STAT 410 A. Stepanov

Examples for Week 02 (Disc1)

STAT 410 does NOT have a discussion section.

STAT 400 (an easier class) DOES have a discussion section.
STAT 410 (a harder class) does NOT.

There is a simple explanation for this: About 20 years ago, there was only one section of STAT 410 with about 40 students and only two lectures of STAT 400 with about 150 students each (each divided into 3 discussion sections of about 50 students each). The number of students has changed a little bit since then, the course structure has not (yet)…

However, don't you wish there was a discussion section in STAT 410?

If STAT 410 did have a discussion section, this is what your TA would most likely go over during it:

0. Find $E(X)$ and $Var(X)$, if random variable X has the following c.d.f.

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \dfrac{x^3 + 1}{9} & -1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

1. Find $E(X)$ and $Var(X)$, if random variable X has the following c.d.f.

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \dfrac{x^3 + 3}{15} & -1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

2. Consider a mixed random variable X with

the p.m.f. of the discrete portion of the probability distribution

$p(2) = 0.08, \qquad p(4) = c, \qquad$ zero otherwise,

and the p.d.f. of the continuous portion of the probability distribution

$f(x) = \dfrac{x^3}{100}, \qquad 2 \le x \le 4, \qquad$ zero elsewhere.

a) Find the value of c that would make this a valid probability distribution.

b) Find $\mu_X = E(X)$.

c) Find $\sigma_X^2 = Var(X)$.

3. The probability that a loss will occur is 0.15. If the loss occurs, the amount of the loss has the density function

$f(x) = c\,x^{-5}, \quad x > 1, \qquad$ zero otherwise.

(That is, loss is a mixed random variable with $p(0) = 1 - 0.15 = 0.85$.)

An insurance company will pay the entire amount of loss.

a) Find the value of c that would make this a valid probability distribution.

b) Calculate the expected value of the payment.

c) Calculate the variance of the payment.

4. Consider a random variable X with the **p**.d.f.

$$\boldsymbol{f}(x) = \begin{cases} 0 & x < 2 \\ 0.12\,x & 2 \le x < 3 \\ 0.20\,x & 3 \le x < 4 \\ \mathbf{0} & x \ge 4 \end{cases}$$

a) Find $\mu = \mathrm{E}(\mathrm{X})$.

b) Find $\sigma^2 = \mathrm{Var}(\mathrm{X})$.

c) Find $\mathrm{P}(2.5 < \mathrm{X} < 3)$, $\mathrm{P}(2.5 \le \mathrm{X} < 3)$, $\mathrm{P}(2.5 < \mathrm{X} \le 3)$, $\mathrm{P}(2.5 \le \mathrm{X} \le 3)$.

5. Consider a random variable X with the **c**.d.f.

$$\mathbf{F}(x) = \begin{cases} 0 & x < 2 \\ 0.12\,x & 2 \le x < 3 \\ 0.20\,x & 3 \le x < 4 \\ \mathbf{1} & x \ge 4 \end{cases}$$

a) Find $\mu = \mathrm{E}(\mathrm{X})$.

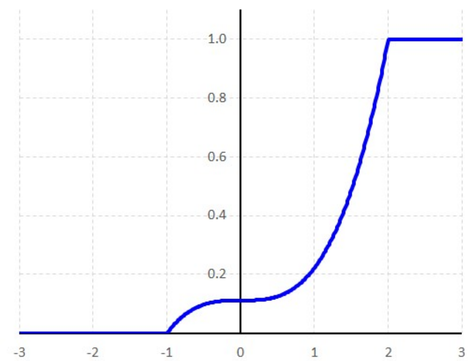
b) Find $\sigma^2 = \mathrm{Var}(\mathrm{X})$.

c) Find $\mathrm{P}(2.5 < \mathrm{X} < 3)$, $\mathrm{P}(2.5 \le \mathrm{X} < 3)$, $\mathrm{P}(2.5 < \mathrm{X} \le 3)$, $\mathrm{P}(2.5 \le \mathrm{X} \le 3)$.

Answers:

0. Find $\mathrm{E}(\mathrm{X})$ and $\mathrm{Var}(\mathrm{X})$, if random variable X has the following c.d.f.

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \dfrac{x^3+1}{9} & -1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$



X is a continuous random variable with the p.d.f.

$$f_X(x) = F_X'(x) = \frac{x^2}{3}, \qquad -1 < x < 2, \qquad \text{zero otherwise.}$$

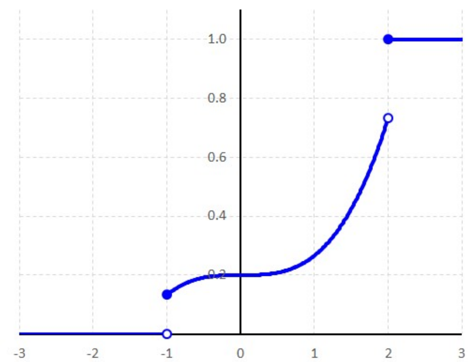
$$\mathrm{E}(\mathrm{X}) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_{-1}^{2} \frac{x^3}{3}\,dx = \frac{x^4}{12}\Bigg|_{-1}^{2} = \frac{16-1}{12} = \mathbf{\frac{5}{4}} = \mathbf{1.25}.$$

$$\mathrm{E}(\mathrm{X}^2) = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\,dx = \int_{-1}^{2} \frac{x^4}{3}\,dx = \frac{x^5}{15}\Bigg|_{-1}^{2} = \frac{32+1}{15} = 2.2.$$

$$\mathrm{Var}(\mathrm{X}) = \mathrm{E}(\mathrm{X}^2) - [\mathrm{E}(\mathrm{X})]^2 = 2.2 - 1.25^2 = \mathbf{0.6375}.$$

1. Find $\mathrm{E}(\mathrm{X})$ and $\mathrm{Var}(\mathrm{X})$, if random variable X has the following c.d.f.

$$\mathrm{F}_{\mathrm{X}}(x) = \begin{cases} 0 & x < -1 \\ \dfrac{x^3+3}{15} & -1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$



X is a mixed random variable.

Discrete:

$\mathrm{F}_{\mathrm{X}}(x)$ "jumps" at $x = -1$ from 0 to $\dfrac{2}{15}$, $p(-1) = \dfrac{2}{15}$,

and $x = 2$ from $\dfrac{11}{15}$ to 1, $p(2) = \dfrac{4}{15}$.

Continuous:

$$f_{\mathrm{X}}(x) = \mathrm{F}_{\mathrm{X}}'(x) = \frac{x^2}{5}, \qquad -1 < x < 2, \qquad \text{zero otherwise.}$$

$$\mathrm{E}(\mathrm{X}) = \sum_{\text{all } x} x \cdot p(x) + \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$= -1 \cdot \frac{2}{15} + 2 \cdot \frac{4}{15} + \int_{-1}^{2} x \cdot \frac{x^2}{5}\,dx = \frac{6}{15} + \frac{x^4}{20}\Bigg|_{-1}^{2}$$

$$= \frac{2}{5} + \frac{16-1}{20} = \mathbf{\frac{23}{20}} = \mathbf{1.15}.$$

$$E(X^2) = \sum_{\text{all } x} x^2 \cdot p(x) + \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$$

$$= (-1)^2 \cdot \frac{2}{15} + (2)^2 \cdot \frac{4}{15} + \int_{-1}^{2} x^2 \cdot \frac{x^2}{5}\, dx = \frac{18}{15} + \frac{x^5}{25}\Bigg|_{-1}^{2}$$

$$= 1.2 + \frac{32+1}{25} = 2.52.$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 2.52 - 1.15^2 = \mathbf{1.1975}.$$

2. Consider a mixed random variable X with

the p.m.f. of the discrete portion of the probability distribution

$p(2) = 0.08,$ $\quad p(4) = c,$ $\quad$ zero otherwise,

and the p.d.f. of the continuous portion of the probability distribution

$f(x) = \dfrac{x^3}{100},$ $\quad 2 \le x \le 4,$ $\quad$ zero elsewhere.

a) Find the value of c that would make this a valid probability distribution.

$$1 = \sum_{\text{all } x} p(x) + \int_{-\infty}^{\infty} f(x)\, dx$$

$$= [0.08 + c] + \int_{2}^{4} \frac{x^3}{100}\, dx = 0.08 + c + \frac{x^4}{400}\Bigg|_{2}^{4}$$

$$= 0.08 + c + \frac{256-16}{400} = 0.08 + c + 0.60 = c + 0.68.$$

$$\Rightarrow \quad c = \mathbf{0.32}.$$

b) Find $\mu_X = E(X)$.

$$E(X) = \sum_{\text{all } x} x \cdot p(x) + \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= 2 \cdot 0.08 + 4 \cdot 0.32 + \int_{2}^{4} x \cdot \frac{x^3}{100}\, dx = 0.16 + 1.28 + \left.\frac{x^5}{500}\right|_{2}^{4}$$

$$= 1.44 + \frac{1024 - 32}{500} = \mathbf{3.424}.$$

c) Find $\sigma_X^2 = \mathrm{Var}(X)$.

$$E(X^2) = \sum_{\text{all } x} x^2 \cdot p(x) + \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$$

$$= (2)^2 \cdot 0.08 + (4)^2 \cdot 0.32 + \int_{2}^{4} x^2 \cdot \frac{x^3}{100}\, dx = 0.32 + 5.12 + \left.\frac{x^6}{600}\right|_{2}^{4}$$

$$= 5.44 + \frac{4096 - 64}{600} = 12.16.$$

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 12.16 - 3.424^2 = \mathbf{0.436224}.$$

3. The probability that a loss will occur is 0.15. If the loss occurs, the amount of the loss has the density function

$$f(x) = c\,x^{-5}, \quad x > 1, \qquad \text{zero otherwise.}$$

(That is, loss is a mixed random variable with $p(0) = 1 - 0.15 = 0.85$.)
An insurance company will pay the entire amount of loss.

Loss is a mixed random variable.

Discrete portion:

$$p(0) = 0.85.$$

Continuous portion:

$$f(x) = c\,x^{-5}, \quad x > 1.$$

a) Find the value of c that would make this a valid probability distribution.

$$0.85 + \int_1^\infty c\,x^{-5}\,dx = 1. \qquad \Rightarrow \qquad 0.85 + \frac{c}{4} = 1. \qquad \Rightarrow \qquad c = \mathbf{0.60}.$$

b) Calculate the expected value of the payment.

$$\mathrm{E}(\text{Payment}) = \mathrm{E}(\text{Loss}) = 0 \times 0.85 + \int_1^\infty x \cdot 0.60\,x^{-5}\,dx = \mathbf{0.20}.$$

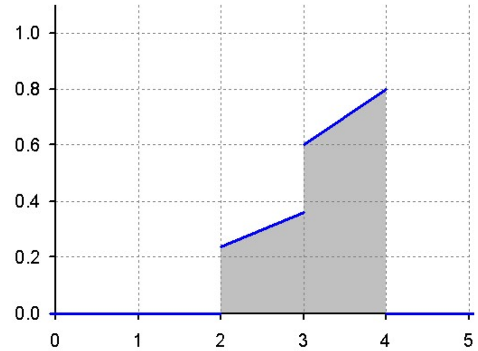
c) Calculate the variance of the payment.

$$\mathrm{E}(\text{Payment}^2) = \mathrm{E}(\text{Loss}^2) = 0^2 \times 0.85 + \int_1^\infty x^2 \cdot 0.60\,x^{-5}\,dx = 0.30.$$

$$\mathrm{Var}(\text{Payment}) = \mathrm{Var}(\text{Loss}) = 0.30 - 0.20^2 = \mathbf{0.26}.$$

4. Consider a random variable X with the **p**.d.f.

$$\boldsymbol{f}(x) = \begin{cases} 0 & x < 2 \\ 0.12\,x & 2 \le x < 3 \\ 0.20\,x & 3 \le x < 4 \\ \mathbf{0} & x \ge 4 \end{cases}$$



a) Find $\mu = \mathrm{E}(\mathrm{X})$.

$$\mathrm{E}(\mathrm{X}) = \int_2^3 x \cdot 0.12\,x\,dx + \int_3^4 x \cdot 0.20\,x\,dx = \left(0.04x^3\right)\Big|_2^3 + \left(\frac{0.20}{3}x^3\right)\Big|_3^4$$

$$= \frac{19}{25} + \frac{37}{15} = \mathbf{\frac{242}{75}} \approx 3.226667.$$

b) Find $\sigma^2 = \mathrm{Var}(\mathrm{X})$.

$$\mathrm{E}(\mathrm{X}^2) = \int_2^3 x^2 \cdot 0.12\,x\,dx + \int_3^4 x^2 \cdot 0.20\,x\,dx = \left(0.03x^4\right)\Big|_2^3 + \left(0.05x^4\right)\Big|_3^4$$

$$= 1.95 + 8.75 = 10.7.$$

$$\mathrm{Var}(\mathrm{X}) = \mathrm{E}(\mathrm{X}^2) - [\mathrm{E}(\mathrm{X})]^2 = 10.7 - \left(\frac{242}{75}\right)^2 = \mathbf{\frac{3247}{11250}} \approx 0.288622.$$

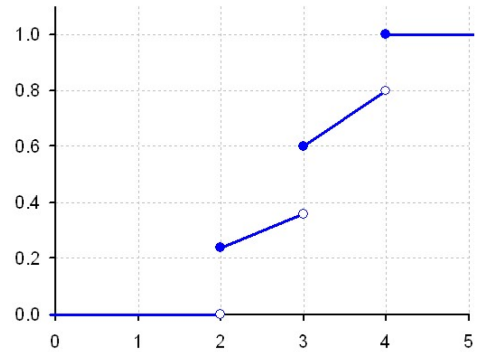
c) Find $\mathrm{P}(2.5 < \mathrm{X} < 3)$, $\mathrm{P}(2.5 \le \mathrm{X} < 3)$, $\mathrm{P}(2.5 < \mathrm{X} \le 3)$, $\mathrm{P}(2.5 \le \mathrm{X} \le 3)$.

Since $\mathrm{P}(\mathrm{X} = 2.5) = \mathrm{P}(\mathrm{X} = 3) = 0$, all four probabilities are equal.

$$\mathrm{P}(2.5 < \mathrm{X} < 3) = \int_{2.5}^3 0.12\,x\,dx = \left(0.06x^2\right)\Big|_{2.5}^3 = \mathbf{0.165}.$$

5. Consider a random variable X with the **c**.d.f.

$$\mathbf{F}(x) = \begin{cases} 0 & x < 2 \\ 0.12\,x & 2 \le x < 3 \\ 0.20\,x & 3 \le x < 4 \\ \mathbf{1} & x \ge 4 \end{cases}$$



$\mathrm{F}(x)$ "jumps" at $x = 2$ from 0 to 0.24, size of the "jump" = 0.24 – 0 = 0.24,
at $x = 3$ from 0.36 to 0.6, size of the "jump" = 0.6 – 0.36 = 0.24,
at $x = 4$ from 0.80 to 1.0, size of the "jump" = 1.0 – 0.8 = 0.20.

$\Rightarrow$ Discrete portion of the probability distribution of X:

$$p(2) = 0.24, \qquad p(3) = 0.24, \qquad p(4) = 0.20.$$

Continuous portion of the probability distribution of X:

$$f(x) = \mathrm{F}'(x) = \begin{cases} 0 & x < 2 \\ 0.12 & 2 < x < 3 \\ 0.20 & 3 < x < 4 \\ 0 & x \ge 4 \end{cases}.$$

a) Find $\mu = \mathrm{E}(\mathrm{X})$.

$$\mathrm{E}(\mathrm{X}) = 2 \cdot 0.24 + 3 \cdot 0.24 + 4 \cdot 0.20 + \int_2^3 x \cdot 0.12\, dx + \int_3^4 x \cdot 0.20\, dx$$

$$= 0.48 + 0.72 + 0.80 + 0.30 + 0.70 = \mathbf{3}.$$

OR

Since X is a nonnegative random variable,

$$\mathrm{E}(\mathrm{X}) = \int_0^\infty (1 - \mathrm{F}(x))\, dx = \int_0^2 1\, dx + \int_2^3 (1 - 0.12\,x)\, dx + \int_3^4 (1 - 0.20\,x)\, dx$$

$$= 2 + (1 - 0.30) + (1 - 0.70) = \mathbf{3}.$$

b) Find $\sigma^2 = \mathrm{Var}(X)$.

$$E(X^2) = 2^2 \cdot 0.24 + 3^2 \cdot 0.24 + 4^2 \cdot 0.20 + \int_2^3 x^2 \cdot 0.12\,dx + \int_3^4 x^2 \cdot 0.20\,dx$$

$$= 0.96 + 2.16 + 3.20 + 0.76 + \frac{37}{15} = \mathbf{\frac{716}{75}} \approx 9.546667.$$

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \mathbf{\frac{41}{75}} \approx 0.546667.$$

c) Find $P(2.5 < X < 3)$, $P(2.5 \le X < 3)$, $P(2.5 < X \le 3)$, $P(2.5 \le X \le 3)$.

Since $P(X = 2.5) = 0$,

$$P(2.5 < X < 3) = P(2.5 \le X < 3)$$

and $$P(2.5 < X \le 3) = P(2.5 \le X \le 3).$$

$$P(2.5 \le X < 3) = P(2.5 < X < 3) = F(3\text{-}) - F(2.5) = 0.36 - 0.30 = \mathbf{0.06}.$$

$$P(2.5 \le X \le 3) = P(2.5 < X \le 3) = F(3) - F(2.5) = 0.60 - 0.30 = \mathbf{0.30}.$$