Examples for Week 01 (2)

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4. The p.d.f. of X is $f_X(x) = \theta\, x^{\theta-1}$, $0 < x < 1$, $0 < \theta < \infty$. Let $Y = -2\,\theta \ln X$. How is Y distributed?

a) Determine the probability distribution of Y by finding the c.d.f. of Y

$$F_Y(y) = P(Y \le y) = P(-2\,\theta \ln X \le y).$$

"Hint": Find $F_X(x)$ first.

$F_X(x) = x^{\theta}$, $0 < x < 1$.

$0 < x < 1$ $\quad y = -2\,\theta \ln x$ $\quad \Rightarrow \quad y > 0$

$$F_Y(y) = P(Y \le y) = P(-2\,\theta \ln X \le y) = P(X \ge e^{-y/2\theta}) = 1 - e^{-y/2}, \quad y > 0.$$

$$\Rightarrow \quad f_Y(y) = F_Y'(y) = \frac{1}{2} e^{-y/2}, \quad y > 0.$$

$\Rightarrow$ Y has Exponential distribution with mean 2.

b) Determine the probability distribution of Y by finding the m.g.f. of Y

$$M_Y(t) = E(e^{Y \cdot t}) = E(e^{-2\,\theta \ln X \cdot t}).$$

$$M_Y(t) = E(e^{Y \cdot t}) = E(e^{-2\,\theta \ln X \cdot t}) = E(X^{-2\,\theta\, t}) = \int_0^1 \left(x^{-2\theta t} \cdot \theta\, x^{\theta-1} \right) dx$$

$$= \int_0^1 \theta\, x^{\theta - 2\theta t - 1}\, dx = \frac{\theta}{\theta - 2\theta t} = \frac{1}{1 - 2t}, \qquad t < \frac{1}{2}.$$

$\Rightarrow$ Y has Exponential distribution with mean 2.

c) Determine the probability distribution of Y by finding the p.d.f. of Y, $f_Y(y)$, using the change-of-variable technique.

$$y = g(x) = -2\theta \ln x \qquad \Rightarrow \qquad x = g^{-1}(y) = \; = e^{-y/2\theta}$$

$$\Rightarrow \qquad \frac{dx}{dy} = -\frac{1}{2\theta} e^{-y/2\theta}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \theta \left(e^{-y/2\theta} \right)^{\theta - 1} \times \left| -\frac{1}{2\theta} e^{-y/2\theta} \right| = \frac{1}{2} e^{-y/2},$$

$$y > 0.$$

$\Rightarrow$ Y has Exponential distribution with mean 2.

Exponential
$0 < \theta$

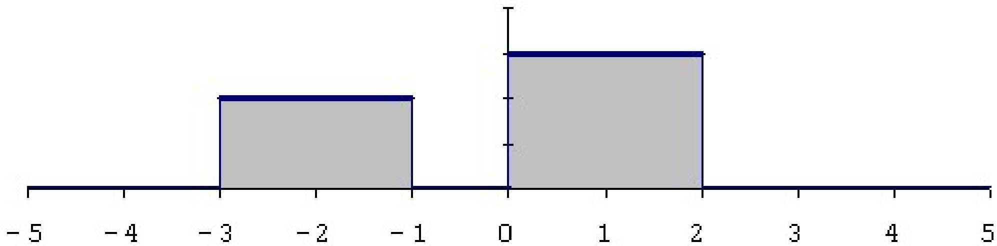
$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \qquad 0 \le x < \infty$$

$$M(t) = \frac{1}{1 - \theta t}, \qquad t < \frac{1}{\theta}$$

$$\mu = \theta, \qquad \sigma^2 = \theta^2$$

5. Consider a continuous random variable X with p.d.f.

$$f_X(x) = \begin{cases} 0.2 & -3 < x < -1 \\ 0.3 & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$



Find the probability distribution of $Y = X^2$.

-3, -1, 0, and 2 are "important" for X. 0 is "important" for $g(x) = x^2$.

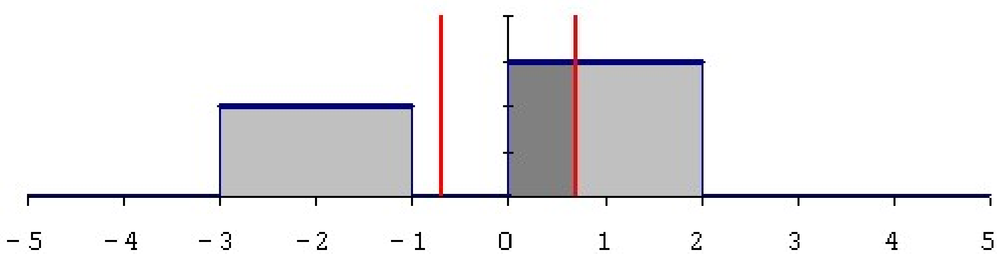
$\Rightarrow$ $(-3)^2$, $(-1)^2$, $(0)^2$, and $(2)^2$ are "important" for $Y = X^2$.

$\Rightarrow$ 0, 1, 4, and 9 are "important" for Y.

$y < 0$ $\quad P(X^2 \le y) = 0 \quad F_Y(y) = 0.$

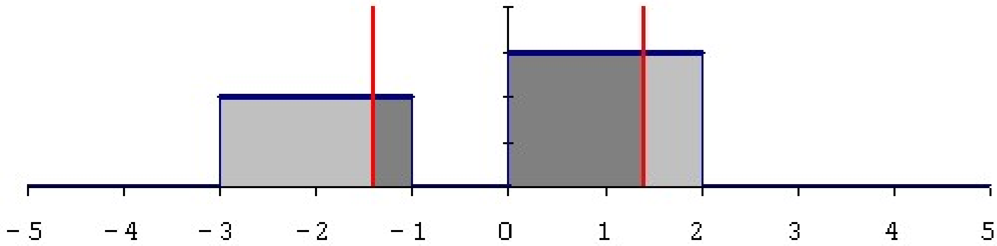
$y \ge 0$ $\quad F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \ldots$

Case 1: $\quad 0 \le y < 1 \quad \Rightarrow \quad 0 \le \sqrt{y} < 1$



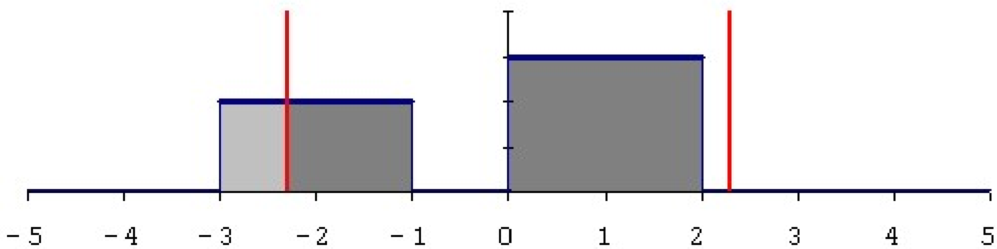
$F_Y(y) = 0.3\sqrt{y}.$

Case 2: $1 \le y < 4 \quad \Rightarrow \quad 1 \le \sqrt{y} < 2$



$$F_Y(y) = 0.2\,(-1 + \sqrt{y}\,) + 0.3\sqrt{y}\,.$$

Case 3: $4 \le y < 9 \quad \Rightarrow \quad 2 \le \sqrt{y} < 3$



$$F_Y(y) = 0.2\,(-1 + \sqrt{y}\,) + 0.6.$$

Case 4: $y \ge 9 \qquad F_Y(y) = 1.$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 0.3\sqrt{y} & 0 \le y < 1 \\ 0.5\sqrt{y} - 0.2 & 1 \le y < 4 \\ 0.2\sqrt{y} + 0.4 & 4 \le y < 9 \\ 1 & y \ge 9 \end{cases} \qquad f_Y(y) = \begin{cases} \dfrac{0.15}{\sqrt{y}} & 0 < y < 1 \\ \dfrac{0.25}{\sqrt{y}} & 1 < y < 4 \\ \dfrac{0.10}{\sqrt{y}} & 4 < y < 9 \\ 0 & \text{otherwise} \end{cases}$$

OR

$$F_X(x) = \begin{cases} 0 & x < -3 \\ 0.2(x+3) & -3 \le x < -1 \\ 0.4 & -1 \le x < 0 \\ 0.3x + 0.4 & 0 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$= \begin{cases} 0 & y < 0 & \\ (0.3\sqrt{y} + 0.4) - (0.4) & 0 \le y < 1 & 0 \le \sqrt{y} < 1 \\ (0.3\sqrt{y} + 0.4) - (0.2(-\sqrt{y} + 3)) & 1 \le y < 4 & 1 \le \sqrt{y} < 2 \\ (1) - (0.2(-\sqrt{y} + 3)) & 4 \le y < 9 & 2 \le \sqrt{y} < 3 \\ (1) - (0) & y \ge 9 & \sqrt{y} \ge 3 \end{cases}$$

$$= \begin{cases} 0 & y < 0 \\ 0.3\sqrt{y} & 0 \le y < 1 \\ 0.5\sqrt{y} - 0.2 & 1 \le y < 4 \\ 0.2\sqrt{y} + 0.4 & 4 \le y < 9 \\ 1 & y \ge 9 \end{cases}$$

OR

$$-3 < x < -1$$

$$f_X(x) = 0.2$$

$$Y = g(X) = X^2$$

$$x = -\sqrt{y} = g^{-1}(y)$$

$$\frac{dx}{dy} = -\frac{1}{2\sqrt{y}}$$

$$9 > y > 1$$

$$f_X(g^{-1}(y))\left|\frac{dx}{dy}\right|$$

$$0.2\cdot\left|-\frac{1}{2\sqrt{y}}\right| = \frac{0.10}{\sqrt{y}}$$

$$0 < x < 2$$

$$f_X(x) = 0.3$$

$$Y = g(X) = X^2$$

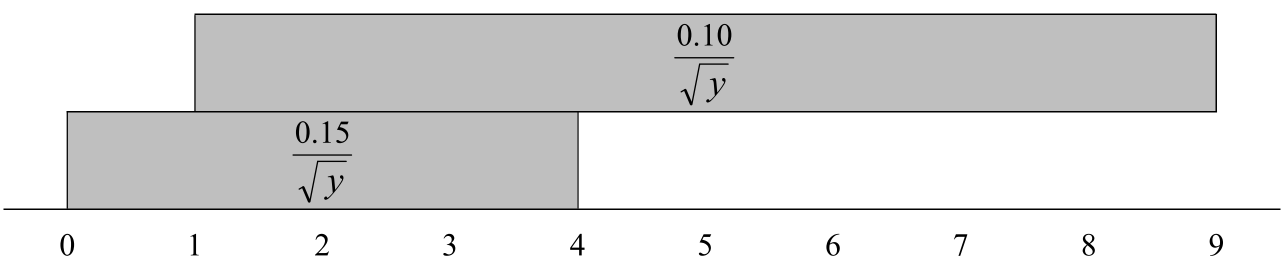
$$x = \sqrt{y} = g^{-1}(y)$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

$$0 < y < 4$$

$$f_X(g^{-1}(y))\left|\frac{dx}{dy}\right|$$

$$0.3\cdot\left|\frac{1}{2\sqrt{y}}\right| = \frac{0.15}{\sqrt{y}}$$



$$f_Y(y) = \begin{cases} 0 + \dfrac{0.15}{\sqrt{y}} & 0 < y < 1 \\[2ex] \dfrac{0.10}{\sqrt{y}} + \dfrac{0.15}{\sqrt{y}} & 1 < y < 4 \\[2ex] \dfrac{0.10}{\sqrt{y}} + 0 & 4 < y < 9 \\[2ex] 0 & \text{otherwise} \end{cases} = \begin{cases} \dfrac{0.15}{\sqrt{y}} & 0 < y < 1 \\[2ex] \dfrac{0.25}{\sqrt{y}} & 1 < y < 4 \\[2ex] \dfrac{0.10}{\sqrt{y}} & 4 < y < 9 \\[2ex] 0 & \text{otherwise} \end{cases}$$