

6.* Let Z be a $N(0, 1)$ standard normal random variable.

Show that $X = Z^2$ has a chi-square distribution with 1 degree of freedom.

$$\begin{aligned}
 M_X(t) &= E(e^{tX}) = E(e^{tZ^2}) = \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
 &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(z^2/2\right) \cdot (1-2t)} dz = \frac{1}{(1-2t)^{1/2}}, \quad t < \frac{1}{2}, \\
 \text{since } \frac{(1-2t)^{1/2}}{\sqrt{2\pi}} e^{-\left(z^2/2\right) \cdot (1-2t)} &\text{ is the p.d.f. of } N\left(0, \frac{1}{1-2t}\right) \text{ distribution.} \\
 \Rightarrow X \text{ has a } \chi^2(1) \text{ distribution.}
 \end{aligned}$$

OR

$$\begin{aligned}
 F_X(x) &= P(X \leq x) = P(Z^2 \leq x) = P(-\sqrt{x} \leq Z \leq \sqrt{x}) \\
 &= \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = F_Z(\sqrt{x}) - F_Z(-\sqrt{x}), \quad x > 0.
 \end{aligned}$$

$$\begin{aligned}
 f_X(x) &= F'_X(x) = \left(\frac{1}{2\sqrt{x}}\right) f_Z(\sqrt{x}) - \left(-\frac{1}{2\sqrt{x}}\right) f_Z(-\sqrt{x}) \\
 &= \left(\frac{1}{2\sqrt{x}}\right) \left(\frac{1}{\sqrt{2\pi}} e^{-x/2}\right) - \left(-\frac{1}{2\sqrt{x}}\right) \left(\frac{1}{\sqrt{2\pi}} e^{-x/2}\right) \\
 &= \frac{1}{\sqrt{\pi} 2^{1/2}} x^{-1/2} e^{-x/2} \\
 &= \frac{1}{\Gamma(1/2) 2^{1/2}} x^{(1/2)-1} e^{-x/2}, \quad x > 0.
 \end{aligned}$$

$\Rightarrow X$ has a $\chi^2(1)$ distribution.

OR

$g(z) = z^2$ is NOT a one-to-one function on the support of Z : $-\infty < z < \infty$.

However, $g(z) = z^2$ is a one-to-one function on $-\infty < z < 0$ and on $0 < z < \infty$.

$-\infty < z < 0$ $X = g(Z) = Z^2$ $\infty > x > 0$ $z = -\sqrt{x} = g^{-1}(x)$ $\frac{dz}{dx} = -\frac{1}{2\sqrt{x}}$ $f_Z(g^{-1}(x)) \left \frac{dz}{dx} \right $ $\frac{1}{\sqrt{2\pi}} e^{-x/2} \cdot \left -\frac{1}{2\sqrt{x}} \right $ $\left(\frac{1}{2\sqrt{x}} \right) \left(\frac{1}{\sqrt{2\pi}} e^{-x/2} \right)$	$0 < z < \infty$ $X = g(Z) = Z^2$ $0 < x < \infty$ $z = \sqrt{x} = g^{-1}(x)$ $\frac{dz}{dx} = \frac{1}{2\sqrt{x}}$ $f_Z(g^{-1}(x)) \left \frac{dz}{dx} \right $ $\frac{1}{\sqrt{2\pi}} e^{-x/2} \cdot \left \frac{1}{2\sqrt{x}} \right $ $\left(\frac{1}{2\sqrt{x}} \right) \left(\frac{1}{\sqrt{2\pi}} e^{-x/2} \right)$
$f_X(x) = \left(\frac{1}{2\sqrt{x}} \right) \left(\frac{1}{\sqrt{2\pi}} e^{-x/2} \right) + \left(\frac{1}{2\sqrt{x}} \right) \left(\frac{1}{\sqrt{2\pi}} e^{-x/2} \right)$ $= \frac{1}{\sqrt{\pi}} \frac{1}{2^{1/2}} x^{-1/2} e^{-x/2}$ $= \frac{1}{\Gamma(1/2)} \frac{1}{2^{1/2}} x^{(1/2)-1} e^{-x/2}, \quad 0 < x < \infty.$	
$\Rightarrow X$ has a $\chi^2(1)$ distribution.	

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du, \quad x > 0 \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(x) = (x-1)\Gamma(x-1), \quad x > 1 \quad \Gamma(n) = (n-1)! \quad \text{if } n \text{ is an integer}$$

Consider a continuous random variable X , with p.d.f. f and c.d.f. F , where F is strictly increasing on some interval I , $F=0$ to the left of I , and $F=1$ to the right of I . I may be a bounded interval or an unbounded interval such as the whole real line. $F^{-1}(u)$ is then well defined for $0 < u < 1$.

Fact 1: Let $U \sim \text{Uniform}(0, 1)$, and let $X = F^{-1}(U)$. Then the c.d.f. of X is F .

Proof: $P(X \leq x) = P(F^{-1}(U) \leq x) = P(U \leq F(x)) = F(x)$.

Fact 2: Let $U = F(X)$; then U has a Uniform(0, 1) distribution.

Proof: $P(U \leq u) = P(F(X) \leq u) = P(X \leq F^{-1}(u)) = F(F^{-1}(u)) = u$.

7. Let X have a uniform distribution on the interval $(0, 1)$.

- a) Find the c.d.f. and the p.d.f. of $Y = \frac{X}{1-X}$.

$$y = \frac{x}{1-x} \quad 0 < x < 1 \quad \Rightarrow \quad 0 < y < \infty.$$

$$y = \frac{x}{1-x} \quad x = \frac{y}{1+y} = g^{-1}(y) \quad \frac{dx}{dy} = \frac{(1+y)-y}{(1+y)^2} = \frac{1}{(1+y)^2}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = 1 \times \left| \frac{1}{(1+y)^2} \right| = \frac{1}{(1+y)^2}, \quad 0 < y < \infty.$$

$$F_Y(y) = \int_0^y \frac{1}{(1+u)^2} du = -\frac{1}{1+u} \Big|_0^y = 1 - \frac{1}{1+y}, \quad 0 < y < \infty.$$

OR

$$F_X(x) = x, \quad 0 < x < 1.$$

$$F_Y(y) = P(Y \leq y) = P\left(\frac{X}{1-X} \leq y\right) = P\left(X \leq \frac{y}{1+y}\right) = \frac{y}{1+y}, \quad y > 0.$$

$$f_Y(y) = F'_Y(y) = \frac{(1+y)-y}{(1+y)^2} = \frac{1}{(1+y)^2}, \quad y > 0.$$

- b) Find the c.d.f. and the p.d.f. of $W = \ln Y$.

$$w = \ln y \quad 0 < y < \infty \quad \Rightarrow \quad -\infty < w < \infty.$$

$$w = \ln y \quad y = e^w = g^{-1}(w) \quad \frac{dy}{dw} = e^w$$

$$\begin{aligned} f_W(w) &= f_Y(g^{-1}(w)) \left| \frac{dy}{dw} \right| = \frac{1}{(1+e^w)^2} \times |e^w| \\ &= \frac{e^w}{(1+e^w)^2} = \frac{e^{-w}}{(1+e^{-w})^2}, \quad -\infty < w < \infty. \end{aligned}$$

$$\begin{aligned} F_W(w) &= \int_{-\infty}^w \frac{e^u}{(1+e^u)^2} du = -\frac{1}{1+e^u} \Big|_{-\infty}^w \\ &= 1 - \frac{1}{1+e^w} = \frac{e^w}{1+e^w} = \frac{1}{1+e^{-w}}, \quad -\infty < w < \infty. \end{aligned}$$

OR

$$F_Y(y) = 1 - \frac{1}{1+y} = \frac{y}{1+y}, \quad 0 < y < \infty.$$

$$F_W(w) = P(W \leq w) = P(\ln Y \leq w) = P(Y \leq e^w) = F_Y(e^w)$$

$$= \frac{e^w}{1+e^w} = \frac{1}{1+e^{-w}}, \quad -\infty < w < \infty.$$

$$\begin{aligned} f_W(w) &= F'_W(w) = \frac{e^w(1+e^w) - e^w \cdot e^w}{(1+e^w)^2} = \frac{e^w}{(1+e^w)^2} = \frac{e^{-w}}{(1+e^{-w})^2}, \\ &\quad -\infty < w < \infty. \end{aligned}$$

These are the transformations in the logistic regression:

$$\text{First,} \quad \text{odds} = \frac{\text{probability}}{1 - \text{probability}}.$$

$$\text{Then} \quad \ln(\text{odds}) = \beta_0 + \beta_1 x_1 + \dots + \beta_k x_k + \varepsilon.$$

The distribution of W is called the (standard) logistic distribution.

8. Let X have a **logistic distribution** with p.d.f.

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \quad -\infty < x < \infty.$$

Show that

$$Y = \frac{1}{1+e^{-X}}$$

has a $U(0, 1)$ distribution.

$$F_X(x) = \frac{1}{1+e^{-x}}, \quad -\infty < x < \infty.$$

$\Rightarrow Y = F_X(X)$ has a Uniform($0, 1$) distribution by Fact 2.

OR

It is easier to note that

$$\frac{dy}{dx} = \frac{e^{-x}}{(1+e^{-x})^2} \quad \text{and} \quad \frac{dx}{dy} = \frac{(1+e^{-x})^2}{e^{-x}}.$$

Say the solution of x in terms of y is given by x^* . Then the p.d.f. of Y is

$$g(y) = \frac{e^{-x^*}}{(1+e^{-x^*})^2} \left| \frac{(1+e^{-x^*})^2}{e^{-x^*}} \right| = 1, \quad 0 < y < 1,$$

as $-\infty < x < \infty$ maps onto $0 < y < 1$. Thus Y is $U(0, 1)$.