

# UESTC 3018 - Communication Systems and Principles

Lecture 12 — Angle Modulation in the Frequency Domain

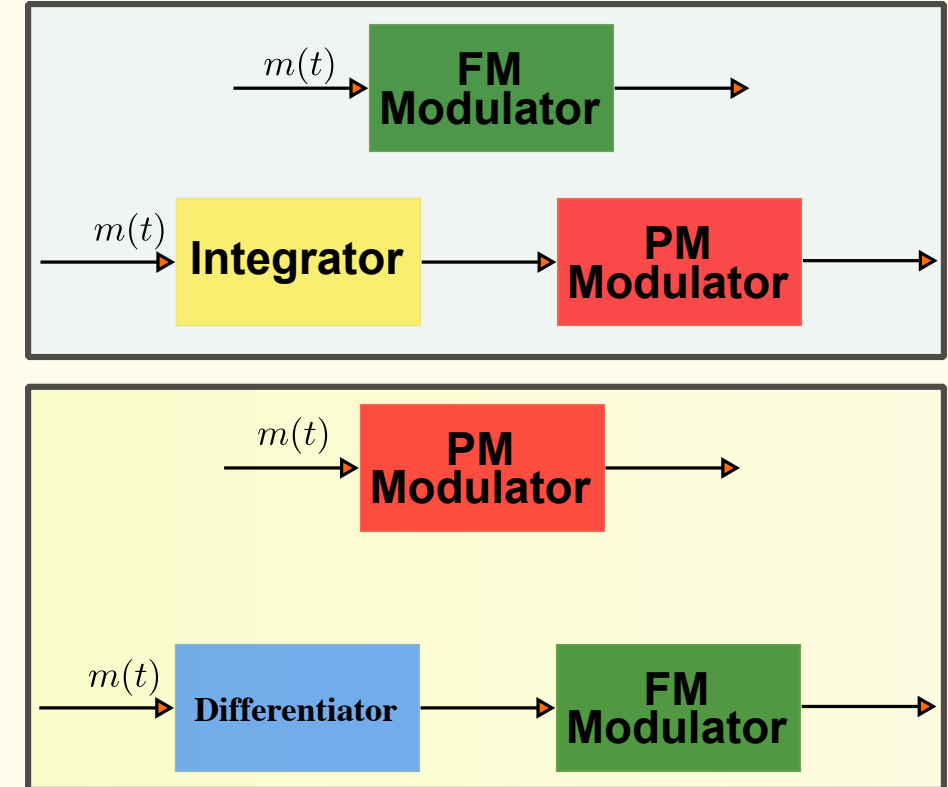
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## From Last Time

- Angle Modulation is a non-linear process
- We don't change the amplitude
- In PM, we vary the phase  $\theta(t)$  **linearly** with  $m(t)$
- In FM, we vary the frequency  $\omega(t)$  **linearly** with  $m(t)$ ,
- PM and FM are very similar - a  $90^\circ$  phase-shift



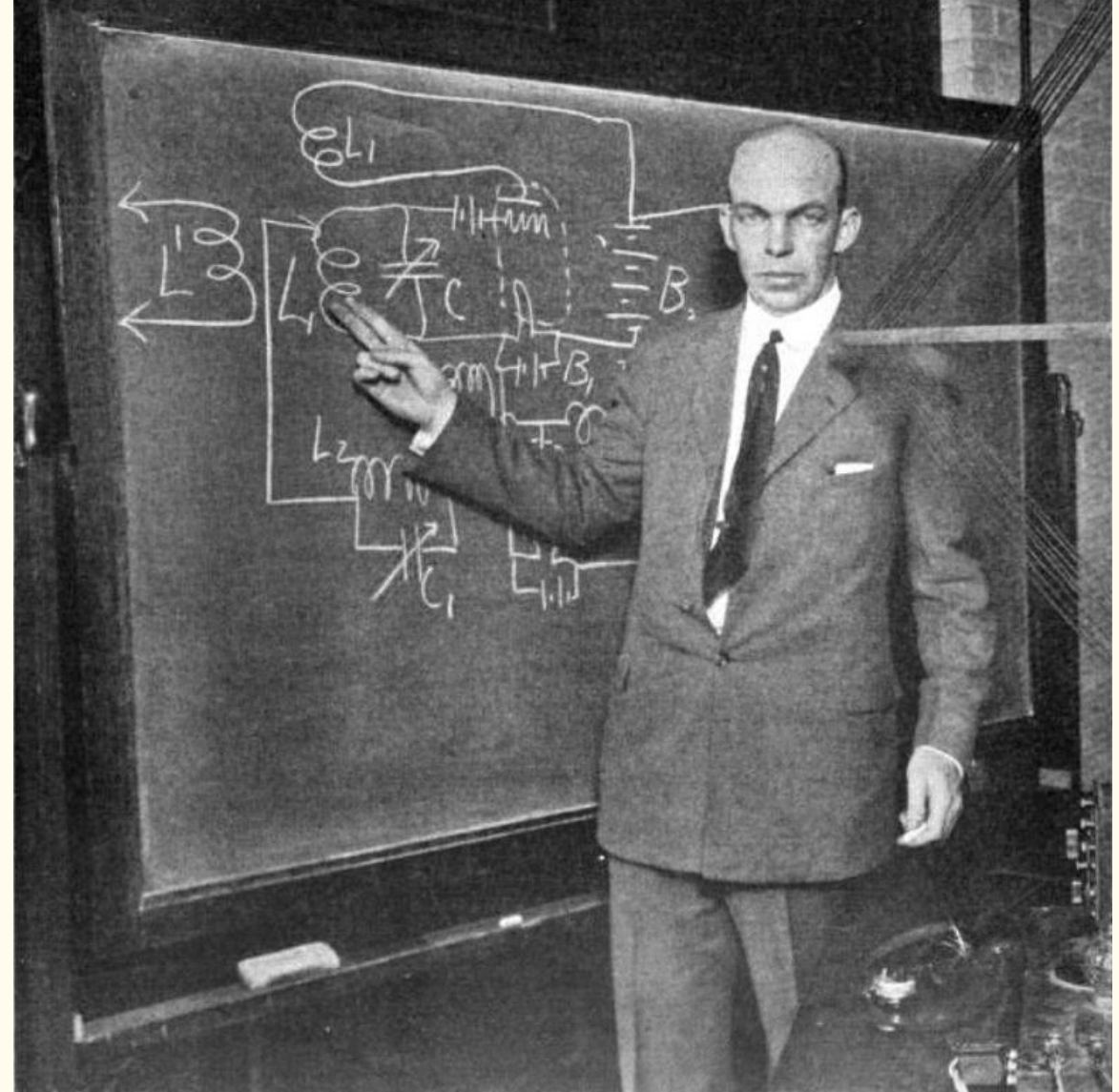
# This Lecture

- Bandwidth in FM
- Carson's Rule
- Spectral Analysis

# Down the History Lane



- Due to inherent non-linearity, FM is hard to analyse
- Can't really apply Fourier transform tools
- Motivation was to reduce the bandwidth
- Turns out FM has infinite theoretical bandwidth



# The Historical Irony: The "Bandwidth Fallacy" 🤯

FM was originally designed to **SAVE** space

- **The 1920s Goal:** AM radios were crowded. Engineers wanted a "Narrowband" system to squeeze more stations onto the dial.
- **The Intuition:** "If I only wiggle the frequency by  $\pm 50$  Hz, surely the bandwidth is tiny!"
- **The Mathematical Reality:**
  - In 1922, mathematician **John Carson** proved that FM actually generates **infinite sidebands**.
  - He famously declared FM "a nuisance" and static.
- **The Pivot:** Edwin Armstrong realised the "failure" was a "feature."
  - *New Idea:* "Stop trying to save bandwidth. Let's **waste** bandwidth to destroy noise!"

## Recall from the Previous Lecture ...

- In FM, we vary the frequency  $\omega_i(t)$  **linearly** with  $m(t)$ ,

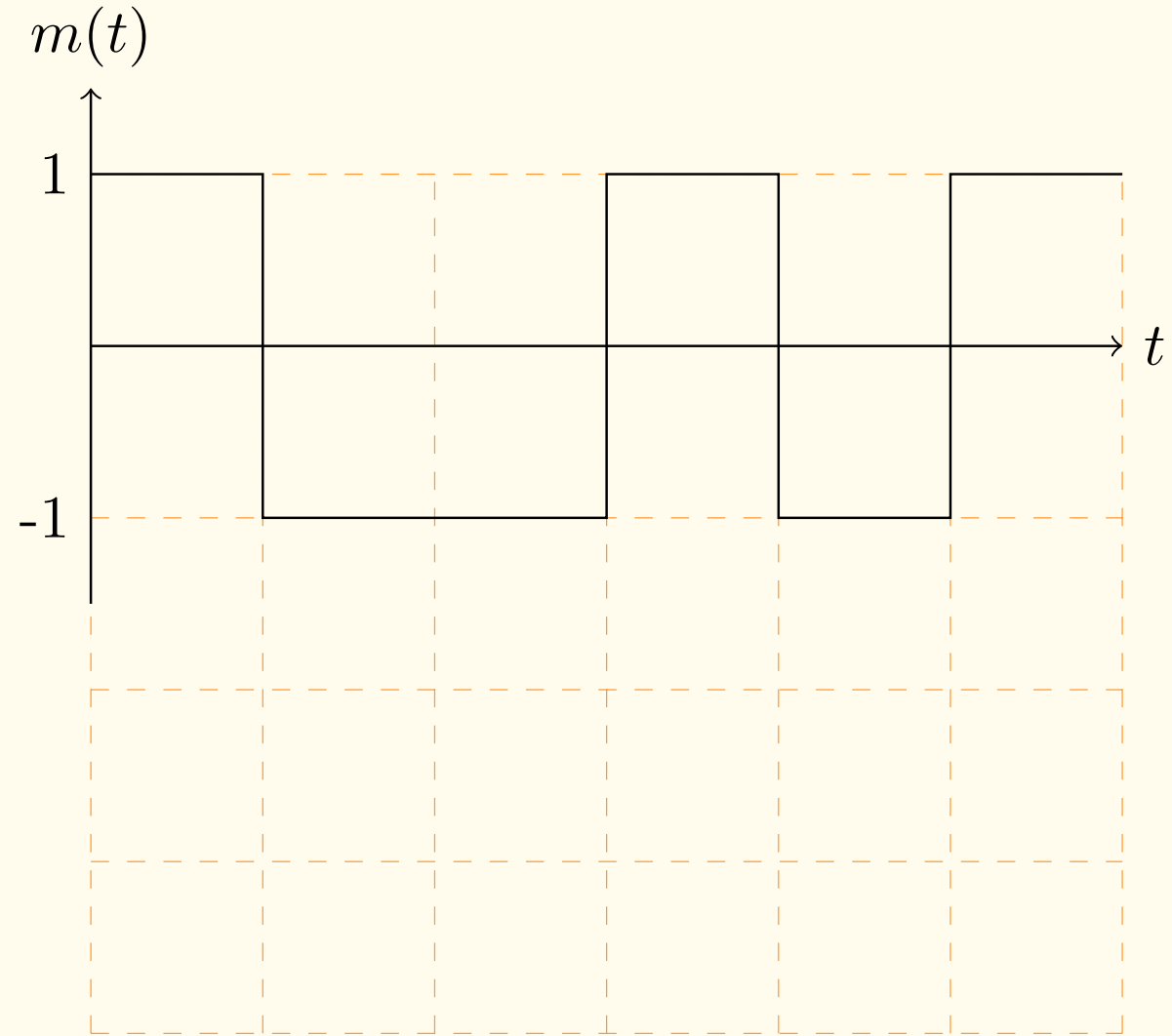
$$\omega_i^{FM}(t) = \omega_c + k_f m(t)$$

- The phase  $\theta^{FM}$  is,

$$\theta^{FM}(t) = \int_{-\infty}^t \omega_i^{FM}(u) du = \omega_c t + k_f \int_{-\infty}^t m(u) du$$

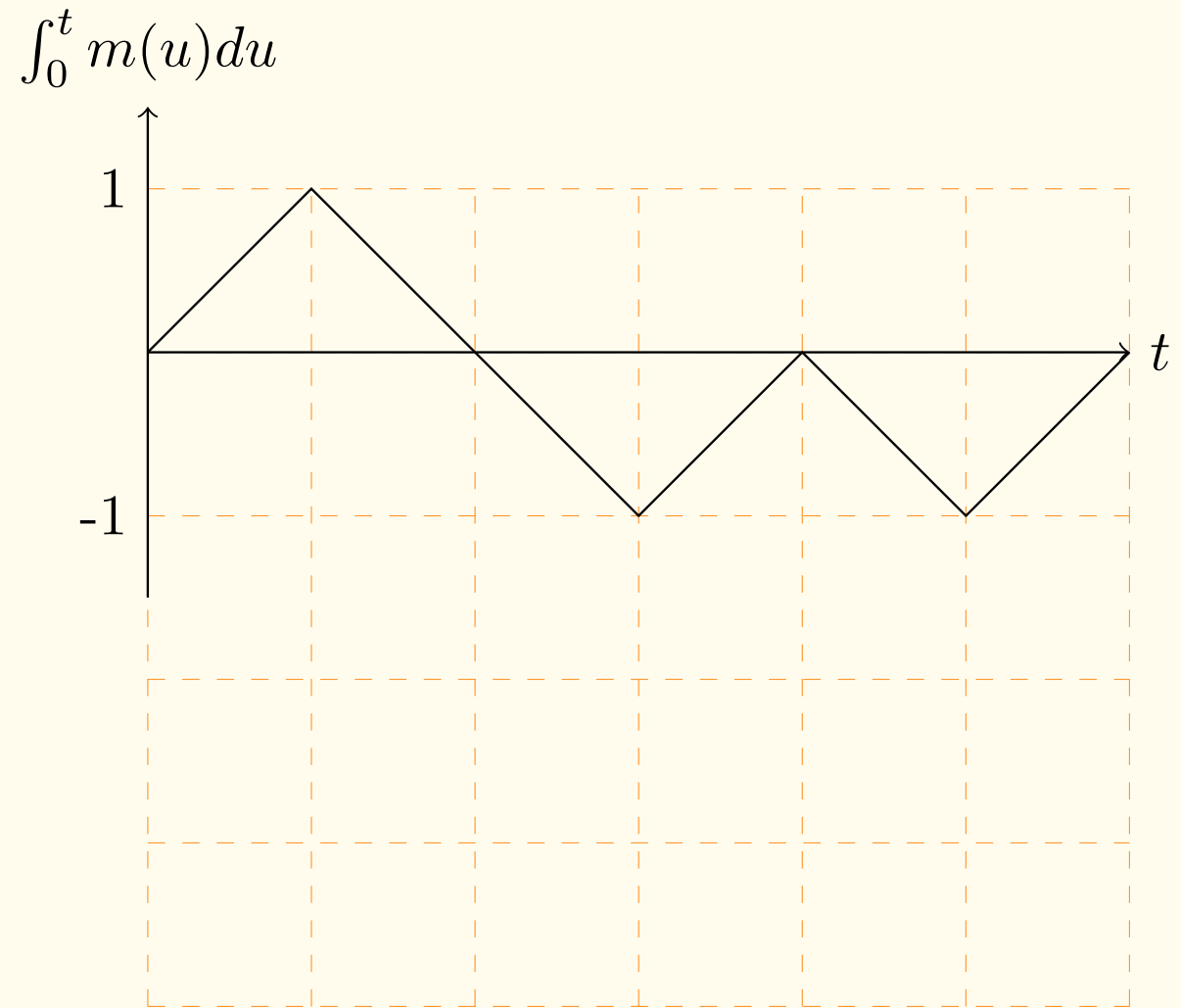
## First an Example - FSK

- Idea is to send a string of bits as two different frequencies
- Commonly used in digital radio, electronics and modems



## What about PM?

- What if we integrated the waveform?
- Essentially achieving the same waveform through phase modulator
- Input is now  $\int_0^t m(u)du$





# Bandwidth Analysis



# Analysing Bandwidth

- To simplify the expression,  $a(t) = \int_{-\infty}^t m(u) du$
- Let's define,

$$\hat{\varphi}^{\text{FM}}(t) = A e^{j[\omega_c t + k_f a(t)]} = A e^{j k_f a(t)} e^{j \omega_c t}$$

from where,

$$\varphi^{\text{FM}}(t) = \Re [\hat{\varphi}^{\text{FM}}(t)] .$$

- Expanding the complex exponential through the Maclaurin power series,

$$\hat{\varphi}^{\text{FM}}(t) = A \left[ 1 + j k_f a(t) - \frac{k_f^2}{2!} a^2(t) + \dots + j^n \frac{k_f^n}{n!} a^n(t) \right] \times e^{j \omega_c t}$$

# Some Observations

- If  $m(t)$  or  $M(\omega)$  has a bandwidth of  $B$
- Then  $a(t)$  also has a bandwidth of  $B$  Hz (integration is a linear operator).
- The  $n^{th}$  term,  $\frac{k_f^n}{n!} a^n(t)$  will have a bandwidth of  $n \times B$
- This is due to convolution principle, i.e.
- $A(\omega) * A(\omega)$  spreads the Fourier transform to  $2B$
- Essentially, we have **infinite bandwidth**
- But...
- $\frac{k_f^n}{n!} a^n(t) \rightarrow 0$ , meaning we only care about the **first few terms**.

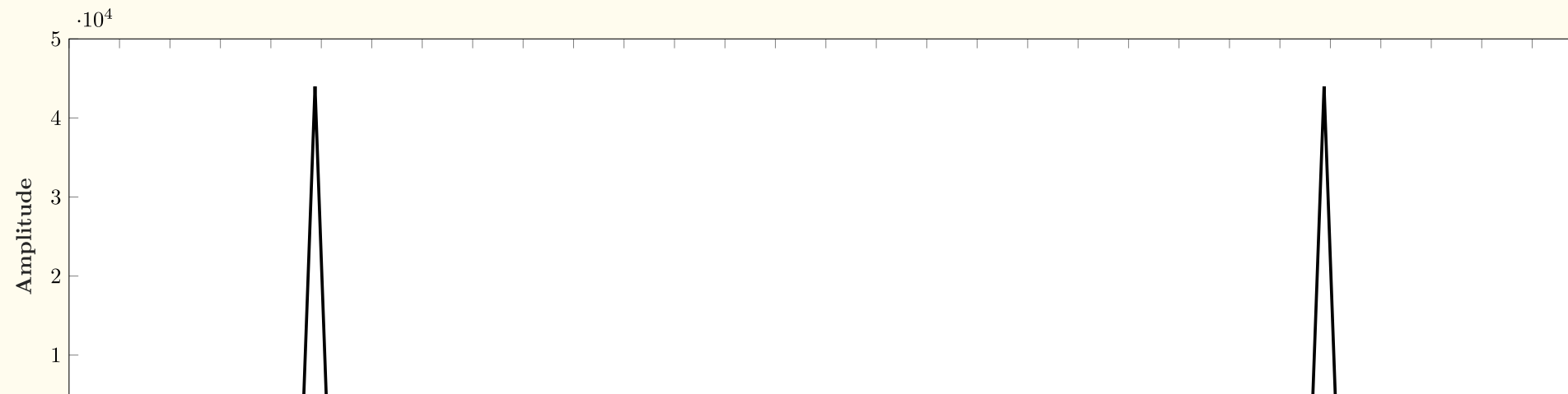
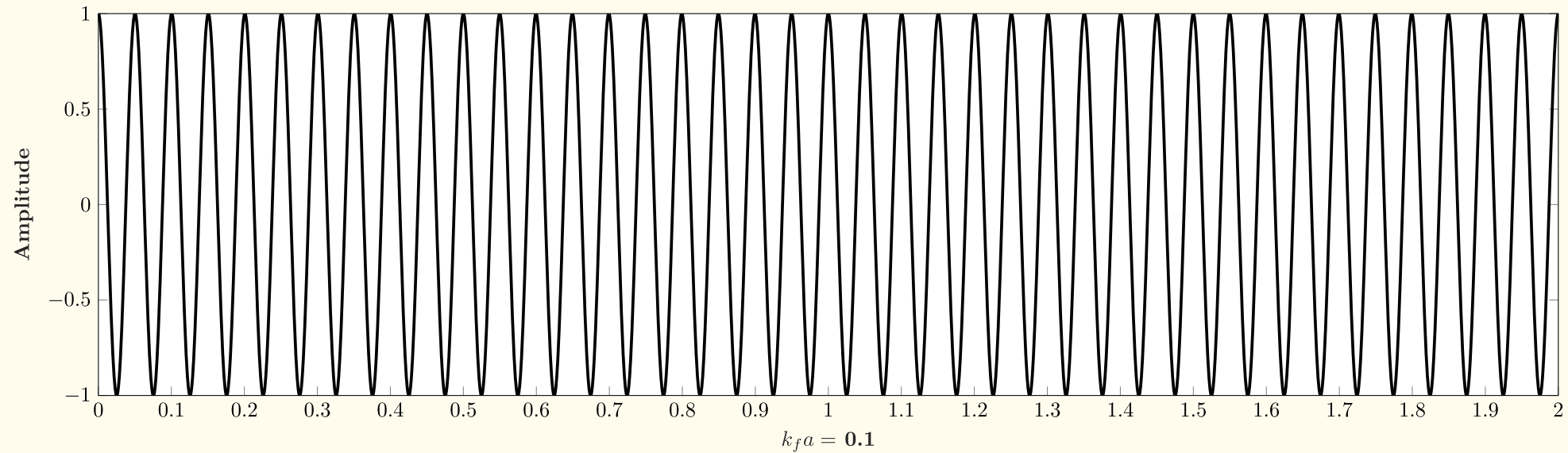
# FM Signal Representation

- Using the  $\varphi^{\text{FM}}(t) = \Re [\hat{\varphi}^{\text{FM}}(t)]$  representation, we get,

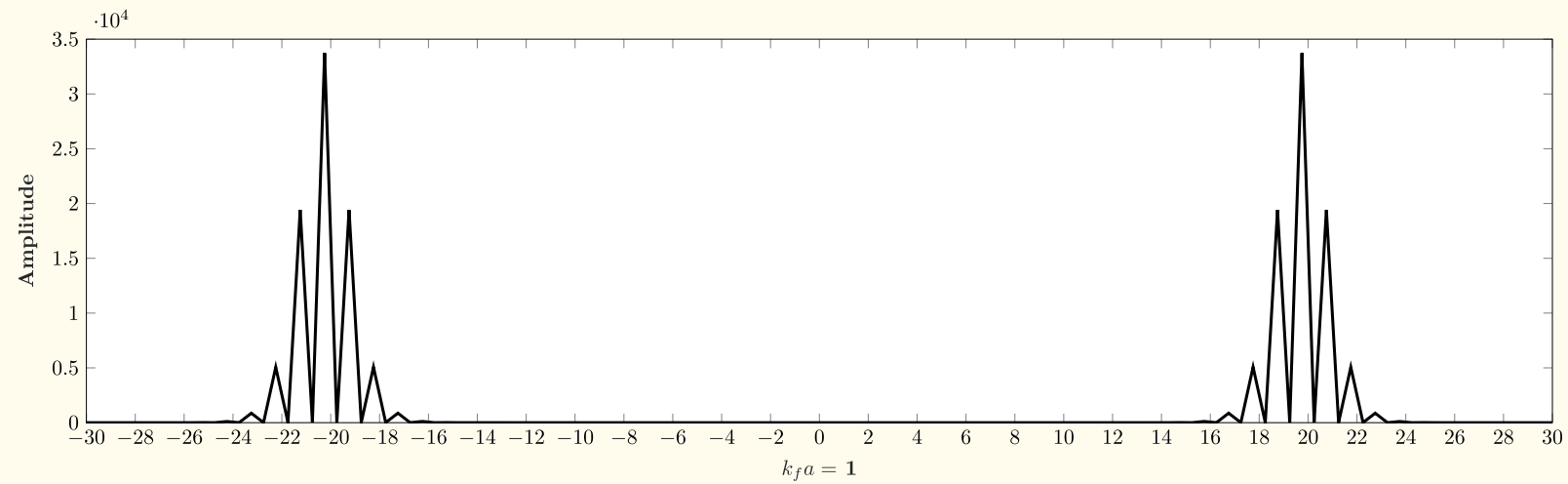
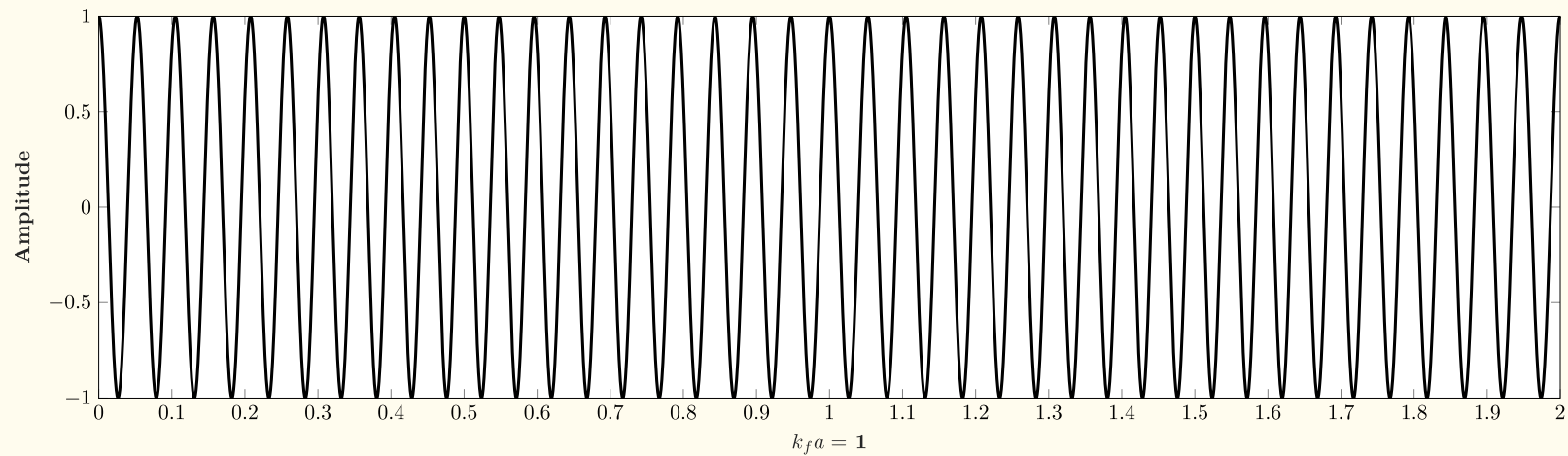
$$\begin{aligned}\varphi^{\text{FM}}(t) &= \Re \left\{ A \left[ 1 + j k_f a(t) - \frac{k_f^2}{2!} a^2(t) + \dots + j^n \frac{k_f^n}{n!} a^n(t) \right] \times [\cos(2\pi f_c t) + j \sin(2\pi f_c t)] \right\} \\ &= A \left( \cos(2\pi f_c t) - k_f a(t) \sin(2\pi f_c t) - \frac{k_f^2}{2!} a^2(t) \cos(2\pi f_c t) + \dots \right) \\ &\approx A (\cos(2\pi f_c t) - k_f a(t) \sin(2\pi f_c t))\end{aligned}$$

- This is a narrowband FM signal representation
- The approximation is good when  $|k_f a(t)| \ll 1$
- Generally, we consider 2B bandwidth as narrowband
- PM has a similar expression

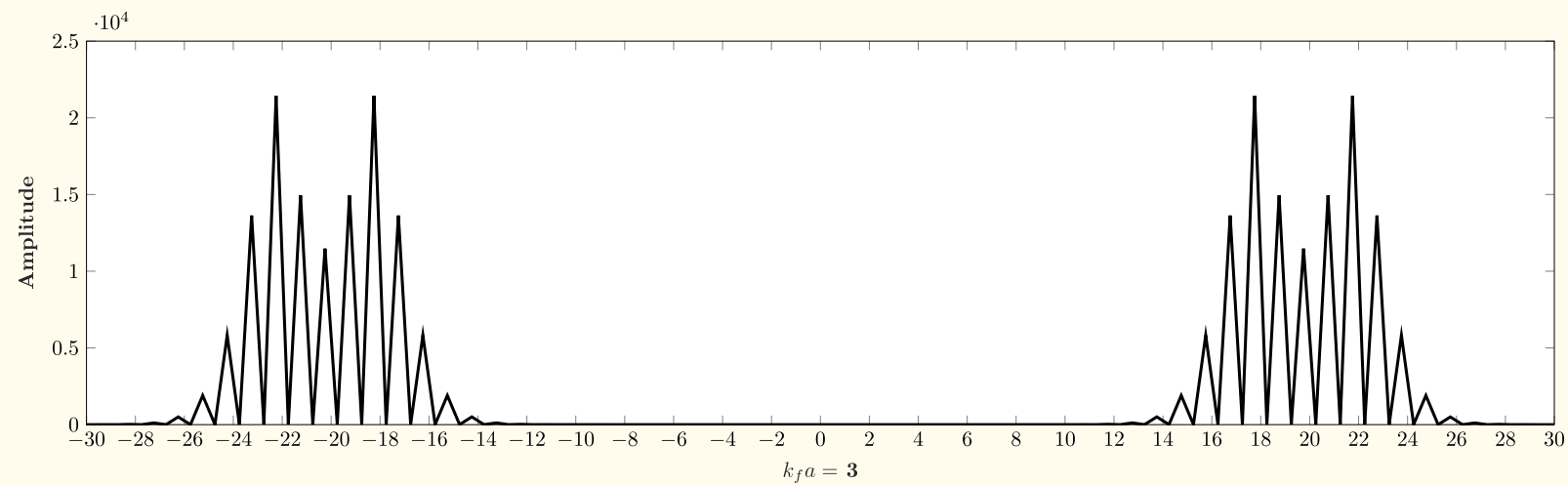
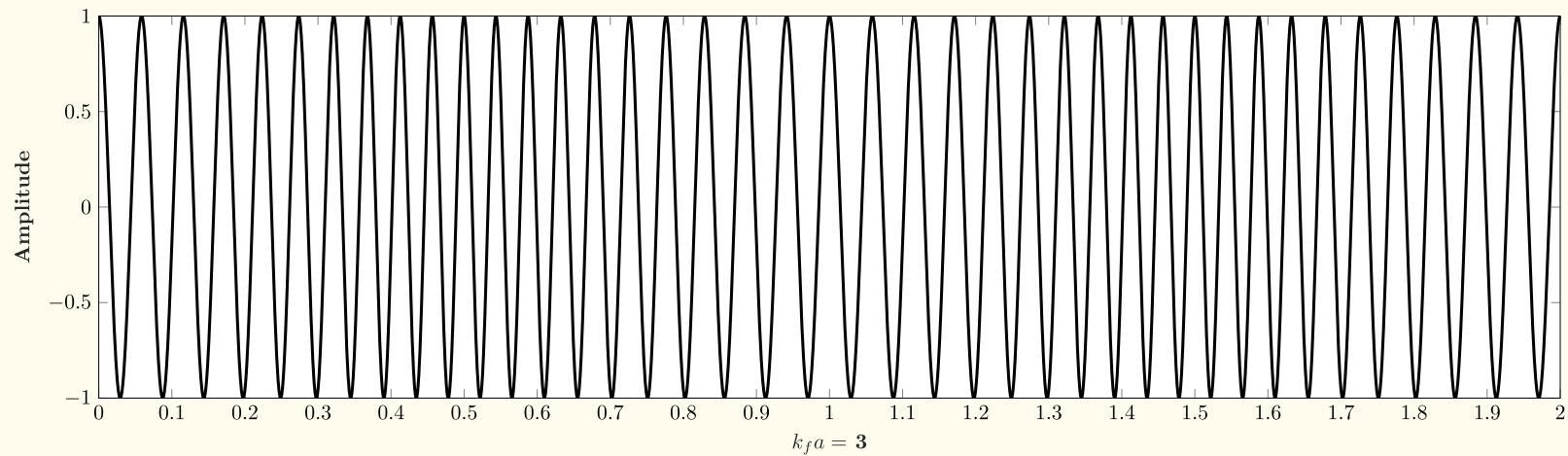
# Playing with the tones



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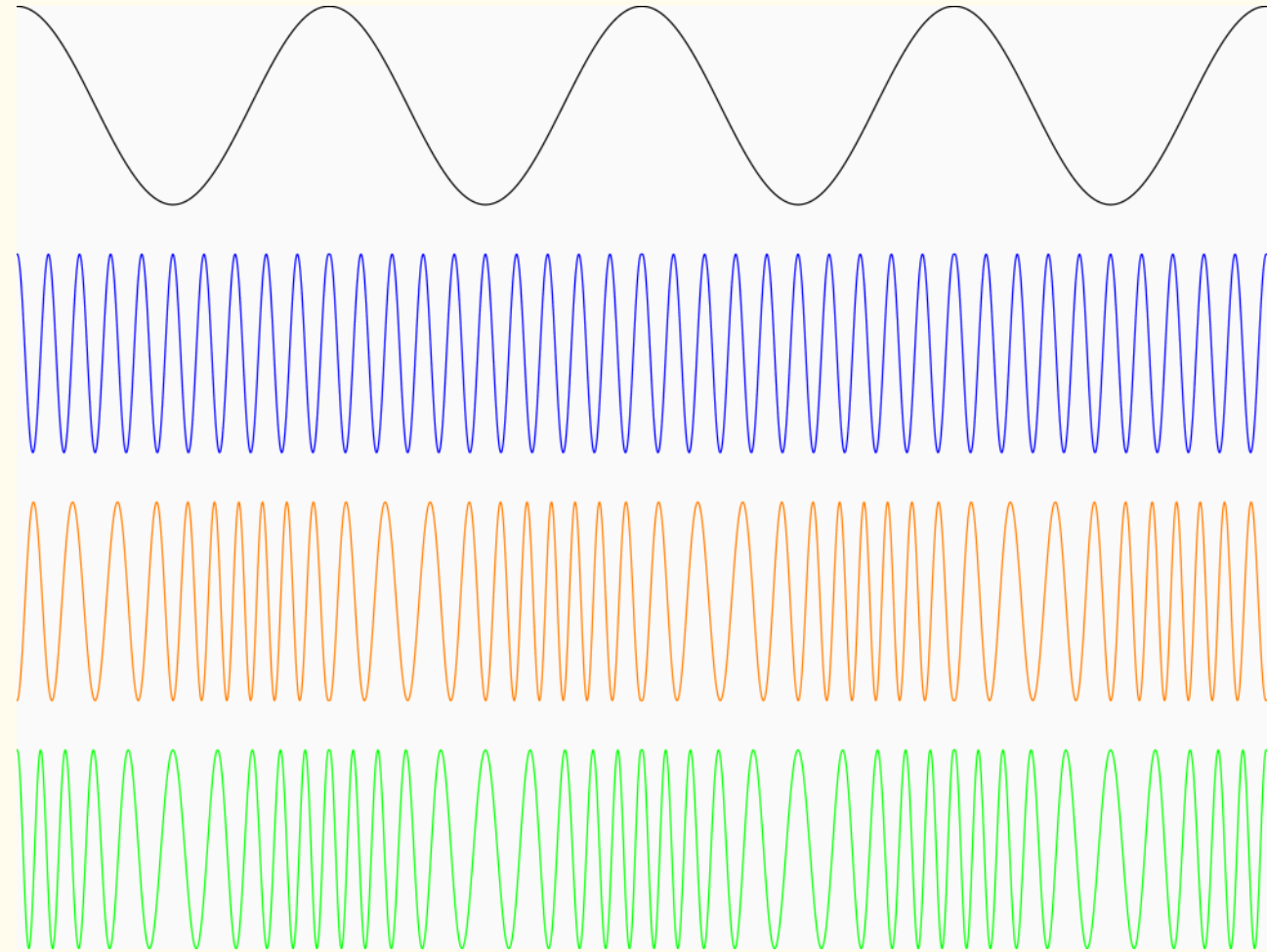
# A Dilemma 🌀

- To make the best of FM (or PM), we need make the frequency deviation large enough
- Need to choose a large enough  $k_f$  to break the  $|k_f a(t)| \ll 1$  condition
- This is the **wideband** FM case
- ⚠️ We can't ignore the higher order terms in the power series anymore
- We need to rely on empirical formulas to estimate the bandwidth



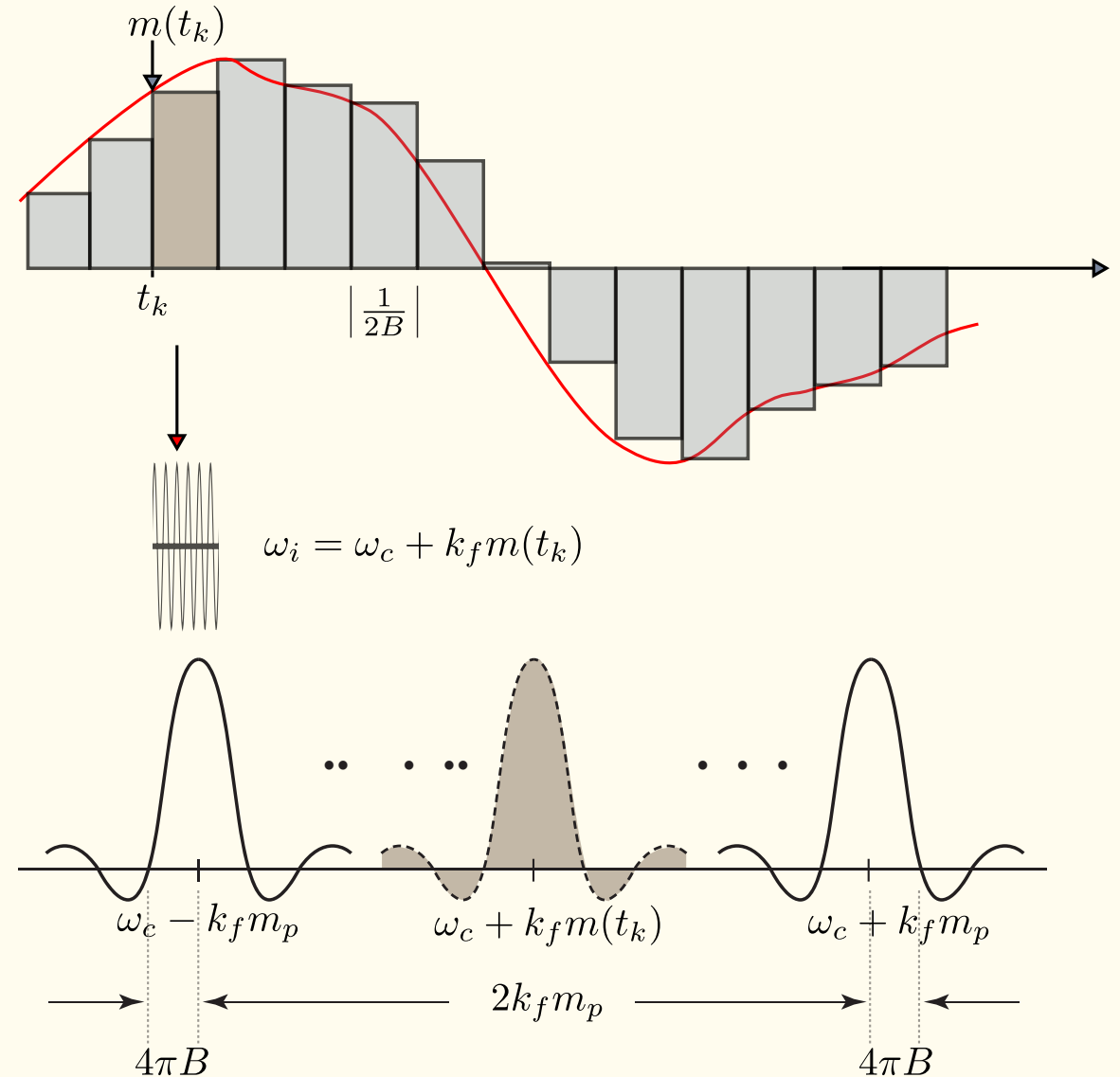
# Wideband FM

- **Context** We need frequency deviation for the FM signal to be meaningful
- But what is the bandwidth of an FM signal?
- **Answer** We use empirical methods / estimations



# An Example

- For an FM signal  $m(t)$ , the max and min centre frequencies are  $\omega_c + k_f m_p$  and  $\omega_c - k_f m_p$  respectively
- Taking into account the bandwidth of the sinc lobe ( $4\pi B$ )
- Total Bandwidth is the difference



# Carson's Rule

- The estimated bandwidth is,

$$B^{\text{FM}} = \frac{1}{2\pi} (2k_f m_p + 8\pi B)$$

- The frequency deviation  $\Delta f$  is given by,

$$\Delta f = k_f \frac{m_{\text{max}} - m_{\text{min}}}{2 \times 2\pi} \text{ Hz}$$

- The estimated bandwidth (Hz) is,

$$B^{\text{FM}} \approx 2(\Delta f + 2B)$$

# Carson's Rule - Narrowband Formula

- Remember for narrowband FM,  $|k_f a(t)| \ll 1$
- $\Delta f$  is also very small

$$B^{\text{FM}} \approx 4B$$

- Earlier we analysed that for narrowband, the bandwidth is 2B Hz.
- A better estimate is then,

$$B^{\text{FM}} = 2(\Delta f + B) = 2\left(\frac{k_f m_p}{2\pi} + B\right)$$

- A commonly used expression is in terms of deviation ratio,

$$B^{\text{FM}} = 2B(\beta + 1)$$

where  $\beta$  is the deviation ratio  $\Delta f / B$

## Further Reading

- Section 4.6 - Bandwidth Analysis of Angle Modulations  
Modern Digital and Analog Communication Systems, 5<sup>th</sup> Edition
- B P Lathi and Zhi Ding

# Get in touch

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