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Examples for Week 01 (4)

1. The Lomax distribution, also called the Pareto Type II distribution, is a heavy-tail probability distribution used in business, economics, actuarial science, queueing theory and Internet traffic modeling.

Let $\delta > 0$ and consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{\delta}{(1+x)^{\delta+1}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

a) Consider $Y = \frac{1}{1+X}$. Find the probability distribution of Y.

b) Consider $W = \ln(1+X)$. Find the probability distribution of W.

2. Let X be a random variable with probability density function

$$f_X(x) = \frac{x+1}{8}, \qquad -1 < x < 3, \qquad \text{zero otherwise.}$$

Find the probability distribution of $Y = X^2$.

Q: What if function $g(x)$ is not one-to-one and $f_X(x)$ is piecewise-defined?

A: Same, carefully.

3. Consider a continuous random variable X with the probability density function

$$f_X(x) = \begin{cases} \dfrac{2x+2}{3C} & -2 < x < 2 \\ \dfrac{4-x}{C} & 2 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

b) Consider $Y = \dfrac{16}{X^2}$. Find the probability distribution of Y.

Q: What if function $g(x)$ is $\cos(x)$ or something similar?

A: Same, carefully.

4. Suppose that X has a (continuous) Uniform distribution on interval $\left(-\dfrac{\pi}{3}, \dfrac{2\pi}{3}\right)$.

Consider $Y = g(X) = \cos(X)$. Find the probability distribution of Y.

Answers:

1. The Lomax distribution, also called the Pareto Type II distribution, is a heavy-tail probability distribution used in business, economics, actuarial science, queueing theory and Internet traffic modeling.

Let $\delta > 0$ and consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{\delta}{(1+x)^{\delta+1}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

a) Consider $Y = \dfrac{1}{1+X}$. Find the probability distribution of Y.

$$x > 0 \qquad Y = \frac{1}{1+X} \qquad \Rightarrow \qquad 0 < y < 1.$$

$$y = \frac{1}{1+x} \qquad x = \frac{1}{y} - 1 = g^{-1}(y) \qquad \frac{dx}{dy} = -\frac{1}{y^2}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \frac{\delta}{\left(1 + \frac{1}{y} - 1\right)^{\delta+1}} \cdot \left| -\frac{1}{y^2} \right| = \delta\, y^{\delta-1}, \qquad 0 < y < 1.$$

OR

$$F_X(x) = \int_0^x \frac{\delta}{(1+u)^{\delta+1}}\,du = -\frac{1}{(1+u)^{\delta}}\Bigg|_0^x = 1 - \frac{1}{(1+x)^{\delta}}, \qquad x > 0.$$

$$F_Y(y) = P(Y \le y) = P\left(\frac{1}{1+X} \le y\right) = P\left(X \ge \frac{1}{y} - 1\right)$$

$$= 1 - F_X\left(\frac{1}{y} - 1\right) = \frac{1}{\left(1 + \frac{1}{y} - 1\right)^{\delta}} = y^{\delta}, \qquad 0 \le y < 1.$$

$$F_Y(y) = 0, \quad y < 0, \qquad\qquad F_Y(y) = 1, \quad y \ge 1.$$

Indeed, $\quad \dfrac{d}{dy}\left(y^{\delta}\right) = \delta\, y^{\delta-1}.$ ☺

b) Consider $W = \ln(1 + X)$. Find the probability distribution of W.

$x > 0 \qquad W = \ln(1 + X) \qquad \Rightarrow \qquad w > 0.$

$W = \ln(1 + X) \qquad x = e^{w} - 1 \qquad \dfrac{dx}{dw} = e^{w}$

$$f_W(w) = f_X(g^{-1}(w)) \left|\frac{dx}{dw}\right| = \delta\, e^{-(\delta+1)w} \cdot \left|e^{w}\right| = \delta\, e^{-\delta w}, \qquad w > 0.$$

$\Rightarrow$ W has Exponential distribution with mean $\theta = \dfrac{1}{\delta}$.

OR

$$F_X(x) = \int_0^x \frac{\delta}{(1+u)^{\delta+1}}\,du = -\frac{1}{(1+u)^{\delta}}\Bigg|_0^x = 1 - \frac{1}{(1+x)^{\delta}}, \qquad x > 0.$$

$$F_W(w) = P(\ln(1+X) \le w) = P(X \le e^{w} - 1) = F_X(e^{w}-1) = 1 - e^{-\delta w},$$

$$w > 0.$$

$\Rightarrow$ W has Exponential distribution with mean $\theta = \frac{1}{\delta}$.

OR

$$M_W(t) = E(e^{tW}) = E(e^{t\ln(1+X)}) = E((1+X)^{t})$$

$$= \int_0^\infty (1+x)^{t} \cdot \frac{\delta}{(1+x)^{\delta+1}}\,dx = \int_0^\infty \frac{\delta}{(1+x)^{\delta-t+1}}\,dx$$

$$= -\frac{\delta}{(\delta-t)(1+x)^{\delta-t}}\Bigg|_0^\infty = \frac{\delta}{\delta-t} = \frac{1}{1-\frac{1}{\delta}t}, \qquad t < \delta.$$

[$M_W(t)$ does not exist if $t \ge \delta$ (the integral diverges).]

$\Rightarrow$ W has Exponential distribution with mean $\theta = \frac{1}{\delta}$.

Exponential
$0 < \theta$

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \qquad 0 \le x < \infty$$

$$M(t) = \frac{1}{1-\theta t}, \qquad t < \frac{1}{\theta}$$

$$\mu = \theta, \qquad \sigma^2 = \theta^2$$

2. Let X be a random variable with probability density function

$$f_X(x) = \frac{x+1}{8}, \qquad -1 < x < 3, \qquad \text{zero otherwise.}$$

Find the probability distribution of $Y = X^2$.

-1 and 3 are "important" for X. 0 is "important" for $g(x) = x^2$.

$\Rightarrow$ $(-1)^2$, $(3)^2$, and $(0)^2$ are "important" for $Y = X^2$.

$\Rightarrow$ 1, 9, and 0 are "important" for Y.

$y < 0$ ("boring"), $0 < y < 1$, $1 < y < 9$, $y > 9$ ("boring") should be considered separately.

Case 0: $y < 0$.

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = 0. \qquad \text{("boring")}$$

For $y > 0$,

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

$$F_X(x) = P(X \le x) = \int_{-1}^{x} \frac{u+1}{8}\,du = \left.\frac{u^2 + 2u}{16}\right|_{-1}^{x}$$

$$= \frac{x^2 + 2x + 1}{16} = \frac{(x+1)^2}{16}, \qquad -1 \le x < 3.$$

Obviously, $F_X(x) = 0$, $x < -1$, $F_X(x) = 1$, $x \ge 3$.

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \dfrac{(x+1)^2}{16} & -1 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

Case 1: $0 < y < 1$.

$-1 < -\sqrt{y} < 0,$ $\qquad$ $0 < \sqrt{y} < 1,$

$$F_X(-\sqrt{y}) = \frac{(-\sqrt{y}+1)^2}{16}, \qquad F_X(\sqrt{y}) = \frac{(\sqrt{y}+1)^2}{16}.$$

$$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \frac{(\sqrt{y}+1)^2}{16} - \frac{(-\sqrt{y}+1)^2}{16}$$

$$= \frac{y + 2\sqrt{y} + 1 - y + 2\sqrt{y} - 1}{16} = \frac{\sqrt{y}}{4}.$$

Case 2: $1 < y < 9$.

$-3 < -\sqrt{y} < -1,$ $\qquad$ $1 < \sqrt{y} < 3,$

$$F_X(-\sqrt{y}) = 0, \qquad F_X(\sqrt{y}) = \frac{(\sqrt{y}+1)^2}{16}.$$

$$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \frac{(\sqrt{y}+1)^2}{16} - 0 = \frac{(\sqrt{y}+1)^2}{16}.$$

Case 3: $y > 9$.

$-\sqrt{y} < -3,$ $\qquad$ $\sqrt{y} > 3,$

$$F_X(-\sqrt{y}) = 0, \qquad F_X(\sqrt{y}) = 1.$$

$$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) = 1 - 0 = 1. \qquad \text{("boring")}$$

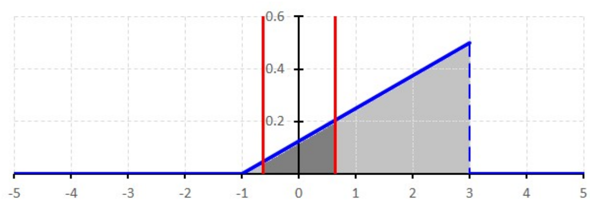
$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \dfrac{\sqrt{y}}{4} & 0 \le y < 1 \\ \dfrac{\left(\sqrt{y}+1\right)^2}{16} & 1 \le y < 9 \\ 1 & y \ge 9 \end{cases}$$

Indeed, $\quad 0 = \dfrac{\sqrt{0}}{4}, \qquad \dfrac{\sqrt{1}}{4} = \dfrac{1}{4} = \dfrac{\left(\sqrt{1}+1\right)^2}{16}, \qquad \dfrac{\left(\sqrt{4}+1\right)^2}{16} = 1.$ ☺

OR

Case 1: $\quad 0 < y < 1.$

$$-1 < -\sqrt{y} < 0,$$
$$0 < \sqrt{y} < 1.$$

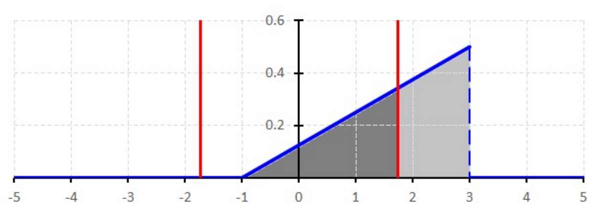


$$F_Y(y) = P(-\sqrt{y} \le X \le \sqrt{y}\,) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{x+1}{8}\,dx = \left(\frac{x^2}{16} + \frac{x}{8}\right)\Bigg|_{-\sqrt{y}}^{\sqrt{y}}$$

$$= \frac{y}{16} + \frac{\sqrt{y}}{8} - \frac{y}{16} + \frac{\sqrt{y}}{8} = \frac{\sqrt{y}}{4}.$$

Case 2: $\quad 1 < y < 9.$

$$-3 < -\sqrt{y} < -1,$$
$$1 < \sqrt{y} < 3.$$



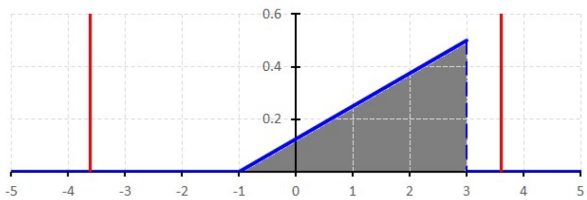
$$F_Y(y) = P(-\sqrt{y} \le X \le \sqrt{y}\,) = \int_{-1}^{\sqrt{y}} \frac{x+1}{8}\,dx = \left(\frac{x^2}{16} + \frac{x}{8}\right)\Bigg|_{-1}^{\sqrt{y}}$$

$$= \frac{y}{16} + \frac{\sqrt{y}}{8} - \frac{1}{16} + \frac{1}{8} = \frac{y + 2\sqrt{y} + 1}{16} = \frac{\left(\sqrt{y}+1\right)^2}{16}.$$

Case 3: $y > 9$.

$-\sqrt{y} < -3$,

$\sqrt{y} > 3$.



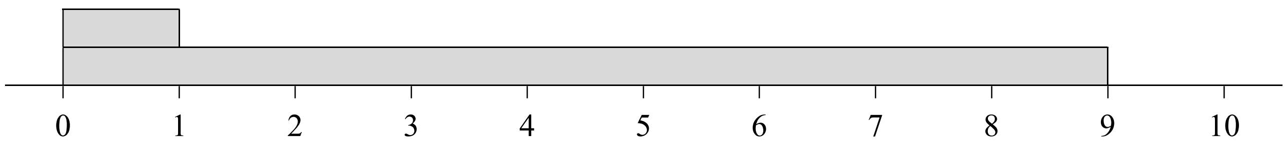
$$F_Y(y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-1}^{3} \frac{x+1}{8}\, dx = 1.$$ ("boring")

OR

$g(x) = x^2$ is NOT a one-to-one function on the support of X: $-1 < x < 3$.

However, $g(x) = x^2$ is a one-to-one function on $-1 < x < 0$ and on $0 < x < 3$.

$-1 < x < 0$	$0 < x < 3$
$Y = g(X) = X^2$	$Y = g(X) = X^2$
$1 > y > 0$	$0 < y < 9$
$x = -\sqrt{y} = g^{-1}(y)$	$x = \sqrt{y} = g^{-1}(y)$
$\frac{dx}{dy} = -\frac{1}{2\sqrt{y}}$	$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$
$f_X(g^{-1}(y)) \left\lvert \frac{dx}{dy} \right\rvert$	$f_X(g^{-1}(y)) \left\lvert \frac{dx}{dy} \right\rvert$
$\frac{-\sqrt{y}+1}{8} \cdot \left\lvert -\frac{1}{2\sqrt{y}} \right\rvert$	$\frac{\sqrt{y}+1}{8} \cdot \left\lvert \frac{1}{2\sqrt{y}} \right\rvert$
$\frac{-\sqrt{y}+1}{16\sqrt{y}}$	$\frac{\sqrt{y}+1}{16\sqrt{y}}$



For $y < 0$, $\quad f_Y(y) = 0.$

For $0 < y < 1$, $\quad f_Y(y) = \dfrac{-\sqrt{y}+1}{16\sqrt{y}} + \dfrac{\sqrt{y}+1}{16\sqrt{y}} = \dfrac{1}{8\sqrt{y}}.$

For $1 < y < 9$, $\quad f_Y(y) = \dfrac{\sqrt{y}+1}{16\sqrt{y}}.$

For $y > 9$, $\quad f_Y(y) = 0.$

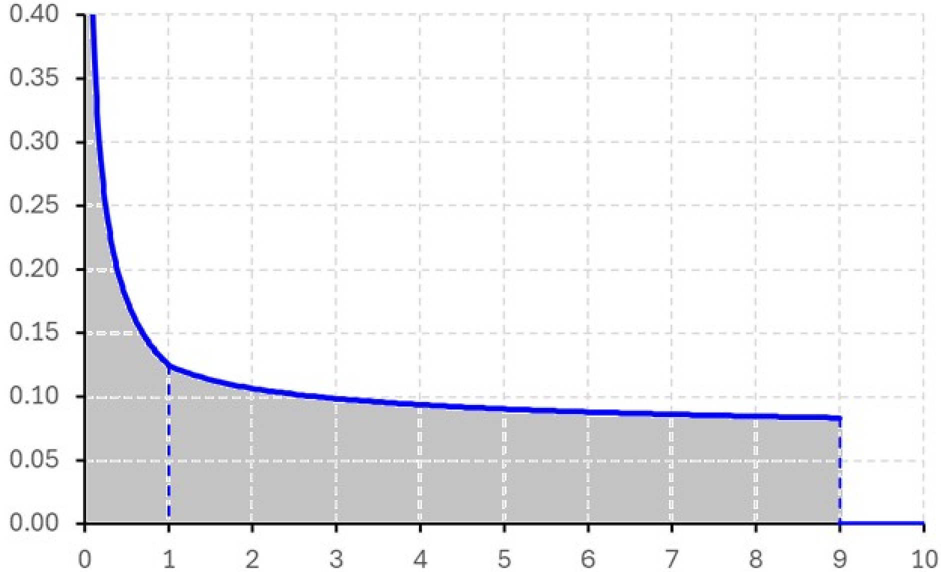
$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \dfrac{\sqrt{y}}{4} & 0 \le y < 1 \\ \dfrac{\left(\sqrt{y}+1\right)^2}{16} & 1 \le y < 9 \\ 1 & y \ge 9 \end{cases} \qquad f_Y(y) = \begin{cases} \dfrac{1}{8\sqrt{y}} & 0 < y < 1 \\ \dfrac{\sqrt{y}+1}{16\sqrt{y}} & 1 < y < 9 \\ 0 & \text{o.w.} \end{cases}$$

Indeed, $\quad \dfrac{d}{dy}\left(\dfrac{\sqrt{y}}{4}\right) = \dfrac{1}{8\sqrt{y}},$

$$\frac{d}{dy}\left(\frac{\left(\sqrt{y}+1\right)^2}{16}\right) = \frac{\sqrt{y}+1}{16\sqrt{y}},$$

That is, $\quad F_Y'(y) = f_Y(y).$ ☺

$f_{\mathrm{Y}}(y)$:



3. Consider a continuous random variable X with the probability density function

$$f_X(x) = \begin{cases} \dfrac{2x+2}{3C} & -1 < x < 2 \\ \dfrac{4-x}{C} & 2 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

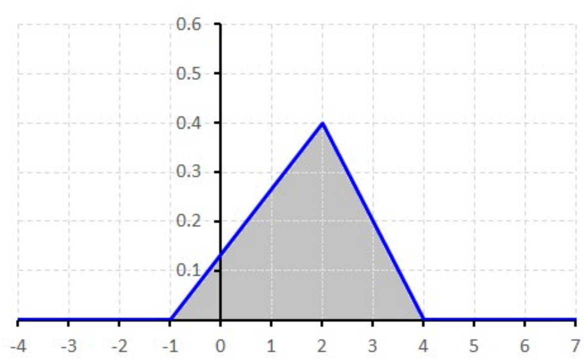
$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-1}^{2} \frac{2x+2}{3C}\,dx + \int_{2}^{4} \frac{4-x}{C}\,dx$$

$$= \left.\frac{x^2+2x}{3C}\right|_{-1}^{2} + \left.\frac{8x-x^2}{2C}\right|_{2}^{4}$$

$$= \frac{9}{3C} + \frac{4}{2C} = \frac{5}{C}.$$

$\Rightarrow$ $C = \mathbf{5}$.

$$f_X(x) = \begin{cases} \dfrac{2x+2}{15} & -1 < x < 2 \\ \dfrac{4-x}{5} & 2 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$



b) Consider $Y = \dfrac{16}{X^2}$. Find the probability distribution of Y.

-1, 2, and 4 are "important" for X. 0 is "important" for $g(x) = \dfrac{16}{x^2}$.

$\Rightarrow$ $\dfrac{16}{(-1)^2}$, $\dfrac{16}{(2)^2}$, $\dfrac{16}{(4)^2}$, and $\dfrac{16}{(0)^2}$ are "important" for $Y = \dfrac{16}{X^2}$.

$\Rightarrow$ 16, 4, 1, and ∞ are "important" for Y.

$y < 1$ ("boring"), $1 < y < 4$, $4 < y < 16$, $y > 16$ should be considered separately.

Case -1: $y < 0$.

$$F_Y(y) = P(Y \le y) = P\left(\frac{16}{X^2} \le y\right) = 0.$$

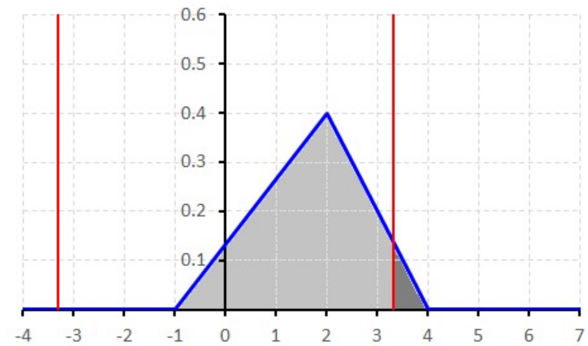
For $y > 0$,

$$F_Y(y) = P(Y \le y) = P\left(\frac{16}{X^2} \le y\right) = P\left(X^2 \ge \frac{16}{y}\right) = P\left(X \le -\frac{4}{\sqrt{y}}\right) + P\left(X \ge \frac{4}{\sqrt{y}}\right).$$

Case 0: $0 < y < 1$. $\dfrac{4}{\sqrt{y}} > 4$, $-\dfrac{4}{\sqrt{y}} < -4$.

$$F_Y(y) = P\left(X \le -\frac{4}{\sqrt{y}}\right) + P\left(X \ge \frac{4}{\sqrt{y}}\right) = 0 + 0 = 0.$$

Case 1: $\quad 1 < y < 4.$



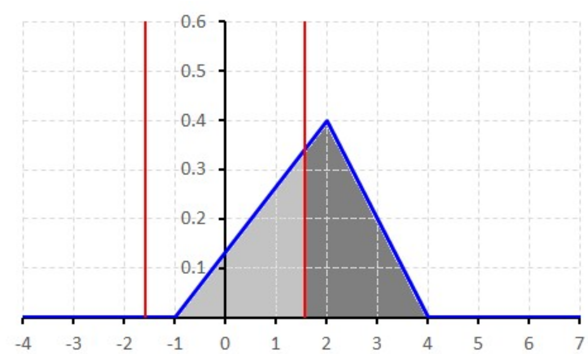
$$2 < \frac{4}{\sqrt{y}} < 4,$$

$$-4 < -\frac{4}{\sqrt{y}} < -2.$$

$$F_Y(y) = P(X \le -\frac{4}{\sqrt{y}}) + P(X \ge \frac{4}{\sqrt{y}}) = 0 + \int_{\frac{4}{\sqrt{y}}}^{4} \frac{4-x}{5}\,dx = \left.\frac{8x - x^2}{10}\right|_{\frac{4}{\sqrt{y}}}^{4}$$

$$= \frac{16 - \frac{32}{\sqrt{y}} + \frac{16}{y}}{10} = \frac{16y - 32\sqrt{y} + 16}{10y}, \qquad 1 < y < 4.$$

Case 2: $\quad 4 < y < 16.$



$$1 < \frac{4}{\sqrt{y}} < 2,$$

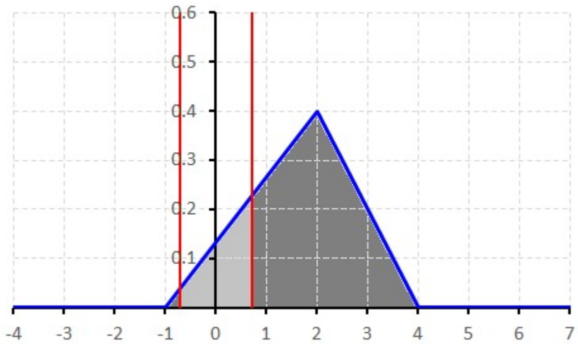
$$-2 < -\frac{4}{\sqrt{y}} < -1.$$

$$F_Y(y) = P(X \le -\frac{4}{\sqrt{y}}) + P(X \ge \frac{4}{\sqrt{y}}) = 1 - \int_{-1}^{\frac{4}{\sqrt{y}}} \frac{2x+2}{15}\,dx = 1 - \left.\frac{x^2 + 2x}{15}\right|_{-1}^{\frac{4}{\sqrt{y}}}$$

$$= 1 - \frac{\frac{16}{y} + \frac{8}{\sqrt{y}} + 1}{15} = 1 - \frac{y + 8\sqrt{y} + 16}{15y} = \frac{14y - 8\sqrt{y} - 16}{15y},$$

$$4 < y < 16.$$

Case 3: $y > 16.$



$$0 < \frac{4}{\sqrt{y}} < 1,$$

$$-1 < -\frac{4}{\sqrt{y}} < 0.$$

$$F_Y(y) = P(X \le -\frac{4}{\sqrt{y}}) + P(X \ge \frac{4}{\sqrt{y}}) = 1 - \int_{-\frac{4}{\sqrt{y}}}^{\frac{4}{\sqrt{y}}} \frac{2x+2}{15}\,dx = 1 - \frac{x^2+2x}{15}\Bigg|_{-\frac{4}{\sqrt{y}}}^{\frac{4}{\sqrt{y}}}$$

$$= 1 - \frac{\frac{16}{y} + \frac{8}{\sqrt{y}} - \frac{16}{y} + \frac{8}{\sqrt{y}}}{15} = 1 - \frac{16}{15\sqrt{y}}, \qquad y > 16.$$

Indeed,

$$0 = \frac{16 \cdot 1 - 32\sqrt{1} + 16}{10 \cdot 1},$$

$$\frac{16 \cdot 4 - 32\sqrt{4} + 16}{10 \cdot 4} = \frac{2}{5} = \frac{14 \cdot 4 - 8\sqrt{4} - 16}{15 \cdot 4},$$

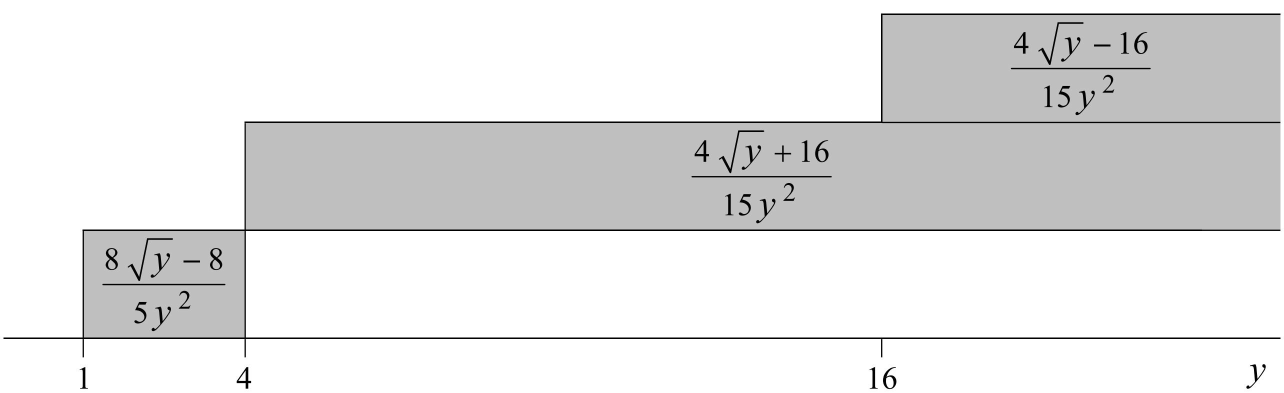
$$\frac{14 \cdot 16 - 8\sqrt{16} - 16}{15 \cdot 16} = \frac{11}{15} = 1 - \frac{16}{15\sqrt{16}},$$

$$1 - \frac{16}{15\sqrt{\infty}} = 1.$$

☺

OR

$-1 < x < 0$	$0 < x < 2$	$2 < x < 4$
$Y = g(X) = \dfrac{16}{X^2}$	$Y = g(X) = \dfrac{16}{X^2}$	$Y = g(X) = \dfrac{16}{X^2}$
$16 < y < \infty$	$\infty > y > 4$	$4 > y > 1$
$x = -\dfrac{4}{\sqrt{y}} = g^{-1}(y)$	$x = \dfrac{4}{\sqrt{y}} = g^{-1}(y)$	$x = \dfrac{4}{\sqrt{y}} = g^{-1}(y)$
$\dfrac{dx}{dy} = \dfrac{2}{y^{3/2}}$	$\dfrac{dx}{dy} = -\dfrac{2}{y^{3/2}}$	$\dfrac{dx}{dy} = -\dfrac{2}{y^{3/2}}$
$f_X(g^{-1}(y)) \left\lvert \dfrac{dx}{dy} \right\rvert$	$f_X(g^{-1}(y)) \left\lvert \dfrac{dx}{dy} \right\rvert$	$f_X(g^{-1}(y)) \left\lvert \dfrac{dx}{dy} \right\rvert$
$\dfrac{-\frac{8}{\sqrt{y}} + 2}{15} \cdot \dfrac{2}{y^{3/2}}$	$\dfrac{\frac{8}{\sqrt{y}} + 2}{15} \cdot \dfrac{2}{y^{3/2}}$	$\dfrac{4 - \frac{4}{\sqrt{y}}}{5} \cdot \dfrac{2}{y^{3/2}}$
$\dfrac{4\sqrt{y} - 16}{15y^2}$	$\dfrac{4\sqrt{y} + 16}{15y^2}$	$\dfrac{8\sqrt{y} - 8}{5y^2}$

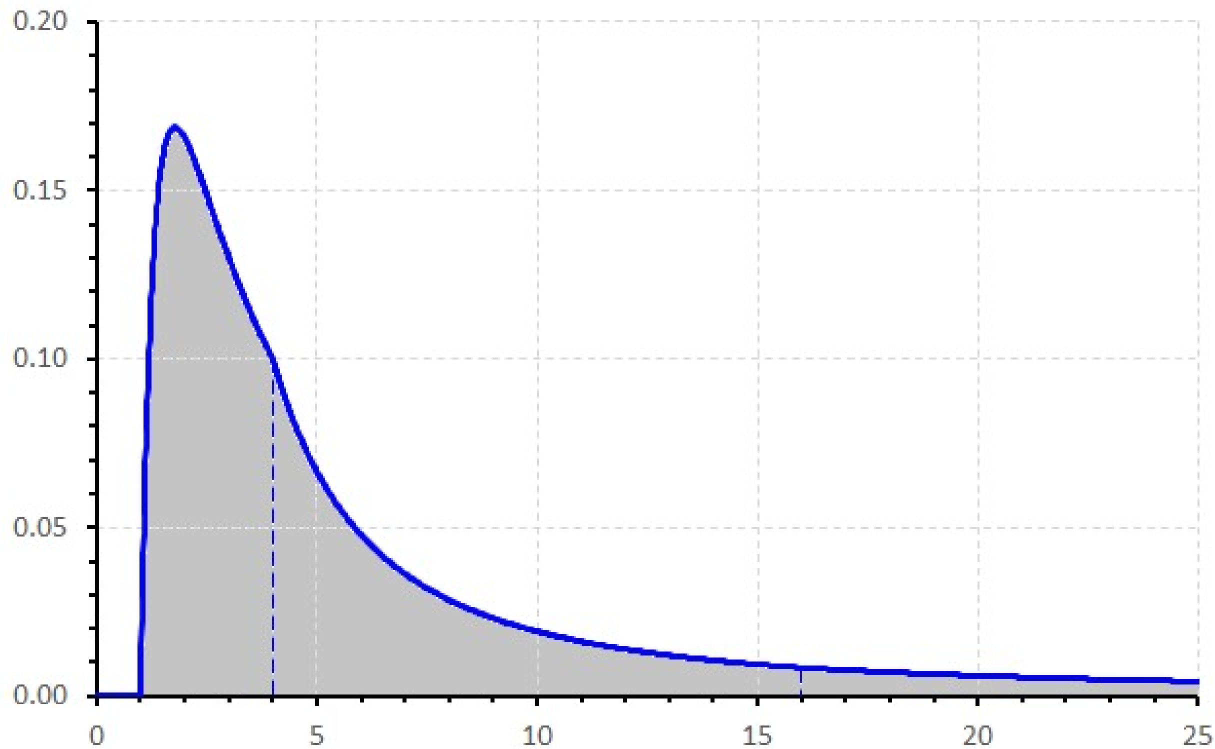


For $y < 1$, $\qquad f_Y(y) = 0.$

For $1 < y < 4$, $\qquad f_Y(y) = \dfrac{8\sqrt{y} - 8}{5y^2}.$

For $4 < y < 16$, $$f_Y(y) = \frac{4\sqrt{y} + 16}{15\,y^2}.$$

For $y > 16$, $$f_Y(y) = \frac{4\sqrt{y} - 16}{15\,y^2} + \frac{4\sqrt{y} + 16}{15\,y^2} = \frac{8\sqrt{y}}{15\,y^2} = \frac{8}{15\,y^{3/2}}.$$



Indeed, $$\frac{d}{dy}\left(\frac{16\,y - 32\sqrt{y} + 16}{10\,y}\right) = \frac{8\sqrt{y} - 8}{5\,y^2},$$

$$\frac{d}{dy}\left(\frac{14\,y - 8\sqrt{y} - 16}{15\,y}\right) = \frac{4\sqrt{y} + 16}{15\,y^2},$$

$$\frac{d}{dy}\left(1 - \frac{16}{15\sqrt{y}}\right) = \frac{8}{15\,y^{3/2}}.$$

That is, $F_Y'(y) = f_Y(y)$. ☺

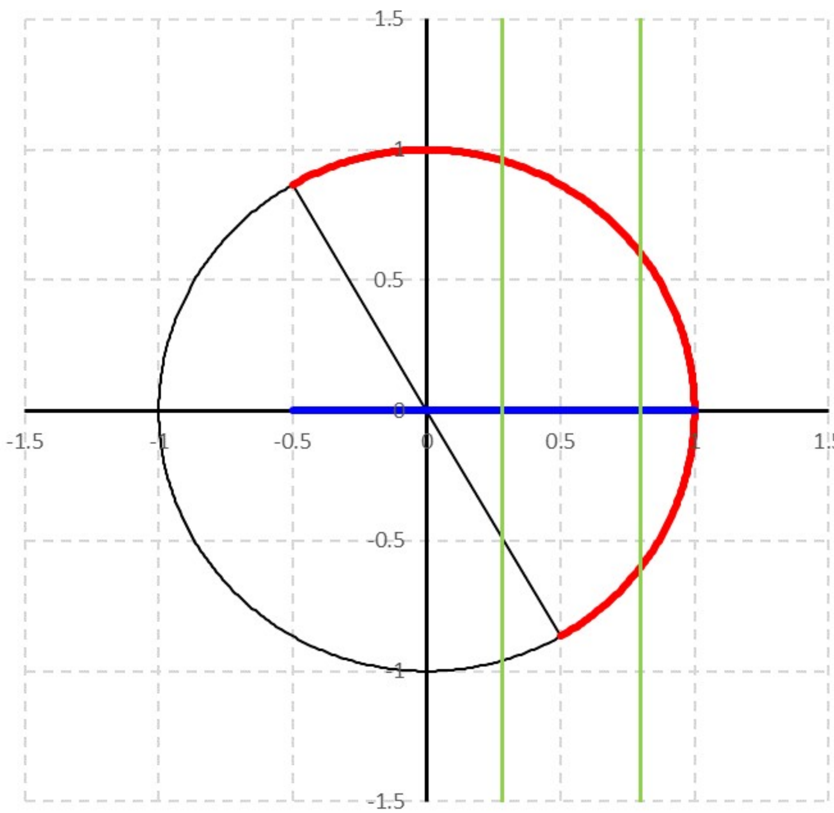
4. Suppose that X has a (continuous) Uniform distribution on interval $\left(-\frac{\pi}{3}, \frac{2\pi}{3}\right)$.

Consider $Y = g(X) = \cos(X)$. Find the probability distribution of Y.

$f_X(x) = \frac{1}{\pi}, \qquad -\frac{\pi}{3} < x < \frac{2\pi}{3}, \qquad$ zero elsewhere.

$F_X(x) = P(X \le x) = \frac{x}{\pi} + \frac{1}{3}, \qquad -\frac{\pi}{3} \le x < \frac{2\pi}{3}.$

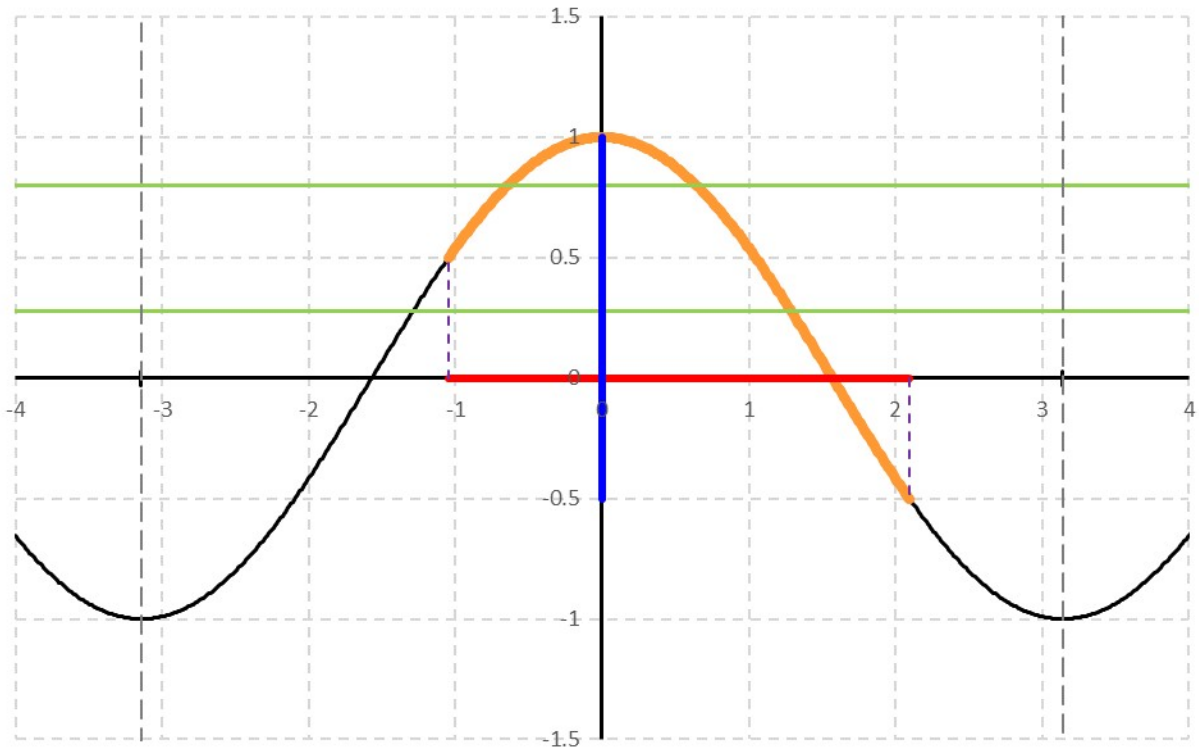
$F_X(x) = 0, \qquad x < -\frac{\pi}{3}, \qquad\qquad F_X(x) = 1, \qquad x \ge \frac{2\pi}{3}.$



The support of Y (the range of possible values of Y) is $\mathbf{-0.5 < y < 1}$.

TWO cases need to be considered separately: $-0.5 < y < 0.5, \qquad 0.5 < y < 1.$

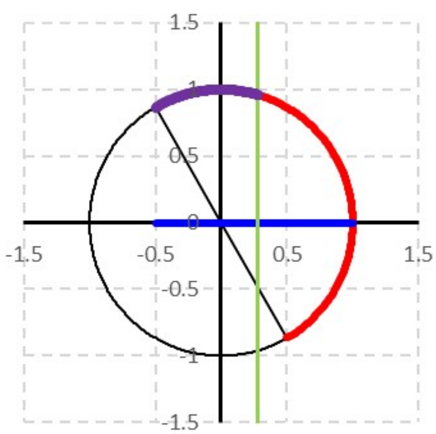
OR

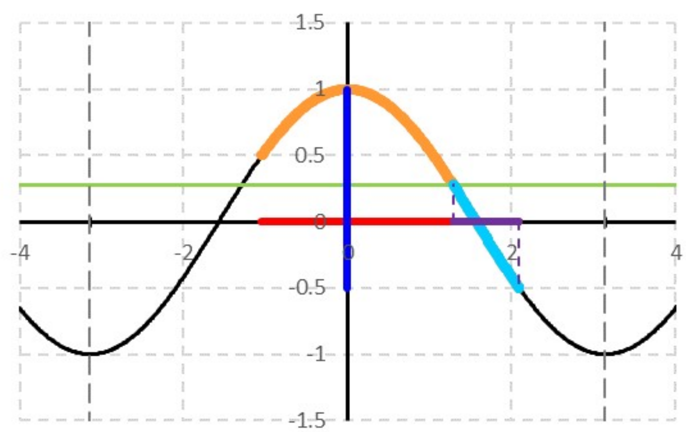


The support of Y (the range of possible values of Y) is **$-0.5 < y < 1$**.

TWO cases need to be considered separately: $-0.5 < y < 0.5$, $0.5 < y < 1$.

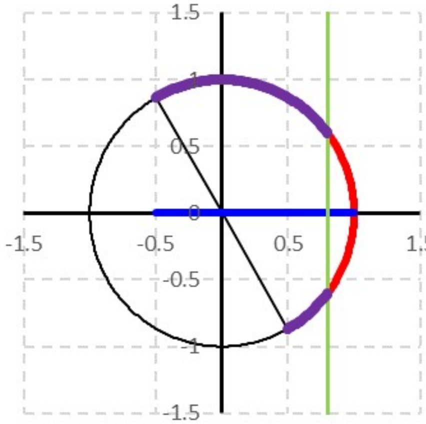
Case 1: $-0.5 < y < 0.5$. $F_Y(y) = P(Y \le y) = P(\cos(X) \le y) = \ldots$

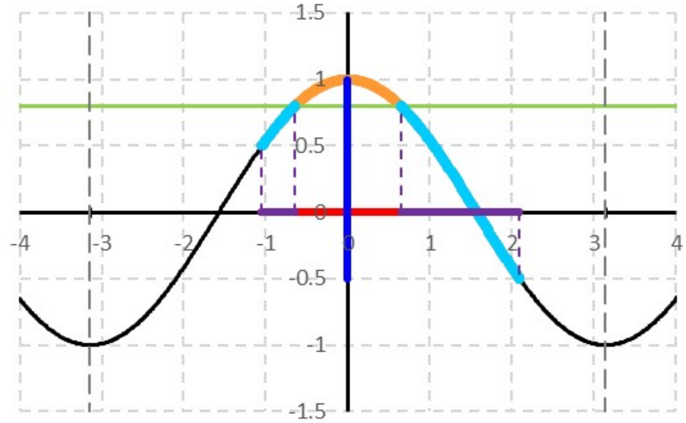




$$\ldots = P\left(\arccos(y) \le X < \frac{2\pi}{3}\right) = \frac{2}{3} - \frac{1}{\pi}\arccos(y), \qquad -0.5 < y < 0.5.$$

Case 2: $\quad 0.5 < y < 1. \qquad F_Y(y) = P(Y \le y) = P(\cos(X) \le y) = \ldots$





$$\ldots = P\left(-\frac{\pi}{3} \le X < -\arccos(y)\right) + P\left(\arccos(y) \le X < \frac{2\pi}{3}\right)$$

$$= \left(-\frac{1}{\pi}\arccos(y) + \frac{1}{3}\right) + \left(\frac{2}{3} - \frac{1}{\pi}\arccos(y)\right)$$

$$= 1 - \frac{2}{\pi}\arccos(y), \qquad 0.5 < y < 1.$$

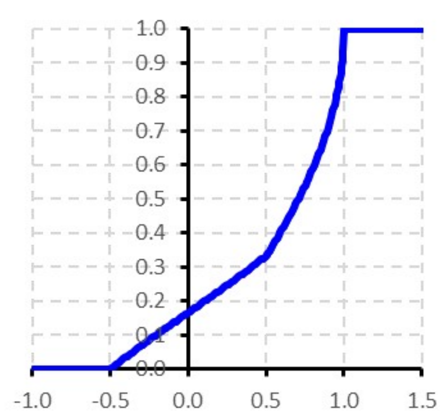
OR

$$\ldots = 1 - P(-\arccos(y) < X < \arccos(y))$$

$$= 1 - \frac{2}{\pi}\arccos(y), \qquad 0.5 < y < 1.$$

c.d.f.

$$F_Y(y) = \begin{cases} 0 & y < -0.5 \\ \frac{2}{3} - \frac{1}{\pi}\arccos(y) & -0.5 \le y < 0.5 \\ 1 - \frac{2}{\pi}\arccos(y) & 0.5 \le y < 1 \\ 1 & y \ge 1 \end{cases}$$



OR

$$-\frac{\pi}{3} < x < 0$$

$$Y = g(X) = \cos(X)$$

$$x = -\arccos(y)$$

$$\frac{dx}{dy} = \frac{1}{\sqrt{1-y^2}}$$

$$0.5 < y < 1$$

$$f_X(g^{-1}(y))\left|\frac{dx}{dy}\right|$$

$$\frac{1}{\pi} \times \left|\frac{1}{\sqrt{1-y^2}}\right|$$

$$\frac{1}{\pi\sqrt{1-y^2}}$$

$$0 < x < \frac{2\pi}{3}$$

$$Y = g(X) = \cos(X)$$

$$x = \arccos(y)$$

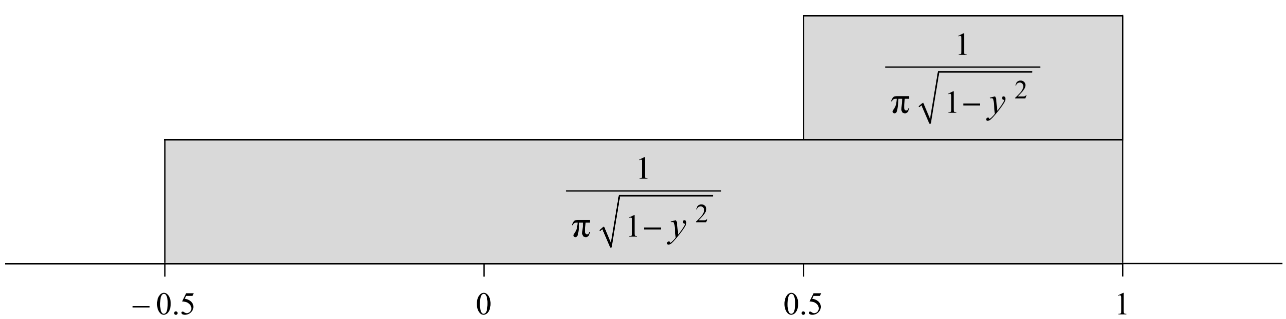
$$\frac{dx}{dy} = -\frac{1}{\sqrt{1-y^2}}$$

$$1 > y > -0.5$$

$$f_X(g^{-1}(y))\left|\frac{dx}{dy}\right|$$

$$\frac{1}{\pi} \times \left|-\frac{1}{\sqrt{1-y^2}}\right|$$

$$\frac{1}{\pi\sqrt{1-y^2}}$$



$$f_Y(y) = \frac{1}{\pi\sqrt{1-y^2}}, \quad -0.5 < y < 0.5.$$

$$f_Y(y) = \frac{2}{\pi\sqrt{1-y^2}}, \quad 0.5 < y < 1.$$

p.d.f.

$$f_Y(y) = \begin{cases} \dfrac{1}{\pi\sqrt{1-y^2}} & -0.5 < y < 0.5 \\ \dfrac{2}{\pi\sqrt{1-y^2}} & 0.5 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$