Examples for Week 03 (1)

2.3 Conditional Distributions and Expectations.

1. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

	y			
x	0	1	2	$p_X(x)$
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
$p_Y(y)$	0.40	0.40	0.20	

a) Find the conditional probability distributions $p_{X|Y}(x|y) = \dfrac{p(x,y)}{p_Y(y)}$ of X given $Y = y$, conditional expectation $E(X|Y = y)$ of X given $Y = y$, and $E(E(X|Y))$.

x	$p_{X\|Y}(x\|0)$
1	$0.15/0.40 = 0.375$
2	$0.25/0.40 = 0.625$

x	$p_{X\|Y}(x\|1)$
1	$0.10/0.40 = 0.25$
2	$0.30/0.40 = 0.75$

x	$p_{X\|Y}(x\|2)$
1	$0.00/0.20 = 0.00$
2	$0.20/0.20 = 1.00$

$E(X|Y = 0) = 1.625$ $\quad$ $E(X|Y = 1) = 1.75$ $\quad$ $E(X|Y = 2) = 2.0$

Def

$$E(X|Y = y) = \sum_x x\, P(X = x|Y = y) = \sum_x x\, p_{X|Y}(x|y) \qquad \text{– discrete}$$

$$E(X|Y = y) = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\,dx \qquad \text{– continuous}$$

Denote by $E(X|Y)$ that function of the random variable Y whose value at $Y = y$ is $E(X|Y = y)$. Note that $E(X|Y)$ is itself a random variable, it depends on the (random) value of Y that occurs.

y	$\mathrm{E}(\mathrm{X} \mid \mathrm{Y}=y)$	$p_{\mathrm{Y}}(y)$	
0	1.625	0.40	0.65
1	1.75	0.40	0.70
2	2.0	0.20	0.40

$\mathrm{E}(\mathrm{E}(\mathrm{X} \mid \mathrm{Y})) = 1.75.$

Recall: $\mathrm{E}(\mathrm{X}) = 1.75.$

- $\mathrm{E}(a_1 \mathrm{X}_1 + a_2 \mathrm{X}_2 \mid \mathrm{Y}) = a_1 \mathrm{E}(\mathrm{X}_1 \mid \mathrm{Y}) + a_2 \mathrm{E}(\mathrm{X}_2 \mid \mathrm{Y})$
- $\mathrm{E}[g(\mathrm{Y}) \mid \mathrm{Y}] = g(\mathrm{Y})$
- $\mathrm{E}(\mathrm{E}(\mathrm{X} \mid \mathrm{Y})) = \mathrm{E}(\mathrm{X})$
- $\mathrm{E}[\mathrm{E}(\mathrm{X} \mid \mathrm{Y}) \mid \mathrm{Y}] = \mathrm{E}(\mathrm{X} \mid \mathrm{Y})$
- $\mathrm{E}[g(\mathrm{Y})\mathrm{X} \mid \mathrm{Y}] = g(\mathrm{Y})\,\mathrm{E}(\mathrm{X} \mid \mathrm{Y})$

b) Find the conditional probability distributions $p_{\mathrm{Y}|\mathrm{X}}(y \mid x) = \dfrac{p(x,y)}{p_{\mathrm{X}}(x)}$ of Y given $\mathrm{X} = x$, conditional expectation $\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = x)$ of Y given $\mathrm{X} = x$, and $\mathrm{E}(\mathrm{E}(\mathrm{Y} \mid \mathrm{X}))$.

y	$p_{\mathrm{Y}|\mathrm{X}}(y \mid 1)$
0	$0.15/0.25 = 0.60$
1	$0.10/0.25 = 0.40$
2	$0.00/0.25 = 0.00$

$\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = 1) = 0.4 = 6/15$

y	$p_{\mathrm{Y}|\mathrm{X}}(y \mid 2)$
0	$0.25/0.75 = 5/15$
1	$0.30/0.75 = 6/15$
2	$0.20/0.75 = 4/15$

$\mathrm{E}(\mathrm{Y} \mid \mathrm{X} = 2) = 14/15$

x	$\mathrm{E}(\mathrm{Y} \mid \mathrm{X}=x)$	$p_{\mathrm{X}}(x)$
1	$6/15$	0.25
2	$14/15$	0.75

$$\mathrm{E}(\mathrm{E}(\mathrm{Y} \mid \mathrm{X})) = \frac{6}{15} \cdot 0.25 + \frac{14}{15} \cdot 0.75 = 0.10 + 0.70 = 0.80.$$

Recall: $\mathrm{E}(\mathrm{Y}) = 0.80.$

2. Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} 60\,x^2\,y & 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Recall:

$$f_{\mathrm{X}}(x) = 30\,x^2\,(1-x)^2, \quad 0 < x < 1, \qquad \mathrm{E}(\mathrm{X}) = \frac{1}{2},$$

$$f_{\mathrm{Y}}(y) = 20\,y\,(1-y)^3, \quad 0 < y < 1, \qquad \mathrm{E}(\mathrm{Y}) = \frac{1}{3}.$$

a) Find the conditional probability density function $f_{\mathrm{Y}|\mathrm{X}}(y\,|\,x) = \dfrac{f(x,y)}{f_{\mathrm{X}}(x)}$ of Y given $\mathrm{X} = x$, $0 < x < 1$.

$$f_{\mathrm{Y}|\mathrm{X}}(y\,|\,x) = \frac{f(x,y)}{f_{\mathrm{X}}(x)} = \frac{60\,x^2\,y}{30\,x^2\,(1-x)^2} = \frac{2\,y}{(1-x)^2}, \qquad 0 < y < 1-x.$$

b) Find $\mathrm{P}(\mathrm{Y} > 1/3 \,|\, \mathrm{X} = 1/2)$, $\mathrm{P}(\mathrm{Y} > 1/4 \,|\, \mathrm{X} = 1/3)$, $\mathrm{P}(\mathrm{Y} < 1/2 \,|\, \mathrm{X} = 1/3)$.

$$\mathrm{P}(\mathrm{Y} > 1/3 \,|\, \mathrm{X} = 1/2) = \int_{1/3}^{1/2} \frac{2\,y}{(1/2)^2}\,dy = \int_{1/3}^{1/2} 8\,y\,dy = \mathbf{\frac{5}{9}}.$$

$$\mathrm{P}(\mathrm{Y} > 1/4 \,|\, \mathrm{X} = 1/3) = \int_{1/4}^{2/3} \frac{2\,y}{(2/3)^2}\,dy = \mathbf{\frac{55}{64}}.$$

$$\mathrm{P}(\mathrm{Y} < 1/2 \,|\, \mathrm{X} = 1/3) = \int_{0}^{1/2} \frac{2\,y}{(2/3)^2}\,dy = \mathbf{\frac{9}{16}} = \mathbf{0.5625}.$$

$$\mathrm{P}(\mathrm{Y} < 1/2 \,|\, \mathrm{X} = 2/3) = \int_{0}^{1/3} \frac{2\,y}{(1/3)^2}\,dy = \mathbf{1}.$$

c) Find $\mathrm{E}(\mathrm{Y} \mid \mathrm{X}=x)$, $\mathrm{E}(\mathrm{Y} \mid \mathrm{X})$, and $\mathrm{E}(\mathrm{E}(\mathrm{Y} \mid \mathrm{X}))$.

Def $$\mathrm{E}(\mathrm{Y} \mid \mathrm{X}=x) = \int_{-\infty}^{\infty} y \cdot f_{\mathrm{Y}|\mathrm{X}}(y \mid x)\, dy.$$

$$\mathrm{E}(\mathrm{X} \mid \mathrm{Y}=y) = \int_{-\infty}^{\infty} x \cdot f_{\mathrm{X}|\mathrm{Y}}(x \mid y)\, dx.$$

$$\mathrm{E}(\mathrm{Y} \mid \mathrm{X}=x) = \int_{0}^{1-x} y \cdot \frac{2y}{(1-x)^2}\, dy = \frac{2}{(1-x)^2} \cdot \int_{0}^{1-x} y^2\, dy = \frac{2}{3} \cdot (1-x), \quad 0 < x < 1.$$

$$\mathrm{E}(\mathrm{Y} \mid \mathrm{X}) = \frac{2}{3}(1-\mathrm{X}).$$

$$\mathrm{E}(\mathrm{E}(\mathrm{Y} \mid \mathrm{X})) = \frac{2}{3}(1-\mathrm{E}(\mathrm{X})) = \frac{2}{3}\left(1-\frac{1}{2}\right) = \frac{1}{3} = \mathrm{E}(\mathrm{Y}).$$

d) Find the conditional probability density function $f_{\mathrm{X}|\mathrm{Y}}(x \mid y) = \dfrac{f(x,y)}{f_{\mathrm{Y}}(y)}$ of X given $\mathrm{Y}=y$, $0 < y < 1$.

$$f_{\mathrm{X}|\mathrm{Y}}(x \mid y) = \frac{f(x,y)}{f_{\mathrm{Y}}(y)} = \frac{60x^2 y}{20y(1-y)^3} = \frac{3x^2}{(1-y)^3}, \qquad 0 < x < 1-y.$$

e) Find $\mathrm{P}\left(\mathrm{X} > \frac{1}{2} \,\middle|\, \mathrm{Y} = \frac{1}{3}\right)$.

$$f_{\mathrm{X}|\mathrm{Y}}\left(x \,\middle|\, \frac{1}{3}\right) = \frac{81x^2}{8}, \qquad 0 < x < \frac{2}{3}.$$

$$\mathrm{P}\left(\mathrm{X} > \frac{1}{2} \,\middle|\, \mathrm{Y} = \frac{1}{3}\right) = \int_{1/2}^{2/3} \frac{81x^2}{8}\, dx = \left(\frac{27x^3}{8}\right)\Bigg|_{1/2}^{2/3} = \mathbf{\frac{37}{64}}.$$

f) Find $\mathrm{E}(\mathrm{X} \mid \mathrm{Y}=y)$, $\mathrm{E}(\mathrm{X} \mid \mathrm{Y})$, and $\mathrm{E}(\mathrm{E}(\mathrm{X} \mid \mathrm{Y}))$.

$$\mathrm{E}(\mathrm{X} \mid \mathrm{Y}=y) = \int_0^{1-y} x \cdot \frac{3x^2}{(1-y)^3}\,dx = \frac{3}{(1-y)^3} \cdot \int_0^{1-y} x^3\,dx = \frac{3}{4} \cdot (1-y), \quad 0 < y < 1.$$

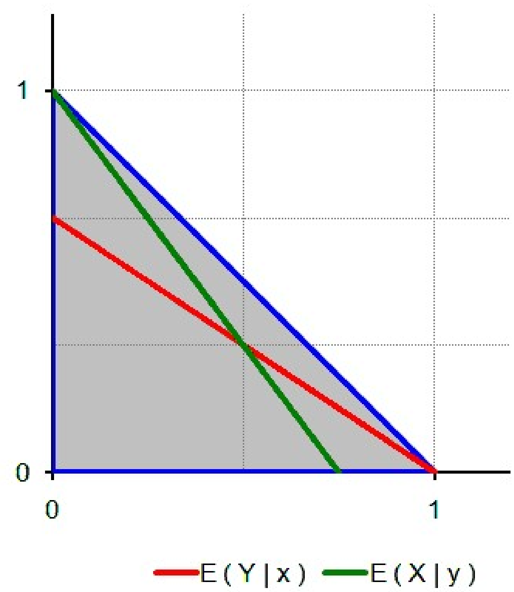
$$\mathrm{E}(\mathrm{X} \mid \mathrm{Y}) = \frac{3}{4}(1-\mathrm{Y}).$$

$$\mathrm{E}(\mathrm{E}(\mathrm{X} \mid \mathrm{Y})) = \frac{3}{4}(1-\mathrm{E}(\mathrm{Y})) = \frac{3}{4}\left(1-\frac{1}{3}\right) = \frac{1}{2} = \mathrm{E}(\mathrm{X}).$$

Recall: $\mathrm{Var}(\mathrm{X}) = \dfrac{9}{252}$,

$\mathrm{Var}(\mathrm{Y}) = \dfrac{8}{252}$,

$\rho_{\mathrm{XY}} = -\dfrac{1}{\sqrt{2}}$.



If $\mathrm{E}(\mathrm{Y} \mid \mathrm{X}=x)$ is linear in x, then

$$\mathrm{E}(\mathrm{Y} \mid \mathrm{X}=x) = \mu_{\mathrm{Y}} + \rho \frac{\sigma_{\mathrm{Y}}}{\sigma_{\mathrm{X}}}(x-\mu_{\mathrm{X}}).$$

$$\mathrm{E}(\mathrm{Y} \mid \mathrm{X}=x) = \frac{1}{3} - \frac{1}{\sqrt{2}} \frac{\sqrt{8/252}}{\sqrt{9/252}}\left(x-\frac{1}{2}\right)$$

$$= \frac{1}{3} - \frac{2}{3}\left(x-\frac{1}{2}\right) = \frac{2}{3} - \frac{2}{3}x.$$

$$\mathrm{E}(\mathrm{X} \mid \mathrm{Y}=y) = \frac{1}{2} - \frac{1}{\sqrt{2}} \frac{\sqrt{9/252}}{\sqrt{8/252}}\left(y-\frac{1}{3}\right) = \frac{1}{2} - \frac{3}{4}\left(y-\frac{1}{3}\right) = \frac{3}{4} - \frac{3}{4}y.$$

3. Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} x + y & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Recall: $f_X(x) = x + \frac{1}{2}, \quad 0 < x < 1.$ $\qquad f_Y(y) = y + \frac{1}{2}, \quad 0 < y < 1.$

a) Find the conditional p.d.f. $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)}$ of Y given $X = x$, $\quad 0 < x < 1.$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{x+y}{x+\frac{1}{2}}, \qquad 0 < y < 1.$$

b) Find $P(Y < {}^1\!/_2 \mid X = {}^3\!/_4).$

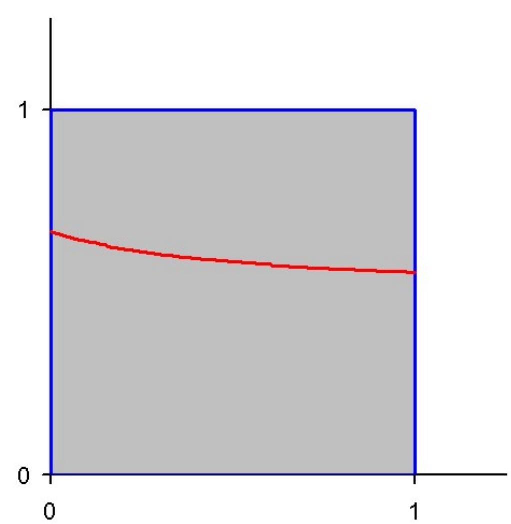
$$P(Y < {}^1\!/_2 \mid X = {}^3\!/_4) = \int_0^{1/2} \frac{\frac{3}{4}+y}{\frac{3}{4}+\frac{1}{2}}\,dy = \left(\frac{3y+2y^2}{5}\right)\Bigg|_0^{1/2} = 0.40.$$

c) Find $E(Y | X = x).$

$$E(Y | X = x) = \int_0^1 y \cdot \frac{x+y}{x+\frac{1}{2}}\,dy$$

$$= \frac{\frac{1}{2}x + \frac{1}{3}}{x+\frac{1}{2}} = \frac{3x+2}{6x+3},$$

$$0 < x < 1.$$

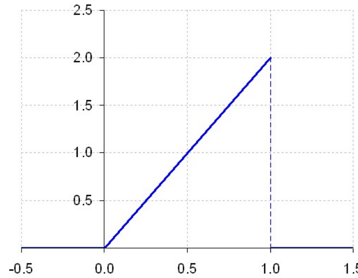


Recall: $\text{Cov}(X,Y) = -\frac{1}{144},$

$$\rho_{XY} = -\frac{1}{11}.$$

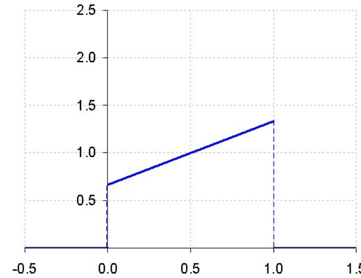
$f_{Y|X}(y|x)$:

$x = 0$



$\Rightarrow$

$x = 1$



4. Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} 12\,x(1-x)\,e^{-2y} & 0 \le x \le 1,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Recall: $f_X(x) = 6\,x(1-x),\ \ 0 < x < 1,$ $\quad E(X) = \dfrac{1}{2},$

$f_Y(y) = 2\,e^{-2y},\ \ y > 0,$ $\quad E(Y) = \dfrac{1}{2}.$ $\quad$ X and Y are independent.

Find $f_{X|Y}(x|y),\ E(X|Y=y),\ f_{Y|X}(y|x),\ E(Y|X=x).$

Since X and Y are independent, and $f(x,y) = f_X(x) \cdot f_Y(y)$,

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = f_X(x) = 6\,x(1-x), \qquad 0 < x < 1,$$

$$E(X|Y=y) = E(X) = \frac{1}{2},$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = f_Y(y) = 2\,e^{-2y}, \qquad y > 0,$$

$$E(Y|X=x) = E(Y) = \frac{1}{2}.$$

5. Let X_1, X_2 be two random variables with joint pdf $f(x_1, x_2) = x_1 \exp\{-x_2\}$, for $0 < x_1 < x_2 < \infty$, zero elsewhere.

a) Find the conditional p.d.f. $f_{1|2}(x_1|x_2)$ of X_1 given $X_2 = x_2$, $0 < x_2 < \infty$.

$$f_2(x_2) = \int_0^{x_2} x_1 e^{-x_2}\, dx_1 = \frac{x_2^2}{2} e^{-x_2}, \qquad 0 < x_2 < \infty.$$

X_2 has a Gamma distribution with $\alpha = 3$, $\theta = 1$.

$$f_{1|2}(x_1|x_2) = \frac{x_1 e^{-x_2}}{\frac{x_2^2}{2} e^{-x_2}} = \frac{2x_1}{x_2^2}, \qquad 0 < x_1 < x_2.$$

For example, $P(X_1 > 3 \mid X_2 = 5) = \int_3^5 \frac{2x_1}{25}\, dx_1 = \frac{16}{25} = 0.64.$

$$P(X_1 < 2 \mid X_2 = 5) = \int_0^2 \frac{2x_1}{25}\, dx_1 = \frac{4}{25} = 0.16.$$

b) Find the conditional p.d.f. $f_{2|1}(x_2|x_1)$ of X_2 given $X_1 = x_1$, $0 < x_1 < \infty$.

$$f_1(x_1) = \int_{x_1}^{\infty} x_1 e^{-x_2}\, dx_2 = x_1 e^{-x_1}, \qquad 0 < x_1 < \infty.$$

X_1 has a Gamma distribution with $\alpha = 2$, $\theta = 1$.

$$f_{2|1}(x_2|x_1) = \frac{x_1 e^{-x_2}}{x_1 e^{-x_1}} = e^{x_1 - x_2}, \qquad x_1 < x_2 < \infty.$$

For example, $P(X_2 < 8 \mid X_1 = 5) = \int_5^8 e^{5-x_2}\, dx_2 = 1 - e^{-3} \approx 0.9502.$

$$P(X_2 > 6 \mid X_1 = 5) = \int_6^{\infty} e^{5-x_2}\, dx_2 = e^{-1} \approx 0.3679.$$

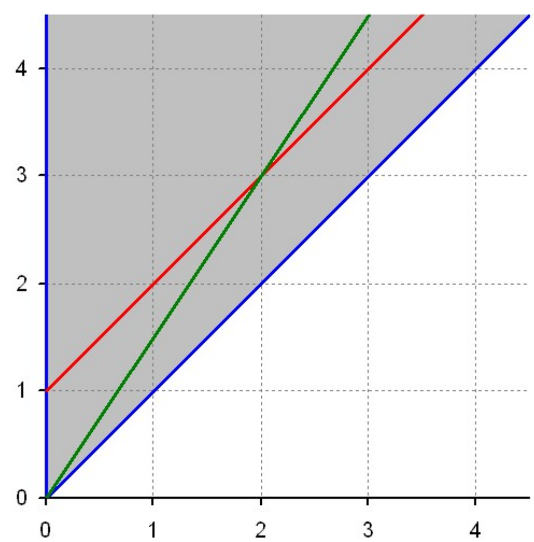
c) Find $E(X_1 \mid X_2 = x_2)$, $E(X_2 \mid X_1 = x_1)$.

$$E(X_1 \mid X_2 = x_2) = \int_0^{x_2} x_1 \cdot \frac{2x_1}{x_2^2}\,dx_1 = \frac{2}{3}x_2, \qquad 0 < x_2 < \infty.$$

$$E(X_2 \mid X_1 = x_1) = \int_{x_1}^{\infty} x_2 \cdot e^{x_1 - x_2}\,dx_2 = x_1 + 1, \qquad 0 < x_1 < \infty.$$

If $E(Y \mid X = x)$ is linear in x, then

$$E(Y \mid X = x) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X).$$



$\mu_1 = 2$, $\sigma_1^2 = 2$, $\mu_2 = 3$, $\sigma_2^2 = 3$.

$$\Rightarrow \qquad \rho = \frac{\sqrt{2}}{\sqrt{3}}.$$

OR

$$E(X_1 X_2) = \int_0^{\infty}\left(\int_0^{x_2} x_1^2\, x_2\, e^{-x_2}\,dx_1\right)dx_2 = \int_0^{\infty}\frac{x_2^4}{3}e^{-x_2}\,dx_2 = \frac{\Gamma(5)}{3} = 8.$$

$\mathrm{Cov}(X_1, X_2) = 8 - 2\cdot 3 = 2.$ $\qquad \rho = \dfrac{2}{\sqrt{2}\cdot\sqrt{3}} = \dfrac{\sqrt{2}}{\sqrt{3}}.$