Examples for Week 01   (3)

1.   Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{x^3}{C}, \qquad\qquad 2 \le x \le 4, \qquad\qquad \text{zero elsewhere.}$$

a)   Find the value of C that makes $f_X(x)$ a valid probability density function.

b)   Find the cumulative distribution function of X, $F_X(x)$.

"Hint":   To double-check your answer: should be $F_X(2) = 0$, $F_X(4) = 1$.

1.   (continued)

Consider $Y = g(X) = X^2$.

c)   Find the support (the range of possible values) of the probability distribution of Y.

d)   Use part (b) and the c.d.f. approach to find the c.d.f. of Y, $F_Y(y)$.

"Hint": $F_Y(y) = P(Y \le y) = P(g(X) \le y) = \ldots$.

e)   Use the change-of-variable technique to find the p.d.f. of Y, $f_Y(y)$.

"Hint": $f_Y(y) = f_X(g^{-1}(y)) \left| \dfrac{dx}{dy} \right|$.

"Hint":   To double-check your answer: should be $f_Y(y) = F_Y'(y)$.

f)   Is μ_Y equal to $g(\mu_X)$? $\mu_X = E(X)$, $\mu_Y = E(Y)$.

1. (continued)

Consider $W = \dfrac{1}{X+6}$.

g)   Find the support (the range of possible values) of the probability distribution of W.

h)   Use part (b) and the c.d.f. approach to find the c.d.f. of W, $F_W(w)$.

i)   Use the change-of-variable technique to find the p.d.f. of W, $f_W(w)$.

j)   Find the moment-generating function of X, $M_X(t)$.

2.   Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{x^3}{C}, \qquad x = 1, 2, 3, 4, \qquad\qquad \text{zero elsewhere.}$$

a)   Find the value of C that makes $p_X(x)$ a valid probability mass function.

2.   (continued)

Consider $Y = g(X) = X^2$.

b)   Find the probability distribution of Y.

c)   Is μ_Y equal to $g(\mu_X)$? $\mu_X = E(X), \ \mu_Y = E(Y)$.

d)   Find the moment-generating function of X, $M_X(t)$.

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1. Consider a **continuous** random variable X with the probability **density** function

$$f_X(x) = \frac{x^3}{C}, \qquad 2 \le x \le 4, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_{2}^{4} \frac{x^3}{C}\,dx = \frac{x^4}{4C}\Big|_{2}^{4} = \frac{256 - 16}{4C} = \frac{60}{C}.$$

$$\Rightarrow \quad C = \mathbf{60}.$$

b) Find the cumulative distribution function of X, $F_X(x)$.

"Hint":   To **double-check** your answer: should be $F_X(2) = 0$, $F_X(4) = 1$.

$$F_X(x) = 0, \qquad x < 2,$$

$$F_X(x) = P(X \le x) = \int_{2}^{x} \frac{u^3}{60}\,du = \frac{u^4}{240}\Big|_{2}^{x} = \frac{x^4 - 16}{240}, \qquad 2 \le x < 4,$$

$$F_X(x) = 1, \qquad x \ge 4.$$

1. (continued)

Consider $Y = g(X) = X^2$.

c) Find the support (the range of possible values) of the probability distribution of Y.

$2 \le x \le 4.$ $\qquad\qquad 4 \le x^2 \le 16.$ $\qquad\qquad \mathbf{4 \le y \le 16}.$

d) Use part (b) and the c.d.f. approach to find the c.d.f. of Y, $F_Y(y)$.

"Hint": $F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = \ldots$.

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = F_X(\sqrt{y})$$

$$= \frac{\left(\sqrt{y}\right)^4 - 16}{240} = \frac{y^2 - 16}{240}, \qquad 4 \leq y < 16.$$

$$F_Y(y) = 0, \qquad y < 4, \qquad\qquad F_Y(y) = 1, \qquad y \geq 16.$$

e) Use the change-of-variable technique to find the p.d.f. of Y, $f_Y(y)$.

"Hint": $f_Y(y) = f_X(g^{-1}(y)) \left| \dfrac{dx}{dy} \right|$.

"Hint": To double-check your answer: should be $f_Y(y) = F_Y'(y)$.

$$y = x^2 \qquad\qquad x = \sqrt{y} = g^{-1}(y) \qquad\qquad \frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \frac{\left(\sqrt{y}\right)^3}{60} \cdot \left| \frac{1}{2\sqrt{y}} \right| = \frac{y}{120}, \qquad 4 \leq y \leq 16.$$

Indeed, $\dfrac{d}{dy}\left(\dfrac{y^2 - 16}{240}\right) = \dfrac{y}{120}$. ☺

f)   Is μ_Y equal to $g(\mu_X)$? $\qquad\qquad$ $\mu_X = E(X),\ \mu_Y = E(Y)$.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_2^4 \frac{x^4}{60}\, dx = \frac{x^5}{300}\Big|_2^4 = \frac{1{,}024 - 32}{300}$$

$$= \frac{992}{300} = \frac{248}{75} \approx 3.306667.$$

$$E(Y) = \int_4^{16} \frac{y^2}{120}\, dy = \frac{y^3}{360}\Big|_4^{16} = \frac{4{,}096 - 64}{360} = \frac{4{,}032}{360} = \frac{56}{5} = 11.2.$$

$$11.2 \neq \left(\frac{248}{75}\right)^2. \qquad\qquad \mu_Y \neq g(\mu_X).$$

Recall: IF $g(x)$ is a linear function, that is, IF $g(x) = a\,x + b$,

$\qquad\qquad$ then $E(g(X)) = E(a\,X + b) = a\,E(X) + b = g(E(X))$.

$\qquad\qquad$ However, in general, if $g(x)$ is NOT a linear function,

$\qquad\qquad$ then $E(g(X)) \neq g(E(X))$.

Spoiler:

$\qquad$ Here, $E(g(X)) > g(E(X))$ since $g(x) = x^2$ "curves up".

For fun:

Moment-generating function approach:

$$M_Y(t) = E(e^{tY}) = E(e^{tX^2}) = \int_{-\infty}^{\infty} e^{tx^2} \cdot f_X(x)\,dx = \int_2^4 e^{tx^2} \cdot \frac{x^3}{60}\,dx$$

$$u = x^2 \qquad du = 2x\,dx$$

$$= \int_4^{16} e^{tu} \cdot \frac{u}{120}\,du = \frac{u}{120\,t} e^{tu} - \frac{1}{120\,t^2} e^{tu} \Big|_4^{16}$$

$$= \frac{16}{120\,t} e^{16t} - \frac{1}{120\,t^2} e^{16t} - \frac{4}{120\,t} e^{4t} + \frac{1}{120\,t^2} e^{4t}, \qquad\qquad t \neq 0.$$

$$M_Y(0) = 1.$$

Fortunately, in this case, we know that this particular moment-generating function belongs to a probability distribution with the probability density function

$$f(u) = \frac{u}{120}, \qquad 4 \le u \le 16, \qquad\qquad \text{zero otherwise.}$$

Indeed,

$$\int_{-\infty}^{\infty} f(u)\,du = \int_4^{16} \frac{u}{120}\,du = \frac{y^2}{240} \Big|_4^{16} = \frac{256 - 16}{240} = 1,$$

$$M(t) = \int_4^{16} e^{tu} \cdot \frac{u}{120}\,du = \frac{u}{120\,t} e^{tu} - \frac{1}{120\,t^2} e^{tu} \Big|_4^{16}$$

$$= \frac{16}{120\,t} e^{16t} - \frac{1}{120\,t^2} e^{16t} - \frac{4}{120\,t} e^{4t} + \frac{1}{120\,t^2} e^{4t}, \qquad\qquad t \neq 0.$$

$$M(0) = 1.$$

Since there is one-to-one correspondence between probability distributions and their moment-generating functions, we now know that

$$f_Y(y) = \frac{y}{120}, \qquad 4 \le y \le 16, \qquad\qquad \text{zero otherwise.}$$

In this particular case, we were observant, and (fortunately) noticed the

$$\ldots \; = \; \int_4^{16} e^{tu} \cdot \frac{u}{120} \, du \; = \; \ldots \quad \text{step.}$$

IF all we had was the answer:

$$M_Y(t) \; = \; \frac{16}{120 \, t} e^{16 t} \; - \; \frac{1}{120 \, t^2} e^{16 t} \; - \; \frac{4}{120 \, t} e^{4 t} \; + \; \frac{1}{120 \, t^2} e^{4 t}, \qquad\qquad t \neq 0,$$

$$M_Y(0) \; = \; 1,$$

it could have been difficult to find the probability density function that matches this moment-generating function.

For continuous random variables,

$$E(h(X)) \; = \; \int_{-\infty}^{\infty} h(x) \cdot f(x) \, dx.$$

Fun fact: Probability is also an expected value.

Define $I_A(x) \; = \; \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$

Then $P(X \in A) \; = \; \int_A f(x) \, dx \; = \; \int_{-\infty}^{\infty} I_A(x) \cdot f(x) \, dx \; = \; E(I_A(X)).$

Another fun fact: $I_A(x) \cdot I_B(x) \; = \; I_{A \cap B}(x).$

1.   (continued)

Consider $W = \dfrac{1}{X+6}$.

g)   Find the support (the range of possible values) of the probability distribution of W.

$2 \le x \le 4.$ $\dfrac{1}{8} \ge \dfrac{1}{x+6} \ge \dfrac{1}{10}.$ **$0.10 \le w \le 0.125$.**

h)   Use part (b) and the c.d.f. approach to find the c.d.f. of W, $F_W(w)$.

$$F_W(w) = P(W \le w) = P\left(\frac{1}{X+6} \le w\right) = P\left(X \ge \frac{1}{w} - 6\right) = 1 - F_X\left(\frac{1}{w} - 6\right)$$

$$= 1 - \frac{\left(\dfrac{1}{w} - 6\right)^4 - 16}{240} = \frac{256 - \left(\dfrac{1}{w} - 6\right)^4}{240} = \frac{256\,w^4 - (1 - 6w)^4}{240\,w^4}$$

$$= 1 - \frac{1280\,w^4 - 864\,w^3 + 216\,w^2 - 24\,w + 1}{240\,w^4}$$

$$= \frac{-1040\,w^4 + 864\,w^3 - 216\,w^2 + 24\,w - 1}{240\,w^4}, \qquad 0.10 \le w < 0.125.$$

$F_W(w) = 0,$ $w < 0.10,$ $F_W(w) = 1,$ $w \ge 0.125.$

OR $\ldots = P\left(X \ge \dfrac{1}{w} - 6\right) = \displaystyle\int_{\frac{1}{w} - 6}^{4} \dfrac{x^3}{60}\,dx = \dfrac{256 - \left(\dfrac{1}{w} - 6\right)^4}{240} = \ldots$

i)   Use the change-of-variable technique to find the p.d.f. of W, $f_W(w)$.

$$w = \frac{1}{x+6} \qquad\qquad x = \frac{1}{w} - 6 \qquad\qquad \frac{dx}{dw} = -\frac{1}{w^2}$$

$$f_W(w) = \frac{\left(\frac{1}{w}-6\right)^3}{60} \cdot \left| -\frac{1}{w^2}\right| = \frac{(1-6w)^3}{60\,w^5}$$

$$= \frac{-216\,w^3 + 108\,w^2 - 18\,w + 1}{60\,w^5}, \qquad\qquad 0.10 \le w \le 0.125.$$

Indeed, $$\qquad \frac{d}{dw}\left(\frac{256\,w^4 - (1-6w)^4}{240\,w^4}\right) = \frac{(1-6w)^3}{60\,w^5}, \qquad ☺$$

$$\frac{d}{dw}\left(\frac{-1040\,w^4 + 864\,w^3 - 216\,w^2 + 24\,w - 1}{240\,w^4}\right) = \frac{-216\,w^3 + 108\,w^2 - 18\,w + 1}{60\,w^5}.$$

j)   Find the moment-generating function of X, $M_X(t)$.

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx}\cdot f_X(x)\,dx = \int_{2}^{4} e^{tx}\cdot \frac{x^3}{60}\,dx = \begin{array}{c}\text{by parts}\\ \text{three times}\end{array}$$

$$= \frac{\left(32\,t^3 - 24\,t^2 + 12\,t - 3\right)e^{4t} - \left(4\,t^3 - 6\,t^2 + 6\,t - 3\right)e^{2t}}{30\,t^4},$$

$$t \ne 0.$$

$$M_X(0) = 1.$$

2. Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{x^3}{C}, \qquad x = 1, 2, 3, 4, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $p_X(x)$ a valid probability mass function.

$$1 = \sum_{\text{all } x} p_X(x) = \frac{1^3}{C} + \frac{2^3}{C} + \frac{3^3}{C} + \frac{4^3}{C} = \frac{100}{C}.$$

$$\Rightarrow \quad C = \mathbf{100}.$$

2. (continued)

Consider $Y = g(X) = X^2$.

b) Find the probability distribution of Y.

x	$p_X(x)$
1	$\dfrac{1}{100} = 0.01$
2	$\dfrac{8}{100} = 0.08$
3	$\dfrac{27}{100} = 0.27$
4	$\dfrac{64}{100} = 0.64$

$\Rightarrow$

y	$p_Y(y)$
$1^2 = 1$	0.01
$2^2 = 4$	0.08
$3^2 = 9$	0.27
$4^2 = 16$	0.64

OR $\qquad p_Y(y) = \dfrac{y^{3/2}}{100}, \qquad y = 1, 4, 9, 16.$

c)   Is μ_Y equal to $g(\mu_X)$? $\mu_X = E(X), \ \mu_Y = E(Y).$

$$E(X) = \sum_{\text{all } x} x \cdot p_X(x).$$

x	$p_X(x)$	$x \cdot p_X(x)$
1	0.01	0.01
2	0.08	0.16
3	0.27	0.81
4	0.64	2.56

3.54

E(X)

y	$p_Y(y)$	$y \cdot p_Y(y)$
1	0.01	0.01
4	0.08	0.32
9	0.27	2.43
16	0.64	10.24

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E(Y)

$13 \neq 3.54^2.$ $\mu_Y \neq g(\mu_X).$

Recall: IF $g(x)$ is a linear function, that is, IF $g(x) = a\,x + b,$

then $E(g(X)) = E(a\,X + b) = a\,E(X) + b = g(E(X)).$

However, in general, if $g(x)$ is NOT a linear function,

then $E(g(X)) \neq g(E(X)).$

Spoiler:

Here, $E(g(X)) > g(E(X))$ since $g(x) = x^2$ "curves up".

d)   Find the moment-generating function of X, $M_X(t)$.

$$M_X(t) = E(e^{tX}) = \sum_{\text{all } x} e^{tx} \cdot p_X(x)$$

$$= 0.01\, e^t + 0.08\, e^{2t} + 0.27\, e^{3t} + 0.64\, e^{4t}.$$

For fun:

Moment-generating function approach:

$$M_Y(t) = E(e^{tY}) = E(e^{tX^2}) = \sum_{\text{all } x} e^{tx^2} \cdot p_X(x)$$

$$= 0.01\, e^t + 0.08\, e^{4t} + 0.27\, e^{9t} + 0.64\, e^{16t}.$$

$\Rightarrow$

y	$p_Y(y)$
1	0.01
4	0.08
9	0.27
16	0.64

For discrete random variables, the possible values are isolated points on the number line.

$\Rightarrow$ no derivatives. $\Rightarrow$ no $\dfrac{dx}{dy}$.