# UESTC 3018 - Communication Systems and Principles

Lecture 12 — Angle Modulation in the Frequency Domain

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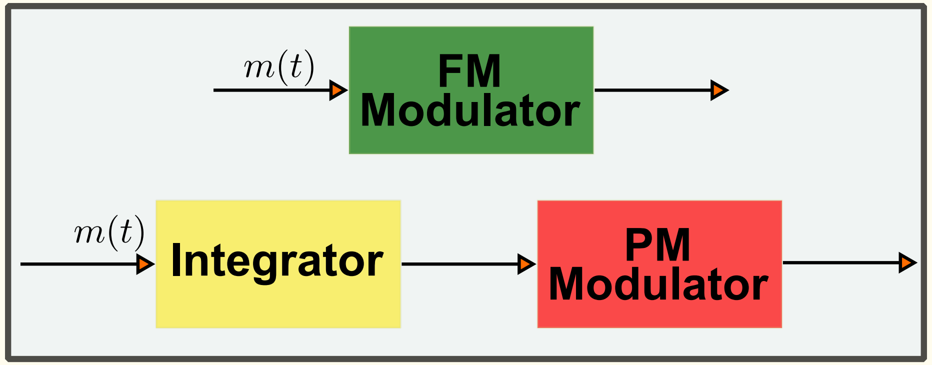
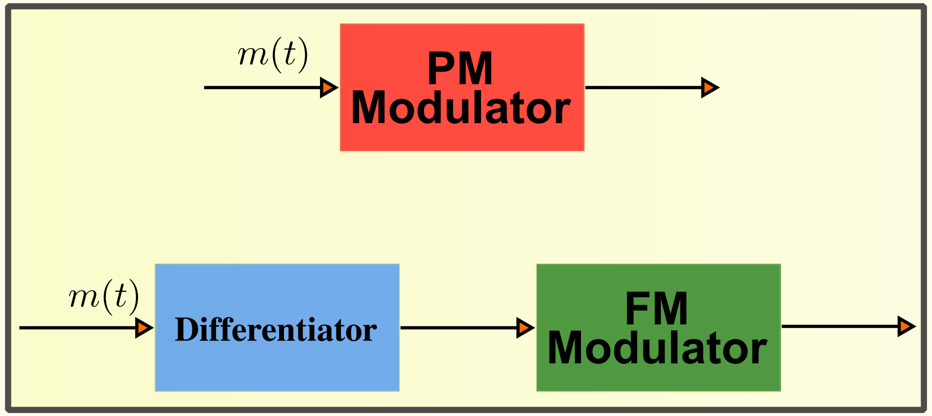
# From Last Time ⏳

- Angle Modulation is a non-linear process
- We don't change the amplitude
- In PM, we vary the phase $\theta(t)$ **linearly** with $m(t)$
- In FM, we vary the frequency $\omega(t)$ **linearly** with $m(t)$,
- PM and FM are very similar - a $90^\circ$ phase-shift



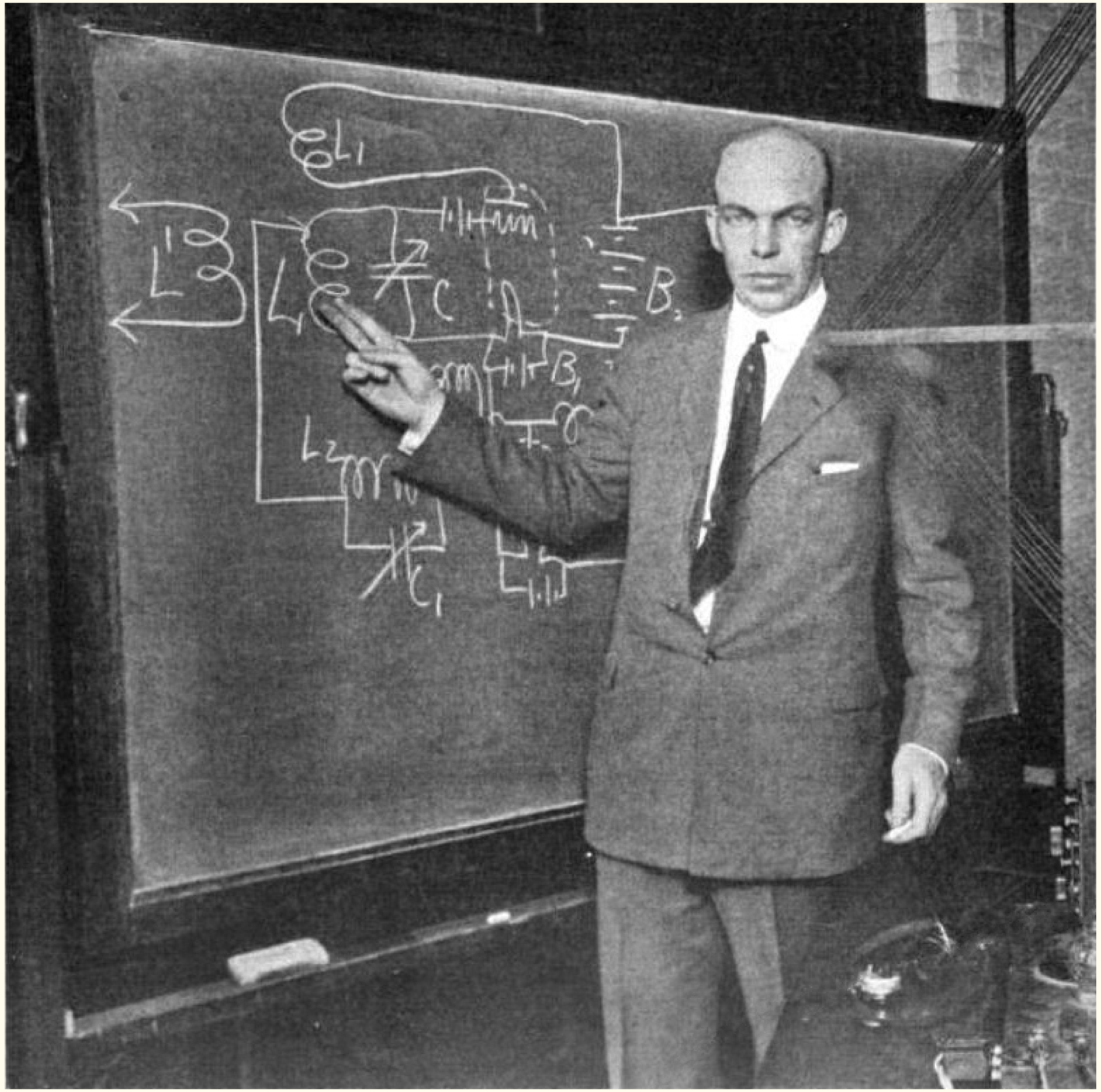



# This Lecture 📆

- Bandwidth in FM
- Carson's Rule
- Spectral Analysis

# Down the History Lane

- Due to inherent non-linearity, FM is hard to analyse
- Can't really apply Fourier transform tools
- Motivation was to reduce the bandwidth
- Turns out FM has infinite theoretical bandwidth



# The Historical Irony: The "Bandwidth Fallacy" 🤯

**FM was originally designed to SAVE space**

- **The 1920s Goal:** AM radios were crowded. Engineers wanted a "Narrowband" system to squeeze more stations onto the dial.
- **The Intuition:** "If I only wiggle the frequency by $\pm 50$ Hz, surely the bandwidth is tiny!"
- **The Mathematical Reality:**
  - In 1922, mathematician **John Carson** proved that FM actually generates **infinite sidebands**.
  - He famously declared FM "a nuisance" and static.
- **The Pivot:** Edwin Armstrong realised the "failure" was a "feature."
  - *New Idea:* "Stop trying to save bandwidth. Let's **waste** bandwidth to destroy noise!"

# Recall from the Previous Lecture ...

- In FM, we vary the frequency $\omega_i(t)$ **linearly** with $m(t)$,

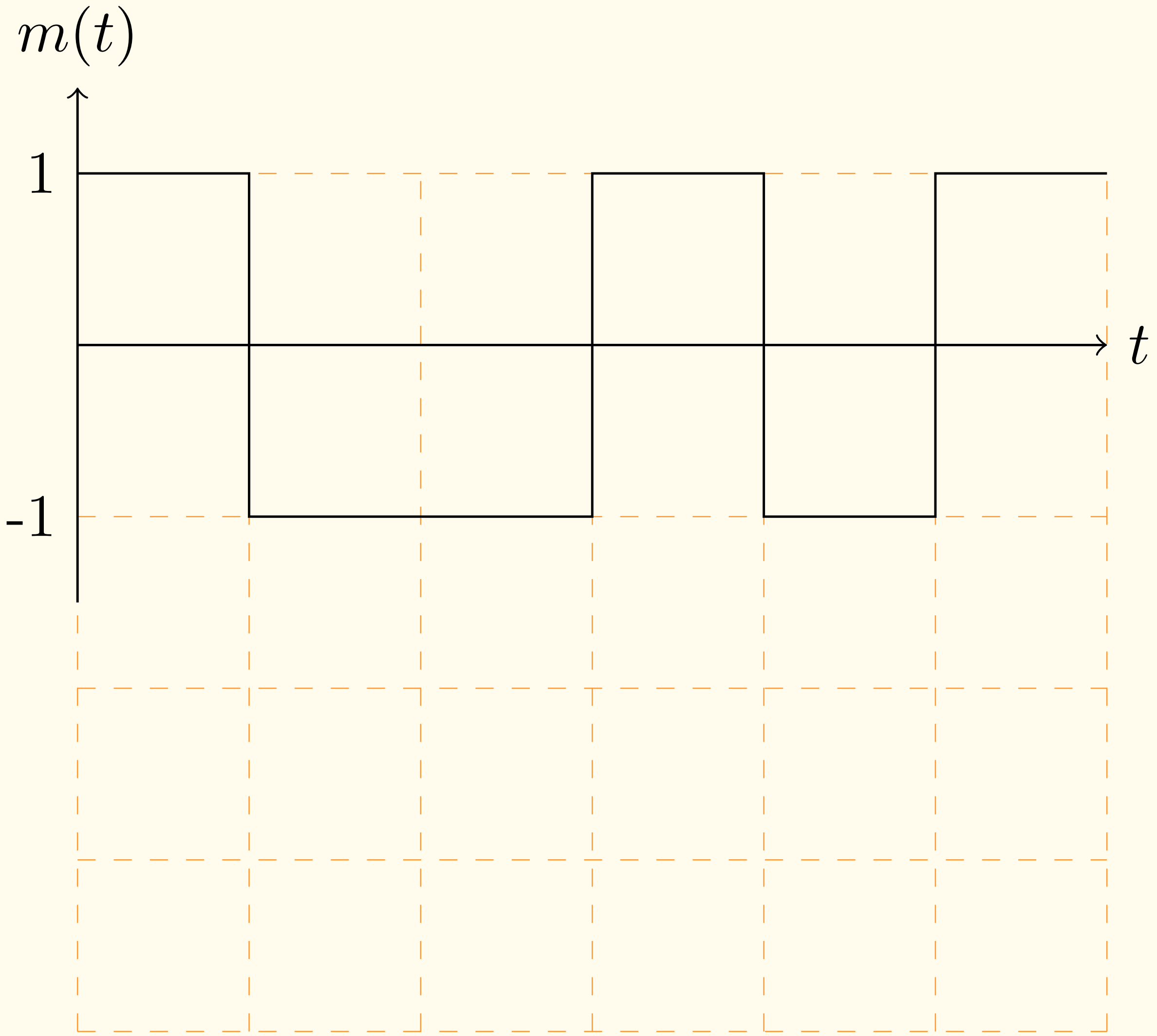
$$\omega_i^{FM}(t) = \omega_c + k_f m(t)$$

- The phase $\theta^{FM}$ is,

$$\theta^{FM}(t) = \int_{-\infty}^{t} \omega_i^{FM}(u)du = \omega_c t + k_f \int_{-\infty}^{t} m(u)du$$

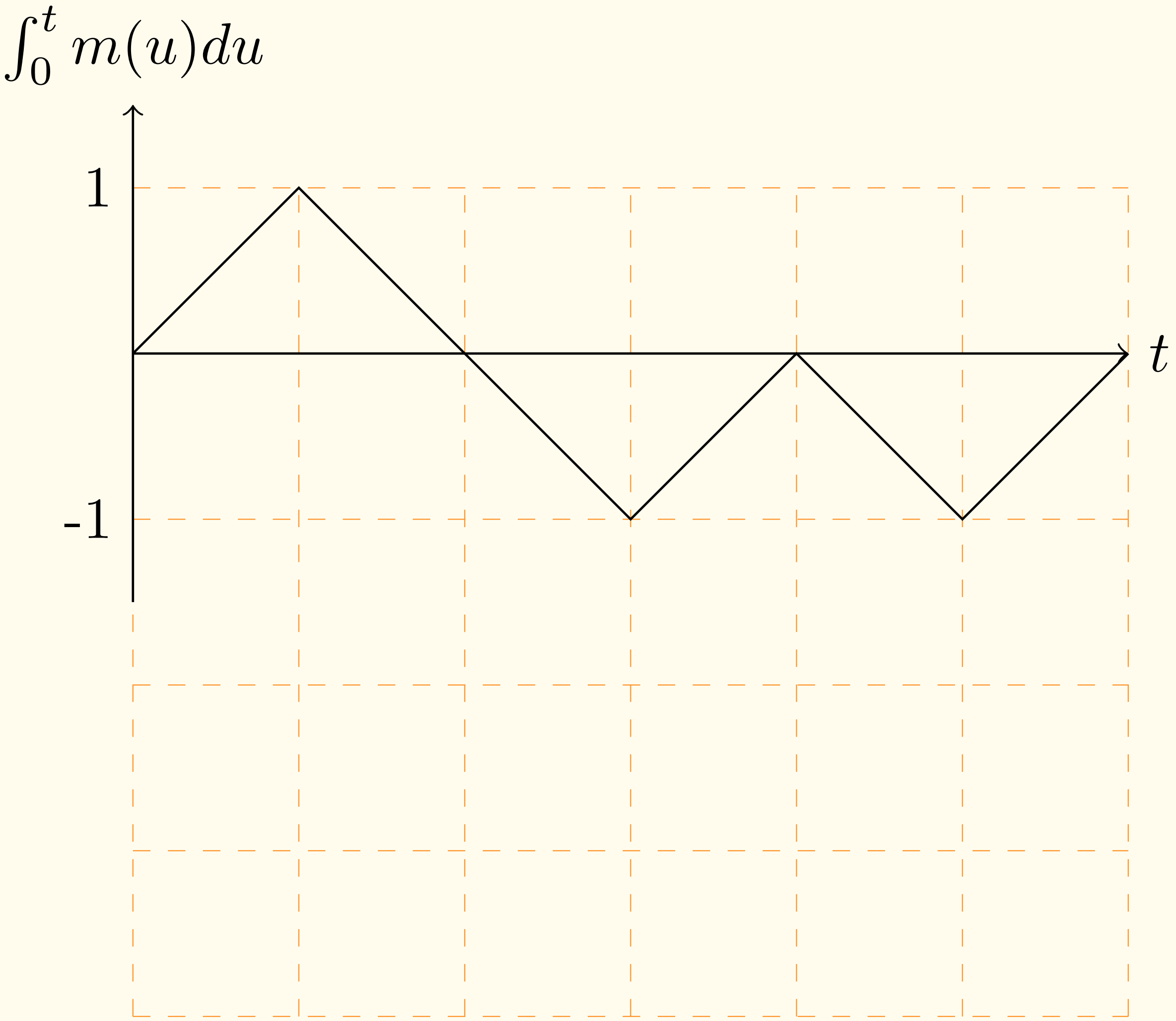
# First an Example - FSK

- Idea is to send a string of bits as two different frequencies
- Commonly used in digital radio, electronics and modems



# What about PM?

- What if we integrated the waveform?
- Essentially achieving the same waveform through phase modulator
- Input is now $\int_0^t m(u)du$



# Bandwidth Analysis ∞

# Analysing Bandwidth

- To simplify the expression, $a(t) = \int_{-\infty}^{t} m(u)\, du$
- Let's define,

$$\hat{\varphi}^{\mathrm{FM}}(t) = A\, e^{j[\omega_c t + k_f a(t)]} = A e^{j k_f a(t)} e^{j\omega_c t}$$

from where,

$$\varphi^{\mathrm{FM}}(t) = \Re\left[\hat{\varphi}^{\mathrm{FM}}(t)\right].$$

- Expanding the complex exponential through the Maclaurin power series,

$$\hat{\varphi}^{\mathrm{FM}}(t) = A\left[1 + j k_f a(t) - \frac{k_f^2}{2!} a^2(t) + \cdots + j^n \frac{k_f^n}{n!} a^n(t)\right] \times e^{j\omega_c t}$$

## Some Observations

- If $m(t)$ or $M(\omega)$ has a bandwidth of B
- Then $a(t)$ also has a bandwidth of $B$ Hz (integration is a linear operator).
- The $n^{th}$ term, $\frac{k_f^n}{n!} a^n(t)$ will have a bandwidth of $n \times B$
- This is due to convolution principle, i.e.
- $A(\omega) * A(\omega)$ spreads the Fourier transform to $2B$
- Essentially, we have **infinite bandwidth**
- But...
- $\frac{k_f^n}{n!} a^n(t) \to 0$, meaning we only care about the first few terms.

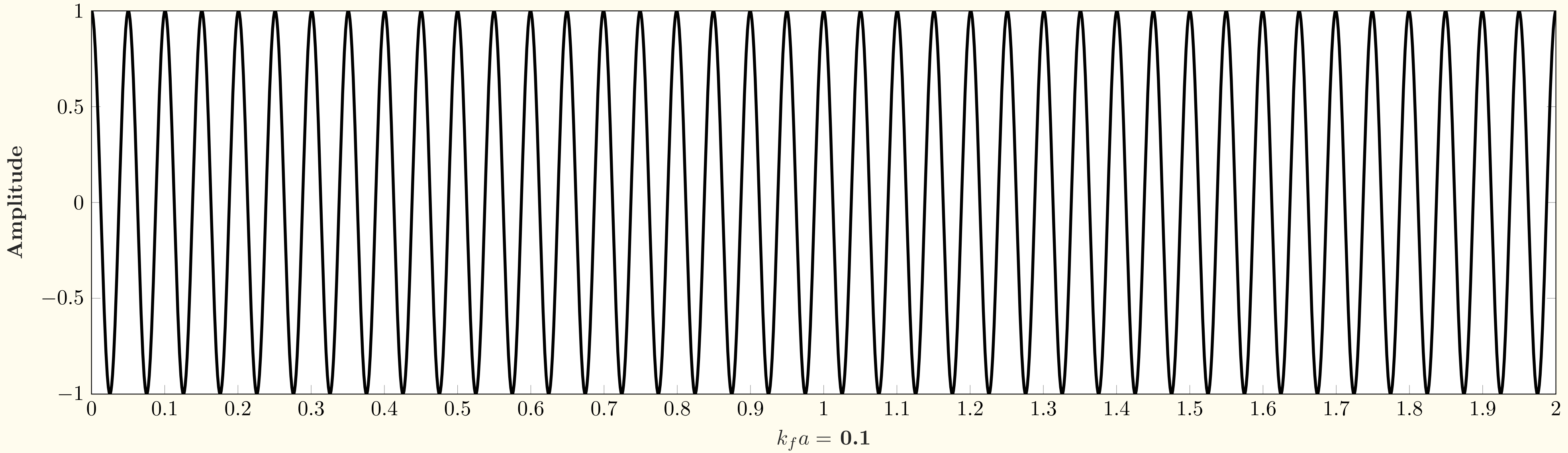
# FM Signal Representation

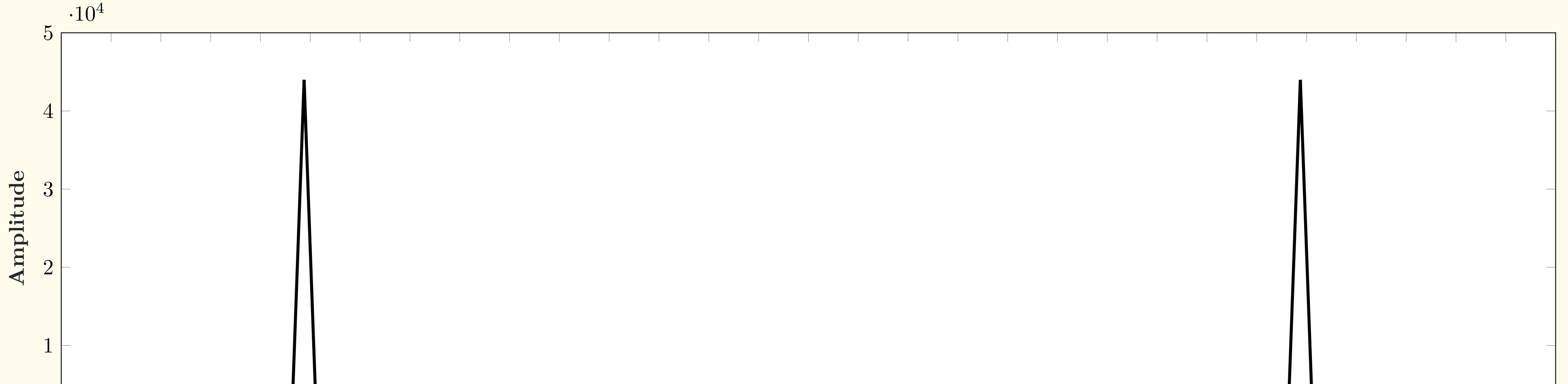
- Using the $\varphi^{\mathrm{FM}}(t) = \Re\left[\hat{\varphi}^{\mathrm{FM}}(t)\right]$ representaion, we get,

$$
\begin{aligned}
\varphi^{\mathrm{FM}}(t) &= \Re\left\{A\left[1 + jk_f a(t) - \frac{k_f^2}{2!}a^2(t) + \cdots + j^n\frac{k_f^n}{n!}a^n(t)\right] \times \left[\cos(2\pi f_c t) + j\sin(2\pi f_c t)\right]\right\} \\
&= A\left(\cos(2\pi f_c t) - k_f a(t)\sin(2\pi f_c t) - \frac{k_f^2}{2!}a^2(t)\cos(2\pi f_c t) + \dots\right) \\
&\approx A\left(\cos(2\pi f_c t) - k_f a(t)\sin(2\pi f_c t)\right)
\end{aligned}
$$

- This is a narrowband FM signal representation
- The approximation is good when $|k_f a(t)| \ll 1$
- Generally, we consider 2B bandwidth as narrowband
- PM has a similar expression

# Playing with the tones



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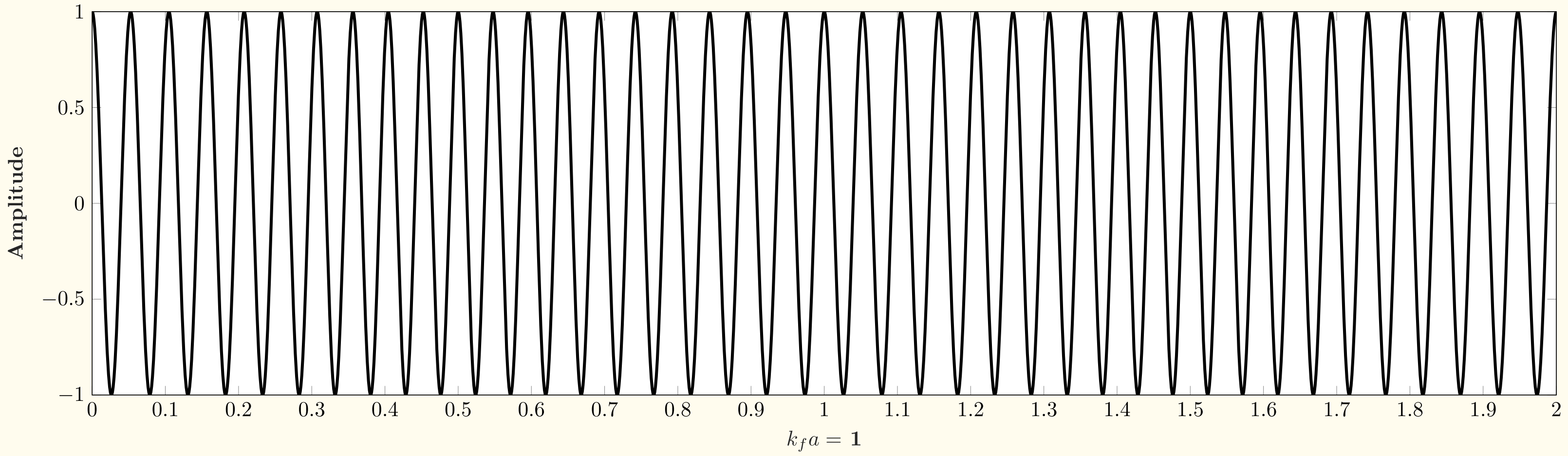


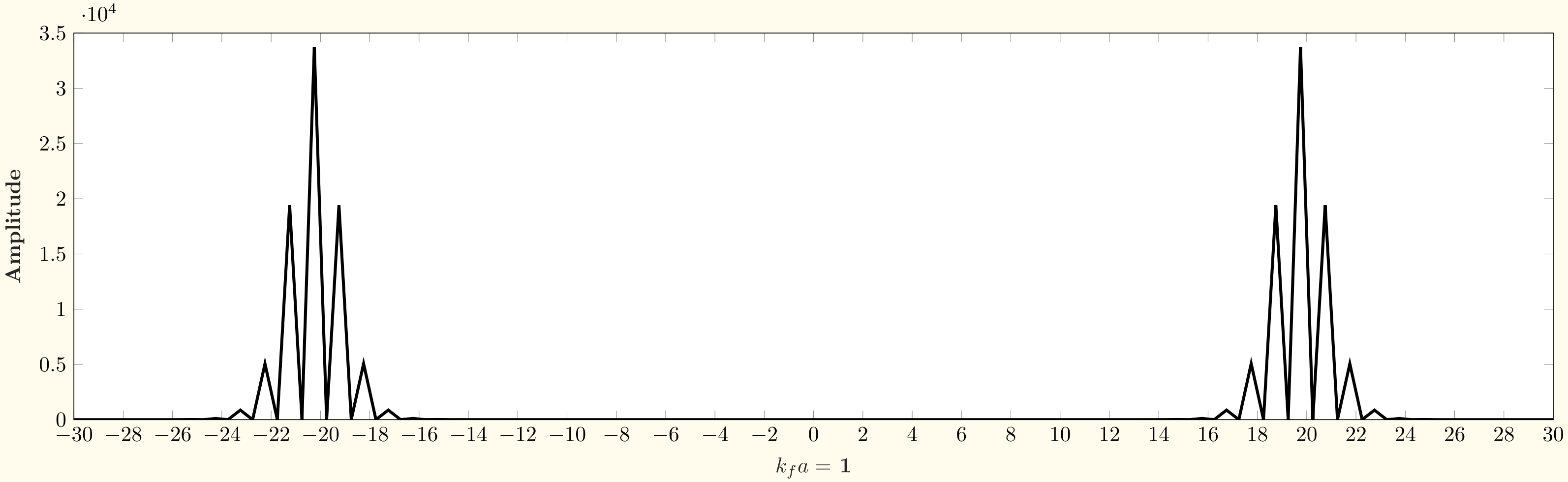

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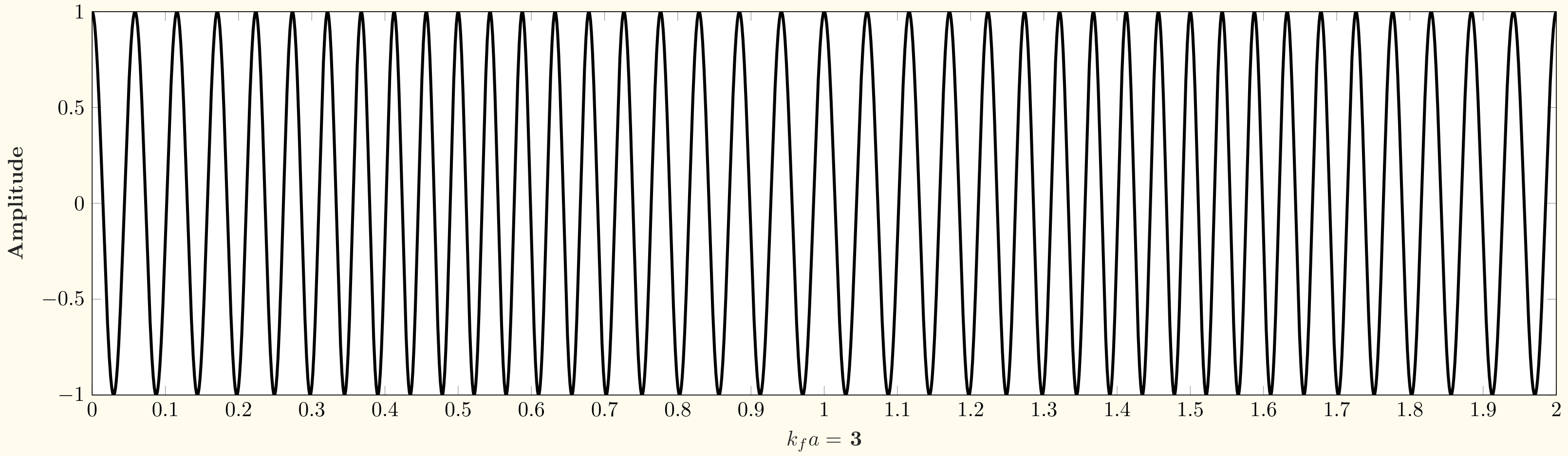
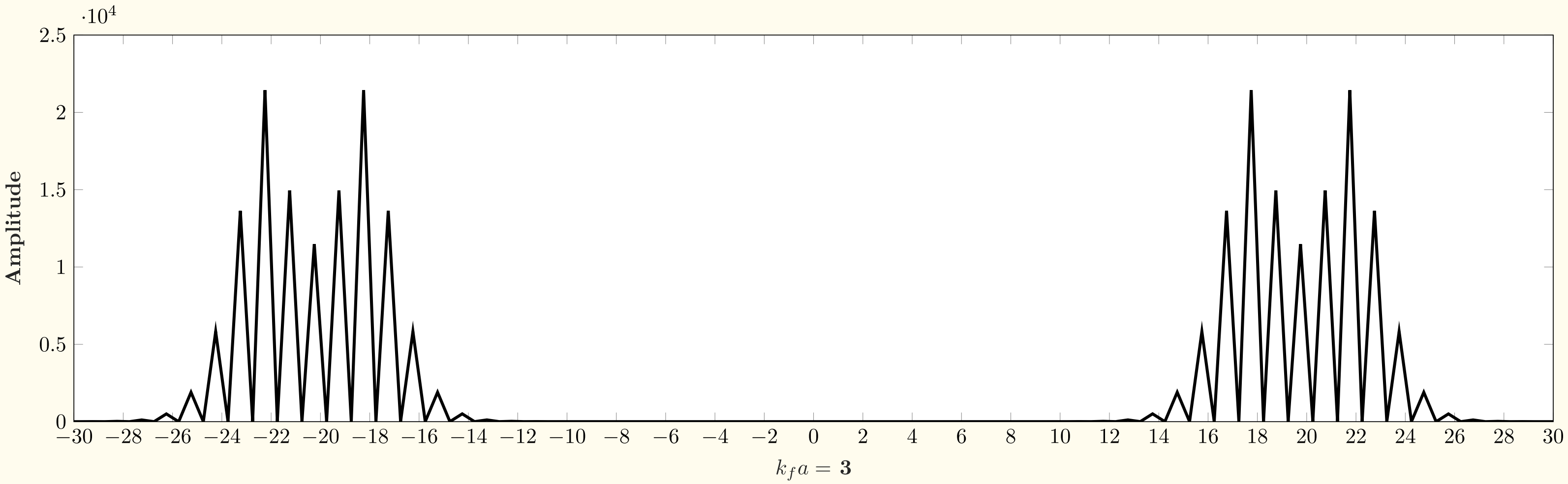


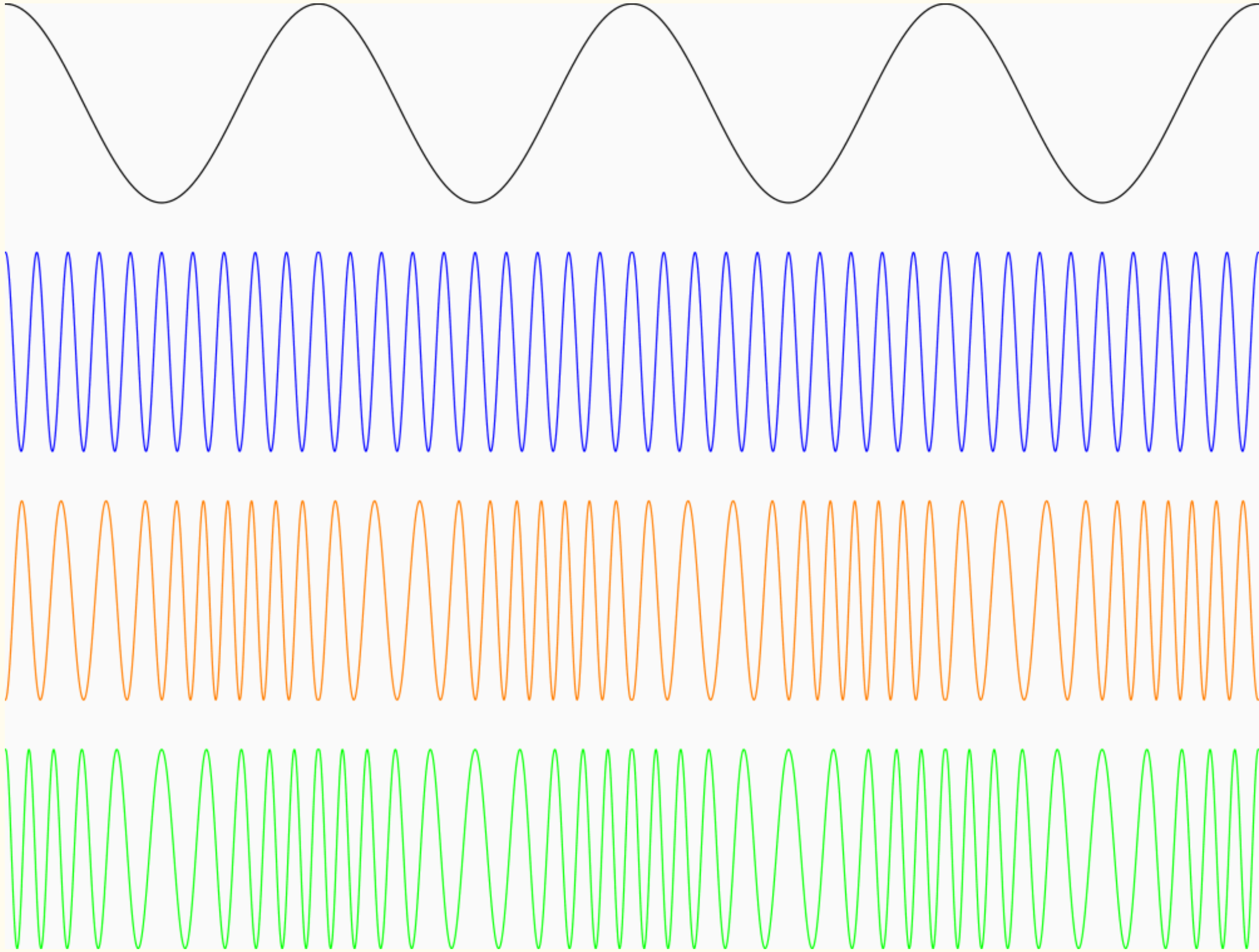



## A Dilemma 😵‍💫

- To make the best of FM (or PM), we need make the frequency deviation large enough
- Need to choose a large enough $k_f$ to break the $|k_f a(t)| \ll 1$ condition
- This is the **wideband** FM case
- ⚠️ We can't ignore the higher order terms in the power series anymore
- We need to rely on empirical formulas to estimate the bandwidth

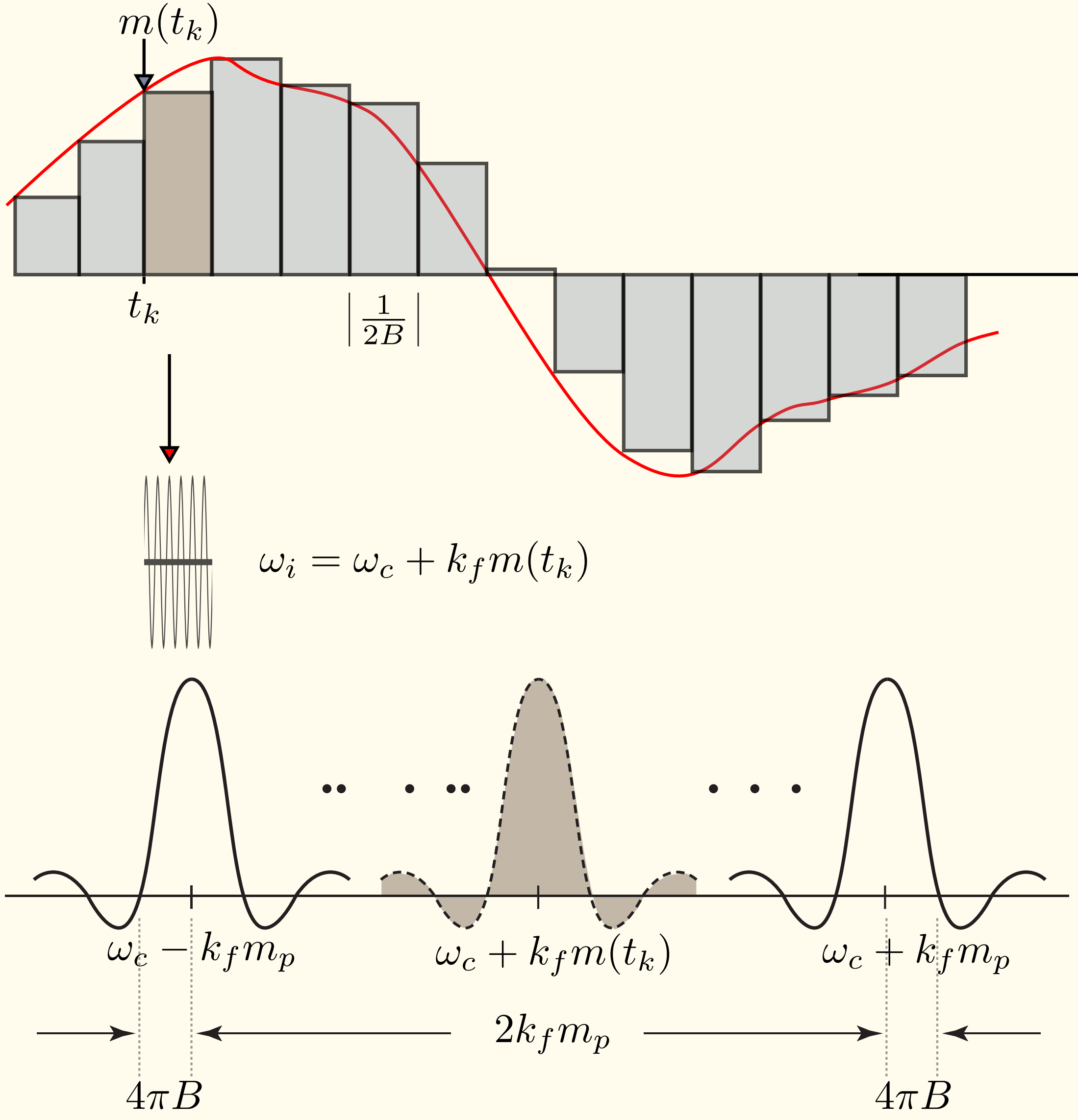
# Wideband FM

- **Context** We need frequency deviation for the FM signal to be meaningful
- But what is the bandwidth of an FM signal?
- *Answer* We use empirical methods / estimations



# An Example

- For an FM signal $m(t)$, the max and min centre frequencies are $\omega_c + k_f m_p$ and $\omega_c - k_f m_p$ respectively
- Taking into account the bandwidth of the sinc lobe ($4\pi B$)
- Total Bandwidth is the difference



# Carson's Rule 🔑

- The estimated bandwidth is,

$$B^{\mathrm{FM}} = \frac{1}{2\pi}\left(2k_f m_p + 8\pi B\right)$$

- The frequency deviation $\Delta f$ is given by,

$$\Delta f = k_f \frac{m_{\max} - m_{\min}}{2 \times 2\pi}\ \mathrm{Hz}$$

- The estimated bandwidth (Hz) is,

$$B^{\mathrm{FM}} \approx 2(\Delta f + 2B)$$

# Carson's Rule - Narrowband Formula

- Remember for narrowband FM, $|k_f a(t)| \ll 1$
- $\Delta f$ is also very small

$$B^{\text{FM}} \approx 4B$$

- Earlier we analysed that for narrowband, the bandwidth is 2B Hz.
- A better estimate is then,

$$B^{\text{FM}} = 2\left(\Delta f + B\right) = 2\left(\frac{k_f m_p}{2\pi} + B\right)$$

- A commonly used expression is in terms of deviation ratio,

$$B^{\text{FM}} = 2B(\beta + 1)$$

where $\beta$ is the deviation ratio $\Delta f / B$

# Further Reading

- Section 4.6 - Bandwidth Analysis of Angle Modulations
  Modern Digital and Analog Communication Systems, $5^{th}$ Edition
- B P Lathi and Zhi Ding

# Get in touch

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