STAT 410 A. Stepanov

Examples for Week 01 (Disc1)

STAT 410 does NOT have a discussion section.

STAT 400 (an easier class) DOES have a discussion section.
STAT 410 (a harder class) does NOT.

There is a simple explanation for this: About 20 years ago, there was only one section of STAT 410 with about 40 students and only two lectures of STAT 400 with about 150 students each (each divided into 3 discussion sections of about 50 students each). The number of students has changed a little bit since then, the course structure has not (yet)…

However, don't you wish there was a discussion section in STAT 410?

If STAT 410 did have a discussion section, this is what your TA would most likely go over during it:

1. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{6x+7}{C}, \qquad 3 \le x \le 8, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

b) Find the cumulative distribution function of X, $F_X(x) = P(X \le x)$.

"Hint": To double-check your answer: should be $F_X(3) = 0$, $F_X(8) = 1$.

c) Find the expected value of X, $E(X) = \mu_X$.

1. (continued)

Consider $Y = g(X) = \sqrt{X+1}$. Find the probability distribution of Y.

d) Find the support (the range of possible values) of the probability distribution of Y.

e) Use part (b) and the c.d.f. approach to find the c.d.f. of Y, $F_Y(y)$.

"Hint": $F_Y(y) = P(Y \le y) = P(g(X) \le y) = \dots$.

f) Use the change-of-variable technique to find the p.d.f. of Y, $f_Y(y)$.

"Hint": $f_Y(y) = f_X(g^{-1}(y)) \left| \dfrac{dx}{dy} \right|$.

"Hint": To double-check your answer: should be $f_Y(y) = F_Y'(y)$.

g) Is μ_Y equal to $g(\mu_X)$? $E(X) = \mu_X$, $E(Y) = \mu_Y$.

"Hint": "equal" means "exactly equal" here.
Not "close" or "sort of close", but "equal".

1. (continued)

Consider $W = h(X) = \dfrac{1}{X+2}$. Find the probability distribution of W.

h) Find the support (the range of possible values) of the probability distribution of W.

i) Use part (b) and the c.d.f. approach to find the c.d.f. of W, $F_W(w)$.

"Hint": $F_W(w) = P(W \le w) = P(h(X) \le w) = \dots$.

j) Use the change-of-variable technique to find the p.d.f. of W, $f_W(w)$.

"Hint": $f_W(w) = f_X(h^{-1}(w)) \left| \dfrac{dx}{dw} \right|$.

"Hint": To double-check your answer: should be $f_W(w) = F_W'(w)$.

k) Is μ_W equal to $h(\mu_X)$? $E(X) = \mu_X$, $E(W) = \mu_W$.

"Hint": "equal" means "exactly equal" here.
Not "close" or "sort of close", but "equal".

2. Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{3x-1}{C}, \qquad x = 3, 4, 5, 6.$$

a) Find the value of C that makes $f_X(x)$ a valid probability mass function.

b) Consider $Y = \frac{12}{X-2}$. Find the probability distribution of Y.

c) Is μ_Y equal to $g(\mu_X)$? $\qquad g(x) = \frac{12}{x-2}$.

3. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{x}{C}, \qquad 4 \le x \le 28, \qquad \text{zero elsewhere.}$$

a) Find the value of C that would make this a valid probability distribution.

b) Find the cumulative distribution function of X, $F_X(x) = P(X \le x)$.

"Hint": To double-check your answer: should be $F_X(4) = 0$, $F_X(28) = 1$.

3. (continued)

Consider $Y = g(X) = \sqrt{X-3}$.

c) Find the support (the range of possible values) of the probability distribution of Y.

d) Use part (b) and the c.d.f. approach to find the c.d.f. of Y, $F_Y(y)$.

"Hint": $F_Y(y) = P(Y \le y) = P(g(X) \le y) = \ldots$.

e) Use the change-of-variable technique to find the p.d.f. of Y, $f_{\mathrm{Y}}(y)$.

"Hint": $f_{\mathrm{Y}}(y) = f_{\mathrm{X}}(g^{-1}(y)) \left| \dfrac{dx}{dy} \right|$.

"Hint": To double-check your answer: should be $f_{\mathrm{Y}}(y) = \mathrm{F}'_{\mathrm{Y}}(y)$.

3. (continued)

Consider $\mathrm{W} = h(\mathrm{X}) = \dfrac{8}{\mathrm{X}+12}$.

f) Find the support (the range of possible values) of the probability distribution of W.

g) Use part (b) and the c.d.f. approach to find the c.d.f. of W, $\mathrm{F}_{\mathrm{W}}(w)$.

"Hint": $\mathrm{F}_{\mathrm{W}}(w) = \mathrm{P}(\mathrm{W} \le w) = \mathrm{P}(h(\mathrm{X}) \le w) = \dots$.

h) Use the change-of-variable technique to find the p.d.f. of W, $f_{\mathrm{W}}(w)$.

"Hint": $f_{\mathrm{W}}(w) = f_{\mathrm{X}}(h^{-1}(w)) \left| \dfrac{dx}{dw} \right|$.

"Hint": To double-check your answer: should be $f_{\mathrm{W}}(w) = \mathrm{F}'_{\mathrm{W}}(w)$.

3. (continued)

i) Find the expected value of X, $\mu_{\mathrm{X}} = \mathrm{E}(\mathrm{X})$.

j) (i) Find the expected value of Y, $\mu_{\mathrm{Y}} = \mathrm{E}(\mathrm{Y})$.

(ii) Does μ_{Y} equal to $g(\mu_{\mathrm{X}})$?

"Hint": "equal" means "exactly equal" here. Not "close" or "sort of close", but "equal".

k) (i) Find the expected value of W, $\mu_{\mathrm{W}} = \mathrm{E}(\mathrm{W})$.

(ii) Does μ_{W} equal to $h(\mu_{\mathrm{X}})$?

"Hint": "equal" means "exactly equal" here. Not "close" or "sort of close", but "equal".

4. The distribution on the GPA of the students at Anytown State University can be nicely approximated by the following probability density function:

$$f_X(x) = \frac{x^3\,(21 - 5x)}{C}, \qquad 1.0 \le x \le 4.0, \qquad \text{zero elsewhere.}$$

(The students with the GPA below 1.0 are "asked" to leave the university.)

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

b) Find the cumulative distribution function of X, $F_X(x) = P(X \le x)$.

"Hint": To double-check your answer: should be $F_X(1) = 0$, $F_X(4) = 1$.

c) Find the average GPA of the students at Anytown State University, $E(X) = \mu_X$.

4. (continued)

The following relationship is proposed to estimate the average amount of awake time per day, in hours, a student spends on activities that are not related to academics, y, from the student's GPA, x:

$$y = g(x) = \frac{24}{x+2}.$$

Consider $Y = g(X) = \dfrac{24}{X+2}$.

d) Find the support (the range of possible values) of the probability distribution of Y.

e) Use part (b) and the c.d.f. approach to find the c.d.f. of Y, $F_Y(y)$.

"Hint": $F_Y(y) = P(Y \le y) = P(g(X) \le y) = \ldots$.

f) Use the change-of-variable technique to find the p.d.f. of Y, $f_Y(y)$.

"Hint": $f_Y(y) = f_X(g^{-1}(y)) \left| \dfrac{dx}{dy} \right|$.

"Hint": To double-check your answer: should be $f_Y(y) = F_Y'(y)$.

g) Find $E(Y) = \mu_Y$.

"Hint": While this integral is "fightable", it will be somewhat annoying. After you set it up, I would recommend using your favorite online integral calculator instead of a hand-to-hand combat.

h) Is μ_Y equal to $g(\mu_X)$?

"Hint": "equal" means "exactly equal" here. Not "close" or "sort of close", but "equal".

5. Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{x^3(21-5x)}{C}, \qquad x = 1, 2, 3, 4, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability mass function.

b) Find $E(X) = \mu_X$.

5. (continued)

Consider $Y = g(X) = \dfrac{24}{X+2}$.

c) Obtain the probability distribution of Y.

d) Find $E(Y) = \mu_Y$.

e) Is μ_Y equal to $g(\mu_X)$?

"Hint": "equal" means "exactly equal" here. Not "close" or "sort of close", but "equal".

Answers:

1. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{6x+7}{C}, \qquad 3 \le x \le 8, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_3^8 \frac{6x+7}{C}\,dx = \frac{3x^2+7x}{C}\Bigg|_3^8 = \frac{(192+56)-(27+21)}{C} = \frac{200}{C}.$$

$\Rightarrow \qquad C = \mathbf{200}.$

$$f_X(x) = \frac{6x+7}{200}, \qquad 3 \le x \le 8.$$

b) Find the cumulative distribution function of X, $F_X(x) = P(X \le x)$.

"Hint": To double-check your answer: should be $F_X(3) = 0$, $F_X(8) = 1$.

$$F_X(x) = P(X \le x) = \int_3^x \frac{6u+7}{200}\,du = \frac{3u^2+7u}{200}\Bigg|_3^x$$

$$= \frac{3x^2+7x-48}{200} = \frac{(x-3)(3x+16)}{200}, \qquad 3 \le x < 8.$$

Obviously, $F_X(x) = 0, \quad x < 3, \qquad F_X(x) = 1, \quad x \ge 8.$

Indeed, $\frac{(3-3)(3\cdot 3+16)}{200} = 0, \qquad \frac{(8-3)(3\cdot 8+16)}{200} = 1.$ ☺

c) Find the expected value of X, $E(X) = \mu_X$.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_3^8 x \cdot \frac{6x+7}{200}\,dx = \left. \frac{2x^3 + \frac{7}{2}x^2}{200} \right|_3^8$$

$$= \frac{(1024+224)-(54+31.5)}{200} = \frac{1{,}162.5}{200} = \mathbf{\frac{93}{16}} = \mathbf{5.8125}.$$

1. (continued)

Consider $Y = g(X) = \sqrt{X+1}$. Find the probability distribution of Y.

d) Find the support (the range of possible values) of the probability distribution of Y.

$3 \le x \le 8 \quad \Rightarrow \quad 4 \le x+1 \le 9$

$\Rightarrow \quad 2 \le \sqrt{x+1} \le 3 \quad \Rightarrow \quad \mathbf{2 \le y \le 3}.$

e) Use part (b) and the c.d.f. approach to find the c.d.f. of Y, $F_Y(y)$.

"Hint": $F_Y(y) = P(Y \le y) = P(g(X) \le y) = \dots$.

$$F_Y(y) = P(Y \le y) = P(\sqrt{X+1} \le y) = P(X \le y^2 - 1) = F_X(y^2 - 1)$$

$$= \frac{(y^2-4)(3y^2+13)}{200} = \frac{(y-2)(y+2)(3y^2+13)}{200}$$

$$= \frac{3(y^2-1)^2 + 7(y^2-1) - 48}{200} = \frac{3y^4 + y^2 - 52}{200}, \qquad 2 \le y < 3.$$

Obviously, $F_Y(y) = 0, \quad y < 2, \qquad F_Y(y) = 1, \quad y \ge 3.$

Indeed,
$$\frac{(2^2-4)(3 \cdot 2^2 + 13)}{200} = 0,$$
$$\frac{(3^2-4)(3 \cdot 3^2 + 13)}{200} = 1. \qquad ☺$$

f) Use the change-of-variable technique to find the p.d.f. of Y, $f_Y(y)$.

"Hint": $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$.

"Hint": To double-check your answer: should be $f_Y(y) = F_Y'(y)$.

$$y = \sqrt{x+1}, \qquad x = y^2 - 1, \qquad \frac{dx}{dy} = 2y.$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \frac{6\left(y^2 - 1\right) + 7}{200} \cdot |\,2y\,|$$

$$= \frac{12y^3 + 2y}{200} = \frac{6y^3 + y}{100}, \qquad 2 < y < 3.$$

Indeed, $\quad \dfrac{d}{dy}\left(\dfrac{3y^4 + y^2 - 52}{200}\right) = \dfrac{6y^3 + y}{100}.$ ☺

g) Is μ_Y equal to $g(\mu_X)$? $\qquad E(X) = \mu_X, \quad E(Y) = \mu_Y.$

"Hint": "equal" means "exactly equal" here.
Not "close" or "sort of close", but "equal".

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy = \int_2^3 y \cdot \frac{6y^3 + y}{100}\,dy = \left. \frac{\frac{6}{5}y^5 + \frac{1}{3}y^3}{100} \right|_2^3 = \left. \frac{18y^5 + 5y^3}{1,500} \right|_2^3$$

$$= \frac{(4,374 + 135) - (576 + 40)}{1,500} = \mathbf{\frac{3,893}{1,500}} \approx 2.59533333.$$

OR

$$E(Y) = E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\,dx$$

$$= \int_3^8 \sqrt{x+1} \cdot \frac{6x+7}{200}\,dx = \ldots$$

$\int_3^8 \sqrt{x+1}\,\frac{6x+7}{200}\,dx$

Solution

$$\int_3^8 \frac{1}{200}\sqrt{x+1}\,(6x+7)\,dx = \frac{3893}{1500} \approx 2.5953$$

$\frac{3893}{1500}$

$g(\mu_X) = \sqrt{5.8125+1} \approx 2.61007663.$

$\mu_Y = E(Y) \neq g(E(X)) = g(\mu_X).$ μ_Y is **NOT** equal to $g(\mu_X)$.

Recall: IF $g(x)$ is a linear function, that is, IF $g(x) = a\,x + b$,

then $E(g(X)) = E(a\,X + b) = a\,E(X) + b = g(E(X)).$

However, in general, if $g(x)$ is NOT a linear function,

then $E(g(X)) \neq g(E(X)).$

Spoiler:

Here, $E(g(X)) < g(E(X))$ since $g(x) = \sqrt{x+1}$ "curves down".

1. (continued)

Consider $W = h(X) = \dfrac{1}{X+2}$. Find the probability distribution of W.

h) Find the support (the range of possible values) of the probability distribution of W.

$3 \le x \le 8 \quad \Rightarrow \quad 5 \le x+2 \le 10$

$\Rightarrow \quad \dfrac{1}{5} \ge \dfrac{1}{x+2} \ge \dfrac{1}{10} \quad \Rightarrow \quad \mathbf{0.10 = \dfrac{1}{10} \le w \le \dfrac{1}{5} = 0.20}.$

i) Use part (b) and the c.d.f. approach to find the c.d.f. of W, $F_W(w)$.

"Hint": $F_W(w) = P(W \le w) = P(h(X) \le w) = \dots .$

$$F_W(w) = P(W \le w) = P\left(\frac{1}{X+2} \le w\right) = P\left(X \ge \frac{1}{w} - 2\right) = 1 - F_X\left(\frac{1}{w} - 2\right)$$

$$= 1 - \frac{3\left(\frac{1}{w} - 2\right)^2 + 7\left(\frac{1}{w} - 2\right) - 48}{200}$$

$$= 1 - \frac{\left(\frac{1}{w} - 2 - 3\right)\left(\frac{3}{w} - 6 + 16\right)}{200} = 1 - \frac{\left(\frac{1}{w} - 5\right)\left(\frac{3}{w} + 10\right)}{200}$$

$$= 1 - \frac{\frac{3}{w^2} - \frac{5}{w} - 50}{200} = \frac{250 + \frac{5}{w} - \frac{3}{w^2}}{200}$$

$$= \frac{250\,w^2 + 5\,w - 3}{200\,w^2} = \frac{5}{4} + \frac{1}{40\,w} - \frac{3}{200\,w^2}, \qquad \frac{1}{10} \le w < \frac{1}{5}.$$

Obviously, $F_W(w) = 0, \quad w < \frac{1}{10}, \qquad F_W(w) = 1, \quad w \ge \frac{1}{5}.$

Indeed, $\dfrac{250 + 5\cdot 10 - 3\cdot 10^2}{200} = 0, \qquad \dfrac{250 + 5\cdot 5 - 3\cdot 5^2}{200} = 1.$ ☺

j) Use the change-of-variable technique to find the p.d.f. of W, $f_W(w)$.

"Hint": $f_W(w) = f_X(h^{-1}(w)) \left| \frac{dx}{dw} \right|$.

"Hint": To double-check your answer: should be $f_W(w) = F_W'(w)$.

$$w = \frac{1}{x+2}, \qquad x = \frac{1}{w} - 2, \qquad \frac{dx}{dw} = -\frac{1}{w^2}.$$

$$f_W(w) = f_X(h^{-1}(w)) \left| \frac{dx}{dw} \right| = \frac{6\left(\frac{1}{w} - 2\right) + 7}{200} \cdot \left| -\frac{1}{w^2} \right|$$

$$= \frac{6 - 5w}{200\,w^3} = \frac{3}{100\,w^3} - \frac{1}{40\,w^2}, \qquad \frac{1}{10} < w < \frac{1}{5}.$$

Indeed, $\frac{d}{dw}\left(\frac{5}{4} + \frac{1}{40\,w} - \frac{3}{200\,w^2} \right) = \frac{3}{100\,w^3} - \frac{1}{40\,w^2}$. ☺

k) Is μ_W equal to $h(\mu_X)$? $E(X) = \mu_X$, $E(W) = \mu_W$.

"Hint": "equal" means "exactly equal" here.
Not "close" or "sort of close", but "equal".

$$E(W) = \int_{-\infty}^{\infty} w \cdot f_W(w)\,dw = \int_{1/10}^{1/5} w \cdot \frac{6 - 5w}{200\,w^3}\,dw = \int_{1/10}^{1/5} \left(\frac{3}{100\,w^2} - \frac{1}{40\,w} \right) dw$$

$$= \left(-\frac{3}{100\,w} - \frac{\ln w}{40} \right) \Bigg|_{1/10}^{1/5} = -\frac{15 - 30}{100} - \frac{\ln\left(\frac{1}{5}\right) - \ln\left(\frac{1}{10}\right)}{40} = \frac{15}{100} - \frac{\ln\left(\frac{10}{5}\right)}{40}$$

$$= \frac{15}{100} - \frac{\ln(2)}{40} = \frac{3}{20} - \frac{\ln(2)}{40} = \frac{6 - \ln(2)}{40} \approx 0.13267132.$$

OR

$$\mathrm{E}(\mathrm{W}) = \mathrm{E}(h(\mathrm{X})) = \int_{-\infty}^{\infty} h(x) \cdot f_{\mathrm{X}}(x)\,dx = \int_{3}^{8} \frac{1}{x+2} \cdot \frac{6x+7}{200}\,dx$$

$$u = x+2 \qquad x = u-2 \qquad dx = du$$

$$= \int_{5}^{10} \frac{1}{u} \cdot \frac{6(u-2)+7}{200}\,du = \int_{5}^{10} \frac{1}{u} \cdot \frac{6u-5}{200}\,du$$

$$= \int_{5}^{10} \left(\frac{3}{100} - \frac{1}{40u} \right) du = \left(\frac{3u}{100} - \frac{\ln u}{40} \right) \Bigg|_{5}^{10}$$

$$= \frac{30-15}{100} - \frac{\ln(10) - \ln(5)}{40} = \frac{30-15}{100} - \frac{\ln\left(\frac{10}{5}\right)}{40}$$

$$= \frac{15}{100} - \frac{\ln(2)}{40} = \frac{3}{20} - \frac{\ln(2)}{40} = \frac{6 - \ln(2)}{40} \approx 0.13267132.$$

$$\int_3^8 \frac{6x+7}{(x+2)\,200}\,dx = \frac{1}{40}(6 - \log(2)) \approx 0.13267$$

$$\int_3^8 \frac{1}{x+2} \cdot \frac{6x+7}{200}\,dx$$

Solution

$$\frac{30 - 5\ln(2)}{200}$$

$$h(\mu_{\mathrm{X}}) = \frac{1}{5.8125+2} = 0.128.$$

$\mu_{\mathrm{W}} = \mathrm{E}(\mathrm{W}) \neq h(\mathrm{E}(\mathrm{X})) = h(\mu_{\mathrm{X}})$. μ_{W} is **NOT** equal to $h(\mu_{\mathrm{X}})$.

Recall: IF $h(x)$ is a linear function, that is, IF $h(x) = a\,x + b$,

then $\mathrm{E}(h(\mathrm{X})) = \mathrm{E}(a\,\mathrm{X} + b) = a\,\mathrm{E}(\mathrm{X}) + b = h(\mathrm{E}(\mathrm{X}))$.

However, in general, if $h(x)$ is NOT a linear function,

then $\mathrm{E}(h(\mathrm{X})) \neq h(\mathrm{E}(\mathrm{X}))$.

Spoiler:

Here, $\mathrm{E}(h(\mathrm{X})) > h(\mathrm{E}(\mathrm{X}))$ since $h(x) = \dfrac{1}{x+2}$ "curves up".

2. Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{3x-1}{C}, \qquad x = 3, 4, 5, 6.$$

a) Find the value of C that makes $f_X(x)$ a valid probability mass function.

$$p_X(3) = \frac{8}{C}, \qquad p_X(4) = \frac{11}{C}, \qquad p_X(5) = \frac{14}{C}, \qquad p_X(6) = \frac{17}{C}.$$

$$p_X(3) + p_X(4) + p_X(5) + p_X(6) = 1.$$

$$\frac{8}{C} + \frac{11}{C} + \frac{14}{C} + \frac{17}{C} = \frac{50}{C} = 1. \qquad \Rightarrow \qquad C = \mathbf{50}.$$

b) Consider $Y = \dfrac{12}{X-2}$. Find the probability distribution of Y.

x	$p_X(x)$	$\frac{12}{x-2}$
3	$\frac{8}{50} = 0.16$	12
4	$\frac{11}{50} = 0.22$	6
5	$\frac{14}{50} = 0.28$	4
6	$\frac{17}{50} = 0.34$	3

y	$p_Y(y)$
3	**0.34**
4	**0.28**
6	**0.22**
12	**0.16**

OR

$$y = \frac{12}{x-2}, \qquad x = \frac{12}{y} + 2.$$

$$p_Y(y) = p_X\left(\frac{12}{y}+2\right) = \frac{\frac{36}{y}+5}{50} = \frac{36+5y}{50y} = \frac{18}{25y} + \frac{1}{10}, \qquad y = 3, 4, 6, 12.$$

Indeed, $p_Y(3) = \frac{18}{25 \cdot 3} + \frac{1}{10} = 0.34$, $\quad p_Y(4) = \frac{18}{25 \cdot 4} + \frac{1}{10} = 0.28$,

$p_Y(6) = \frac{18}{25 \cdot 6} + \frac{1}{10} = 0.22$, $\quad p_Y(12) = \frac{18}{25 \cdot 12} + \frac{1}{10} = 0.16$.

For discrete random variables, the possible values are isolated points on the number line.

$\Rightarrow$ no derivatives. $\Rightarrow$ no $\frac{dx}{dy}$.

c) Is μ_Y equal to $g(\mu_X)$? $\qquad g(x) = \frac{12}{x-2}$.

$$E(X) = \sum_{\text{all } x} x \cdot p_X(x) = 3 \cdot 0.16 + 4 \cdot 0.22 + 5 \cdot 0.28 + 6 \cdot 0.34 = 4.8.$$

$$E(Y) = \sum_{\text{all } y} y \cdot p_Y(y) = 3 \cdot 0.34 + 4 \cdot 0.28 + 6 \cdot 0.22 + 12 \cdot 0.16 = 5.38.$$

OR

$$E(Y) = E(g(X)) = \sum_{\text{all } x} g(x) \cdot p_X(x)$$

$$= \frac{12}{3-2} \cdot \frac{8}{50} + \frac{12}{4-2} \cdot \frac{11}{50} + \frac{12}{5-2} \cdot \frac{14}{50} + \frac{12}{6-2} \cdot \frac{17}{50} = 5.38.$$

$g(\mu_X) = \dfrac{12}{4.8 - 2} \approx 4.285714.$

$\mu_Y = \mathrm{E}(Y) \neq g(\mathrm{E}(X)) = g(\mu_X).$ μ_Y is NOT equal to $g(\mu_X)$.

Recall: IF $g(x)$ is a linear function, that is, IF $g(x) = a\,x + b$,

then $\mathrm{E}(g(X)) = \mathrm{E}(a\,X + b) = a\,\mathrm{E}(X) + b = g(\mathrm{E}(X)).$

However, in general, if $g(x)$ is NOT a linear function,

then $\mathrm{E}(g(X)) \neq g(\mathrm{E}(X)).$

Spoiler:

Here, $\mathrm{E}(g(X)) > g(\mathrm{E}(X))$ since $g(x) = \dfrac{12}{x-2}$ “curves up” for $3 \le x \le 6$.

3. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{x}{C}, \qquad 4 \le x \le 28, \qquad \text{zero elsewhere.}$$

a) Find the value of C that would make this a valid probability distribution.

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_4^{28} \frac{x}{C}\,dx = \left.\frac{x^2}{2C}\right|_4^{28} = \frac{784-16}{2C} = \frac{384}{C}.$$

$\Rightarrow \quad C = \mathbf{384}.$

$$f_X(x) = \frac{x}{384}, \qquad 4 \le x \le 28.$$

b) Find the cumulative distribution function of X, $F_X(x) = P(X \le x)$.

"Hint": To double-check your answer: should be $F_X(4) = 0$, $F_X(28) = 1$.

$$F_X(x) = P(X \le x) = \int_4^x \frac{u}{384}\,du = \left.\frac{u^2}{768}\right|_4^x = \frac{x^2 - 16}{768} = \frac{x^2}{768} - \frac{1}{48}, \qquad 4 \le x < 28.$$

Obviously, $F_X(x) = 0, \quad x < 4, \qquad F_X(x) = 1, \quad x \ge 28.$

Indeed, $\frac{4^2 - 16}{768} = 0, \qquad \frac{28^2 - 16}{768} = 1.$ ☺

3. (continued)

Consider $Y = g(X) = \sqrt{X-3}$.

c) Find the support (the range of possible values) of the probability distribution of Y.

$4 \le x \le 28 \quad \Rightarrow \quad 1 \le x-3 \le 25 \quad \Rightarrow \quad 1 \le \sqrt{x-3} \le 5.$

$\mathbf{1 \le y \le 5}$.

d) Use part (b) and the c.d.f. approach to find the c.d.f. of Y, $F_Y(y)$.

"Hint": $F_Y(y) = P(Y \le y) = P(g(X) \le y) = \dots$.

$$F_Y(y) = P(Y \le y) = P(\sqrt{X-3} \le y) = P(X \le y^2+3) = F_X(y^2+3)$$

$$= \frac{\left(y^2+3\right)^2 - 16}{768} = \frac{y^4 + 6y^2 - 7}{768} = \frac{\left(y^2-1\right)\left(y^2+7\right)}{768}$$

$$= \frac{(y-1)(y+1)\left(y^2+7\right)}{768}, \qquad 1 \le y < 5.$$

Obviously, $F_Y(y) = 0, \quad y < 1, \qquad F_Y(y) = 1, \quad y \ge 5.$

Indeed, $\dfrac{1^4 + 6\cdot 1^2 - 7}{768} = 0, \qquad \dfrac{5^4 + 6\cdot 5^2 - 7}{768} = 1.$ ☺

e) Use the change-of-variable technique to find the p.d.f. of Y, $f_Y(y)$.

"Hint": $f_Y(y) = f_X(g^{-1}(y)) \left| \dfrac{dx}{dy} \right|$.

"Hint": To double-check your answer: should be $f_Y(y) = F_Y'(y)$.

$y = \sqrt{x-3}$ $\qquad x = y^2 + 3$ $\qquad \dfrac{dx}{dy} = 2y$

$$f_Y(y) = \frac{\left(y^2 + 3\right)}{384} \cdot |2y| = \frac{y^3 + 3y}{192}, \qquad 1 \le y \le 5.$$

Indeed, $$\frac{d}{dy}\left(\frac{y^4 + 6y^2 - 7}{768}\right) = \frac{y^3 + 3y}{192}.$$ ☺

3. (continued)

Consider $W = h(X) = \dfrac{8}{X+12}$.

f) Find the support (the range of possible values) of the probability distribution of W.

$4 \le x \le 28 \quad \Rightarrow \quad 16 \le x + 12 \le 40 \quad \Rightarrow \quad \dfrac{1}{2} \ge \dfrac{8}{x+12} \ge \dfrac{1}{5}$.

$$\mathbf{0.2 = \frac{1}{5} \le \boldsymbol{w} \le \frac{1}{2} = 0.5}.$$

g) Use part (b) and the c.d.f. approach to find the c.d.f. of W, $F_W(w)$.

"Hint": $F_W(w) = P(W \le w) = P(h(X) \le w) = \ldots$.

$$F_W(w) = P(W \le w) = P\left(\frac{8}{X+12} \le w\right) = P\left(X \ge \frac{8}{w} - 12\right) = 1 - F_X\left(\frac{8}{w} - 12\right)$$

$$= 1 - \frac{\left(\frac{8}{w} - 12\right)^2 - 16}{768} = 1 - \frac{\frac{64}{w^2} - \frac{192}{w} + 128}{768} = 1 - \frac{\frac{1}{w^2} - \frac{3}{w} + 2}{12}$$

$$= 1 - \frac{\left(\frac{1}{w} - 1\right)\left(\frac{1}{w} - 2\right)}{12} = \frac{10 + \frac{3}{w} - \frac{1}{w^2}}{12} = \frac{10w^2 + 3w - 1}{12w^2}$$

$$= \frac{(5w-1)(2w+1)}{12w^2} = \frac{5}{6} + \frac{1}{4w} - \frac{1}{12w^2}, \qquad \frac{1}{5} \le w < \frac{1}{2}.$$

Obviously, $F_W(w) = 0, \quad w < \frac{1}{5}, \qquad F_W(w) = 1, \quad w \ge \frac{1}{2}.$

Indeed,
$$\frac{10 + \frac{3}{0.2} - \frac{1}{0.2^2}}{12} = \frac{10 + 15 - 25}{12} = 0,$$

$$\frac{10 + \frac{3}{0.5} - \frac{1}{0.5^2}}{12} = \frac{10 + 6 - 4}{12} = 1.$$

☺

h) Use the change-of-variable technique to find the p.d.f. of W, $f_W(w)$.

"Hint": $f_W(w) = f_X(h^{-1}(w)) \left| \frac{dx}{dw} \right|$.

"Hint": To double-check your answer: should be $f_W(w) = F'_W(w)$.

$$w = \frac{8}{x+12} \qquad x = \frac{8}{w} - 12 \qquad \frac{dx}{dw} = -\frac{8}{w^2}.$$

$$f_W(w) = f_X(h^{-1}(w)) \left| \frac{dx}{dw} \right| = \frac{\left(\frac{8}{w} - 12\right)}{384} \cdot \left| -\frac{8}{w^2} \right|$$

$$= \frac{\frac{64}{w} - 96}{384\,w^2} = \frac{\frac{2}{w} - 3}{12\,w^2} = \frac{2 - 3w}{12\,w^3} = \frac{1}{6\,w^3} - \frac{1}{4\,w^2}, \qquad \frac{1}{5} \le w \le \frac{1}{2}.$$

Indeed, $\frac{d}{dw}\left(\frac{5}{6} + \frac{1}{4\,w} - \frac{1}{12\,w^2}\right) = \frac{1}{6\,w^3} - \frac{1}{4\,w^2}$. ☺

3. (continued)

i) Find the expected value of X, $\mu_X = E(X)$.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_4^{28} x \cdot \frac{x}{384}\,dx = \int_4^{28} \frac{x^2}{384}\,dx = \left. \frac{x^3}{1{,}152} \right|_4^{28}$$

$$= \frac{21{,}952 - 64}{1{,}152} = \mathbf{19}.$$

For fun:

i ½) Find the median of the probability distribution of X.

Need m such that $\quad P(X \le m) = P(X \ge m) = \frac{1}{2}.$

$$P(X \le m) = F_X(m) = \frac{m^2 - 16}{768} = \frac{1}{2}. \qquad \Rightarrow \qquad m^2 - 16 = 384.$$

$$\Rightarrow \qquad m^2 = 400. \qquad \Rightarrow \qquad m = \mathbf{20}.$$

j) (i) Find the expected value of Y, $\mu_Y = E(Y)$.

(ii) Does μ_Y equal to $g(\mu_X)$?

"Hint": "equal" means "exactly equal" here. Not "close" or "sort of close", but "equal".

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy = \int_1^5 y \cdot \frac{y^3 + 3y}{192}\, dy = \int_1^5 \frac{y^4 + 3y^2}{192}\, dy$$

$$= \left(\frac{\frac{1}{5} y^5 + y^3}{192} \right) \Bigg|_1^5 = \frac{625 + 125 - 0.2 - 1}{192} = \mathbf{3.9}.$$

OR

$$\mathrm{E}(\mathrm{Y}) = \mathrm{E}(g(\mathrm{X})) = \int_{-\infty}^{\infty} g(x) \cdot f_{\mathrm{X}}(x)\,dx = \int_{4}^{28} \sqrt{x-3} \cdot \frac{x}{384}\,dx$$

$u = x - 3 \qquad x = u + 3 \qquad dx = du$

$$= \int_{1}^{25} \sqrt{u} \cdot \frac{u+3}{384}\,du = \frac{1}{384} \cdot \int_{1}^{25} u^{1.5}\,du + \frac{3}{384} \cdot \int_{1}^{25} u^{0.5}\,du$$

$$= \frac{1}{384} \cdot \frac{u^{2.5}}{2.5} \Bigg|_{1}^{25} + \frac{3}{384} \cdot \frac{u^{1.5}}{1.5} \Bigg|_{1}^{25}$$

$$= \frac{1}{384} \cdot \frac{3{,}125 - 1}{2.5} + \frac{3}{384} \cdot \frac{125 - 1}{1.5} = \mathbf{3.9}.$$

$g(\mu_{\mathrm{X}}) = \sqrt{19-3} = 4.$

$\mu_{\mathrm{Y}} = \mathrm{E}(\mathrm{Y}) \neq g(\mathrm{E}(\mathrm{X})) = g(\mu_{\mathrm{X}}).$ μ_{Y} is **NOT** equal to $g(\mu_{\mathrm{X}})$.

Recall: IF $g(x)$ is a linear function, that is, IF $g(x) = a\,x + b$,

then $\mathrm{E}(g(\mathrm{X})) = \mathrm{E}(a\,\mathrm{X} + b) = a\,\mathrm{E}(\mathrm{X}) + b = g(\mathrm{E}(\mathrm{X})).$

However, in general, if $g(x)$ is NOT a linear function,

then $\mathrm{E}(g(\mathrm{X})) \neq g(\mathrm{E}(\mathrm{X})).$

Spoiler:

Here, $\mathrm{E}(g(\mathrm{X})) < g(\mathrm{E}(\mathrm{X}))$ since $g(x) = \sqrt{x-3}$ "curves down" for $4 < x < 28$.

k) (i) Find the expected value of W, $\mu_W = E(W)$.

(ii) Does μ_W equal to $h(\mu_X)$?

"Hint": "equal" means "exactly equal" here. Not "close" or "sort of close", but "equal".

$$E(W) = \int_{-\infty}^{\infty} w \cdot f_W(w)\,dw = \int_{1/5}^{1/2} w \cdot \frac{2-3w}{12w^3}\,dw = \int_{1/5}^{1/2} \left(\frac{1}{6w^2} - \frac{1}{4w} \right) dw$$

$$= \left(-\frac{1}{6w} - \frac{\ln w}{4} \right) \Bigg|_{1/5}^{1/2} = -\frac{2}{6} + \frac{\ln(2)}{4} + \frac{5}{6} - \frac{\ln(5)}{4}$$

$$= \frac{1}{2} - \frac{\ln(2.5)}{4} = \frac{2-\ln(2.5)}{4} \approx 0.2709273.$$

OR

$$E(W) = E(h(X)) = \int_{-\infty}^{\infty} h(x) \cdot f_X(x)\,dx = \int_{4}^{28} \frac{8}{x+12} \cdot \frac{x}{384}\,dx$$

$u = x + 12$ $\qquad$ $x = u - 12$ $\qquad$ $dx = du$

$$= \int_{16}^{40} \frac{8}{u} \cdot \frac{u-12}{384}\,du = \frac{1}{48} \cdot \int_{16}^{40} \left(1 - \frac{12}{u} \right) du = \frac{1}{48} \cdot (u - 12 \ln u) \Big|_{16}^{40}$$

$$= \frac{1}{48} \cdot \big(40 - 12 \ln(40) - 16 + 12 \ln(16) \big) = \frac{1}{48} \cdot \left(24 - 12 \ln\left(\frac{40}{16} \right) \right)$$

$$= \frac{1}{2} - \frac{\ln(2.5)}{4} = \frac{2-\ln(2.5)}{4} \approx 0.2709273.$$

$$h(\mu_X) = \frac{8}{19+12} \approx 0.2580645.$$

$\mu_W = E(W) \neq h(E(X)) = h(\mu_X)$. μ_W is **NOT** equal to $h(\mu_X)$.

Recall: IF $h(x)$ is a linear function, that is, IF $h(x) = ax + b$,

then $E(h(X)) = E(aX + b) = aE(X) + b = h(E(X))$.

However, in general, if $h(x)$ is NOT a linear function,

then $E(h(X)) \neq h(E(X))$.

Spoiler:

Here, $E(h(X)) > h(E(X))$ since $h(x) = \dfrac{8}{x+12}$ "curves up"

for $4 < x < 28$.

4. The distribution on the GPA of the students at Anytown State University can be nicely approximated by the following probability density function:

$$f_X(x) = \frac{x^3(21-5x)}{C}, \qquad 1.0 \le x \le 4.0, \qquad \text{zero elsewhere.}$$

(The students with the GPA below 1.0 are "asked" to leave the university.)

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

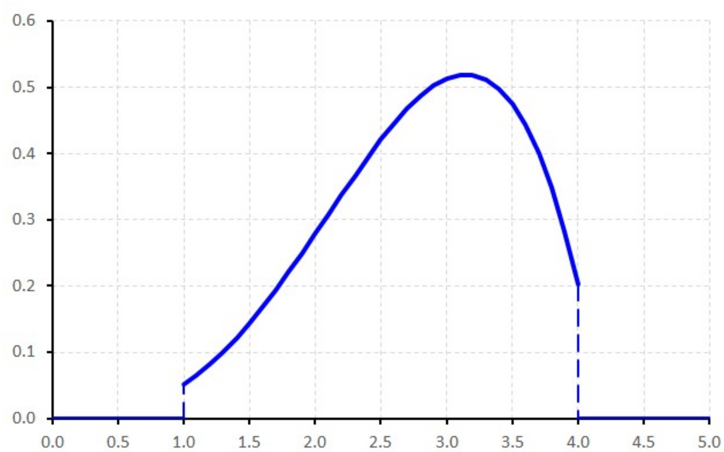
$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_1^4 \frac{x^3(21-5x)}{C}\,dx = \int_1^4 \frac{21x^3 - 5x^4}{C}\,dx = \left.\frac{21x^4 - 4x^5}{4C}\right|_1^4$$

$$= \frac{1{,}263}{4C} = \frac{315.75}{C} = 1.$$

$\Rightarrow$ $C = \mathbf{315.75}$

$$= \frac{\mathbf{1{,}263}}{\mathbf{4}}.$$

$$f_X(x) = \frac{x^3(21-5x)}{315.75} = \frac{4x^3(21-5x)}{1{,}263}, \qquad 1 \le x \le 4.$$



b) Find the cumulative distribution function of X, $F_X(x) = P(X \le x)$.

"Hint": To double-check your answer: should be $F_X(1) = 0$, $F_X(4) = 1$.

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(u)\,du = \int_{1}^{x} \frac{u^3(21-5u)}{315.75}\,du$$

$$= \frac{21u^4 - 4u^5}{1{,}263} \Bigg|_1^x = \frac{21x^4 - 4x^5 - 17}{1{,}263}, \qquad 1 \le x < 4.$$

Obviously, $F_X(x) = 0, \quad x < 1,$ $\qquad F_X(x) = 1, \quad x \ge 4.$

Indeed, $\dfrac{21 \cdot 1^4 - 4 \cdot 1^5 - 17}{1{,}263} = 0, \qquad \dfrac{21 \cdot 4^4 - 4 \cdot 4^5 - 17}{1{,}263} = 1.$ ☺

c) Find the average GPA of the students at Anytown State University, $E(X) = \mu_X$.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_{1}^{4} x \cdot \frac{x^3(21-5x)}{315.75}\,dx = \int_{1}^{4} \frac{21x^4 - 5x^5}{315.75}\,dx$$

$$= \frac{126x^5 - 25x^6}{9{,}472.5} \Bigg|_1^4 = \frac{26{,}523}{9{,}472.5} = \mathbf{2.8}.$$

4. (continued)

The following relationship is proposed to estimate the average amount of awake time per day, in hours, a student spends on activities that are not related to academics, y, from the student's GPA, x:

$$y = g(x) = \frac{24}{x+2}.$$

Consider $\mathrm{Y} = g(\mathrm{X}) = \dfrac{24}{\mathrm{X}+2}$.

d) Find the support (the range of possible values) of the probability distribution of Y.

$1 \le x \le 4 \quad \Rightarrow \quad 3 \le x+2 \le 6$

$\Rightarrow \quad 8 \ge \dfrac{24}{x+2} \ge 4 \quad \Rightarrow \quad \mathbf{4 \le y \le 8}.$

e) Use part (b) and the c.d.f. approach to find the c.d.f. of Y, $\mathrm{F_Y}(y)$.

"Hint": $\mathrm{F_Y}(y) = \mathrm{P}(\mathrm{Y} \le y) = \mathrm{P}(g(\mathrm{X}) \le y) = \ldots$.

$$\mathrm{F_Y}(y) = \mathrm{P}(\mathrm{Y} \le y) = \mathrm{P}\left(\frac{24}{\mathrm{X}+2} \le y\right) = \mathrm{P}\left(\mathrm{X} \ge \frac{24}{y} - 2\right) = 1 - \mathrm{F_X}\left(\frac{24}{y} - 2\right)$$

$$= 1 - \frac{21\left(\frac{24}{y}-2\right)^4 - 4\left(\frac{24}{y}-2\right)^5 - 17}{1{,}263} = \frac{1{,}280 - 21\left(\frac{24}{y}-2\right)^4 + 4\left(\frac{24}{y}-2\right)^5}{1{,}263}$$

$$= \frac{816\,y^5 + 23{,}808\,y^4 - 474{,}624\,y^3 + 4{,}534{,}272\,y^2 - 20{,}238{,}336\,y + 31{,}850{,}496}{1{,}263\,y^5}$$

$$= \frac{272\,y^5 + 7{,}936\,y^4 - 158{,}208\,y^3 + 1{,}511{,}424\,y^2 - 6{,}746{,}112\,y + 10{,}616{,}832}{421\,y^5}$$

$$= \frac{16\,(y-4)\left(17y^4 + 564y^3 - 7{,}632y^2 + 63{,}936y - 165{,}888\right)}{421y^5}, \qquad 4 \le y < 8.$$

Obviously, $\quad F_Y(y) = 0, \quad y < 4, \qquad F_Y(y) = 1, \quad y \ge 8.$

Indeed,
$$\frac{1{,}280 - 21\left(\frac{24}{4}-2\right)^4 + 4\left(\frac{24}{4}-2\right)^5}{1{,}263} = 0,$$

$$\frac{1{,}280 - 21\left(\frac{24}{8}-2\right)^4 + 4\left(\frac{24}{8}-2\right)^5}{1{,}263} = 1.$$

☺

f) Use the change-of-variable technique to find the p.d.f. of Y, $f_Y(y)$.

"Hint": $\quad f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right|.$

"Hint": To double-check your answer: should be $f_Y(y) = F_Y'(y)$.

$$y = \frac{24}{x+2}, \qquad x = \frac{24}{y} - 2, \qquad \frac{dx}{dy} = -\frac{24}{y^2}.$$

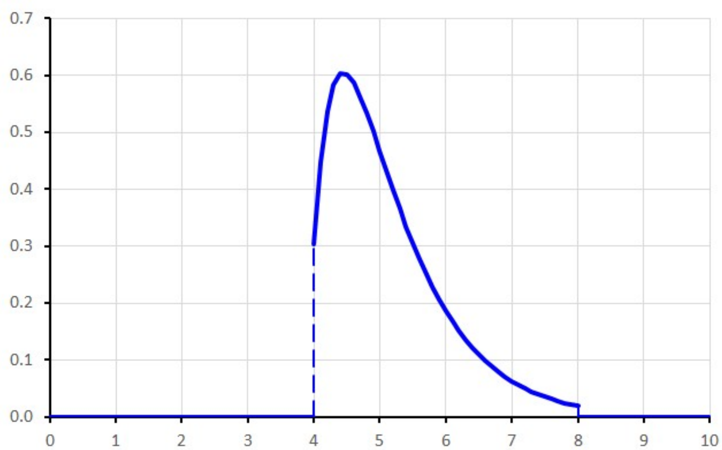
$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right| = \frac{\left(\frac{24}{y}-2\right)^3\left(21 - 5\left(\frac{24}{y}-2\right)\right)}{315.75} \cdot \left|-\frac{24}{y^2}\right|$$

$$= \frac{24\left(\frac{24}{y}-2\right)^3\left(31-\frac{120}{y}\right)}{315.75\,y^2} = \frac{96\left(\frac{24}{y}-2\right)^3\left(31-\frac{120}{y}\right)}{1,263\,y^2}$$

$$= \frac{32\left(\frac{24}{y}-2\right)^3\left(31-\frac{120}{y}\right)}{421\,y^2} = \frac{256\,(12-y)^3\,(31y-120)}{421\,y^6}$$

$$= \frac{-7,936\,y^4+316,416\,y^3-4,534,272\,y^2+26,984,448\,y-53,084,160}{421\,y^6},$$

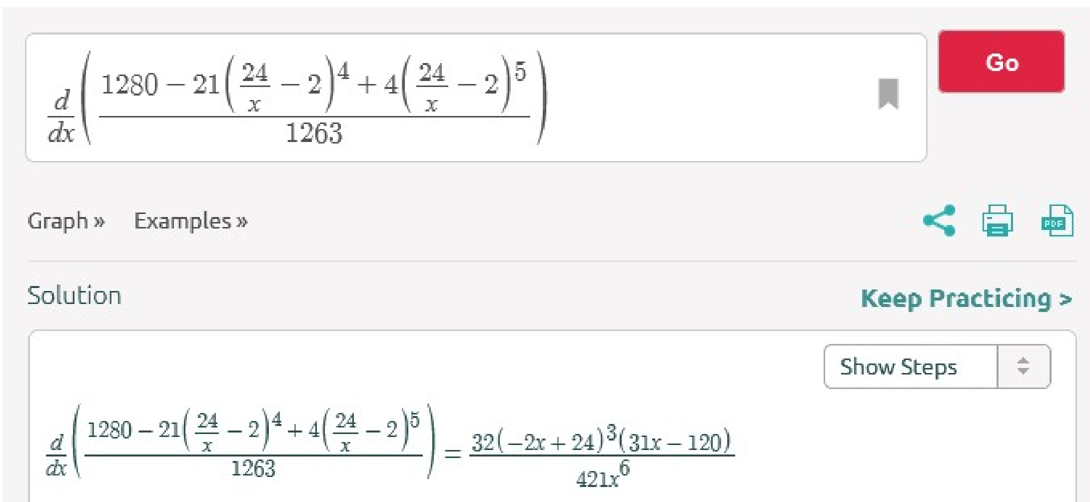
$$4 \le y < 8.$$



Indeed,

$$\frac{d}{dy}\left(\frac{272\,y^5+7{,}936\,y^4-158{,}208\,y^3+1{,}511{,}424\,y^2-6{,}746{,}112\,y+10{,}616{,}832}{421\,y^5}\right)$$

$$=\frac{-7{,}936\,y^4+316{,}416\,y^3-4{,}534{,}272\,y^2+26{,}984{,}448\,y-53{,}084{,}160}{421\,y^6}.$$ ☺



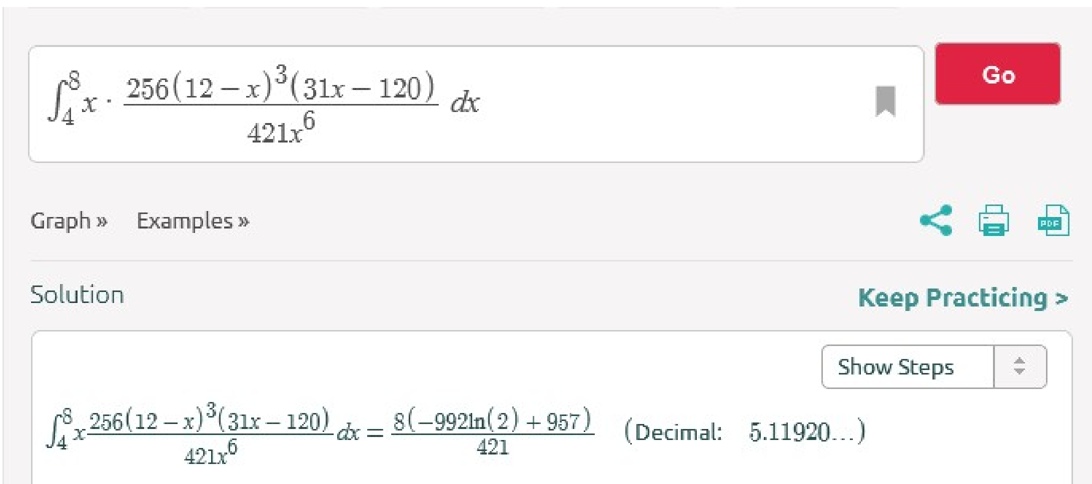
$$\frac{32\,(-2\,y+24)^3\,(31\,y-120)}{421\,y^6}=\frac{32\left(\frac{24}{y}-2\right)^3\left(31-\frac{120}{y}\right)}{421\,y^2}.$$ ☺

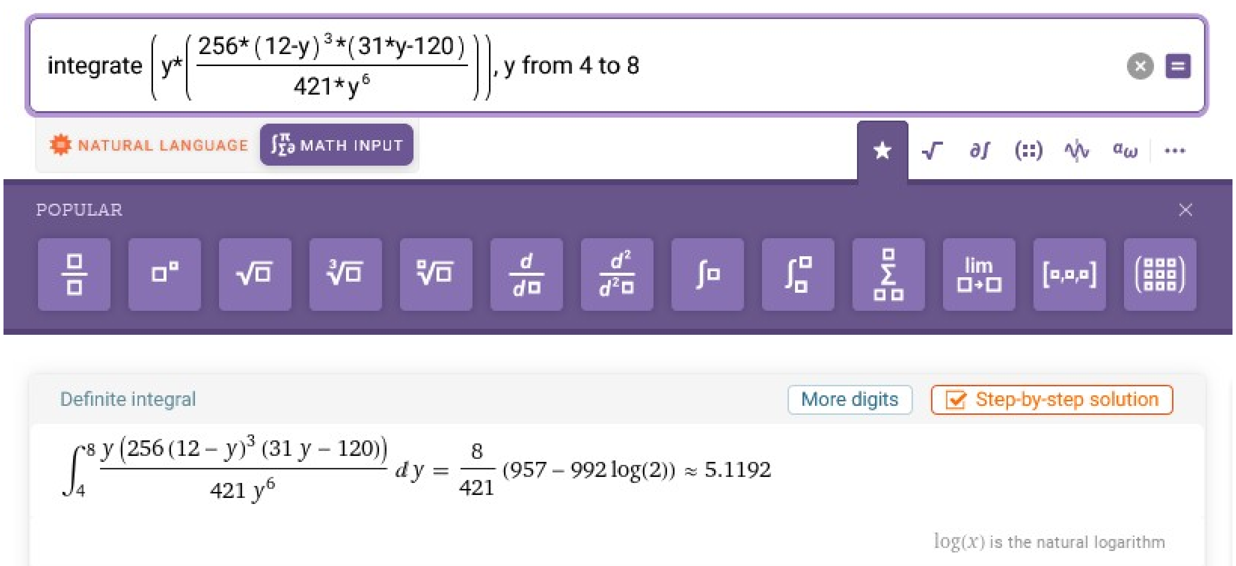
g) Find $E(Y) = \mu_Y$.

"Hint": While this integral is "fightable", it will be somewhat annoying. After you set it up, I would recommend using your favorite online integral calculator instead of a hand-to-hand combat.

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy = \int_4^8 y \cdot \frac{256\,(12-y)^3\,(31y-120)}{421\,y^6}\,dy = \ldots$$

… indeed "fightable", but annoying …

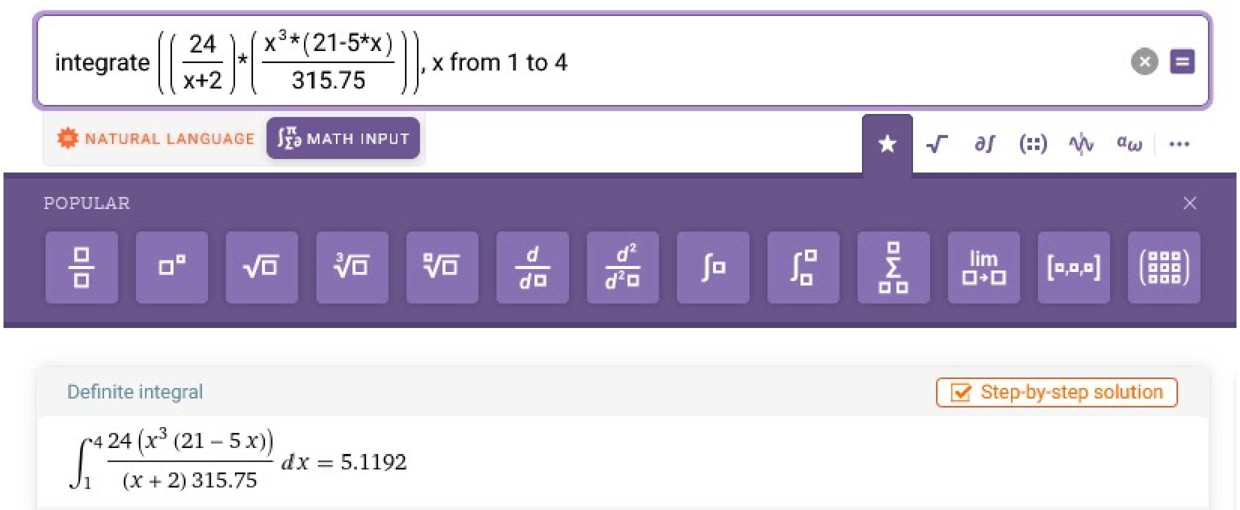




$\approx$ **5.1192**.

OR

$$\mathrm{E}(\mathrm{Y}) = \mathrm{E}(g(\mathrm{X})) = \int_{-\infty}^{\infty} g(x) \cdot f_{\mathrm{X}}(x)\, dx = \int_{1}^{4} \frac{24}{x+2} \cdot \frac{x^{3}(21-5x)}{315.75}\, dx = \ldots$$



$\approx$ **5.1192**.

h) Is μ_Y equal to $g(\mu_X)$?

"Hint": "equal" means "exactly equal" here. Not "close" or "sort of close", but "equal".

$\mu_Y \approx 5.1192.$ $\qquad g(\mu_X) = \dfrac{24}{2.8 + 2} = 5.$

$\mu_Y = \mathrm{E}(Y) \neq g(\mathrm{E}(X)) = g(\mu_X).$ $\qquad \mu_Y$ is NOT equal to $g(\mu_X)$.

Recall: IF $g(x)$ is a linear function, that is, IF $g(x) = a\,x + b$,

then $\mathrm{E}(g(X)) = \mathrm{E}(a\,X + b) = a\,\mathrm{E}(X) + b = g(\mathrm{E}(X)).$

However, in general, if $g(x)$ is NOT a linear function,

then $\mathrm{E}(g(X)) \neq g(\mathrm{E}(X)).$

Spoiler:

Here, $\mathrm{E}(g(X)) > g(\mathrm{E}(X))$ since $g(x) = \dfrac{24}{x+2}$ "curves up" for $1 \le x \le 4$.

5. Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{x^3(21-5x)}{C}, \qquad x = 1, 2, 3, 4, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability mass function.

$$p_X(1) = \frac{16}{C}, \qquad p_X(2) = \frac{88}{C}, \qquad p_X(3) = \frac{162}{C}, \qquad p_X(4) = \frac{64}{C}.$$

$$p_X(1) + p_X(2) + p_X(3) + p_X(4) = 1.$$

$$\frac{16}{C} + \frac{88}{C} + \frac{162}{C} + \frac{64}{C} = \frac{330}{C} = 1. \qquad \Rightarrow \qquad C = \mathbf{330}.$$

b) Find $E(X) = \mu_X$.

$$E(X) = \sum_{\text{all } x} x \cdot p_X(x) = 1 \cdot \frac{16}{330} + 2 \cdot \frac{88}{330} + 3 \cdot \frac{162}{330} + 4 \cdot \frac{64}{330} = \frac{934}{330} \approx \mathbf{2.8303}.$$

5. (continued)

Consider $Y = g(X) = \dfrac{24}{X+2}$.

c) Obtain the probability distribution of Y.

x	$p_X(x)$	$g(x)$
1	$\dfrac{16}{330} \approx 0.0485$	8
2	$\dfrac{88}{330} \approx 0.2667$	6
3	$\dfrac{162}{330} \approx 0.4909$	4.8
4	$\dfrac{64}{330} \approx 0.1939$	4

y	$p_Y(y)$
4	$\mathbf{\dfrac{64}{330} = \dfrac{32}{165}}$
4.8	$\mathbf{\dfrac{162}{330} = \dfrac{81}{165}}$
6	$\mathbf{\dfrac{88}{330} = \dfrac{44}{165}}$
8	$\mathbf{\dfrac{16}{330} = \dfrac{8}{165}}$

d) Find $E(Y) = \mu_Y$.

$$E(Y) = \sum_{\text{all } y} y \cdot p_Y(y) = 4 \cdot \frac{64}{330} + 4.8 \cdot \frac{162}{330} + 6 \cdot \frac{88}{330} + 8 \cdot \frac{16}{330} = \frac{1{,}689.6}{330} = \mathbf{5.12}.$$

OR

$$E(Y) = E(g(X)) = \sum_{\text{all } x} g(x) \cdot p_X(x)$$

$$= \frac{24}{1+2} \cdot \frac{16}{330} + \frac{24}{2+2} \cdot \frac{88}{330} + \frac{24}{3+2} \cdot \frac{162}{330} + \frac{24}{4+2} \cdot \frac{64}{330} = \mathbf{5.12}.$$

e) Is μ_Y equal to $g(\mu_X)$?

"Hint": "equal" means "exactly equal" here. Not "close" or "sort of close", but "equal".

$\mu_Y = 5.12.$ $\qquad g(\mu_X) = \dfrac{24}{\frac{934}{330}+2} \approx 4.9686.$

$\mu_Y = E(Y) \neq g(E(X)) = g(\mu_X).$ $\qquad \mu_Y$ is NOT equal to $g(\mu_X)$.

Recall: IF $g(x)$ is a linear function, that is, IF $g(x) = a\,x + b$,

then $E(g(X)) = E(a\,X + b) = a\,E(X) + b = g(E(X)).$

However, in general, if $g(x)$ is NOT a linear function,

then $E(g(X)) \neq g(E(X)).$

Spoiler:

Here, $E(g(X)) > g(E(X))$ since $g(x) = \dfrac{24}{x+2}$ "curves up" for $1 \le x \le 4$.