

STAT 410 does NOT have a discussion section.

STAT 400 (an easier class) DOES have a discussion section.

STAT 410 (a harder class) does NOT.

There is a simple explanation for this: About 20 years ago, there was only one section of STAT 410 with about 40 students and only two lectures of STAT 400 with about 150 students each (each divided into 3 discussion sections of about 50 students each). The number of students has changed a little bit since then, the course structure has not (yet)...

However, don't you wish there was a discussion section in STAT 410?

If STAT 410 did have a discussion section, this is what your TA would most likely go over during it:

- 1.** Consider a continuous random variable  $X$  with the probability density function

$$f_X(x) = \frac{6x+7}{C}, \quad 3 \leq x \leq 8, \quad \text{zero elsewhere.}$$

- a) Find the value of  $C$  that makes  $f_X(x)$  a valid probability density function.
- b) Find the cumulative distribution function of  $X$ ,  $F_X(x) = P(X \leq x)$ .

“Hint”: To double-check your answer: should be  $F_X(3) = 0$ ,  $F_X(8) = 1$ .

- c) Find the expected value of  $X$ ,  $E(X) = \mu_X$ .

- 1.** (continued)

Consider  $Y = g(X) = \sqrt{X+1}$ . Find the probability distribution of  $Y$ .

- d) Find the support (the range of possible values) of the probability distribution of  $Y$ .

e) Use part (b) and the c.d.f. approach to find the c.d.f. of Y,  $F_Y(y)$ .

“Hint”:  $F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = \dots$

f) Use the change-of-variable technique to find the p.d.f. of Y,  $f_Y(y)$ .

“Hint”:  $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$ .

“Hint”: To double-check your answer: should be  $f_Y(y) = F_Y'(y)$ .

g) Is  $\mu_Y$  equal to  $g(\mu_X)$ ?  $E(X) = \mu_X, E(Y) = \mu_Y$ .

“Hint”: “equal” means “exactly equal” here.  
Not “close” or “sort of close”, but “equal”.

1. (continued)

Consider  $W = h(X) = \frac{1}{X+2}$ . Find the probability distribution of W.

h) Find the support (the range of possible values) of the probability distribution of W.

i) Use part (b) and the c.d.f. approach to find the c.d.f. of W,  $F_W(w)$ .

“Hint”:  $F_W(w) = P(W \leq w) = P(h(X) \leq w) = \dots$

j) Use the change-of-variable technique to find the p.d.f. of W,  $f_W(w)$ .

“Hint”:  $f_W(w) = f_X(h^{-1}(w)) \left| \frac{dx}{dw} \right|$ .

“Hint”: To double-check your answer: should be  $f_W(w) = F_W'(w)$ .

k) Is  $\mu_W$  equal to  $h(\mu_X)$ ?  $E(X) = \mu_X, E(W) = \mu_W$ .

“Hint”: “equal” means “exactly equal” here.  
Not “close” or “sort of close”, but “equal”.

**2.** Consider a discrete random variable  $X$  with the probability mass function

$$p_X(x) = \frac{3x-1}{C}, \quad x=3, 4, 5, 6.$$

a) Find the value of  $C$  that makes  $f_X(x)$  a valid probability mass function.

b) Consider  $Y = \frac{12}{X-2}$ . Find the probability distribution of  $Y$ .

c) Is  $\mu_Y$  equal to  $g(\mu_X)$ ? 
$$g(x) = \frac{12}{x-2}.$$

**3.** Consider a continuous random variable  $X$  with the probability density function

$$f_X(x) = \frac{x}{C}, \quad 4 \leq x \leq 28, \quad \text{zero elsewhere.}$$

a) Find the value of  $C$  that would make this a valid probability distribution.

b) Find the cumulative distribution function of  $X$ ,  $F_X(x) = P(X \leq x)$ .

“Hint”: To double-check your answer: should be  $F_X(4) = 0$ ,  $F_X(28) = 1$ .

**3.** (continued)

Consider  $Y = g(X) = \sqrt{X-3}$ .

c) Find the support (the range of possible values) of the probability distribution of  $Y$ .

d) Use part (b) and the c.d.f. approach to find the c.d.f. of  $Y$ ,  $F_Y(y)$ .

“Hint”:  $F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = \dots$

e) Use the change-of-variable technique to find the p.d.f. of Y,  $f_Y(y)$ .

“Hint”:  $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$ .

“Hint”: To double-check your answer: should be  $f_Y'(y) = F_Y'(y)$ .

3. (continued)

Consider  $W = h(X) = \frac{8}{X+12}$ .

f) Find the support (the range of possible values) of the probability distribution of W.

g) Use part (b) and the c.d.f. approach to find the c.d.f. of W,  $F_W(w)$ .

“Hint”:  $F_W(w) = P(W \leq w) = P(h(X) \leq w) = \dots$

h) Use the change-of-variable technique to find the p.d.f. of W,  $f_W(w)$ .

“Hint”:  $f_W(w) = f_X(h^{-1}(w)) \left| \frac{dx}{dw} \right|$ .

“Hint”: To double-check your answer: should be  $f_W'(w) = F_W'(w)$ .

3. (continued)

i) Find the expected value of X,  $\mu_X = E(X)$ .

j) (i) Find the expected value of Y,  $\mu_Y = E(Y)$ .

(ii) Does  $\mu_Y$  equal to  $g(\mu_X)$ ?

“Hint”: “equal” means “exactly equal” here. Not “close” or “sort of close”, but “equal”.

k) (i) Find the expected value of W,  $\mu_W = E(W)$ .

(ii) Does  $\mu_W$  equal to  $h(\mu_X)$ ?

“Hint”: “equal” means “exactly equal” here. Not “close” or “sort of close”, but “equal”.

- 4.** The distribution on the GPA of the students at Anytown State University can be nicely approximated by the following probability density function:

$$f_X(x) = \frac{x^3(21-5x)}{C}, \quad 1.0 \leq x \leq 4.0, \quad \text{zero elsewhere.}$$

(The students with the GPA below 1.0 are “asked” to leave the university.)

- a) Find the value of  $C$  that makes  $f_X(x)$  a valid probability density function.

- b) Find the cumulative distribution function of  $X$ ,  $F_X(x) = P(X \leq x)$ .

“Hint”: To double-check your answer: should be  $F_X(1) = 0$ ,  $F_X(4) = 1$ .

- c) Find the average GPA of the students at Anytown State University,  $E(X) = \mu_X$ .

- 4.** (continued)

The following relationship is proposed to estimate the average amount of awake time per day, in hours, a student spends on activities that are not related to academics,  $y$ , from the student’s GPA,  $x$ :

$$y = g(x) = \frac{24}{x+2}.$$

Consider  $Y = g(X) = \frac{24}{X+2}$ .

- d) Find the support (the range of possible values) of the probability distribution of  $Y$ .

- e) Use part (b) and the c.d.f. approach to find the c.d.f. of  $Y$ ,  $F_Y(y)$ .

“Hint”:  $F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = \dots$

- f) Use the change-of-variable technique to find the p.d.f. of  $Y$ ,  $f_Y(y)$ .

“Hint”:  $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$ .

“Hint”: To double-check your answer: should be  $f_Y(y) = F_Y'(y)$ .

g) Find  $E(Y) = \mu_Y$ .

“Hint”: While this integral is “fightable”, it will be somewhat annoying. After you set it up, I would recommend using your favorite online integral calculator instead of a hand-to-hand combat.

h) Is  $\mu_Y$  equal to  $g(\mu_X)$ ?

“Hint”: “equal” means “exactly equal” here. Not “close” or “sort of close”, but “equal”.

5. Consider a discrete random variable  $X$  with the probability mass function

$$p_X(x) = \frac{x^3(21-5x)}{C}, \quad x = 1, 2, 3, 4, \quad \text{zero elsewhere.}$$

a) Find the value of  $C$  that makes  $f_X(x)$  a valid probability mass function.

b) Find  $E(X) = \mu_X$ .

5. (continued)

$$\text{Consider } Y = g(X) = \frac{24}{X+2}.$$

c) Obtain the probability distribution of  $Y$ .

d) Find  $E(Y) = \mu_Y$ .

e) Is  $\mu_Y$  equal to  $g(\mu_X)$ ?

“Hint”: “equal” means “exactly equal” here. Not “close” or “sort of close”, but “equal”.

**Answers:**

1. Consider a continuous random variable  $X$  with the probability density function

$$f_X(x) = \frac{6x+7}{C}, \quad 3 \leq x \leq 8, \quad \text{zero elsewhere.}$$

- a) Find the value of  $C$  that makes  $f_X(x)$  a valid probability density function.

$$1 = \int_{-\infty}^{\infty} f_X(x) dx = \int_3^8 \frac{6x+7}{C} dx = \frac{3x^2 + 7x}{C} \Big|_3^8 = \frac{(192 + 56) - (27 + 21)}{C} = \frac{200}{C}.$$

$$\Rightarrow C = 200.$$

$$f_X(x) = \frac{6x+7}{200}, \quad 3 \leq x \leq 8.$$

- b) Find the cumulative distribution function of  $X$ ,  $F_X(x) = P(X \leq x)$ .

“Hint”: To double-check your answer: should be  $F_X(3) = 0$ ,  $F_X(8) = 1$ .

$$\begin{aligned} F_X(x) &= P(X \leq x) = \int_3^x \frac{6u+7}{200} du = \frac{3u^2 + 7u}{200} \Big|_3^x \\ &= \frac{3x^2 + 7x - 48}{200} = \frac{(x-3)(3x+16)}{200}, \quad 3 \leq x < 8. \end{aligned}$$

Obviously,  $F_X(x) = 0, \quad x < 3,$   $F_X(x) = 1, \quad x \geq 8.$

Indeed,  $\frac{(3-3)(3 \cdot 3 + 16)}{200} = 0, \quad \frac{(8-3)(3 \cdot 8 + 16)}{200} = 1.$  ☺

c) Find the expected value of  $X, E(X) = \mu_X.$

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x \cdot f_X(x) dx = \int_3^8 x \cdot \frac{6x+7}{200} dx = \frac{2x^3 + \frac{7}{2}x^2}{200} \Big|_3^8 \\ &= \frac{(1024 + 224) - (54 + 31.5)}{200} = \frac{1,162.5}{200} = \frac{93}{16} = 5.8125. \end{aligned}$$

1. (continued)

Consider  $Y = g(X) = \sqrt{X+1}.$  Find the probability distribution of  $Y.$

d) Find the support (the range of possible values) of the probability distribution of  $Y.$

$$3 \leq x \leq 8 \quad \Rightarrow \quad 4 \leq x+1 \leq 9$$

$$\Rightarrow 2 \leq \sqrt{x+1} \leq 3 \quad \Rightarrow \quad 2 \leq y \leq 3.$$

e) Use part (b) and the c.d.f. approach to find the c.d.f. of Y,  $F_Y(y)$ .

“Hint”:  $F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = \dots$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(\sqrt{X+1} \leq y) = P(X \leq y^2 - 1) = F_X(y^2 - 1) \\ &= \frac{(y^2 - 4)(3y^2 + 13)}{200} = \frac{(y-2)(y+2)(3y^2 + 13)}{200} \\ &= \frac{3(y^2 - 1)^2 + 7(y^2 - 1) - 48}{200} = \frac{3y^4 + y^2 - 52}{200}, \quad 2 \leq y < 3. \end{aligned}$$

Obviously,  $F_Y(y) = 0, \quad y < 2, \quad F_Y(y) = 1, \quad y \geq 3.$

Indeed,

$$\begin{aligned} \frac{(2^2 - 4)(3 \cdot 2^2 + 13)}{200} &= 0, \\ \frac{(3^2 - 4)(3 \cdot 3^2 + 13)}{200} &= 1. \quad \text{☺} \end{aligned}$$

f) Use the change-of-variable technique to find the p.d.f. of Y,  $f_Y(y)$ .

“Hint”:  $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|.$

“Hint”: To double-check your answer: should be  $f_Y(y) = F_Y'(y).$

$$y = \sqrt{x+1}, \quad x = y^2 - 1, \quad \frac{dx}{dy} = 2y.$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \frac{6(y^2 - 1) + 7}{200} \cdot |2y|$$

$$= \frac{12y^3 + 2y}{200} = \frac{6y^3 + y}{100}, \quad 2 < y < 3.$$

Indeed,  $\frac{d}{dy} \left( \frac{3y^4 + y^2 - 52}{200} \right) = \frac{6y^3 + y}{100}$ . 

g) Is  $\mu_Y$  equal to  $g(\mu_X)$ ?  $E(X) = \mu_X, E(Y) = \mu_Y$ .

“Hint”: “equal” means “exactly equal” here.  
Not “close” or “sort of close”, but “equal”.

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y) dy = \int_2^3 y \cdot \frac{6y^3 + y}{100} dy = \frac{\frac{6}{5}y^5 + \frac{1}{3}y^3}{100} \Big|_2^3 = \frac{18y^5 + 5y^3}{1,500} \Big|_2^3$$

$$= \frac{(4,374 + 135) - (576 + 40)}{1,500} = \frac{3,893}{1,500} \approx 2.59533333.$$

OR

$$E(Y) = E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) dx$$

$$= \int_3^8 \sqrt{x+1} \cdot \frac{6x+7}{200} dx = \dots$$

$$\int_3^8 \sqrt{x+1} \frac{6x+7}{200} dx$$

Solution

$$\int_3^8 \frac{1}{200} \sqrt{x+1} (6x+7) dx = \frac{3893}{1500} \approx 2.5953$$

$$\frac{3893}{1500}$$

$$g(\mu_X) = \sqrt{5.8125+1} \approx 2.61007663.$$

$$\mu_Y = E(Y) \neq g(E(X)) = g(\mu_X). \quad \mu_Y \text{ is NOT equal to } g(\mu_X).$$

Recall: IF  $g(x)$  is a linear function, that is, IF  $g(x) = ax + b$ ,  
then  $E(g(X)) = E(ax + b) = aE(X) + b = g(E(X))$ .

However, in general, if  $g(x)$  is NOT a linear function,

$$\text{then } E(g(X)) \neq g(E(X)).$$

Spoiler:

Here,  $E(g(X)) < g(E(X))$  since  $g(x) = \sqrt{x+1}$  “curves down”.

## 1. (continued)

Consider  $W = h(X) = \frac{1}{X+2}$ . Find the probability distribution of  $W$ .

- h) Find the support (the range of possible values) of the probability distribution of  $W$ .

$$3 \leq x \leq 8 \quad \Rightarrow \quad 5 \leq x + 2 \leq 10$$

$$\Rightarrow \quad \frac{1}{5} \geq \frac{1}{x+2} \geq \frac{1}{10} \quad \Rightarrow \quad \mathbf{0.10 = \frac{1}{10} \leq w \leq \frac{1}{5} = 0.20.}$$

i) Use part (b) and the c.d.f. approach to find the c.d.f. of  $W$ ,  $F_W(w)$ .

“Hint”:  $F_W(w) = P(W \leq w) = P(h(X) \leq w) = \dots$

$$F_W(w) = P(W \leq w) = P\left(\frac{1}{X+2} \leq w\right) = P\left(X \geq \frac{1}{w} - 2\right) = 1 - F_X\left(\frac{1}{w} - 2\right)$$

$$= 1 - \frac{3\left(\frac{1}{w} - 2\right)^2 + 7\left(\frac{1}{w} - 2\right) - 48}{200}$$

$$= 1 - \frac{\left(\frac{1}{w} - 2 - 3\right)\left(\frac{3}{w} - 6 + 16\right)}{200} = 1 - \frac{\left(\frac{1}{w} - 5\right)\left(\frac{3}{w} + 10\right)}{200}$$

$$= 1 - \frac{\frac{3}{w^2} - \frac{5}{w} - 50}{200} = \frac{250 + \frac{5}{w} - \frac{3}{w^2}}{200}$$

$$= \frac{250w^2 + 5w - 3}{200w^2} = \frac{5}{4} + \frac{1}{40w} - \frac{3}{200w^2}, \quad \frac{1}{10} \leq w < \frac{1}{5}.$$

Obviously,  $F_W(w) = 0, \quad w < \frac{1}{10}, \quad F_W(w) = 1, \quad w \geq \frac{1}{5}.$

Indeed,  $\frac{250 + 5 \cdot 10 - 3 \cdot 10^2}{200} = 0, \quad \frac{250 + 5 \cdot 5 - 3 \cdot 5^2}{200} = 1.$  

j) Use the change-of-variable technique to find the p.d.f. of  $W$ ,  $f_W(w)$ .

“Hint”:  $f_W(w) = f_X(h^{-1}(w)) \left| \frac{dx}{dw} \right|.$

“Hint”: To double-check your answer: should be  $f_W(w) = F_W'(w).$

$$w = \frac{1}{x+2}, \quad x = \frac{1}{w} - 2, \quad \frac{dx}{dw} = -\frac{1}{w^2}.$$

$$\begin{aligned} f_W(w) &= f_X(h^{-1}(w)) \left| \frac{dx}{dw} \right| = \frac{6 \left( \frac{1}{w} - 2 \right) + 7}{200} \cdot \left| -\frac{1}{w^2} \right| \\ &= \frac{6 - 5w}{200w^3} = \frac{3}{100w^3} - \frac{1}{40w^2}, \quad \frac{1}{10} < w < \frac{1}{5}. \end{aligned}$$

Indeed,  $\frac{d}{dw} \left( \frac{5}{4} + \frac{1}{40w} - \frac{3}{200w^2} \right) = \frac{3}{100w^3} - \frac{1}{40w^2}.$  ☺

k) Is  $\mu_W$  equal to  $h(\mu_X)$ ?  $E(X) = \mu_X, E(W) = \mu_W.$

“Hint”: “equal” means “exactly equal” here.  
Not “close” or “sort of close”, but “equal”.

$$\begin{aligned} E(W) &= \int_{-\infty}^{\infty} w \cdot f_W(w) dw = \int_{1/10}^{1/5} w \cdot \frac{6 - 5w}{200w^3} dw = \int_{1/10}^{1/5} \left( \frac{3}{100w^2} - \frac{1}{40w} \right) dw \\ &= \left( -\frac{3}{100w} - \frac{\ln w}{40} \right) \Big|_{1/10}^{1/5} = -\frac{15 - 30}{100} - \frac{\ln\left(\frac{1}{5}\right) - \ln\left(\frac{1}{10}\right)}{40} = \frac{15}{100} - \frac{\ln\left(\frac{10}{5}\right)}{40} \end{aligned}$$

$$= \frac{15}{100} - \frac{\ln(2)}{40} = \frac{3}{20} - \frac{\ln(2)}{40} = \frac{6 - \ln(2)}{40} \approx 0.13267132.$$

OR

$$\begin{aligned} E(W) &= E(h(X)) = \int_{-\infty}^{\infty} h(x) \cdot f_X(x) dx = \int_3^8 \frac{1}{x+2} \cdot \frac{6x+7}{200} dx \\ &= \int_5^{10} \frac{1}{u} \cdot \frac{6(u-2)+7}{200} du = \int_5^{10} \frac{1}{u} \cdot \frac{6u-5}{200} du \\ &= \int_5^{10} \left( \frac{3}{100} - \frac{1}{40u} \right) du = \left( \frac{3u}{100} - \frac{\ln u}{40} \right) \Big|_5^{10} \\ &= \frac{30-15}{100} - \frac{\ln(10)-\ln(5)}{40} = \frac{30-15}{100} - \frac{\ln\left(\frac{10}{5}\right)}{40} \\ &= \frac{15}{100} - \frac{\ln(2)}{40} = \frac{3}{20} - \frac{\ln(2)}{40} = \frac{6 - \ln(2)}{40} \approx 0.13267132. \end{aligned}$$

$$\begin{aligned} \int_3^8 \frac{1}{x+2} \cdot \frac{6x+7}{200} dx &= \frac{1}{40} (6 - \log(2)) \approx 0.13267 \\ \text{Solution} & \\ \frac{30-5\ln(2)}{200} & \end{aligned}$$

$$h(\mu_X) = \frac{1}{5.8125+2} = 0.128.$$

$$\mu_W = E(W) \neq h(E(X)) = h(\mu_X). \quad \mu_W \text{ is NOT equal to } h(\mu_X).$$

Recall: IF  $h(x)$  is a linear function, that is, IF  $h(x) = ax + b$ ,  
then  $E(h(X)) = E(ax + b) = aE(X) + b = h(E(X))$ .

However, in general, if  $h(x)$  is NOT a linear function,  
then  $E(h(X)) \neq h(E(X))$ .

Spoiler:

Here,  $E(h(X)) > h(E(X))$  since  $h(x) = \frac{1}{x+2}$  “curves up”.

2. Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{3x-1}{C}, \quad x=3, 4, 5, 6.$$

a) Find the value of C that makes  $f_X(x)$  a valid probability mass function.

$$p_X(3) = \frac{8}{C}, \quad p_X(4) = \frac{11}{C}, \quad p_X(5) = \frac{14}{C}, \quad p_X(6) = \frac{17}{C}.$$

$$p_X(3) + p_X(4) + p_X(5) + p_X(6) = 1.$$

$$\frac{8}{C} + \frac{11}{C} + \frac{14}{C} + \frac{17}{C} = \frac{50}{C} = 1. \Rightarrow C = 50.$$

b) Consider  $Y = \frac{12}{X-2}$ . Find the probability distribution of Y.

$x$	$p_X(x)$	$\frac{12}{x-2}$
3	$\frac{8}{50} = 0.16$	12
4	$\frac{11}{50} = 0.22$	6
5	$\frac{14}{50} = 0.28$	4
6	$\frac{17}{50} = 0.34$	3

$y$	$p_Y(y)$
3	0.34
4	0.28
6	0.22
12	0.16

OR

$$y = \frac{12}{x-2}, \quad x = \frac{12}{y} + 2.$$

$$p_Y(y) = p_X\left(\frac{12}{y} + 2\right) = \frac{\frac{36}{y} + 5}{50} = \frac{36 + 5y}{50y} = \frac{18}{25y} + \frac{1}{10}, \quad y = 3, 4, 6, 12.$$

Indeed,

$p_Y(3) = \frac{18}{25 \cdot 3} + \frac{1}{10} = 0.34,$	$p_Y(4) = \frac{18}{25 \cdot 4} + \frac{1}{10} = 0.28,$
$p_Y(6) = \frac{18}{25 \cdot 6} + \frac{1}{10} = 0.22,$	$p_Y(12) = \frac{18}{25 \cdot 12} + \frac{1}{10} = 0.16.$

For discrete random variables, the possible values are isolated points on the number line.

$$\Rightarrow \text{ no derivatives.} \quad \Rightarrow \quad \text{no } \frac{dx}{dy}.$$

c) Is  $\mu_Y$  equal to  $g(\mu_X)$ ? 
$$g(x) = \frac{12}{x-2}.$$

$$E(X) = \sum_{\text{all } x} x \cdot p_X(x) = 3 \cdot 0.16 + 4 \cdot 0.22 + 5 \cdot 0.28 + 6 \cdot 0.34 = 4.8.$$

$$E(Y) = \sum_{\text{all } y} y \cdot p_Y(y) = 3 \cdot 0.34 + 4 \cdot 0.28 + 6 \cdot 0.22 + 12 \cdot 0.16 = 5.38.$$

OR

$$\begin{aligned} E(Y) &= E(g(X)) = \sum_{\text{all } x} g(x) \cdot p_X(x) \\ &= \frac{12}{3-2} \cdot \frac{8}{50} + \frac{12}{4-2} \cdot \frac{11}{50} + \frac{12}{5-2} \cdot \frac{14}{50} + \frac{12}{6-2} \cdot \frac{17}{50} = 5.38. \end{aligned}$$

$$g(\mu_X) = \frac{12}{4.8 - 2} \approx 4.285714.$$

$$\mu_Y = E(Y) \neq g(E(X)) = g(\mu_X). \quad \mu_Y \text{ is NOT equal to } g(\mu_X).$$

Recall: IF  $g(x)$  is a linear function, that is, IF  $g(x) = ax + b$ ,  
then  $E(g(X)) = E(ax + b) = aE(X) + b = g(E(X))$ .

However, in general, if  $g(x)$  is NOT a linear function,

then  $E(g(X)) \neq g(E(X))$ .

Spoiler:

Here,  $E(g(X)) > g(E(X))$  since  $g(x) = \frac{12}{x-2}$  “curves up” for  $3 \leq x \leq 6$ .

3. Consider a continuous random variable  $X$  with the probability density function

$$f_X(x) = \frac{x}{C}, \quad 4 \leq x \leq 28, \quad \text{zero elsewhere.}$$

- a) Find the value of  $C$  that would make this a valid probability distribution.

$$1 = \int_{-\infty}^{\infty} f_X(x) dx = \int_4^{28} \frac{x}{C} dx = \frac{x^2}{2C} \Big|_4^{28} = \frac{784 - 16}{2C} = \frac{384}{C}.$$

$$\Rightarrow C = 384.$$

$$f_X(x) = \frac{x}{384}, \quad 4 \leq x \leq 28.$$

- b) Find the cumulative distribution function of  $X$ ,  $F_X(x) = P(X \leq x)$ .

“Hint”: To double-check your answer: should be  $F_X(4) = 0$ ,  $F_X(28) = 1$ .

$$F_X(x) = P(X \leq x) = \int_4^x \frac{u}{384} du = \frac{u^2}{768} \Big|_4^x = \frac{x^2 - 16}{768} = \frac{x^2}{768} - \frac{1}{48}, \quad 4 \leq x < 28.$$

$$\text{Obviously, } F_X(x) = 0, \quad x < 4, \quad F_X(x) = 1, \quad x \geq 28.$$

$$\text{Indeed, } \frac{4^2 - 16}{768} = 0, \quad \frac{28^2 - 16}{768} = 1. \quad \text{😊}$$

**3.** (continued)

Consider  $Y = g(X) = \sqrt{X-3}$ .

- c) Find the support (the range of possible values) of the probability distribution of  $Y$ .

$$4 \leq x \leq 28 \quad \Rightarrow \quad 1 \leq x - 3 \leq 25 \quad \Rightarrow \quad 1 \leq \sqrt{x-3} \leq 5.$$

$$1 \leq y \leq 5.$$

- d) Use part (b) and the c.d.f. approach to find the c.d.f. of  $Y$ ,  $F_Y(y)$ .

“Hint”:  $F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = \dots$

$$F_Y(y) = P(Y \leq y) = P(\sqrt{X-3} \leq y) = P(X \leq y^2 + 3) = F_X(y^2 + 3)$$

$$\begin{aligned} &= \frac{(y^2 + 3)^2 - 16}{768} = \frac{y^4 + 6y^2 - 7}{768} = \frac{(y^2 - 1)(y^2 + 7)}{768} \\ &= \frac{(y-1)(y+1)(y^2 + 7)}{768}, \quad 1 \leq y < 5. \end{aligned}$$

$$\text{Obviously, } F_Y(y) = 0, \quad y < 1, \quad F_Y(y) = 1, \quad y \geq 5.$$

$$\text{Indeed, } \frac{1^4 + 6 \cdot 1^2 - 7}{768} = 0, \quad \frac{5^4 + 6 \cdot 5^2 - 7}{768} = 1. \quad \text{😊}$$

e) Use the change-of-variable technique to find the p.d.f. of Y,  $f_Y(y)$ .

“Hint”:  $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|.$

“Hint”: To double-check your answer: should be  $f_Y'(y) = F_Y'(y).$

$$y = \sqrt{x-3} \quad x = y^2 + 3 \quad \frac{dx}{dy} = 2y$$

$$f_Y(y) = \frac{(y^2 + 3)}{384} \cdot |2y| = \frac{y^3 + 3y}{192}, \quad 1 \leq y \leq 5.$$

Indeed,  $\frac{d}{dy} \left( \frac{y^4 + 6y^2 - 7}{768} \right) = \frac{y^3 + 3y}{192}$ . 

3. (continued)

Consider  $W = h(X) = \frac{8}{X+12}.$

f) Find the support (the range of possible values) of the probability distribution of W.

$$4 \leq x \leq 28 \quad \Rightarrow \quad 16 \leq x + 12 \leq 40 \quad \Rightarrow \quad \frac{1}{2} \geq \frac{8}{x+12} \geq \frac{1}{5}.$$

$$0.2 = \frac{1}{5} \leq w \leq \frac{1}{2} = 0.5.$$

g) Use part (b) and the c.d.f. approach to find the c.d.f. of  $W$ ,  $F_W(w)$ .

“Hint”:  $F_W(w) = P(W \leq w) = P(h(X) \leq w) = \dots$

$$\begin{aligned}
F_W(w) &= P(W \leq w) = P\left(\frac{8}{X+12} \leq w\right) = P\left(X \geq \frac{8}{w} - 12\right) = 1 - F_X\left(\frac{8}{w} - 12\right) \\
&= 1 - \frac{\left(\frac{8}{w} - 12\right)^2 - 16}{768} = 1 - \frac{\frac{64}{w^2} - \frac{192}{w} + 128}{768} = 1 - \frac{\frac{1}{w^2} - \frac{3}{w} + 2}{12} \\
&= 1 - \frac{\left(\frac{1}{w} - 1\right)\left(\frac{1}{w} - 2\right)}{12} = \frac{10 + \frac{3}{w} - \frac{1}{w^2}}{12} = \frac{10w^2 + 3w - 1}{12w^2} \\
&= \frac{(5w-1)(2w+1)}{12w^2} = \frac{5}{6} + \frac{1}{4w} - \frac{1}{12w^2}, \quad \frac{1}{5} \leq w < \frac{1}{2}.
\end{aligned}$$

Obviously,  $F_W(w) = 0, \quad w < \frac{1}{5}, \quad F_W(w) = 1, \quad w \geq \frac{1}{2}$ .

Indeed,  $\frac{10 + \frac{3}{0.2} - \frac{1}{0.2^2}}{12} = \frac{10 + 15 - 25}{12} = 0,$

$$\frac{10 + \frac{3}{0.5} - \frac{1}{0.5^2}}{12} = \frac{10 + 6 - 4}{12} = 1. \quad \text{☺}$$

h) Use the change-of-variable technique to find the p.d.f. of  $W$ ,  $f_W(w)$ .

“Hint”:  $f_W(w) = f_X(h^{-1}(w)) \left| \frac{dx}{dw} \right|$ .

“Hint”: To double-check your answer: should be  $f_W(w) = F'_W(w)$ .

$$w = \frac{8}{x+12} \quad x = \frac{8}{w} - 12 \quad \frac{dx}{dw} = -\frac{8}{w^2}.$$

$$\begin{aligned} f_W(w) &= f_X(h^{-1}(w)) \left| \frac{dx}{dw} \right| = \frac{\left( \frac{8}{w} - 12 \right)}{384} \cdot \left| -\frac{8}{w^2} \right| \\ &= \frac{\frac{64}{w} - 96}{384w^2} = \frac{\frac{2}{w} - 3}{12w^2} = \frac{2 - 3w}{12w^3} = \frac{1}{6w^3} - \frac{1}{4w^2}, \quad \frac{1}{5} \leq w \leq \frac{1}{2}. \end{aligned}$$

Indeed,  $\frac{d}{dw} \left( \frac{5}{6} + \frac{1}{4w} - \frac{1}{12w^2} \right) = \frac{1}{6w^3} - \frac{1}{4w^2}$ . 

3. (continued)

i) Find the expected value of  $X$ ,  $\mu_X = E(X)$ .

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x \cdot f_X(x) dx = \int_4^{28} x \cdot \frac{x}{384} dx = \int_4^{28} \frac{x^2}{384} dx = \frac{x^3}{1,152} \Big|_4^{28} \\ &= \frac{21,952 - 64}{1,152} = 19. \end{aligned}$$

For fun:

i½) Find the median of the probability distribution of X.

Need  $m$  such that  $P(X \leq m) = P(X \geq m) = \frac{1}{2}$ .

$$P(X \leq m) = F_X(m) = \frac{m^2 - 16}{768} = \frac{1}{2} \Rightarrow m^2 - 16 = 384.$$

$$\Rightarrow m^2 = 400 \Rightarrow m = \mathbf{20}.$$

j) (i) Find the expected value of Y,  $\mu_Y = E(Y)$ .

(ii) Does  $\mu_Y$  equal to  $g(\mu_X)$ ?

“Hint”: “equal” means “exactly equal” here. Not “close” or “sort of close”, but “equal”.

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y) dy = \int_1^5 y \cdot \frac{y^3 + 3y}{192} dy = \int_1^5 \frac{y^4 + 3y^2}{192} dy$$

$$= \left( \frac{\frac{1}{5}y^5 + y^3}{192} \right) \Big|_1^5 = \frac{625 + 125 - 0.2 - 1}{192} = \mathbf{3.9}.$$

OR

$$\begin{aligned}
E(Y) = E(g(X)) &= \int_{-\infty}^{\infty} g(x) \cdot f_X(x) dx = \int_4^{28} \sqrt{x-3} \cdot \frac{x}{384} dx \\
&\quad u = x - 3 \qquad \qquad \qquad x = u + 3 \qquad \qquad \qquad dx = du \\
&= \int_1^{25} \sqrt{u} \cdot \frac{u+3}{384} du = \frac{1}{384} \cdot \int_1^{25} u^{1.5} du + \frac{3}{384} \cdot \int_1^{25} u^{0.5} du \\
&= \frac{1}{384} \cdot \frac{u^{2.5}}{2.5} \Big|_1^{25} + \frac{3}{384} \cdot \frac{u^{1.5}}{1.5} \Big|_1^{25} \\
&= \frac{1}{384} \cdot \frac{3,125 - 1}{2.5} + \frac{3}{384} \cdot \frac{125 - 1}{1.5} = 3.9.
\end{aligned}$$

$$g(\mu_X) = \sqrt{19-3} = 4.$$

$$\mu_Y = E(Y) \neq g(E(X)) = g(\mu_X). \qquad \mu_Y \text{ is NOT equal to } g(\mu_X).$$

Recall: IF  $g(x)$  is a linear function, that is, IF  $g(x) = a x + b$ ,  
then  $E(g(X)) = E(a X + b) = a E(X) + b = g(E(X))$ .

However, in general, if  $g(x)$  is NOT a linear function,  
then  $E(g(X)) \neq g(E(X))$ .

Spoiler:

Here,  $E(g(X)) < g(E(X))$  since  $g(x) = \sqrt{x-3}$  “curves down”  
for  $4 < x < 28$ .

k) (i) Find the expected value of  $W$ ,  $\mu_W = E(W)$ .

(ii) Does  $\mu_W$  equal to  $h(\mu_X)$ ?

“Hint”: “equal” means “exactly equal” here. Not “close” or “sort of close”, but “equal”.

$$\begin{aligned}
 E(W) &= \int_{-\infty}^{\infty} w \cdot f_W(w) dw = \int_{1/5}^{1/2} w \cdot \frac{2-3w}{12w^3} dw = \int_{1/5}^{1/2} \left( \frac{1}{6w^2} - \frac{1}{4w} \right) dw \\
 &= \left( -\frac{1}{6w} - \frac{\ln w}{4} \right) \Big|_{1/5}^{1/2} = -\frac{2}{6} + \frac{\ln(2)}{4} + \frac{5}{6} - \frac{\ln(5)}{4} \\
 &= \frac{1}{2} - \frac{\ln(2.5)}{4} = \frac{2 - \ln(2.5)}{4} \approx 0.2709273.
 \end{aligned}$$

OR

$$\begin{aligned}
 E(W) &= E(h(X)) = \int_{-\infty}^{\infty} h(x) \cdot f_X(x) dx = \int_4^{28} \frac{8}{x+12} \cdot \frac{x}{384} dx \\
 u &= x + 12 & x &= u - 12 & dx &= du \\
 &= \int_{16}^{40} \frac{8}{u} \cdot \frac{u-12}{384} du = \frac{1}{48} \cdot \int_{16}^{40} \left( 1 - \frac{12}{u} \right) du = \frac{1}{48} \cdot (u - 12 \ln u) \Big|_{16}^{40} \\
 &= \frac{1}{48} \cdot (40 - 12 \ln(40) - 16 + 12 \ln(16)) = \frac{1}{48} \cdot \left( 24 - 12 \ln\left(\frac{40}{16}\right) \right) \\
 &= \frac{1}{2} - \frac{\ln(2.5)}{4} = \frac{2 - \ln(2.5)}{4} \approx 0.2709273.
 \end{aligned}$$

$$h(\mu_X) = \frac{8}{19+12} \approx 0.2580645.$$

$$\mu_W = E(W) \neq h(E(X)) = h(\mu_X). \quad \mu_W \text{ is NOT equal to } h(\mu_X).$$

Recall: IF  $h(x)$  is a linear function, that is, IF  $h(x) = ax + b$ ,  
then  $E(h(X)) = E(ax + b) = aE(X) + b = h(E(X))$ .

However, in general, if  $h(x)$  is NOT a linear function,  
then  $E(h(X)) \neq h(E(X))$ .

Spoiler:

Here,  $E(h(X)) > h(E(X))$  since  $h(x) = \frac{8}{x+12}$  “curves up”  
for  $4 < x < 28$ .

4. The distribution on the GPA of the students at Anytown State University can be nicely approximated by the following probability density function:

$$f_X(x) = \frac{x^3(21-5x)}{C}, \quad 1.0 \leq x \leq 4.0, \quad \text{zero elsewhere.}$$

( The students with the GPA below 1.0 are “asked” to leave the university. )

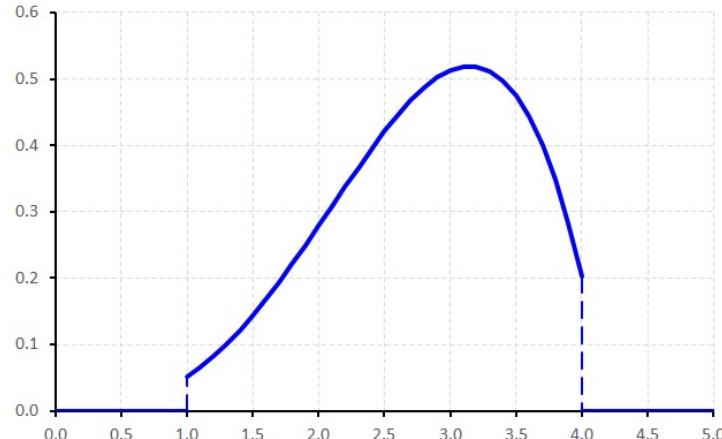
- a) Find the value of  $C$  that makes  $f_X(x)$  a valid probability density function.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(x) dx = \int_1^4 \frac{x^3(21-5x)}{C} dx = \int_1^4 \frac{21x^3 - 5x^4}{C} dx = \frac{21x^4 - 4x^5}{4C} \Big|_1^4 \\ &= \frac{1,263}{4C} = \frac{315.75}{C} = 1. \end{aligned}$$

$$\Rightarrow C = \mathbf{315.75}$$

$$= \frac{1,263}{4}.$$

$$\begin{aligned} f_X(x) &= \frac{x^3(21-5x)}{315.75} \\ &= \frac{4x^3(21-5x)}{1,263}, \\ 1 \leq x &\leq 4. \end{aligned}$$



b) Find the cumulative distribution function of  $X$ ,  $F_X(x) = P(X \leq x)$ .

“Hint”: To double-check your answer: should be  $F_X(1) = 0$ ,  $F_X(4) = 1$ .

$$\begin{aligned} F_X(x) = P(X \leq x) &= \int_{-\infty}^x f_X(u) du = \int_1^x \frac{u^3(21-5u)}{315.75} du \\ &= \left. \frac{21u^4 - 4u^5}{1,263} \right|_1^x = \frac{21x^4 - 4x^5 - 17}{1,263}, \quad 1 \leq x < 4. \end{aligned}$$

Obviously,  $F_X(x) = 0, \quad x < 1, \quad F_X(x) = 1, \quad x \geq 4.$

Indeed,  $\frac{21 \cdot 1^4 - 4 \cdot 1^5 - 17}{1,263} = 0, \quad \frac{21 \cdot 4^4 - 4 \cdot 4^5 - 17}{1,263} = 1.$  ☺

c) Find the average GPA of the students at Anytown State University,  $E(X) = \mu_X$ .

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x \cdot f_X(x) dx = \int_1^4 x \cdot \frac{x^3(21-5x)}{315.75} dx = \int_1^4 \frac{21x^4 - 5x^5}{315.75} dx \\ &= \left. \frac{126x^5 - 25x^6}{9,472.5} \right|_1^4 = \frac{26,523}{9,472.5} = 2.8. \end{aligned}$$

**4.** (continued)

The following relationship is proposed to estimate the average amount of awake time per day, in hours, a student spends on activities that are not related to academics,  $y$ , from the student's GPA,  $x$ :

$$y = g(x) = \frac{24}{x+2}.$$

Consider  $Y = g(X) = \frac{24}{X+2}$ .

- d) Find the support (the range of possible values) of the probability distribution of  $Y$ .

$$1 \leq x \leq 4 \quad \Rightarrow \quad 3 \leq x+2 \leq 6$$

$$\Rightarrow 8 \geq \frac{24}{x+2} \geq 4 \quad \Rightarrow \quad 4 \leq y \leq 8.$$

- e) Use part (b) and the c.d.f. approach to find the c.d.f. of  $Y$ ,  $F_Y(y)$ .

“Hint”:  $F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = \dots$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P\left(\frac{24}{X+2} \leq y\right) = P\left(X \geq \frac{24}{y} - 2\right) = 1 - F_X\left(\frac{24}{y} - 2\right) \\ &= 1 - \frac{21\left(\frac{24}{y} - 2\right)^4 - 4\left(\frac{24}{y} - 2\right)^5 - 17}{1,263} = \frac{1,280 - 21\left(\frac{24}{y} - 2\right)^4 + 4\left(\frac{24}{y} - 2\right)^5}{1,263} \\ &= \frac{816y^5 + 23,808y^4 - 474,624y^3 + 4,534,272y^2 - 20,238,336y + 31,850,496}{1,263y^5} \\ &= \frac{272y^5 + 7,936y^4 - 158,208y^3 + 1,511,424y^2 - 6,746,112y + 10,616,832}{421y^5} \end{aligned}$$

$$= \frac{16(y-4) \left( 17y^4 + 564y^3 - 7,632y^2 + 63,936y - 165,888 \right)}{421y^5}, \quad 4 \leq y < 8.$$

Obviously,  $F_Y(y) = 0, \quad y < 4, \quad F_Y(y) = 1, \quad y \geq 8.$

Indeed,  $\frac{1,280 - 21\left(\frac{24}{4} - 2\right)^4 + 4\left(\frac{24}{4} - 2\right)^5}{1,263} = 0,$

$$\frac{1,280 - 21\left(\frac{24}{8} - 2\right)^4 + 4\left(\frac{24}{8} - 2\right)^5}{1,263} = 1. \quad \text{☺}$$

f) Use the change-of-variable technique to find the p.d.f. of Y,  $f_Y(y).$

“Hint”:  $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|.$

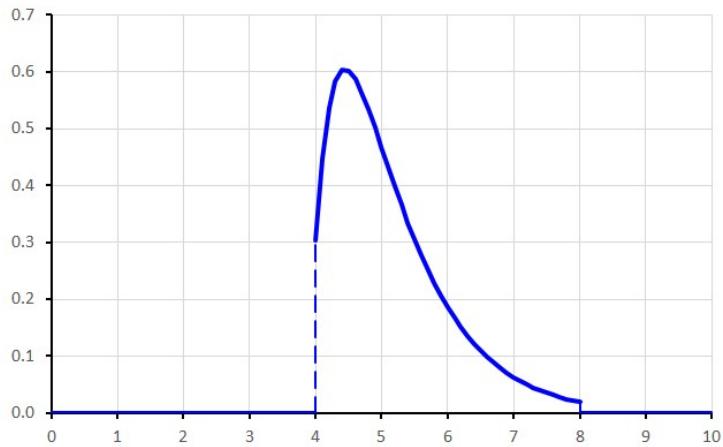
“Hint”: To double-check your answer: should be  $f_Y'(y) = F_Y'(y).$

$$y = \frac{24}{x+2}, \quad x = \frac{24}{y} - 2, \quad \frac{dx}{dy} = -\frac{24}{y^2}.$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \frac{\left( \frac{24}{y} - 2 \right)^3 \left( 21 - 5 \left( \frac{24}{y} - 2 \right) \right)}{315.75} \cdot \left| -\frac{24}{y^2} \right|$$

$$\begin{aligned}
&= \frac{24 \left( \frac{24}{y} - 2 \right)^3 \left( 31 - \frac{120}{y} \right)}{315.75 y^2} = \frac{96 \left( \frac{24}{y} - 2 \right)^3 \left( 31 - \frac{120}{y} \right)}{1,263 y^2} \\
&= \frac{32 \left( \frac{24}{y} - 2 \right)^3 \left( 31 - \frac{120}{y} \right)}{421 y^2} = \frac{256 (12-y)^3 (31y-120)}{421 y^6} \\
&= \frac{-7,936 y^4 + 316,416 y^3 - 4,534,272 y^2 + 26,984,448 y - 53,084,160}{421 y^6},
\end{aligned}$$

$$4 \leq y < 8.$$



Indeed,

$$\frac{d}{dy} \left( \frac{272y^5 + 7,936y^4 - 158,208y^3 + 1,511,424y^2 - 6,746,112y + 10,616,832}{421y^5} \right)$$

$$= \frac{-7,936y^4 + 316,416y^3 - 4,534,272y^2 + 26,984,448y - 53,084,160}{421y^6}.$$



$$\frac{d}{dx} \left( \frac{1280 - 21\left(\frac{24}{x} - 2\right)^4 + 4\left(\frac{24}{x} - 2\right)^5}{1263} \right)$$

Go

Graph » Examples »

Solution

**Keep Practicing >**

Show Steps ▼

$$\frac{d}{dx} \left( \frac{1280 - 21\left(\frac{24}{x} - 2\right)^4 + 4\left(\frac{24}{x} - 2\right)^5}{1263} \right) = \frac{32(-2x+24)^3(31x-120)}{421x^6}$$

$$\frac{32(-2y+24)^3(31y-120)}{421y^6} = \frac{32\left(\frac{24}{y}-2\right)^3\left(31-\frac{120}{y}\right)}{421y^2}.$$



g) Find  $E(Y) = \mu_Y$ .

“Hint”: While this integral is “fightable”, it will be somewhat annoying. After you set it up, I would recommend using your favorite online integral calculator instead of a hand-to-hand combat.

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y) dy = \int_4^8 y \cdot \frac{256(12-y)^3(31y-120)}{421y^6} dy = \dots$$

... indeed “fightable”, but annoying ...

The screenshot shows a web-based integral calculator interface. The input field contains the integral  $\int_4^8 x \cdot \frac{256(12-x)^3(31x-120)}{421x^6} dx$ . To the right of the input field is a red "Go" button. Below the input field, there are links for "Graph" and "Examples". On the right side of the screen, there are sharing icons (link, print, download). The word "Solution" is visible above a box containing the result of the integration. To the right of the result is a "Keep Practicing >" link. A "Show Steps" button is located at the top of the result box. The result itself is  $\frac{8(-992\ln(2) + 957)}{421}$  (Decimal: 5.11920...).

The screenshot shows a mobile or tablet-based integral calculator. The input field contains the integral  $\int_4^8 y \cdot \frac{256(12-y)^3(31y-120)}{421y^6} dy$ . Below the input field are buttons for "NATURAL LANGUAGE" and "MATH INPUT". To the right of the input field are buttons for a star, square root, derivative, limit, summation, and other mathematical operations. A purple bar labeled "POPULAR" contains buttons for square root, square, cube root, fourth root, derivative, second derivative, integral, double integral, triple integral, limit, summation, and matrix operations. The background of the calculator is purple.

The screenshot shows a web-based calculator interface. The input field contains the integral  $\int_4^8 y \cdot \frac{256(12-y)^3(31y-120)}{421y^6} dy$ . To the right of the input field are buttons for "More digits" and "Step-by-step solution". The result of the integral is  $\frac{8}{421}(957 - 992 \log(2)) \approx 5.1192$ . Below the result, a note states "log(x) is the natural logarithm".

$\approx 5.1192$ .

OR

$$E(Y) = E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) dx = \int_1^4 \frac{24}{x+2} \cdot \frac{x^3(21-5x)}{315.75} dx = \dots$$

The screenshot shows a math solver interface. At the top, there is a search bar containing the integral expression  $\int_1^4 \frac{24}{x+2} \cdot \frac{x^3(21-5x)}{315.75} dx$ . To the right of the search bar are a bookmark icon, a red "Go" button, and sharing/printing options. Below the search bar, there are links for "Graph" and "Examples". On the left, a "Solution" section displays the integral  $\int_1^4 \frac{24}{x+2} \cdot \frac{x^3(21-5x)}{315.75} dx = 5.11920\dots$ . To the right of the solution is a "Keep Practicing >" link. A "Show Steps" button is located at the bottom of the solution section.

The screenshot shows a mobile calculator application. At the top, there is a text input field containing the integral expression  $\text{integrate}\left(\left(\frac{24}{x+2}\right) * \left(\frac{x^3*(21-5*x)}{315.75}\right)\right), x \text{ from } 1 \text{ to } 4$ . Below the input field are two buttons: "NATURAL LANGUAGE" and "MATH INPUT". To the right of the input field are various mathematical operators and symbols. A numeric keypad is visible at the bottom, with rows for numbers 1-9, 0, decimal point, and arithmetic operators (+, -, ×, ÷, =).

The screenshot shows a web-based calculator interface. At the top, there is a text input field containing the integral expression  $\int_1^4 \frac{24(x^3(21-5x))}{(x+2)315.75} dx = 5.1192$ . To the right of the input field is a checkbox labeled "Step-by-step solution". Below the input field, there is a "Definite integral" label and a numeric keypad.

$\approx 5.1192.$

h) Is  $\mu_Y$  equal to  $g(\mu_X)$ ?

“Hint”: “equal” means “exactly equal” here. Not “close” or “sort of close”, but “equal”.

$$\mu_Y \approx 5.1192. \quad g(\mu_X) = \frac{24}{2.8 + 2} = 5.$$

$$\mu_Y = E(Y) \neq g(E(X)) = g(\mu_X). \quad \mu_Y \text{ is NOT equal to } g(\mu_X).$$

Recall: IF  $g(x)$  is a linear function, that is, IF  $g(x) = ax + b$ ,

$$\text{then } E(g(X)) = E(ax + b) = aE(X) + b = g(E(X)).$$

However, in general, if  $g(x)$  is NOT a linear function,

$$\text{then } E(g(X)) \neq g(E(X)).$$

Spoiler:

Here,  $E(g(X)) > g(E(X))$  since  $g(x) = \frac{24}{x+2}$  “curves up” for  $1 \leq x \leq 4$ .

5. Consider a discrete random variable  $X$  with the probability mass function

$$p_X(x) = \frac{x^3(21-5x)}{C}, \quad x=1, 2, 3, 4, \quad \text{zero elsewhere.}$$

a) Find the value of  $C$  that makes  $f_X(x)$  a valid probability mass function.

$$p_X(1) = \frac{16}{C}, \quad p_X(2) = \frac{88}{C}, \quad p_X(3) = \frac{162}{C}, \quad p_X(4) = \frac{64}{C}.$$

$$p_X(1) + p_X(2) + p_X(3) + p_X(4) = 1.$$

$$\frac{16}{C} + \frac{88}{C} + \frac{162}{C} + \frac{64}{C} = \frac{330}{C} = 1. \Rightarrow C = 330.$$

b) Find  $E(X) = \mu_X$ .

$$E(X) = \sum_{\text{all } x} x \cdot p_X(x) = 1 \cdot \frac{16}{330} + 2 \cdot \frac{88}{330} + 3 \cdot \frac{162}{330} + 4 \cdot \frac{64}{330} = \frac{934}{330} \approx 2.8303.$$

5. (continued)

Consider  $Y = g(X) = \frac{24}{X+2}$ .

c) Obtain the probability distribution of  $Y$ .

$x$	$p_X(x)$	$g(x)$
1	$\frac{16}{330} \approx 0.0485$	8
2	$\frac{88}{330} \approx 0.2667$	6
3	$\frac{162}{330} \approx 0.4909$	4.8
4	$\frac{64}{330} \approx 0.1939$	4

$y$	$p_Y(y)$
4	$\frac{64}{330} = \frac{32}{165}$
4.8	$\frac{162}{330} = \frac{81}{165}$
6	$\frac{88}{330} = \frac{44}{165}$
8	$\frac{16}{330} = \frac{8}{165}$

d) Find  $E(Y) = \mu_Y$ .

$$E(Y) = \sum_{\text{all } y} y \cdot p_Y(y) = 4 \cdot \frac{64}{330} + 4.8 \cdot \frac{162}{330} + 6 \cdot \frac{88}{330} + 8 \cdot \frac{16}{330} = \frac{1,689.6}{330} = 5.12.$$

OR

$$\begin{aligned} E(Y) &= E(g(X)) = \sum_{\text{all } x} g(x) \cdot p_X(x) \\ &= \frac{24}{1+2} \cdot \frac{16}{330} + \frac{24}{2+2} \cdot \frac{88}{330} + \frac{24}{3+2} \cdot \frac{162}{330} + \frac{24}{4+2} \cdot \frac{64}{330} = 5.12. \end{aligned}$$

e) Is  $\mu_Y$  equal to  $g(\mu_X)$ ?

“Hint”: “equal” means “exactly equal” here. Not “close” or “sort of close”, but “equal”.

$$\mu_Y = 5.12. \quad g(\mu_X) = \frac{24}{\frac{934}{330} + 2} \approx 4.9686.$$

$$\mu_Y = E(Y) \neq g(E(X)) = g(\mu_X). \quad \mu_Y \text{ is NOT equal to } g(\mu_X).$$

Recall: IF  $g(x)$  is a linear function, that is, IF  $g(x) = ax + b$ ,

then  $E(g(X)) = E(ax + b) = aE(X) + b = g(E(X))$ .

However, in general, if  $g(x)$  is NOT a linear function,

then  $E(g(X)) \neq g(E(X))$ .

Spoiler:

Here,  $E(g(X)) > g(E(X))$  since  $g(x) = \frac{24}{x+2}$  “curves up” for  $1 \leq x \leq 4$ .