

2.3 Conditional Distributions and Expectations. (continued)

Def $\text{Var}(X|Y) = E[(X - E(X|Y))^2|Y] = E(X^2|Y) - [E(X|Y)]^2$

Theorem $E(E(X|Y)) = E(X)$

$$\text{Var}(E(X|Y)) \leq \text{Var}(X)$$

Furthermore, $\text{Var}(X) = \text{Var}(E(X|Y)) + E[\text{Var}(X|Y)]$

$\text{Var}(X)$	$=$	$\text{Var}(E(X Y))$	$+$	$E[\text{Var}(X Y)]$
		Portion of the distribution of X that is related to Y		Portion of the distribution of X that is independent of Y

If X is a function of Y , then

$$E(X|Y) = X \quad \text{and} \quad \text{Var}(E(X|Y)) = \text{Var}(X),$$

$$\text{Var}(X|Y) = 0 \quad \text{and} \quad E[\text{Var}(X|Y)] = 0.$$

If X and Y are independent, then

$$E(X|Y) = E(X) \quad \text{and} \quad \text{Var}(E(X|Y)) = 0 \quad \text{since } E(X) \text{ is a constant,}$$

$$\text{Var}(X|Y) = \text{Var}(X) \quad \text{and} \quad E[\text{Var}(X|Y)] = \text{Var}(X).$$

2. Let the joint probability density function for (X, Y) be

$$f(x, y) = \begin{cases} 60x^2y & 0 \leq x \leq 1, 0 \leq y \leq 1, x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Recall: $f_X(x) = 30x^2(1-x)^2, \quad 0 < x < 1,$

$$E(X) = \frac{1}{2}, \quad \text{Var}(X) = \frac{9}{252}, \quad E(X^2) = \frac{72}{252} = \frac{2}{7};$$

$$f_Y(y) = 20y(1-y)^3, \quad 0 < y < 1,$$

$$E(Y) = \frac{1}{3}, \quad \text{Var}(Y) = \frac{8}{252}, \quad E(Y^2) = \frac{36}{252} = \frac{1}{7};$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{2y}{(1-x)^2}, \quad 0 < y < 1-x, \quad 0 < x < 1.$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{3x^2}{(1-y)^3}, \quad 0 < x < 1-y, \quad 0 < y < 1.$$

$$E(Y|X=x) = \int_0^{1-x} y \cdot \frac{2y}{(1-x)^2} dy = \frac{2}{3}(1-x), \quad 0 < x < 1.$$

$$E(Y|X) = \frac{2}{3}(1-X). \quad E(E(Y|X)) = \frac{1}{3} = E(Y).$$

$$E[(E(Y|X))^2] = \frac{4}{9} \left(1 - 2E(X) + E(X^2) \right) = \frac{4}{9} \left(1 - 2 \cdot \frac{1}{2} + \frac{2}{7} \right) = \frac{8}{63}.$$

$$\text{Var}(E(Y|X)) = \frac{8}{63} - \left(\frac{1}{3} \right)^2 = \frac{1}{63}.$$

OR

$$\text{Var}(E(Y|X)) = \text{Var}\left(\frac{2}{3}(1-X)\right) = \frac{4}{9} \text{Var}(X) = \frac{4}{9} \cdot \frac{9}{252} = \frac{4}{252} = \frac{1}{63}.$$

$$\text{Var}(\text{E}(\text{Y}|\text{X})) = \frac{1}{63} = \frac{4}{252} < \frac{8}{252} = \text{Var}(\text{Y}).$$

$$\text{E}(\text{Y}^2|\text{X}=x) = \int_0^{1-x} y^2 \cdot \frac{2y}{(1-x)^2} dy = \frac{1}{2}(1-x)^2, \quad 0 < x < 1.$$

$$\begin{aligned} \text{Var}(\text{Y}|\text{X}=x) &= \text{E}(\text{Y}^2|\text{X}=x) - [\text{E}(\text{Y}|\text{X}=x)]^2 \\ &= \frac{1}{2}(1-x)^2 - \frac{4}{9}(1-x)^2 = \frac{1}{18}(1-x)^2, \quad 0 < x < 1. \end{aligned}$$

$$\text{Var}(\text{Y}|\text{X}) = \frac{1}{18}(1-\text{X})^2.$$

$$\text{E}[\text{Var}(\text{Y}|\text{X})] = \frac{1}{18} \left(1 - 2\text{E}(\text{X}) + \text{E}(\text{X}^2) \right) = \frac{1}{18} \left(1 - 2 \cdot \frac{1}{2} + \frac{2}{7} \right) = \frac{1}{63}.$$

$$\text{Var}(\text{E}(\text{Y}|\text{X})) + \text{E}[\text{Var}(\text{Y}|\text{X})] = \frac{1}{63} + \frac{1}{63} = \frac{8}{252} = \text{Var}(\text{Y}).$$

$$E(X|Y=y) = \int_0^{1-y} x \cdot \frac{3x^2}{(1-y)^3} dx = \frac{3}{4}(1-y), \quad 0 < y < 1.$$

$$E(X|Y) = \frac{3}{4}(1-Y).$$

$$E(E(X|Y)) = \frac{3}{4}(1-E(Y)) = \frac{3}{4}\left(1-\frac{1}{3}\right) = \frac{1}{2} = E(X).$$

$$\text{Var}(E(X|Y)) = \text{Var}\left(\frac{3}{4}(1-Y)\right) = \frac{9}{16}\text{Var}(Y) = \frac{9}{16} \cdot \frac{8}{252} = \frac{1}{56}.$$

$$\text{Var}(E(X|Y)) = \frac{1}{56} = \frac{4.5}{252} < \frac{9}{252} = \text{Var}(X).$$

$$E(X^2|Y=y) = \int_0^{1-y} x^2 \cdot \frac{3x^2}{(1-y)^3} dx = \frac{3}{5}(1-y)^2, \quad 0 < y < 1.$$

$$\begin{aligned} \text{Var}(X|Y=y) &= E(X^2|Y=y) - [E(X|Y=y)]^2 \\ &= \frac{3}{5}(1-y)^2 - \frac{9}{16}(1-y)^2 = \frac{3}{80}(1-y)^2, \quad 0 < y < 1. \end{aligned}$$

$$\text{Var}(X|Y) = \frac{3}{80}(1-Y)^2.$$

$$E[\text{Var}(X|Y)] = \frac{3}{80}\left(1-2E(Y)+E(Y^2)\right) = \frac{3}{80}\left(1-2\cdot\frac{1}{3}+\frac{1}{7}\right) = \frac{1}{56}.$$

$$\text{Var}(E(X|Y)) + E[\text{Var}(X|Y)] = \frac{1}{56} + \frac{1}{56} = \frac{9}{252} = \text{Var}(X).$$

1. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y :

	y			
x	0	1	2	$p_X(x)$
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
$p_Y(y)$	0.40	0.40	0.20	

x	$p_{X Y}(x 0)$	x	$p_{X Y}(x 1)$	x	$p_{X Y}(x 2)$
1	$0.15/0.40 = 0.375$	1	$0.10/0.40 = 0.25$	1	$0.00/0.20 = 0.00$
2	$0.25/0.40 = 0.625$	2	$0.30/0.40 = 0.75$	2	$0.20/0.20 = 1.00$

$$E(X|Y=0) = 1.625$$

$$E(X|Y=1) = 1.75$$

$$E(X|Y=2) = 2$$

$$\text{Var}(X|Y=0) = 0.234375$$

$$\text{Var}(X|Y=1) = 0.1875$$

$$\text{Var}(X|Y=2) = 0$$

$$E(X|Y):$$

y	$E(X Y=y)$	$p_Y(y)$
0	1.625	0.40
1	1.75	0.40
2	2	0.20

$$\text{Var}(X|Y):$$

$\text{Var}(X Y=y)$	$p_Y(y)$
0.234375	0.40
0.1875	0.40
0	0.20

$$E(E(X|Y)) = 1.75 = E(X)$$

$$E(\text{Var}(X|Y)) = 0.16875$$

$$\text{Var}(E(X|Y)) = 0.01875 < 0.1875 = \text{Var}(X).$$

$$\text{Var}(E(X|Y)) + E(\text{Var}(X|Y)) = 0.01875 + 0.16875 = 0.1875 = \text{Var}(X).$$

y	$p_{Y X}(y 1)$
0	$0.15/0.25 = 0.60$
1	$0.10/0.25 = 0.40$
2	$0.00/0.25 = 0.00$

$$E(Y|X=1) = 0.4 = 6/15$$

$$\text{Var}(Y|X=1) = 0.24 = 54/225$$

$E(Y|X):$

x	$E(Y X=x)$	$p_X(x)$
1	$6/15$	0.25
2	$14/15$	0.75

$$E(E(Y|X)) = 12/15 = 0.80 = E(Y)$$

$$\text{Var}(E(Y|X)) = 4/75 \approx 0.053333 < 0.56 = \text{Var}(Y)$$

$$\text{Var}(Y) = 0.56 = 4/75 + 38/75 = \text{Var}(E(Y|X)) + E(\text{Var}(Y|X))$$

y	$p_{Y X}(y 2)$
0	$0.25/0.75 = 5/15$
1	$0.30/0.75 = 6/15$
2	$0.20/0.75 = 4/15$

$$E(Y|X=2) = 14/15$$

$$\text{Var}(Y|X=2) = 134/225$$

$\text{Var}(Y|X):$

$\text{Var}(Y X=x)$	$p_X(x)$
$54/225$	0.25
$134/225$	0.75

$$E(\text{Var}(Y|X)) = 38/75$$