Sam Grayson's Notebook (with LaTeX)

January 14, 2015

1.1	$ma = b$	Definition of 'divides'
	$na = c$	Definition of 'divides'
	$na + ma = b + c$	Algebra
	$(n + m)a = b + c$	Algebra
	$a\|(b + c)$	Definition of 'divides' ∎
1.2	Let $d = -c$	
	$a\|(b + d)$	Theorem 1.1
	$a\|(b - c)$	substitution ∎
1.3	$ma = b$	Definition of 'divides'
	$na = c$	Definition of 'divides'
	$mana = bc$	Algebra
	$a\|bc$	Definition of 'divides' ∎
1.4	$mana = bc$	see last proof
	$a^2\|bc$	Definition of 'divides' ∎

1.5 *If $a|b$ then $a|b^n$*

	$b = ka$	Definition of 'divides'
	$b^n = (ka)^n = kk^{(n-1)}a^n$	Algebra
	$k\|b^n$	Definition of 'divides' ∎
1.6	$ka = b$	Definition of 'divides'
	$ack = bc$	Algebra
	$a\|bc$	Definition of 'divides' ∎

1.7

1. $45 - 9 = 36 = 9 \cdot 4$. True
2. $37 - 2 = 35 = 7 \cdot 5$. True
3. $37 - 3 = 34$. False
4. $37 - (-3) = 40 = 8 \cdot 5$. True

1.8	let k be all the numbers	
	where $k \equiv b \pmod 3$	
	$3\|(k - b)$	Definition of 'mod'
	$3n = k - b$	Definition of 'divides'
	$3n + k = n$	Algebra ∎

1. $3n$
2. $3n + 1$
3. $3n + 2$
4. $3n$
5. $3n + 1$

1.9	$a - a = 0 = 0n$	Arithmetic
	$n\|(a - a)$	Definition of 'divides'
	$a \equiv 0 \pmod n$	Definition of 'mod' ∎

1.10	$n\|(a-b)$	Definition of 'mod'
	$kn = a-b$	Definition of 'divides'
	$-kn = b-a$	Algebra
	$n\|(b-a)$	Definition of 'divides'
	$b \equiv a \pmod{n}$ ■	
1.11	$n\|(a-b)$	Definition of 'mod'
	$n\|(b-c)$	Definition of 'mod'
	$n\|(a-b+b-c)$	Theorem 1.1
	$n\|(a-c)$	Algebra
	$a \equiv c \pmod{n}$	Definition of 'mod' ■
1.12	$n\|(a-b)$	Definition of 'mod'
	$n\|(c-d)$	Definition of 'mod'
	$n\|(a+c-b-d))$	Theorem 1.1
	$n\|((a+c)-(b+d))$	Algebra
	$a+c \equiv b+d \pmod{n}$	definion 'mod' ■
1.13	let $e=-c$ and $f=-d$	
	$a+e \equiv b+f$	Theorem 1.12
	$a-c \equiv b-d$	substitution ■
1.14	$n\|(a-b)$	Definition of 'mod'
	$n\|(c-d)$	Definition of 'mod'
	$n\|(a-b)(c-d)$	Theorem 1.3 ■
1.15	$a \equiv b \pmod{n}$	Premise
	$a^2 \equiv b^2 \pmod{n}$	Theorem 1.14 ■
1.16	$a \equiv b \pmod{n}$	Premise
	$a^2 \equiv b^2 \pmod{n}$	Theorem 1.15
	$a^2 a \equiv b^2 b \pmod{n}$	Theorem 1.14
	$a^3 \equiv b^3 \pmod{n}$	Algebra ■
1.17	$a \equiv b \pmod{n}$	Premise
	$a^{k-1} \equiv b^{k-1} \pmod{n}$	Premise
	$a^{k-1}a \equiv b^{k-1}b \pmod{n}$	Theorem 1.14
	$a^k \equiv b^k \pmod{n}$	Algebra ■
1.18	Base case:	
	$a \equiv b \pmod{n}$	Premise
	Inductive Hypothesis:	
	$a^{k-1} \equiv b^{k-1} \pmod{n}$	(assumption)
	Inductive step:	
	$a^{k-1}a \equiv b^{k-1}b \pmod{n}$	Theorem 1.14
	$a^k \equiv b^k \pmod{n}$	Algebra
	Conclusion:	
	$a^k \equiv b^k \pmod{n}$	inductively ■

1.19 12. $6 \equiv 2 \pmod{4}$
$5 \equiv 1 \pmod{4}$

$6+5 \equiv 2+1 \pmod 4$

13. $6-5 \equiv 2-1 \pmod 4$

14. $6 \cdot 5 \equiv 2 \cdot 1$

15. $6^2 \equiv 2^2 \pmod 4$

16. $6^3 \equiv 2^3 \pmod 4$

17. $6^4 \equiv 2^4 \pmod 4$

18. $6^k \equiv 2^k \pmod 4$

1.20 No
Consider the case wehre $n=4$, $c=0$, $a=1$, and $b=2$.
$ac \equiv bc \pmod n$
$a \neq b$

1.21 See 1.22 and 1.23

1.22

$3 \mid a$	Premise (Base Case)
$3 \mid b$	Let b be an integer where. . . (In
$3 \mid 9$	Arithmetic
$3 \mid (9b_k 10^{k-1})$	Theorem 1.3
$3 \mid (b - 9b_k 10^{k-1})$	Theorem 1.2
$3 \mid (b_{k-1} + b_k)b_{k-2} \ldots b_0$	Algebra* (Inductive Step)
$3 \mid (a_k + a_{k-1} + a_{k-2} + \ldots a_1 + a_0)$	Inductive axiom ■

Here is the algebra I used in the step labeled 'Algebra*':

$$\begin{array}{r}
b - b_k 910^{k-1} \quad = \\
b - b_k(10-1)10^{k-1} \quad = \\
b + (-b_k 10 \cdot 10^{k-1} + b_k 110^{k-1}) \quad = \\
b + (-b_k 10^k + b_k 10^{k-1}) \quad =
\end{array}$$

$$\begin{array}{cccccccc}
 & b_k & b_{k-1} & b_{k-2} & \ldots & b_0 & \\
+ & (-b_k) & b_k & 0 & \ldots & 0 & = \\
\hline
 & & (b_k + b_{k-1}) & b_{k-2} & \ldots & b_0 &
\end{array}$$

1.23

$3 \mid a$	Premise (Base Case)
$3 \mid (b_k + b_{k-1} + \ldots + b_0)$	Assumption (Inductive Hypothe
$3 \mid 9$	Arithmetic
$3 \mid (b_k 9c)$ where c is k ones in a row	Theorem 1.3
$3 \mid (b_k + b_{k-1} + \ldots + b_0 + b_k 9c)$	Theorem 1.2
$3 \mid (b_k 10^k + b_{k-1} + \ldots + b_0)$	Algebra*
$3 \mid (a_k 10^k + a_{k-1} 10^{k-1} + \ldots + a_0 10^0)$	Inductive Axiom
$3 \mid (a_k a_{k-1} \ldots a_0)$	Definition of digits ■

Here is the algebra I used in the step labeled 'Algebra*':

$$
\begin{aligned}
b_k + b_{k-1} + \ldots + b_0 + b_k 9c &= \\
b_k + b_{k-1} + \ldots + b_0 + b_k d &= \quad \text{where d is a number with } k \text{ nines} \\
b_k + b_{k-1} + \ldots + b_0 + b_k(10^k - 1) &= \\
b_k + b_{k-1} + \ldots + b_0 + b_k 10^k - b_k &= \\
b_{k-1} + \ldots + b_0 + b_k 10^k &
\end{aligned}
$$

1.24 $4|a$ if and only if $4|(a_1 + a_3 + \ldots)(a_0 + a_2 + a_4 + \ldots)$

1.25
1. $m = nq + r$ where $m = 25$, $n = 7$, $q = 3$, and $r = 4$
2. $m = 277$, $n = 4$, $q = 66$, and $r = 1$
3. $m = 33$, $n = 11$, $q = 3$, $r = 0$
4. $m = 33$, $n = 45$, $q = 0$, $r = 33$

1.26 Setup:

(Make a list of multiples of n that are greater than m and choose the smallest one to define $n(q+1)$.)	
$A := \{k \mid k \in \mathbb{N} \ \wedge kn > m\}$	
$\exists a \ni (a \in A \wedge an > m \wedge \forall k \in A(a \leq k))$	Well-ordering Principle
$q := a - 1$	
$r := m - nq$	

Proving r satisfies upper bound

(If it didn't, then a wouldn't be an element of A, but we know that a is in A.) $r > n - 1$	Assume for contradiction
$r \geq n$	Inequality over integers
$\exists j \ni (r - n = j \wedge j \geq 0)$	Property of inequalities
$nq + r = m$	Algebra (from definition of r)
$nq + (n + j) = m$	Algebra
$n(q+1) + j = m$	Algebra
$n(q+1) \leq m$	Property of inequalities
$n(q+1) > m$	Algebra (from definition of a)
$\therefore r \leq n - 1$	Contradiction

Proving r satisfies lower bound

(If it didn't, then there would be another element smaller than a in A, but a is the least element in A.) $r < 0$	Assume for contradiction
$nq + r = m$	Algebra (from definition of r)
$nq > m$	Property of inequalities
$q \in A$	$q \in mathbbN \wedge nq > m$ is the c
$\forall k(k \in A \rightarrow q + 1 \leq k)$	Definition of a (it's the smallest
$q \leq q + 1$	Universal instantiation
$\therefore r \geq 0$	Contradiction

■