Solutions to Quantum Field Theory and the Standard Model

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April 19, 2022

Some observations.

If you find error, typos or if you have a different idea to solve some of the exercises, I beg you to contact and tell it on my personal e-mail: sr.glaucofsr@gmail.com

I have several issues to keep a convention while writting, I'm so sorry for confusions and English errors, if something has confused you, contact me and I'll try to give a better explanation.

I'll update this archive every weekend (supposing I made some advance during the week), my goal is to add all the exercises from Schwartz's book.

Quantum Field Theory and de Standard Model (Matthew D. Schwartz)

0.0.1 Chapter2

Solution 1. (2.1)
Let's write the Transformations into an first approximation:

$$x \to x + vt, \quad t \to t + \delta t$$

We should ask it to keep the space-time interval invariant, so, up to first order in v and assuming δt is linear in x and t:

$$t^2 - x^2 = (t + \delta t)^2 - (x + vt)^2 = t^2 + 2t\delta t + \delta t^2 - x^2 - 2xvt - \mathcal{O}\big(v^2\big)$$

$$\delta t^2 + 2t\delta t - 2xvt = 0$$

$$\delta t = \frac{-2t \pm \sqrt{4t^2 + 8xvt}}{2}$$

Expanding the root in taylor expansion we have:

$$\sqrt{4t^2 + 8xvt} \approx 2t + \frac{8xvt}{4t} + \mathcal{O}\big(v^2\big)$$

We want time to increase in positive direction then:

$$\delta t = xv$$

We can go further and increase a little bit the approximation:

$$x \to x + vt, \quad t \to t + \delta t$$

Again, asking invariance of the space-time interval and up to second order in v:

$$t^2 - x^2 = (t + \delta t)^2 - (x + vt)^2 = t^2 + 2t\delta t + \delta t^2 - x^2 - 2xvt - v^2t^2$$

$$\delta t^2 + 2t\delta t - (2x + vt)vt = 0$$

$$\delta t = -t \pm \sqrt{t^2 + (2x + vt)vt}$$

Again, expanding the root:

$$\sqrt{t^2 + (2x + vt)vt} = t + \frac{(2x + vt)vt}{2t} + \mathcal{O}\big(v^3\big) = t + vx + \frac{v^2t}{2}$$

$$\delta t = vt + \frac{xv^2}{2}$$

Now for the spatial correction, without loss of generality:

$$x \to x + \delta x, \quad t \to t + vx$$

From space time interval invariance comes:

$$t^2 - x^2 = (t + xv)^2 - (x + \delta x)^2 = t^2 + 2t\delta t + \delta t^2 - x^2 - \delta x^2 - 2x\delta x$$

$$\delta x^2 + 2x\delta x - (2t + vx)vx = 0$$

$$\delta x = \frac{-2x \pm 2\sqrt{x^2 + (2t + vx)vt}}{2}$$

Expanding the root:

$$\sqrt{x^2 + (2t + vx)vx} = x - \frac{(2t + vx)vx}{2x} + \mathcal{O}\big(v^3\big) = x + \frac{(2t + xv)xv}{2x} = x + vt + \frac{v^2x}{2}$$

$$\delta x = vt + \frac{v^2x}{2}$$

Both agree to the second order expansion of the full transformation.

Solution 2. Since we're going to take values from Wikipedia, we'll work with $c \approx 3 \cdot 10^8 m/s$, in these coordinates $7Tev \approx 1.12 \cdot 10^{-12} J$ and $m_p \approx 1.7 \cdot 10^-27$, we have then:

$$E_{tot} = m_p \gamma c^2 \Rightarrow \gamma = \frac{E}{m_p c^2}$$

We can then write $\beta^2 = \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2} = 1 - \frac{m_p^2 c^4}{E^2}$

So, difference between the final velocity and the light velocity, expanding the second part of the root will appear ($m_p/E << 1$), will be:

$$c - v = c(1-\beta) \approx \frac{m_p^2 c^5}{2E^2} = 2.44 m/s$$

A more accurate calculation gives $c - v \approx 3m/s$

To the other part, notice that:

$$V_r el = \frac{\beta^2}{1-\beta} c \approx c$$

Solution 3. (2.6)

a) We need to evaluate the integral:

$$\int_{-\infty}^{\infty} dk^0 \delta(k^2 - m^2)\theta(k^0)$$

First we need to setup a propertie of the Dirac's delta:

$$\delta(x^2 - \alpha^2) = \frac{1}{2|\alpha|}[\delta(x-a) + \delta(x+a)]$$

$$\delta(k^2 - m^2) = \delta((k^0)^2 - \vec{k}^2 - m^2) = \delta((k^0)^2 - \omega_k^2) = \frac{1}{2\omega_k}\left[\delta(k^0 - \omega_k) + \delta(k^0 + \omega_k)\right]$$

Then, it's easily to achieve:

$$\int_{-\infty}^{\infty} dk^0 \delta(k^2 - m^2)\theta(k^0) = \int_{-\infty}^{\infty} dk^0 \frac{1}{2\omega_k}\left[\delta(k^0 - \omega_k) + \delta(k^0 + \omega_k)\right]\theta(k^0) = \int_0^{\infty} dk^0 \frac{1}{2\omega_k}\delta(k^0 - \omega_k)$$

$$\int_{-\infty}^{\infty} dk^0 \delta(k^2 - m^2)\theta(k^0) = \frac{1}{2\omega_k}$$

b) Notice that $dk^4 \to \Lambda dk^4$, the jacobian is $|\Lambda| = 1$, then dk^4 is Lorentz invariant.

c) We already showed that dk^4 is a Lorentz invariant, then:

$$\int \frac{d^3k}{2\omega_k} = \int dk^4 \delta(k^2 - m^2)\theta(k^0)$$

We just need to show that $\delta(k^2 - m^2)\theta(k^0)$ is a Lorentz scalar, when we do: $k'^\mu = \Lambda^\mu_\nu k^\nu$, notice that, by definition $k'^2 = k^2$ then the Dirac's delta is already a Lorentz scalar, now on the step function, if k isn't time-like or space-like, then none orthocronos Lorentz transformation can change the k^o sign, in this case the Step-function is also a Lorentz-invariant, then the integral itself is a Lorentz-invariant.

0.0.2 Chapter 3

Exercise 1. (3.4) Find the generalization of the Euler-Lagrange equations for general Lagrangians of the form $\mathcal{L}[\phi, \partial_\mu\phi, \partial_\nu\partial_\mu\phi, ...]$

Solution 4. We start writing the action for this Lagrangian:

$$S = \int \mathrm{d}\, x^4 \mathcal{L}(\phi, \partial_\mu\phi, \partial_\nu\partial_\mu\phi, ...)$$

To the ϕ be a extremum for the action, we expect that for small variations around ϕ the action remains the same, i.e. doing $\phi(x) \to \phi(x) + \eta(x)$ in such way that η is infinitesimally small and vanishes in the borders (or infinity) we should have:

$$\delta S = \int \mathrm{d}\, x^4 \mathcal{L}(\phi + \eta, \partial_\mu(\phi + \eta), \partial_\nu\partial_\mu(\phi + \eta), ...) - \int \mathrm{d}\, x^4 \mathcal{L}(\phi, \partial_\mu\phi, \partial_\nu\partial_\mu\phi, ...) = 0$$

We're physicists and don't know what to do, then we "Taylor expands" to first order in η omitting the arguments:

$$\mathcal{L}(\phi+\eta, \partial_\mu(\phi+\eta), \partial_\nu\partial_\mu(\phi+\eta), ...) = \mathcal{L} + \frac{\partial\mathcal{L}}{\partial\phi}\eta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\eta + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\partial_\nu\partial_\mu\eta + \cdots + \mathcal{O}\big(\eta^2, (\partial_\mu\eta)^2, (\partial_\nu\partial_\mu\eta)^2, ...\big)$$

Substituing this into the expression for δS we have:

$$\delta S = \int \mathrm{d}\, x^4 \left(\frac{\partial\mathcal{L}}{\partial\phi}\eta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\eta + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\partial_\nu\partial_\mu\eta + \dots\right)$$

If we ask the derivatives of η to go to zero in the infinity, getting ride of the terms who are clearly zero we have:

$$\delta S = \int \mathrm{d}\, x^4 \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} + \partial_\nu\partial_\mu\frac{\mathcal{L}}{(\partial_\nu\partial_\mu\phi)} + \dots\right)\eta$$

To second order derivatives we have the Euler-Lagrange equations in form:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} + \partial_\nu\partial_\mu\frac{\mathcal{L}}{(\partial_\nu\partial_\mu\phi)} = 0$$

Now I need a clearly way to write it to general higher derivatives terms, keeping the characteristic $(-1)^n$ term where n is the greater derivative of ϕ we're including

Solution 5. 2.2

a) We start by writing the Lorentz transformation in a way that:

$$\Lambda_{\mu\nu} = \delta_{\mu\nu} + \omega_{\mu\nu}$$

With $\omega_{\mu\nu}$ an antisymmetric tensor, then the Lorentz transformation of x_μ will be just:

$$x_\mu \to \Lambda_{\mu\nu}x_\nu = (\delta_{\mu\nu} + \omega_{\mu\nu})x_\nu = x_\mu + \omega_{\mu\nu}x_\nu$$

This transformation law gives us the change in the field:

$$\phi_n(x) \to \phi_n(x+x\omega) = \phi_n(x)+\partial_\mu\phi_n(x)\omega_{\mu\nu}x_\nu+\mathcal{O}(\omega^2) = \phi_n(x)+\frac{(\partial_\mu\phi_n(x)x_\nu - \partial_\nu\phi_n(x)x_\mu)}{2}\omega_{\mu\nu}$$

$$\frac{\delta\phi_n}{\delta\omega_{\mu\nu}} = \frac{(\partial_\mu\phi_n(x)x_\nu - \partial_\nu\phi_n(x)x_\mu)}{2}$$

Note that the Lagrangian, being an scalar, should change in the same way, so:

$$\frac{\delta\mathcal{L}}{\delta\omega^{\mu\nu}} = \frac{(\partial_\mu\mathcal{L}x_\nu - \partial_\nu\mathcal{L}x_\mu)}{2}$$

We know that those changes in the Lagrangian (inducted directly by the change in the coordinates because our Lagrangian is a scalar and the other caused by the change in the scalar fields ϕ_n due the change in coordinates) must be equal, then, if $\mathcal{L}$ satisfies the E.L. equations we have:

$$\frac{\delta\mathcal{L}}{\delta\omega^{\mu\nu}} = \frac{(\partial_\mu\mathcal{L}x_\nu - \partial_\nu\mathcal{L}x_\mu)}{2} = \partial_\alpha\left(\sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi_n)}\left(\frac{(\partial_\mu\phi_n(x)x_\nu - \partial_\nu\phi_n(x)x_\mu)}{2}\right)\right)$$

We can rewrite it then finally as a conserved current:

$$\partial_\alpha\left(x_\nu\left\{\sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi_n)}\partial_\mu\phi_n(x) - g_{\alpha\mu}\mathcal{L}\right\} - x_\mu\left\{\sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi_n)}\partial_\nu\phi_n(x) - g_{\alpha\nu}\mathcal{L}\right\}\right) = 0$$

Notice the terms inside the brackets are just the Energy-momentum tensors, then we can define this conserved current as:

$$K_{\mu\nu\alpha} = x_\nu\mathcal{T}_{\alpha\mu} - x_\mu\mathcal{T}_{\alpha\nu}$$

b) Setting a Lagrangian as the enunciate dictates: $\mathcal{L} = \frac{1}{2}\phi(\Box^2+m^2)\phi$, since it satisfies the E.L. equations we should be able to calculate the energy-momentum tensor and the current, as it follows:

$$\mathcal{T}_{\alpha\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\partial_\mu\phi - g_{\alpha\mu}\mathcal{L} = \frac{1}{2}\partial_\alpha\phi$$

$$K_{\mu\nu\alpha} = x_\nu(\phi\partial_\alpha\phi\partial_\mu\phi - g_{\alpha\mu}\mathcal{L}) - x_\mu(\phi\partial_\alpha\phi\partial_\nu\phi - g_{\alpha\nu}\mathcal{L})$$

Now we need to see if the current vanishes, notice first that, since $\mathcal{T}_{\alpha\mu}$ is clearly symmetric:

$$\partial_\alpha x_\nu\mathcal{T}_{\alpha\mu} - \partial_\alpha x_\mu\mathcal{T}_{\alpha\mu} = \delta_{\alpha\nu}\mathcal{T}_{\alpha\mu} - \delta_{\alpha\mu}\mathcal{T}_{\alpha\nu} = \mathcal{T}_{\mu\nu} - \mathcal{T}_{\nu\mu} = 0$$

Now, if the field ϕ satisfies the Euler-Lagrange equations for the field, then $\mathcal{T}_{\alpha\mu}$ itself is a current, associated with $x^\gamma \to x^\gamma + \epsilon^\gamma$, then should be true that $\partial_\alpha \mathcal{T}_{\alpha\mu} = 0$. From these considerations, we obtain:

$$\partial_\alpha K_{\mu\nu\alpha} = \partial_\alpha(x_\nu \mathcal{T}_{\alpha\mu} - x_\mu \mathcal{T}_\alpha \nu) = [\partial_\alpha x_\nu \mathcal{T}_{\alpha\mu} - \partial_\alpha x_\mu \mathcal{T}_{\alpha\mu}] + x_\nu \partial_\alpha \mathcal{T}_{\alpha\mu} - x_\mu \partial_\alpha \mathcal{T}_{\alpha\nu} = 0$$

c) This quantity has the form:

$$K_{0i0} = x_i \mathcal{T}_{00} - x_0 \mathcal{T}_{0i} = x_i \mathcal{E} - x_0 \mathcal{P}_i$$

Then, if we treat x_i as a small volume with $\mathcal{E}$ energy, we can say that:

$$Q_i = \int dx^3 K_{0i0} = \int dx^3 x_i \mathcal{E} - \int dx^3 x_0 \mathcal{P}_i = \bar{x}_i E - x_0 P_i$$

If we interpret $\bar{x}_i$ as a "energy center" analogous to the "mass center", we can say that this energy center moves in a straight line as $x_0 = t$ advances:

$$\bar{x}_i = \frac{Q_i}{E} + \frac{tP_i}{E}$$

d) From the Heisenberg equation of motion we have:

$$\frac{\mathrm{d}Q_i}{\mathrm{d}t} = i[Q_i, H] + \frac{\partial Q_i}{\partial t}$$

Since Q_i involve a dependence on x or p and H does not, generally $[Q_i, H] \neq 0$, in such case:

$$\frac{\mathrm{d}Q_i}{\mathrm{d}t} = 0 \Rightarrow i\frac{\partial Q_i}{\partial t} = [Q_i, H]$$

Solution 6. 3.3 Well, this is straight calculation $\mathcal{T}_{\alpha\beta} \to \mathcal{T}'_{\alpha\beta}$

$$\mathcal{T}'_{\alpha\beta} = \frac{\partial \mathcal{L} + \partial_\mu X^m u}{\partial(\partial_\alpha \phi)} \partial_\beta \phi - g_{\alpha\beta} \mathcal{L} - g_{\alpha\beta} \partial_\mu X^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} \partial_\beta \phi - g_{\alpha\beta} \mathcal{L} - g_{\alpha\beta} \partial_\mu X^\mu = \mathcal{T}_{\alpha\beta} - g_{\alpha\beta} \partial_\mu X^\mu$$

b)

$$\int dx^3 \mathcal{T}'_{00} = \int dx^3 \mathcal{T}_{00} - g_{00} \int dx^3 \partial_\mu X^\mu = \int dx^3 \mathcal{T}_{00}$$

Where the last step is due the fact all fields must be null at the infinity. It's clearly a conserved quantity.

c) By its definition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with A_μ being the dynamical variables, we can calculate then the Energy-momentum tensor:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\epsilon)} = \frac{\partial F_{\mu\nu} F_{\mu\nu}}{\partial(\partial_\alpha A_\epsilon)} = 2(\delta_{\mu\alpha}\delta_{\nu\epsilon} - \delta_{\nu\alpha}\delta_{\mu\epsilon}) F_{\mu\nu} = 4F_{\alpha\epsilon}$$

$$\mathcal{T}_{\alpha\beta} = -F_{\alpha\epsilon}\partial_\beta A_\epsilon + \frac{g_{\alpha\beta}F_{\mu\nu}^2}{4}$$

This tensor is clearly non-symmetrical since:

$$\mathcal{T}_{\alpha\beta} - \mathcal{T}_{\beta\alpha} = F_{\beta\epsilon}\partial_\alpha A_\epsilon - F_{\alpha\epsilon}\partial_\beta A_\epsilon \neq 0$$

To keep it symmetric we need to add a term $\partial_\lambda K_{\lambda\alpha\beta} = F_{\beta\epsilon}\partial_\alpha A_\epsilon$, since for definition $\partial_\mu F_{\mu\nu} = j_\nu$, in a case without sources we can choose $X_\lambda = K_{\lambda\alpha\beta} = F_{\beta\lambda}A_\lambda$. Under those changes, doing $\lambda \to \epsilon$ the Energy-momentum tensor will be:

$$T_{\mu\nu} = F_{\beta\epsilon}\partial_\alpha A_\epsilon - F_{\alpha\epsilon}\partial_\beta A_\epsilon + \frac{g_{\alpha\beta}F_{\mu\nu}^2}{4}$$

That's symmetric now, as desired.

Solution 7. 3.5

a) Setting the Lagrangian $\mathcal{L} = -\frac{1}{2}\phi\Box\phi + m^2\phi^2 - \frac{\lambda}{4!}\phi^4$, we can calculate the motion equations:

$$\phi\Box\phi = -\partial_\mu\phi\partial_\mu\phi + \partial_\mu(\phi\partial_\mu\phi)$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} = \partial_\nu\phi \Rightarrow \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\right) = \Box\phi$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = m^2\phi - \frac{\lambda}{3!}\phi^3$$

Then, the motion equation is

$$-\Box\phi + m^2\phi - \frac{\lambda}{3!}\phi^3$$

If we're searching a constant solution, then any derivative of ϕ is zero, setting $\phi = c$, $\phi = 0$ is a trivial one, then we have:

$$m^2c - \frac{\lambda}{6}c^3 = 0 \Rightarrow m^2 - \frac{\lambda}{6}c^2 = 0$$

$$c_\pm = \pm m\sqrt{\frac{6}{\lambda}}$$

Now, to evaluate the energy, since the kinetic part is null, we need to evaluate the potential part, the groud state will emerge from the minimum of the potential. Obviously $V(0) = 0$, now for $V(c_\pm)$ we have (since both are quadratical powers of ϕ):

$$V(c_\pm) = \frac{\lambda}{24}\frac{36}{\lambda^2}m^4 - m^4\frac{6}{\lambda} = \frac{m^4}{\lambda}\left(\frac{3}{2} - 6\right) < 0$$

Then, $\phi_\pm(x) = \pm m\sqrt{6/\lambda}$ are the ground states of the system.

b) Since there are two minima, or vacuum, one of them must be specified to be the ground state, so the $\mathbb{Z}_2$ symmetry no more holds.

c) Instering $\phi \to c + \pi(c)$ into the Lagrangian we have

$$\mathcal{L}' = -\frac{1}{2}\left[(c+\pi)\Box\pi + m^2(c^2 + \pi^2 + 2c\pi) - \frac{\lambda}{4!}(c^4 + 4c^3\pi + 6c^2\pi^2 + 4c\pi^3 + \pi^4)\right]$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\pi)}\right) = \Box\pi$$

$$\frac{\partial\mathcal{L}}{\partial\pi} = \frac{1}{2}\big(2m^2\pi + 2m^2c\big) - \frac{\lambda}{4!}\big(4c^3 + 12c^2\pi + 12c\pi^2 + 3\pi^3\big)$$

The equations of motion will be

$$-\Box\pi + m^2\pi + m^2c - \frac{\lambda}{4!}\big(4c^3 + 12c^2\pi + 12c\pi^2 + 3\pi^3\big) = 0$$

For $\pi = 0$, since there will be no difference in the signs of the remaining terms, we have:

$$\pm m^3\sqrt{\frac{6}{\lambda}} - \pm\frac{\lambda}{4!}\frac{6*4}{\lambda}m^3\sqrt{\frac{6}{\lambda}} = 0$$

Now notice that, under the transformation $\phi \to -\phi$, π should transform as $\pi \to -\pi - 2c$. Fell free to see the Lagrangian is invariant.

Solution 8. (**3.6**)

a) It's direct calculation. For the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2A_\mu^2 - A_\mu J_\mu$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\kappa A_\epsilon)} = -\frac{1}{4}\frac{\partial F_{\mu\nu}^2}{\partial(\partial_\kappa A_\epsilon)} = -\frac{1}{2}(\delta_{\kappa\mu}\delta_{\epsilon\nu} - \delta_{\kappa\nu}\delta_{\mu\epsilon})(\partial_\mu A_\nu - \partial_\nu A_\mu) = -\frac{1}{2}(\partial_\kappa A_\epsilon - \partial_\epsilon A_\kappa - \partial_\epsilon A_\kappa - \partial_\kappa + \partial_\kappa A_\epsilon) = -$$

$$\frac{\partial\mathcal{L}}{\partial A_\epsilon} = m^2A_\epsilon - J_\epsilon$$

The equations of motion are then

$$\partial_\kappa F_{\kappa\epsilon} + m^2A_\epsilon - J_\epsilon = 0$$

We can derive everything by x^ϵ letting:

$$\partial_\epsilon\partial_\kappa F_{\kappa\epsilon} + m^2\partial_\epsilon A_\epsilon = \partial_\epsilon J_\epsilon = 0$$

Notice that, since $F_{\kappa\epsilon}$ is anti-symmetric

$$\partial_\epsilon \partial_\kappa F_{\kappa\epsilon} = \frac{1}{2}(\partial_\epsilon \partial_\kappa - \partial_\kappa \partial_\epsilon) F_{\kappa\epsilon} = 0$$

Then, the constraint to A_μ is just:

$$\partial_\mu A_\mu = 0$$

b) To calculate A_0, from the equation of movement:

$$\partial_\mu(\partial_\mu A_0 - \partial_0 A_\mu) + m^2 A_0 = q\delta^{(3)}(r)$$

Since the field is static $\partial_0 A_m u = 0$, then the motion equations reduces to:

$$(\Box + m^2) A_0(r) = e\delta^{(3)}(r)$$

$$A_0(r) = -\frac{e}{m - \triangle}\delta^{(3)}(r) = \frac{e}{(2\pi)^3}\int dk^3 \frac{\mathrm{e}^{-ikr}}{k^2 + m^2}$$

$$= \frac{e}{(2\pi)^3}\int_0^\infty dk \frac{k^2}{k^2+m^2}\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos(\theta)\mathrm{e}^{ikr\cos(\theta)}$$

$$= \frac{e}{(2\pi)^2 ir}\int_0^\infty dk \frac{k}{k^2+m^2}(\mathrm{e}^{ikr} - \mathrm{e}^{-ikr}) = \frac{e}{4\pi^2 ir}\int_{-\infty}^{\infty} dk \frac{k}{k^2+m^2}\mathrm{e}^{ikr}$$

Just to make explicit the equation:

$$A_0(r) = \frac{e}{4\pi^2 ir}\int_{-\infty}^{\infty} dk \frac{k}{k^2+m^2}\mathrm{e}^{ikr}$$

c) To evaluate this integral, notice that it belongs to the sum below in the limit $R \to \infty$:

$$\int_\Gamma dk \frac{k}{k^2+m^2}\mathrm{e}^{ikr} = \int_S dk \frac{k}{k^2+m^2}\mathrm{e}^{ikr} + \int_{-R}^{R} dk \frac{k}{k^2+m^2}\mathrm{e}^{ikr}$$

Where the curves are defined according to: S is a semi-circular contour with radius R which englobes im, there are a theorem in complex analisys who says:

"Suppose $f(z) = p(z)/q(z)$ is a rational function, where the degree of p(z) is n and the degree of q(z) is $m \geq n+2$. Given S and with $z = R\mathrm{e}^{i\theta}, 0 \leq \theta \leq \pi$ and $\alpha > 0$, then $\int_S f(z)\mathrm{e}^{i\alpha z}dz \to 0$ as $R \to \infty$".

In our limit then the first integral in the right side will vanish, taking the limit $R \to \infty$ and evaluating the left integral by residues theorem:

$$\int_{-\infty}^{\infty} dk \frac{k}{k^2+m^2} \mathrm{e}^{ikr} = \int_{\Gamma} dk \frac{k}{k^2+m^2} \mathrm{e}^{ikr} = \frac{k}{2\pi i(k+im)} \mathrm{e}^{ikr}\bigg|_{k=im} = \pi i \mathrm{e}^{-mr}$$

Then, finnaly we get that:

$$A_0(r) = \frac{e}{4\pi^2 ir} \pi i \mathrm{e}^{-mr} = \frac{e}{4\pi r} \mathrm{e}^{-mr}$$

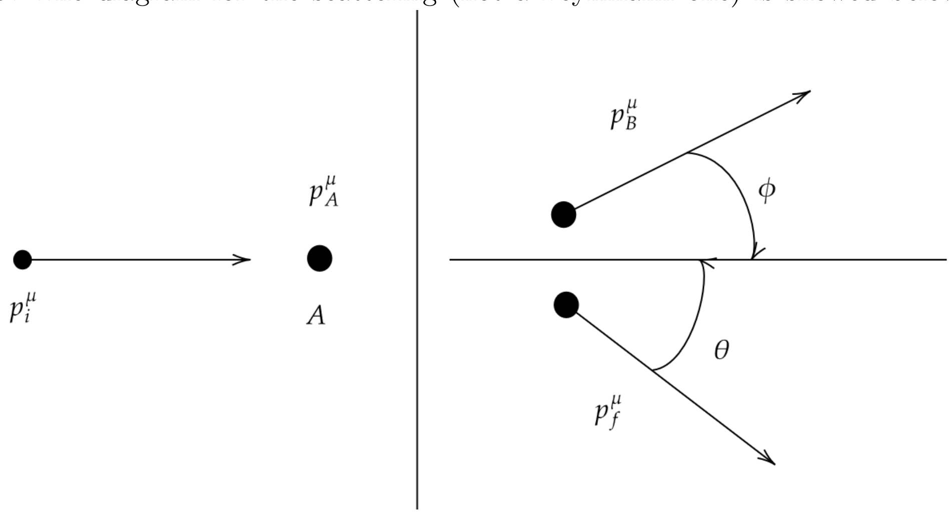
d) In the limit $m \to 0$ we have the Coulumb potential:

$$A_0(r) = \frac{e}{4\pi r}$$

e) It has a low range dictated by the order of $\sim\ 1/m$. In order to keep protons together it should have at least the range of a proton radius, then $m\ \sim\ 1/0,84 fm^{-1} \approx 223 MeV$

Chapter 5

Solution 9. The diagram for the scattering (not a Feynmann one) is showed below:



We start by calculatint the $d\Pi_{LIPS}$ which should be:

$$d\Pi_{LIPS} = \frac{(2\pi)^4}{(2\pi)^6} \frac{d^3p_f}{2E_f} \frac{d^3p_B}{2E_B} \delta^{(4)}(\Sigma p)$$

Conservation laws give us $E_i + E_A = E_f + E_B$ and $\vec{p}_i = \vec{p}_f + \vec{p}_B$
Integrating over $\vec{p}_B$ we have $p_B = -p_f + p_i$

$$d\Pi_{LIPS} = \frac{1}{16\pi^2} \frac{d^3p_f}{E_f E_B} \delta(E_f + E_B - E_i - m_A)$$

The condition imposes that $E_f = \sqrt{(m_f)^2 + (p_f)^2},\ E_B = \sqrt{(m_B)^2 + (p_f - p_i)^2}$

$$d\Pi_{LIPS} = \frac{1}{16\pi^2} d\Omega \int dp_f \frac{(p_f)^2}{E_f E_B} \delta(E_f + E_B - E_i - m_A)$$

Changing variables to $x(p_f) = E_f(p_f) + E_B(p_f) - E_i - m_A$

$$\frac{\mathrm{d}x(p_f)}{\mathrm{d}p_f} = \frac{\mathrm{d}E_f}{\mathrm{d}p_f} + \frac{\mathrm{d}E_B}{\mathrm{d}p_f} = \frac{p_f}{E_f} + \frac{p_f - p_i\cos(\theta)}{E_B}$$

$$d\Pi_{LIPS} = \frac{d\Omega}{16\pi^2} \int_{m_f+m_B-E_i-m_A}^{\infty} dx p_f \left[E_B + E_f\left(1 - \frac{p_i}{p_f}\cos(\theta)\right)\right]^{-1} \delta(x)$$

Plugging it in the cross section equation:

$$d\sigma = \frac{1}{4m_A E_1(\vec{v}_1)} \frac{d\Omega}{16\pi^2} \int_{m_f+m_B-E_i-m_A}^{\infty} dx p_f \left[E_B + E_f\left(1 - \frac{p_i}{p_f}\cos(\theta)\right)\right]^{-1} \delta(x)$$

Using that $\vec{v}_i = \vec{p}_i/p_i^0$, going on:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 m_A} \frac{|\vec{p}_f|}{|\vec{p}_i|} \left[E_B + E_f\left(1 - \frac{|\vec{p}_i|}{|\vec{p}_f|}\cos(\theta)\right)\right]^{-1} |\mathcal{M}|^2 \theta(E_i + m_A - m_f - m_B)$$

In the right mass region, the cross section is:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 m_A} \frac{|\vec{p}_f|}{|\vec{p}_i|} \left[E_B + E_f\left(1 - \frac{|\vec{p}_i|}{|\vec{p}_f|}\cos(\theta)\right)\right]^{-1} |\mathcal{M}|^2$$

Solution 10. (**5.2**) It's directly from what was done before since $\int d^3p \frac{1}{2(2\pi)^3\omega_p}$ is Lorentz invariant and the delta is obviously Lorentz invariant.

Solution 11. **(5.3)** Again we start by calculating the $d\Pi_{LIPS}$, here we'll assume all particles are described by its moments labeled: $p_\mu^{\mu^-}, p_\mu^{e^-}, p_\mu^{\nu_\mu}, p_\mu^{\nu_e}$ for the muon, electron, muon neutrino and electron neutrino, respectively. The $d\Pi_{LIPS}$ read

$$d\Pi_{LIPS} = \frac{(2\pi)^4}{(2\pi)^9} \delta^{(4)}(\Sigma p) \frac{d^3p_{e^-}}{2E_{e^-}} \frac{d^3p_{\nu_\mu}}{2E_{\nu_\mu}} \frac{d^3p_{\nu_e}}{2E_{\nu_e}} = \frac{1}{(2\pi)^5} \frac{d^3p_{e^-}}{2E_{e^-}} \frac{d^3p_{\nu_\mu}}{2E_{\nu_\mu}} \frac{d^3p_{\nu_e}}{2E_{\nu_e}} \delta^{(4)}(p_\mu^{\mu^-} - p_\mu^{e^-} - p_\mu^{\nu_\mu} - p_\mu^{\nu_e})$$

Integrating over $\vec{p}_e$, due the delta we have the constraint $\vec{p}_{\mu^-} - \vec{p}_{\nu_e} - \vec{p}_{\nu_\mu} = \vec{p}_e$. We can then change to the muon's rest frame where $\vec{p}_\mu = 0$.

$$d\Pi_{LIPS} = \frac{1}{8(2\pi)^5} \frac{d^3p_{\nu_e}}{E_{\nu_e}} \frac{d^3p_{\nu_\mu}}{E_{\nu_\mu}} \frac{1}{E_e} \delta(m_\mu - p_{\nu_e} - p_{\nu_\mu} - E_e)$$

As we are treating $m_e \approx 0$, $m_{\mu_\nu} \approx 0$ and $m_{\nu_e} \approx 0$ then $E_e = \sqrt{p_e^2} = \sqrt{(p_{\nu_\mu} + p_{\nu_e})^2}$, $p_{\nu_\mu} = E_{\nu_\mu}$ and $p_{\nu_e} = E_{\nu_e}$. And lets define, as in the enunciate $m_\mu = m$ and $p_{\nu_e} = E_{\nu_e} = E$. The differential may be rewritten as

$$d\Pi_{LIPS} = \frac{1}{8(2\pi)^5} d\Omega_{\nu_\mu} d\Omega_E \frac{dp_{\nu_\mu} dE}{p_{\nu_\mu} E} \frac{p_{\nu_\mu}^2 E^2}{\sqrt{(p_{\nu_\mu}^2 + p_{\nu_e}^2 + 2p_{\nu_\mu} p_{\nu_e} \cos(\theta_{\nu_\mu e})}} \delta(m_\mu - p_{\nu_e} - p_{\nu_\mu} - E_e)$$

Where, just to avoid even more confusion, p_{ν_μ} and p_{ν_e} are shorthands for $|\vec{p}_{\nu_\mu}|$ and $|\vec{p}_{\nu_e}|$ and θ_{μ_ν} is the angle between the momentum $\vec{p}_{\nu_\mu}$ and $\vec{p}_{\nu_e}$. It's convenient now to assume that one of these momentum are parallel to the z axis (if it's not the case we can just rotate the axis), let's take the electron momentum parallel to z, then integrating $d\Omega_E$ gives 4π and $\theta_{\nu_\mu e}$ is the θ angle between the z-axis and p_{ν_μ} contained in $d\Omega_{\nu_\mu}$.

$$d\Pi_{LIPS} = \delta(m_\mu - p_{\nu_e} - p_{\nu_\mu} - E_e) \frac{4\pi}{8(2\pi)^5} dp_{\nu_\mu} dE \int_0^{2\pi} d\varphi \int_{-1}^{1} dcos(\theta) \frac{p_{\nu_\mu} E}{\sqrt{p_{\nu_\mu}^2 + E^2 + 2p_{\nu_\mu} E \cos(\theta)}}$$

It's easy to evaluate this integral through a variable change $x = p_{\nu_\mu}^2 + E^2 + 2p_{\nu_\mu} E \cos(\theta)$, then:

$$d\Pi_{LIPS} = \frac{1}{2(2\pi)^3} dp_{\nu_\mu} E dE \delta(m_\mu - p_{\nu_e} - p_{\nu_\mu} - E_e)$$

Define a new variable: $y(p_{\nu_\mu}) = E + p_{\nu_\mu} + \sqrt{(E + p_{\nu_\mu})^2} - m_\mu$

$$\frac{\mathrm{d}y}{\mathrm{d}p_{\nu_\mu}} = 1 + \frac{p_{\nu_\mu}}{E_e} + \frac{E}{E_e}$$

$$\left(1 + \frac{p_{\nu_\mu}}{E_e} + \frac{E}{E_e}\right)^{-1} dy = dp_{\nu_\mu}$$

$$d\Pi_{LIPS} = \frac{1}{2(2\pi)^3} E dE \left[\left(1 + \frac{p_{\nu_\mu}}{E_e} + \frac{E}{E_e}\right)^{-1} \int_{2E-m}^{\infty} dy \delta(y)\right]$$

Now doing $y = -\gamma$ to put the right symbol in the delta's argument and inverting the up and low limits we have

$$d\Pi_{LIPS} = \frac{1}{2(2\pi)^3} E dE \left(1 + \frac{p_{\nu_\mu}}{E_e} + \frac{E}{E_e}\right)^{-1} \int_{-\infty}^{m-2E} d\gamma \delta(\gamma)$$

$$d\Pi_{LIPS} = \frac{1}{2(2\pi)^3} E dE \left(1 + \frac{p_{\nu_\mu}}{E_e} + \frac{E}{E_e}\right)^{-1} \theta(m - 2E)$$

But notice that

$$\frac{p_{\nu_\mu}}{E_e} + \frac{E}{E_e} = 1$$

Then, finnaly we get

$$d\Pi_{LIPS} = \frac{1}{4(2\pi)^3} E dE \theta(m - 2E)$$

The decay rate is then

$$d\Gamma = \frac{1}{2m} 32 G_F^{\,2}(m^2 - 2mE)mE \frac{1}{4(2\pi)^3} E dE \theta(m - 2E)$$

$$d\Gamma = \frac{32}{8(2\pi)^3} G_F^2 (m^2 - 2mE)E^2 \theta(m - 2E) dE$$

$$\Gamma = \frac{32}{8(2\pi)^3} G_F^2 \int_0^{m/2} (m^2 - 2mE)E^2 dE = \frac{1}{8(2\pi)^3} G_F^2 \frac{m^5}{3} = \frac{m^5 G_F^2}{192\pi^3}$$

$$\Gamma \approx 3.06 \cdot 10^{-16} MeV \approx 4.9 \cdot 10^{-29} J \approx 4.67 \cdot 10^5 s^{-1}$$

$$\tau = \frac{1}{\Gamma} \approx 2.14 \cdot 10^{-6} s = 2.14 \ \mu s$$

Our error is about 3%, I think it's greatly due to treat the electron mass as null. It should be a hard and nice exercise to solve it with the electron rest mass and see how it accounts to the final result.

Extras: An Introduction to Quantum Field Theory (Peskin Schroeder)

Exercise 2. Classical Electromagnetism (with no sources) follows from the action

$$S = \int \mathrm{d}^4x \Big(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} \Big), \qquad \textit{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

a) Derive Maxwell's equations as the Euler-Lagrange equations of this action, treating the components $A_\nu(x)$ as the dynamical variables. Write the equations in standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$

b) Construct the energy-momentum tensor for this theory. Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction with

$$K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu$$

leads to an energy-momentum tensor $\hat{T}$ that is symmetric and yields the standard formulae fot the electromagnetic energy and momentum densities

$$\varepsilon = \frac{1}{2}(E^2 + B^2), \quad \mathbf{S} = \mathbf{E} \times \mathbf{B}$$

Solution 12. a)

From the Euler-Lagrange field equations (2.3) applied in the Lagrangean we have:

$$\partial_\eta \bigg(\frac{\partial \mathcal{L}}{\partial(\partial_\eta A_\epsilon)} \bigg) - \frac{\partial \mathcal{L}}{\partial A_\epsilon} = -\frac{1}{4} \bigg[\partial_\eta \bigg(\frac{\partial F^{\mu\nu}F_{\mu\nu}}{\partial(\partial_\eta A_\epsilon)} \bigg) - \frac{\partial F^{\mu\nu}F_{\mu\nu}}{\partial A_\epsilon} \bigg] = 0$$

So, doing the partial derivatives[1]:

$$\frac{\partial F_{\mu\nu}}{\partial A_\epsilon} = \frac{\partial}{\partial A_\epsilon}\partial_\nu A_\mu - \frac{\partial}{\partial A_\epsilon}\partial_\mu A_\nu = \partial_\nu \bigg(\frac{\partial A_\mu}{\partial A_\epsilon} \bigg) - \partial_\mu \bigg(\frac{\partial A_\nu}{\partial A_\epsilon} \bigg) = \partial_\nu \delta^\epsilon_\mu - \partial_\mu \delta^\epsilon_\nu = 0$$

$$\frac{\partial F_{\mu\nu}}{\partial(\partial_\eta A_\epsilon)} = \frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\eta A_\epsilon)} - \frac{\partial(\partial_\nu A_\mu)}{\partial(\partial_\eta A_\epsilon)} = \delta^\eta_\mu \delta^\epsilon_\nu - \delta^\eta_\nu \delta^\epsilon_\mu$$

[1] Here we impose use the field's continuity.

Substituting them in the E.L. we have the dynamical equation for the field:

$$\partial_\eta F^{\eta\epsilon} = 0$$

Straight from this last equation we have:

$$-\partial_0 F^{0j} = -\partial_t E^j = \partial_i F^{ij} = -\epsilon^{ijk}\partial_i B^k$$

Written in the most usual way:

$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$$

Even $F^{\mu\nu}$ being antisymmetric, in the dynamical equation we can exchange those dumb index 'cause of the equality to 0, then:

$$\partial_\mu F^{\nu\mu} = 0 \quad \textit{or more useful} \quad \partial_\mu F^{0\mu} = 0$$

Hence $F^{00} = 0$ we have our second Maxwell equation:

$$\partial_i T^{0i} = \nabla \cdot \mathbf{E} = 0$$

The other two cames from the Bianchi's identity[2]

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\mu\lambda} = 0$$

b) We can construct the energy-momentum tensor directly from the equation (2.17) and the relations found in the last item.

$$-4T^\eta_\omega = \frac{\partial F^{\mu\nu}F_{\mu\nu}}{\partial(\partial_\eta A_\epsilon)}\partial_\omega A_\epsilon - F^{\mu\nu}F_{\mu\nu}\delta^\eta_\omega = 4F^{\eta\epsilon}\partial_\omega A_\epsilon - F^{\mu\nu}F_{\mu\nu}\delta^\eta_\omega$$

$$T^{\eta\omega} = -F^{\eta\epsilon}\partial^\omega A_\epsilon + \frac{1}{4}F^{\mu\nu}F_{\mu\nu}g^{\eta\omega}$$

This tensor is clearly antisymmetric, we can make it symmetric by adding up a term $\partial_\lambda K^{\lambda\eta\omega}$, antisymmetric in the two first index, with $K^{\lambda\eta\omega} = F^{\eta\lambda}A^{\omega}$[3].
We got then, calling this new tensor $\hat{T}^{\eta\omega}$e:

$$\hat{T}^{\eta\omega} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu}g^{\eta\omega} - F^{\eta\epsilon}\partial^\omega A_\epsilon + \partial_\lambda(F^{\eta\lambda}A^\omega) = \frac{1}{4}F^{\mu\nu}F_{\mu\nu}g^{\eta\omega} - F^{\eta\epsilon}\partial^\omega A_\epsilon + F^{\eta\lambda}\partial_\lambda A^\omega$$

Where the last term can be transformed with a easy algebra (by doing $\epsilon \to \lambda$):

$$F^{\eta\lambda}\partial_\lambda A^\omega - F^{\eta\epsilon}\partial^\omega A_\epsilon = F^{\eta\lambda}\partial_\lambda A^\omega - F^{\eta\lambda}\partial^\omega A_\lambda = F^{\eta\lambda}F_\lambda{}^\omega$$

[2]Adicionar depois
[3]Remembering that $\partial_\lambda(F^{\mu\lambda}) = -ej^\mu = 0$ since we are considering no sources

So we have this last expression for the energy-momentum tensor, which is symmetric as you can see:

$$\hat{T}^{\eta\omega} = F^{\eta\lambda}F_{\lambda}{}^{\omega} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}g^{\eta\omega}$$

The conserved quantity can be easily obtained since:

$$T^{00} = -F^{0\lambda}F^{0}_{\ \lambda} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

Exercise 3. The Complex Scalar Field. Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is:

$$S = \int \mathrm{d}^4\, x(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi)$$

It is easy to analyze this theory by considering $\phi(x)$ and $\phi^*(x)$, rather than the real and imaginary parts of $\phi(x)$, as the basic dynamical variables.

a) Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is

$$H = \int \mathrm{d}^3\, x(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi).$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

b) Diagonalize H by introducting creation and annihilation operators. Show that the theory contains two sets of particles of mass m.

c) Rewrite the conserved charge

$$Q = \int \mathrm{d}^3\, x \frac{i}{2}(\phi^*\pi^* - \pi\phi)$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

d) Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_a^*(x)$, where $a = 1, 2$. Show that there are now four conserved charges, one given by the generalization of part (c), and the other three given by

$$Q^i = \int \mathrm{d}^3\, x \frac{i}{2}(\phi_a^*(\sigma^i)_{ab}\pi_b^* - \pi_a(\sigma^i)_{ab}\phi_b).$$

where σ^i are the Pauli sigma matrices. Show that these three charges have the same commutation relations of angular momentum (SU(2)). Generalize these results to the case of n identical complex scalar fields.[4]

Solution 13. a) We start finding the canonical momenta for both variables, we will omit the x dependence:

$$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^*)} = \frac{\partial(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi)}{\partial(\partial_0 \phi^*)} = \partial^0 \phi = \dot{\phi}$$

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \frac{\partial(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi)}{\partial(\partial_0 \phi)} = \delta^0{}_0 \partial_0 \phi^* = \dot{\phi}^*$$

The canonical commutation relations must be then:

$$[\phi(x,t), \pi(x',t)] = [\phi(x,t), \dot{\phi}^*(x',t)] = i\hbar\delta(x-x')$$

$$[\phi^*(x,t), \pi^*(x',t)] = [\phi^*(x,t), \dot{\phi}(x',t)] = i\hbar\delta(x-x')$$

$$[\phi(x,t), \phi^*(x',t)] = [\dot{\phi}(x,t), \dot{\phi}^*(x',t)] = 0$$

Now the Hamiltonian density can be written as:

$$H = \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L} = \pi\dot{\phi} + \pi^*\dot{\phi}^* - \partial_0\phi^*\partial^0\phi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi$$

$$\mathcal{H} = \int \mathrm{d}^3\, xH = \int \mathrm{d}^3\, x(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi)$$

We can then persue the time evolution of the operators using the Heisenberg picture and realize they obey the Klein-Gordon equation[5]

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = -i[\dot{\phi}, \mathcal{H}] = -i[\pi^*(x), \mathcal{H}]$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = -i\int \mathrm{d}^3\, y([\pi^*(y), \pi^*(x)])\pi(x) + [\pi^*(x), \nabla\phi^*(y)]\nabla\phi(y) + m^2[\pi^*(x), \phi^*(y)]\phi(y))$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = -\int \mathrm{d}^3\, y(\nabla(\delta(x-y))\nabla\phi(y) + m^2\delta(x-y)\phi(y))$$

[4] With some additional work you can show that there are actually six conserved charges in the case of two complex fields, and $n(2n-1)$ in the case of n fields, corresponding to the generators of the rotation group in four and 2n dimensions, respectively. The extras symmetries often do not survive when nonlinear interactions of the fields are included.

[5] It's reciprocal to the Complex field

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = -\int \mathrm{d}^3\, y(-\delta(x-y)\nabla^2\phi(y) + m^2\delta(x-y)\phi(y)) = \nabla^2\phi(x) - m^2\phi(x)$$

$$(\partial_\mu\partial^\mu + m^2)\phi(x) = 0, \quad (\partial_\mu\partial^\mu + m^2)\phi^*(x) = 0$$

b) Both the parts (real and complex) follow an equation who seems like a harmonic oscillator, we can then go to the p-space and work with:

$$\phi(\mathbf{x},t) = \int \frac{\mathrm{d}^3\, p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{p},t) \qquad \phi^*(\mathbf{x},t) = \int \frac{\mathrm{d}^3\, p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\phi^*(\mathbf{p},t)$$

$$\frac{\partial^2\phi(\mathbf{p},t)}{\partial t^2} + \omega_p^2\phi(\mathbf{p},t) \qquad\qquad \frac{\partial^2\phi^*(\mathbf{p},t)}{\partial t^2} + \omega_p^2\phi^*(\mathbf{p},t)$$

With $\omega_p^2 = |\mathbf{p}|^2 + m^2$. We can rewrite the expressions using the ladder operators, the same way Dirac's solution to the quantum mechanical analogous. Setting $E_\mathbf{p} = \omega_\mathbf{p}$ we can write the operators in the form:

$$\phi = \frac{1}{\sqrt{2E_\mathbf{p}}}(a_\mathbf{p} + a_\mathbf{p}^\dagger), \quad \pi = -i\sqrt{\frac{E_\mathbf{p}}{2}}(a_\mathbf{p} - a_\mathbf{p}^\dagger)$$
$$\phi^* = \frac{1}{\sqrt{2E_\mathbf{p}}}(b_\mathbf{p} + b_\mathbf{p}^\dagger), \quad \pi^* = -i\sqrt{\frac{E_\mathbf{p}}{2}}(b_\mathbf{p} - b_\mathbf{p}^\dagger)$$

Which we can convert completely to fit in the integral before, this way:

$$\phi(\mathbf{x}) = \int \frac{\mathrm{d}^3\, p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}(a_\mathbf{p} + a_{\text{-}\mathbf{p}}^\dagger)e^{i\mathbf{p}\cdot\mathbf{x}}, \quad \pi(\mathbf{x}) = -i\int \frac{\mathrm{d}^3\, p}{(2\pi)^3}\sqrt{\frac{E_\mathbf{p}}{2}}(a_\mathbf{p} - a_{\text{-}\mathbf{p}}^\dagger)e^{i\mathbf{p}\cdot\mathbf{x}}$$

$$\phi^*(\mathbf{x}) = \int \frac{\mathrm{d}^3\, p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}(b_\mathbf{p} + b_{\text{-}\mathbf{p}}^\dagger)e^{i\mathbf{p}\cdot\mathbf{x}}, \quad \pi^*(\mathbf{x}) = -i\int \frac{\mathrm{d}^3\, p}{(2\pi)^3}\sqrt{\frac{E_\mathbf{p}}{2}}(b_\mathbf{p} - b_{\text{-}\mathbf{p}}^\dagger)e^{i\mathbf{p}\cdot\mathbf{x}}$$

Setting this equations in the Hamiltonian we get

$$\mathcal{H} = \int \mathrm{d}^3\, x \iint \frac{\mathrm{d}^3\, p\, \mathrm{d}^3\, p'}{(2\pi)^3}\Bigg((-1)\frac{\sqrt{E_\mathbf{p}E_{\mathbf{p}'}}}{2}(b_\mathbf{p} - b_\mathbf{p}^\dagger)(a_{\mathbf{p}'} - a_{\mathbf{p}'}^\dagger)$$
$$+\frac{-\mathbf{p}'\cdot\mathbf{p} + m^2}{2\sqrt{E_\mathbf{p}E_{\mathbf{p}'}}}(b_\mathbf{p} + b_\mathbf{p}^\dagger)(a_{\mathbf{p}'} + a_{\mathbf{p}'}^\dagger)\Bigg)e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}}$$

Integrating in the x-space we'll obtain:

$$\mathcal{H} = \iint \frac{\mathrm{d}^3 p\, \mathrm{d}^3 p'}{2(2\pi)^3} \left((-1)\sqrt{E_{\mathbf{p}}E_{\mathbf{p}'}}(b_{\mathbf{p}} - b^\dagger_{\mathbf{p}})(a_{\mathbf{p}'} - a^\dagger_{\mathbf{p}'}) + \frac{-\mathbf{p}' \cdot \mathbf{p} + m^2}{\sqrt{E_{\mathbf{p}}E_{\mathbf{p}'}}}(b_{\mathbf{p}} + b^\dagger_{\mathbf{p}})(a_{\mathbf{p}'} + a^\dagger_{\mathbf{p}'}) \right) \delta^{(3)}(\mathbf{p} + \mathbf{p}')$$

$$\mathcal{H} = \frac{1}{2(2\pi)^3} \int \mathrm{d}^3 p E_{\mathbf{p}} \left[(b_{\mathbf{p}} + b^\dagger_{-\mathbf{p}})(a_{-\mathbf{p}} + a^\dagger_{\mathbf{p}}) - (b_{\mathbf{p}} - b^\dagger_{-\mathbf{p}})(a_{-\mathbf{p}} - a^\dagger_{\mathbf{p}}) \right]$$

So we'll encounter the final form for the Hamiltonian

$$\mathcal{H} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3 p (b_{\mathbf{p}} a^\dagger_{\mathbf{p}} + b^\dagger_{-\mathbf{p}} a_{-\mathbf{p}}) E_{\mathbf{p}}$$

Exercise 4. Evaluate the function

$$\langle 0| \phi(x)\phi(y) |0\rangle = D(x-y) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)}$$

for $(x-y)$ spacelike so that $(x-y)^2 = -r^2$, explicitly in terms of Bessel functions.