STAT 410 | **Homework #01** | Spring 2026

(due Friday, January 30, by 11:59 p.m.)

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No credit will be given without supporting work.

1. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{2x+3}{C}, \qquad 1 < x < 6, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_1^6 \frac{2x+3}{C}\,dx = \left.\frac{x^2+3x}{C}\right|_1^6 = \frac{(36+18)-(1+3)}{C} = \frac{50}{C}.$$

$\Rightarrow \qquad C = \mathbf{50}.$

$$f_X(x) = \frac{2x+3}{50}, \qquad 1 < x < 6.$$

b) Find the cumulative distribution function of X, $F_X(x) = P(X \le x)$.

"Hint": To double-check your answer: should be $F_X(1) = 0$, $F_X(6) = 1$.

$$F_X(x) = P(X \le x) = \int_1^x \frac{2u+3}{50}\,du = \left.\frac{u^2+3u}{50}\right|_1^x$$

$$= \frac{x^2+3x-4}{50} = \frac{(x-1)(x+4)}{50}, \qquad 1 \le x < 6.$$

Obviously, $F_X(x) = 0, \quad x < 1, \qquad F_X(x) = 1, \quad x \ge 6.$

Indeed, $\dfrac{(1-1)(1+4)}{50} = 0, \qquad \dfrac{(6-1)(6+4)}{50} = 1.$ ☺

c) Find the expected value of X, $E(X) = \mu_X$.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_1^6 x \cdot \frac{2x+3}{50}\,dx = \left.\frac{\frac{2}{3}x^3 + \frac{3}{2}x^2}{50}\right|_1^6$$

$$= \frac{(144+54) - \left(\frac{2}{3} + \frac{3}{2}\right)}{50} = \frac{1{,}175}{300} = \mathbf{\frac{47}{12}} \approx 3.916667.$$

1. (continued)

Consider $Y = g(X) = \sqrt{X+3}$. Find the probability distribution of Y:

d) Find the support (the range of possible values) of the probability distribution of Y.

$1 < x < 6 \quad \Rightarrow \quad 4 < x + 3 < 9$

$\Rightarrow \quad 2 < \sqrt{x+3} < 3 \quad \Rightarrow \quad \mathbf{2 < y < 3}.$

e) Use part (b) and the c.d.f. approach to find the c.d.f. of Y, $F_Y(y)$.

"Hint": $F_Y(y) = P(Y \le y) = P(g(X) \le y) = \dots$.

$$F_Y(y) = P(Y \le y) = P(\sqrt{X+3} \le y) = P(X \le y^2 - 3) = F_X(y^2 - 3)$$

$$= \frac{\left(y^2 - 4\right)\left(y^2 + 1\right)}{50} = \frac{(y-2)(y+2)\left(y^2+1\right)}{50}$$

$$= \frac{\left(y^2 - 3\right)^2 + 3\left(y^2 - 3\right) - 4}{50} = \frac{y^4 - 3y^2 - 4}{50}, \qquad 2 \le y < 3.$$

Obviously, $F_Y(y) = 0, \quad y < 2, \qquad F_Y(y) = 1, \quad y \ge 3.$

Indeed, $\dfrac{\left(2^2 - 4\right)\left(2^2 + 1\right)}{50} = 0, \qquad \dfrac{\left(3^2 - 4\right)\left(3^2 + 1\right)}{50} = 1.$ ☺

f) Use the change-of-variable technique to find the p.d.f. of Y, $f_Y(y)$.

"Hint": $f_Y(y) = f_X(g^{-1}(y)) \left| \dfrac{dx}{dy} \right|$.

"Hint": To double-check your answer: should be $f_Y(y) = F_Y'(y)$.

$$y = \sqrt{x+3}, \qquad x = y^2 - 3, \qquad \frac{dx}{dy} = 2y.$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \frac{2\left(y^2-3\right)+3}{50} \cdot |2y|$$

$$= \frac{4y^3-6y}{50} = \frac{2y^3-3y}{25}, \qquad 2 < y < 3.$$

Indeed, $$\frac{d}{dy}\left(\frac{y^4-3y^2-4}{50}\right) = \frac{4y^3-6y}{50}.$$ ☺

1. (continued)

Consider $W = h(X) = \dfrac{10}{X+4}$. Find the probability distribution of W:

g) Find the support (the range of possible values) of the probability distribution of W.

$1 < x < 6 \quad \Rightarrow \quad 5 < x + 4 < 10$

$\Rightarrow \quad 2 > \dfrac{10}{x+4} > 1 \quad \Rightarrow \quad \mathbf{1 < w < 2}.$

h) Use part (b) and the c.d.f. approach to find the c.d.f. of W, $F_W(w)$.

"Hint": $F_W(w) = P(W \le w) = P(h(X) \le w) = \dots$.

$$F_W(w) = P(W \le w) = P\left(\frac{10}{X+4} \le w\right) = P\left(X \ge \frac{10}{w} - 4\right) = 1 - F_X\left(\frac{10}{w} - 4\right)$$

$$= 1 - \frac{\left(\frac{10}{w} - 4\right)^2 + 3\left(\frac{10}{w} - 4\right) - 4}{50} = 1 - \frac{\frac{100}{w^2} - \frac{50}{w}}{50}$$

$$= 1 - \frac{2-w}{w^2} = \frac{w^2 + w - 2}{w^2} = \frac{(w-1)(w+2)}{w^2}$$

$$= 1 - \frac{\left(\frac{10}{w} - 5\right)\frac{10}{w}}{50} = 1 + \frac{1}{w} - \frac{2}{w^2}, \qquad 1 \le w < 2.$$

Obviously, $F_W(w) = 0, \quad w < 1, \qquad F_W(w) = 1, \quad w \ge 2.$

Indeed, $\quad \dfrac{1^2+1-2}{1^2} = 0, \qquad \dfrac{2^2+2-2}{2^2} = 1.$ ☺

i) Use the change-of-variable technique to find the p.d.f. of W, $f_W(w)$.

"Hint": $\quad f_W(w) = f_X(h^{-1}(w)) \left| \dfrac{dx}{dw} \right|.$

"Hint": To double-check your answer: should be $f_W(w) = F'_W(w)$.

$$w = \frac{10}{x+4}, \qquad x = \frac{10}{w} - 4, \qquad \frac{dx}{dw} = -\frac{10}{w^2}.$$

$$f_W(w) = f_X(h^{-1}(w)) \left| \frac{dx}{dw} \right| = \frac{2\left(\frac{10}{w} - 4\right) + 3}{50} \cdot \left| -\frac{10}{w^2} \right|$$

$$= \frac{\frac{200}{w} - 50}{50\, w^2} = \frac{4-w}{w^3} = \frac{4}{w^3} - \frac{1}{w^2}, \qquad 1 < w < 2.$$

Indeed, $\quad \dfrac{d}{dw}\left(1 + \dfrac{1}{w} - \dfrac{2}{w^2}\right) = \dfrac{4}{w^3} - \dfrac{1}{w^2}.$ ☺

For fun:

j) (i) Find the expected value of Y, $\mu_Y = E(Y)$.

(ii) Does μ_Y equal to $g(\mu_X)$?

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy = \int_2^3 y \cdot \frac{2y^3 - 3y}{25}\,dy = \int_2^3 \frac{2y^4 - 3y^2}{25}\,dy$$

$$= \left(\frac{\frac{2}{5}y^5 - y^3}{25} \right) \Bigg|_2^3 = \frac{\left(\frac{486}{5} - 27 \right) - \left(\frac{64}{5} - 8 \right)}{25} = \mathbf{\frac{327}{125}} = \mathbf{2.616}.$$

OR

$$E(Y) = E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\,dx = \int_1^6 \sqrt{x+3} \cdot \frac{2x+3}{50}\,dx$$

$$u = x + 3 \qquad x = u - 3 \qquad dx = du$$

$$= \int_4^9 \sqrt{u} \cdot \frac{2u-3}{50}\,du = \frac{1}{25} \cdot \int_4^9 u^{1.5}\,du - \frac{3}{50} \cdot \int_4^9 u^{0.5}\,du$$

$$= \frac{1}{25} \cdot \frac{u^{2.5}}{2.5} \Bigg|_4^9 - \frac{3}{50} \cdot \frac{u^{1.5}}{1.5} \Bigg|_4^9$$

$$= \frac{1}{25} \cdot \frac{243 - 32}{2.5} - \frac{3}{50} \cdot \frac{27 - 8}{1.5} = 3.376 - 0.76 = \mathbf{2.616}.$$

$$g(\mu_X) = \sqrt{\frac{47}{12} + 3} = \sqrt{\frac{83}{12}} \approx 2.62995564.$$

$\mu_Y = E(Y) \neq g(E(X)) = g(\mu_X)$. μ_Y does **NOT** equal to $g(\mu_X)$.

Recall: IF $g(x)$ is a linear function, that is, IF $g(x) = ax + b$,

then $E(g(X)) = E(aX + b) = aE(X) + b = g(E(X))$.

However, in general, if $g(x)$ is NOT a linear function,

then $E(g(X)) \neq g(E(X))$.

Spoiler:

Here, $E(g(X)) < g(E(X))$ since $g(x) = \sqrt{x+3}$ "curves down" for $1 < x < 6$.

k) (i) Find the expected value of W, $\mu_W = E(W)$.

(ii) Does μ_W equal to $h(\mu_X)$?

$$E(W) = \int_{-\infty}^{\infty} w \cdot f_W(w)\,dw = \int_1^2 w \cdot \frac{4-w}{w^3}\,dw = \int_1^2 \left(\frac{4}{w^2} - \frac{1}{w} \right) dw$$

$$= \left(-\frac{4}{w} - \ln w \right) \Bigg|_1^2 = -\frac{4}{2} - \ln(2) + \frac{4}{1} + \ln(1)$$

$$= 2 - \ln(2) \approx 1.30685282.$$

OR

$$\mathrm{E}(\mathrm{W}) = \mathrm{E}(h(\mathrm{X})) = \int_{-\infty}^{\infty} h(x) \cdot f_{\mathrm{X}}(x)\,dx = \int_{1}^{6} \frac{10}{x+4} \cdot \frac{2x+3}{50}\,dx$$

$u = x + 4 \qquad x = u - 4 \qquad dx = du$

$$= \int_{5}^{10} \frac{10}{u} \cdot \frac{2u-5}{50}\,du = \int_{5}^{10} \left(\frac{2}{5} - \frac{1}{u} \right) du = \left(\frac{2}{5} \cdot u - \ln u \right) \Big|_{5}^{10}$$

$$= 4 - \ln(10) - 2 + \ln(5) = 2 - \ln\left(\frac{10}{5} \right)$$

$$= 2 - \ln(2) \approx 1.30685282.$$

$$h(\mu_{\mathrm{X}}) = \frac{10}{\frac{47}{12} + 4} = \frac{120}{95} = \frac{24}{19} \approx 1.2631579.$$

$\mu_{\mathrm{W}} = \mathrm{E}(\mathrm{W}) \neq h(\mathrm{E}(\mathrm{X})) = h(\mu_{\mathrm{X}})$. $\qquad$ μ_{W} does **NOT** equal to $h(\mu_{\mathrm{X}})$.

Recall: IF $h(x)$ is a linear function, that is, IF $h(x) = a\,x + b$,

then $\mathrm{E}(h(\mathrm{X})) = \mathrm{E}(a\,\mathrm{X} + b) = a\,\mathrm{E}(\mathrm{X}) + b = h(\mathrm{E}(\mathrm{X}))$.

However, in general, if $h(x)$ is NOT a linear function,

then $\mathrm{E}(h(\mathrm{X})) \neq h(\mathrm{E}(\mathrm{X}))$.

Spoiler:

Here, $\mathrm{E}(h(\mathrm{X})) > h(\mathrm{E}(\mathrm{X}))$ since $h(x) = \dfrac{10}{x+4}$ "curves up"

for $1 < x < 6$.

l) Simulate 1,000 values of X.

Make a probability histogram of the simulated values.

Superimpose $f_X(x)$ to the histogram.

$$u = \mathrm{F}(x) = \frac{x^2 + 3x - 4}{50} \qquad \Rightarrow \qquad x = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot (-4 - 50u)}}{2 \cdot 1}.$$

$$1 < x < 6 \qquad \Rightarrow \qquad x = \frac{\sqrt{200u + 25} - 3}{2} = \mathrm{F}^{-1}(u).$$

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> u = runif(1000)
> x = (sqrt(200*u+25)-3)/2
> hist(x,prob=TRUE)
> curve((2*x+3)/50,add=TRUE)
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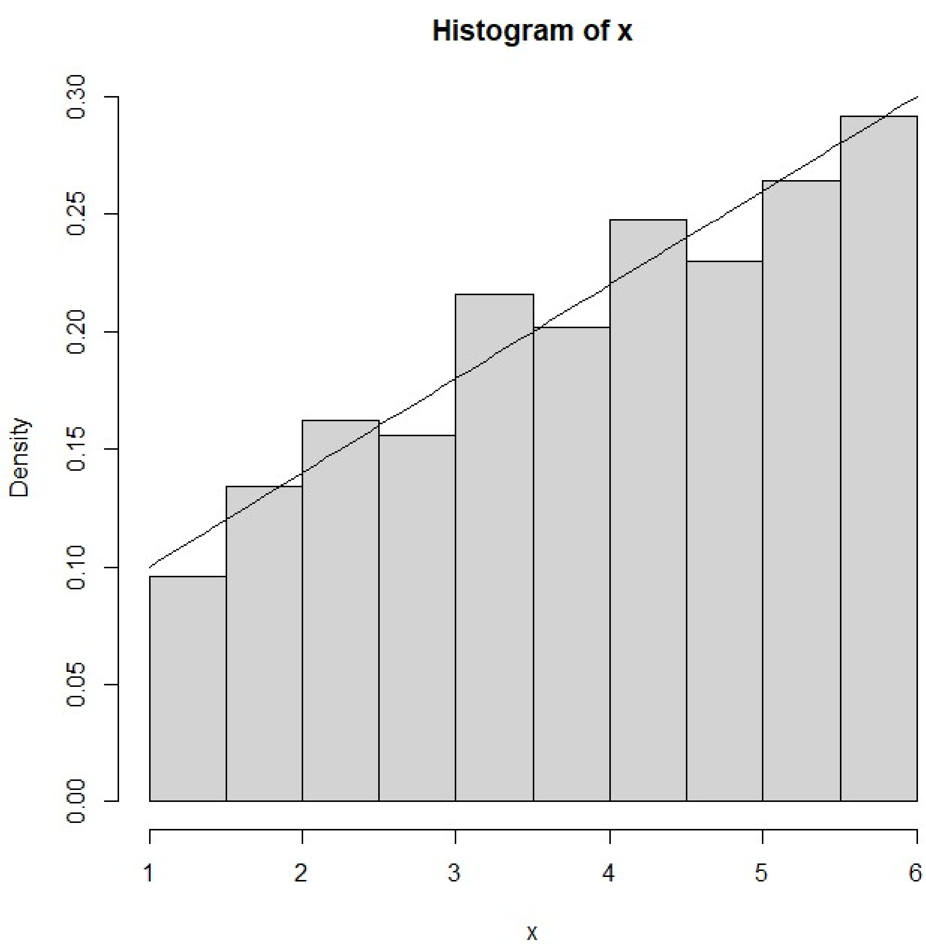

2. Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{2x+3}{C}, \qquad x = 2, 3, 4, 5, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $p_X(x)$ a valid probability mass function.

$$p_X(2) = \frac{7}{C}, \qquad p_X(3) = \frac{9}{C}, \qquad p_X(4) = \frac{11}{C}, \qquad p_X(5) = \frac{13}{C}.$$

$$p_X(2) + p_X(3) + p_X(4) + p_X(5) = 1.$$

$$\frac{7}{C} + \frac{9}{C} + \frac{11}{C} + \frac{13}{C} = \frac{40}{C} = 1. \qquad \Rightarrow \qquad C = \mathbf{40}.$$

b) Consider $Y = \dfrac{12}{X-1}$. Find the probability distribution of Y.

x	$p_X(x)$	$\frac{12}{x-1}$
2	$\frac{7}{40}$	12
3	$\frac{9}{40}$	6
4	$\frac{11}{40}$	4
5	$\frac{13}{40}$	3

$\Rightarrow$

y	$p_Y(y)$
3	$\mathbf{\frac{13}{40} = 0.325}$
4	$\mathbf{\frac{11}{40} = 0.275}$
6	$\mathbf{\frac{9}{40} = 0.225}$
12	$\mathbf{\frac{7}{40} = 0.175}$

OR

$$y = \frac{12}{x-1}, \qquad x = \frac{12}{y} + 1.$$

$$p_Y(y) = p_X\left(\frac{12}{y} + 1\right) = \frac{\frac{24}{y} + 5}{40} = \frac{24 + 5y}{40y} = \frac{3}{5y} + \frac{1}{8}, \qquad y = 3, 4, 6, 12.$$

Indeed, $p_Y(3) = \frac{3}{15} + \frac{1}{8} = 0.325,$ $\qquad p_Y(4) = \frac{3}{20} + \frac{1}{8} = 0.275,$

$p_Y(6) = \frac{3}{30} + \frac{1}{8} = 0.225,$ $\qquad p_Y(12) = \frac{3}{60} + \frac{1}{8} = 0.175.$

For discrete random variables, the possible values are isolated points on the number line.

$\Rightarrow$ no derivatives. $\qquad \Rightarrow$ no $\frac{dx}{dy}$.

For fun:

c) (i) Find the expected value of X, $\mu_X = E(X)$.

(ii) Find the expected value of Y, $\mu_Y = E(Y)$.

(ii) Does μ_Y equal to $\dfrac{12}{\mu_X - 1}$?

$$E(X) = \sum_{\text{all } x} x \cdot p_X(x) = 2 \cdot \frac{7}{40} + 3 \cdot \frac{9}{40} + 4 \cdot \frac{11}{40} + 5 \cdot \frac{13}{40} = \frac{150}{40} = \mathbf{3.75}.$$

$$E(Y) = \sum_{\text{all } y} y \cdot p_Y(y) = 3 \cdot \frac{13}{40} + 4 \cdot \frac{11}{40} + 6 \cdot \frac{9}{40} + 12 \cdot \frac{7}{40} = \frac{221}{40} = \mathbf{5.525}.$$

$$\frac{12}{\mu_X - 1} = \frac{12}{3.75 - 1} = \frac{48}{11} \approx 4.36363636.$$

μ_Y does **NOT** equal to $\dfrac{12}{\mu_X - 1}$.

Recall: IF $g(x)$ is a linear function, that is, IF $g(x) = a\,x + b$,

then $E(g(X)) = E(a\,X + b) = a\,E(X) + b = g(E(X))$.

However, in general, if $g(x)$ is NOT a linear function,

then $E(g(X)) \neq g(E(X))$.

Spoiler:

Here, $E(g(X)) > g(E(X))$ since $g(x) = \dfrac{12}{x-1}$ "curves up"

for $2 \le x \le 5$.