STAT 410 **Examples for Week 02 (Disc4)** A. Stepanov

STAT 410 does NOT have a discussion section.

STAT 400 (an easier class) DOES have a discussion section.
STAT 410 (a harder class) does NOT.

There is a simple explanation for this: About 20 years ago, there was only one section of STAT 410 with about 40 students and only two lectures of STAT 400 with about 150 students each (each divided into 3 discussion sections of about 50 students each). The number of students has changed a little bit since then, the course structure has not (yet)…

However, don't you wish there was a discussion section in STAT 410?

If STAT 410 did have a discussion section, this is what your TA would most likely go over during it:

1. Let the joint probability density function for (X, Y) be

$$f(x,y) = C\,x\,y^3, \qquad 0<y<1, \quad y<x<2, \qquad \text{zero otherwise.}$$

a) Sketch the support of (X, Y).
That is, sketch $\{\,(x,y):\ 0<y<1,\ \ y<x<2\,\}$.

b) Find the value of C so that $f(x,y)$ is a valid joint probability density function.

c) Find the marginal probability density function of X, $f_{\mathrm{X}}(x)$.
Be sure to include its support.

"Hint": It would be **wise** to break this question into **piece**s.

d) Find the marginal probability density function of Y, $f_{\mathrm{Y}}(y)$.
Be sure to include its support.

e) Find the probability $P(X + Y \le 2)$.

f) Find the probability $P(X \cdot Y \le 1)$.

g) Find the probability $P(\frac{Y}{X} \le \frac{1}{2})$.

h) Find the probability $P(2X + 3Y \ge 4)$.

i) Are X and Y independent? *Justify your answer.*
If X and Y are not independent, find $\text{Cov}(X, Y)$.

2. Every month, the government of Neverland spends X million dollars purchasing guns and Y million dollars purchasing butter. Since the people become disobedient if they do not have food, the amount spent monthly on butter is always at least 2 million dollars. However, there is a law that prevents the government of Neverland spending more than 5 million dollars on guns in one month. In addition, the government of Neverland cannot afford to spend more than 8 million dollars per month on guns and butter. Suppose that the joint density function for (X, Y) is

$$f(x,y) = \frac{7x + 2y}{C}, \qquad x \ge 0, \quad y \ge 2, \quad x \le 5, \quad x + y \le 8, \qquad \text{zero otherwise.}$$

X – guns, Y – butter.

a) Sketch the support of (X, Y).
That is, sketch $\{ (x,y) : x \ge 0, \quad y \ge 2, \quad x \le 5, \quad x + y \le 8 \}$.

b) Find the value of C so that $f(x,y)$ is a valid joint probability density function.

c) Find the marginal probability density function of X, $f_X(x)$.

d) Find the marginal probability density function of Y, $f_Y(y)$.

e) Are X and Y independent?

f) Find the probability that the total amount spent monthly on guns and butter exceeds 6.5 million dollars. That is, find $\mathrm{P}(\mathrm{X}+\mathrm{Y}>6.5)$.

g) Find the probability that the total amount spent monthly on guns and butter exceeds 7.4 million dollars. That is, find $\mathrm{P}(\mathrm{X}+\mathrm{Y}>7.4)$.

h) Find the probability that the government of Neverland spends more purchasing guns than purchasing butter in a given month. That is, find $\mathrm{P}(\mathrm{X}>\mathrm{Y})$.

3. Consider two continuous random variables X and Y with joint p.d.f.

$$f(x,y) = \begin{cases} \frac{2}{81}x^2 y & 0<x<K,\ 0<y<K \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of K so that $f(x,y)$ is a valid joint p.d.f.

b) Find $\mathrm{P}(\mathrm{X}>3\,\mathrm{Y})$.

c) Find $\mathrm{P}(\mathrm{X}+\mathrm{Y}>3)$.

d) Are X and Y independent? *Justify your answer.*

4. Suppose that (X,Y) is uniformly distributed over the region defined by $-1\le x\le 1$ and $0\le y\le 1-x^2$. That is,

$$f(x,y) = C, \qquad -1\le x\le 1,\ 0\le y\le 1-x^2, \qquad \text{zero elsewhere.}$$

a) What is the joint probability density function of X and Y? That is, find C.

b) Find the marginal probability density function of X, $f_{\mathrm{X}}(x)$.

c) Find the marginal probability density function of Y, $f_{\mathrm{Y}}(y)$.

d) Are X and Y independent? If not, find $\mathrm{Cov}(\mathrm{X},\mathrm{Y})$.

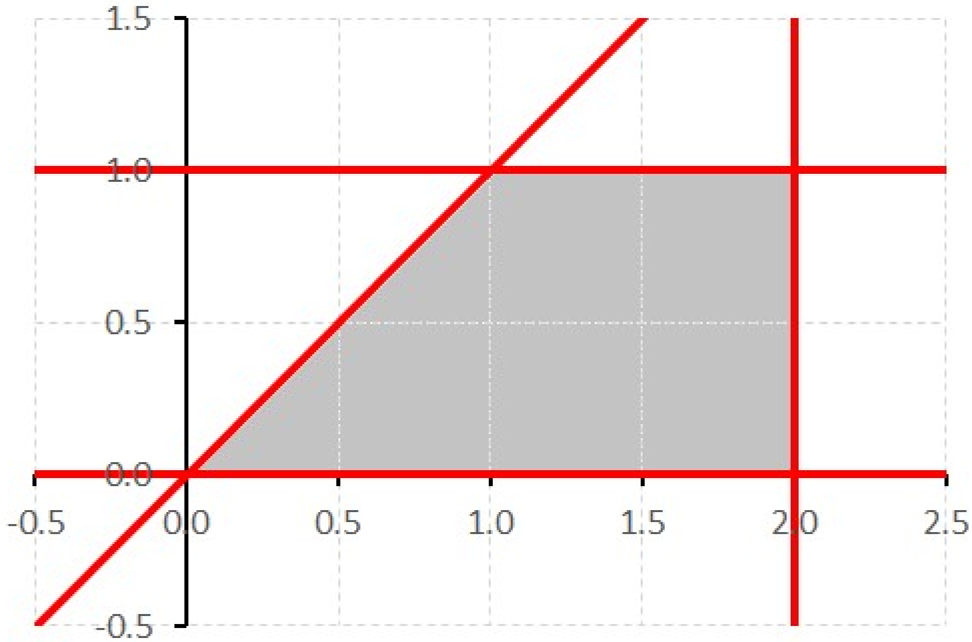
Answers:

1. Let the joint probability density function for (X, Y) be

$$f(x,y) = C\,x\,y^3, \qquad 0 < y < 1, \quad y < x < 2, \qquad \text{zero otherwise.}$$

a) Sketch the support of (X, Y).

That is, sketch $\{(x,y) : 0 < y < 1, \; y < x < 2\}$.



b) Find the value of C so that $f(x,y)$ is a valid joint probability density function.

$$1 = \int_0^1 \left(\int_y^2 C\,x\,y^3\,dx \right) dy = \int_0^1 \left(\frac{C}{2}\,x^2\,y^3 \right) \Bigg|_{x=y}^{x=2} dy = \int_0^1 \left(2\,C\,y^3 - \frac{C}{2}\,y^5 \right) dy$$

$$= \left(\frac{C}{2}\,y^4 - \frac{C}{12}\,y^6 \right) \Bigg|_{y=0}^{y=1} = \frac{C}{2} - \frac{C}{12} = \frac{5\,C}{12} = 1.$$

$$\Rightarrow \qquad C = \mathbf{\frac{12}{5}} = \mathbf{2.4}.$$

OR

$$1 = \int_0^1 \left(\int_0^x C\,x\,y^3\,dy \right) dx + \int_1^2 \left(\int_0^1 C\,x\,y^3\,dy \right) dx$$

$$= \int_0^1 \left(\frac{C}{4}\,x\,y^4 \right) \Bigg|_{y=0}^{y=x} dx + \int_1^2 \left(\frac{C}{4}\,x\,y^4 \right) \Bigg|_{y=0}^{y=1} dx$$

$$= \int_0^1 \frac{C}{4}\,x^5\,dx + \int_1^2 \frac{C}{4}\,x\,dx = \left(\frac{C}{24}\,x^6 \right) \Bigg|_{x=0}^{x=1} + \left(\frac{C}{8}\,x^2 \right) \Bigg|_{x=1}^{x=2}$$

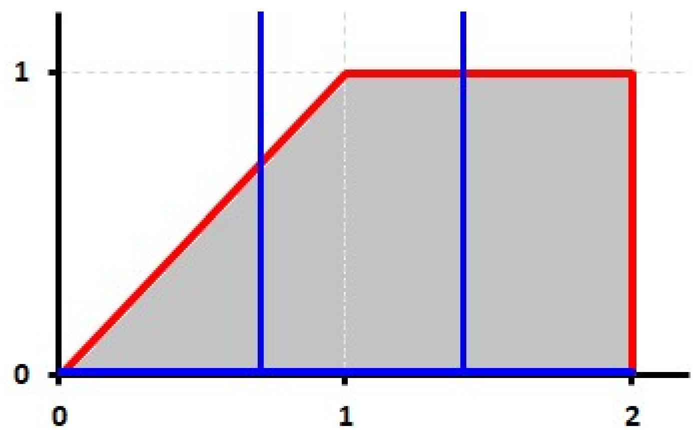
$$= \frac{C}{24} + \frac{3\,C}{8} = \frac{10\,C}{24} = \frac{5\,C}{12} = 1.$$

$$\Rightarrow \qquad C = \mathbf{\frac{12}{5}} = \mathbf{2.4}.$$

c) Find the marginal probability density function of X, $f_X(x)$.

Be sure to include its support.

"Hint": It would be **wise** to break this question into **piece**s.



For $0 < x < 1$, $$f_X(x) = \int_0^x \frac{12}{5}\, x\, y^3\, dy = \left(\frac{3}{5}\, x\, y^4 \right) \Bigg|_{y=0}^{y=x} = \frac{3}{5}\, x^5.$$

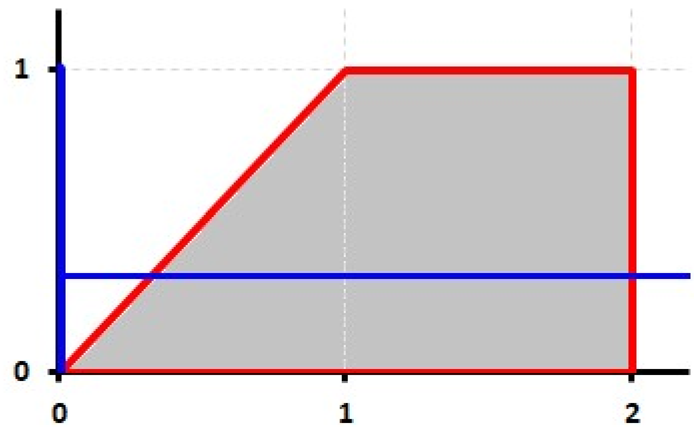
For $1 < x < 2$, $$f_X(x) = \int_0^1 \frac{12}{5}\, x\, y^3\, dy = \left(\frac{3}{5}\, x\, y^4 \right) \Bigg|_{y=0}^{y=1} = \frac{3}{5}\, x.$$

Check: $$\int_0^1 \frac{3}{5}\, x^5\, dx + \int_1^2 \frac{3}{5}\, x\, dx = \left(\frac{1}{10}\, x^6 \right) \Bigg|_{x=0}^{x=1} + \left(\frac{3}{10}\, x^2 \right) \Bigg|_{x=1}^{x=2}$$

$$= \frac{1}{10} + \frac{9}{10} = 1.$$

d) Find the marginal probability density function of Y, $f_Y(y)$.

Be sure to include its support.



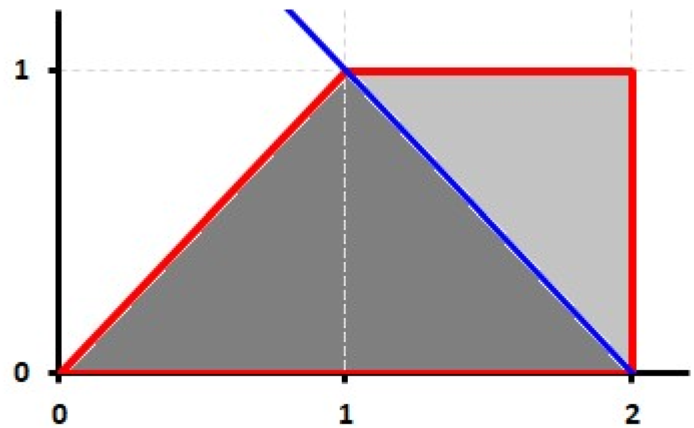
For $0 < y < 1$,

$$f_Y(y) = \int_y^2 \frac{12}{5}\, x\, y^3 \, dx = \left(\frac{6}{5} x^2 y^3 \right) \Bigg|_{x=y}^{x=2} = \frac{24}{5} y^3 - \frac{6}{5} y^5 .$$

Check:

$$\int_0^1 \left(\frac{24}{5} y^3 - \frac{6}{5} y^5 \right) dy = \left(\frac{6}{5} y^4 - \frac{1}{5} y^6 \right) \Bigg|_{y=0}^{y=1} = \frac{6}{5} - \frac{1}{5} = 1.$$

e) Find the probability $P(X + Y \le 2)$.



$$\int_0^1 \left(\int_y^{2-y} \frac{12}{5} x y^3 \, dx \right) dy = \int_0^1 \left(\frac{6}{5} x^2 y^3 \right) \Bigg|_{x=y}^{x=2-y} dy = \int_0^1 \left(\frac{24}{5} y^3 - \frac{24}{5} y^4 \right) dy$$

$$= \left(\frac{6}{5} y^4 - \frac{24}{25} y^5 \right) \Bigg|_{y=0}^{y=1} = \mathbf{\frac{6}{25}} = \mathbf{0.24}.$$

OR

$$1 - \int_0^1 \left(\int_{2-y}^{2} \frac{12}{5} x y^3 \, dx \right) dy = 1 - \int_0^1 \left(\frac{6}{5} x^2 y^3 \right) \Bigg|_{x=2-y}^{x=2} dy$$

$$= 1 - \int_0^1 \left(\frac{24}{5} y^4 - \frac{6}{5} y^5 \right) dy = 1 - \left(\frac{24}{25} y^5 - \frac{1}{5} y^6 \right) \Bigg|_{y=0}^{y=1} = \mathbf{\frac{6}{25}}.$$

OR

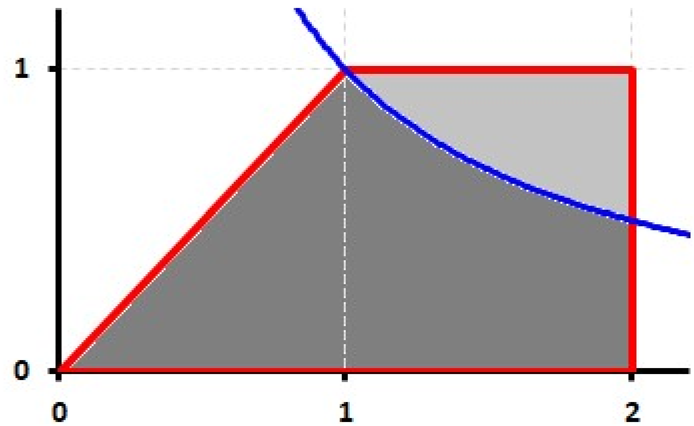
$$\int_0^1 \left(\int_0^x \frac{12}{5} x y^3 \, dy \right) dx + \int_1^2 \left(\int_0^{2-x} \frac{12}{5} x y^3 \, dy \right) dx$$

$$= \int_0^1 \frac{3}{5} x^5 \, dx + \int_1^2 \frac{3}{5} x (2-x)^4 \, dx = \frac{1}{10} + \int_1^2 \frac{3}{5} x (2-x)^4 \, dx = \ldots$$

OR

$$1-\int_{1}^{2}\left(\int_{2-x}^{1}\frac{12}{5}\,x\,y^{3}\,dy\right)dx \;=\; 1-\int_{1}^{2}\left(\frac{3}{5}\,x-\frac{3}{5}\,x\,(2-x)^{4}\right)dx \;=\; \ldots$$

f) Find the probability $P(X \cdot Y \le 1)$.



$$\int_{0}^{1}\left(\int_{0}^{x}\frac{12}{5}\,x\,y^{3}\,dy\right)dx \;+\; \int_{1}^{2}\left(\int_{0}^{1/x}\frac{12}{5}\,x\,y^{3}\,dy\right)dx \;=\; \int_{0}^{1}\frac{3}{5}\,x^{5}\,dx+\int_{1}^{2}\frac{3}{5}\,\frac{1}{x^{3}}\,dx$$

$$=\left(\frac{1}{10}\,x^{6}\right)\Bigg|_{x=0}^{x=1}+\left(-\frac{3}{10}\,\frac{1}{x^{2}}\right)\Bigg|_{x=1}^{x=2} \;=\; \frac{1}{10}-\frac{3}{40}+\frac{3}{10} \;=\; \mathbf{\frac{13}{40}} \;=\; \mathbf{0.325}.$$

OR

$$\int_{0}^{1/2}\left(\int_{y}^{2}\frac{12}{5}\,x\,y^{3}\,dx\right)dy \;+\; \int_{1/2}^{1}\left(\int_{y}^{1/y}\frac{12}{5}\,x\,y^{3}\,dx\right)dy$$

$$=\int_{0}^{1/2}\left(\frac{24}{5}\,y^{3}-\frac{6}{5}\,y^{5}\right)dy \;+\; \int_{1/2}^{1}\left(\frac{6}{5}\,y-\frac{6}{5}\,y^{5}\right)dy$$

$$= \left(\frac{6}{5} y^4 - \frac{1}{5} y^6 \right) \Big|_{0}^{1/2} + \left(\frac{3}{5} y^2 - \frac{1}{5} y^6 \right) \Big|_{1/2}^{1}$$

$$= \frac{3}{40} - \frac{1}{320} + \frac{3}{5} - \frac{1}{5} - \frac{3}{20} + \frac{1}{320} = \frac{2}{5} - \frac{3}{40} = \mathbf{\frac{13}{40}}.$$

OR

$$1 - \int_1^2 \left(\int_{1/x}^{1} \frac{12}{5} x y^3 \, dy \right) dx = 1 - \int_1^2 \left(\frac{3}{5} x - \frac{3}{5} \frac{1}{x^3} \right) dx = 1 - \left(\frac{3}{10} x^2 + \frac{3}{10} \frac{1}{x^2} \right) \Big|_{x=1}^{x=2}$$

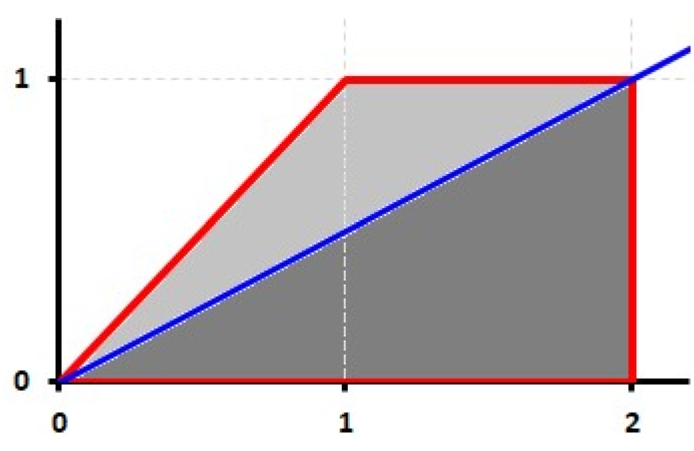
$$= 1 - \left(\frac{6}{5} + \frac{3}{40} - \frac{3}{10} - \frac{3}{10} \right) = 1 - \left(\frac{3}{5} + \frac{3}{40} \right) = 1 - \frac{27}{40} = \mathbf{\frac{13}{40}}.$$

OR

$$1 - \int_{1/2}^{1} \left(\int_{1/y}^{2} \frac{12}{5} x y^3 \, dx \right) dy = 1 - \int_{1/2}^{1} \left(\frac{24}{5} y^3 - \frac{6}{5} y \right) dy = 1 - \left(\frac{6}{5} y^4 - \frac{3}{5} y^2 \right) \Big|_{1/2}^{1}$$

$$= 1 - \left(\frac{6}{5} - \frac{3}{5} - \frac{3}{40} + \frac{3}{20} \right) = 1 - \left(\frac{3}{5} + \frac{3}{40} \right) = 1 - \frac{27}{40} = \mathbf{\frac{13}{40}}.$$

g) Find the probability $P\left(\frac{Y}{X} \le \frac{1}{2}\right)$.



$$\int_0^2 \left(\int_0^{x/2} \frac{12}{5}\, x\, y^3\, dy \right) dx \;=\; \int_0^2 \frac{3}{80}\, x^5\, dx \;=\; \left(\frac{1}{160}\, x^6 \right) \Bigg|_{x=0}^{x=2} \;=\; \mathbf{\frac{2}{5}} \;=\; \mathbf{0.40}.$$

OR

$$\int_0^1 \left(\int_{2y}^{2} \frac{12}{5}\, x\, y^3\, dx \right) dy \;=\; \int_0^1 \left(\frac{24}{5}\, y^3 - \frac{24}{5}\, y^5 \right) dy \;=\; \left(\frac{6}{5}\, y^4 - \frac{4}{5}\, y^6 \right) \Bigg|_{y=0}^{y=1} \;=\; \mathbf{\frac{2}{5}}.$$

OR

$$1 - \int_0^1 \left(\int_{y}^{2y} \frac{12}{5}\, x\, y^3\, dx \right) dy \;=\; 1 - \int_0^1 \left(\frac{24}{5}\, y^5 - \frac{6}{5}\, y^5 \right) dy \;=\; 1 - \int_0^1 \frac{18}{5}\, y^5\, dy$$

$$=\; 1 - \left(\frac{3}{5}\, y^6 \right) \Bigg|_{y=0}^{y=1} \;=\; 1 - \frac{3}{5} \;=\; \mathbf{\frac{2}{5}}.$$

OR

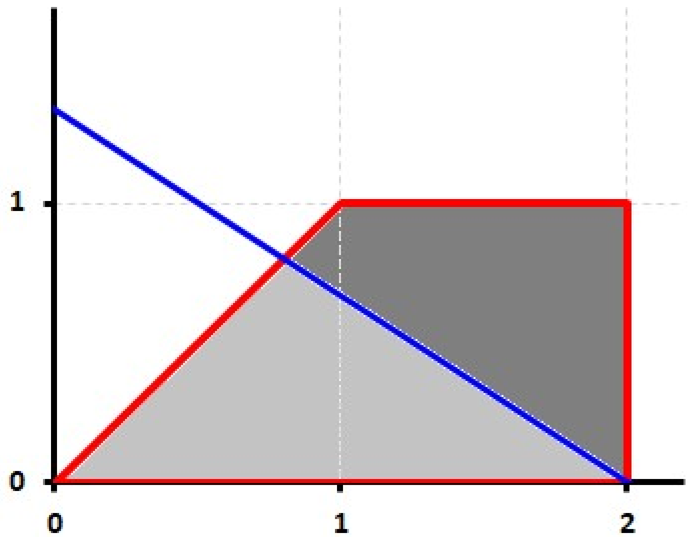
$$1 \;-\; \int_0^1 \left(\int_{x/2}^{x} \frac{12}{5}\, x\, y^3\, dy \right) dx \;-\; \int_1^2 \left(\int_{x/2}^{1} \frac{12}{5}\, x\, y^3\, dy \right) dx$$

$$=\; 1 \;-\; \int_0^1 \left(\frac{3}{5}\, x^5 - \frac{3}{80}\, x^5 \right) dx \;-\; \int_1^2 \left(\frac{3}{5}\, x - \frac{3}{80}\, x^5 \right) dx$$

$$=\; 1 \;-\; \left(\frac{1}{10}\, x^6 - \frac{1}{160}\, x^6 \right) \Bigg|_{x=0}^{x=1} \;-\; \left(\frac{3}{10}\, x^2 - \frac{1}{160}\, x^6 \right) \Bigg|_{x=1}^{x=2}$$

$$=\; 1 - \left(\frac{1}{10} - \frac{1}{160} \right) - \left(\frac{6}{5} - \frac{2}{5} \right) + \left(\frac{3}{10} - \frac{1}{160} \right) \;=\; \mathbf{\frac{2}{5}}.$$

h) Find the probability $P(2X + 3Y \geq 4)$.



$2x + 3y = 4 \quad \& \quad x = y \qquad \Rightarrow \qquad x = y = 0.8.$

$$1 - \int_0^{0.8} \left(\int_y^{2-1.5y} \frac{12}{5} x y^3 \, dx \right) dy = 1 - \int_0^{0.8} \left(\frac{6}{5} x^2 y^3 \right) \Bigg|_{x=y}^{x=2-1.5y} dy$$

$$= 1 - \int_0^{0.8} \left(4.8 y^3 - 7.2 y^4 + 1.5 y^5 \right) dy$$

$$= 1 - \left(1.2 y^4 - 1.44 y^5 + 0.25 y^6 \right) \Bigg|_{y=0}^{y=0.8} = \mathbf{0.9148032}.$$

OR

$$1 - \int_0^{0.8} \left(\int_0^x \frac{12}{5} x y^3 \, dy \right) dx - \int_{0.8}^2 \left(\int_0^{\frac{4-2x}{3}} \frac{12}{5} x y^3 \, dy \right) dx$$

$$= \int_0^{0.8} \frac{3}{5} x^5 \, dx + \int_{0.8}^2 \frac{3}{5} x \left(\frac{4-2x}{3} \right)^4 dx = \ldots$$

OR

$$\int_{0.8}^{1}\left(\int_{\frac{4-2x}{3}}^{x}\frac{12}{5}\,x\,y^{3}\,dy\right)dx \;+\; \int_{1}^{2}\left(\int_{\frac{4-2x}{3}}^{1}\frac{12}{5}\,x\,y^{3}\,dy\right)dx \;=\; \ldots$$

OR

$$\int_{0}^{0.8}\left(\int_{2-1.5y}^{2}\frac{12}{5}\,x\,y^{3}\,dx\right)dy \;+\; \int_{0.8}^{1}\left(\int_{y}^{2}\frac{12}{5}\,x\,y^{3}\,dx\right)dy \;=\; \ldots$$

i) Are X and Y independent? *Justify your answer.*
If X and Y are not independent, find $\operatorname{Cov}(X, Y)$.

$f(x,y) \neq f_X(x)\cdot f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

OR

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

$$E(X) = \int_{0}^{1} x\cdot\frac{3}{5}x^{5}\,dx + \int_{1}^{2} x\cdot\frac{3}{5}x\,dx = \left(\frac{3}{35}x^{7}\right)\Bigg|_{x=0}^{x=1} + \left(\frac{1}{5}x^{3}\right)\Bigg|_{x=1}^{x=2}$$

$$= \frac{3}{35} + \frac{7}{5} = \frac{52}{35}.$$

$$E(Y) = \int_{0}^{1} y\cdot\left(\frac{24}{5}y^{3} - \frac{6}{5}y^{5}\right)dy = \left(\frac{24}{25}y^{5} - \frac{6}{35}y^{7}\right)\Bigg|_{y=0}^{y=1} = \frac{24}{25} - \frac{6}{35} = \frac{138}{175}.$$

$$E(XY) = \int_0^1 \left(\int_y^2 x\,y \cdot \frac{12}{5} x\,y^3 \, dx \right) dy = \int_0^1 \left(\frac{4}{5} x^3 y^4 \right) \Bigg|_{x=y}^{x=2} dy$$

$$= \int_0^1 \left(\frac{32}{5} y^4 - \frac{4}{5} y^7 \right) dy = \left(\frac{32}{25} y^5 - \frac{1}{10} y^8 \right) \Bigg|_{y=0}^{y=1}$$

$$= \frac{32}{25} - \frac{1}{10} = \frac{59}{50} = 1.18.$$

OR

$$E(XY) = \int_0^1 \left(\int_0^x x\,y \cdot \frac{12}{5} x\,y^3 \, dy \right) dx + \int_1^2 \left(\int_0^1 x\,y \cdot \frac{12}{5} x\,y^3 \, dy \right) dx$$

$$= \int_0^1 \left(\frac{12}{25} x^2 y^5 \right) \Bigg|_{y=0}^{y=x} dx + \int_1^2 \left(\frac{12}{25} x^2 y^5 \right) \Bigg|_{y=0}^{y=1} dx$$

$$= \int_0^1 \frac{12}{25} x^7 \, dx + \int_1^2 \frac{12}{25} x^2 \, dx = \left(\frac{3}{50} x^2 \right) \Bigg|_{x=0}^{x=1} + \left(\frac{4}{25} x^3 \right) \Bigg|_{x=1}^{x=2}$$

$$= \frac{3}{50} + \frac{28}{25} = \frac{59}{50} = 1.18.$$

$$\mathrm{Cov}(X, Y) = E(XY) - E(X) \cdot E(Y)$$

$$= \frac{59}{50} - \frac{52}{35} \cdot \frac{138}{175} = \mathbf{\frac{103}{12{,}250}} \approx \mathbf{0.008408}.$$

$\mathrm{Cov}(X, Y) \neq 0.$ $\Rightarrow$ X and Y are **NOT independent**.

2. Every month, the government of Neverland spends X million dollars purchasing guns and Y million dollars purchasing butter. Since the people become disobedient if they do not have food, the amount spent monthly on butter is always at least 2 million dollars. However, there is a law that prevents the government of Neverland spending more than 5 million dollars on guns in one month. In addition, the government of Neverland cannot afford to spend more than 8 million dollars per month on guns and butter. Suppose that the joint density function for (X, Y) is

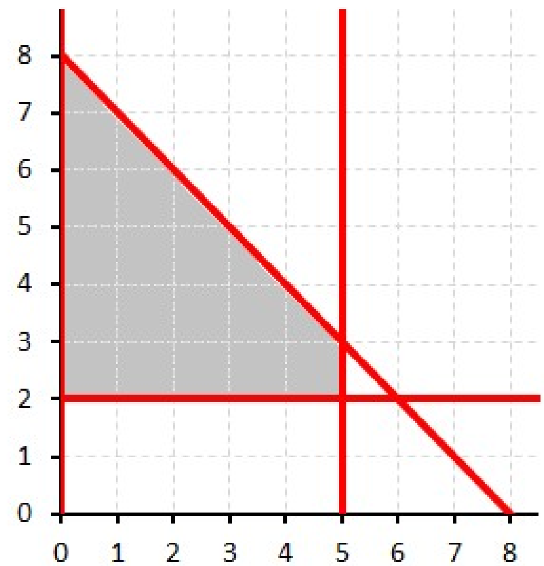
$$f(x,y) = \frac{7x+2y}{C}, \qquad x \ge 0, \quad y \ge 2, \quad x \le 5, \quad x+y \le 8, \qquad \text{zero otherwise.}$$

X – guns, Y – butter.

a) Sketch the support of (X, Y).

That is, sketch

$$\{ (x,y): x \ge 0, \quad y \ge 2, \quad x \le 5, \quad x+y \le 8 \}.$$



b) Find the value of C so that $f(x,y)$ is a valid joint probability density function.

Must have $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$

$$\int_0^5 \left(\int_2^{8-x} \frac{7x+2y}{C}\,dy \right) dx = \int_0^5 \left(\frac{7xy+y^2}{C} \right) \Bigg|_{y=2}^{y=8-x} dx$$

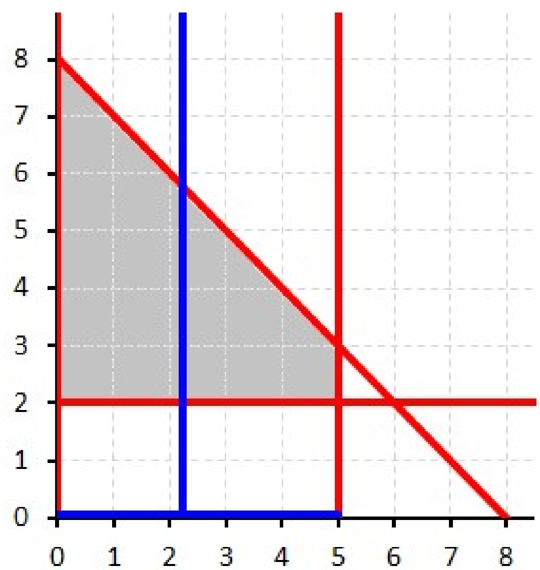
$$= \int_0^5 \frac{7x(8-x)+(8-x)^2 - 14x - 4}{C}\,dx$$

$$= \int_0^5 \frac{56x-7x^2+64-16x+x^2-14x-4}{C}\,dx = \int_0^5 \frac{60+26x-6x^2}{C}\,dx$$

$$= \left(\frac{60x + 13x^2 - 2x^3}{C} \right) \Bigg|_0^5 = \frac{300 + 325 - 250}{C} = \frac{375}{C} = 1.$$

$\Rightarrow \qquad C = \mathbf{375}.$

c) Find the marginal probability density function of X, $f_X(x)$.



$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy.$$

The range of possible values for X is $0 \le x \le 5$.

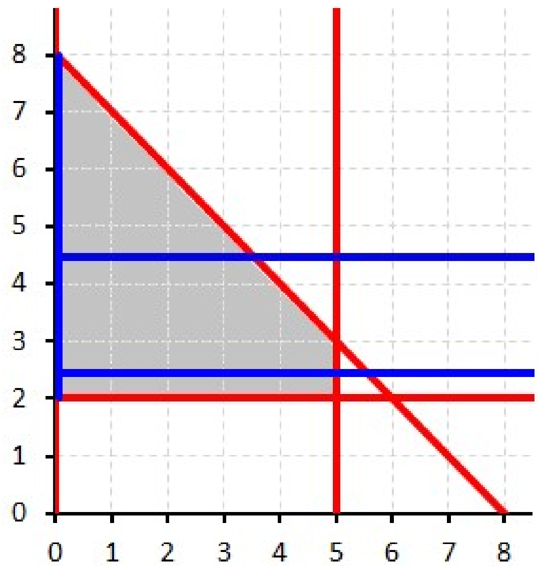
$$f_X(x) = \int_2^{8-x} \frac{7x + 2y}{375}\, dy = \left(\frac{7xy + y^2}{375} \right) \Bigg|_{y=2}^{y=8-x}$$

$$= \frac{7x(8-x) + (8-x)^2 - 14x - 4}{375}$$

$$= \frac{56x - 7x^2 + 64 - 16x + x^2 - 14x - 4}{375}$$

$$= \frac{60 + 26x - 6x^2}{375} = \frac{2(5+3x)(6-x)}{375}, \qquad 0 \le x \le 5.$$

d) Find the marginal probability density function of Y, $f_Y(y)$.



$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx.$$

The range of possible values for Y is $2 \le y \le 8$.

$f_Y(y)$ will be a **piecewise-defined** function, since the limits of this integral will not be the same for $2 \le y \le 3$ and for $3 \le y \le 8$.

If $2 \le y \le 3$,

$$f_Y(y) = \int_0^5 \frac{7x + 2y}{375}\, dx = \left(\frac{7x^2 + 4xy}{750} \right) \Bigg|_{x=0}^{x=5} = \frac{175 + 20y}{750} = \frac{35 + 4y}{150}, \qquad 2 \le y \le 3.$$

If $3 \le y \le 8$,

$$f_Y(y) = \int_0^{8-y} \frac{7x + 2y}{375}\, dx = \left(\frac{7x^2 + 4xy}{750} \right) \Bigg|_{x=0}^{x=8-y}$$

$$= \frac{7(8-y)^2 + 4y(8-y)}{750} = \frac{448 - 112y + 7y^2 + 32y - 4y^2}{750}$$

$$= \frac{448 - 80y + 3y^2}{750} = \frac{(56 - 3y)(8 - y)}{750}, \qquad 3 \le y \le 8.$$

$$f_Y(y) = \begin{cases} \dfrac{35+4y}{150} & 2 \le y \le 3 \\[2ex] \dfrac{448-80y+3y^2}{750} & 3 \le y \le 8 \\[2ex] 0 & \text{otherwise} \end{cases}$$

e) Are X and Y independent? *Justify your answer.*

$f(x,y) \neq f_X(x) \cdot f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

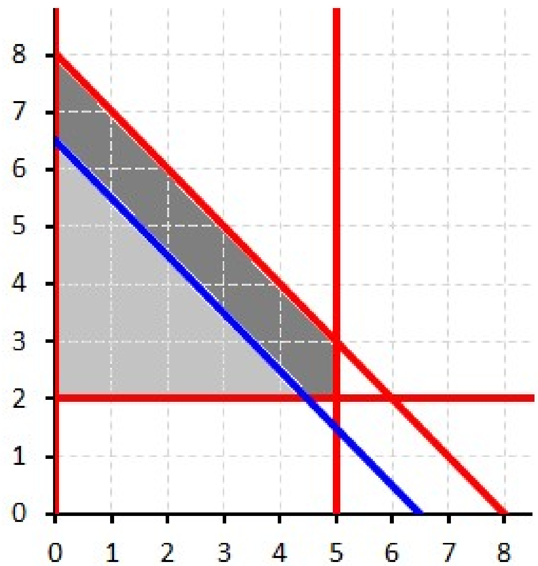
OR

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

OR

$f(x,y)$ cannot be written as a product of two functions, one of x only, the other of y only. $\Rightarrow$ X and Y are **NOT independent**.

f) Find the probability that the total amount spent monthly on guns and butter exceeds 6.5 million dollars. That is, find $P(X+Y>6.5)$.



$$P(X+Y>6.5) = \ldots$$

$$\ldots = 1 - \int_0^{4.5} \left(\int_2^{6.5-x} \frac{7x+2y}{375}\, dy \right) dx$$

$$\ldots = 1 - \int_2^{6.5} \left(\int_0^{6.5-y} \frac{7x+2y}{375}\, dx \right) dy$$

$$\ldots = \int_0^{4.5} \left(\int_{6.5-x}^{8-x} \frac{7x+2y}{375}\, dy \right) dx + \int_{4.5}^{5} \left(\int_2^{8-x} \frac{7x+2y}{375}\, dy \right) dx$$

$$\ldots = \int_2^{3} \left(\int_{6.5-y}^{5} \frac{7x+2y}{375}\, dx \right) dy + \int_3^{6.5} \left(\int_{6.5-y}^{8-y} \frac{7x+2y}{375}\, dx \right) dy + \int_{6.5}^{8} \left(\int_0^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

$$1 - \int_0^{4.5} \left(\int_2^{6.5-x} \frac{7x+2y}{375}\, dy \right) dx = 1 - \int_0^{4.5} \frac{38.25+18.5x-6x^2}{375}\, dx$$

$$= 1 - 0.4725 = \mathbf{0.5275}.$$

$$1 - \int_2^{6.5} \left(\int_0^{6.5-y} \frac{7x+2y}{375}\, dx \right) dy = 1 - \int_2^{6.5} \frac{147.875-32.5y+1.5y^2}{375}\, dy$$

$$= 1 - 0.4725 = \mathbf{0.5275}.$$

$$\int_{0}^{4.5}\left(\int_{6.5-x}^{8-x}\frac{7x+2y}{375}\,dy\right)dx+\int_{4.5}^{5}\left(\int_{2}^{8-x}\frac{7x+2y}{375}\,dy\right)dx$$

$$=\int_{0}^{4.5}\frac{21.75+7.5x}{375}\,dx+\int_{4.5}^{5}\frac{60+26x-6x^2}{375}\,dx$$

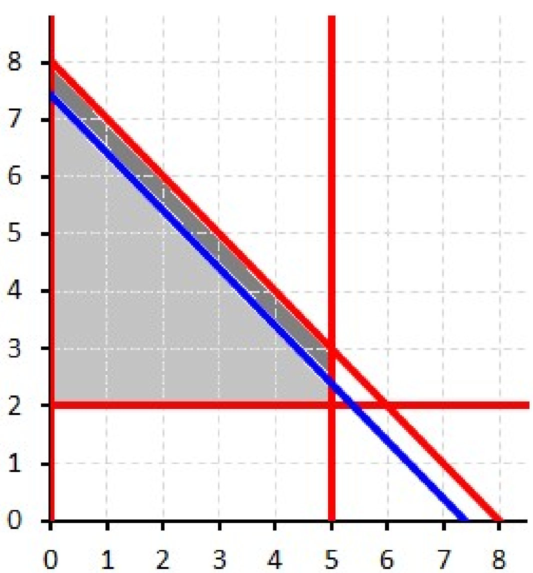
$$=0.4635+0.0640=\mathbf{0.5275}.$$

$$\int_{2}^{3}\left(\int_{6.5-y}^{5}\frac{7x+2y}{375}\,dx\right)dy+\int_{3}^{6.5}\left(\int_{6.5-y}^{8-y}\frac{7x+2y}{375}\,dx\right)dy+\int_{6.5}^{8}\left(\int_{0}^{8-y}\frac{7x+2y}{375}\,dx\right)dy$$

$$=\int_{2}^{3}\frac{-60.375+42.5y-1.5y^2}{375}\,dy+\int_{3}^{6.5}\frac{76.125-7.5y}{375}\,dy+\int_{6.5}^{8}\frac{224-40y+1.5y^2}{375}\,dy$$

$$=0.0970+0.3780+0.0525=\mathbf{0.5275}.$$

g) Find the probability that the total amount spent monthly on guns and butter exceeds 7.4 million dollars. That is, find $P(X+Y>7.4)$.



$$P(X+Y>7.4)=\ldots$$

$$\ldots=\int_{0}^{5}\left(\int_{7.4-x}^{8-x}\frac{7x+2y}{375}\,dy\right)dx$$

$$\ldots=1-\int_{0}^{5}\left(\int_{2}^{7.4-x}\frac{7x+2y}{375}\,dy\right)dx$$

$$\ldots=1-\int_{2}^{2.4}\left(\int_{0}^{5}\frac{7x+2y}{375}\,dx\right)dy$$

$$-\int_{2.4}^{7.4}\left(\int_{0}^{7.4-y}\frac{7x+2y}{375}\,dx\right)dy$$

$$\ldots = \int_{2.4}^{3} \left(\int_{7.4-y}^{5} \frac{7x+2y}{375}\, dx \right) dy + \int_{3}^{7.4} \left(\int_{7.4-y}^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

$$+ \int_{7.4}^{8} \left(\int_{0}^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

$$\int_{0}^{5} \left(\int_{7.4-x}^{8-x} \frac{7x+2y}{375}\, dy \right) dx = \int_{0}^{5} \frac{9.24+3x}{375}\, dx = \mathbf{0.2232}.$$

$$1 - \int_{0}^{5} \left(\int_{2}^{7.4-x} \frac{7x+2y}{375}\, dy \right) dx = 1 - \int_{0}^{5} \frac{50.76+23x-6x^2}{375}\, dx = 1 - 0.7768 = \mathbf{0.2232}.$$

$$1 - \int_{2}^{2.4} \left(\int_{0}^{5} \frac{7x+2y}{375}\, dx \right) dy - \int_{2.4}^{7.4} \left(\int_{0}^{7.4-y} \frac{7x+2y}{375}\, dx \right) dy$$

$$= 1 - \int_{2}^{2.4} \frac{87.5+10y}{375}\, dy - \int_{2.4}^{7.4} \frac{191.66-37y+1.5y^2}{375}\, dy$$

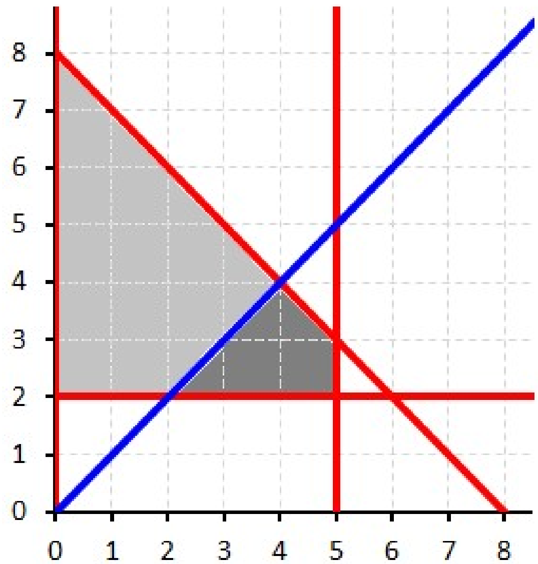
$$= 1 - 0.1168 - 0.6600 = \mathbf{0.2232}.$$

$$\int_{2.4}^{3} \left(\int_{7.4-y}^{5} \frac{7x+2y}{375}\, dx \right) dy + \int_{3}^{7.4} \left(\int_{7.4-y}^{8-y} \frac{7x+2y}{375}\, dx \right) dy + \int_{7.4}^{8} \left(\int_{0}^{8-y} \frac{7x+2y}{375}\, dx \right) dy$$

$$= \int_{2.4}^{3} \frac{-104.16+47y-1.5y^2}{375}\, dy + \int_{3}^{7.4} \frac{32.34-3y}{375}\, dy + \int_{7.4}^{8} \frac{224-40y+1.5y^2}{375}\, dy$$

$$= 0.018816 + 0.196416 + 0.007968 = \mathbf{0.2232}.$$

h) Find the probability that the government of Neverland spends more purchasing guns than purchasing butter in a given month. That is, find $P(X > Y)$.



$$P(X > Y) = \ldots$$

$$\ldots = \int_2^4 \left(\int_2^x \frac{7x + 2y}{375}\, dy \right) dx + \int_4^5 \left(\int_2^{8-x} \frac{7x + 2y}{375}\, dy \right) dx$$

$$\ldots = \int_2^3 \left(\int_y^5 \frac{7x + 2y}{375}\, dx \right) dy + \int_3^4 \left(\int_y^{8-y} \frac{7x + 2y}{375}\, dx \right) dy$$

$$\ldots = 1 - \int_0^2 \left(\int_2^{8-x} \frac{7x + 2y}{375}\, dy \right) dx - \int_2^4 \left(\int_x^{8-x} \frac{7x + 2y}{375}\, dy \right) dx$$

$$\ldots = 1 - \int_2^4 \left(\int_0^y \frac{7x + 2y}{375}\, dx \right) dy - \int_4^8 \left(\int_0^{8-y} \frac{7x + 2y}{375}\, dx \right) dy$$

$$\int_2^4 \left(\int_2^x \frac{7x + 2y}{375}\, dy \right) dx + \int_4^5 \left(\int_2^{8-x} \frac{7x + 2y}{375}\, dy \right) dx$$

$$= \int_2^4 \frac{-4 - 14x + 8x^2}{375}\, dx + \int_4^5 \frac{60 + 26x - 6x^2}{375}\, dx$$

$$= \frac{172}{1125} + \frac{11}{75} = \mathbf{\frac{337}{1125}} \approx 0.29955555\ldots.$$

$$\int_2^3 \left(\int_y^5 \frac{7x+2y}{375}\,dx \right) dy + \int_3^4 \left(\int_y^{8-y} \frac{7x+2y}{375}\,dx \right) dy$$

$$= \int_2^3 \frac{87.5+10y-5.5y^2}{375}\,dy + \int_3^4 \frac{224-40y-4y^2}{375}\,dy$$

$$= \frac{233}{1125} + \frac{104}{1125} = \mathbf{\frac{337}{1125}} \approx 0.29955555\ldots.$$

$$1 - \int_0^2 \left(\int_2^{8-x} \frac{7x+2y}{375}\,dy \right) dx - \int_2^4 \left(\int_x^{8-x} \frac{7x+2y}{375}\,dy \right) dx$$

$$= 1 - \int_0^2 \frac{60+26x-6x^2}{375}\,dx - \int_2^4 \frac{64+40x-14x^2}{375}\,dx$$

$$= 1 - \frac{52}{125} - \frac{64}{225} = \mathbf{\frac{337}{1125}} \approx 0.29955555\ldots.$$

$$1 - \int_2^4 \left(\int_0^y \frac{7x+2y}{375}\,dx \right) dy - \int_4^8 \left(\int_0^{8-y} \frac{7x+2y}{375}\,dx \right) dy$$

$$= 1 - \int_2^4 \frac{5.5y^2}{375}\,dy - \int_4^8 \frac{224-40y+1.5y^2}{375}\,dy$$

$$= 1 - \frac{308}{1125} - \frac{32}{75} = \mathbf{\frac{337}{1125}} \approx 0.29955555\ldots.$$

3. Consider two continuous random variables X and Y with joint p.d.f.

$$f(x,y) = \begin{cases} \dfrac{2}{81}x^2 y & 0 < x < K,\ 0 < y < K \\ 0 & \text{otherwise} \end{cases}$$

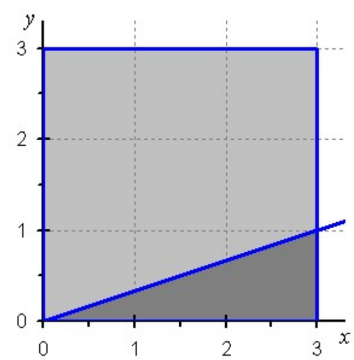
a) Find the value of K so that $f(x,y)$ is a valid joint p.d.f.

$$1 = \int_0^K \int_0^K \frac{2}{81}x^2\, y\, dx\, dy = \frac{K^5}{243}. \qquad \Rightarrow \qquad K = \mathbf{3}.$$

b) Find $\mathrm{P}(\mathrm{X} > 3\,\mathrm{Y})$.

$$\mathrm{P}(\mathrm{X} > 3\,\mathrm{Y}) = \int_0^3 \left(\int_0^{x/3} \frac{2}{81}x^2\, y\, dy \right) dx$$

$$= \int_0^3 \frac{1}{729}x^4\, dx = \mathbf{\frac{1}{15}}.$$



OR

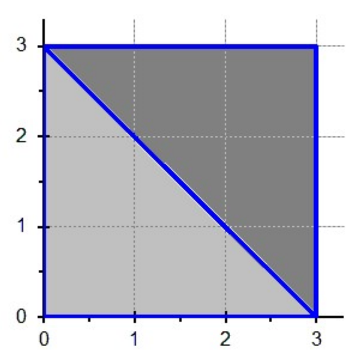
$$\mathrm{P}(\mathrm{X} > 3\,\mathrm{Y}) = \int_0^1 \left(\int_{3y}^{3} \frac{2}{81}x^2\, y\, dx \right) dy = \ldots = \mathbf{\frac{1}{15}}.$$

c) Find $\mathrm{P}(\mathrm{X} + \mathrm{Y} > 3)$.

$$\mathrm{P}(\mathrm{X} + \mathrm{Y} > 3) = \int_0^3 \left(\int_{3-x}^{3} \frac{2}{81}x^2\, y\, dy \right) dx$$

$$= \int_0^3 \frac{1}{81}x^2 \left[9 - (3-x)^2 \right] dx$$

$$= \frac{1}{81} \cdot \int_0^3 \left(6x^3 - x^4 \right) dx$$



$$= \frac{1}{81} \cdot \left(\frac{3}{2} x^4 - \frac{1}{5} x^5 \right) \Bigg|_0^3 = \frac{1}{81} \cdot \left(\frac{243}{2} - \frac{243}{5} \right) = \mathbf{0.90}.$$

OR

$$\mathrm{P}(\mathrm{X} + \mathrm{Y} > 3) = 1 - \int_0^3 \left(\int_0^{3-x} \frac{2}{81} x^2 \, y \, dy \right) dx = 1 - \int_0^3 \frac{1}{81} x^2 (3-x)^2 \, dx$$

$$= 1 - \frac{1}{81} \cdot \int_0^3 \left(9x^2 - 6x^3 + x^4 \right) dx = 1 - \frac{1}{81} \cdot \left(3x^3 - \frac{3}{2} x^4 + \frac{1}{5} x^5 \right) \Bigg|_0^3$$

$$= 1 - \frac{1}{81} \cdot \left(81 - \frac{243}{2} + \frac{243}{5} \right) = \mathbf{0.90}.$$

d) Are X and Y independent? *Justify your answer.*

$$f_{\mathrm{X}}(x) = \int_0^3 \frac{2}{81} x^2 \, y \, dy = \frac{1}{9} x^2, \quad 0 < x < 3,$$

$$f_{\mathrm{Y}}(y) = \int_0^3 \frac{2}{81} x^2 \, y \, dx = \frac{2}{9} y, \quad 0 < y < 3.$$

$f(x, y) = f_{\mathrm{X}}(x) \cdot f_{\mathrm{Y}}(y)$. $\Rightarrow$ X and Y are **independent**.

OR

The support of (X, Y) is a rectangle.

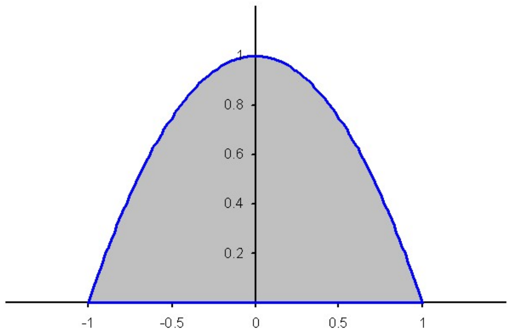
$f_{\mathrm{X,Y}}(x, y)$ can be written as a product of two functions,
one of x only, the other of y only.

$\Rightarrow$ X and Y are **independent**.

4. Suppose that (X, Y) is uniformly distributed over the region defined by $-1 \le x \le 1$ and $0 \le y \le 1 - x^2$. That is,

$$f(x, y) = C, \qquad -1 \le x \le 1,\ 0 \le y \le 1 - x^2, \qquad \text{zero elsewhere.}$$

a) What is the joint probability density function of X and Y? That is, find C.



$$\int_{-1}^{1}\left(\int_{0}^{1-x^2} dy\right)dx = \int_{-1}^{1}\left(1-x^2\right)dx$$

$$= \left(x - \frac{x^3}{3}\right)\Bigg|_{-1}^{1} = \frac{4}{3}.$$

$$\Rightarrow \quad f_{\mathrm{X,Y}}(x, y) = \mathbf{\frac{3}{4}}, \quad -1 \le x \le 1, \quad 0 \le y \le 1 - x^2.$$

b) Find the marginal probability density function of X, $f_{\mathrm{X}}(x)$.

$$f_{\mathrm{X}}(x) = \int_{0}^{1-x^2} \frac{3}{4}\,dy = \frac{3}{4}\left(1-x^2\right), \qquad -1 \le x \le 1.$$

c) Find the marginal probability density function of Y, $f_{\mathrm{Y}}(y)$.

$y = 1 - x^2$ $\qquad\qquad$ $x = \pm\sqrt{1-y}$

$$f_{\mathrm{Y}}(y) = \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{3}{4}\,dx = \frac{3}{2}\sqrt{1-y}, \qquad 0 \le y \le 1.$$

d) Are X and Y independent? If not, find $\mathrm{Cov}(\mathrm{X}, \mathrm{Y})$.

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

OR

$f_{\mathrm{X,Y}}(x, y) \neq f_{\mathrm{X}}(x) \times f_{\mathrm{Y}}(y)$. $\Rightarrow$ X and Y are **NOT independent**.

$\mathrm{E}(\mathrm{X}) = 0,$ since the distribution of X is symmetric about 0.

$$\mathrm{E}(\mathrm{Y}) = \int_0^1 y \cdot \frac{3}{2} \sqrt{1-y}\, dy = 0.4.$$

$$\mathrm{E}(\mathrm{XY}) = \int_{-1}^{1} \left(\int_0^{1-x^2} xy \cdot \frac{3}{4}\, dy \right) dx = 0.$$

$$\mathrm{Cov}(\mathrm{X}, \mathrm{Y}) = \mathrm{E}(\mathrm{XY}) - \mathrm{E}(\mathrm{X}) \times \mathrm{E}(\mathrm{Y}) = \mathbf{0}.$$

Recall:			
	Independent	$\Rightarrow$	Cov = 0
	Cov = 0	$\not\Rightarrow$	Independent