslant 1$ and $\alpha_{ii} \Delta t \leqslant (\sqrt{1 - a_i} + \sqrt{\gamma_i \Delta t})^2$, where $a_i = \sum_{k \neq i} \alpha_{ik} \Delta t N^k / N^i$.*

Assume $I_0^i \geqslant 0$ and $S_0^i \geqslant 0$ for all $i$. We show that $S_1^i \geqslant 0$ and $I_1^i \geqslant 0$ if and only if the above conditions hold. First note that $I_1^i \geqslant 0$ if and only if $\gamma_i \Delta t \leqslant 1$. Let $S_1^i = \epsilon$ and $I_1^i = N^i - \epsilon$, $0 \leqslant \epsilon \leqslant N^i$. Now

$$S_1^i \geqslant \epsilon\left(1 - a_i - \alpha_{ii} \Delta t + \epsilon \frac{\alpha_{ii} \Delta t}{N^i}\right) + \gamma_i \Delta t (N^i - \epsilon)$$

$$= \epsilon^2 \frac{\alpha_{ii} \Delta t}{N^i} + \epsilon(1 - a_i - \gamma_i \Delta t - \alpha_{ii} \Delta t) + \gamma_i \Delta t N^i$$

$$= p(\epsilon).$$

Note that $p(0) = \gamma_i \Delta t N_i$ and $p(N^i) = N^i(1 - a_i)$. Then for $S_1^i \geqslant 0$ we must have $a_i \leqslant 1$. The minimum of $p$ occurs at the vertex $(\epsilon^*, p^*)$, where $\epsilon^* = N^i(\alpha_{ii} \Delta t + a_i + \gamma_i \Delta t - 1)/(2\alpha_{ii} \Delta t)$ and $p^* = \gamma_i \Delta t N^i - N^i(1 - a_i - \gamma_i \Delta t + \alpha_{ii} \Delta t)^2/(4\alpha_{ii} \Delta t)$. If $\alpha_{ii} \Delta t \leqslant 1 - a_i - \gamma_i \Delta t$ or if $\alpha_{ii} \Delta t \leqslant a_i + \gamma_i \Delta t - 1$, then the minimum of $p$ occurs outside the interval $(0, N^i)$. If the minimum of $p$ occurs on the interval $(0, N^i)$, then we must have $p^* \geqslant 0$. The latter inequality is equivalent to $\alpha_{ii} \Delta t \leqslant (\sqrt{1 - a_i} + \sqrt{\gamma_i \Delta t})^2$. Since $S_1^i \geqslant 0$ for all initial conditions if and only if $p(\epsilon) \geqslant 0$, the lemma follows.

Note that the discrete-time *SIR* model with births and deaths can be shown to have nonnegative solutions by applying an argument similar to the one in Lemma 3. Note that $I_n \geqslant 0$ if and only if $(\gamma + \beta)\Delta t \leqslant 1$; $\beta \Delta t \leqslant 1$ implies $R_n \geqslant 0$. By applying an argument similar to the one above: $S_1^i \geqslant \epsilon(1 - \alpha \Delta t + \epsilon \alpha \Delta t / N) + \beta \Delta t(N - \epsilon) = \epsilon^2 \alpha \Delta t / N + \epsilon(1 - \alpha \Delta t - \beta \Delta t) + \beta \Delta t N = p(\epsilon)$. In the proof above, let $a_i = 0$, replace $\alpha_{ii}$ by $\alpha$ and $\gamma_i$ by $\beta$, then the required condition on $\alpha \Delta t$ follows: $\alpha \Delta t \leqslant (1 + \sqrt{\beta \Delta t})^2$.

Nonnegativity of solutions in the multi-population, discrete-time *SIS* and *SIR* models with births and deaths follows in a manner similar to the proof of Lemma 3. In the *SIS* model replace $\gamma_i$ by $\gamma_i + \beta_i$. In the *SIR* model, first note that it is necessary that $\gamma_i \Delta t + \beta_i \Delta t \leqslant 1$ for $I_{n+1}^i \geqslant 0$ and $\beta_i \Delta t \leqslant 1$ implies $R_{n+1}^i \geqslant 0$. In the proof of Lemma 3, if $\gamma_i \Delta t$ is replaced by $\beta_i \Delta t$, the nonnegativity condition follows: $\alpha_{ii} \Delta t \leqslant (\sqrt{1 - a_i} + \sqrt{\beta_i \Delta t})^2$.

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