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Examples for Week 02 (2)

1. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{x^3}{60}, \qquad 2 \le x \le 4, \qquad \text{zero elsewhere.}$$

Recall: The cumulative distribution function of X is

$$F_X(x) = 0, \qquad x < 2,$$

$$F_X(x) = P(X \le x) = \int_2^x \frac{u^3}{60}\,du = \frac{u^4}{240}\Bigg|_2^x = \frac{x^4 - 16}{240}, \qquad 2 \le x < 4,$$

$$F_X(x) = 1, \qquad x \ge 4.$$

Consider a continuous random variable X, with p.d.f. f and c.d.f. F, where F is strictly increasing on some interval I, F = 0 to the left of I, and F = 1 to the right of I. I may be a bounded interval or an unbounded interval such as the whole real line. $F^{-1}(u)$ is then well defined for $0 < u < 1$.

Fact 1: Let U ~ Uniform(0, 1), and let $X = F^{-1}(U)$. Then the c.d.f. of X is F.

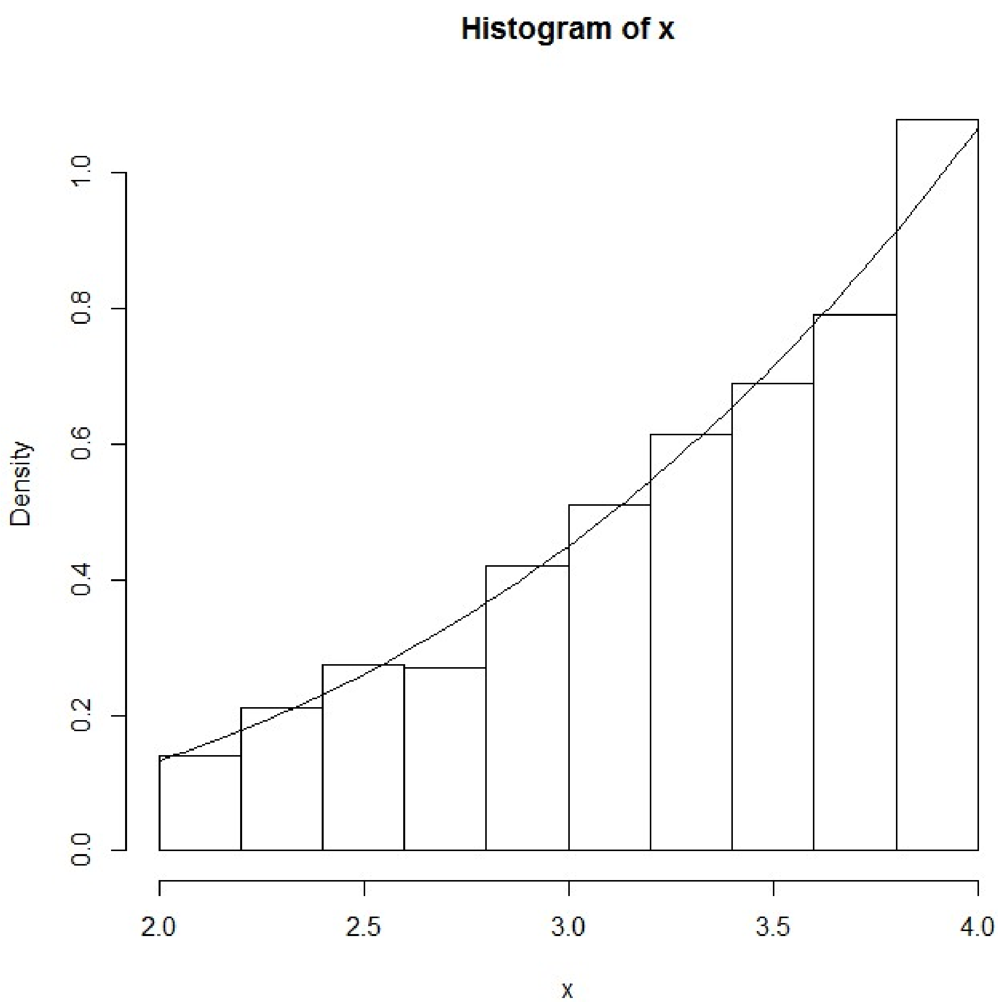
Proof: $P(X \le x) = P(F^{-1}(U) \le x) = P(U \le F(x)) = F(x)$.

Fact 2: Let $U = F(X)$; then U has a Uniform(0, 1) distribution.

Proof: $P(U \le u) = P(F(X) \le u) = P(X \le F^{-1}(u)) = F(F^{-1}(u)) = u$.

$$u = F(x) = \frac{x^4 - 16}{240} \qquad \Rightarrow \qquad x = (240\,u + 16)^{0.25} = F^{-1}(u).$$

```
> u = runif(1000)
> x = (240*u+16)^0.25
>
> hist(x, prob=TRUE)
> curve(x^3/60, add=TRUE)
```



Probability histogram of 1,000 simulated values of X with the probability density function $f_X(x)$ superimposed. They are indeed very close.

2. Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{x^3}{100}, \qquad x = 1, 2, 3, 4, \qquad \text{zero elsewhere.}$$

x	$p_X(x)$	$F_X(x)$
1	0.01	0.01
2	0.08	0.09
3	0.27	0.36
4	0.64	1.00

$$F_X(x) = \begin{cases} 0 & x < 1 \\ 0.01 & 1 \le x < 2 \\ 0.09 & 2 \le x < 3 \\ 0.36 & 3 \le x < 4 \\ 1 & x \ge 4 \end{cases}$$

U ~ Uniform(0, 1).

$$0 < u < 0.01 \quad \Rightarrow \quad x = 1$$

$$0.01 < u < 0.09 \quad \Rightarrow \quad x = 2$$

$$0.09 < u < 0.36 \quad \Rightarrow \quad x = 3$$

$$0.36 < u < 1 \quad \Rightarrow \quad x = 4$$

```
> u = runif(1000)
> x = rep(1,1000)
> for (i in 1:1000) {
+     if (u[i]>0.01) { x[i] = x[i]+1 }
+     if (u[i]>0.09) { x[i] = x[i]+1 }
+     if (u[i]>0.36) { x[i] = x[i]+1 }
+ }
> length(subset(x,x==1))/1000
[1] 0.009
> length(subset(x,x==2))/1000
[1] 0.078
> length(subset(x,x==3))/1000
[1] 0.278
> length(subset(x,x==4))/1000
[1] 0.635
```

These are indeed close to 0.01, 0.08, 0.27, 0.64.