STAT 400 Review (2)

A. Stepanov

Let X and Y be two discrete random variables. The **joint probability mass function** $p(x, y)$ is defined for each pair of numbers (x, y) by

$$p(x, y) = \mathrm{P}(\mathrm{X} = x \text{ and } \mathrm{Y} = y).$$

Let A be any set consisting of pairs of (x, y) values. Then

$$\mathrm{P}\big((\mathrm{X}, \mathrm{Y}) \in A\big) = \sum_{(x,y)\in A}\sum p(x, y).$$

Let X and Y be two continuous random variables. Then $f(x, y)$ is the **joint probability density function** for X and Y if for any two-dimensional set A

$$\mathrm{P}\big((\mathrm{X}, \mathrm{Y}) \in A\big) = \iint_A f(x, y)\, dx\, dy.$$

1. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

x \ y	0	1	2	
1	0.15	0.10	0	
2	0.25	0.30	0.20	

a) Find $\mathrm{P}(\mathrm{X} + \mathrm{Y} = 2)$.

$\mathrm{P}(\mathrm{X} + \mathrm{Y} = 2) = p(1, 1) + p(2, 0) = 0.10 + 0.25 = \mathbf{0.35}$.

b) Find $\mathrm{P}(\mathrm{X} > \mathrm{Y})$.

$\mathrm{P}(\mathrm{X} > \mathrm{Y}) = p(1, 0) + p(2, 0) + p(2, 1) = 0.15 + 0.25 + 0.30 = \mathbf{0.70}$.

The **marginal probability mass functions** of X and of Y are given by

$$p_{\mathrm{X}}(x) = \sum_{\text{all } y} p(x, y), \qquad p_{\mathrm{Y}}(y) = \sum_{\text{all } x} p(x, y).$$

The **marginal probability density functions** of X and of Y are given by

$$f_{\mathrm{X}}(x) = \int_{-\infty}^{\infty} f(x, y)\, dy, \qquad f_{\mathrm{Y}}(y) = \int_{-\infty}^{\infty} f(x, y)\, dx.$$

c) Find the (marginal) probability distributions $p_X(x)$ of X and $p_Y(y)$ of Y.

x	$p_X(x)$
1	0.25
2	0.75

y	$p_Y(y)$
0	0.40
1	0.40
2	0.20

If $p(x, y)$ is the joint probability mass function of (X, Y) OR $f(x, y)$ is the joint probability density function of (X, Y), then

discrete continuous

$$E(g(X, Y)) = \sum_{\text{all } x} \sum_{\text{all } y} g(x, y) \cdot p(x, y) \qquad E(g(X, Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \cdot f(x, y)\, dx\, dy$$

d) Find $E(X), E(Y), E(X+Y), E(X \cdot Y)$.

$E(X) = 1 \times 0.25 + 2 \times 0.75 = \mathbf{1.75}$.

$E(Y) = 0 \times 0.40 + 1 \times 0.40 + 2 \times 0.20 = \mathbf{0.8}$.

$E(X+Y) = 1 \times 0.15 + 2 \times 0.25 + 2 \times 0.10 + 3 \times 0.30 + 3 \times 0 + 4 \times 0.20 = \mathbf{2.55}$.

OR

$E(X+Y) = E(X) + E(Y) = 1.75 + 0.8 = \mathbf{2.55}$.

$E(X \cdot Y) = 0 \times 0.15 + 0 \times 0.25 + 1 \times 0.10 + 2 \times 0.30 + 2 \times 0 + 4 \times 0.20 = \mathbf{1.5}$.

Moment-generating function

$$M_{X,Y}(t_1, t_2) = E\left(e^{t_1 X + t_2 Y}\right), \qquad \text{if it exists for } |t_1| < h_1, |t_2| < h_2.$$

$$M_{X,Y}(t_1, 0) = M_X(t_1), \qquad M_{X,Y}(0, t_2) = M_Y(t_2).$$

e) Find the moment-generating function $M_{X,Y}(t_1, t_2)$.

$$M_{X,Y}(t_1, t_2) = 0.15\, e^{t_1} + 0.25\, e^{2t_1} + 0.10\, e^{t_1 + t_2} + 0.30\, e^{2t_1 + t_2} + 0.20\, e^{2t_1 + 2t_2}.$$

1.5. Consider two random variables X and Y with the moment-generating function

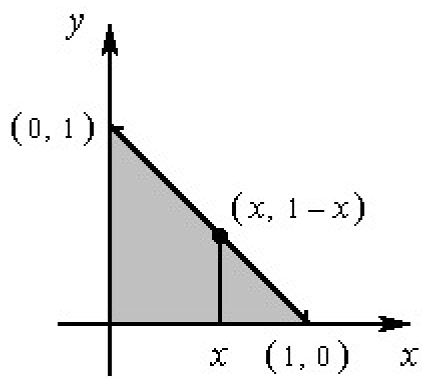
$$M(t_1, t_2) = 0.10 + 0.20\, e^{t_1} + 0.30\, e^{2t_2} + 0.40\, e^{2t_1 + t_2}.$$

Find the joint probability mass function $p(x, y)$.

$$\begin{aligned} M(t_1, t_2) = {} & 0.10\, e^{0t_1 + 0t_2} \\ & + 0.20\, e^{1t_1 + 0t_2} \\ & + 0.30\, e^{0t_1 + 2t_2} \\ & + 0.40\, e^{2t_1 + 1t_2}. \end{aligned}$$

$x \setminus y$	0	1	2
0	0.10	0	0.30
1	0.20	0	0
2	0	0.40	0

2. Alexis Nuts, Inc. markets cans of deluxe mixed nuts containing almonds, cashews, and peanuts. Suppose the net weight of each can is exactly 1 lb, but the weight contribution of each type of nut is random. Because the three weights sum to 1, a joint probability model for any two gives all necessary information about the weight of the third type. Let X = the weight of almonds in a selected can and Y = the weight of cashews.



Then the region of positive density is $D = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1, x + y \le 1\}$.
Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} 60\,x^2\,y & 0 \le x \le 1,\ 0 \le y \le 1,\ x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Verify that $f(x,y)$ is a legitimate probability density function.

1. $f(x,y) \ge 0$ for all (x, y). ✓

2. $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1 \left(\int_0^{1-x} 60\,x^2\,y\,dy \right) dx = \int_0^1 \left(30\,x^2\,(1-x)^2 \right) dx$$

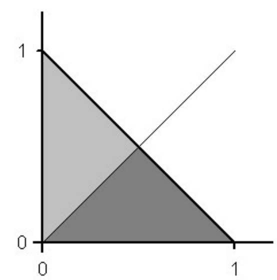
$$= \int_0^1 \left(30\,x^2 - 60\,x^3 + 30\,x^4 \right) dx = \left(10\,x^3 - 15\,x^4 + 6\,x^5 \right) \Big|_0^1 = 1. \checkmark$$

b) Find the probability that the two types of nuts together make up less than 50% of the can. That is, find the probability $P(X+Y<0.50)$. (Find the probability that peanuts make up over 50% of the can.)

$$P(X+Y<0.50) = \int_0^{0.5}\left(\int_0^{0.5-x} 60\,x^2\,y\,dy\right)dx = \int_0^{0.5} 30\,x^2(0.5-x)^2\,dx$$

$$= \int_0^{0.5}\left(7.5\,x^2-30\,x^3+30\,x^4\right)dx = \left(2.5\,x^3-7.5\,x^4+6\,x^5\right)\Big|_0^{0.5} = \mathbf{\frac{1}{32}} = \mathbf{0.03125}.$$

c) Find the probability that there are more almonds than cashews in a can. That is, find the probability $P(X>Y)$.



$$P(X>Y) = \int_0^{1/2}\left(\int_y^{1-y} 60\,x^2\,y\,dx\right)dy$$

$$= \int_0^{1/2} 20\,y\left(\int_y^{1-y} 3\,x^2\,dx\right)dy$$

$$= \int_0^{1/2} 20\,y\left((1-y)^3-y^3\right)dy$$

$$= \int_0^{1/2} 20\,y\left(1-3\,y+3\,y^2-2\,y^3\right)dy = \int_0^{1/2}\left(20\,y-60\,y^2+60\,y^3-40\,y^4\right)dy$$

$$= \left(10\,y^2-20\,y^3+15\,y^4-8\,y^5\right)\Big|_0^{1/2} = \mathbf{\frac{11}{16}} = \mathbf{0.6875}.$$

OR

$$P(X>Y) = 1-\int_0^{1/2}\left(\int_x^{1-x} 60\,x^2\,y\,dy\right)dx = 1-\int_0^{1/2} 30\,x^2\left(\int_x^{1-x} 2\,y\,dy\right)dx$$

$$= 1-\int_0^{1/2} 30\,x^2\left((1-x)^2-x^2\right)dx = 1-\int_0^{1/2} 30\,x^2(1-2\,x)\,dx$$

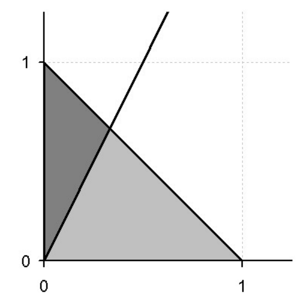
$$= 1-\int_0^{1/2}\left(30\,x^2-60\,x^3\right)dx = 1-\left(10\,x^3-15\,x^4\right)\Big|_0^{1/2} = \mathbf{\frac{11}{16}} = \mathbf{0.6875}.$$

OR

$$P(X > Y) = \int_0^{1/2}\left(\int_0^{x} 60x^2\, y\, dy\right)dx + \int_{1/2}^{1}\left(\int_0^{1-x} 60x^2\, y\, dy\right)dx = \ldots$$

d) Find the probability that there are at least twice as many cashews as there are almonds. That is, find the probability $P(2X \le Y)$.

$$P(Y \ge 2X) = \int_0^{1/3}\left(\int_{2x}^{1-x} 60x^2\, y\, dy\right)dx$$



$$= \int_0^{1/3}\left(30x^2\left[(1-x)^2 - (2x)^2\right]\right)dx$$

$$= \int_0^{1/3}\left(30x^2 - 60x^3 - 90x^4\right)dx$$

$$= \int_0^{1/3}\left(30x^2 - 60x^3 - 90x^4\right)dx = \left(10x^3 - 15x^4 - 18x^5\right)\Big|_0^{1/3}$$

$$= \frac{10}{27} - \frac{15}{81} - \frac{18}{243} = \mathbf{\frac{1}{9}}.$$

e) Find the marginal probability density function for X.

$$f_X(x) = \int_0^{1-x} 60x^2\, y\, dy = 30x^2\int_0^{1-x} 2y\, dy = 30x^2(1-x)^2, \qquad 0 < x < 1.$$

f) Find the marginal probability density function for Y.

$$f_Y(y) = \int_0^{1-y} 60x^2\, y\, dx = 20y\int_0^{1-y} 3x^2\, dx = 20y(1-y)^3, \qquad 0 < y < 1.$$

g) Find $E(X)$, $E(Y)$, $E(X+Y)$, $E(X \cdot Y)$.

$$E(X) = \int_0^1 x \cdot 30x^2(1-x)^2\, dx = \int_0^1\left(30x^3 - 60x^4 + 30x^5\right)dx$$

$$= \left(7.5\,x^4 - 12\,x^5 + 5\,x^6 \right) \Big|_0^1 = \mathbf{0.5} = \mathbf{\frac{1}{2}}.$$

$$\mathrm{E(Y)} = \int_0^1 y \cdot 20\,y\,(1-y)^3\,dy = \int_0^1 \left(20\,y^2 - 60\,y^3 + 60\,y^4 - 20\,y^5 \right) dy$$

$$= \left(\frac{20}{3}\,y^3 - 15\,y^4 + 12\,y^5 - \frac{20}{6}\,y^6 \right) \Big|_0^1 = \mathbf{\frac{1}{3}}.$$

$$\mathrm{E(X+Y)} = \mathrm{E(X)} + \mathrm{E(Y)} = \mathbf{\frac{5}{6}}.$$

$$\mathrm{E(X \cdot Y)} = \int_0^1 \left(\int_0^{1-x} x\,y \cdot 60\,x^2\,y\,dy \right) dx = \int_0^1 \left(20\,x^3\,(1-x)^3 \right) dx$$

$$= \int_0^1 \left(20\,x^3 - 60\,x^4 + 60\,x^5 - 20\,x^6 \right) dx$$

$$= \left(5\,x^4 - 12\,x^5 + 10\,x^6 - \frac{20}{7}\,x^7 \right) \Big|_0^1 = \mathbf{\frac{1}{7}}.$$

h) If 1 lb of almonds costs the company \$1.00, 1 lb of cashews costs \$1.50, and 1 lb of peanuts costs \$0.60, what is the expected total cost of the content of a can?

$$\text{Total cost} = (1.00)\,\mathrm{X} + (1.50)\,\mathrm{Y} + (0.60)\,(1 - \mathrm{X} - \mathrm{Y}) = 0.6 + 0.4\,\mathrm{X} + 0.9\,\mathrm{Y}.$$

$$\mathrm{E}(\text{Total cost}) = 0.6 + 0.4\,\mathrm{E(X)} + 0.9\,\mathrm{E(Y)} = 0.60 + 0.40 \cdot \frac{1}{2} + 0.90 \cdot \frac{1}{3} = \mathbf{\$1.10}.$$

OR

$$\mathrm{E}(\text{Total cost}) = (1.00)\,\mathrm{E(X)} + (1.50)\,\mathrm{E(Y)} + (0.60)\,\mathrm{E}(1 - \mathrm{X} - \mathrm{Y})$$

$$= 1.00 \cdot \frac{1}{2} + 1.50 \cdot \frac{1}{3} + 0.60 \cdot \left(1 - \frac{1}{2} - \frac{1}{3} \right) = \mathbf{\$1.10}.$$

i) Find the moment-generating function $\mathrm{M}_{\mathrm{X,Y}}(t_1, t_2)$.

$$\mathrm{M}(t_1, t_2) = \int_0^1 \left(\int_0^{1-x} e^{t_1 x + t_2 y} \cdot 60\,x^2\,y\,dy \right) dx = \ldots$$