STAT 410 **Examples for Week 02 (Disc2)** A. Stepanov

STAT 410 does NOT have a discussion section.

STAT 400 (an easier class) DOES have a discussion section.
STAT 410 (a harder class) does NOT.

There is a simple explanation for this: About 20 years ago, there was only one section of STAT 410 with about 40 students and only two lectures of STAT 400 with about 150 students each (each divided into 3 discussion sections of about 50 students each). The number of students has changed a little bit since then, the course structure has not (yet)…

However, don't you wish there was a discussion section in STAT 410?

If STAT 410 did have a discussion section, this is what your TA would most likely go over during it:

1. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{15 + 2x^3}{100}, \qquad -1 \le x \le 3, \qquad \text{zero elsewhere.}$$

Find the probability distribution of $Y = g(X) = X^2$.

"Hint": $g(x)$ is NOT a 1-1 function on the support of X.

2. Consider a discrete random variable X with the probability mass function

$$f_X(x) = \frac{15 + 2x^3}{145}, \qquad -1 \le x \le 3, \qquad x \text{ is an integer,}$$

zero elsewhere.

Find the probability distribution of $Y = g(X) = X^2$.

3. Find $\mathrm{E}(\mathrm{X})$ for a mixed random variable X with the following cumulative distribution function:

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \dfrac{15 + 2x^3}{100} & -1 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

4. Consider a mixed random variable X with

the p.m.f. of the discrete portion of the probability distribution

$$p(-1) = 0.09, \qquad p(3) = c, \qquad \text{zero otherwise,}$$

and the p.d.f. of the continuous portion of the probability distribution

$$f(x) = \frac{15 + 2x^3}{160}, \qquad -1 \le x \le 3, \qquad \text{zero elsewhere.}$$

a) Find the value of c that would make this a valid probability distribution.

b) Find $\mathrm{E}(\mathrm{X})$.

5. **1.9.18** (8th edition) **1.9.18** (7th edition) **1.9.17** (6th edition)

Find the mean and the variance of the distribution that has the cdf

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{8} & 0 \le x < 2 \\ \dfrac{x^2}{16} & 2 \le x < 4 \\ 1 & 4 \le x. \end{cases}$$

6. **1.9.24** (8th edition) **1.9.23** (7th edition) **1.9.22** (6th edition)

Let X have the c.d.f. $\mathrm{F}(x)$ that is a mixture of the continuous and discrete types, namely

$$\mathrm{F}(x)=\begin{cases} 0 & x<0 \\ \dfrac{x+1}{4} & 0\le x<1 \\ 1 & 1\le x. \end{cases}$$

Determine reasonable definitions of $\mu=\mathrm{E}(\mathrm{X})$ and $\sigma^2=\mathrm{Var}(\mathrm{X})$ and compute each.

Hint: Determine the parts of the p.m.f. and the p.d.f. associated with each of the discrete and continuous parts, and them sum for the discrete part and integrate for the continuous part.

7. Find $\mathrm{E}(\mathrm{X})$ for a mixed random variable with c.d.f.

$$\mathrm{F}_{\mathrm{X}}(x) = \begin{cases} 0 & x<1 \\ \dfrac{x}{5} & 1\le x<1.5 \\ \ln x & 1.5\le x<2.25 \\ 1 & x\ge 2.25 \end{cases}$$

8. Consider a mixed random variable X with the cumulative distribution function

$$\mathrm{F}_{\mathrm{X}}(x) = \begin{cases} 0 & x<0 \\ x^3 & 0\le x<0.5 \\ \dfrac{8x+7}{32} & 0.5\le x<2 \\ 1-\dfrac{4}{x^5} & x\ge 2 \end{cases}$$

a) Identify the discrete portion of the probability distribution.

b) Identify the continuous portion of the probability distribution.

c) Find $\mathrm{E}(\mathrm{X})$.

9.* The policyholder's loss, Y, follows a distribution with density function:

$$f(y) = \begin{cases} \dfrac{2}{y^3} & \text{if } y > 1 \\ 0 & \text{otherwise} \end{cases}$$

a) What is the expected value and the variance of the policyholder's loss?

b) An insurance policy reimburses a loss up to a benefit limit of 10. What is the expected value and the variance of the benefit paid under the insurance policy?

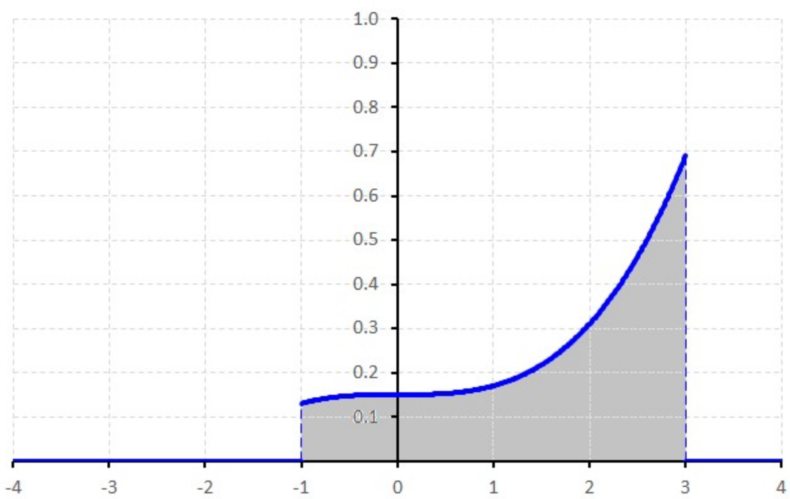
c) An insurance policy has a deductible of 2. What is the expected value and the variance of the benefit paid under the insurance policy?

1. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{15+2x^3}{100}, \qquad -1 \le x \le 3, \qquad \text{zero elsewhere.}$$

Find the probability distribution of $Y = g(X) = X^2$.

"Hint": $g(x)$ is NOT a 1-1 function on the support of X.



"important" points for X: $x = -1$ and $x = 3$.

"important" point for $g(x)$: $x = 0$.

Therefore, "important" points for Y: $(-1)^2 = 1$, $(3)^2 = 9$, $(0)^2 = 0$.

There are two cases to be considered separately: $0 < y < 1$, $1 < y < 9$.

Technically, there are four cases: $y < 0$, $0 < y < 1$, $1 < y < 9$, $y > 9$,

but $y < 0$ and $y > 9$ are boring.

Case 0: $y < 0$.

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = 0.$$

If $y \ge 0$, then

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

$x < -1, \qquad\qquad F_X(x) = 0,$

$$-1 \;\; x < 3, \qquad\qquad F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(u)\,du = \int_{-1}^{x} \frac{15 + 2u^3}{100}\,du$$

$$= \frac{30u + u^4}{200} \Bigg|_{-1}^{x} = \frac{x^4 + 30x + 29}{200},$$

$x \ge 3, \qquad\qquad F_X(x) = 1.$

Case 1: $\quad 0 \le y < 1 \qquad \Rightarrow \qquad 0 \le \sqrt{y} < 1.$

$$F_X(-\sqrt{y}\,) = \frac{\left(-\sqrt{y}\right)^4 + 30\left(-\sqrt{y}\right) + 29}{200} = \frac{y^2 - 30\sqrt{y} + 29}{200},$$

$$F_X(\sqrt{y}\,) = \frac{\left(\sqrt{y}\right)^4 + 30\left(\sqrt{y}\right) + 29}{200} = \frac{y^2 + 30\sqrt{y} + 29}{200}.$$

$$F_Y(y) = F_X(\sqrt{y}\,) - F_X(-\sqrt{y}\,) = \frac{y^2 + 30\sqrt{y} + 29}{200} - \frac{y^2 - 30\sqrt{y} + 29}{200}$$

$$= \frac{60\sqrt{y}}{200} = 0.30\sqrt{y}, \qquad\qquad 0 \le y < 1.$$

Case 2: $\quad 1 \le y < 9 \qquad \Rightarrow \qquad 1 \le \sqrt{y} < 3.$

$$F_X(-\sqrt{y}\,) = 0, \qquad\qquad F_X(\sqrt{y}\,) = \frac{\left(\sqrt{y}\right)^4 + 30\left(\sqrt{y}\right) + 29}{200} = \frac{y^2 + 30\sqrt{y} + 29}{200}.$$

$$F_Y(y) = F_X(\sqrt{y}\,) - F_X(-\sqrt{y}\,) = \frac{y^2 + 30\sqrt{y} + 29}{200} - 0 = \frac{y^2 + 30\sqrt{y} + 29}{200},$$

$$1 \le y < 9.$$

Case 3: $\quad y \ge 9 \quad \Rightarrow \quad \sqrt{y} \ge 3.$

$F_X(-\sqrt{y}) = 0, \qquad F_X(\sqrt{y}) = 1.$

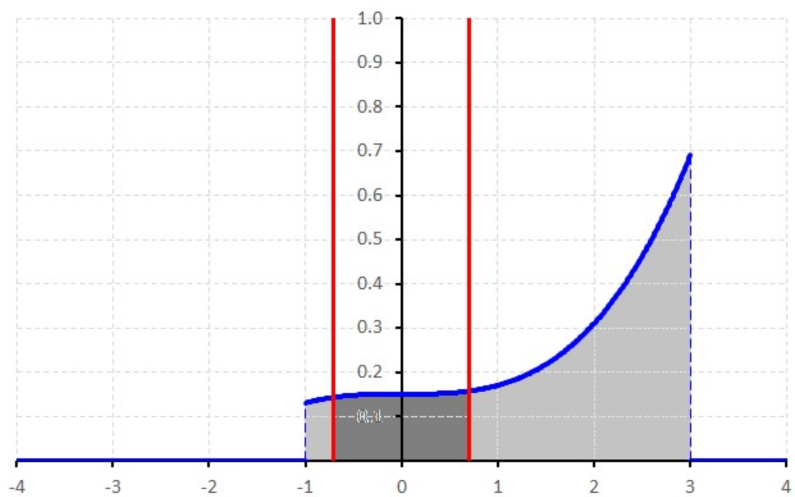
$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) = 1 - 0 = 1, \qquad y \ge 9.$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \dfrac{60\sqrt{y}}{200} & 0 \le y < 1 \\ \dfrac{y^2 + 30\sqrt{y} + 29}{200} & 1 \le y < 9 \\ 1 & y \ge 9 \end{cases}$$

OR

Case 1: $\quad 0 \le y < 1$

$\Rightarrow \quad 0 \le \sqrt{y} < 1.$

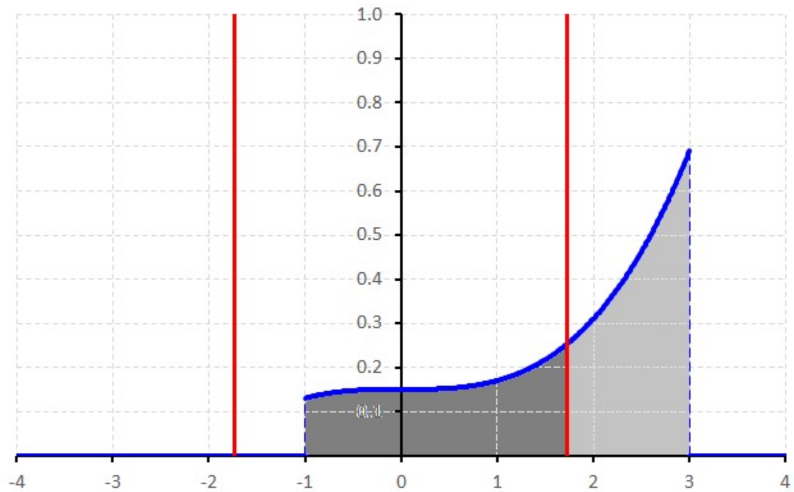


$$F_Y(y) = P(-\sqrt{y} \le X \le \sqrt{y}\,) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{15+2x^3}{100}\,dx = \frac{30x+x^4}{200}\Bigg|_{-\sqrt{y}}^{\sqrt{y}}$$

$$= \frac{60\sqrt{y}}{200} = 0.30\sqrt{y}, \qquad 0 \le y < 1.$$

Case 2: $\quad 1 \le y < 9$

$\Rightarrow \quad 1 \le \sqrt{y} < 3.$



$$F_Y(y) = P(-\sqrt{y} \le X \le \sqrt{y}\,) = \int_{-1}^{\sqrt{y}} \frac{15+2x^3}{100}\,dx = \frac{30x+x^4}{200}\Bigg|_{-1}^{\sqrt{y}}$$

$$= \frac{y^2+30\sqrt{y}+29}{200}, \qquad 1 \le y < 9.$$

OR

$-1 \le x < 0$

$$f_X(x) = \frac{15 + 2x^3}{100}$$

$$Y = g(X) = X^2$$

$1 \ge y > 0$

$$x = -\sqrt{y} = g^{-1}(y)$$

$$\frac{dx}{dy} = -\frac{1}{2\sqrt{y}}$$

$$f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$$

$$\frac{15 + 2\left(-\sqrt{y}\right)^3}{100} \cdot \frac{1}{2\sqrt{y}}$$

$$\frac{15}{200\sqrt{y}} - \frac{y}{100}$$

$0 < x \le 3$

$$f_X(x) = \frac{15 + 2x^3}{100}$$

$$Y = g(X) = X^2$$

$0 < y \le 9$

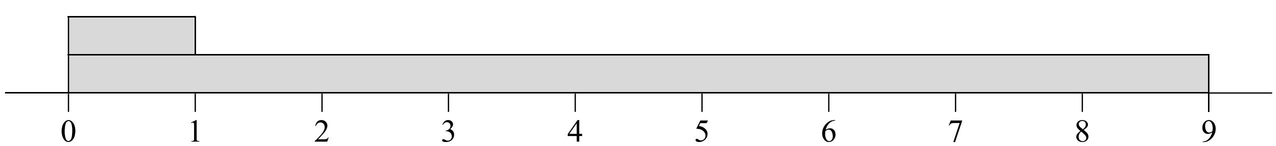
$$x = \sqrt{y} = g^{-1}(y)$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

$$f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$$

$$\frac{15 + 2\left(\sqrt{y}\right)^3}{100} \cdot \frac{1}{2\sqrt{y}}$$

$$\frac{15}{200\sqrt{y}} + \frac{y}{100}$$



If $0 < y < 1$, $\quad f_Y(y) = \dfrac{15}{200\sqrt{y}} - \dfrac{y}{100} + \dfrac{15}{200\sqrt{y}} + \dfrac{y}{100} = \dfrac{15}{100\sqrt{y}}$.

If $1 < y < 9$, $\quad f_Y(y) = 0 + \dfrac{15}{200\sqrt{y}} + \dfrac{y}{100} = \dfrac{15}{200\sqrt{y}} + \dfrac{y}{100}$.

$$f_Y(y) = \begin{cases} \dfrac{15}{100\sqrt{y}} & 0 < y < 1 \\[2ex] \dfrac{15}{200\sqrt{y}} + \dfrac{y}{100} & 1 < y < 9 \\[2ex] 0 & \text{otherwise} \end{cases} = \begin{cases} \dfrac{0.15}{\sqrt{y}} & 0 < y < 1 \\[2ex] \dfrac{0.075}{\sqrt{y}} + 0.01\,y & 1 < y < 9 \\[2ex] 0 & \text{otherwise} \end{cases}$$

Indeed,
$$\frac{d}{dy}\left(\frac{60\sqrt{y}}{200}\right) = \frac{15}{100\sqrt{y}},$$

$$\frac{d}{dy}\left(\frac{y^2 + 30\sqrt{y} + 29}{200}\right) = \frac{15}{200\sqrt{y}} + \frac{y}{100}.$$

☺

2. Consider a discrete random variable X with the probability mass function

$$p_X(x) = \frac{15 + 2x^3}{145}, \qquad -1 \le x \le 3, \qquad x \text{ is an integer},$$

zero elsewhere.

Find the probability distribution of $Y = g(X) = X^2$.

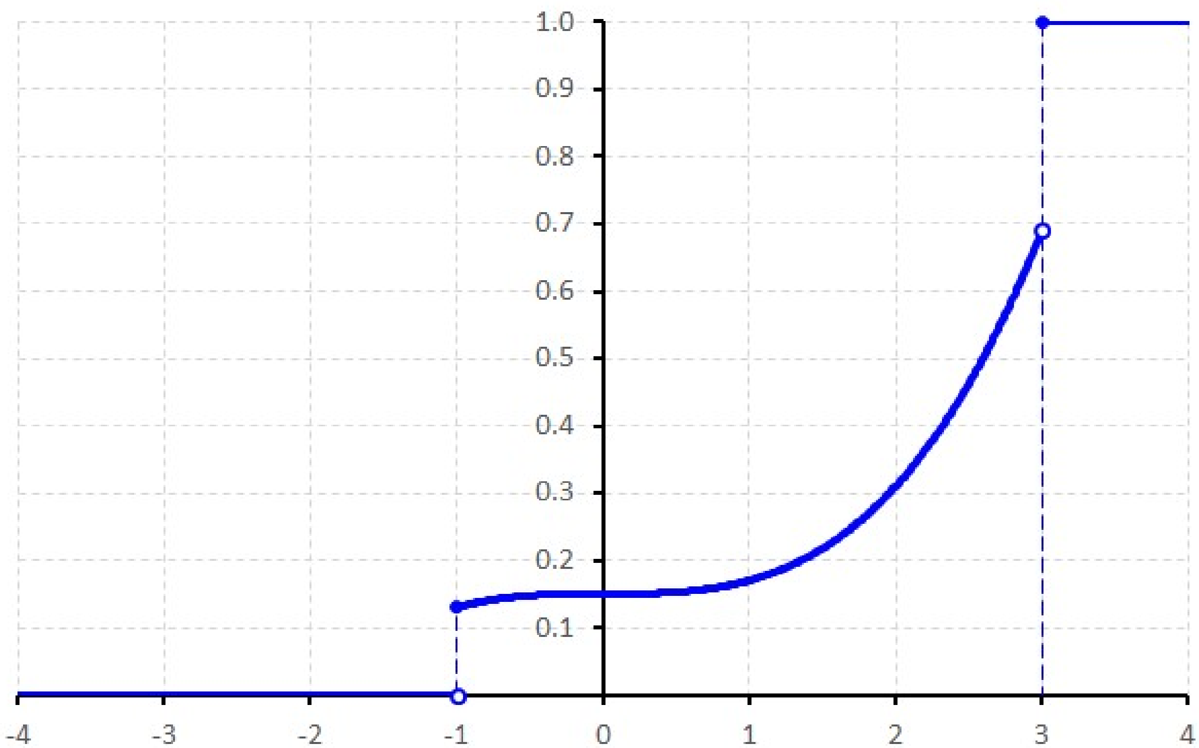
x	$p_X(x)$	y
-1	$\frac{13}{145}$	1
0	$\frac{15}{145}$	0
1	$\frac{17}{145}$	1
2	$\frac{31}{145}$	4
3	$\frac{69}{145}$	9
	1	

y	$p_Y(y)$
0	$\frac{15}{145}$
1	$\frac{30}{145}$
4	$\frac{31}{145}$
9	$\frac{69}{145}$
	1

3. Find $E(X)$ for a mixed random variable X with the following cumulative distribution function:

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \dfrac{15 + 2x^3}{100} & -1 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

"Hint": X is a mixed random variable.
Identify the discrete portion of the probability distribution and the continuous portion of the probability distribution first.



Discrete:

$F_X(x)$ "jumps" at $x = -1$ from 0 to 0.13, $p(-1) = 0.13$,

and $x = 3$ from 0.69 to 1, $p(3) = 0.31$.

Continuous:

$$f_X(x) = F_X'(x) = \frac{3x^2}{50}, \qquad -1 < x < 3, \qquad \text{zero otherwise.}$$

$$E(X) = \sum_{\text{all } x} x \cdot p(x) + \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= -1 \cdot 0.13 + 3 \cdot 0.31 + \int_{-1}^{3} x \cdot \frac{3x^2}{50}\, dx = 0.80 + \left. \frac{3x^4}{200} \right|_{-1}^{3}$$

$$= 0.80 + \frac{243-3}{200} = \mathbf{2}.$$

4. Consider a mixed random variable X with

the p.m.f. of the discrete portion of the probability distribution

$$p(-1) = 0.09, \qquad p(3) = c, \qquad \text{zero otherwise},$$

and the p.d.f. of the continuous portion of the probability distribution

$$f(x) = \frac{15 + 2x^3}{160}, \qquad -1 \le x \le 3, \qquad \text{zero elsewhere.}$$

a) Find the value of c that would make this a valid probability distribution.

$$1 = \sum_{\text{all } x} p(x) + \int_{-\infty}^{\infty} f(x)\, dx$$

$$= [\,0.09 + c\,] + \int_{-1}^{3} \frac{15 + 2x^3}{160}\, dx = 0.09 + c + \left. \frac{30x + x^4}{320} \right|_{-1}^{3}$$

$$= 0.09 + c + 0.625 = c + 0.715.$$

$$\Rightarrow \qquad c = \mathbf{0.285}.$$

Side note: A "clever" thing would have been to notice that in Problem 1,

$$f_X(x) = \frac{15 + 2x^3}{100}, \quad -1 \le x \le 3, \quad \text{was a valid p.d.f. of a random variable,}$$

and $\displaystyle\int_{-1}^{3} \frac{15 + 2x^3}{100}\, dx = 1$. Therefore, $\displaystyle\int_{-1}^{3} \frac{15 + 2x^3}{160}\, dx = \frac{100}{160} = 0.625$.

b) Find $E(X)$.

$$E(X) = \sum_{\text{all } x} x \cdot p(x) + \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

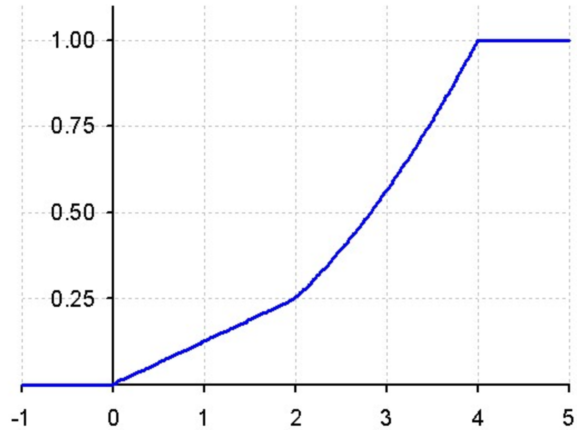
$$= -1 \cdot 0.09 + 3 \cdot 0.285 + \int_{-1}^{3} x \cdot \frac{15 + 2x^3}{160}\, dx$$

$$= 0.765 + \left. \frac{75x^2 + 4x^5}{1600} \right|_{-1}^{3} = 0.765 + 0.985 = \mathbf{1.75}.$$

5. **1.9.18** (8th edition) **1.9.18** (7th edition) **1.9.17** (6th edition)

Find the mean and the variance of the distribution that has the cdf

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{8} & 0 \le x < 2 \\ \dfrac{x^2}{16} & 2 \le x < 4 \\ 1 & 4 \le x. \end{cases}$$



X is a continuous random variable.

$$f(x) = F'(x) = \begin{cases} \dfrac{1}{8} & 0 < x < 2 \\ \dfrac{x}{8} & 2 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = E(X) = \int_0^2 x \cdot \frac{1}{8}\,dx + \int_2^4 x \cdot \frac{x}{8}\,dx = \left(\frac{x^2}{16}\right)\Bigg|_0^2 + \left(\frac{x^3}{24}\right)\Bigg|_2^4 = \mathbf{{}^{31}\!/\!_{12}}.$$

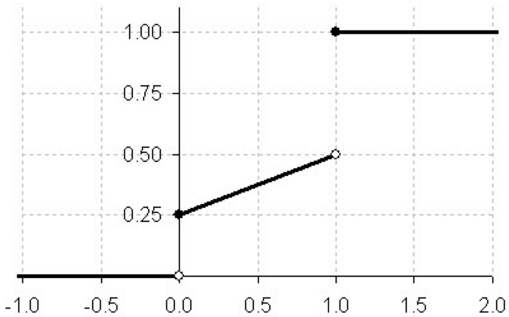
$$E(X^2) = \int_0^2 x^2 \cdot \frac{1}{8}\,dx + \int_0^2 x^2 \cdot \frac{x}{8}\,dx = \left(\frac{x^3}{24}\right)\Bigg|_0^2 + \left(\frac{x^4}{32}\right)\Bigg|_2^4 = {}^{94}\!/\!_{12}.$$

$$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2 = {}^{94}\!/\!_{12} - \left({}^{31}\!/\!_{12}\right)^2 = \mathbf{{}^{167}\!/\!_{144}}.$$

6. **1.9.24** (8th edition) **1.9.23** (7th edition) **1.9.22** (6th edition)

Let X have the c.d.f. $F(x)$ that is a mixture of the continuous and discrete types, namely

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x+1}{4} & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$$



Determine reasonable definitions of $\mu = E(X)$ and $\sigma^2 = \mathrm{Var}(X)$ and compute each.

Hint: Determine the parts of the p.m.f. and the p.d.f. associated with each of the discrete and continuous parts, and them sum for the discrete part and integrate for the continuous part.

Discrete portion of the probability distribution of X:

$$p(0) = 1/4, \qquad p(1) = 1/2.$$

Continuous portion of the probability distribution of X:

$$f(x) = F'(x) = \begin{cases} \dfrac{1}{4} & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}.$$

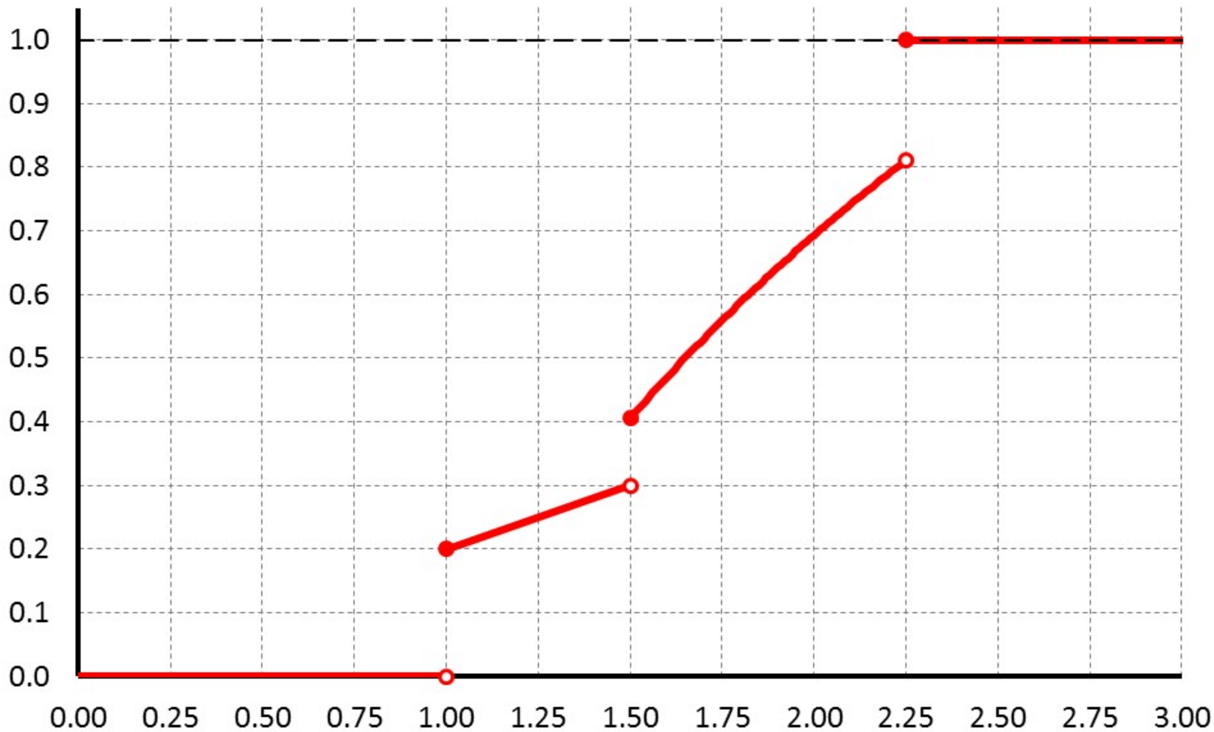
$$\mu = E(X) = 0 \cdot 1/4 + 1 \cdot 1/2 + \int_0^1 x \cdot \frac{1}{4}\,dx = \mathbf{5/8}.$$

$$E(X^2) = 0^2 \cdot 1/4 + 1^2 \cdot 1/2 + \int_0^1 x^2 \cdot \frac{1}{4}\,dx = 7/12.$$

$$\sigma^2 = \mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 7/12 - \left(5/8\right)^2 = \mathbf{37/192}.$$

7. Find $\mathrm{E}(\mathrm{X})$ for a mixed random variable with c.d.f.

$$\mathrm{F}_{\mathrm{X}}(x) = \begin{cases} 0 & x < 1 \\ \dfrac{x}{5} & 1 \le x < 1.5 \\ \ln x & 1.5 \le x < 2.25 \\ 1 & x \ge 2.25 \end{cases}$$



Discrete portion of the probability distribution of X:

$$p(1) = 0.2 - 0 = 0.2, \qquad p(1.5) = \ln 1.5 - 0.3,$$

$$p(2.25) = 1 - \ln 2.25 = 1 - 2\ln 1.5.$$

Continuous portion of the probability distribution of X:

$$f(x) = \mathrm{F}'(x) = \begin{cases} \dfrac{1}{5} & 1 < x < 1.5 \\ \dfrac{1}{x} & 1.5 < x < 2.25 \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = 1 \cdot 0.2 + 1.5 \cdot (\ln 1.5 - 0.3) + 2.25 \cdot (1 - 2\ln 1.5) + \int_1^{1.5} x \cdot \frac{1}{5}\, dx + \int_{1.5}^{2.25} x \cdot \frac{1}{x}\, dx$$

$$= 1 \cdot 0.2 + 1.5 \cdot (\ln 1.5 - 0.3) + 2.25 \cdot (1 - 2\ln 1.5) + \left.\frac{x^2}{10}\right|_1^{1.5} + \left. x \right|_{1.5}^{2.25}$$

$$= 0.2 + 1.5 \ln 1.5 - 0.45 + 2.25 - 4.5 \ln 1.5 + \frac{(2.25 - 1)}{10} + (2.25 - 1.5)$$

$$= \mathbf{2.875 - 3 \ln 1.5} \approx 1.6586.$$

OR

Since X is a non-negative random variable,

$$E(X) = \int_0^{\infty} (1 - F(x))\, dx = \int_0^1 1\, dx + \int_1^{1.5} \left(1 - \frac{x}{5}\right) dx + \int_{1.5}^{2.25} (1 - \ln x)\, dx$$

$$= 1 + \left.\left(x - \frac{x^2}{10}\right)\right|_1^{1.5} + \left.(2x - x \ln x)\right|_{1.5}^{2.25}$$

$$= 1 + \left(1.5 - \frac{1.5^2}{10}\right) - \left(1 - \frac{1^2}{10}\right) + (4.5 - 2.25 \ln 2.25) - (3 - 1.5 \ln 1.5)$$

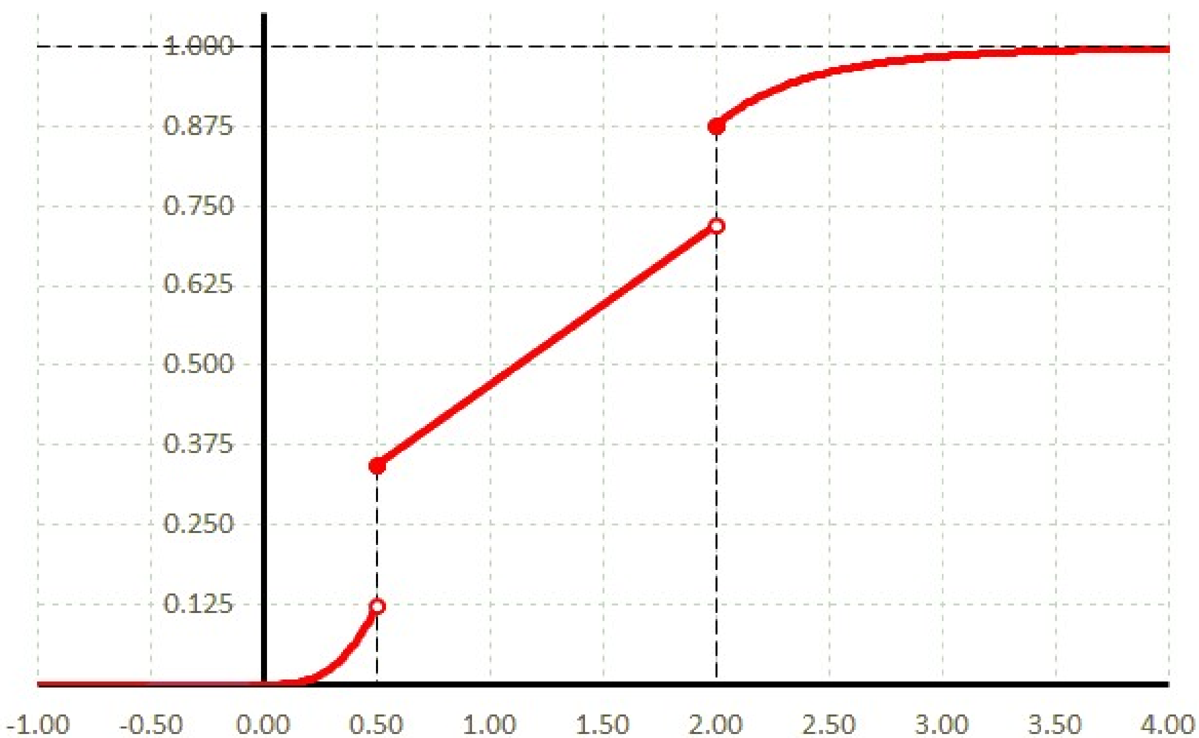
$$= 1 + 1.275 - 0.90 + (4.5 - 4.5 \ln 1.5) - (3 - 1.5 \ln 1.5)$$

$$= 1.375 + 1.5 - 3 \ln 1.5$$

$$= \mathbf{2.875 - 3 \ln 1.5} \approx 1.6586.$$

8. Consider a mixed random variable X with the cumulative distribution function

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \le x < 0.5 \\ \dfrac{8x+7}{32} & 0.5 \le x < 2 \\ 1 - \dfrac{4}{x^5} & x \ge 2 \end{cases}$$



a) Identify the discrete portion of the probability distribution.

$F_X(x)$ "jumps" at $x = 0.5$ from $\frac{1}{8}$ to $\frac{11}{32}$, size of the "jump" $= \frac{11}{32} - \frac{1}{8} = \frac{7}{32}$,

at $x = 2$ from $\frac{23}{32}$ to $\frac{7}{8}$, size of the "jump" $= \frac{7}{8} - \frac{23}{32} = \frac{5}{32}$.

$\Rightarrow$ Discrete portion of the probability distribution of X:

$$p(0.5) = \frac{7}{32} = 0.21875, \qquad\qquad p(2) = \frac{5}{32} = 0.15625.$$

b) Identify the continuous portion of the probability distribution.

Continuous portion of the probability distribution of X:

$$f(x) = F'(x) = \begin{cases} 0 & x < 0 \\ 3x^2 & 0 < x < 0.5 \\ \dfrac{1}{4} & 0.5 < x < 2 \\ \dfrac{20}{x^6} & x > 2 \end{cases}$$

c) Find $E(X)$.

$$E(X) = 0.5 \cdot \frac{7}{32} + 2 \cdot \frac{5}{32} + \int_0^{0.5} x \cdot 3x^2\, dx + \int_{0.5}^{2} x \cdot \frac{1}{4}\, dx + \int_2^{\infty} x \cdot \frac{20}{x^6}\, dx$$

$$= \frac{7}{64} + \frac{20}{64} + \frac{3}{4}x^4 \Big|_0^{0.5} + \frac{1}{8}x^2 \Big|_{0.5}^{2} + \left(-\frac{5}{x^4}\right) \Big|_2^{\infty}$$

$$= \frac{27}{64} + \frac{3}{64} + \left(\frac{1}{2} - \frac{1}{32}\right) + \left(0 - \frac{5}{16}\right) = \frac{80}{64} = \mathbf{\frac{5}{4}} = \mathbf{1.25}.$$

OR

Since X is a non-negative random variable,

$$\mathrm{E}(\mathrm{X}) = \int_0^{\infty} \left(1 - \mathrm{F}(x)\right) dx$$

$$= \int_0^{0.5} \left(1 - x^3\right) dx + \int_{0.5}^{2} \left(1 - \frac{8x+7}{32}\right) dx + \int_2^{\infty} \frac{4}{x^5}\, dx$$

$$= \left(x - \frac{1}{4} x^4\right) \Bigg|_0^{0.5} + \left(\frac{25}{32} x - \frac{1}{8} x^2\right) \Bigg|_{0.5}^{2} + \left(-\frac{1}{x^4}\right) \Bigg|_2^{\infty}$$

$$= \frac{31}{64} + \frac{45}{64} + \frac{1}{16} = \frac{80}{64} = \mathbf{\frac{5}{4}} = \mathbf{1.25}.$$

9.* The policyholder's loss, Y, follows a distribution with density function:

$$f(y) = \begin{cases} \dfrac{2}{y^3} & \text{if } y > 1 \\ 0 & \text{otherwise} \end{cases}$$

a) What is the expected value and the variance of the policyholder's loss?

$$\mathrm{E}(\text{Loss}) = \int_1^{\infty} y \cdot \frac{2}{y^3}\,dy = -\frac{2}{y}\Big|_1^{\infty} = \mathbf{2}.$$

$$\mathrm{E}(\text{Loss}^2) = \int_1^{\infty} y^2 \cdot \frac{2}{y^3}\,dy = 2\ln y\Big|_1^{\infty} \text{ is not finite.} \Rightarrow \mathrm{Var}(\text{Loss}) \text{ is not finite.}$$

b) An insurance policy reimburses a loss up to a benefit limit of 10. What is the expected value and the variance of the benefit paid under the insurance policy?

$$\text{The benefit paid under the insurance policy} = \begin{cases} y & \text{for } 1 < y \le 10 \\ 10 & \text{for } y \ge 10 \end{cases}$$

$$\mathrm{E}(\text{Benefit Paid}) = \int_1^{10} y \cdot \frac{2}{y^3}\,dy + \int_{10}^{\infty} 10 \cdot \frac{2}{y^3}\,dy = -\frac{2}{y}\Big|_1^{10} - \frac{10}{y^2}\Big|_{10}^{\infty} = \mathbf{1.9}.$$

$$\mathrm{E}(\text{Benefit Paid}^2) = \int_1^{10} y^2 \cdot \frac{2}{y^3}\,dy + \int_{10}^{\infty} 10^2 \cdot \frac{2}{y^3}\,dy = 2\ln y\Big|_1^{10} - \frac{100}{y^2}\Big|_{10}^{\infty}$$

$$= 2\ln 10 + 1.$$

$$\mathrm{Var}(\text{Benefit Paid}) = 2\ln 10 + 1 - 1.9^2 = \mathbf{2\ln 10 - 2.61} \approx \mathbf{1.99517}.$$

c) An insurance policy has a deductible of 2. What is the expected value and the variance of the benefit paid under the insurance policy?

$$\text{The benefit paid under the insurance policy} = \begin{cases} 0 & \text{for } 1 < y \le 2 \\ y-2 & \text{for } y \ge 2 \end{cases}$$

$$\mathrm{E}(\text{Benefit Paid}) = \int_2^\infty (y-2)\cdot\frac{2}{y^3}\,dy = \int_2^\infty \frac{2}{y^2}\,dy - \int_2^\infty \frac{4}{y^3}\,dy$$

$$= -\frac{2}{y}\Big|_2^\infty + \frac{2}{y^2}\Big|_2^\infty = \mathbf{0.5}.$$

$$\mathrm{E}(\text{Benefit Paid}^2) = \int_2^\infty (y-2)^2\cdot\frac{2}{y^3}\,dy \quad \text{is not finite.}$$

$\Rightarrow$ $\mathrm{Var}(\text{Benefit Paid})$ is not finite.