Stability Analysis

Given a system of first-order differential equations

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t)),$$

the equilibrium points are determined by $\frac{d\mathbf{x}(t)}{dt} = 0$, i.e.,

$$\mathbf{f}(\mathbf{x}(t)) = 0,$$

from which we solve the equilibrium point $\mathbf{x}(t) = \mathbf{x}_0$

Linear Stability Analysis

The system of differential equations is linearized using perturbations. For simplicity, we consider the following two-dimensional system,

$$\begin{aligned} \frac{x_1(t)}{dt} &= f_1(x_1, x_2) \\ \frac{x_2(t)}{dt} &= f_2(x_1, x_2). \end{aligned} \tag{1}$$

The equilibrium point $x_{1,0}, x_{2,0}$ is determined by

$$\begin{aligned} f_1(x_{1,0}, x_{2,0}) &= 0 \\ f_2(x_{1,0}, x_{2,0}) &= 0. \end{aligned}$$

We perturb the equations around its equilibrium point,

$$\begin{aligned} x_1(t) &\to x_{1,0} + \delta x_1(t) \\ x_2(t) &\to x_{2,0} + \delta x_2(t), \end{aligned}$$

where $\delta x_1(t)$ and $\delta x_2(t)$ are small quantities.

The equation (1) becomes

$$\begin{aligned} \frac{d\delta x_1(t)}{dt} &= f_1(x_{1,0} + \delta x_1(t), x_{2,0} + \delta x_2(t)) \\ \frac{d\delta x_2(t)}{dt} &= f_2(x_{1,0} + \delta x_1(t), x_{2,0} + \delta x_2(t)). \end{aligned} \tag{2}$$

We apply Talor series to $f_1(x_{1,0} + \delta x_1(t), x_{2,0} + \delta x_2(t))$,

$$\begin{aligned} & f_1(x_{1,0} + \delta x_1(t), x_{2,0} + \delta x_2(t)) \\ =& \frac{df_1(x_1, x_2)}{dx_1}\bigg|_{x_1=x_{1,0}, x_2=x_{2,0}} \delta x_1 + \frac{df_1(x_1, x_2)}{dx_2}\bigg|_{x_1=x_{1,0}, x_2=x_{2,0}} \delta x_2 + \mathcal{O}, \end{aligned}$$

where we have used $f_1(x_{1,0}, x_{2,0}) = 0$. The perturbed equation (2) is linearized

$$\frac{d\delta x_1(t)}{dt} = \left.\frac{df_1(x_1,x_2)}{dx_1}\right|_{x_1=x_{1,0},x_2=x_{2,0}} \delta x_1 + \left.\frac{df_1(x_1,x_2)}{dx_2}\right|_{x_1=x_{1,0},x_2=x_{2,0}} \delta x_2$$
$$\frac{d\delta x_2(t)}{dt} = \left.\frac{df_2(x_1,x_2)}{dx_1}\right|_{x_1=x_{1,0},x_2=x_{2,0}} \delta x_1 + \left.\frac{df_2(x_1,x_2)}{dx_2}\right|_{x_1=x_{1,0},x_2=x_{2,0}} \delta x_2. \quad (3)$$

The equation (3) can be rewritten into a matrix form

$$\frac{d}{dt}\begin{pmatrix} \delta x_1(t) \\ \delta x_2(t) \end{pmatrix} = \left.\begin{pmatrix} \frac{df_1(x_1,x_2)}{dx_1} & \frac{df_1(x_1,x_2)}{dx_2} \\ \frac{df_2(x_1,x_2)}{dx_1} & \frac{df_2(x_1,x_2)}{dx_2} \end{pmatrix}\right|_{x_1=x_{1,0},x_2=x_{2,0}} \begin{pmatrix} \delta x_1(t) \\ \delta x_2(t) \end{pmatrix}. \quad (4)$$

For simplicity, we define the Jacobian matrix

$$\mathbf{J} = \left.\begin{pmatrix} \frac{df_1(x_1,x_2)}{dx_1} & \frac{df_1(x_1,x_2)}{dx_2} \\ \frac{df_2(x_1,x_2)}{dx_1} & \frac{df_2(x_1,x_2)}{dx_2} \end{pmatrix}\right|_{x_1=x_{1,0},x_2=x_{2,0}},$$

so that the matrix form of the linearized equation (4) becomes,

$$\frac{d}{dt}\begin{pmatrix} \delta x_1(t) \\ \delta x_2(t) \end{pmatrix} = \mathbf{J}\begin{pmatrix} \delta x_1(t) \\ \delta x_2(t) \end{pmatrix}.$$

To investigate the stability of the differential system, we assume that

$$\begin{pmatrix} \delta x_1(t) \\ \delta x_2(t) \end{pmatrix} = \begin{pmatrix} \delta x_1(t_0) \\ \delta x_2(t_0) \end{pmatrix} e^{\lambda t},$$

which leads to linear equations

$$\begin{pmatrix} \delta x_1(t_0) \\ \delta x_2(t_0) \end{pmatrix} \lambda = \mathbf{J}\begin{pmatrix} \delta x_1(t_0) \\ \delta x_2(t_0) \end{pmatrix}.$$

For non-trivial solutions, we require

$$\mathrm{Det}(\mathbf{J} - \lambda \mathbf{I}) = 0,$$

which is also the eigen value problem of the Jacobian matrix. We expand the determinant

$$\lambda^2 - (J_{11} + J_{22})\lambda + (J_{11}J_{22} - J_{12}J_{21}) = 0. \quad (5)$$

For real positive solutions $\lambda > 0$, we get an exponential growing result for the linearized equation. Any deviation from the equilibrium point leads to a run-away process and the system moves further away from the equilibrium point. For real negative solutions $\lambda < 0$, the system will move back to the equilibrium point given any deviations from the equilibrium. Imaginary solutions of λ leads to oscillations.