2.5   Independent Random Variables

1.   Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

x \ y	0	1	2	
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
	0.40	0.40	0.20	

Recall: A and B are independent if and only if $P(A \cap B) = P(A) \cdot P(B)$.

a)   Are events $\{X = 1\}$ and $\{Y = 1\}$ independent?

$$P(X = 1 \cap Y = 1) = p(1, 1) = 0.10 = 0.25 \times 0.40 = P(X = 1) \times P(Y = 1).$$

$\{X = 1\}$ and $\{Y = 1\}$ are **independent**.

Def   Random variables X and Y are **independent** if and only if

discrete $p(x, y) = p_X(x) \cdot p_Y(y)$ for all x, y.

continuous $f(x, y) = f_X(x) \cdot f_Y(y)$ for all x, y.

$F(x, y) = P(X \le x, Y \le y)$. $f(x, y) = \partial^2 F(x, y) / \partial x \, \partial y$.

Def   Random variables X and Y are **independent** if and only if

$$F(x, y) = F_X(x) \cdot F_Y(y)$$ for all x, y.

b)   Are random variables X and Y independent?

$$p(1, 0) = 0.15 \neq 0.25 \times 0.40 = p_X(1) \times p_Y(0).$$

X and Y are **NOT independent**.

2. Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} 60\, x^2\, y & 0 \le x \le 1,\ 0 \le y \le 1,\ x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Recall: $f_X(x) = 30\, x^2\, (1-x)^2, \quad 0 < x < 1,$

$f_Y(y) = 20\, y\, (1-y)^3, \quad 0 < y < 1.$

Are random variables X and Y independent?

The support of (X, Y) is not a rectangle.

X and Y are **NOT independent**.

OR

Since $f(x,y) \ne f_X(x) \cdot f_Y(y),$

X and Y are **NOT independent**.

3. Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} x + y & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

$$\begin{aligned}
f_1(x) &= \int_0^1 (x + y)\, dy \\
&= \left[xy + \frac{1}{2} y^2 \right]_0^1 = x + \frac{1}{2}, \qquad 0 \le x \le 1\ ; \\
f_2(y) &= \int_0^1 (x + y)\, dx = y + \frac{1}{2}, \qquad 0 \le y \le 1; \\
f(x,y) &= x + y \ne \left(x + \frac{1}{2} \right) \left(y + \frac{1}{2} \right) = f_1(x) f_2(y).
\end{aligned}$$

X and Y are **NOT independent**.

4. Let the joint probability density function for (X, Y) be

$$f(x, y) = \begin{cases} 12\,x(1-x)e^{-2y} & 0 \le x \le 1, \ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

$$f_X(x) = \int_0^\infty 12\,x(1-x)e^{-2y}\,dy = 6\,x(1-x), \quad 0 < x < 1.$$

$$f_Y(y) = \int_0^1 12\,x(1-x)e^{-2y}\,dx = 2\,e^{-2y}, \quad y > 0.$$

Since $f(x, y) = f_X(x) \cdot f_Y(y)$ for all x, y, X and Y are **independent**.

If random variables X and Y are independent, then

$$E(g(X) \cdot h(Y)) = E(g(X)) \cdot E(h(Y)).$$

$\Rightarrow$ $\qquad M_{X,Y}(t_1, t_2) = M_{X,Y}(t_1, 0) \cdot M_{X,Y}(0, t_2) = M_X(t_1) \cdot M_Y(t_2).$

5. Suppose the probability density functions of T_1 and T_2 are

$$f_{T_1}(x) = \alpha\,e^{-\alpha x}, \quad x > 0, \qquad f_{T_2}(y) = \beta\,e^{-\beta y}, \quad y > 0,$$

respectively. Suppose T_1 and T_2 are independent. Find $P(2\,T_1 > T_2)$.

$$P(2\,T_1 > T_2) = \int_0^\infty \left(\int_{y/2}^\infty \left(\alpha\beta\,e^{-\alpha x - \beta y} \right) dx \right) dy = \int_0^\infty \beta\,e^{-\beta y} \left(\int_{y/2}^\infty \left(\alpha\,e^{-\alpha x} \right) dx \right) dy$$

$$= \int_0^\infty \beta\,e^{-\beta y} \left(e^{-\alpha y/2} \right) dy = \int_0^\infty \beta\,e^{-(\alpha/2 + \beta)y}\,dy = \frac{2\beta}{\alpha + 2\beta}.$$

6.   Let X and Y be two independent random variables, X has a Geometric distribution with the probability of "success" $p = 1/3$, Y has a Poisson distribution with mean 3. That is,

$$p_X(x) = \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^{x-1}, \qquad x = 1, 2, 3, \dots,$$

$$p_Y(y) = \frac{3^y e^{-3}}{y!}, \qquad y = 0, 1, 2, 3, \dots.$$

a)   Find $P(X = Y)$.

$$P(X = Y) = \sum_{k=1}^{\infty} p_X(k) \cdot p_Y(k) = \sum_{k=1}^{\infty} \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^{k-1} \cdot \frac{3^k e^{-3}}{k!}$$

$$= e^{-3} \cdot \sum_{k=1}^{\infty} \frac{2^{k-1}}{k!} = \frac{e^{-3}}{2} \cdot \left[\sum_{k=0}^{\infty} \frac{2^k}{k!} - 1\right] = \frac{e^{-3}}{2} \cdot \left[e^2 - 1\right]$$

$$= \frac{e^{-1} - e^{-3}}{2} \approx 0.159.$$

b)   Find $P(X = 2\,Y)$.

$$P(X = 2\,Y) = \sum_{k=1}^{\infty} p_X(2k) \cdot p_Y(k) = \sum_{k=1}^{\infty} \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^{2k-1} \cdot \frac{3^k e^{-3}}{k!}$$

$$= \frac{e^{-3}}{2} \cdot \sum_{k=1}^{\infty} \left(\frac{4}{3}\right)^k \cdot \frac{1}{k!} = \frac{e^{-3}}{2} \cdot \left[e^{4/3} - 1\right] \approx 0.069544.$$

c)   Find $P(X > Y)$.

$$P(X > Y) = \sum_{y=0}^{\infty} \sum_{x=y+1}^{\infty} \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^{x-1} \cdot \frac{3^y e^{-3}}{y!} = \sum_{y=0}^{\infty} \left(\frac{2}{3}\right)^y \cdot \frac{3^y e^{-3}}{y!}$$

$$= e^{-3} \cdot \sum_{y=0}^{\infty} \frac{2^y}{y!} = e^{-1} \approx 0.368.$$

2.4   Covariance and Correlation Coefficient

Covariance of X and Y

$$\sigma_{XY} = \text{Cov}(X,Y) = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - \mu_X \mu_Y$$

(a) $\text{Cov}(X,X) = \text{Var}(X);$

(b) $\text{Cov}(X,Y) = \text{Cov}(Y,X);$

(c) $\text{Cov}(aX + b, Y) = a\,\text{Cov}(X,Y);$

(d) $\text{Cov}(X + Y, W) = \text{Cov}(X,W) + \text{Cov}(Y,W).$

$$\text{Cov}(aX + bY, cX + dY)$$
$$= ac\,\text{Var}(X) + (ad + bc)\,\text{Cov}(X,Y) + bd\,\text{Var}(Y).$$

$$\text{Var}(aX + bY) = \text{Cov}(aX + bY, aX + bY)$$
$$= a^2\,\text{Var}(X) + 2ab\,\text{Cov}(X,Y) + b^2\,\text{Var}(Y).$$

0.   Find in terms of σ_X^2, σ_Y^2, and σ_{XY}:

a) $\text{Cov}(2X + 3Y, X - 2Y),$

$$\text{Cov}(2X + 3Y, X - 2Y) = 2\,\text{Var}(X) - \text{Cov}(X,Y) - 6\,\text{Var}(Y).$$

b) $\text{Var}(2X + 3Y),$

$$\text{Var}(2X + 3Y) = \text{Cov}(2X + 3Y, 2X + 3Y)$$
$$= 4\,\text{Var}(X) + 12\,\text{Cov}(X,Y) + 9\,\text{Var}(Y).$$

c) $\text{Var}(X - 2Y).$

$$\text{Var}(X - 2Y) = \text{Cov}(X - 2Y, X - 2Y)$$
$$= \text{Var}(X) - 4\,\text{Cov}(X,Y) + 4\,\text{Var}(Y).$$

$$E(aX + bY) = aE(X) + bE(Y),$$

$$Var(aX + bY) = a^2 Var(X) + 2ab\, Cov(X, Y) + b^2 Var(Y).$$

If $X_1, X_2, \ldots, X_n$ are n random variables and $a_0, a_1, a_2, \ldots, a_n$ are $n+1$ constants, then the random variable $U = a_0 + a_1 X_1 + a_2 X_2 + \ldots + a_n X_n$ has mean

$$E(U) = a_0 + a_1 E(X_1) + a_2 E(X_2) + \ldots + a_n E(X_n)$$

and variance

$$Var(U) = \sum_{i=1}^{n} a_i^2 Var(X_i) + \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} a_i a_j Cov(X_i, X_j)$$

$$= \sum_{i=1}^{n} a_i^2 Var(X_i) + 2 \sum_{i<j} \sum a_i a_j Cov(X_i, X_j)$$

Correlation coefficient of X and Y

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{Cov(X, Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}} = E\left[\left(\frac{X - \mu_X}{\sigma_X} \right) \cdot \left(\frac{Y - \mu_Y}{\sigma_Y} \right) \right]$$

(a) $-1 \leq \rho_{XY} \leq 1$;

(b) ρ_{XY} is either $+1$ or -1 if and only if X and Y are linear functions of one another.

If random variables X and Y are independent, then

$$E(g(X) \cdot h(Y)) = E(g(X)) \cdot E(h(Y)).$$

$$\Rightarrow \qquad Cov(X, Y) = \sigma_{XY} = 0, \qquad Corr(X, Y) = \rho_{XY} = 0.$$

If $X_1, X_2, \ldots, X_n$ are n independent random variables and $a_0, a_1, a_2, \ldots, a_n$ are $n+1$ constants, then the random variable $U = a_0 + a_1 X_1 + a_2 X_2 + \ldots + a_n X_n$ has variance

$$Var(U) = a_1^2 Var(X_1) + a_2^2 Var(X_2) + \ldots + a_n^2 Var(X_n)$$

1. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

	y			
x	0	1	2	$p_X(x)$
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
$p_Y(y)$	0.40	0.40	0.20	1.00

Recall:

$E(X) = 1.75,$

$E(Y) = 0.8,$

$E(XY) = 1.5.$

Find $\text{Cov}(X, Y) = \sigma_{XY}$ and $\text{Corr}(X, Y) = \rho_{XY}$.

$$\text{Cov}(X, Y) = \sigma_{XY} = 1.5 - 1.75 \cdot 0.8 = \mathbf{0.10}.$$

$$E(X^2) = 1 \times 0.25 + 4 \times 0.75 = 3.25.$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 3.25 - 1.75^2 = 0.1875.$$

$$E(Y^2) = 0 \times 0.40 + 1 \times 0.40 + 4 \times 0.20 = 1.2.$$

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = 1.2 - 0.8^2 = 0.56.$$

$$\text{Corr}(X, Y) = \rho_{XY} = \frac{0.10}{\sqrt{0.1875} \cdot \sqrt{0.56}} \approx \mathbf{0.3086}.$$

2. Let the joint probability density function for (X, Y) be

$$f(x, y) = \begin{cases} 60\, x^2\, y & 0 \le x \le 1,\ 0 \le y \le 1,\ x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\text{Cov}(X, Y) = \sigma_{XY}$ and $\text{Corr}(X, Y) = \rho_{XY}$.

Recall:
$$f_X(x) = 30\, x^2 (1 - x)^2, \quad 0 < x < 1, \qquad E(X) = \frac{1}{2},$$

$$f_Y(y) = 20\, y (1 - y)^3, \qquad 0 < y < 1, \qquad E(Y) = \frac{1}{3}, \qquad E(XY) = \frac{1}{7}.$$

$$\text{Cov}(X, Y) = \frac{1}{7} - \frac{1}{2} \cdot \frac{1}{3} = -\frac{1}{42}. \qquad \text{Var}(X) = \frac{9}{252}. \qquad \text{Var}(Y) = \frac{8}{252}.$$

$$\rho_{XY} = \frac{-1/42}{\sqrt{9/252} \cdot \sqrt{8/252}} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \approx -\mathbf{0.7071}.$$

3. Let the joint probability density function for (X, Y) be

$$f(x, y) = \begin{cases} x+y & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\text{Cov}(X, Y) = \sigma_{XY}$ and $\text{Corr}(X, Y) = \rho_{XY}$.

Recall: $f_X(x) = x + \dfrac{1}{2}, \quad 0 < x < 1.$ $\qquad f_Y(y) = y + \dfrac{1}{2}, \quad 0 < y < 1.$

$$\mu_X = \int_0^1 x\left(x + \frac{1}{2}\right) dx = \left[\frac{1}{3}x^3 + \frac{1}{4}x^2\right]_0^1 = \frac{7}{12};$$

$$\mu_Y = \int_0^1 y\left(y + \frac{1}{2}\right) dy = \frac{7}{12};$$

$$E(X^2) = \int_0^1 x^2\left(x + \frac{1}{2}\right) dx = \left[\frac{1}{4}x^4 + \frac{1}{6}x^3\right]_0^1 = \frac{5}{12},$$

$$\sigma_X^2 = E(X^2) - \mu_X^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}.$$

Similarly, $\sigma_Y^2 = \dfrac{11}{144}.$

$$E(XY) = \int_0^1\int_0^1 xy(x+y)\,dx\,dy = \int_0^1\int_0^1 \left(x^2 y + xy^2\right)dx\,dy = \int_0^1\left(\frac{y}{3} + \frac{y^2}{2}\right)dy = \frac{1}{3}.$$

$$\text{Cov}(X, Y) = \frac{1}{3} - \frac{7}{12}\cdot\frac{7}{12} = -\frac{1}{144}. \qquad \rho_{XY} = \frac{-1/144}{\sqrt{11/144}\cdot\sqrt{11/144}} = -\frac{1}{11}.$$

4. Let the joint probability density function for (X, Y) be

$$f(x, y) = \begin{cases} 12x(1-x)e^{-2y} & 0 \le x \le 1, y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $\text{Cov}(X, Y) = \sigma_{XY}$ and $\text{Corr}(X, Y) = \rho_{XY}$.

Recall: $f_X(x) = 6x(1-x), \quad 0 < x < 1.$ $\qquad f_Y(y) = 2e^{-2y}, \quad y > 0.$

Since X and Y are independent,

$$\text{Cov}(X, Y) = \sigma_{XY} = \mathbf{0}, \qquad \text{Corr}(X, Y) = \rho_{XY} = \mathbf{0}.$$