Examples for Week 02 (4)

A. Stepanov

1. Consider a mixed random variable X with

the p.m.f. of the discrete portion of the probability distribution

$p(-1) = 0.10,$ $\quad p(2) = 0.15,$ $\quad$ zero otherwise,

and the p.d.f. of the continuous portion of the probability distribution

$$f(x) = \frac{2x+c}{20}, \qquad -1 < x < 2, \qquad \text{zero elsewhere.}$$

a) Find the value of c that would make this a valid probability distribution.

b) Find $\mathrm{E}(\mathrm{X})$.

c) Find $\mathrm{Var}(\mathrm{X})$.

2. Find $\mathrm{E}(\mathrm{X})$ and $\mathrm{Var}(\mathrm{X})$ for a mixed random variable X with the following cumulative distribution function:

$$\mathrm{F}_\mathrm{X}(x) = \begin{cases} 0 & x < -1 \\ \dfrac{2x+3}{16} & -1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

\+ Find probabilities

$\mathrm{P}(-1 < \mathrm{X} \le 1)$, $\mathrm{P}(-1 < \mathrm{X} < 1)$, $\mathrm{P}(-1 \le \mathrm{X} \le 1)$, $\mathrm{P}(-1 \le \mathrm{X} < 1)$.

Bonus:

3. Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{2x+3}{16}, \qquad x = -1, 0, 1, 2, \qquad \text{zero elsewhere.}$$

Consider $Y = g(X) = 4 - X^2$. Find the probability distribution of Y.

4. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{2x+3}{12}, \qquad -1 \le x \le 2, \qquad \text{zero elsewhere.}$$

Consider $Y = g(X) = 4 - X^2$. Find the probability distribution of Y.

1. Consider a mixed random variable X with

the p.m.f. of the discrete portion of the probability distribution

$p(-1) = 0.10,\qquad p(2) = 0.15,\qquad$ zero otherwise,

and the p.d.f. of the continuous portion of the probability distribution

$f(x) = \dfrac{2x+c}{20},\qquad -1 < x < 2,\qquad$ zero elsewhere.

a) Find the value of c that would make this a valid probability distribution.

$$1 = \sum_{\text{all } x} p(x) + \int_{-\infty}^{\infty} f(x)\,dx$$

$$= [\,0.10 + 0.15\,] + \int_{-1}^{2} \frac{2x+c}{20}\,dx = 0.25 + \left.\frac{x^2+cx}{20}\right|_{-1}^{2}$$

$$= 0.25 + \frac{3+3c}{20} = 0.40 + 0.15\,c.$$

$\Rightarrow\qquad c = \mathbf{4}.$

b) Find E(X).

$$\mathrm{E}(\mathrm{X}) = \sum_{\text{all } x} x\cdot p(x) + \int_{-\infty}^{\infty} x\cdot f(x)\,dx$$

$$= -1\cdot 0.10 + 2\cdot 0.15 + \int_{-1}^{2} x\cdot\frac{2x+4}{20}\,dx$$

$$= 0.20 + \left.\frac{2x^3+6x^2}{60}\right|_{-1}^{2} = 0.20 + \frac{40-4}{60} = 0.20 + 0.60 = \mathbf{0.80}.$$

c) Find $\operatorname{Var}(X)$.

$$E(X^2) = \sum_{\text{all } x} x^2 \cdot p(x) + \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$$

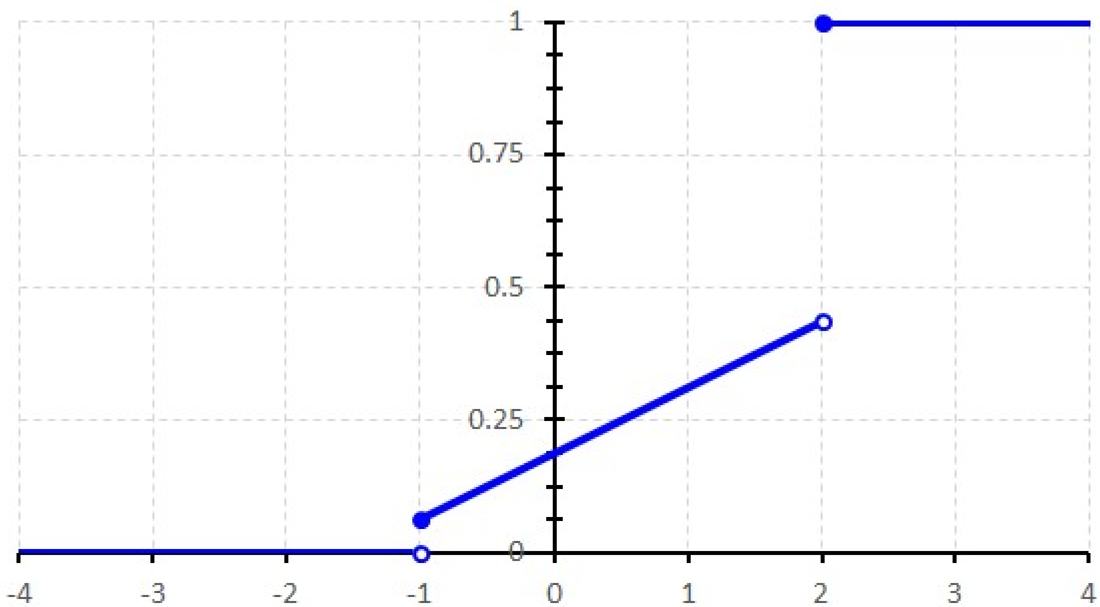
$$= (-1)^2 \cdot 0.10 + (2)^2 \cdot 0.15 + \int_{-1}^{2} x^2 \cdot \frac{2x+4}{20}\, dx$$

$$= 0.70 + \left. \frac{3x^4 + 8x^3}{120} \right|_{-1}^{2} = 0.70 + \frac{112+5}{120} = 0.70 + 0.975 = 1.675.$$

$$\operatorname{Var}(X) = E(X^2) - [E(X)]^2 = 1.675 - 0.80^2 = \mathbf{1.035}.$$

2. Find $\mathrm{E}(\mathrm{X})$ and $\mathrm{Var}(\mathrm{X})$ for a mixed random variable X with the following cumulative distribution function:

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \dfrac{2x+3}{16} & -1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$



Discrete:

$F_X(x)$ "jumps" at $x = -1$ from 0 to $\frac{1}{16}$, $\quad p(-1) = \frac{1}{16}$,

and at $x = 2$ from $\frac{7}{16}$ to 1, $\quad p(2) = \frac{9}{16}$.

Continuous:

$f_X(x) = F'_X(x) = \frac{2}{16} = \frac{1}{8}$, $\quad -1 < x < 2$, $\quad$ zero otherwise.

$$\mathrm{E}(\mathrm{X}) = \sum_{\text{all } x} x \cdot p(x) + \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= -1 \cdot \frac{1}{16} + 2 \cdot \frac{9}{16} + \int_{-1}^{2} x \cdot \frac{1}{8}\, dx = \frac{17}{16} + \frac{x^2}{16} \Bigg|_{-1}^{2}$$

$$= \frac{17}{16} + \frac{3}{16} = \mathbf{\frac{20}{16}} = \mathbf{\frac{5}{4}} = \mathbf{1.25}.$$

$$\mathrm{E}(\mathrm{X}^2) = \sum_{\text{all } x} x^2 \cdot p(x) + \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$$

$$= (-1)^2 \cdot \frac{1}{16} + (2)^2 \cdot \frac{9}{16} + \int_{-1}^{2} x^2 \cdot \frac{1}{8}\, dx = \frac{37}{16} + \frac{x^3}{24} \Bigg|_{-1}^{2}$$

$$= \frac{37}{16} + \frac{9}{24} = \frac{43}{16} = 2.6875.$$

$$\mathrm{Var}(\mathrm{X}) = \mathrm{E}(\mathrm{X}^2) - [\mathrm{E}(\mathrm{X})]^2 = 2.6875 - 1.25^2 = \mathbf{\frac{9}{8}} = \mathbf{1.125}.$$

Notations:

$$g(a-) = \lim_{x \nearrow a} g(x) = \lim_{\substack{x \to a \\ x < a}} g(x) \quad - \text{ limit from the left.}$$

$$g(a+) = \lim_{x \searrow a} g(x) = \lim_{\substack{x \to a \\ x > a}} g(x) \quad - \text{ limit from the right.}$$

Def Cumulative distribution function: $F_X(x) = P(X \le x)$.

Then $P(X \le x) = F(x)$,

$P(X < x) = F(x-)$.

$P(X = x) = P(X \le x) - P(X < x) = F(x) - F(x-)$.

If c.d.f. $F(x)$ is continuous at a, if $F(a) = F(a-)$, then $P(X = a) = 0$.

If c.d.f. $F(x)$ jumps at a, then there is positive probability "attached" to a,

$P(X = a) = F(a) - F(a-)$ – size of the jump.

$$P(-1 < X \le 1) = P(X \le 1) - P(X \le -1) = F(1) - F(-1) = \frac{5}{16} - \frac{1}{16} = \frac{4}{16}.$$

$$P(-1 < X < 1) = P(X < 1) - P(X \le -1) = F(1-) - F(-1) = \frac{5}{16} - \frac{1}{16} = \frac{4}{16}.$$

$$P(-1 \le X \le 1) = P(X \le 1) - P(X < -1) = F(1) - F(-1-) = \frac{5}{16} - 0 = \frac{5}{16}.$$

$$P(-1 \le X < 1) = P(X < 1) - P(X < -1) = F(1-) - F(-1-) = \frac{5}{16} - 0 = \frac{5}{16}.$$

OR

$$P(-1 < X \le 1) = P(-1 < X < 1) = \int_{-1}^{1} \frac{1}{8}\,dx = \frac{2}{8}.$$

$$P(-1 \le X \le 1) = P(-1 \le X < 1) = p(-1) + \int_{-1}^{1} \frac{1}{8}\,dx = \frac{1}{16} + \frac{2}{8} = \frac{5}{16}.$$

3. Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{2x+3}{16}, \qquad x = -1, 0, 1, 2, \qquad \text{zero elsewhere.}$$

Consider $Y = g(X) = 4 - X^2$. Find the probability distribution of Y.

x	$p_X(x)$	$g(x)$
-1	$\frac{1}{16} = 0.0625$	3
0	$\frac{3}{16} = 0.1875$	4
1	$\frac{5}{16} = 0.3125$	3
2	$\frac{7}{16} = 0.4375$	0

y	$p_Y(y)$
0	$\mathbf{\frac{7}{16} = 0.4375}$
3	$\mathbf{\frac{6}{16} = 0.3750}$
4	$\mathbf{\frac{3}{16} = 0.1875}$

For fun:

$$E(X) = (-1)\cdot\frac{1}{16} + 0\cdot\frac{3}{16} + 1\cdot\frac{5}{16} + 2\cdot\frac{7}{16} = \frac{18}{16} = 1.125.$$

$$E(Y) = 0\cdot\frac{7}{16} + 3\cdot\frac{6}{16} + 4\cdot\frac{3}{16} = \frac{30}{16} = 1.875.$$

Note that

$$g(E(X)) = 4 - [E(X)]^2 = 2.734375 \neq E(g(X)) = E(Y) = 1.875.$$

Recall: IF $g(x)$ is a linear function, that is, IF $g(x) = ax + b$,

then $E(g(X)) = E(aX+b) = a\,E(X) + b = g(E(X))$.

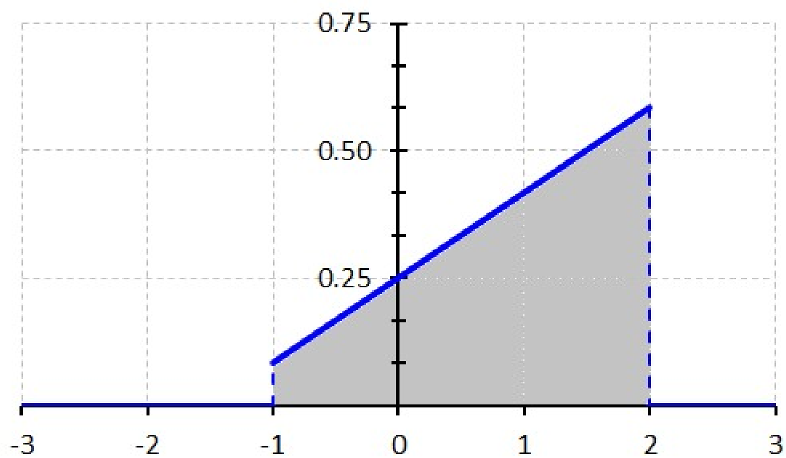
However, in general, if $g(x)$ is NOT a linear function,

then $E(g(X)) \neq g(E(X))$.

4. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{2x+3}{12}, \qquad -1 \le x \le 2, \qquad \text{zero elsewhere.}$$

Consider $Y = g(X) = 4 - X^2$. Find the probability distribution of Y.



"important" points for X: $x = -1$ and $x = 2$.

"important" point for $g(x)$: $x = 0$.

Therefore, "important" points for Y:

$4 - (-1)^2 = 3,$

$4 - (2)^2 = 0,$

$4 - (0)^2 = 4.$

There are two cases to be considered separately: $0 < y < 3, \quad 3 < y < 4.$

Technically, there are four cases: $y < 0, \quad 0 < y < 3, \quad 3 < y < 4, \quad y > 4,$

but $y < 0$ and $y > 4$ are boring.

$x < -1$, $\qquad F_X(x) = 0,$

$-1 \;\; x < 2$, $\qquad F_X(x) = P(X \le x) = \int\limits_{-\infty}^{x} f_X(u)\,du = \int\limits_{-1}^{x} \frac{2u+3}{12}\,du$

$$= \frac{u^2+3u}{12}\Bigg|_{-1}^{x} = \frac{x^2+3x+2}{12} = \frac{(x+1)(x+2)}{12},$$

$x \ge 2$, $\qquad F_X(x) = 1.$

Case 0: $y \geq 4.$ $4 - y \leq 0.$

$$F_Y(y) = P(Y \leq y) = P(4 - X^2 \leq y) = P(X^2 \geq 4 - y) = 1.$$

If $y < 4$, then $4 - y > 0.$

$$F_Y(y) = P(Y \leq y) = P(4 - X^2 \leq y) = P(X^2 \geq 4 - y)$$
$$= P(X \leq -\sqrt{4-y}\,) + P(X \geq \sqrt{4-y}\,)$$
$$= F_X(-\sqrt{4-y}\,) + 1 - F_X(\sqrt{4-y}\,).$$

Case 1: $0 \leq y < 3.$ $4 \geq 4 - y > 1,$ $1 < \sqrt{4-y} \leq 2,$
$-2 \leq -\sqrt{4-y} < -1.$

$$F_X(-\sqrt{4-y}\,) = 0, \qquad F_X(\sqrt{4-y}\,) = \frac{4 - y + 3\sqrt{4-y} + 2}{12}.$$

$$F_Y(y) = F_X(-\sqrt{4-y}\,) + 1 - F_X(\sqrt{4-y}\,) = \frac{6 + y - 3\sqrt{4-y}}{12}.$$

Case 2: $3 \leq y < 4.$ $1 \geq 4 - y > 0,$ $0 < \sqrt{4-y} \leq 1,$
$-1 \leq -\sqrt{4-y} < 0.$

$$F_X(-\sqrt{4-y}\,) = \frac{4 - y - 3\sqrt{4-y} + 2}{12}, \quad F_X(\sqrt{4-y}\,) = \frac{4 - y + 3\sqrt{4-y} + 2}{12}.$$

$$F_Y(y) = F_X(-\sqrt{4-y}\,) + 1 - F_X(\sqrt{4-y}\,) = 1 - \frac{6\sqrt{4-y}}{12} = \frac{12 - 6\sqrt{4-y}}{12}.$$

Case 3: $y < 0.$ $4 - y > 4,$ $\sqrt{4-y} > 2.$
$-\sqrt{4-y} < -2.$

$$F_X(-\sqrt{4-y}\,) = 0, \qquad F_X(\sqrt{4-y}\,) = 1.$$

$$F_Y(y) = F_X(-\sqrt{4-y}\,) + 1 - F_X(\sqrt{4-y}\,) = 0.$$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \dfrac{6 + y - 3\sqrt{4-y}}{12} & 0 \le y < 3 \\ 1 - \dfrac{\sqrt{4-y}}{2} & 3 \le y < 4 \\ 1 & y \ge 4 \end{cases}$$

Indeed, $\quad 0 = \dfrac{6 + 0 - 3\sqrt{4-0}}{12}$,

$$\frac{6 + 3 - 3\sqrt{4-3}}{12} = \frac{1}{2} = 1 - \frac{\sqrt{4-3}}{2},$$

$$1 - \frac{\sqrt{4-4}}{2} = 1.$$

☺

OR

If $0 \le y < 4$, then

$$F_Y(y) = P(Y \le y) = P(4 - X^2 \le y) = P(X^2 \ge 4 - y)$$

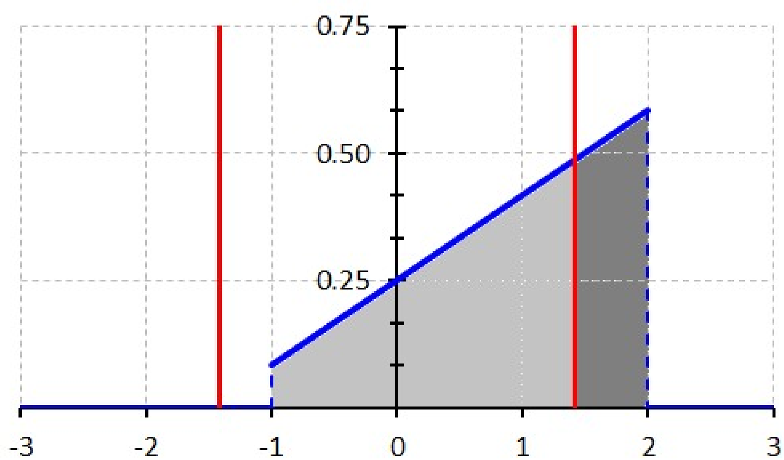
$$= P(X \le -\sqrt{4-y}\,) + P(X \ge \sqrt{4-y}\,).$$

Case 1: $\quad 0 \le y < 3.$

$4 \ge 4 - y > 1,$

$1 < \sqrt{4-y} \le 2,$

$-2 \le -\sqrt{4-y} < -1.$



$$F_Y(y) = P(X \le -\sqrt{4-y}\,) + P(X \ge \sqrt{4-y}\,)$$

$$= 0 + \int_{\sqrt{4-y}}^{2} \frac{2x+3}{12}\,dx = \left.\frac{x^2+3x}{12}\right|_{\sqrt{4-y}}^{2} = \frac{10}{12} - \frac{4-y+3\sqrt{4-y}}{12}$$

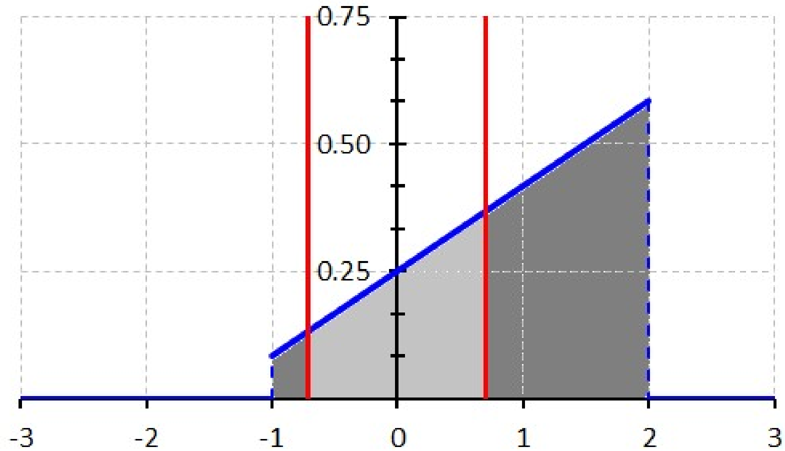
$$= \frac{6+y-3\sqrt{4-y}}{12} = \frac{1}{2} + \frac{y}{12} - \frac{\sqrt{4-y}}{4}, \qquad 0 \le y < 3.$$

Case 2: $3 \le y < 4$.

$1 \ge 4-y > 0$,

$0 < \sqrt{4-y} \le 1$,

$-1 \le -\sqrt{4-y} < 0$.



$$F_Y(y) = P(X \le -\sqrt{4-y}\,) + P(X \ge \sqrt{4-y}\,)$$

$$= \int_{-1}^{-\sqrt{4-y}} \frac{2x+3}{12}\,dx + \int_{\sqrt{4-y}}^{2} \frac{2x+3}{12}\,dx$$

$$= \left.\frac{x^2+3x}{12}\right|_{-1}^{-\sqrt{4-y}} + \left.\frac{x^2+3x}{12}\right|_{\sqrt{4-y}}^{2}$$

$$= \frac{4-y-3\sqrt{4-y}}{12} - \frac{-2}{12} + \frac{10}{12} - \frac{4-y+3\sqrt{4-y}}{12}$$

$$= \frac{12-6\sqrt{4-y}}{12} = 1 - \frac{\sqrt{4-y}}{2}, \qquad 3 \le y < 4.$$

OR

$-1 < x < 0$

$$f_X(x) = \frac{2x+3}{12}$$

$$Y = g(X) = 4 - X^2$$

$$3 < y < 4$$

$$x = -\sqrt{4-y} = g^{-1}(y)$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{4-y}}$$

$$f_X(g^{-1}(y))\left|\frac{dx}{dy}\right|$$

$$\frac{-2\sqrt{4-y}+3}{12} \cdot \frac{1}{2\sqrt{4-y}}$$

$$\frac{-2\sqrt{4-y}+3}{24\sqrt{4-y}}$$

$0 < x < 2$

$$f_X(x) = \frac{2x+3}{12}$$

$$Y = g(X) = 4 - X^2$$

$$4 > y > 0$$

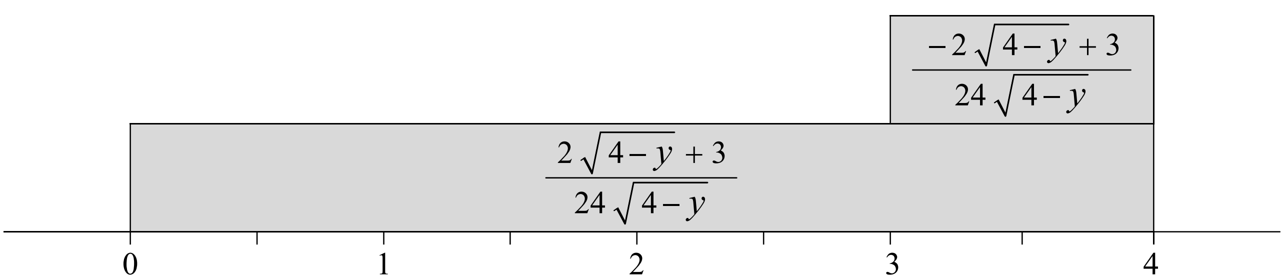
$$x = \sqrt{4-y} = g^{-1}(y)$$

$$\frac{dx}{dy} = -\frac{1}{2\sqrt{4-y}}$$

$$f_X(g^{-1}(y))\left|\frac{dx}{dy}\right|$$

$$\frac{2\sqrt{4-y}+3}{12} \cdot \frac{1}{2\sqrt{4-y}}$$

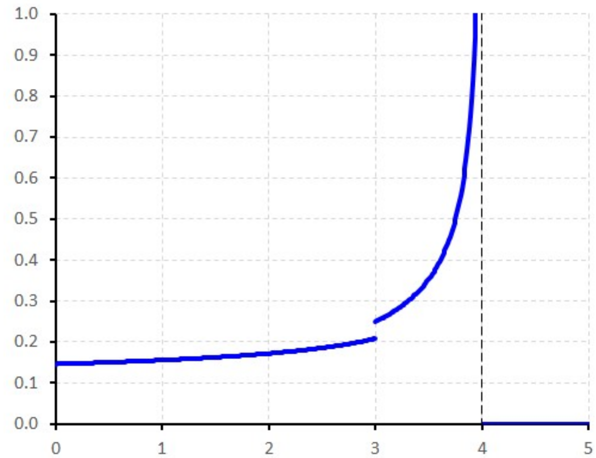
$$\frac{2\sqrt{4-y}+3}{24\sqrt{4-y}}$$



If $0 < y < 3$, $\qquad f_Y(y) = 0 + \dfrac{2\sqrt{4-y}+3}{24\sqrt{4-y}} = \dfrac{2\sqrt{4-y}+3}{24\sqrt{4-y}} = \dfrac{1}{12} + \dfrac{1}{8\sqrt{4-y}}.$

If $3 < y < 4$, $\qquad f_Y(y) = \dfrac{-2\sqrt{4-y}+3}{24\sqrt{4-y}} + \dfrac{2\sqrt{4-y}+3}{24\sqrt{4-y}} = \dfrac{1}{4\sqrt{4-y}}.$

$$f_Y(y) = \begin{cases} \dfrac{1}{12} + \dfrac{1}{8\sqrt{4-y}} & 0 < y < 3 \\[2ex] \dfrac{1}{4\sqrt{4-y}} & 3 < y < 4 \\[2ex] 0 & \text{otherwise} \end{cases}$$



Indeed,
$$\frac{d}{dy}\left(\frac{1}{2} + \frac{y}{12} - \frac{\sqrt{4-y}}{4} \right) = \frac{1}{12} + \frac{1}{8\sqrt{4-y}},$$
$$\frac{d}{dy}\left(1 - \frac{\sqrt{4-y}}{2} \right) = \frac{1}{4\sqrt{4-y}}.$$

☺

Graph of
$y = g(x) = 4 - x^2$,
$-1 \le x \le 2$:

(to convince yourself that the possible range of y values is $0 < y < 4$, and that $0 < y < 3$ and $3 < y < 4$ are different)