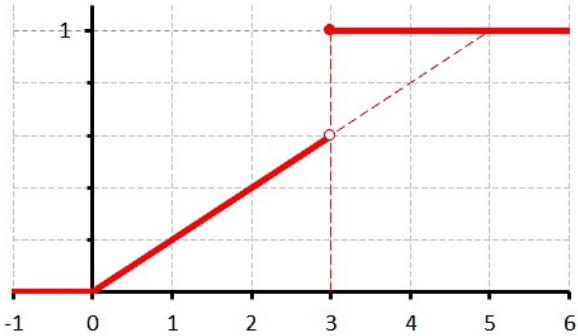


Mixed Random Variables

1. Consider a random variable X with c.d.f.

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{5} & 0 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$



a) Find $\mu = E(X)$.

b) Find $\sigma^2 = \text{Var}(X)$.

For example, a light bulb's lifetime has a Uniform distribution on $(0, 5)$, and it is replaced at failure or at age 3, whichever occurs first. X is the age of the light bulb at the time of replacement.

Discrete portion of the probability distribution of X :

$$F(x) \text{ jumps at } x=3 \text{ from } \frac{3}{5} \text{ to } 1. \quad p(3) = 1 - \frac{3}{5} = \frac{2}{5} = 0.4.$$

Continuous portion of the probability distribution of X :

$$f(x) = F'(x) = \begin{cases} \frac{1}{5} & 0 < x < 3 \\ 0 & \text{o.w.} \end{cases}$$

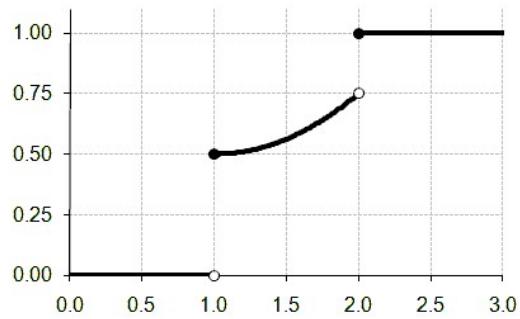
a) $\mu = E(X) = 3 \cdot 0.4 + \int_0^3 x \cdot \frac{1}{5} dx = 1.2 + \frac{x^2}{10} \Big|_0^3 = 1.2 + 0.9 = 2.1.$

b) $E(X^2) = 3^2 \cdot 0.4 + \int_0^3 x^2 \cdot \frac{1}{5} dx = 3.6 + \frac{x^3}{15} \Big|_0^3 = 3.6 + 1.8 = 5.4.$

$$\sigma^2 = \text{Var}(X) = E(X^2) - [E(X)]^2 = 5.4 - 2.1^2 = 0.99.$$

2. Consider a random variable X with c.d.f.

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{x^2 - 2x + 3}{4} & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$



a) Find $\mu = E(X)$.

b) Find $\sigma^2 = \text{Var}(X)$.

Discrete portion of the probability distribution of X:

$$p(1) = 1/2, \quad p(2) = 1/4.$$

Continuous portion of the probability distribution of X:

$$f(x) = \begin{cases} \frac{x-1}{2} & 1 < x < 2 \\ 0 & \text{o.w.} \end{cases}.$$

$$\text{a)} \quad \mu = E(X) = 1 \cdot 1/2 + 2 \cdot 1/4 + \int_1^2 x \cdot \frac{x-1}{2} dx = 17/12.$$

$$\text{b)} \quad E(X^2) = 1^2 \cdot 1/2 + 2^2 \cdot 1/4 + \int_1^2 x^2 \cdot \frac{x-1}{2} dx = 53/24.$$

$$\sigma^2 = \text{Var}(X) = E(X^2) - [E(X)]^2 = 29/144.$$

$$P(1 < X < 1.5) = F(1.5-) - F(1) = 0.5625 - 0.5000 = 0.0625.$$

$$P(1 < X \leq 1.5) = F(1.5) - F(1) = 0.5625 - 0.5000 = 0.0625.$$

$$P(1 \leq X < 1.5) = F(1.5-) - F(1-) = 0.5625 - 0.0000 = 0.5625.$$

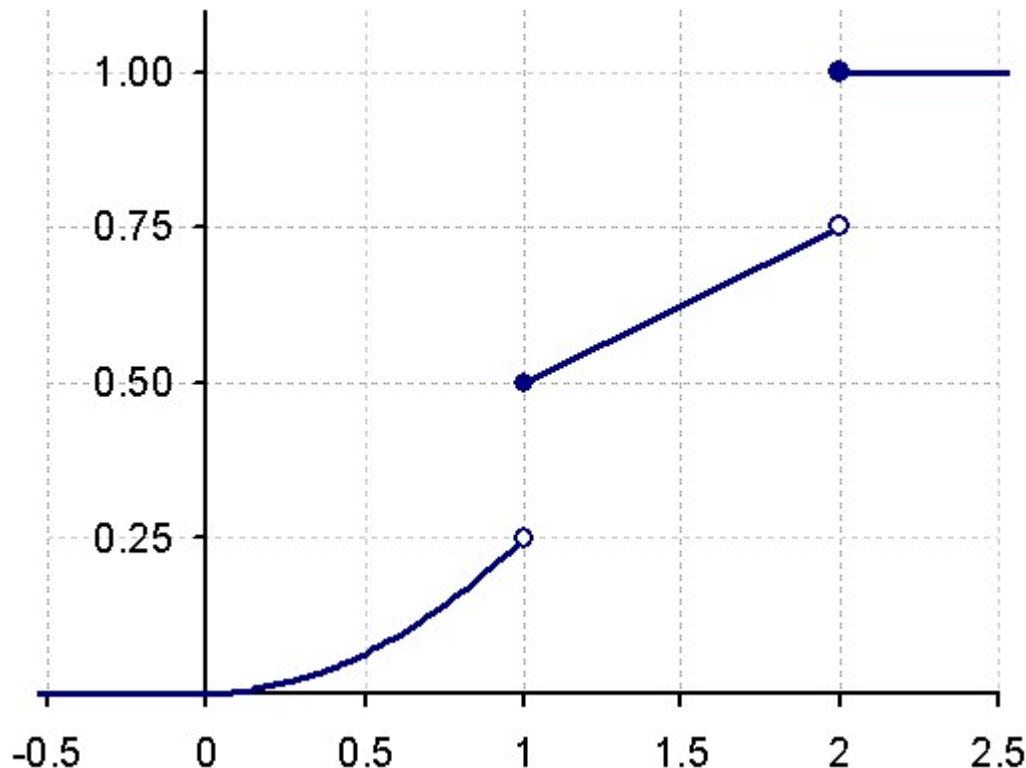
$$P(1 \leq X \leq 1.5) = F(1.5) - F(1-) = 0.5625 - 0.0000 = 0.5625.$$

3. Let X be a random variable of the mixed type having the distribution function

$$F(x) = \begin{cases} 0 & x < 0, \\ \frac{x^2}{4} & 0 \leq x < 1, \\ \frac{x+1}{4} & 1 \leq x < 2, \\ 1 & 2 \leq x. \end{cases}$$

a) Carefully sketch the graph of $F(x)$.
 b) Find the mean and the variance of X .
 c) Find $P(X=1)$, $P(X=\frac{1}{2})$,
 $P(\frac{1}{4} < X < 1)$, $P(\frac{1}{4} < X \leq 1)$,
 $P(\frac{1}{2} \leq X < 2)$, $P(\frac{1}{2} \leq X \leq 2)$.

a)



Discrete portion of the probability distribution of X :

$$p(1) = \frac{1}{4}, \quad p(2) = \frac{1}{4}.$$

Continuous portion of the probability distribution of X:

$$f(x) = F'(x) = \begin{cases} x/2 & 0 < x < 1 \\ 1/4 & 1 < x < 2 \\ 0 & \text{o.w.} \end{cases}$$

b) $\mu = E(X) = 1 \cdot 1/4 + 2 \cdot 1/4 + \int_0^1 x \cdot \frac{x}{2} dx + \int_1^2 x \cdot \frac{1}{4} dx = 31/24.$

$$E(X^2) = 1^2 \cdot 1/4 + 2^2 \cdot 1/4 + \int_0^1 x^2 \cdot \frac{x}{2} dx + \int_1^2 x^2 \cdot \frac{1}{4} dx = 47/24.$$

$$\sigma^2 = \text{Var}(X) = E(X^2) - [E(X)]^2 = 47/24 - (31/24)^2 = 167/576.$$

c) $P(X=1) = F(1) - F(1-) = 1/2 - 1/4 = 1/4.$

$$P(X=1/2) = F(1/2) - F(1/2-) = 1/16 - 1/16 = 0.$$

$$P(1/4 < X < 1) = F(1-) - F(1/4) = 1/4 - 1/64 = 15/64.$$

OR

$$P(1/4 < X < 1) = \int_{1/4}^1 \frac{x}{2} dx = 15/64.$$

$$P(1/4 < X \leq 1) = F(1) - F(1/4) = 1/2 - 1/64 = 31/64.$$

$$P(1/2 \leq X < 2) = F(2) - F(1/2-) = 3/4 - 1/16 = 11/16.$$

$$P(1/2 \leq X \leq 2) = F(2) - F(1/2-) = 1 - 1/16 = 15/16.$$

- 4.* An insurance policy reimburses a loss up to a benefit limit of C and has a deductible of d . The policyholder's loss, X , follows a distribution with density function $f(x)$. Find the expected value of the benefit paid under the insurance policy?

$$\text{Benefit Paid} = \begin{cases} 0 & x < d \\ x - d & d \leq x < C + d \\ C & x \geq C + d \end{cases}$$

$$E(\text{Benefit Paid}) = \int_0^d 0 \cdot f_X(x) dx + \int_d^{C+d} (x - d) \cdot f_X(x) dx + \int_{C+d}^{\infty} C \cdot f_X(x) dx.$$

For example, if X has an Exponential distribution with mean θ ,

$$\begin{aligned} E(\text{Benefit Paid}) &= \int_d^{C+d} (x - d) \cdot f_X(x) dx + \int_{C+d}^{\infty} C \cdot f_X(x) dx \\ &= \int_d^{C+d} (x - d) \cdot \frac{1}{\theta} e^{-x/\theta} dx + \int_{C+d}^{\infty} C \cdot \frac{1}{\theta} e^{-x/\theta} dx \\ &= \left[-(x - d) \cdot e^{-x/\theta} - \theta e^{-x/\theta} \right] \Big|_d^{C+d} + \left[-C e^{-x/\theta} \right] \Big|_{C+d}^{\infty} \\ &= \theta e^{-d/\theta} - \theta e^{-(C+d)/\theta}. \end{aligned}$$

For example, if $d = 2$, $C = 10$, X has an Exponential distribution with mean $\theta = 5$,

$$\begin{aligned} E(\text{Benefit Paid}) &= \int_2^{12} (x - 2) \cdot f_X(x) dx + \int_{12}^{\infty} 10 \cdot f_X(x) dx \\ &= \int_2^{12} (x - 2) \cdot \frac{1}{5} e^{-x/5} dx + \int_{12}^{\infty} 10 \cdot \frac{1}{5} e^{-x/5} dx \\ &= \left[-(x - 2) \cdot e^{-x/5} - 5 e^{-x/5} \right] \Big|_2^{12} + \left[-10 e^{-x/5} \right] \Big|_{12}^{\infty} \\ &= 5 e^{-2/5} - 5 e^{-12/5} = 5 e^{-0.4} - 5 e^{-2.4} \approx \mathbf{2.898}. \end{aligned}$$

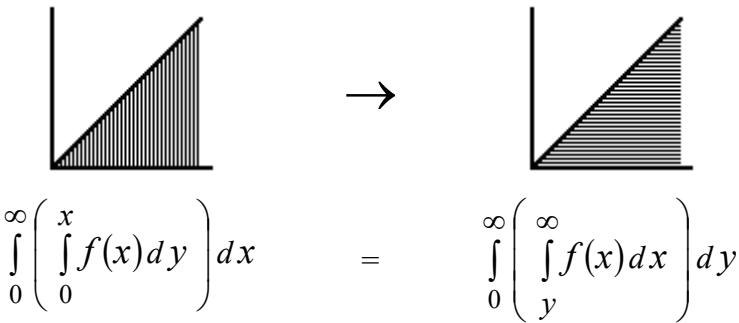
Fact: Let X be a nonnegative continuous random variable with p.d.f. $f(x)$ and c.d.f. $F(x)$. Then

$$E(X) = \int_0^\infty (1 - F(x)) dx.$$

~1.9.21 (8th edition) ~1.9.20 (7th edition) ~1.9.19 (6th edition)

Proof:

$$E(X) = \int_0^\infty x f(x) dx = \int_0^\infty \left(\int_0^x dy \right) f(x) dx = \int_0^\infty \left(\int_0^x f(x) dy \right) dx$$



$$\Rightarrow E(X) = \int_0^\infty \left(\int_y^\infty f(x) dx \right) dy = \int_0^\infty P(X > y) dy = \int_0^\infty (1 - F(y)) dy.$$

Example: Find the expected value of an $\text{Exponential}(\theta)$ distribution.

For $\text{Exponential}(\theta)$, $1 - F(x) = P(X > x) = e^{-x/\theta}$, $t > 0$.

$$E(X) = \int_0^\infty (1 - F(x)) dx = \int_0^\infty e^{-x/\theta} dx = \theta.$$

Fact: Let X be a random variable of the discrete type with pmf $p(x)$ that is positive on the nonnegative integers and is equal to zero elsewhere. Then

$$E(X) = \sum_{x=0}^{\infty} [1 - F(x)],$$

where $F(x)$ is the cdf of X .

~1.9.22 (8th edition) ~1.9.21 (7th edition) ~1.9.20 (6th edition)

Proof:

$$1 - F(x) = P(X > x) = p(x+1) + p(x+2) + p(x+3) + p(x+4) + \dots$$

$$1 - F(0) = p(1) + p(2) + p(3) + p(4) + p(5) + p(6) + p(7) + \dots$$

$$1 - F(1) = p(2) + p(3) + p(4) + p(5) + p(6) + p(7) + \dots$$

$$1 - F(2) = p(3) + p(4) + p(5) + p(6) + p(7) + \dots$$

$$1 - F(3) = p(4) + p(5) + p(6) + p(7) + \dots$$

$$1 - F(4) = p(5) + p(6) + p(7) + \dots$$

...

...

$$\Rightarrow \sum_{x=0}^{\infty} [1 - F(x)] = 1 \times p(1) + 2 \times p(2) + 3 \times p(3) + 4 \times p(4) + \dots = E(X).$$

Example: Find the expected value of a Geometric(p) distribution.

For Geometric(p), $1 - F(x) = P(X > x) = (1 - p)^x, x = 0, 1, 2, \dots$

$$E(X) = \sum_{x=0}^{\infty} [1 - F(x)] = \sum_{x=0}^{\infty} [1 - p]^x = \frac{1}{1 - [1 - p]} = \frac{1}{p}.$$

1.

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{5} & 0 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

$$E(X) = \int_0^\infty (1 - F(x)) dx = \int_0^3 \left(1 - \frac{x}{5}\right) dx = \left(x - \frac{x^2}{10}\right) \Big|_0^3 = 2.1.$$

2.

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{x^2 - 2x + 3}{4} & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

$$\begin{aligned} E(X) &= \int_0^\infty (1 - F(x)) dx = \int_0^1 (1) dx + \int_1^2 \left(1 - \frac{x^2 - 2x + 3}{4}\right) dx \\ &= 1 + \int_1^2 \left(\frac{1}{4} + \frac{x}{2} - \frac{x^2}{4}\right) dx = 1 + \left(\frac{x}{4} + \frac{x^2}{4} - \frac{x^3}{12}\right) \Big|_1^2 = \frac{17}{12}. \end{aligned}$$