STAT 410

Examples for Week 01 (Disc3)

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STAT 410 does NOT have a discussion section.

STAT 400 (an easier class) DOES have a discussion section.
STAT 410 (a harder class) does NOT.

There is a simple explanation for this: About 20 years ago, there was only one section of STAT 410 with about 40 students and only two lectures of STAT 400 with about 150 students each (each divided into 3 discussion sections of about 50 students each). The number of students has changed a little bit since then, the course structure has not (yet)…

However, don't you wish there was a discussion section in STAT 410?

If STAT 410 did have a discussion section, this is what your TA would most likely go over during it:

1. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{x}{C}, \qquad 3 \le x \le 7, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

b) Find the cumulative distribution function of X, $F_X(x)$.

"Hint": To double-check your answer: should be $F_X(3) = 0$, $F_X(7) = 1$.

1. (continued)

Consider $Y = g(X) = \dfrac{100}{X^2 + 1}$.

c) Find the support (the range of possible values) of the probability distribution of Y.

d) Use part (b) and the c.d.f. approach to find the c.d.f. of Y, $F_Y(y)$.

"Hint": $F_Y(y) = P(Y \le y) = P(g(X) \le y) = \dots$.

e) Use the change-of-variable technique to find the p.d.f. of Y, $f_Y(y)$.

"Hint": $f_Y(y) = f_X(g^{-1}(y)) \left| \dfrac{dx}{dy} \right|$.

"Hint": To double-check your answer: should be $f_Y(y) = F_Y'(y)$.

2. Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{x}{C}, \qquad x = 3, 4, 5, 6, 7, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $p_X(x)$ a valid probability mass function.

b) Consider $Y = g(X) = \dfrac{100}{X^2 + 1}$. Find the probability distribution of Y.

3. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{|x|}{C}, \qquad -2 \le x \le 1, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

b) Find the cumulative distribution function of X, $F_X(x)$.

"Hint": To double-check your answer: should be $F_X(-2) = 0$, $F_X(1) = 1$.

c) Consider $Y = g(X) = \dfrac{100}{X^2 + 1}$. Find the probability distribution of Y.

4. Let X be a random variable with probability density function

$$f_X(x) = -\ln x, \qquad 0 < x < 1, \qquad \text{zero otherwise.}$$

a) Find the probability distribution of $Y = -\ln X$.

b) Find the probability distribution of $Y = \sqrt{X}$.

c) Find the probability distribution of $Y = \dfrac{1}{\sqrt[3]{X}}$.

5. Let $\theta > 1$ and let X be a random variable with probability density function

$$f_X(x) = \frac{1}{x \ln\theta}, \qquad 1 < x < \theta.$$

a) Let $U = \ln X$. What is the probability distribution of U?

b) Let $a > 0$ and let $V = X^a$. What is the probability distribution of V?

c) Let $W = \dfrac{X}{\theta}$. What is the probability distribution of W?

d) Let $Y = \dfrac{1}{X}$. What is the probability distribution of Y?

Answers:

1. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{x}{C}, \qquad 3 \le x \le 7, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_3^7 \frac{x}{C}\,dx = \left.\frac{x^2}{2C}\right|_3^7 = \frac{49-9}{2C} = \frac{20}{C}.$$

$$\Rightarrow \qquad C = \mathbf{20}.$$

b) Find the cumulative distribution function of X, $F_X(x)$.

"Hint": To double-check your answer: should be $F_X(3) = 0$, $F_X(7) = 1$.

$$F_X(x) = 0, \qquad x < 3,$$

$$F_X(x) = P(X \le x) = \int_3^x \frac{u}{20}\,du = \left.\frac{x^2}{40}\right|_3^x = \frac{x^2-9}{40}, \qquad 3 \le x < 7,$$

$$F_X(x) = 1, \qquad x \ge 7.$$

1. (continued)

Consider $Y = g(X) = \dfrac{100}{X^2 + 1}$.

c) Find the support (the range of possible values) of the probability distribution of Y.

$g(x) = \dfrac{100}{x^2 + 1}$ – strictly decreasing on $(3, 7)$.

$g(3) = 10,$ $\qquad g(7) = 2.$ $\qquad \mathbf{2 \le y \le 10}.$

$3 \le x \le 7.$ $\qquad 9 \le x^2 \le 49.$ $\qquad 10 \le x^2 + 1 \le 50.$

$10 \ge \dfrac{100}{x^2 + 1} \ge 2.$ $\qquad \mathbf{2 \le y \le 10}.$

d) Use part (b) and the c.d.f. approach to find the c.d.f. of Y, $F_Y(y)$.

"Hint": $F_Y(y) = P(Y \le y) = P(g(X) \le y) = \dots$.

$$F_Y(y) = P(Y \le y) = P\left(\frac{100}{X^2 + 1} \le y\right) = P\left(X \ge \sqrt{\frac{100}{y} - 1}\right) = 1 - F_X\left(\sqrt{\frac{100}{y} - 1}\right)$$

$$= 1 - \frac{\left(\frac{100}{y} - 1\right) - 9}{40} = \frac{50 - \frac{100}{y}}{40} = \frac{5y - 10}{4y} = 1.25 - \frac{2.50}{y}, \qquad 2 \le y < 10.$$

$F_Y(y) = 0, \qquad y < 2,$ $\qquad F_Y(y) = 1, \qquad y \ge 10.$

e) Use the change-of-variable technique to find the p.d.f. of Y, $f_Y(y)$.

"Hint": $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$.

"Hint": To double-check your answer: should be $f_Y(y) = F_Y'(y)$.

$$y = \frac{100}{x^2+1} \qquad x = \sqrt{\frac{100}{y} - 1} \qquad \frac{dx}{dy} = \frac{1}{2\sqrt{\frac{100}{y}-1}} \cdot \left(-\frac{100}{y^2}\right)$$

$$f_Y(y) = \frac{\sqrt{\frac{100}{y}-1}}{20} \times \left| \frac{1}{2\sqrt{\frac{100}{y}-1}} \cdot \left(-\frac{100}{y^2}\right) \right| = \frac{5}{2y^2} = \frac{2.50}{y^2},$$

$$2 \le y \le 10.$$

Indeed, $\frac{d}{dy}\left(1.25 - \frac{2.50}{y}\right) = \frac{2.50}{y^2}$. ☺

To check: $\int_{-\infty}^{\infty} f_Y(y)\,dy = \int_2^{10} \frac{2.50}{y^2}\,dy = -\frac{2.50}{y}\Big|_2^{10}$

$= 1.25 - 0.25 = 1.$ ☺

2. Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{x}{C}, \qquad x = 3, 4, 5, 6, 7, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $p_X(x)$ a valid probability mass function.

$$1 = \sum_{\text{all } x} p_X(x) = \frac{3}{C} + \frac{4}{C} + \frac{5}{C} + \frac{6}{C} + \frac{7}{C} = \frac{25}{C}.$$

$\Rightarrow$ $C = \mathbf{25}$.

b) Consider $Y = g(X) = \dfrac{100}{X^2 + 1}$. Find the probability distribution of Y.

x	$p_X(x)$
3	$\frac{3}{25} = 0.12$
4	$\frac{4}{25} = 0.16$
5	$\frac{5}{25} = 0.20$
6	$\frac{6}{25} = 0.24$
7	$\frac{7}{25} = 0.28$

$\Rightarrow$

y	$p_Y(y)$
$\frac{100}{10} = 10$	0.12
$\frac{100}{17} \approx 5.882$	0.16
$\frac{100}{26} \approx 3.846$	0.20
$\frac{100}{37} \approx 2.703$	0.24
$\frac{100}{50} = 2$	0.28

OR

$$p_Y(y) = \frac{\sqrt{\frac{100}{y} - 1}}{25}, \qquad y = 2, \frac{100}{37}, \frac{100}{26}, \frac{100}{17}, 10.$$

3. Consider a continuous random variable X with the probability density function

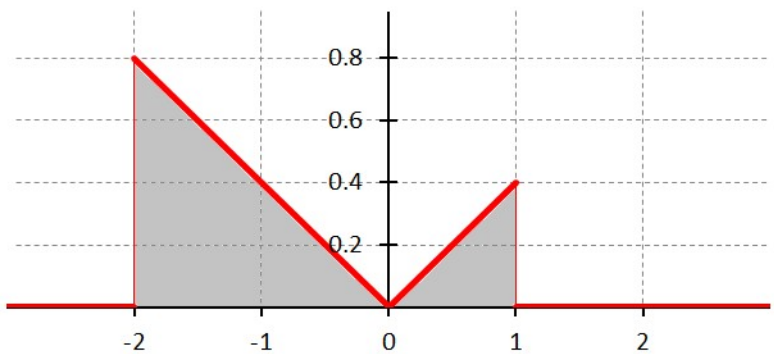
$$f_X(x) = \frac{|x|}{C}, \qquad -2 \le x \le 1, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-2}^{1} \frac{|x|}{C}\,dx = \int_{-2}^{0} \frac{-x}{C}\,dx + \int_{0}^{1} \frac{x}{C}\,dx$$

$$= \frac{-x^2}{2C}\Bigg|_{-2}^{0} + \frac{x^2}{2C}\Bigg|_{0}^{1} = \frac{4}{2C} + \frac{1}{2C} = \frac{5}{2C} = \frac{2.5}{C}.$$

$\Rightarrow \qquad C = \mathbf{2.5}.$



b) Find the cumulative distribution function of X, $F_X(x)$.

"Hint": To double-check your answer: should be $F_X(-2) = 0$, $F_X(1) = 1$.

$$F_X(x) = 0, \qquad x < -2,$$

$$F_X(x) = \int_{-2}^{x} \left(-\frac{u}{2.5} \right) du = \frac{4 - x^2}{5} = \frac{4}{5} - \frac{x^2}{5}, \qquad -2 \le x < 0,$$

$$F_X(x) = \int_{-2}^{0} \left(-\frac{u}{2.5} \right) du + \int_{0}^{x} \left(\frac{u}{2.5} \right) du = \frac{4}{5} + \frac{x^2}{5}, \qquad 0 \le x < 1,$$

$$F_X(x) = 1, \qquad x \ge 1.$$

c) Consider $Y = g(X) = \dfrac{100}{X^2 + 1}$. Find the probability distribution of Y.

$-2 \le x \le 1$ $\qquad$ $-2 \le x \le 0$ $\qquad$ $0 \le x \le 1$

$y = \dfrac{100}{x^2 + 1}$ $\qquad$ $20 \le y \le 100$ $\qquad$ $100 \ge y \ge 50$

$F_Y(y) = 0, \quad y < 20, \qquad F_Y(y) = 1, \quad y \ge 100.$

$20 \le y \le 100$

$$F_Y(y) = P(Y \le y) = P\left(\frac{100}{X^2+1} \le y\right) = P\left(X^2 \ge \frac{100}{y} - 1\right)$$

$$= P\left(X \le -\sqrt{\frac{100}{y}-1}\right) + P\left(X \ge \sqrt{\frac{100}{y}-1}\right)$$

$$= F_X\left(-\sqrt{\frac{100}{y}-1}\right) + 1 - F_X\left(\sqrt{\frac{100}{y}-1}\right).$$

<u>Case 1</u>. $\quad 20 \le y < 50. \qquad 4 \ge \dfrac{100}{y} - 1 > 1 \qquad 1 < \sqrt{\dfrac{100}{y}-1} \le 2.$

$$F_X\left(-\sqrt{\frac{100}{y}-1}\right) = \frac{4}{5} - \frac{\frac{100}{y}-1}{5} = 1 - \frac{20}{y}, \qquad F_X\left(\sqrt{\frac{100}{y}-1}\right) = 1.$$

$$F_Y(y) = F_X\left(-\sqrt{\frac{100}{y}-1}\right) + 1 - F_X\left(\sqrt{\frac{100}{y}-1}\right) = 1 - \frac{20}{y}, \qquad 20 \le y < 50.$$

Case 2. $\quad 50 \le y < 100. \qquad 1 \ge \frac{100}{y} - 1 > 0 \qquad 0 < \sqrt{\frac{100}{y} - 1} \le 1.$

$$F_X\left(-\sqrt{\frac{100}{y}-1}\right) = \frac{4}{5} - \frac{\frac{100}{y}-1}{5} = 1 - \frac{20}{y},$$

$$F_X\left(\sqrt{\frac{100}{y}-1}\right) = \frac{4}{5} + \frac{\frac{100}{y}-1}{5} = \frac{3}{5} + \frac{20}{y}.$$

$$F_Y(y) = F_X\left(-\sqrt{\frac{100}{y}-1}\right) + 1 - F_X\left(\sqrt{\frac{100}{y}-1}\right) = \frac{7}{5} - \frac{40}{y}, \qquad 50 \le y < 100.$$

c.d.f.

$$F_Y(y) = \begin{cases} 0 & y < 20 \\ 1 - \frac{20}{y} & 20 \le y < 50 \\ \frac{7}{5} - \frac{40}{y} & 50 \le y < 100 \\ 1 & y \ge 100 \end{cases}$$

p.d.f.

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{20}{y^2} & 20 < y < 50 \\ \frac{40}{y^2} & 50 < y < 100 \\ 0 & \text{otherwise} \end{cases}$$

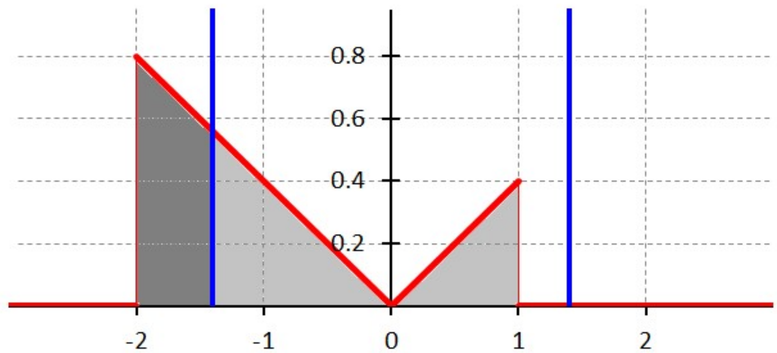
OR

$20 \le y \le 100$ $\qquad F_Y(y) = P\left(X \le -\sqrt{\dfrac{100}{y} - 1}\right) + P\left(X \ge \sqrt{\dfrac{100}{y} - 1}\right).$

Case 1. $\qquad 20 \le y < 50.$

$$4 \ge \frac{100}{y} - 1 > 1.$$

$$1 < \sqrt{\frac{100}{y} - 1} \le 2.$$

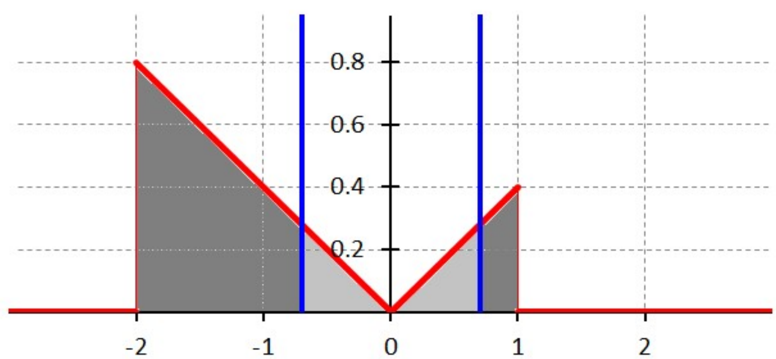


$$F_Y(y) = \int_{-2}^{-\sqrt{\frac{100}{y}-1}} \left(-\frac{x}{2.5}\right) dx = \frac{4 - \left(\frac{100}{y} - 1\right)}{5} = 1 - \frac{20}{y}, \qquad 20 \le y < 50.$$

Case 2. $\qquad 50 \le y < 100.$

$$1 \ge \frac{100}{y} - 1 > 0.$$

$$0 < \sqrt{\frac{100}{y} - 1} \le 1.$$



$$F_Y(y) = \int_{-2}^{-\sqrt{\frac{100}{y}-1}} \left(-\frac{x}{2.5}\right) dx + \int_{\sqrt{\frac{100}{y}-1}}^{1} \left(\frac{x}{2.5}\right) dx$$

$$= \frac{4 - \left(\frac{100}{y} - 1\right)}{5} + \frac{1 - \left(\frac{100}{y} - 1\right)}{5} = \frac{7}{5} - \frac{40}{y}, \qquad 20 \le y < 50.$$

OR

$$-2 \le x < 0$$

$$f_X(x) = -\frac{x}{2.5}$$

$$Y = g(X) = \frac{100}{X^2 + 1}$$

$$20 \le y < 100$$

$$x = -\sqrt{\frac{100}{y} - 1} = g^{-1}(y)$$

$$\frac{dx}{dy} = -\frac{1}{2\sqrt{\frac{100}{y} - 1}} \cdot \left(-\frac{100}{y^2}\right)$$

$$f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$$

$$-\frac{-\sqrt{\frac{100}{y} - 1}}{2.5} \times \left| -\frac{1}{2\sqrt{\frac{100}{y} - 1}} \cdot \left(-\frac{100}{y^2}\right) \right|$$

$$\frac{20}{y^2}$$

$$0 < x \le 1$$

$$f_X(x) = \frac{x}{2.5}$$

$$Y = g(X) = \frac{100}{X^2 + 1}$$

$$100 > y \ge 50$$

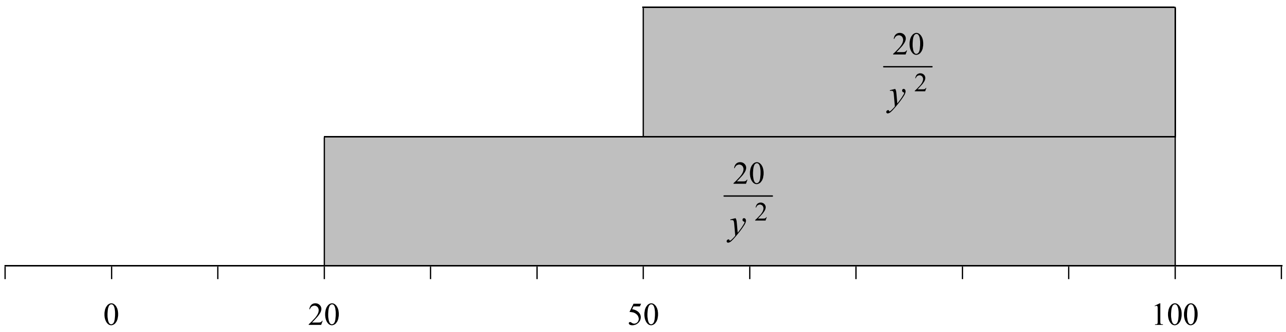
$$x = \sqrt{\frac{100}{y} - 1} = g^{-1}(y)$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{\frac{100}{y} - 1}} \cdot \left(-\frac{100}{y^2}\right)$$

$$f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$$

$$\frac{\sqrt{\frac{100}{y} - 1}}{2.5} \times \left| \frac{1}{2\sqrt{\frac{100}{y} - 1}} \cdot \left(-\frac{100}{y^2}\right) \right|$$

$$\frac{20}{y^2}$$



$$f_Y(y) = \frac{20}{y^2}, \qquad\qquad f_Y(y) = \frac{20}{y^2} + \frac{20}{y^2} = \frac{40}{y^2},$$

$$20 < y < 50. \qquad\qquad\qquad 50 < y < 100.$$

p.d.f.

$$f_Y(y) = F_Y'(y) = \begin{cases} \dfrac{20}{y^2} & 20 < y < 50 \\[2ex] \dfrac{40}{y^2} & 50 < y < 100 \\[2ex] 0 & \text{otherwise} \end{cases}$$

To check:

$$\int_{-\infty}^{\infty} f_Y(y)\,dy = \int_{20}^{50} \frac{20}{y^2}\,dy + \int_{50}^{100} \frac{40}{y^2}\,dy$$

$$= \left(-\frac{20}{y}\right)\Bigg|_{20}^{50} + \left(-\frac{40}{y}\right)\Bigg|_{50}^{100}$$

$$= -\frac{20}{50} + \frac{20}{20} - \frac{40}{100} + \frac{40}{50} = 1.$$

☺

4. Let X be a random variable with probability density function

$$f_X(x) = -\ln x, \qquad 0 < x < 1, \qquad \text{zero otherwise.}$$

a) Find the probability distribution of $Y = -\ln X$.

$$y = -\ln x \qquad 0 < x < 1 \qquad \Rightarrow \qquad 0 < y < \infty.$$

$$y = -\ln x \qquad x = e^{-y} \qquad \frac{dx}{dy} = -e^{-y}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = y \times \left| -e^{-y} \right| = y\,e^{-y}, \qquad 0 < y < \infty.$$

Y has a Gamma distribution with $\alpha = 2$ and $\theta = 1$.

b) Find the probability distribution of $Y = \sqrt{X}$.

$$y = \sqrt{x} \qquad 0 < x < 1 \qquad \Rightarrow \qquad 0 < y < 1.$$

$$y = \sqrt{x} \qquad x = y^2 \qquad \frac{dx}{dy} = 2y$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = -2\ln y \times \left| 2y \right| = -4y\ln y, \qquad 0 < y < 1.$$

c) Find the probability distribution of $Y = \dfrac{1}{\sqrt[3]{X}}$.

$$y = \frac{1}{\sqrt[3]{x}} \qquad 0 < x < 1 \qquad \Rightarrow \qquad 1 < y < \infty.$$

$$y = \frac{1}{\sqrt[3]{x}} \qquad x = y^{-3} \qquad \frac{dx}{dy} = -3y^{-4}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = 3\ln y \times \left| -3y^{-4} \right| = 9y^{-4}\ln y = \frac{9\ln y}{y^4},$$

$$1 < y < \infty.$$

5. Let $\theta > 1$ and let X be a random variable with probability density function

$$f_X(x) = \frac{1}{x \ln\theta}, \qquad 1 < x < \theta.$$

a) Let $U = \ln X$. What is the probability distribution of U?

$1 < x < \theta \qquad u = \ln x \qquad \Rightarrow \qquad 0 < u < \ln\theta$

$x = e^u \qquad {}^{dx}/_{du} = e^u$

$$f_U(u) = f_X(g^{-1}(u)) \left| \frac{dx}{du} \right| = \frac{1}{e^u \ln\theta} \cdot e^u = \frac{1}{\ln\theta}, \qquad 0 < u < \ln\theta.$$

Uniform on $(0, \ln\theta)$.

b) Let $a > 0$ and let $V = X^a$. What is the probability distribution of V?

$1 < x < \theta \qquad v = x^a \qquad \Rightarrow \qquad 1 < v < \theta^a$

$x = v^{1/a} \qquad {}^{dx}/_{dv} = \frac{1}{a} v^{(1/a)-1}$

$$f_V(v) = f_X(g^{-1}(v)) \left| \frac{dx}{dv} \right| = \frac{1}{v^{(1/a)} \ln\theta} \cdot \frac{1}{a} v^{(1/a)-1} = \frac{1}{a\, v \ln\theta} = \frac{1}{v \ln\theta^a},$$

$$1 < v < \theta^a.$$

c) Let $W = \dfrac{X}{\theta}$. What is the probability distribution of W?

$$1 < x < \theta \qquad w = \frac{x}{\theta} \qquad \Rightarrow \qquad \frac{1}{\theta} < w < 1$$

$$x = \theta w \qquad {}^{dx}\!/_{dw} = \theta$$

$$f_W(w) = f_X(g^{-1}(w)) \left| \frac{dx}{dw} \right| = \frac{1}{\theta\, w \ln\theta} \cdot \theta = \frac{1}{w \ln\theta}, \qquad \frac{1}{\theta} < w < 1.$$

d) Let $Y = \dfrac{1}{X}$. What is the probability distribution of Y?

$$1 < x < \theta \qquad y = \frac{1}{x} \qquad \Rightarrow \qquad \frac{1}{\theta} < y < 1$$

$$x = \frac{1}{y} \qquad {}^{dx}\!/_{dy} = -\frac{1}{y^2}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \frac{y}{\ln\theta} \cdot \frac{1}{y^2} = \frac{1}{y \ln\theta}, \qquad \frac{1}{\theta} < y < 1.$$