STAT 410

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Examples for Week 01 (Disc4)

STAT 410 does NOT have a discussion section.

STAT 400 (an easier class) DOES have a discussion section.
STAT 410 (a harder class) does NOT.

There is a simple explanation for this: About 20 years ago, there was only one section of STAT 410 with about 40 students and only two lectures of STAT 400 with about 150 students each (each divided into 3 discussion sections of about 50 students each). The number of students has changed a little bit since then, the course structure has not (yet)…

However, don't you wish there was a discussion section in STAT 410?

If STAT 410 did have a discussion section, this is what your TA would most likely go over during it:

1. Let X have the pdf $f(x) = 4x^3$, for $0 < x < 1$, zero elsewhere.

a) **1.7.25** (8th edition) **1.7.23** (7th and 6th edition)

Find the cdf and the pdf of $Y = -\ln X^4$.

b) Let $Y = e^X$. Find the probability distribution of Y.

c) Let $Y = X^2$. Find the probability distribution of Y.

d) Let $Y = \sqrt{X}$. Find the probability distribution of Y.

2. **1.7.26** (8th edition) **1.7.24** (7th and 6th edition)

Let $f(x) = \frac{1}{3}$, $-1 < x < 2$, zero elsewhere, be the p.d.f. of X. Find the c.d.f. and the p.d.f. of $Y = X^2$.

Hint: Consider $P(X^2 \le y)$ for two cases: $0 \le y < 1$ and $1 \le y < 4$.

3. **1.8.9** (8th edition) **1.8.8** (7th edition) **1.8.10** (6th edition) + (a½)

Let $f(x) = 2x,\ 0 < x < 1,$ zero elsewhere, be the p.d.f. of X.

a) Compute $\mathrm{E}(\frac{1}{\mathrm{X}})$.

a½) Compute $\mathrm{E}(\mathrm{X})$. Does $\mathrm{E}(\frac{1}{\mathrm{X}})$ equal $\frac{1}{\mathrm{E}(\mathrm{X})}$?

b) Find the c.d.f. and the p.d.f. of $\mathrm{Y} = \frac{1}{\mathrm{X}}$.

c) Compute $\mathrm{E}(\mathrm{Y})$ and compare this result with the answer obtained in part (a).

4. **1.7.24** (8th edition) **1.7.22** (7th and 6th edition)

Let X have the uniform pdf $f_{\mathrm{X}}(x) = \frac{1}{\pi}$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the pdf of $\mathrm{Y} = \tan \mathrm{X}$. This is the pdf of a **Cauchy distribution**.

5. Let X be a random variable with the probability density function

$$f_{\mathrm{X}}(x) = \frac{11 - 2x}{50}, \qquad -2 < x < 3, \qquad \text{zero otherwise.}$$

Find the probability distribution of $\mathrm{Y} = \mathrm{X}^2$.

6. Consider a continuous random variable X with the probability density function

$$f_{\mathrm{X}}(x) = \frac{3 - x}{8}, \qquad -1 \le x \le 3, \qquad \text{zero elsewhere.}$$

Consider $\mathrm{Y} = g(\mathrm{X}) = \frac{9}{\mathrm{X}^2}$. Find the probability distribution of Y.

7. Suppose a random variable X has the following probability density function:

$$f_X(x) = \begin{cases} x e^x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the moment-generating function of X, $M_X(t)$.

b) Let $Y = e^X$. Find the probability distribution of Y.

c) Find $\mathrm{Var}(Y)$.

Answers:

1. Let X have the pdf $f(x) = 4x^3$, for $0 < x < 1$, zero elsewhere.

a) **1.7.25** (8th edition) **1.7.23** (7th and 6th edition)

Find the cdf and the pdf of $Y = -\ln X^4$.

$F_X(x) = x^4, \quad 0 < x < 1.$

$0 < x < 1 \qquad y = -4\ln x \qquad \Rightarrow \qquad y > 0$

$F_Y(y) = P(Y \le y) = P(-4\ln X \le y) = P(X \ge e^{-y/4}) = 1 - e^{-y}, \qquad y > 0.$

$\Rightarrow \quad f_Y(y) = F_Y'(y) = e^{-y}, \qquad y > 0.$

$\Rightarrow$ Y has Exponential distribution with mean 1.

OR

$y = g(x) = -4\ln x \qquad \Rightarrow \qquad x = g^{-1}(y) = = e^{-y/4}$

$\Rightarrow \qquad dx/dy = -\frac{1}{4} e^{-y/4}$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = 4\left(e^{-y/4}\right)^3 \times \left| -\frac{1}{4} e^{-y/4} \right| = e^{-y}, \qquad y > 0.$$

$\Rightarrow$ Y has Exponential distribution with mean 1.

OR

$$M_Y(t) = E(e^{Y \cdot t}) = E(e^{-4\ln X \cdot t}) = E(X^{-4t}) = \int_0^1 \left(x^{-4t} \cdot 4x^3\right)dx$$

$$= \int_0^1 4x^{3-4t}\,dx = \frac{4}{4-4t} = \frac{1}{1-t}, \qquad t < 1.$$

$\Rightarrow$ Y has Exponential distribution with mean 1.

b) Let $Y = e^X$. Find the probability distribution of Y.

$0 < x < 1 \qquad y = e^x \qquad \Rightarrow \qquad 1 < y < e.$

$y = g(x) = e^x \qquad \Rightarrow \qquad x = g^{-1}(y) = \ln y$

$\Rightarrow \qquad dx/dy = \dfrac{1}{y}$

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right| = 4(\ln y)^3 \times \left|\frac{1}{y}\right| = \frac{4}{y}(\ln y)^3, \qquad 1 < y < e.$$

OR

$F_X(x) = x^4, \qquad 0 < x < 1.$

$$F_Y(y) = P(Y \le y) = P(e^X \le y) = P(X \le \ln y) = (\ln y)^4, \qquad 1 < y < e.$$

$$\Rightarrow \quad f_Y(y) = F_Y'(y) = 4(\ln y)^3 \times \frac{1}{y} = \frac{4}{y}(\ln y)^3, \qquad 1 < y < e.$$

c) Let $Y = X^2$. Find the probability distribution of Y.

$$F_X(x) = x^4, \qquad 0 < x < 1. \qquad\qquad 0 < x < 1 \quad y = x^2 \quad \Rightarrow \quad 0 < y < 1.$$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y}\,) = F_X(\sqrt{y}\,) = y^2, \quad 0 < y < 1.$$

$$\Rightarrow \quad f_Y(y) = F_Y'(y) = 2y, \qquad 0 < y < 1.$$

OR

$$g(x) = x^2 \qquad\qquad g^{-1}(y) = \sqrt{y} = y^{1/2} \qquad\qquad dx/dy = \frac{1}{2}\, y^{-1/2}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = (4y^{3/2}) \left| \frac{1}{2}\, y^{-1/2} \right| = 2y, \qquad 0 < y < 1.$$

d) Let $Y = \sqrt{X}$. Find the probability distribution of Y.

$$F_X(x) = x^4, \qquad 0 < x < 1. \qquad\qquad 0 < x < 1 \quad y = \sqrt{x} \quad \Rightarrow \quad 0 < y < 1.$$

$$F_Y(y) = P(Y \le y) = P(\sqrt{X} \le y) = P(X \le y^2) = F_X(y^2) = y^8, \quad 0 < y < 1.$$

$$\Rightarrow \quad f_Y(y) = F_Y'(y) = 8y^7, \qquad 0 < y < 1.$$

OR

$$g(x) = \sqrt{x} \qquad\qquad g^{-1}(y) = y^2 \qquad\qquad dx/dy = 2y$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = = (4y^6) \left| 2y \right| = 8y^7, \qquad 0 < y < 1.$$

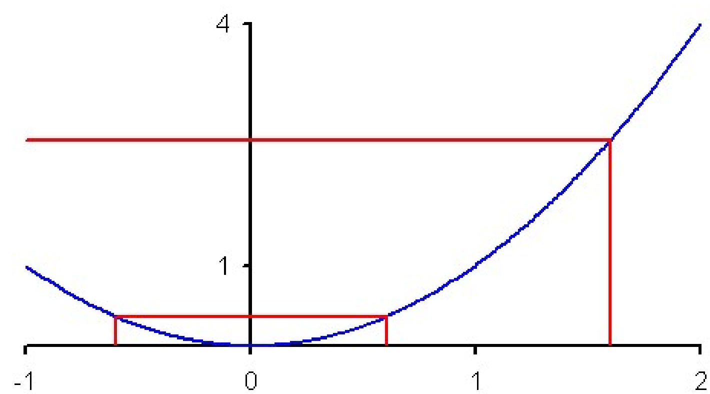
2. **1.7.26** (8th edition) **1.7.24** (7th and 6th edition)

Let $f(x) = \frac{1}{3}$, $-1 < x < 2$, zero elsewhere, be the p.d.f. of X. Find the c.d.f. and the p.d.f. of $Y = X^2$.

Hint: Consider $P(X^2 \le y)$ for two cases: $0 \le y < 1$ and $1 \le y < 4$.

$$f_X(x) = \begin{cases} 1/3 & -1 < x < 2 \\ 0 & \text{o.w.} \end{cases} \qquad F_X(x) = \begin{cases} 0 & x < -1 \\ \dfrac{x+1}{3} & -1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

$Y = X^2$



$0 \le y < 1$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{3}\,dx = \frac{2}{3}\sqrt{y}.$$

$1 \le y < 4$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-1 \le X \le \sqrt{y})$$

$$= \int_{-1}^{\sqrt{y}} \frac{1}{3}\,dx = \frac{1}{3}\left(\sqrt{y} + 1\right).$$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{2}{3}\sqrt{y} & 0 \le y < 1 \\ \frac{1}{3}\left(\sqrt{y}+1\right) & 1 \le y < 4 \\ 1 & y \ge 4 \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{3\sqrt{y}} & 0 < y < 1 \\ \frac{1}{6\sqrt{y}} & 1 < y < 4 \\ 0 & \text{o.w.} \end{cases}$$

OR

$-1 < x < 0$	$0 < x < 2$
$Y = g(X) = X^2$	$Y = g(X) = X^2$
$x = -\sqrt{y} = g^{-1}(y)$	$x = \sqrt{y} = g^{-1}(y)$
$\frac{dx}{dy} = -\frac{1}{2\sqrt{y}}$	$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$
$1 > y > 0$	$0 < y < 4$
$f_X(g^{-1}(y)) \times \left\lvert \frac{dx}{dy} \right\rvert$	$f_X(g^{-1}(y)) \times \left\lvert \frac{dx}{dy} \right\rvert$
$\frac{1}{3} \times \left\lvert -\frac{1}{2\sqrt{y}} \right\rvert = \frac{1}{6\sqrt{y}}$	$\frac{1}{3} \times \left\lvert \frac{1}{2\sqrt{y}} \right\rvert = \frac{1}{6\sqrt{y}}$

$$f_Y(y) = \frac{1}{6\sqrt{y}} + \frac{1}{6\sqrt{y}} = \frac{1}{3\sqrt{y}}, \qquad 0 < y < 1.$$

$$f_Y(y) = 0 + \frac{1}{6\sqrt{y}} = \frac{1}{6\sqrt{y}}, \qquad 1 < y < 4.$$

3. **1.8.9** (8th edition) **1.8.8** (7th edition) **1.8.10** (6th edition) + (a½)

Let $f(x) = 2x,\ 0 < x < 1,$ zero elsewhere, be the p.d.f. of X.

a) Compute $\mathrm{E}(\frac{1}{\mathrm{X}})$.

$$\mathrm{E}(\frac{1}{\mathrm{X}}) = \int_0^1 \frac{1}{x} \cdot 2x\,dx = \int_0^1 2\,dx = \mathbf{2}.$$

a½) Compute $\mathrm{E}(\mathrm{X})$. Does $\mathrm{E}(\frac{1}{\mathrm{X}})$ equal $\frac{1}{\mathrm{E}(\mathrm{X})}$?

$$\mathrm{E}(\mathrm{X}) = \int_0^1 x \cdot 2x\,dx = \int_0^1 2x^2\,dx = \mathbf{\frac{2}{3}}. \qquad \mathrm{E}(\frac{1}{\mathrm{X}}) \neq \frac{1}{\mathrm{E}(\mathrm{X})}.$$

b) Find the c.d.f. and the p.d.f. of $\mathrm{Y} = \frac{1}{\mathrm{X}}$.

$0 < x < 1 \qquad \mathrm{Y} = {}^{1}/_{\mathrm{X}} \qquad \Rightarrow \qquad y > 1.$

$\mathrm{g}(x) = {}^{1}/_{x} \qquad \mathrm{g}^{-1}(y) = {}^{1}/_{y} = y^{-1} \qquad {}^{dx}/_{dy} = -y^{-2}$

$$f_{\mathrm{Y}}(y) = f_{\mathrm{X}}(\mathrm{g}^{-1}(y)) \left| \frac{dx}{dy} \right| = (2y^{-1})(y^{-2}) = 2y^{-3}, \qquad y > 1.$$

OR

$$\mathrm{F_Y}(y) = \mathrm{P}(\mathrm{Y} \le y) = \mathrm{P}({}^{1}/_{\mathrm{X}} \le y) = \mathrm{P}(\mathrm{X} \ge {}^{1}/_{y}) = 1 - \mathrm{P}(\mathrm{X} < {}^{1}/_{y})$$
$$= 1 - \mathrm{F_X}({}^{1}/_{y}) = 1 - {}^{1}/_{y^2}, \qquad y > 1.$$

$$\mathrm{F_Y}(y) = \begin{cases} 0 & y < 1 \\ 1 - {}^{1}/_{y^2} & y \ge 1 \end{cases} \qquad f_{\mathrm{Y}}(y) = \begin{cases} 0 & y < 1 \\ {}^{2}/_{y^3} & y \ge 1 \end{cases}$$

c) Compute $\mathrm{E}(\mathrm{Y})$ and compare this result with the answer obtained in part (a).

$$\mathrm{E}(\mathrm{Y}) = \int_1^\infty y \cdot \frac{2}{y^3}\,dy = \int_1^\infty \frac{2}{y^2}\,dy = \left(-\frac{2}{y}\right)\Bigg|_1^\infty = \mathbf{2}.$$

Same. ☺

4. **1.7.24** (8th edition) **1.7.22** (7th and 6th edition)

Let X have the uniform pdf $f_X(x) = \frac{1}{\pi}$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the pdf of Y = tan X. This is the pdf of a **Cauchy distribution**.

$$f_X(x) = \begin{cases} \frac{1}{\pi} & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & \text{o.w.} \end{cases} \qquad F_X(x) = \begin{cases} 0 & x < -\frac{\pi}{2} \\ \frac{x}{\pi} + \frac{1}{2} & -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ 1 & x \ge \frac{\pi}{2} \end{cases}$$

$$F_Y(y) = P(Y \le y) = P(\tan X \le y) = P(X \le \arctan(y)) = \frac{1}{\pi}\arctan(y) + \frac{1}{2},$$

$$-\infty < y < \infty.$$

$$f_Y(y) = \frac{1}{\pi\left(1+y^2\right)}, \qquad -\infty < y < \infty. \qquad \text{(Standard) Cauchy distribution.}$$

OR

$$g(x) = \tan x \qquad g^{-1}(y) = \arctan(y) \qquad {}^{dx}\!/_{dy} = \frac{1}{1+y^2}$$

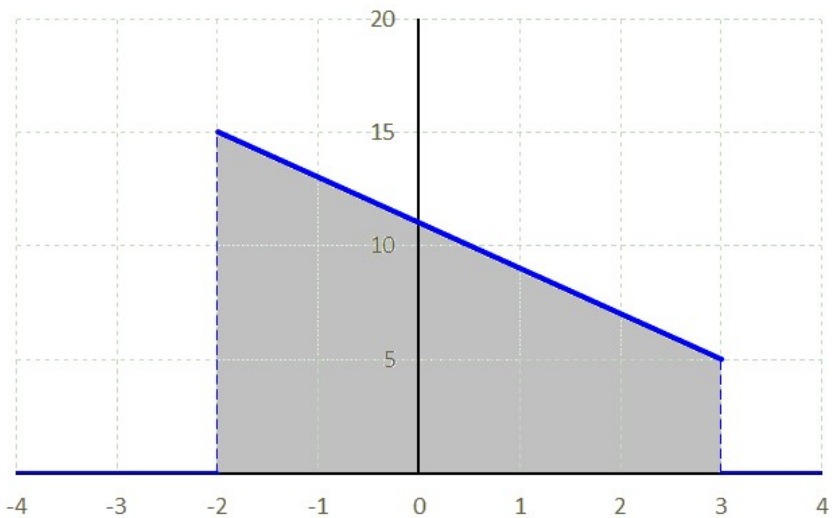
$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right| = \left(\frac{1}{\pi}\right)\left(\frac{1}{1+y^2}\right) = \frac{1}{\pi\left(1+y^2\right)}, \qquad -\infty < y < \infty.$$

$$F_Y(y) = \int_{-\infty}^{y} \frac{1}{\pi\left(1+u^2\right)} du = \frac{1}{\pi}\arctan(y) + \frac{1}{2}, \qquad -\infty < y < \infty.$$

5. Let X be a random variable with the probability density function

$$f_X(x) = \frac{11-2x}{50}, \qquad -2 < x < 3, \qquad \text{zero otherwise.}$$

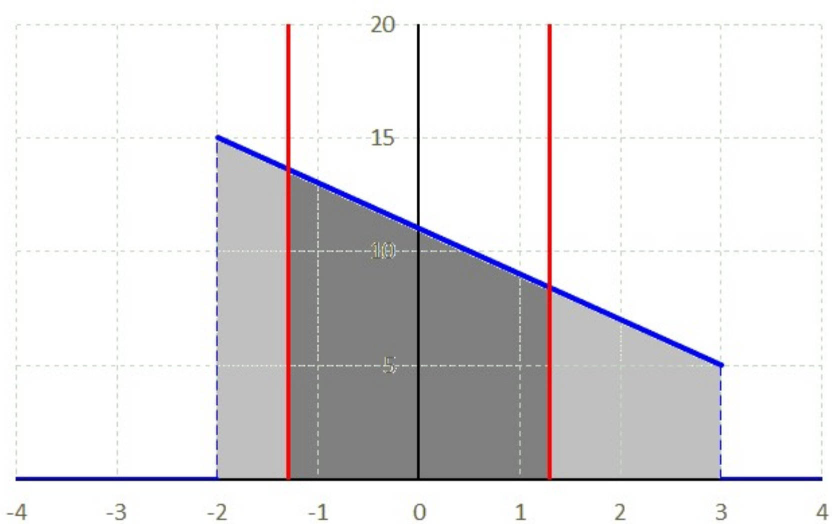
Find the probability distribution of $Y = X^2$.



$y < 0$ $\qquad P(X^2 \le y) = 0 \qquad F_Y(y) = 0.$

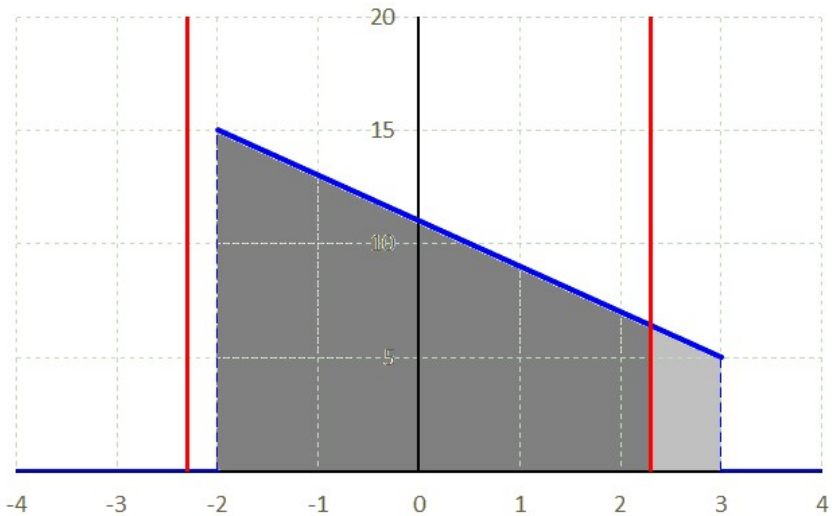
$y \ge 0$ $\qquad F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}\,)$

Case 1: $\quad 0 \le y < 4 \quad \Rightarrow \quad 0 \le \sqrt{y} < 2 \quad \Rightarrow \quad -2 < -\sqrt{y} \le \sqrt{y} < 3$



$$F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{11-2x}{50}\,dx = \left.\frac{11x - x^2}{50}\right|_{-\sqrt{y}}^{\sqrt{y}} = \frac{11\sqrt{y}}{25}, \qquad 0 \le y < 4.$$

Case 2: $4 \le y < 9 \quad \Rightarrow \quad 2 \le \sqrt{y} < 3 \quad \Rightarrow \quad -\sqrt{y} \le -2 < \sqrt{y} < 3$



$$F_Y(y) = \int_{-2}^{\sqrt{y}} \frac{11-2x}{50}\,dx = \left.\frac{11x - x^2}{50}\right|_{-2}^{\sqrt{y}} = \frac{26 + 11\sqrt{y} - y}{50}, \qquad 4 \le y < 9.$$

$y \ge 9 \qquad F_Y(y) = 1.$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \dfrac{11\sqrt{y}}{25} & 0 \le y < 4 \\ \dfrac{26 + 11\sqrt{y} - y}{50} & 4 \le y < 9 \\ 1 & y \ge 9 \end{cases}$$

$$f_Y(y) = F'_Y(y)$$

$$f_Y(y) = \begin{cases} \dfrac{11}{50\sqrt{y}} & 0 < y < 4 \\ \dfrac{11 - 2\sqrt{y}}{100\sqrt{y}} & 4 < y < 9 \\ 0 & \text{otherwise} \end{cases}$$

OR

$-2 < x < 0$

$$f_X(x) = \frac{11-2x}{50}$$

$$Y = g(X) = X^2$$

$$x = -\sqrt{y} = g^{-1}(y)$$

$$\frac{dx}{dy} = -\frac{1}{2\sqrt{y}}$$

$0 < y < 4$

$$f_X(g^{-1}(y))\left|\frac{dx}{dy}\right|$$

$$\frac{11+2\sqrt{y}}{50}\cdot\left|-\frac{1}{2\sqrt{y}}\right| = \frac{11+2\sqrt{y}}{100\sqrt{y}}$$

$0 < x < 3$

$$f_X(x) = \frac{11-2x}{50}$$

$$Y = g(X) = X^2$$

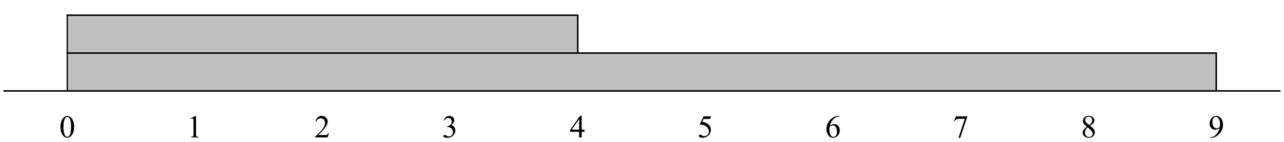
$$x = \sqrt{y} = g^{-1}(y)$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

$0 < y < 9$

$$f_X(g^{-1}(y))\left|\frac{dx}{dy}\right|$$

$$\frac{11-2\sqrt{y}}{50}\cdot\left|\frac{1}{2\sqrt{y}}\right| = \frac{11-2\sqrt{y}}{100\sqrt{y}}$$



$$f_Y(y) = \begin{cases} \dfrac{11+2\sqrt{y}}{100\sqrt{y}} + \dfrac{11-2\sqrt{y}}{100\sqrt{y}} & 0 < y < 4 \\ 0 + \dfrac{11-2\sqrt{y}}{100\sqrt{y}} & 4 < y < 9 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \dfrac{11}{50\sqrt{y}} & 0 < y < 4 \\ \dfrac{11-2\sqrt{y}}{100\sqrt{y}} & 4 < y < 9 \\ 0 & \text{otherwise} \end{cases}$$

OR

$$F_X(x) = 0, \qquad x < -2.$$

$$F_X(x) = \int_{-2}^{x} \frac{11-2u}{50}\,du = \left.\frac{11u-u^2}{50}\right|_{-2}^{x} = \frac{11x-x^2+26}{50}, \qquad -2 \le x < 3.$$

$$F_X(x) = 1, \qquad x \ge 3.$$

$y < 0$ $\qquad P(X^2 \le y) = 0 \qquad F_Y(y) = 0.$

$y \ge 0$ $\qquad F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$= \begin{cases} \dfrac{11\sqrt{y}-y+26}{50} - \dfrac{-11\sqrt{y}-y+26}{50} & 0 \le y < 4 \\[2ex] \dfrac{11\sqrt{y}-y+26}{50} - 0 & 4 \le y < 9 \\[2ex] 1-0 & y \ge 9 \end{cases}$$

$$= \begin{cases} \dfrac{11\sqrt{y}}{25} & 0 \le y < 4 \\[2ex] \dfrac{11\sqrt{y}-y+26}{50} & 4 \le y < 9 \\[2ex] 1 & y \ge 9 \end{cases}$$

6. Consider a continuous random variable X with the probability density function

$$f_X(x) = \frac{3-x}{8}, \qquad -1 \le x \le 3, \qquad \text{zero elsewhere.}$$

Consider $Y = g(X) = \dfrac{9}{X^2}$. Find the probability distribution of Y.

$-1 \le x \le 3$	$-1 \le x < 0$	$0 < x \le 3$
$y = \dfrac{9}{x^2}$	$9 \le y < \infty$	$\infty > y \ge 1$

$y < 1$

$$F_Y(y) = P(Y \le y) = P\left(\frac{9}{X^2} \le y\right) = P\left(X^2 \ge \frac{9}{y}\right) = 0.$$

$y \ge 1$

$$F_Y(y) = P(Y \le y) = P\left(\frac{9}{X^2} \le y\right) = P\left(X^2 \ge \frac{9}{y}\right)$$

$$= P\left(X \le -\frac{3}{\sqrt{y}}\right) + P\left(X \ge \frac{3}{\sqrt{y}}\right)$$

$$= F_X\left(-\frac{3}{\sqrt{y}}\right) + 1 - F_X\left(\frac{3}{\sqrt{y}}\right).$$

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(u)\,du = \int_{-1}^{x} \frac{3-u}{8}\,du = \left.\frac{6u - u^2}{16}\right|_{-1}^{x}$$

$$= \frac{7 + 6x - x^2}{16}, \qquad -1 \le x < 3.$$

Obviously, $\quad F_X(x) = 0, \quad x < -1, \qquad F_X(x) = 1, \quad x \ge 3.$

Case 1: $1 \le y < 9$ $3 \ge \dfrac{3}{\sqrt{y}} > 1.$

$$F_Y(y) = F_X\left(-\frac{3}{\sqrt{y}}\right) + 1 - F_X\left(\frac{3}{\sqrt{y}}\right) = 0 + 1 - \frac{7 + \frac{18}{\sqrt{y}} - \frac{9}{y}}{16}$$

$$= \frac{9 - \frac{18}{\sqrt{y}} + \frac{9}{y}}{16} = \frac{9y - 18\sqrt{y} + 9}{16y} = \frac{9\left(\sqrt{y} - 1\right)^2}{16y}, \qquad 1 \le y < 9.$$

Case 2: $9 \le y < \infty$ $1 \ge \dfrac{3}{\sqrt{y}} > 0.$

$$F_Y(y) = F_X\left(-\frac{3}{\sqrt{y}}\right) + 1 - F_X\left(\frac{3}{\sqrt{y}}\right) = \frac{7 - \frac{18}{\sqrt{y}} - \frac{9}{y}}{16} + 1 - \frac{7 + \frac{18}{\sqrt{y}} - \frac{9}{y}}{16}$$

$$= 1 - \frac{9}{4\sqrt{y}}, \qquad 9 \le y < \infty.$$

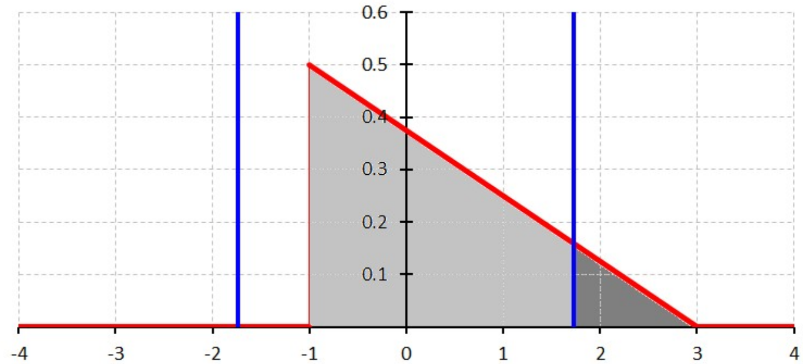
c.d.f. $$F_Y(y) = \begin{cases} 0 & y < 1 \\ \dfrac{9y - 18\sqrt{y} + 9}{16y} & 1 \le y < 9 \\ 1 - \dfrac{9}{4\sqrt{y}} & y \ge 9 \end{cases}$$

p.d.f. $$f_Y(y) = \begin{cases} 0 & y < 1 \\ \dfrac{9}{16y^{1.5}} - \dfrac{9}{16y^2} & 1 < y < 9 \\ \dfrac{9}{8y^{1.5}} & y > 9 \end{cases} = \begin{cases} 0 & y < 1 \\ \dfrac{9\left(\sqrt{y} - 1\right)}{16y^2} & 1 < y < 9 \\ \dfrac{9}{8y^{1.5}} & y > 9 \end{cases}$$

OR

Case 1: $1 \le y < 9$

$\Rightarrow \quad 3 \ge \dfrac{3}{\sqrt{y}} > 1.$



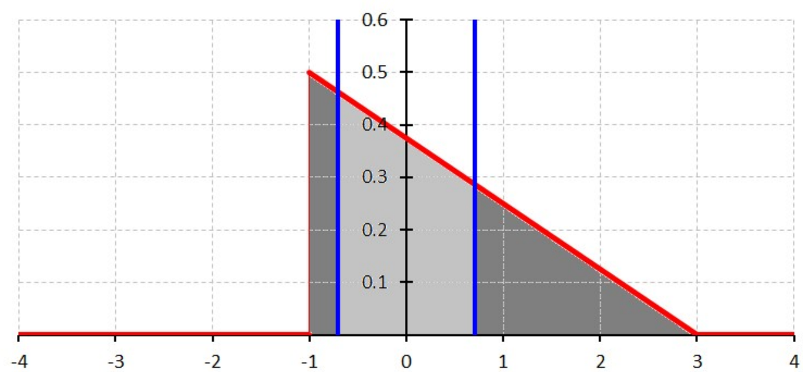
$$F_Y(y) = P(X \le -\frac{3}{\sqrt{y}}) + P(X \ge \frac{3}{\sqrt{y}})$$

$$= 0 + \int_{\frac{3}{\sqrt{y}}}^{3} \frac{3-x}{8}\,dx = \frac{6x - x^2}{16}\Bigg|_{\frac{3}{\sqrt{y}}}^{3} = \frac{9 - \frac{18}{\sqrt{y}} + \frac{9}{y}}{16}$$

$$= \frac{9y - 18\sqrt{y} + 9}{16y} = \frac{9\left(\sqrt{y}-1\right)^2}{16y}, \qquad 1 \le y < 9.$$

Case 2: $9 \le y < \infty$

$\Rightarrow \quad 1 \ge \dfrac{3}{\sqrt{y}} > 0.$



$$F_Y(y) = P(X \le -\frac{3}{\sqrt{y}}) + P(X \ge \frac{3}{\sqrt{y}})$$

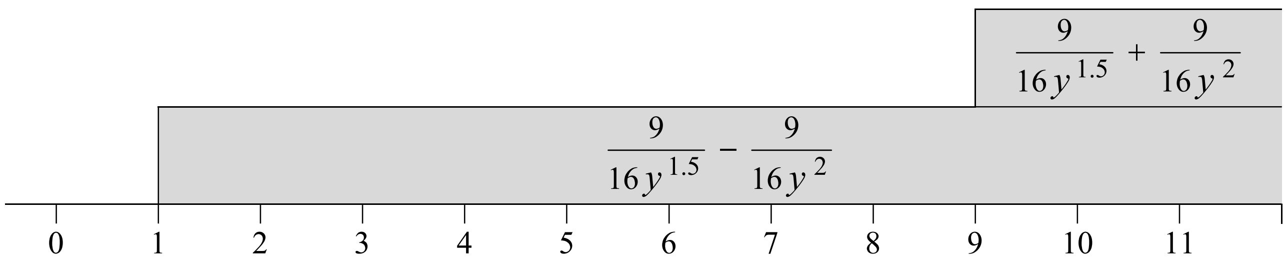
$$= \int_{-1}^{-\frac{3}{\sqrt{y}}} \frac{3-x}{8}\,dx + \int_{\frac{3}{\sqrt{y}}}^{3} \frac{3-x}{8}\,dx$$

$$= \left.\frac{6x-x^2}{16}\right|_{-1}^{-\frac{3}{\sqrt{y}}} + \left.\frac{6x-x^2}{16}\right|_{\frac{3}{\sqrt{y}}}^{3}$$

$$= \frac{-\frac{18}{\sqrt{y}} - \frac{9}{y} + 7}{16} + \frac{9 - \frac{18}{\sqrt{y}} + \frac{9}{y}}{16} = 1 - \frac{9}{4\sqrt{y}}, \qquad 9 \le y < \infty.$$

OR

$-1 \le x < 0$	$0 < x \le 3$
$Y = g(X) = \dfrac{9}{X^2}$	$Y = g(X) = \dfrac{9}{X^2}$
$x = -\dfrac{3}{\sqrt{y}} = g^{-1}(y)$	$x = \dfrac{3}{\sqrt{y}} = g^{-1}(y)$
$\dfrac{dx}{dy} = \dfrac{3}{2y^{1.5}}$	$\dfrac{dx}{dy} = -\dfrac{3}{2y^{1.5}}$
$9 \le y < \infty$	$\infty > y \ge 1$
$f_X(g^{-1}(y)) \left\lvert \dfrac{dx}{dy} \right\rvert$	$f_X(g^{-1}(y)) \left\lvert \dfrac{dx}{dy} \right\rvert$
$\dfrac{3 + \frac{3}{\sqrt{y}}}{8} \times \left\lvert \dfrac{3}{2y^{1.5}} \right\rvert = \dfrac{9}{16y^{1.5}} + \dfrac{9}{16y^2}$	$\dfrac{3 - \frac{3}{\sqrt{y}}}{8} \times \left\lvert -\dfrac{3}{2y^{1.5}} \right\rvert = \dfrac{9}{16y^{1.5}} - \dfrac{9}{16y^2}$



$$f_Y(y) = \frac{9}{16y^{1.5}} - \frac{9}{16y^2}, \qquad 1 < y < 9, \qquad\qquad f_Y(y) = \frac{9}{8y^{1.5}}, \qquad 9 < y < \infty.$$

p.d.f.

$$f_Y(y) = \begin{cases} 0 & y < 1 \\ \dfrac{9}{16y^{1.5}} - \dfrac{9}{16y^2} & 1 < y < 9 \\ \dfrac{9}{8y^{1.5}} & y > 9 \end{cases} = \begin{cases} 0 & y < 1 \\ \dfrac{9\left(\sqrt{y}-1\right)}{16y^2} & 1 < y < 9 \\ \dfrac{9}{8y^{1.5}} & y > 9 \end{cases}$$

7. Suppose a random variable X has the following probability density function:

$$f_X(x) = \begin{cases} x e^x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the moment-generating function of X, $M_X(t)$.

$$M_X(t) = \int_0^1 e^{tx} \cdot x e^x \, dx = \int_0^1 x e^{(t+1)x} \, dx$$

$$= \left[\frac{1}{t+1} x e^{(t+1)x} - \frac{1}{(t+1)^2} e^{(t+1)x} \right] \Bigg|_0^1$$

$$= \frac{1}{t+1} e^{t+1} - \frac{1}{(t+1)^2} e^{t+1} + \frac{1}{(t+1)^2}$$

$$= \frac{t e^{t+1} + 1}{(t+1)^2}, \qquad t \neq -1.$$

$$M_X(-1) = \int_0^1 x \, dx = \frac{1}{2}.$$

b) Let $Y = e^X$. Find the probability distribution of Y.

$$y = g(x) = e^x \qquad x = g^{-1}(y) = \ln y \qquad \frac{dx}{dy} = \frac{1}{y}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = (\ln y \cdot y) \cdot \frac{1}{y} = \ln y, \qquad 1 < y < e.$$

OR

$$F_X(x) = 1 + x\,e^x - e^x, \qquad 0 < x < 1.$$

$$F_Y(y) = P(Y \le y) = P(e^X \le y) = P(X \le \ln y) = F_X(\ln y) = 1 + \ln y \cdot y - y,$$

$$1 < y < e.$$

$$f_Y(y) = F_Y'(y) = \ln y, \qquad 1 < y < e.$$

c) Find $\mathrm{Var}(Y)$.

$$E(Y) = E(e^X) = M_X(1) = \frac{e^2+1}{4}.$$

$$E(Y^2) = E[(e^X)^2] = E(e^{2X}) = M_X(2) = \frac{2e^3+1}{9}.$$

$$\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{2e^3+1}{9} - \left(\frac{e^2+1}{4}\right)^2 \approx 0.176.$$

OR

$$\mathrm{Var}(Y) = \int_1^e y^2 \cdot \ln y\,dy - \left(\int_1^e y \cdot \ln y\,dy\right)^2 = \ldots \quad \text{by parts} \quad \ldots$$