Introduction to Analysis

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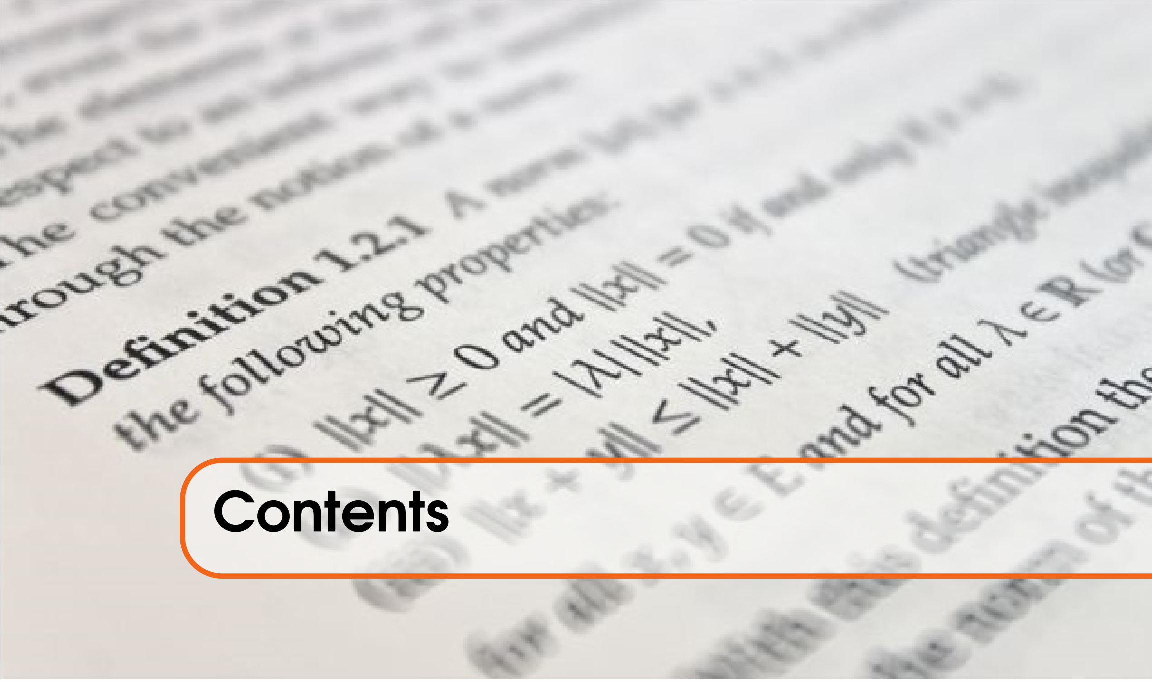
Trigger warning: Mathematics contained herein.



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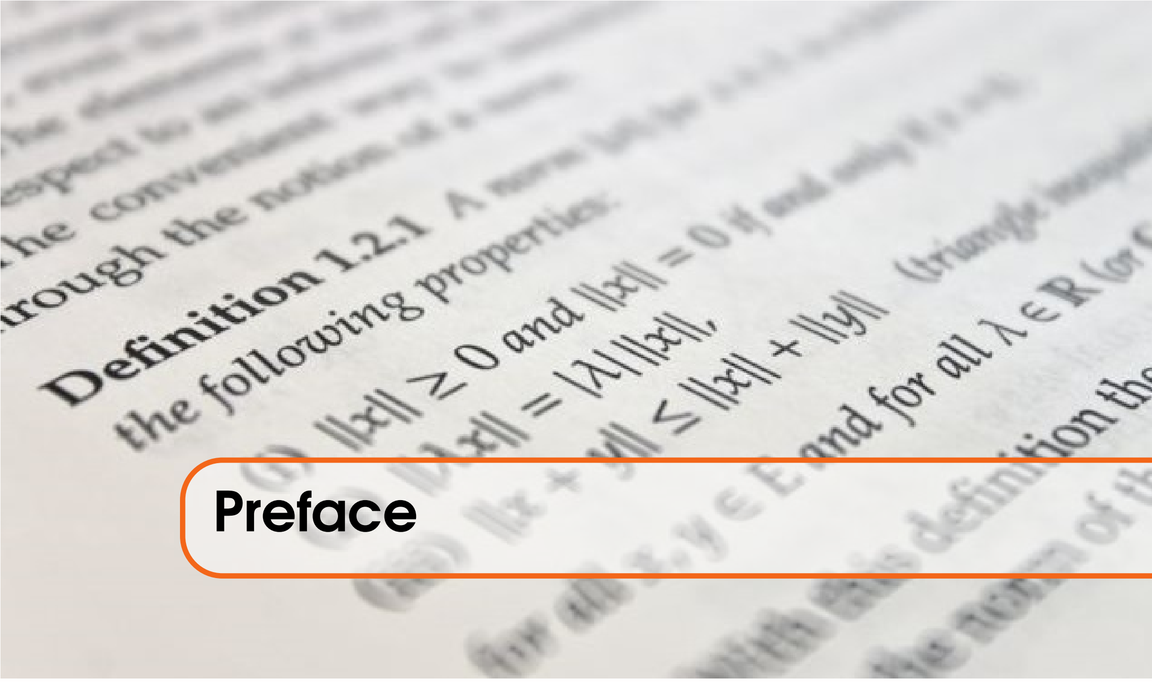

Preface

I started writing these notes in order to prepare for the 'lectures'[1] I would be giving for MATH 104 Introduction to Analysis at the University of California, Berkeley during Summer 2015. My obsessive-compulsive perfectionism and completionism turned them into what they are today.

At first, these notes were really just for me—I wanted to be sure I was ready to teach the class. As I'm sure you're aware if you've ever taught before, there is much more to being able to teach well than simply knowing all the material. For example, it is not enough to simply know Theorems 1 and 2. Among other things, for example, you have to know the order in which they come in the theory. Most of the motivation for starting these notes was to make sure I got all of that straight in my mind before I went up in front of a class.

It almost immediately became apparent, however, that the book I was assigned to use to teach the course (which shall remain unnamed[2]. . .) was not sufficient for what I wanted to do. The target

[1]This is in quotes because they weren't really lectures in the traditional sense of the word.

[2]Lest you think I am referring to Pugh's book [Pug02], let me clarify that I am not. (Pugh is at Berkeley, and so it would be natural for you to think that they made this the required text, but alas, they did not.) Indeed, Pugh's book is one of my favorite real analysis books, and it's likely that had I been assigned to use his book, these notes would not exist.

audience for the course (or so I thought) was students who were at least considering going to graduate school in mathematics, but also who had never seen any analysis before, and the book I was supposed to use was just simply too elementary to serve this purpose, especially for the caliber of students at one of the world's top mathematics departments. What really surprised me, however, is that I could not find a single mathematical analysis book that served *all* of my needs. For example, I wanted to teach the Lebesgue integral in place of the Riemann integral, and nearly all of the mathematical analysis books that teach the Lebesgue integral as the *primary* definition of the integral assume that the student has seen analysis before. Thus, within a week or so after started the course,[3] the notes switched from just being a device I would use to prepare, to what would serve as the effective text for the course.

I now briefly discuss several of the topics / ways of treating topics that I had difficultly finding in standard analysis texts.

Analysis with an eye towards category theory

Given that this is a first course in analysis, I didn't want to go super heavy on the category theory—I don't even use functors, for example—but on the other hand I felt as if it would be a sin to say nothing of it. For example, $\mathbb{R}$ as a set, a field, a topological space, a uniform space, a metric space, an ordered-field, blah blah etc. etc.. . . all of these are *very* different things, and the appropriate way to make these distinctions is of course via the use of morphisms (because morphisms determine the meaning of isomorphism). In **Set**, the mathematics cannot tell the difference between $\mathbb{R}$ and $2^{\mathbb{N}}$, whereas in **Top** they are very different: one is compact and the other is not. On the other hand, in **Top**, the mathematics cannot tell between $(0, 1)$ and $\mathbb{R}$, whereas in **Uni** they are very different: one is complete and the other is not. I can understand one making the argument that you can make these distinctions without using categories per se. On the other hand, the budding mathematician *has* to learn categories at some point, and so they may as well learn the absolute basics now.

[3]The course ran 8 weeks in total.

I also implicitly made use of other more advanced categorical ideas, most notably of 'pseudo-universal' properties.[4] For example, the integers are defined as the unique[5] totally-ordered integral cring which (i) contains $\mathbb{N}$ and (ii) is contained in any other totally-ordered integral cring which contains $\mathbb{N}$. This is just one example on how I tried to stress that *mathematical objects are defined by the properties that uniquely specify them*, instead of the nuts and bolts which happen to make them up. The right way to think of the real numbers is as the unique nonzero[6] Dedekind-complete totally-ordered field, not as Dedekind cuts or equivalence classes of Cauchy sequences or whatever—these are merely tools used to prove the existence of such an object.

Construction of the reals from scratch

Many books go from $\mathbb{N}$ to $\mathbb{R}$; what I actually had difficulty finding was sources that constructed $\mathbb{N}$ from scratch. I very much wanted to say at the end of the course that everything we proved could be reduced to nothing more than pure logic[7]; and the idea that if you have a bunch of things, you can give all those things a single name (X, for example), and now you have a new thing.[8] With these basic tools in place, we can construct $\mathbb{N}$ from the ground-up, and so in turn we can construct $\mathbb{R}$ from the ground-up.

Nets

I really can't believe how many peers of mine, and even professional mathematicians, don't know nets. There is no really getting around them: if you want to do general topology, you *have* to use nets.

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[4]I say "pseudo" because I usually phrase things as "A is the 'smallest' things that satisfies XYZ and contains B." instead of "There is a universal morphism from B to A in category ABC.". Maybe I will change this in the future, but I felt as if this level of mathematical sophistication in Chapter 1 of a first analysis course could lose a lot of people.

[5]Up to isomorphism in the category of preordered rings.

[6]It is common to require that $0 \neq 1$ in fields, but I find this condition awkward and unnecessary—I'm not aware of any serious benefit this has other than to remove the necessity of having to say "nonzero" in places (like here).

[7]In a naive sense. Not in the sense of mathematical logic.

[8]That is, the naive notion of a set.

[9] Sequences are just not enough. For example, the real numbers with the discrete topology and the cocountable topology provide an example of a set equipped with two nonhomeomorphic topologies for which the notions of sequential convergence agree. Thus, you cannot uniquely specify a topology by saying what i means for sequences to converge—if you ever want to define a topology using convergence, you *must* use nets. Moreover, the definition of a uniform space is the way it is because then the uniformity itself, ordered by reverse star-refinement, is a directed set, so that one can conveniently index nets by the uniformity itself. And even if you do only care about $\mathbb{R}$, however, nets can make some arguments a tad bit easier. For example, $\mathbb{R}^+$ both with the usual ordering and reverse ordering provide examples of directed sets, and so you can interpret $\lim_{\varepsilon\to 0^+}$ and $\lim_{x\to\infty}$ both as limits of nets, which can be used to slightly simplify some proofs.

Subnets

I imagine that there are at least some sources that shy away from using nets because subnets are notoriously more difficult than subsequences. Despite how determined I was to use nets, even I struggled with the tedium of the definition of subnets for awhile. Eventually I realized that this difficult was a consequence of using the 'wrong' definition of subnet. There are at least two distinct definitions of subnet in the literature that I was aware—see Propositions 2.4.5.9 and 2.4.5.11—and both of these were 'wrong' in the sense that they did not agree with what the analogous notion for filter bases (filterings—see Definition 3.3.6) was.[10] By examining the definition of a filtering, I was able to formulate a definition of subnet (Definition 2.4.5.1) that agreed (Proposition 3.3.9) with the definition of filter bases. That definition is as follows.

> A *subnet* of a net $x : \Lambda \to X$ is a net $y : \Lambda' \to X$ such that
>
> (i). for all $\mu \in \Lambda'$, $y_\mu = x_{\lambda_\mu}$ for some $\lambda_\mu \in \Lambda$; and

[9] Okay, so that's not quite true. There are alternatives to nets (e.g. filter bases). The point is that sequences themselves are not enough—if not nets, you will need some other generalization/replacement for sequences.

[10] There wasn't really any uncertainly about what the 'right' notion was for filter bases.

(ii). whenever $U \subseteq X$ eventually contains x, it eventually contains y.

That is, it is a net made up from terms of the original net that has the property that any set which eventually contains the original net eventually contains the subnet. I personally find this to be a much cleaner definition than the ones involving awkward conditions on the indices of the nets: The indices themselves don't matter—it is only what *eventually happens* that matters.

Uniform spaces

My inclusion of uniform spaces was essentially a replacement for a treatment of metric spaces (which, for some reason, seems standard). Unfortunately, metric spaces are just not enough. For example, if you care about continuous functions on the real line,[11] you cannot get away with metric spaces. You could generalize to semimetric spaces, but I find this to not be as clean as uniform spaces. Uniform spaces also include *all* topological groups, and so by generalizing to uniform spaces, you include another hugely important example of topological spaces. For some reason, the theory of uniform spaces seems to be very obscure in standard undergraduate and graduate curricula, and I myself was never formally taught it in my own education. Despite this, I've found knowledge of uniform spaces to be incredibly useful,[12] in particular in functional analysis, and they deserve more attention than they get in courses in analysis and general topology.

σ-algebras

In general, I try to place a relatively large amount of emphasis on motivation, and so when I first started writing down my development of measure theory, I decided from the beginning that I would first introduce the notion of an outer-measure, and then introduce the notion of an abstract σ-algebra once I had shown that Carathéodory's Theorem (Theorem 5.1.1.23) says that the measurable sets form a

[11]If you're a mathematician or planning on becoming one, chances are you do, at least a little.

[12]For example, there is a generalization of Haar measure to uniform spaces (see the Haar-Howes Theorem), which, among other things, gives isometric invariance of Lebesgue measure for free.

σ-algebra. That is, you don't have to pull the definition out of thin air—the theory gives it to you. As I continued to develop the theory, however, I found that I never really needed to make use of the notion—working with measure spaces (sets equipped with a measure) was sufficient: all I had to do was add in the hypothesis "measurable" in certain places. I found that this made the theory quite a bit cleaner. For one thing, it was easier to teach, but even in terms of the mathematics itself, it made some things a bit easier. For example, when discussing product measures, you run into the issue of whether or not the product σ-algebra agrees with the σ-algebra of measurable sets on the product. If I recall, in general this fails, but that doesn't matter: if you don't use σ-algebras, it's just not an issue—Mr. Carathéodory's *always* tells you what the measurable sets are.

Measure theory and the Lebesgue integral

Since the first day you learned the 'definition' of the integral in your first calculus class,[13] you have almost certainly thought of the integral as the '(signed) area under the curve'. That's what it is. Measure theory allows one to make this the *definition* of the integral. There is no need for these awkward approximations by rectangles.[14] The counter-argument to this is "Okay, sure, but that just reduces the complication of the Lebesgue integral to measure theory.". To be fair, this is correct—if you take this route, you have to develop measure theory, whereas, with the Riemann integral, you can start talking about partitions, etc., almost immediately. That being said, I take my time developing measure theory more fully because I don't like approaches which 'run straight to the finish line', but in principle, if you really don't like the amount of build-up that measure theory requires, you can probably cut-down the length of my development by two-thirds or so to get to the definition of the integral. To those who would still object to even a minimal measure theoretic treatment, I would argue the following. There are two types of students: those who want to become mathematicians and those who do not. The

[13]Or when self-teaching, for the particularly precocious amongst you.

[14]Of course, in the spirit of defining things by the properties they uniquely specify, I don't *quite* state it like this. Instead I say something like "The area under the curve is the unique extended real-valued function on nonnegative Borel functions such that. . . "—see Definition 5.2.2.1.

former need to learn the Lebesgue integral anyways, and so I don't see much point dicking around with the Riemann integral when they'll have to learn the Lebesgue integral in a graduate analysis course (if not earlier) anyways. In the latter case, simply stating informally "It's the area under the curve." should suffice—in fact, that's not even really a lie, or even a stretch of the truth—if you set things up right, that's *literally* what it is.

Tangent spaces in differentiation in $\mathbb{R}^d$

I actually found differentiation to be quite a bit easier when studying it on manifolds as opposed to in $\mathbb{R}^d$. I think this is because it is easy to mix up objects that should be thought of as *points* and objects that should be thought of as *vectors* in $\mathbb{R}^d$, something that is just plain impossible to do in a general manifold. In an attempt to circumvent this potential confusion, I introduce the term *tangent space* as well as notation for it, $\mathrm{T}_x(\mathbb{R}^d)$. This is really only for conceptual clarity: it is not the definition one would use on a manifold. In fact, it would be essentially impossible to do this because the usual definition of the tangent space on a manifold requires one to already have a theory of differentiation on $\mathbb{R}^d$. Instead, I just take $\mathrm{T}_x(\mathbb{R}^d)$ as a *metric vector space* whereas the 'entire space' $\mathbb{R}^d$ is considered as a *metric space*.[15]

I also made a point to introduce index notation and tensor fields. The motivation for this one really comes from my physics background: index notation and tensor calculus are mandatory background if one wants to do theoretical physics. But this is not the only reason for introducing it of course—I personally find index notation to sometimes be very useful, even in pure Riemannian geometry. In particular, I find that index notation can make some equations much more transparent. For example, compare[16]

$$R_{ab}{}^{c}{}_{d} \coloneqq \nabla_a \nabla_b - \nabla_b \nabla_a$$

[15]Note the implicit elementary categorical ideas coming into play here. Also note the two distinct uses of the term "metric". Who the hell came-up with this terminology anyways? Mathematicians need to get better at naming things. . . Surely you can find a more descriptive adjective than "normal" or "regular" for your fancy new idea, yes?

[16]At least for a torsion-free covariant derivative.

versus

$$R(u,v)w := \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w.$$

Morally, the curvature tensor is supposed to measure the lack of commutativity of the covariant derivative, and the former makes that perfectly clear—it is literally the commutator $[\nabla_a, \nabla_b]$—whereas the latter has this awkward extra term $\nabla_{[u,v]}w$ that feels like it shouldn't be there. Mathematicians then also tend to make this (what I find awkward) distinction between[17] the curvature endomorphism R, the curvature tensor Rm, and the curvature form $\mathfrak{R}$, but in my mind, these are all just different versions of ${R_{ab}}^c{}_d$: ${R_{ab}}^c{}_d$, R_{abcd}, and $[R_{ab}]^c{}_d$ respectively. To each his own, but, in this case, I personally find physicist's notation much clearer, even in the realm of pure mathematics. Note that there is no discussion of riemannian geometry here—this is just a motivating example to justify the teaching of index notation.

Differentiability

Finally, I also use what seems to be a nonstandard definition of differentiable. This was almost by accident. Without consulting a reference, I just wrote down what to me seemed to be the most obvious thing: a real-valued function on $\mathbb{R}^d$ is differentiable at a point x iff (i) all the directional derivatives at x exist and (ii) the map that sends a tangent vector to its directional derivative is linear and continuous. That is, I put the minimal conditions in order to guarantee that the differential was a one-form. It wasn't until I started trying to prove things that I realized this was nonstandard. As it turns out, with this definition, there are *infinitely-differentiable functions which are not continuous*.[18] Oopsies. Despite this pathology, I just decided to roll with it, partially due to time constraints. To give you some idea, the first draft when I had finished it was 344 pages, which had been written in just a little more than 8 weeks. I had completely underestimated how much work this was going to be[19] and at that point in the course, were I try to go back and change the definition of differentiable, I

[17]Using the notation of [Lee97].

[18]See Example 6.5.1.

[19]By the end, I pretty much did nothing besides teach and write these notes.

would have been at serious risk of just simply not being prepared for future classes. In the end, things worked out just fine though—I found that a lot of the time I didn't need differentiable functions to be continuous, and when I did, a simple distinction between smooth and infinitely-differentiable did the trick.[20] In fact, as I find this definition simpler than the one usually given (Fréchet differentiable), I may just stick with it the next time I teach.

A note to the reader

The mathematics in these notes is developed "from the ground up". In particular, in principle, there are no prerequisites. That said, there is a modest amount of basic material that I cover sufficiently fast that it would be very helpful if you had at least passing familiarity with. Essentially all of this 'prerequisite' material is given in the appendices.[21] I recommend you read these notes linearly, and refer to the appendices as needed when you come across concepts you are not familiar with. The notes are written in such a way that I would expect a student with no background to refer to the appendices *very often* in the beginning, but very little by the end.

Of all the statements which are true in these notes, they are roughly divided into two broad categories: the statements which are true by definition and the statements which are true because we can prove them. For the former, we have *definitions*; for the latter, we have *theorems*, *propositions*, *corollaries*, *lemmas*, and *claims*. We also have "meta" versions of (some of) these.

A definition is exactly what you think it is. A "meta-definition" is actually a whole collection of definitions—whenever you plug something in for XYZ, you get an actual definition. This explanation is probably not very lucid now, but I imagine it should be pretty obvious what I mean by this when you actually come to them—see, for example, Meta-definition 2.4.2.1. For the record, "meta-definition"

[20] *Smooth* wound-up meaning that (i) the function was infinitely differentiable and (ii) all the derivatives where continuous.

[21] Though there is also quite a bit of material there that I would not expect you to know.

is not a standard term (and I don't really think there is a standard term for this).[22] Similarly for "'meta-propositions", etc..

There is no hard and fast distinction between what I called theorems, propositions, corollaries, and lemmas. I tried to roughly adhere to the following conventions. If a result is used only in a proof of a single result and nowhere else, it is a *lemma*. If a result follows immediately or almost immediately from another result, it is a *corollary*. Results of particular significance are *theorems*. Everything else is a *proposition*. Claims, on the other hand, are distinct in that, not only only are they used in the proof of a single result like lemmas, but furthermore they wouldn't even make sense as stand-alone results (for example, if they use notation specific to the proof).

In particular, note that the distinction between theorems and propositions has to do with the relative *significance*[23] of the *statement* of the result, and has nothing to to with the *difficulty* of the *proof*. Indeed, there are quite a few rather trivial results labeled as theorems simply because they are important.

There are also statements presented in blue boxes. The blue box is meant to draw attention to the fact contained therein, as it is particularly important for one reason or another. That said, the content in the blue box doesn't always contain the "full story", and is potentially a "watered-down" version of the truth. For example, we will very often omit stating what things are in the box itself and the reader will have to consult the surrounding context to gain the full meaning. This is by design—the boxes are supposed to highlight something important for convenience of the reader, not precise mathematics per se, and 'bogging it down' with details that are probably clear from context defeats the purpose of being quick and convenient.

I should mention that every now and then I give nonstandard names to results which would otherwise not have names. Part of the motivation for this is that I personally find this makes it easier to remember which result is which. For example, would you rather I refer to "Theorem 3.2.14" or the "Kelley's Convergence Axioms"?

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[22]The closest I can think of is *axiom schema*, but "definition schema" sounds quite awkward to my ear.

[23]Obviously this is completely subjective and I would not expect any mathematician to pick out the exact same results which deserve the title of "theorem".

Just be warned that you shouldn't go up to other mathematicians, use these names, and expect them to know what you're talking about. (I will point it out when a name is nonstandard.)

As an unimportant comment, I mention in case you're curious that I used double quotes when I am quoting something, usually a term or phrase either I or people in general use, and single quotes to indicate that the thing is quotes is not literally that thing. For example, mathematicians prove theorems and physicists prove 'theorems'.

Finally, I want to make a comment on the difficulty of these notes. This course is supposed to be hard. Very hard. But not impossible. I do expect you to seriously bust your ass to learn the material. But it's also reasonable enough that, if you do put in the work, you will be able to do decently in the course.

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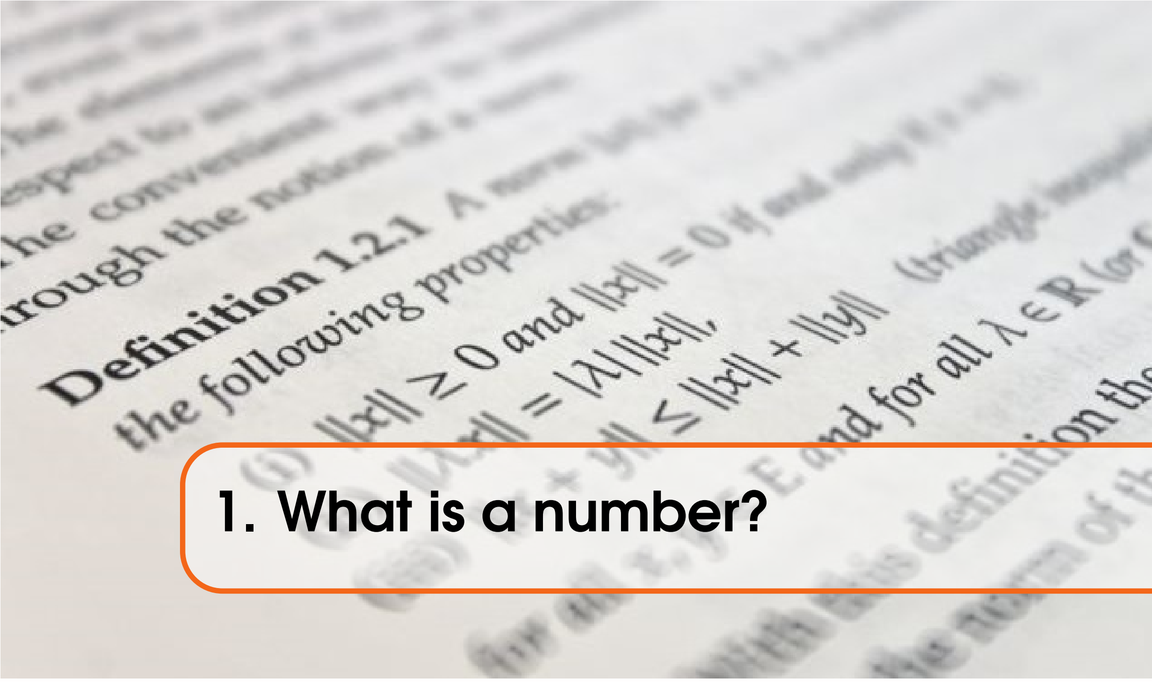


1. What is a number?

Before doing anything else, our first order of business is to answer the question “What is a number?”. Of course, the answer is that the word “number” means whatever we declare it to mean, and indeed, what a mathematician means when he says the word “number” will vary from context to context. There is no single universal type of number that will work in all contexts, so instead of coming up with a single answer to this question, we will instead define and develop several different ‘number systems’, those being the natural numbers, the integers, the rational numbers, and the real numbers. Being the most fundamental, we start with the natural numbers.

1.1 Cardinality of sets and the natural numbers

The motivation for introducing the natural numbers is that these are the things that allow us to *count* things. We must thus first answer the question “What does it mean to ‘count’?”. We will make sense of this by making sense of the notion of the *cardinality* of a set, the cardinality being a sort of measure of how many elements the set contains.

1.1.1 The natural numbers as a set

The first step in defining the cardinality of sets is being able to decide when two sets have the same number of elements. So, suppose we are given two sets X and Y and that we would like to determine whether X and Y have the same number of elements. How would you do this?

Intuitively, you could start by trying to label all the elements in Y by elements of X, without repeating labels. If either (i) you ran out of labels before you finished labeling all elements in Y or (ii) you were forced to assign more than one label to an element of Y, then you could deduce that X and Y did *not* have the same number of elements. To make this precise, we think of this labeling as a function from X to Y. Then, the first case corresponds to this labeling function not being surjective and the second case corresponds to this labeling function not being injective.

The more precise intuition is then that the two sets X and Y have the same number of elements, that is, the same cardinality, iff there is a bijection $f\colon X \to Y$ between them: that f is an injection says that we don't use a label more than once (or equivalently that Y has at least as many elements as X) and that f is a surjection says that we label everything at least once (or equivalently that X has at least as many elements as Y). In other words, according to Exercise B.2.5, two sets should have the same cardinality iff they are isomorphic in **Set**.[1]

Definition 1.1.1.1 — Equinumerous Let X and Y be sets. Then, X and Y are ***equinumerous*** iff $X \cong_{\mathbf{Set}} Y$.

So we've determined what it means for two sets to have the same cardinality, but what actually *is* a cardinality? The trick is to identify a cardinal with the collection of all sets which have that cardinality.

Definition 1.1.1.2 — Cardinal number A ***cardinal number*** is an element of

$$\aleph := \mathrm{Obj}(\mathbf{Set})/{\cong_{\mathbf{Set}}} := \left\{ [X]_{\cong_{\mathbf{Set}}} : X \in \mathrm{Obj}(\mathbf{Set}) \right\}. \quad (1.1.1.3)$$

[1] **Set** is the category of sets. If you are reading this linearly and you see something you don't recognize, chances are it's in the appendix. Use the index and index of notation at the end of the notes to find exactly where.

R In other words, a cardinal is an equivalence class of sets, the equivalence relation being equinumerosity.[a] Furthermore, for X a set, we write $|X| := [X]_{\cong_{\mathbf{Set}}}$.

[a] Recall that the relation of isomorphism is an equivalence relation in every category—see Exercise B.2.9.

The idea then is that the natural numbers are precisely those cardinals which are finite. We thus must now answer the question "What does it mean to be 'finite'?". This is actually a tad bit tricky.

Of course, we don't have a precise definition yet, but everyone has an intuitive idea of what it means to be infinite. So, consider an 'infinite set' X. Now remove one element $x_0 \in X$ to form the set $U := X \setminus \{x_0\}$. For any reasonable definition of "infinite", removing a single element from an infinite set should not change the fact that it is infinite, and so U should still be infinite. In fact, more should be true. Not only should U still be infinite, but it should still have the same cardinality as X.[2] It is this idea that we take as the defining property of being infinite.

Definition 1.1.1.4 — Finite and infinite Let X be a set. Then, X is ***infinite*** iff there is a bijection from X to a proper subset of X. X is ***finite*** iff it is not infinite.

R The keyword here is *proper*—there is a bijection from every set X to some subset of X, namely $X \subseteq X$ itself.

Before getting to the natural numbers themselves, let's discuss a couple of interesting properties about infinite sets.

Proposition 1.1.1.5 Let X be a set and define

$$\mathscr{F}_X := \{S \subseteq X : S \text{ is finite.}\} . \tag{1.1.1.6}$$

[2] We will see in the next chapter that there are infinite sets which are not of the same cardinality. That is, in this sense, there is more than one type of infinity.

Then, if X is infinite, then $|X| = |\mathscr{F}_X|$.

R In words, for infinite sets, the cardinality of the set itself is the same as the cardinality of its collection of finite subsets.

Proof. We leave this as an exercise.

Exercise 1.1.1.7 Prove this yourself.

■

Proposition 1.1.1.8 Let $\mathscr{F}$ be an infinite collection of finite sets. Then,

$$\left|\bigcup_{F\in\mathscr{F}} F\right| = |\mathscr{F}|. \tag{1.1.1.9}$$

R In words, if κ is an infinite cardinal, the union of κ many finite sets still has cardinality κ.

Proof. We leave this as an exercise.

Exercise 1.1.1.10 Prove this yourself.

■

And now finally:

Definition 1.1.1.11 — Natural numbers The ***natural numbers***, $\mathbb{N}$, are defined as

$$\mathbb{N} := \{|X| : X \in \mathrm{Obj}(\mathbf{Set}) \text{ is finite.}\}. \tag{1.1.1.12}$$

In words, the natural numbers are precisely the cardinals of finite sets.

R Some people take the natural numbers to not include 0. This is a bit silly for a couple of reasons. First of all, if you think of the natural numbers as cardinals, as we are doing here, then 0 has to be a natural number as it is the cardinality of the empty-set. Furthermore, as we shall see in the next subsection, it makes the algebraic structure of $\mathbb{N}$ slightly nicer because 0 acts as an additive identity. Indeed, I am not even aware of a term to describe the sort of algebraic object $\mathbb{N}$ would be if it did not contain 0. Finally, regardless of your convention, you already have a symbol to denote $\{1, 2, 3, \ldots\}$, namely $\mathbb{Z}^+$:[a] having the symbol $\mathbb{N}$ denote the same is an inefficient use of notation.

[a]Of course, at this point in the next, we technically don't know what any of these symbols mean. For the purposes of motivating a convention, however, I have no qualms about pretending you are not completely ignorant.

1.1.2 The natural numbers as an integral crig

Great! We now know what the natural numbers are. Our next objective then is to be able to add and multiply natural numbers. In fact, we will define not only what it means to add and multiply natural numbers, but instead we will define what it means to add and multiply *any* cardinal numbers. Then, we will just need to check that the sum and product of two finite cardinals is again finite.[3]

Proposition 1.1.2.1 — Addition and multiplication Let $m, n \in \mathbb{N}$ and let M and N be sets such that $m = |M|$ and $n = |N|$. Then,

$$m + n := |M \sqcup N| \text{ and } mn := |M \times N| \tag{1.1.2.2}$$

are well-defined.

R Recall that $|M|$ means the equivalence class of the set M under the equivalence relation of equinu-

[3]This is required so that addition and multiplication are *binary operations* $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. In other words, what you don't want to happen is add two natural numbers and obtain something that isn't a natural number (i.e. a finite cardinal).

merosity (see the definition of a cardinal, Definition 1.1.1.2). Whenever we define an operation on equivalence classes that makes reference to a specific representative of that equivalence class, we must check that our definition does not depend on this representative. For example, perhaps if I take a different set M' with $|M'| = |M|$, it will turn out that $|M' \sqcup N| \neq |M \sqcup N|$. If this happens, then our definition doesn't make sense. Of course, it doesn't happen, but we need to check that it doesn't happen. This is what it means to be *well-defined*.

Proof. Let M_1, M_2, N_1, N_2 be sets with $|M_1| = |M_2|$ and $|N_1| = |N_2|$. By definition, this means that there are bijections $f\colon M_1 \to M_2$ and $g\colon N_1 \to N_2$. We would like to show that $|M_1 \sqcup N_1| = |M_2 \sqcup N_2|$. To show this, by definition, we need to construct a bijection from $M_1 \sqcup N_1$ to $M_2 \sqcup N_2$. We do this as follows. Define $h\colon M_1 \sqcup N_1 \to M_2 \sqcup N_2$ by

$$h(x) := \begin{cases} f(x) & \text{if } x \in M_1 \\ g(x) & \text{if } x \in N_1. \end{cases} \tag{1.1.2.3}$$

We now check that h is a bijection. Suppose that $h(x_1) = h(x_2)$. This single element must be contained in either M_2 or N_2—without loss of generality suppose that $h(x_1) = h(x_2) \in M_2$. Then, from the definition of h, we have that $f(x_1) = h(x_1) = h(x_2) = f(x_2)$, and so because f is injective, we have that $x_1 = x_2$. Thus, h is injective. To show that h is surjective, let $y \in M_2 \sqcup N_2$. Without loss of generality, suppose that $y \in M_2$. Then, because f is surjective, there is some $x \in M_1$ such that $f(x) = y$, so that $h(x) = f(x) = y$. Thus, h is surjective, and hence bijective.

Thus, we have shown that

$$|M_1 \sqcup N_1| = |M_2 \sqcup N_2|, \tag{1.1.2.4}$$

so that addition is well-defined.

Exercise 1.1.2.5 Complete the proof by showing that multiplication is well-defined.

■

Definition 1.1.2.6 — 0 and 1 Define

$$0 := |\emptyset|, \qquad 1 := |\{\emptyset\}| \in \mathbb{N}. \tag{1.1.2.7}$$

R In words, 0 is the cardinality of the empty-set and 1 is the cardinality of the set that contains the empty-set and only the empty-set.

R In the definition of 1, there isn't anything particularly special about $\{\emptyset\}$—any set that contains 'one' element would have worked just as well (e.g. $\{*\}$ or $\{\text{your mom}\}$[a].

[a] Actually, I take that back—yo momma so fat $|\{\text{your mom}\}| \geq 2$.

Proposition 1.1.2.8 $\langle \aleph, +, \cdot, 0, 1\rangle$ is an integral crig.

Proof. We simply need to verify the properties of the definition of a crig, Definition A.4.16 (and also check that it is integral, Definition A.4.22). We first check that $+$ is associative, so let $m, n, o \in \mathbb{N}$ and write $m = |M|, n = |N|, o = |O|$ for sets M, N, and O. Then,

$$\begin{aligned}(m+n)+o &= (|M| + |N|) + |O| \\ &= |M \sqcup N| + |O| \\ &= |(M \sqcup N) \sqcup O| \\ &= |M \sqcup (N \sqcup O)| \\ &= m + (n+o).\end{aligned} \tag{1.1.2.9}$$

A similar argument shows that additive is commutative. As for the additive identity, we have

$$m + 0 = |M \sqcup \emptyset| = |M| = m = 0 + m. \tag{1.1.2.10}$$

Thus, $(\aleph, +, 0)$ is a commutative monoid.

Exercise 1.1.2.11 Check that $\langle \aleph, \cdot, 1 \rangle$ is a commutative monoid.

Exercise 1.1.2.12 To show that $\aleph$ is a crig, there is one final property to check. What is it? Check it.

To show that $\aleph$ is integral, suppose that $mn = 0$. Then, there must be a bijection between $M \times N$ and the empty-set, which implies that $M \times N$ is empty. But if neither M nor N is empty, then $M \times N$ will be nonempty. Thus, we must have that either M or N is empty, or equivalently, that either $m = 0$ or $n = 0$, so that $\aleph$ is integral. ■

Now we must check that addition and multiplication restrict to binary operations on the collection of finitely cardinalities, namely, $\mathbb{N}$. This amounts to showing that the sum and product of finite cardinalities are both finite.

Proposition 1.1.2.13 If M, N are finite sets, then $M \sqcup N$ and $M \times N$ are finite sets.

Proof. We first show that $M \sqcup \{*\}$ is finite if M is. We proceed by contradiction: suppose there is a proper subset $S \subset M \sqcup \{*\}$ and a bijection $f\colon M \sqcup \{*\} \to S$. If M is empty, then $\{*\}$ is finite because its only proper subset is the empty-set to which there can be no bijection (in fact, there is *no* function from a nonempty-set to the empty-set—see Exercise A.3.17). Thus, we may without loss of generality assume that M is nonempty. So, let $x_0 \in M$. We know that either S is of the form $S = M'$ for $M' \subseteq M$ or $S = M' \sqcup \{*\}$ for $M' \subset M$. By relabeling $*$ as x_0 and x_0 as $*$, we may as well assume that we are in the latter case, so that we have a bijection $f : M \sqcup \{*\} \to M' \sqcup \{*\}$ for $M' \subset M$. (Forget the label x_0; we will want that notation later to refer to something else.) If $*$ maps to $*$ under f, then the restriction of f to M yields a bijection from M to M' showing that M is infinite: a contradiction. Thus, we may as well assume that $f(x_0) = *$ for some $x_0 \in M$. Let $g : M \sqcup \{*\} \to M \sqcup \{*\}$ be any bijection which sends $*$ to x_0 (such a bijection exists by Exercise A.3.30). Then, $f \circ g : M \sqcup \{*\} \to M' \sqcup \{*\}$ is a bijection such that the image of M is M'. Thus, the restriction of $f \circ g$ to M yields a bijection of M onto a proper subset, showing that M is infinite: a contradiction. Thus, $M \sqcup \{*\}$ is finite.

Applying this result inductively shows that $M \sqcup N$ is finite if both M and N are.

To see that $M \times N$ is finite, think of the product as (Exercise A.3.1.12)

$$M \times N = \sqcup_{y \in N} M. \tag{1.1.2.14}$$

That $M \times N$ is finite now follows inductively from the fact that the disjoint union of two finite sets is finite. ■

Corollary 1.1.2.15 $\langle \mathbb{N}, +, \cdot, 0, 1 \rangle$ is an integral crig.

1.1.3 The natural numbers as a well-ordered set

For the moment, we will set aside the algebraic structure we have just defined on $\mathbb{N}$ and equip $\mathbb{N}$ with a preorder (which turns out to be a well-order). Then, in the next subsection, we will show that these two structures are compatible in a way that makes $\mathbb{N}$ into a well-ordered integral crig. Once again, we will in fact define the preorder on all of $\aleph$ and show that it is a well-order on $\aleph$. It will then follow automatically that it restricts to a well-order on $\mathbb{N}$.

As for what the definition of that preorder should be, recall our explanation of thinking of a function $f\colon X \to Y$ as 'labeling elements of Y with elements of X'—see the beginning of Subsection 1.1.1 The natural numbers as a set. We argued that our definition of 'same number of elements' should have the properties that (i) every element of Y is labeled and (ii) no element of Y is labeled more than once. Similarly, our definition of "Y has at least as many element of X" should have the property that are not forced to label an element of Y more than once (i.e. that f is injective), but not necessarily that every element of Y is labeled.

Definition 1.1.3.1 Let $m, n \in \aleph$ and let M and N be sets such that $m = |M|$ and $n = |N|$. Then, we define $m \leq n$ iff there is an injective map from M to N.

Exercise 1.1.3.2 Check that $\leq$ is well-defined.

You might be thinking "Ah, that makes sense. But why use injective? Couldn't we also say that $|X| \geq |Y|$ iff there is a *surjective* function $X \to Y$?". Unfortunately, this is only *almost* correct.

Exercise 1.1.3.3

(i). Show that if X and Y are nonempty sets, then $|X| \leq |Y|$ iff there is a surjective function from Y to X.
(ii). On the other hand, show how this might fail without the assumption of nonemptiness.

Proposition 1.1.3.4 $\langle \aleph, \leq \rangle$ is a preordered set.

Proof. Recall that being a preorder just means that $\leq$ is reflexive and transitive (see Definition A.3.3.1).

Let $m, n, o \in \mathbb{N}$ and let M, N, O be sets such that $m = |M|$, $n = |N|$, $o = |O|$. The identity map from M to M is an injection (and, in fact, a bijection), which shows that $m = |M| \leq |M| = m$, so that $\leq$ is reflexive.

To show transitivity, suppose that $m \leq n$ and $n \leq o$. Then, there is an injection $f\colon M \to N$ and an injection from $g\colon N \to O$. Then, $g \circ f\colon M \to O$ is an injection (this is part of Exercise A.3.28), and so we have $m = |M| \leq |O| = o$, so that $\leq$ is transitive, and hence a preorder. ■

The next result is perhaps the first theorem we have come to that has a nontrivial amount of content to it.

Theorem 1.1.3.5 — Bernstein-Cantor-Schröder Theorem. $\langle \aleph, \leq \rangle$ is a partially-ordered set.

> R This theorem is usually stated as "If there is an injection from X to Y and there is an injection from Y to X, then there is a bijection from X to Y.".

> R This theorem is *incredibly* useful for showing that two sets have the same cardinality—it's often much easier to construct an injection in each direction than it is to construct a single bijection—and it would do you well to not forget it.

Proof. [a] STEP 1: RECALL WHAT IT MEANS TO BE A PARTIAL-ORDER

Recall that being a partial-order just means that $\leq$ is an antisymmetric preorder. We have just shown that $\leq$ is a preorder (see Definition A.3.3.6), so all that remains to be seen is that $\leq$ is antisymmetric.

STEP 2: DETERMINE WHAT EXPLICITLY WE NEED TO SHOW

Let $m, n \in \aleph$ and let M, N be sets such that $m = |M|$ and $n = |N|$. Suppose that $m \leq n$ and $n \leq m$. By definition, this means that there is an injection $f\colon M \to N$ and an injection

$g : N \rightarrow M$. We would like to show that $m = n$. By definition, this means we must show that there is a bijection from M to N.

STEP 3: NOTE THE EXISTENCE OF LEFT-INVERSE TO BOTH f AND g

If M is empty, then as N injects into M, N must also be empty, and we are done. Likewise, if N is empty, we are also done. Thus, we may as well assume that M and N are both nonempty. We can now use the result of Equation (A.3.26) which says that both f and g have left inverses.[b] Denote these inverses by $f^{-1} : N \rightarrow M$ and $g^{-1} : M \rightarrow N$ respectively, so that

$$f^{-1} \circ f = \mathrm{id}_M \text{ and } g^{-1} \circ g = \mathrm{id}_N .^c \qquad (1.1.3.6)$$

STEP 4: DEFINE C_x

Fix an element $x \in M$ and define

$$\begin{aligned} C_x \coloneqq & \Big\{\ldots, g^{-1}\left(f^{-1}\left(g^{-1}(x)\right)\right), f^{-1}\left(g^{-1}(x)\right), g^{-1}(x), x, \\ & \quad f(x), g(f(x)), f(g(f(x))), \ldots\Big\} \qquad (1.1.3.7) \\ & \subseteq M \sqcup N.^d \end{aligned}$$

Note that C_x is 'closed' under application of f, g, f^{-1}, and g^{-1}, in the sense that, if $x' \in C_x$ and $f(x')$ makes sense (i.e. if $x' \in M$), then $f(x') \in C_x$, and similarly for g, f^{-1}, and g^{-1}.

STEP 5: SHOW THAT $\{C_x : x \in M\}$ FORMS A PARTITION OF $M \sqcup N$

We now claim that the collection $\{C_x : x \in M\}$ forms a partition of $M \sqcup N$ (recall that this means that any two given C_xs are either identical or disjoint—see Definition A.3.2.9). If C_{x_1} is disjoint from C_{x_2} we are done, so instead suppose that there is some element x_0 that is in both C_{x_1} and C_{x_2}. First, let us do the case in which $x_0 \in M$. From the definition of C_x (1.1.3.7), we then must have that

$$[g \circ f]^k(x_1) = x_0 = [g \circ f]^l(x_2) \qquad (1.1.3.8)$$

for some $k, l \in \mathbb{Z}$. Without loss of generality, suppose that $k \leq l$. Then, applying $f^{-1} \circ g^{-1}$ to both sides of this equation k times,[e] we find that

$$x_1 = [g \circ f]^{l-k}(x_2). \tag{1.1.3.9}$$

In other words, $x_1 \in C_{x_2}$. Not only this, but $f(x_1) \in C_{x_2}$ as well because $f(x_1) = f\left([g \circ f]^{l-k}(x_2)\right)$. Similarly, $g^{-1}(x_1) \in C_{x_2}$, and so on. It follows that $C_{x_1} \subseteq C_{x_2}$. Switching $1 \leftrightarrow 2$ and applying the same arguments gives us $C_{x_2} \subseteq C_{x_1}$, and hence $C_{x_1} = C_{x_2}$. Thus, indeed, $\{C_x : x \in M\}$ forms a partition of $M \sqcup N$. In particular, it follows that

$$C_x = C_{x'} \text{ for all } x' \in C_x. \tag{1.1.3.10}$$

STEP 6: DEFINE X_1, X_2, Y_1, Y_2

Now define

$$A := \bigcup_{\substack{x \in M \text{ s.t.} \\ C_x \cap N \subseteq f(M)}} C_x \tag{1.1.3.11}$$

as well as

$$X_1 := M \cap A, \qquad Y_1 := N \cap A, \tag{1.1.3.12a}$$
$$X_2 := M \cap A^{\mathsf{C}}, \qquad Y_2 := N \cap A^{\mathsf{C}}. \tag{1.1.3.12b}$$

Note that, as $\{C_x : x \in M\}$ is a partition of $M \sqcup N$, we have that

$$A^{\mathsf{C}} = \bigcup_{\substack{x \in M \text{ s.t.} \\ C_x \cap N \not\subseteq f(M)}} C_x. \tag{1.1.3.13}$$

STEP 7: SHOW THAT $f|_{X_1} : X_1 \to Y_1$ IS A BIJECTION

We claim that $f|_{X_1} : X_1 \to Y_1$ is a bijection. First of all, note the it $x \in X_1$, then in fact $f(x) \in Y_1$, so that this statement indeed does make sense. Of course, it is injective because f is. To show surjectivity, let $y \in Y_1 := N \cap A$. From the definition of A (1.1.3.11), we see that $y \in C_x \cap N$ for some C_x with $C_x \cap N \subseteq f(M)$, so that $y = f(x')$ for some $x' \in M$. We still need to show that $x' \in X_1$. However, we have that $x' = f^{-1}(y)$, and so as $y \in C_x$, we have that $x' = f^{-1}(y) \in C_x$ as well. We already had that $C_x \cap N \subseteq f(M)$, so that indeed $x' \in A$, and hence $x' \in X_1$. Thus, $f|_{X_1} : X_1 \to Y_1$ is a bijection.

Step 8: Show that $g|_{Y_2} : Y_2 \to X_2$ is a bijection

We now show that $g|_{Y_2} : Y_2 \to X_2$ is a bijection. Once again, all we must show is surjectivity, so let $x \in X_2 = M \cap A^{\mathsf{C}}$. From the definition of A (1.1.3.11), it thus cannot be the case that $C_x \cap N$ is contained in $f(M)$, so that there is some $y \in C_x \cap N$ such that $y \notin f(M)$. By virtue of (1.1.3.10), we have that $C_x = C_y$, and in particular $x \in C_y$. From the definition of C_y (1.1.3.7), it follows that either (i) $x = y$, (ii) x is in the image of f^{-1}, or (iii) x is in the image of g (the other possibilities are excluded because $x \in M$). Of course it cannot be the case that $x = y$ because $x \in M$ and $y \in N$. Likewise, it cannot be the case that x is in the image of f^{-1} because $x \in A^{\mathsf{C}}$. Thus, we must have that $x = g(y')$ for some $y' \in N$. Once again, we still must show that $y' \in Y_2$. However, we have that $y' = g^{-1}(x)$, so that $y' \in C_x$. Furthermore, as $C_x \cap N$ is not contained in $f(M)$, from (1.1.3.13) it follows that $C_x \subseteq A^{\mathsf{C}}$. Thus, $y' \in C_x \subseteq A^{\mathsf{C}}$, and so $y' \in Y_2$. Thus, $g|_{Y_2} : Y_2 \to X_2$ is a bijection.

Step 9: Construct the bijection from M to N

Finally, we can define the bijection from M to N. We define $h\colon M \to N$ by

$$h(x) := \begin{cases} f(x) & \text{if } x \in X_1 \\ g^{-1}(x) & \text{if } x \in X_2. \end{cases} \tag{1.1.3.14}$$

Note that $\{X_1, X_2\}$ is a partition of M and $\{Y_1, Y_2\}$ is a partition of N. To show injectivity, suppose that $h(x_1) = h(x_2)$. If this element is in Y_1, then because $f|_{X_1}: X_1 \to Y_1$ is a bijection, it follows that both $x_1, x_2 \in X_1$, so that $f(x_1) = h(x_1) = h(x_2) = f(x_2)$, and hence that $x_1 = x_2$. Similarly if this element is contained in Y_2. To show surjectivity, let $y \in N$. First assume that $y \in Y_1$. Then, $f^{-1}(y) \in X_1$, so that $h\left(f^{-1}(y)\right) = y$. Similarly, if $y \in Y_2$, then $h(g(y)) = y$. Thus, h is surjective, and hence bijective. ■

R I think perhaps the mathematical precision here has obfuscated the core idea of the proof. Briefly, the basic idea is as follows. Once we have defined the chains C_xs, they 'break-up' M and N into 'chunks' in such a way that it suffices to construct a bijection separately on each chunk (that is, they form a *partition*). If the elements of C_x in the codomain are actually contained in the image of f, then f itself can serve as the bijection on that "chunk"—otherwise, we can use g.

[a] Proof adapted from [Abb02, pg. 29].

[b] To use this, we first needed to have that M and N are nonempty.

[c] Note that it is *not* necessarily the case that $f \circ f^{-1} = \mathrm{id}_N$ (and similarly for g). This certainly constitutes an abuse of notation, as we should really be reserving the notation f^{-1} for a *two*-sided inverse, but as this makes the proof quite a bit more readable, we ignore such pedantry for the time being.

[d] The "C" is for "chain".

[e] If k happens to be negative, it is understood that we instead apply $g \circ f$ $-k$ times.

Theorem 1.1.3.15. $\langle \aleph, \leq \rangle$ is well-ordered.

Proof. [a] STEP 1: CONCLUDE THAT IT SUFFICES TO SHOW THAT EVERY NONEMPTY SUBSET HAS A SMALLEST ELEMENT

By Proposition A.3.3.16, we do not need to check totality explicitly, and so it suffices to show that every nonempty subset of $\aleph$ has a smallest element.

STEP 2: DEFINE $\mathcal{T}$ AS A PREORDERED SET

So, let $S \subseteq \aleph$ be a nonempty collection of cardinals and for each $m \in S$ write $m = |M_m|$ for some set M_m. Define

$$M := \prod_{m \in S} M_m \tag{1.1.3.16}$$

and

$$\begin{aligned}\mathcal{T} := \{T \subseteq M : T \in \mathrm{Obj}(\mathbf{Set});\ & \text{for all } x, y \in T, \\ & \text{if } x \neq y \text{ it follows that} \\ & x_m \neq y_m \text{ for all } m \in S.\}\ .\end{aligned} \tag{1.1.3.17}$$

Order $\mathcal{T}$ by inclusion.

STEP 3: VERIFY THAT $\mathcal{T}$ SATISFIES THE HYPOTHESES OF ZORN'S LEMMA

We wish to apply Zorn's Lemma (Theorem A.3.5.9) to $\mathcal{T}$. To do that of course, we must first verify the hypotheses of Zorn's Lemma. $\mathcal{T}$ is a partially-ordered set by Exercise A.3.3.11. Let $\mathcal{W} \subseteq \mathcal{T}$ be a well-ordered subset and define

$$W := \bigcup_{T \in \mathcal{W}} T. \tag{1.1.3.18}$$

It is certainly the case that $T \subseteq W$ for all $T \in \mathcal{W}$. In order to verify that W is indeed an upper-bound of $\mathcal{W}$ in $\mathcal{T}$, however, we need to check that W is actually an element of $\mathcal{T}$. So, let $x, y \in W$ be distinct. Then, there are $T_1, T_2 \in \mathcal{W}$ such that $x \in T_1$ and $y \in T_2$. Because $\mathcal{W}$ is in particular totally-ordered, we may without loss of generality assume that $T_1 \subseteq T_2$. In this case, both $x, y \in T_2$. As $T_2 \in \mathcal{T}$, it then follows that $x_m \neq x_m$ for all $m \in S$. It then follows in turn that $W \in \mathcal{T}$.

STEP 4: CONCLUDE THE EXISTENCE OF A MAXIMAL ELEMENT

The hypotheses of Zorn's Lemma being verified, we deduce that there is a maximal element $T_0 \in \mathcal{T}$.

STEP 5: SHOW THAT THERE IS SOME PROJECTION WHOSE RESTRICTION TO THE MAXIMAL ELEMENT IS SURJECTIVE

Let $\pi_m : M \longrightarrow M_m$ be the canonical projection. We claim that there is some $m_0 \in S$ such that $\pi_{m_0}(T_0) = M_{m_0}$. To show this, we proceed by contradiction: suppose that for all $m \in M$ there is some element $x_m \in M_m \setminus \pi_m(T_0)$. Then, $T_0 \cup \{x\} \in \mathcal{T}$ is strictly larger than T_0: a contradiction of maximality. Therefore, there is some $m_0 \in S$ such that $\pi_{m_0}(T_0) = M_{m_0}$.

STEP 6: CONSTRUCT AN INJECTION FROM M_{m_0} TO M_m FOR ALL $m \in S$

The defining condition of $\mathcal{T}$, (1.1.3.17), is simply the statement that $\pi_m|_T : T \longrightarrow M_m$ is injective for all $T \in \mathcal{T}$. In particular, by the previous step, $\pi_{m_0}\big|_{T_0} : T_0 \longrightarrow M_{m_0}$ is a bijection. And therefore, the composition $\pi_m \circ \pi_{m_0}\big|_{T_0}^{-1} : M_{m_0} \longrightarrow M_m$ is an injection from M_{m_0} to M. Therefore,

$$m_0 = \left|M_{m_0}\right| \leq |M_m| = m \tag{1.1.3.19}$$

for all $m \in S$. That is, m_0 is the smallest element of S, and so $\aleph$ is well-ordered. ■

[a] Proof adapted from [Hön].

Corollary 1.1.3.20 $\langle \mathbb{N}, \leq \rangle$ is a well-ordered set.

1.1.4 The natural numbers as a well-ordered rig

We have shown that $\langle \mathbb{N}, +, \cdot \rangle$ is a crig and that $\langle \mathbb{N}, \leq \rangle$ is a well-ordered. We now finally show how these two different structures, the algebraic structure and the order structure, are compatible. But before we do that, of course, we have to make precise what we mean by the word "compatible".

Definition 1.1.4.1 — Preordered rg A ***preordered rg*** is a set X equipped with two binary operations $+$ and $\cdot$, and a relation $\leq$, so that

(i). $\langle X, +, 0, \cdot \rangle$ is a rg,
(ii). $\langle X, \leq \rangle$ is a preordered set,
(iii). $x \leq y$ implies that $x + z \leq y + z$ for all $x, y, z \in X$,
(iv). $x \leq y$ and $0 \leq z$ implies that $xz \leq yz$ and $zx \leq zy$.

and furthermore, in the case $\langle X, +, 0, \cdot, 1 \rangle$ is a rig, that $0 \leq 1$.

R If X is a *totally*-ordered ring, we automatically have $0 \leq 1$. By totality, we automatically have that either $0 \leq 1$ or $1 \leq 0$. Do you see why the latter cannot happen (if $1 \neq 0$)?

In general, however, we do need to make the requirement that $0 \leq 1$—see the following exercise (Exercise 1.1.4.3).

For X a preordered rg, we write

$$X^+ := \{x \in X : x > 0\} \text{ and } X_0^+ := \{x \in X : x \geq 0\} .$$

R A partially-ordered rg, totally-ordered rg, etc. are just preordered rgs whose underlying preorder is respectively a partial-order, total-order, etc.. Similarly if you replace "rg" with "rig", "ring", etc..

R In a totally-ordered ring, we define $\operatorname{sgn} : X \to \{0, 1, -1\} \subseteq X$ by

$$\operatorname{sgn}(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases} \tag{1.1.4.2}$$

This is the ***signum function*** and is meant merely to return the sign of an element.

Exercise 1.1.4.3 Let $\langle R, +, 0, \cdot, \leq \rangle$ satisfy all the axioms of a preordered rg except $0 \leq 1$.

(i). Show that if R is totally-ordered ring, then $0 \leq 1$.
(ii). Find an example of a partially-ordered ring in which $0 \nleq 1$.
(iii). Find an example of a totally-ordered rig in which $0 \leq\leq 1$.

R Thus, while $0 \leq 1$ may be automatically in a totally-ordered ring, if you drop either the assumptions of totality or the existence of additive inverses, it may very well fail.

One thing to note about preordered rngs is that, to define the order, it suffices only to be able to compare everything with 0 (and in fact, some will even take this as their definition).

Exercise 1.1.4.4 Let $\langle X, +, 0, - \rangle$ be a rng and let P be a subset of X[a] that is (i) closed under addition, (ii) closed under multiplication, and (iii) contains 0. Show that there is a unique preorder $\leq$ on X such that (i) $\langle X, +, 0, -, \leq \rangle$ is a preordered rng and (ii) $P = \{x \in X : x \geq 0\}$. Furthermore, show that $\leq$ is a partial-order iff $x, -x \in P$ implies $x = 0$, and finally show that $\leq$ is a total-order iff, in addition, either $x \in P$ or $-x \in P$ for all $x \in X$.

[a] P is to be thought of as the collection of nonnegative elements (the "P" is for "positive").

Definition 1.1.4.5 — The category of preordered rgs The category of preordered rgs is the category **PreRg**

(i). whose collection of objects $\mathrm{Obj}(\mathbf{PreRg})$ is the collection of all preordered rgs;
(ii). with morphism set $\mathrm{Mor}_{\mathbf{PreRg}}(X, Y)$ precisely the set of nondecreasing homomorphisms from X to Y;
(iii). whose composition is given by ordinary function composition; and
(iv). whose the identities are given by the identity functions.

R We similarly have categories of preordered rigs **PreRig**, preordered rngs **PreRng**, and preordered rings **PreRing**.

R This should be pretty much what you expect: a preordered rg has two different structures on it, namely the rg structure and the preorder structure, and so we require the morphisms in the category of preordered rgs to preserve *both* of these structures.

Proposition 1.1.4.6 $\langle \aleph, +, \cdot, 0, 1, \leq \rangle$ is a well-ordered integral crig.

Proof. We just showed in the last two sections (Proposition 1.1.2.8 and Theorem 1.1.3.15) that $\langle \aleph, +, \cdot, 0, 1 \rangle$ is an integral crig and that $\langle \aleph, \leq \rangle$ is well-ordered, so all that remains to be checked are properties (iii) and (iv) of Definition 1.1.4.1.

We first check (iii). So, let $m, n, o \in \aleph$ and let M, N, O be sets such that $m = |M|$, $n = |N|$, and $o = |O|$. Suppose that $m \leq n$. This means that there is an injection $f : M \to N$. We would like to show that $m + o \leq n + o$. In other words, we want to show that there is an injection from $M \sqcup O$ to $N \sqcup O$. Of course, the function $g : M \sqcup O \to N \sqcup O$ defined by

$$g(x) := \begin{cases} f(x) & \text{if } x \in M \\ x & \text{if } x \in O \end{cases} \tag{1.1.4.7}$$

is an injection because f is, and so $m + o \leq n + o$.

Now we show (iv). So, suppose that $m \leq n$ and that $0 \leq o$.[a] Let $f\colon M \to N$ be the injection the same as before. We wish to show that $mo \leq no$. In other words, we wish to construct an injection from $M \times O$ into $N \times O$. Of course, the function $g\colon M \times O \to N \times O$ defined by

$$g\left(\langle x, y\rangle\right) := \langle f(x), y\rangle \tag{1.1.4.8}$$

is an injection because f is, and so $mo \leq no$. That $om \leq on$ as well follows immediately by commutativity of multiplication.

Finally, $0 \leq 1$ follows because $\emptyset \subseteq \{\emptyset\}$ (the empty-set is a subset of every set).

Thus, $\langle \aleph, +, 0, 1, \leq\rangle$ is a well-ordered integral crig. ■

[a] Of course we don't actually need to assume that $0 \leq o$. Every cardinal is greater than or equal to 0.

Corollary 1.1.4.9 $\langle \mathbb{N}, +, \cdot, 0, 1, \leq\rangle$ is a well-ordered integral crig.

Before moving on, we summarize all the properties of $\mathbb{N}$ that we have shown. This is nothing more than explicitly spelling out what it means for $\langle \mathbb{N}, +, \cdot, 0, 1, \leq\rangle$ to be a well-ordered integral crig.

(i). $+$ is associative,
(ii). $+$ is commutative,
(iii). 0 is an additive identity,
(iv). $\cdot$ is associative,
(v). $\cdot$ is commutative,
(vi). 1 is a multiplicative identity,
(vii). multiplication distributes over addition,
(viii). $mn = 0$ implies either $m = 0$ or $n = 0$,
(ix). $\leq$ is reflexive,
(x). $\leq$ is transitive,
(xi). $\leq$ is antisymmetric,
(xii). $\leq$ is total,
(xiii). every nonempty subset has a smallest element,
(xiv). $m \leq n$ implies $m + o \leq n + o$, and

(xv). $m \leq n$ and $0 \leq o$ implies $mo \leq no$.

This is great and all, but we still can't actually definitively write $\mathbb{N} = \{0, 1, 2, \ldots\}$. That $0, 1, 2$, etc.. are natural numbers is tautological: these are merely symbols we use to denote the cardinalities of particular finite sets. What is less trivial, however, is that there is nothing else.

Proposition 1.1.4.10 Define $s \colon \mathbb{N} \to \mathbb{N} \setminus \{0\}$ by $s(m) := m+1$. Then,

$$\mathbb{N} = \bigcup_{k=0}^{\infty} s^k(0). \tag{1.1.4.11}$$

> R While you can take 2 to explicitly be the cardinality of a certain set, we instead consider 2 to be defined by $2 := 1+1$. Similarly, $3 := 2+1$, and so on. Thus, the appropriate way to make precise the intuitive notion $\mathbb{N} := \{0, 1, 2, \ldots\}$ that every natural number can be obtained from 0 by simply adding 1 sufficiently many times, that is, $\mathbb{N} = \bigcup_{k=0}^{\infty} s^k(0)$.

Proof. We leave this as an exercise.

Exercise 1.1.4.12 Do the proof yourself.

■

Finally, we prove one more property of the natural numbers that we will need when discussing the integers.

Proposition 1.1.4.13 Let $m, n, o \in \aleph$. Then, if o is finite and $m + o \leq n + o$, then $m \leq n$. In particular, because this is a partial order, if $m + o = n + o$, then $m = n$.

> R Note that this is *false* in o is not finite. For example, $0 + \aleph_0 = 1 + \aleph_0$, but $0 \neq 1$. ($\aleph_0 := |\mathbb{N}|$—see Definition 2.1.2.)

Proof. Suppose that $m + o \leq n + o$. Let M, N, O be finite sets such that $m = |M|$, $n = |N|$, and $o = |O|$. Because $m + o \leq n + o$, there is an injection $\phi\colon M \sqcup O \to N \sqcup O$. Define

$$P := \{x \in M : \phi(x) \in O\} =: \phi^{-1}(O) \cap M.^{a} \quad (1.1.4.14)$$

We prove the result by induction on the cardinality of P.[b]

If P is empty, it follows that $\phi(M) \subseteq N$, so that $\phi|_M : M \to N$ is an injection, and hence $m \leq n$.

Now suppose the result is true if $|P| = k$ for $k \geq 1$. We show that it must also be true for $|P| = k+1$. We first show that $N \setminus \phi(M)$ is nonempty. If it were empty, then every element of N would be the image of some element of M, and so if the image of an element of O were also in N, it would necessarily be the case that this element of O had the same image as an element of M, contradicting injectivity. Thus, it would have to be the case that $\phi(O) \subseteq O$, and hence, as O is finite and ϕ is an injection, we would have that $\phi(O) = O$ (injections are bijections onto their image). By injectivity again, we could then not have any element of M being mapped into O, that is, P would the necessarily be empty, a contradiction of the fact that $|P| = k \geq 1$. Thus, it must be the case that $N \setminus \phi(M)$ is nonempty.

So, let $n_0 \in N \setminus \phi(M)$ and let $p_0 \in P$, so that, by the definition of P, $\phi(p_0) \in O$. Let $\psi\colon N \sqcup O \to N \sqcup O$ be the map that exchanges $\phi(p_0)$ and n_0 and leaves everything else fixed. This is a bijection, and so $\psi \circ \phi\colon M \sqcup O \to N \sqcup O$ is an injection. Furthermore, now the image of p_0 is contained in N (and there is no new point in M that gets mapped into O), so that now there are only k elements of M which map into O via the injection $\psi \circ \phi$. By the induction hypothesis, there is then a injection from M to N, and we are done. ■

[a] "P" is for "ploblematic".

[b]We accept induction as valid in the naive sense described at the very beginning of the appendix. If for some reason you're not convinced, we can essentially prove this anyways—see The Peano axioms (Theorem 1.1.4.18).

An axiomatization of $\mathbb{N}$

We have attempted to introduce the natural numbers in such a way that feels, well, natural. Arguably counting is the very first mathematics all of us learn, and it is the natural numbers which allow us to do exactly this. On the other hand, you will find that this treatment is not exactly analogous to our development of the integers, rationals, or reals. Of course, in all cases, I chose to develop the number systems in the way which I felt to be most natural—but this doesn't mean that all the developments will be analogous. One might like to see a result for $\mathbb{N}$ analogous to the defining results for $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{Q}$ (Theorems 1.2.1, 1.3.4 and 1.4.2.9). Unfortunately,[4] the analogous result is not true.

Example 1.1.4.15 — A well-ordered integral crig N for which $\mathbb{N} \not\subseteq N$ Looking ahead at Theorems 1.2.1, 1.3.4 and 1.4.2.9, you will find that the analogous result for the naturals would be

> There exists a unique nonzero well-ordered integral crig $\mathbb{N}'$ that has the property that, if N is any other nonzero well-ordered integral crig, then $\mathbb{N}' \subseteq N$.

with uniqueness and $\subseteq$ interpreted in terms of isomorphism of preordered rigs (see Theorem 1.2.1 for elaboration on this). In this example, we construct a well-ordered integral crig N for which there is no embedding $\mathbb{N} \to N$.[a]

Define $N := \{0, 1\}$, and equip N with the usual ordering and multiplication, but instead define $m +_N n := \max\{m, n\}$.

Exercise 1.1.4.16 Show that $\langle N, +_N, 0\rangle$ is a commutative monoid.

[4]Or perhaps fortunately, as this just solidifies our claim that the presentation in terms of finite cardinals is a natural one.

As $\langle \mathbb{N}, \cdot, 1 \rangle$ it a commutative monoid, it follows immediately that $\langle N, \cdot, 1 \rangle$ is as well.

Exercise 1.1.4.17 To show that $\langle N, +, 0, \cdot, 1 \rangle$ is a crig, there is one final property to check. What is it? Check it

Finally, $m \cdot n = 1$ in fact implies that $m = 1 = n$, so that $\langle N, +, 0, \cdot, 1 \rangle$ is indeed integral.

As we are using the usual ordering, that $\{N, \leq\}$ is well-ordered is immediate. What we must check is compatibility with addition and multiplication.

As our multiplication is the usual one, we in fact have already checked this when constructing $\mathbb{N}$. It is similarly immediate that $0 \leq 1$. What remains to be checked is compatibility with addition. So, let $m, n, o \in N$, and suppose that $m \leq n$. We wish to show that $m +_N o \leq m +_n o$, that is, that $\max\{m, o\} \leq \max\{n, o\}$. Of course, either $o = 0$ or $o = 1$, in which case we must have respectively $o \leq m \leq n$ or $m \leq n \leq o$. In the former case, $\max\{m, o\} = m$ and $\max\{n, o\} = n$, so that indeed $\max\{m, o\} \leq \max\{n, o\}$ as $m \leq n$. Similarly, in the latter case, $\max\{m, o\} = o$ and $\max\{n, o\} = o$, so that indeed $\max\{m, o\} \leq \max\{n, o\}$ (in fact, we have equality). This finish the checks that $\{N, +_N, 0, \cdot, 1, \leq\}$ is a well-ordered integral crig.

On the other hand, not only is there no embedding $\mathbb{N} \to N$, there isn't even an injective function $\mathbb{N} \to N$.

[a] See Definition B.2.16 for the precise definition of embedding. Here, explicitly, an embedding $\iota \colon \mathbb{N} \to N$ is an injective morphism of preordered rigs that has the property that $\iota(m_1) \leq \iota(m_2)$ iff $m_1 \leq m_2$—see Exercises B.2.18 and B.2.19.

The Peano axioms

It is not uncommon for textbooks to introduce the natural numbers via the Peano axioms. We included this material not because it is essential to the development of the real numbers, but rather for the sake of completeness: even though it is not strictly necessary, people will expect every mathematician to know of the Peano axioms.

People who do use the Peano axioms to introduce the natural numbers, instead of constructing the natural numbers and proving they have the desired properties, will simply assume that a structure which satisfies the Peano axioms exists. We, however, will instead *prove* the Peano axioms are true; for us, they are theorems. Before you try to go off and prove them, however, I had probably better tell you what they are.

Theorem 1.1.4.18 — The Peano axioms. There exists a set $\mathbb{N}'$ which contains an element $0' \in \mathbb{N}'$ and a function $s : \mathbb{N}' \to \mathbb{N}'$ (called the ***successor function***), such that

(i). $m \in \mathbb{N}'$ is in the image of S iff $m \neq 0'$;
(ii). s is injective, and
(iii). a subset $S \subseteq \mathbb{N}'$ such that (iii.a) $0' \in S$ and (iii.b) $m \in S$ implies $s(m) \in S$ is equal to all of $\mathbb{N}'$.

R The prime mark on $\mathbb{N}'$ is simply to distinguish the set in this theorem from the collection of finite cardinals (though of course $\mathbb{N}$ itself satisfies these properties (for a suitable obvious definition of s)) Similarly for $0'$.

R The successor of an element is of course 'supposed' to be that element plus one.

R (iii) is of course thought of as *induction*.

Proof. Step 1: Define everything
Define $\mathbb{N}' := \mathbb{N}$, $s(m) := m + 1$, and $0' := 0$.

Step 2: Prove (i)

Exercise 1.1.4.19 Prove that there is no $m \in \mathbb{N}$ such that $m + 1 = 0$.

The remainder of this part is exactly the content of Proposition 1.1.4.10.

STEP 3: PROVE (II)

Exercise 1.1.4.20 Prove that if $m + 1 = n + 1$, then $m = n$.

STEP 4: PROVE (III)

Let $S \subseteq \mathbb{N}'$ have the properties that (iii.a) $0 \in S$ and (iii.b) $m \in S$ implies $m + 1 \in S$. We wish to show that $S = \mathbb{N}'$. We proceed by contradiction: suppose that $S \neq \mathbb{N}'$. Then, S^{C} is nonempty, and as $\mathbb{N}$ is well-ordered, S^{C} has a least element $m_0 \in S^{\mathsf{C}}$. As $0 \in S$, it cannot be the case that $m_0 = 0$. Then, by (i), there is some $n_0 \in \mathbb{N}'$ such that $s(n_0) = m_0$. As we have defined $s(n_0) \coloneqq n_0 + 1$, we in particular have that $n_0 < m_0$. As n_0 is less than m_0 and m_0 is the *least* element of S^{C}, we cannot have $n_0 \in S^{\mathsf{C}}$. Therefore, $n_0 \in S$. But then, by hypothesis, $m_0 = s(n_0) \in S$: a contradiction (as $m_0 \in S^{\mathsf{C}}$). Hence, we must have that $S = \mathbb{N}$. ■

1.2 Additive inverses and the integers

Suppose we have a simple algebraic equation involving three natural numbers, $m + n = o$, we are given n and o, and we would like to find m. Of course, we know what the answer *should* be, namely $m = o - n$, but currently this is nonsensical as we have not defined what this crazy new symbol "$-$" means. The job of the integers is to make sense out of this.

We thus would like to find a set with algebraic structure (which will turn out to be an integral cring whose elements are thought of as a new more general type of "number") that allows us to solve simple equations like $m + n = o$. More precisely, we seek a cring (that is a crig with *additive inverses*) which contains $\mathbb{N}$ and, in some sense, is the 'simplest' cring that will do so.

To understand what it means to be a cring Z that contains $\mathbb{N}$ is quite easy, but how does one make sense of the statement that Z is the 'simplest' cring with this property? The precise sense in which Z should be the simplest cring which contains $\mathbb{N}$ is, if Z' is any other cring which contains $\mathbb{N}$, then Z' contains Z as well. That is, Z is contained in every cring which contains $\mathbb{N}$.

Theorem 1.2.1 — Integers. There exists a unique totally-ordered integral cring $\mathbb{Z}$, the ***integers***, such that

(i). $\mathbb{N} \subseteq \mathbb{Z}$; and
(ii). if Z is any other totally-ordered integral cring such that $\mathbb{N} \subseteq Z$, then $\mathbb{Z} \subseteq Z$.

Furthermore, $\mathbb{Z}$ is additionally the unique ring such that

(i′). $\mathbb{N} \subseteq \mathbb{Z}$; and
(ii′). if Z is any other ring such that $\mathbb{N} \subseteq Z$, then $\mathbb{Z} \subseteq Z$.

R As you read through the proof, you will find that we construct the integers as equivalence classes of ordered pairs of natural numbers. However, you should *not consider this as a definition of the integers*. Instead, the integers are defined above in the statement of the theorem, defined implicitly by the properties that uniquely specify them. The correct perspective is that the construction in terms of of ordered pairs of natural numbers is merely a tool to prove the result (this one) that defines the integers. In particular, if you're ever proving something about $\mathbb{Z}$, *do not use the construction as equivalence classes of ordered pairs*. Not only is this blasphemous, it will almost certainly make the proof more difficult anyways.

R In fact, $\mathbb{Z}$ isn't just the smallest totally-ordered integral cring which contains $\mathbb{N}$: it's the smallest totally-ordered integral cring *period*[a]—see Theorem 1.2.27.

R For the purposes of (hopefully) increasing clarity, we are actually being sloppy in a couple of places here. First of all, when we say "unique", what we actually means is that $\mathbb{Z}$ is "unique up to unique isomorphism" in the sense that, if Z is some other totally-ordered cring which satisfies (i) and (ii), then there is a unique isomorphism (of preordered rings) $\mathbb{Z} \to Z$.

Similarly, when we write $\mathbb{N} \subseteq \mathbb{Z}$ (or $\mathbb{Z} \subseteq Z$), we don't *literally* mean that $\mathbb{N}$ is a subset of $\mathbb{Z}$, but rather, that there is a unique embedding[b] of preordered rigs $\mathbb{N} \to \mathbb{Z}$.

Similar remarks apply to the analogous results for $\mathbb{Q}$ and $\mathbb{R}$ (Theorems 1.3.4 and 1.4.2.9 respectively).

R The order itself doesn't really play a role here. The significance is in going from a rig to a ring; the order just 'comes along for the ride', so to speak. The "Furthermore,. . . " part of the result is the precise statement of this intuition.

R This is actually a special case of a more general construction, the construction of the *Grothendieck Group* of a commutative monoid. We don't need this general construction and we are already pushing the envelope for level of abstraction in an introductory analysis course, so we don't present it in this level of generality.

R Integral crings are usually called ***integral domains***. In fact, this term is the origin of our use of the word "integral".

[a]Besides the zero cring of course.

[b]See Definition B.2.16 for the general definition of embedding. In this context, to be an embedding $\iota\colon \mathbb{N} \to \mathbb{Z}$ means that ι is an injective nondecreasing ring homomorphism that has the property that $\iota(m_1) \leq \iota(m_2)$ iff $m_1 \leq m_2$. This is just a sophisticated way of saying "For almost all intents and purposes, we can pretend that $\mathbb{N}$ is actually a subset of $\mathbb{Z}$.".

Proof. STEP 1: DEFINE AN EQUIVALENCE RELATION ON $\mathbb{N} \times \mathbb{N}$

The idea behind constructing the integers is to think of a pair of *natural* numbers $\langle m, n \rangle$ as representing what should be $m - n$. One issue with this, however, is that, if we do this, then it should be the case that $\langle m+1, n+1 \rangle$ and $\langle m, n \rangle$ both represent the same number. Thus, we put an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

Define $\langle m_1, n_1 \rangle \sim \langle m_2, n_2 \rangle$ iff $m_1 + n_2 = m_2 + n_1$. We came up with this of course because the statement $\langle m_1, n_1 \rangle \sim \langle m_2, n_2 \rangle$ *should* be $m_1 - n_1 = m_2 - n_2$. Of course, this itself doesn't make sense, and so we write this same thing as something that does make sense given what we have already defined, namely $m_1 + n_2 = m_2 + n_1$.

STEP 2: CHECK THAT $\sim$ IS AN EQUIVALENCE RELATION

Exercise 1.2.2 Show that $\sim$ is reflexive and symmetric.

To show that it is transitive, suppose that $\langle m_1, n_1 \rangle \sim \langle m_2, n_2 \rangle$ and $\langle m_2, n_2 \rangle \sim \langle m_3, n_3 \rangle$. Then, $m_1 + n_2 = m_2 + n_1$ and $m_2 + n_3 = m_3 + n_2$, and so

$$m_1 + n_2 + m_2 + n_3 = m_2 + n_1 + m_3 + n_2. \tag{1.2.3}$$

It follows from Proposition 1.1.4.13 that $m_1 + n_3 = n_1 + m_3$, and so $\langle m_1, n_1 \rangle \sim \langle m_3, n_3 \rangle$, so that $\sim$ is transitive.

STEP 3: DEFINE $\mathbb{Z}$ AS A SET

We define

$$\mathbb{Z} := \mathbb{N} \times \mathbb{N} / \sim . \tag{1.2.4}$$

Recall (Definition A.3.2.16) that this is the quotient set with respect to $\sim$, that is, the set of equivalence classes.

STEP 4: DEFINE ADDITION AND MULTIPLICATION ON $\mathbb{Z}$

We define

$$[\langle m_1, n_1\rangle] + [\langle m_2, n_2\rangle] := [\langle m_1 + m_2, n_1 + n_2\rangle] \quad (1.2.5)$$

and

$$[\langle m_1, n_1\rangle] \cdot [\langle m_2, n_2\rangle] := [\langle m_1 m_2 + n_1 n_2, m_1 n_2 + n_1 m_2\rangle].^{a} \quad (1.2.6)$$

Exercise 1.2.7 Show that $+$ and $\cdot$ are both well-defined.

STEP 5: DEFINE THE IDENTITIES

We define

$$0 := [\langle 0, 0\rangle] \text{ and } 1 := [\langle 1, 0\rangle]. \quad (1.2.8)$$

STEP 6: SHOW THAT $\mathbb{Z}$ IS AN INTEGRAL CRING

Exercise 1.2.9 Show that $\mathbb{Z}$ is an integral cring.

STEP 7: DEFINE A PREORDER ON $\mathbb{Z}$

We define

$$[\langle m_1, n_1\rangle] \leq [\langle m_2, n_2\rangle] \text{ iff } m_1 + n_2 \leq m_2 + n_1. \quad (1.2.10)$$

Exercise 1.2.11 Show that $\leq$ is well-defined.

Exercise 1.2.12 Show that $\leq$ is a preorder.

Step 8: Show that $\leq$ is a total-order

Exercise 1.2.13 Show that $\leq$ is a total-order.

Step 9: Show that $\langle \mathbb{Z}, +, \cdot, 0, 1, -, \leq \rangle$ is a totally-ordered integral cring

Exercise 1.2.14 Show that $\langle \mathbb{Z}, +, \cdot, 0, 1, -, \leq \rangle$ is a totally-ordered integral cring.

Step 10: Show that $\mathbb{Z}$ contains $\mathbb{N}$ as a preordered rig

Exercise 1.2.15 Define a function $\iota : \mathbb{N} \to \mathbb{Z}$. Show that it is an embedding of preordered rigs.

R Note that, by Exercises B.2.18 and B.2.19, explicitly this just means that you need to check that ι is (i) injective, (ii) a rig homomorphism, and (iii) satisfies $\iota(m_1) \leq \iota(m_2)$ iff $m_1 \leq m_2$.

(i) injective and (ii) a morphism of preordered rigs.

Step 11: Show that every totally-ordered integral cring which contains $\mathbb{N}$ contains a unique copy of $\mathbb{Z}$

Let Z be a totally-ordered integral cring which contains $\mathbb{N}$. This is slightly informal language for the more precise statement that there is a unique embedding $\iota\colon \mathbb{N} \to Z$ of preordered rigs. To show that in fact Z contains $\mathbb{Z}$, what we really want to show is that there is an embedding of preordered rigs $\mathbb{Z} \to Z$. So, define $i\colon \mathbb{Z} \to Z$ by

$$i([\langle m, n\rangle]) := \iota(m) - \iota(n). \tag{1.2.16}$$

Exercise 1.2.17 Show that i is indeed an embedding preordered rings.

R Note that, from Exercises B.2.18 and B.2.19, it follows quite easily that i is an embedding iff it is an injective nondecreasing rig homomorphism that furthermore satisfies $i(m_1) \leq i(m_2)$ iff $m_1 \leq m_2$.

This shows that Z 'contains a copy' of $\mathbb{Z}$. It still remains to check that this copy is unique.[b] If there were another copy, these two copies would have to share the same multiplicative identity (by uniqueness of identities, Exercise A.4.7). Both copies being rings, it would then follow that $2 := 1 + 1$, $3 := 1 + 1 + 1$, etc. would have to be the same element in each, and hence that by uniqueness of inverses (Exercise A.4.9), -1, -2, , -3, etc. would have to be the same element in each.

Exercise 1.2.18 Finish the proof that Z contains a *unique* copy of $\mathbb{Z}$.

STEP 12: SHOW THAT $\mathbb{Z}$ IS UNIQUE UP TO ISOMORPHISM UNIQUE ISOMORPHISM

Let Z be some other totally-ordered integral cring that satisfies the two properties (i) and (ii). As Z 'contains' $\mathbb{N}$, (ii) applied to

$\mathbb{Z}$ implies that Z contains $\mathbb{Z}$. On the other hand, as $\mathbb{Z}$ contains $\mathbb{N}$, (ii) applied to Z implies likewise that $\mathbb{Z}$ contains Z. As $\mathbb{Z}$ contains Z and Z contains $\mathbb{Z}$, and furthermore, because these copies are *unique*, it must be the case that $\mathbb{Z} = Z$.[c]

STEP 13: SHOW THAT $\mathbb{Z}$ IS THE 'SMALLEST' RING WHICH CONTAINS $\mathbb{N}$

Here, we are referring to the "Furthermore,. . . " part of the statement of the theorem. Of course, we have already shown that $\mathbb{Z}$ is a ring which 'contains' $\mathbb{N}$. Furthermore, if we can show (ii′), then the same argument as before will show uniqueness up to unique isomorphism of rings. So, let Z be some other ring such that $\mathbb{N} \subseteq Z$, that is, for which there is a unique embedding (i.e. injective homomorphism—see Exercise B.2.18) $\iota \colon \mathbb{N} \to Z$ of rigs. Similarly as before, define $i \colon \mathbb{Z} \to Z$ by

$$i([\langle m, n\rangle]) := \iota(m) - \iota(n). \tag{1.2.19}$$

Exercise 1.2.20 Show that i is an injective morphism of rings.

This shows that Z 'contains a copy' of $\mathbb{Z}$ (as a ring, as opposed to as a totally-ordered ring as before).

Exercise 1.2.21 Show that $i \colon \mathbb{Z} \to Z$ is the 'unique copy' of $\mathbb{Z}$ contained in Z, using the proof of Step 11 as guidance.

■

[a] To see how we came up with these definitions, recall that we are thinking of $\langle m_1, n_1\rangle$ and $\langle m_2, n_2\rangle$ respetively as $m_1 - n_1$ and $m_2 - n_2$. Thus, the product of these two integers 'is' $(m_1 - n_1)(m_2 - n_2) = (m_1 m_2 + n_1 n_2) - (m_1 n_2 + n_1 m_2)$.

[b] Precisely, we must show that there is a unique embedding $\mathbb{Z} \to Z$. Nevertheless, we continue to use the term "copy" informally in this sense.

[c]We say that $\mathbb{Z}$ is unique *up to isomorphism* because in fact all we can say is that Z contains an *isomorphic copy* of $\mathbb{Z}$. We abused language and said just "$\mathbb{Z}$" instead of "isomorphic copy of $\mathbb{Z}$". This abuse of language is very common in mathematics. Indeed, you should usually be thinking of two things which are isomorphic as, for all intents and purpose, the same exact thing.

You will notice a common theme through these notes, and indeed, throughout all of mathematics: if ever we want something that isn't there, throw it in. For example, we had the natural numbers, but wanted additive inverses, and so we came up with $\mathbb{Z}$ by just 'adjoining' the additive inverses of all the elements of $\mathbb{N}$. Similarly, we want multiplicative inverses, and so by throwing them in, we obtain $\mathbb{Q}$. Doing the same with limits gives us $\mathbb{R}$, and in turn doing the same with roots of polynomials gives us $\mathbb{C}$.[5]

We now know that $\mathbb{Z}$ is a totally-ordered integral cring. Of course, $\mathbb{Z}$ satisfies other properties as well (a lot of which follow from the fact that $\mathbb{Z}$ is a totally-ordered integral cring).

Exercise 1.2.22 Let X be a preordered ring and let $x_1, x_2 \in X$. Show the following statements.

(i). $0 \leq x_1$ implies $-x_1 \leq 0$.
(ii). $x_1, x_2 \leq 0$ implies $0 \leq x_1 x_2$.
(iii). If X is totally-ordered, then $0 \leq x_1^2$.
(iv). If X is totally-ordered, then $0 \leq 1$.
(v). $x_1 \leq 1$ and $0 \leq x_2$ imply $x_1 x_2 \leq x_2$.
(vi). $x_1 < x_2$ implies $x_1 + x_3 < x_2 + x_3$. Find a counter-example if X is not a ring.
(vii). If X is integral, then $x_1 < x_2$ and $0 < x_3$ implies $x_1 x_3 < x_2 x_3$. Find a counter-example if X is not integral.[a]

[5]Though we won't be touching $\mathbb{C}$ ourselves, let us make one passing comment, When passing from $\mathbb{R}$ to $\mathbb{C}$, something of a miracle happens: by 'throwing in' *just one* root you didn't have before (namely, a root of the polynomial $x^2 + 1$), you obtain *every root of every polynomial, even for polynomials with coefficients made up of the roots you just obtained*!

[a]See Example 1.2.29.

As with the natural numbers, it is of course desirable to know whether this agrees with what you likely think of as the integers, namely $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The analogous result for $\mathbb{N}$ that made this precise was Proposition 1.1.4.10. This time, the precise statement of this is given by the following.

Proposition 1.2.23 $\mathbb{Z} = \mathbb{N} \cup (-\mathbb{N})$.

R $\mathbb{N} \subseteq \mathbb{Z}$, and so, as $\mathbb{Z}$ contains additive inverse of all its elements, $-\mathbb{N} := \{-m : m \in \mathbb{N}\}$ is likewise a subset of $\mathbb{Z}$. The claim is that this is everything, that is, every integer is a natural number or the additive inverse of a natural number.

This should make sense: the integers are, by definition, the 'smallest' (totally-ordered integral c)ring which contains $\mathbb{N}$, and so at the bare minimum, $\mathbb{Z}$ had better contain $\mathbb{N}$ and additive inverses of every element of $\mathbb{N}$. It turns out this works.[a]

[a]Contrast this with the rationals in which there are many more rationals than just $\mathbb{Z}$ and the multiplicative inverses of elements of $\mathbb{Z}$.

Proof. We leave this as an exercise.

Exercise 1.2.24 Do the proof yourself.

∎

Exercise 1.2.25 Let $m \in \mathbb{Z}$ and suppose that $0 \leq m \leq 1$. Show that either $m = 0$ or $m = 1$.

Exercise 1.2.26 Let $m, n \in \mathbb{Z}$. Show that the following are equivalent.

(i). $m < n$.
(ii). $m \leq n - 1$.

(iii). $m + 1 \leq n$.

As was mentioned in a remark of the previous theorem, in fact, $\mathbb{Z}$ is the smallest nonzero totally-ordered cring *period*.

Theorem 1.2.27. $\mathbb{Z}$ is the unique nonzero totally-ordered integral cring that has the property that, if Z is any nonzero other totally-ordered integral cring, then $\mathbb{Z} \subseteq Z$.

R In fact, we could even take this as the definition of $\mathbb{Z}$. Indeed, this is arguably preferable to Theorem 1.2.1, but we as we actually make use of that result in the following proof, to do this, we would either have to rephrase things entirely or 'redefine' $\mathbb{Z}$. Moreover, this doesn't align as well with our perspective that the natural numbers come first, and that $\mathbb{Z}$, $\mathbb{Q}$, etc.., are developed in turn to rectify 'problems' that the natural numbers themselves possess.

Proof. STEP 1: INTRODUCE NOTATION
Let Z be some other nonzero totally-ordered integral cring. Let us write 0_Z and 1_Z respectively for the additive and multiplicative identity in Z, to distinguish them from $0, 1 \in \mathbb{Z}$.

STEP 2: DEFINE $\iota\colon \mathbb{Z} \to Z$
Propositions 1.1.4.10 and 1.2.23[a] imply that every element of $\mathbb{Z}$ is of the form $\underbrace{1 + \cdots + 1}_{m}$ or $-(\underbrace{1 + \cdots + 1}_{m})$ for some $m \in \mathbb{N}$. Note that any homomorphism of rings $\mathbb{Z} \to Z$ must send $\underbrace{1 + \cdots + 1}_{m}$ to $\underbrace{1_Z + \cdots + 1_Z}_{m}$ (and similarly for the additive inverse). There is thus a unique homomorphism $\iota\colon \mathbb{Z} \to Z$ defined in exactly this way. We wish to show that ι is an embedding of totally-ordered rings.

It is a homomorphism of rings by construction, and so we need only check that it nondecreasing, injective, and as the property that $\iota(m) \leq \iota(n)$ iff $m \leq n$.

STEP 3: SHOW THAT ι IS NONDECREASING
We leave this as an exercise.

Exercise 1.2.28 Show that ι is nondecreasing.

STEP 4: SHOW THAT ι IS INJECTIVE
We check that ι is injective. For $m \in \mathbb{Z}$, let us write $m_Z \in Z$ to represent $\underbrace{1_Z + \cdots + 1_Z}_{m}$ if $m \geq 0$ and $-(\underbrace{1_Z + \cdots + 1_Z}_{-m})$ if $m \leq 0$, or more concisely, $m_Z = \operatorname{sgn}(m)(\underbrace{1_Z + \cdots + 1_Z}_{|m|})$. Thus, in this notation, $\iota(m) = m_Z$. Now, suppose that $m_Z = n_Z$. Without loss of generality, $n \leq m$. From this equation, it follows that $(m-n)_Z = 0_Z$.

If $m = n$, we are done, so suppose that $m - n \geq 1$. From the axioms of a preordered rig, it follows that $1_Z \leq (m-n)_Z$. But then $0_Z \leq 1_Z \leq (m-n)_Z = 0_Z$, and so $0_Z = 1_Z$. By Exercise A.4.21, it follows that Z is the zero cring: a contradiction. Hence, ι is injective.

STEP 5: SHOW THAT $\iota(m) \leq \iota(n)$ IFF $m \leq n$
Finally, we check that $\iota(m) \leq \iota(n)$ iff $m \leq n$. One of these directions is just the statement that ι is nondecreasing. For the other direction, suppose that $\iota(m) \leq \iota(n)$. In different notation, this is just the statement that $m_Z \leq n_Z$, from which we see that $0 \leq (n-m)_Z$. If $m > n$, then $(n-m)_Z = -(m-n)_Z$ is the additive inverse of a finite sum of 1_Zs. As $1_Z \geq 0_Z$, $(m-n)_Z \geq 0_Z$ as well, and so $-(m-n)_Z \leq 0$. We then have that $0 \leq (n-m)_Z = -(m-n)_Z \leq 0_Z$, which forces $(n-m)_Z = 0$, which, by injectivity, gives us that $n - m = 0$,

that is, $m = n$, a contradiction of our assumption that $m < n$. Thus, it must be that $m \geq n$.

STEP 6: DEDUCE THAT ι IS AN EMBEDDING OF PREORDERED RINGS

We have shown that ι is an injective nondecreasing ring homomorphism such that $\iota(m) \leq \iota(n)$ iff $m \leq n$, and so by Exercises B.2.18 and B.2.19, it follows that ι is an embedding of preordered rings.

STEP 7: SHOW THAT ι IS THE UNIQUE EMBEDDING OF $\mathbb{Z}$ INTO Z

We mentioned before that there is a unique homomorphism $\mathbb{Z} \to Z$, and so in particular this embedding is unique.

STEP 8: SHOW THAT $\mathbb{Z}$ IS THE UNIQUE SUCH NONZERO TOTALLY-ORDERED INTEGRAL CRING

Uniqueness up to unique isomorphism now follows from the usual argument: if Z is some other nonzero totally-ordered integral cring which satisfies this property, then $\mathbb{Z}$ embeds uniquely into Z and Z embeds uniquely into $\mathbb{Z}$, whence it follows that these unique embeddings are in fact isomorphisms. ■

[a] The latter says that every integer is either a natural number or the additive inverse of a natural number, and the former says that every natural number can be obtained from 0 by adding 1 sufficiently many times.

We end this section with an example that is a bit off track: when writing this section, the question came to mind whether it was true or not that a totally-ordered cring was necessarily integral (in particular, I wanted to know whether we could get integrality for free from other properties of the integers). It turns out this is false.

Example 1.2.29 — A totally-ordered cring that is not integral Define $R := \mathbb{Z} \times \mathbb{Z}$, and for $\langle m_1, n_1 \rangle, \langle m_2, n_2 \rangle \in R$,[a]

define

$$\begin{aligned}\langle m_1, n_1\rangle + \langle m_2, n_2\rangle &:= \langle m_1 + m_2, n_1 + n_2\rangle \\ \langle m_1, n_1\rangle \cdot \langle m_2, n_2\rangle &:= \langle m_1 m_2, m_1 n_2 + n_1 m_2\rangle,\end{aligned}$$

as well as

$$\langle m_1, n_1\rangle \leq \langle m_2, n_2\rangle \text{ iff } m_1 < m_2 \text{ or } (m_1 = m_2 \text{ and } n_1 \leq n_2).^b$$

Exercise 1.2.31 Show that $\langle R, +, \langle 0,0\rangle, -, \cdot, \langle 1,0\rangle\rangle$ is a cring.

Exercise 1.2.32 Show that $\langle R, \leq\rangle$ is a totally-ordered set.

We check that $\langle R, +, \langle 0,0\rangle, -, \cdot, \langle 1,0\rangle, \leq\rangle$ is a totally-ordered cring ourselves. It remains only to check that compatibility axioms.

Suppose that $\langle m_1, n_1\rangle \leq \langle m_2, n_2\rangle$, so that, $m_1 < m_2$, or $m_1 = m_2$ and $n_1 \leq n_2$. In the former case, we have that $m_1 + m_3 < m_2 + m_3$, and so we would in turn have that $\langle m_1, n_1\rangle + \langle m_3, n_3\rangle \leq \langle m_2, n_2\rangle + \langle m_3, n_3\rangle$. In the latter case, we have that $n_1 + n_3 \leq n_2 + n_3$, and so we would have in turn once again have that $\langle m_1, n_1\rangle + \langle m_3, n_3\rangle \leq \langle m_2, n_2\rangle + \langle m_3, n_3\rangle$.

Now suppose additionally that $\langle 0,0\rangle \leq \langle m_3, n_3\rangle$, so that $0 < m_3$, or $0 = m_3$ and $0 \leq n_3$. Explicitly, we are assuming

$$\begin{aligned}&(m_1 < m_2 \text{ or } (m_1 = m_2 \text{ and } n_1 \leq n_2)) \\ &(0 < m_3 \text{ or } (0 = m_3 \text{ and } 0 \leq n_3)).\end{aligned} \tag{1.2.33}$$

As $\langle m_1, n_1\rangle\langle m_3, n_3\rangle = \langle m_1 m_3, m_1 n_3 + n_1 m_3\rangle$ and $\langle m_2, n_2\rangle\langle m_3, n_3\rangle = \langle m_2 m_3, m_2 n_3 + n_2 m_3\rangle$, we wish to show that

$$\begin{aligned}&m_1 m_3 < m_2 m_3 \text{ or} \\ &\quad (m_1 m_3 = m_2 m_3 \text{ and } m_1 n_3 + n_1 m_3 \leq m_2 n_3 + n_2 m_3).\end{aligned}$$

In case $m_1 < m_2$ and $0 < m_3$, it follows that $m_1 m_3 < m_2 m_3$, as desired. In case $m_1 < m_2$, and $0 = m_3$ and $0 \leq n_3$, it follows that $m_1 m_3 = 0 = m_2 m_3$ and $m_1 n_3 + n_1 m_3 = m_1 n_3 < m_2 n_3 = m_2 n_3 + n_2 m_3$, as desired. In case $m_1 = m_2$ and $n_1 \leq n_2$, and $0 < m_3$, we have that $m_1 m_3 = m_2 m_3$ and $m_1 n_3 + n_1 m_3 \leq m_2 n_3 + n_2 m_3$, as desired. Finally, in case $m_1 = m_2$ and $n_1 \leq n_2$, and $0 = m_3$ and $0 \leq n_3$, we have that $m_1 m_3 = 0 = m_2 m_3$ and $m_1 n_3 + n_1 m_3 \leq m_2 n_3 + n_2 m_3$, as desired. Thus, this is indeed a totally-ordered cring.

On the other hand, it is not integral as $\langle 0, 1\rangle \cdot \langle 0, 1\rangle = \langle 0, 0\rangle$.

---

[a] Secretly, I am thinking of $\langle m, n\rangle$ as $m + nx$ where x is some 'variable' that satisfies $x^2 = 0$.

[b] This is often called the ***lexicographic order*** because the order is defined similarly to alphabetical order, that is, you compare the first entry first, and if those are equal, you look at the next entry, etc..

1.3 Multiplicative inverses and the rationals

The motivation of the introduction to the rationals is essentially the same as the motivation for the introduction of the integers, just with multiplication instead of division. That is, we would like to be able to solve equations of the form $mn = o$ for $m, n, o \in \mathbb{Z}$. There is one significant difference however. Unlike with addition, we cannot invert everything: in particular, we cannot invert 0.

Exercise 1.3.1 Let R be a ring. Show that if 0 is invertible in R, then $R = 0 := \{0\}$.

That is, the only ring in which 0 is invertible is the 0 ring. On the other hand, there are many interesting *rigs* in which 0 is invertible.

Example 1.3.2 — Tropical integers The tropical integers are an example of a nonzero integral crig in which 0 is invertible.[a] Thus, you do indeed need additive inverses for the result of the previous exercise to hold.

Define $R := \mathbb{N}$, $m +_R n := \max\{m, n\}$, $m \cdot_R n := m + n$, and $0_R := 0 =: 1_R$. Then, $\langle R, +_R, 0_R, \cdot_R, 1_R\rangle$ are the ***tropical integers***. The subscript R is meant to distinguish 'tropical' version from the usual version (e.g. $+_R := \max$ vs. $+$).

Exercise 1.3.3 Show that indeed $\langle R, +_R, 0_R, \cdot_R, 1_R\rangle$ is a crig. Furthermore, show that $0_R \cdot_R 0_R = 1_R$, so that indeed 0_R is invertible with multiplicative inverse $0_R^{-1} = 0_R$.

[a] As a matter of fact, 0 is the *only* invertible element.

Besides 0, however, we wind up being able to invert everything we would like.

Theorem 1.3.4 — Rational numbers. There exists a unique totally-ordered field[a] $\mathbb{Q}$, the *rational numbers*, such that

(i). $\mathbb{Z} \subseteq \mathbb{Q}$; and
(ii). if Q is any other totally-ordered field such that $\mathbb{Z} \subseteq Q$, then $\mathbb{Q} \subseteq Q$.

Furthermore, $\mathbb{Q}$ is additionally the unique field such that

(i). $\mathbb{Z} \subseteq \mathbb{Q}$; and
(ii). if Q is any other field such that $\mathbb{Z} \subseteq Q$, then $\mathbb{Q} \subseteq Q$.

R Similarly as with the integers, you will find that in the proof you will be constructing $\mathbb{Q}$ as equivalence classes of ordered pairs of integers (probably with one of the elements in the pair being positive, or at least nonzero[b]). However, once again, you should *not consider this as a definition of the rational numbers*. The rational numbers are defined implicitly in the statement of this result, and that construction is merely a tool used to prove this result. In particular, once again, don't you dare consider using equivalence classes of integers in proving something about the rational numbers!

R In fact, $\mathbb{Q}$ isn't just the smallest totally-ordered field which contains $\mathbb{Z}$: it's the smallest totally-ordered field *period*[c]—see Theorem 1.3.7.

R Just as with the integers, several things here are to be interpreted only up to isomorphism—see the remark in Theorem 1.2.1 for elaboration on this.

R This is also a special case of a more general construction known as the construction of *fraction fields*.[d] For the same reason as with the Grothendieck group construction, we do not present this in its full generality.

[a] Recall that a field is a cring in which every *nonzero* element has a multiplicative inverse—see Definition A.4.25.

[b] Because it should be representing the denominator of course.

[c] Besides the zero cring of course.

[d] Or even more generally, *localization*.

Proof. We leave this as an exercise.

Exercise 1.3.5 Try to do the entire proof yourself, using the proof of Theorem 1.2.1 as guidance.

■

As was mentioned in a remark of the previous theorem, in fact, $\mathbb{Q}$ is the smallest nonzero totally-ordered field *period*.

Exercise 1.3.6 Let F be a totally-ordered field. Show that $\mathrm{Char}(F) = 0$.

Theorem 1.3.7. $\mathbb{Q}$ is the unique nonzero totally-ordered field that has the property that, if F is any other nonzero totally-ordered field, then $\mathbb{Q} \subseteq F$.

R Once again, uniqueness and $\subseteq$ should be interpreted "up to isomorphism".

R In fact, we could even take this as the definition of $\mathbb{Q}$. Indeed, this is arguably preferable to Theorem 1.3.4, but we as we actually make use of that result in the following proof, to do this, we would either have to rephrase things entirely or 'redefine' $\mathbb{Q}$. Moreover, this doesn't align as well with our perspective that the natural numbers come first, and that $\mathbb{Z}$, $\mathbb{Q}$, etc.., are developed in turn to rectify 'problems' that the natural numbers themselves possess.

Proof. As $\mathrm{Char}(F) = 0$, F contains a copy of $\mathbb{N}$, namely, $\{0, 1, 2, 3, \ldots\}$.[a] Recall that $\mathbb{Z}$ is the smallest totally-ordered integral cring which contains $\mathbb{N}$. Thus, as F is in particular now a totally-ordered integral cring which contains $\mathbb{N}$, we have that in fact F contains $\mathbb{Z}$. Similarly, as $\mathbb{Q}$ was the smallest totally-ordered field that contained $\mathbb{Z}$, and we have just showed that F is a totally-ordered field that contains $\mathbb{Z}$, it follows that $\mathbb{Q} \subseteq F$. ■

[a] We needed that $\mathrm{Char}(F) = 0$ in order that this be a copy of $\mathbb{N}$. For example, think of the integers modulo m—see Example A.4.1.18, as well as the sequence of examples starting with Example A.3.2.2. For example, in the case $m = 3$, we have that $1 + 1 + 1 = 0$, and so this set $\{0, 1, 2, 3 \ldots\}$ would just be $\{0, 1, 2, 0, \ldots\} = \{0, 1, 2\}$, certainly not a copy of the natural numbers.

Just as with the the natural numbers (Proposition 1.1.4.10) and the integers (Proposition 1.2.23), we would like to know that this agrees with our naive idea of what the rational numbers are.

Proposition 1.3.8 For all $x \in \mathbb{Q}$, there exist unique $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ such that (i) $\gcd(m, n) = 1$ and (ii) $x = \frac{m}{n}$.

R m is the ***numerator*** of x and n is the ***denominator*** of x.[a]

R Thus, this theorem says that not only can we write every rational number as the quotient of two integers, but furthermore, if we require that the denominator be positive and the two integers have "no common factors", then there is only one way to write a given rational number in this form.

[a] More generally, you might say that any integers $m, n \in \mathbb{Z}$ for which $x = \frac{m}{n}$ are numerators and denominators of x respectively, regardless of whether or not n is positive or $\gcd(m, n) = 1$.

Proof. By (i) of Theorem 1.3.4 (the defining theorem of $\mathbb{Q}$), $\mathbb{Q}$ contains $\mathbb{Z}$. As $\mathbb{Q}$ is a field, it thus must contain multiplicative inverses of every nonzero integer. For $m \in \mathbb{Z}$ not zero, denote its multiplicative inverse in $\mathbb{Q}$ by $\frac{1}{m}$. For $n \in \mathbb{Z}$, denote $\frac{n}{m} := n \cdot \frac{1}{m}$, and define

$$Q := \left\{ \tfrac{n}{m} \in \mathbb{Q} : m, n \in \mathbb{Z}, m \neq 0 \right\} . \tag{1.3.9}$$

Note that Q is a field which contains $\mathbb{Z}$ as a subpreordered ring. By (ii) of Theorem 1.3.4, we thus have that $\mathbb{Q} \subseteq Q$. Of course we already knew that $Q \subseteq \mathbb{Q}$, and so we have that $Q = \mathbb{Q}$.

Now for $x \in \mathbb{Q}$ arbitrary, we can write $x = \frac{n}{m}$ for some $m, n \in \mathbb{Z}$ with $m \neq 0$. If $m < 0$, then we can write $x = \frac{-n}{-m}$, so that now the denominator is positive. Define $d := \gcd(m, n)$ and write $m = m'd$ and $n = n'd$, so that $x = \frac{n'}{m'}$.

Exercise 1.3.10 Show that $\gcd(m', n') = 1$.

This shows existence.

We now prove uniqueness. Suppose that $\frac{n_1}{m_1} = \frac{n_2}{m_2}$ for $m_1, n_1, m_2, n_2 \in \mathbb{Z}$ with $m_1, m_2 > 0$, $\gcd(m_1, n_1) = 1 = \gcd(m_2, n_2)$. Rearranging, we have $m_2 n_1 = m_1 n_2$, so that $m_2 \mid m_1 n_2$. As $\gcd(m_2, n_2) = 1$, it follows that $m_2 \mid m_1$. On the other hand, this same equation implies that $m_1 \mid m_2 n_1$, and therefore $m_1 \mid m_2$. That $m_1 \mid m_2$ and $m_2 \mid m_1$ implies that $m_1 = \pm m_2$. However, as they are both positive, we have that $m_1 = m_2$. It then follows that $n_1 = n_2$ from the equation $m_2 n_1 = m_1 n_2$. ∎

Exercise 1.3.11 Let X be a totally-ordered field and let $x_1, x_2 \in X$ be nonzero. Show that the following statements are true.

(i). x_1^{-1} has the same sign as x_1.
(ii). $0 < x_1 \leq x_2$ implies $0 < x_2^{-1} \leq x_1^{-1}$.

1.4 Least upper-bounds and the real numbers

Finally we come to the first material that might be considered 'the point' of this course.

Just as the integers corrected the 'deficiency' of the natural numbers that was the lack of additive inverses, and the rationals corrected the 'deficiency' of the integers that was the lack of multiplicative inverses, the real numbers will correct a 'deficiency' of the rational numbers. But what is this "deficiency"? The answer turns out to be that this 'deficiency' is a lack of what are called least upper-bounds.

1.4.1 Least upper-bounds and why you should care

Definition 1.4.1.1 — Suprema Let $\langle X, \leq\rangle$ be a preordered set, let $S \subseteq X$, and let $x \in X$. Then, x is a ***supremum*** of S iff

(i). x is an upper-bound of S, and
(ii). if x' is any other upper-bound of S, then $x \leq x'$.

R Supremum is synonymous with ***least upper-bound*** (because they are an upper-bound that is less than or equal to every other upper-bound).

R If S has a supremum x, then we write $x := \sup(S)$. This is justified by the following exercise.

Exercise 1.4.1.2 Let $\langle X, \leq\rangle$ be a partially-ordered set, let $S \subseteq X$, and let $x_1, x_2 \in S$ be two suprema of S. Show that $x_1 = x_2$.

We extend the definition of supremum as follows.

$$\sup(S) := \begin{cases} \infty & \text{if } S \text{ is not bounded above} \\ -\infty & \text{if } S = \emptyset. \end{cases} \quad [a] \tag{1.4.1.3}$$

R To remember this convention, note that the supremum of 'smaller' sets is smaller, precisely $S \subseteq T$ implies $\sup(S) \subseteq \sup(T)$. Therefore, the "smallest" set, namely $\emptyset$, should have supremum $-\infty$.

R

[a] In general, we don't want to consider $\pm\infty$ as elements of X, and if you like, we are extending the order on X to the set $X \sqcup \{\pm\infty\}$ in such a way so that $\pm\infty$ are the maximum and minimum of X respectively. On the other hand, if X already had a maximum or minimum, then you should consider $\pm\infty$ as synonyms for the previously existing maximum and minimum.

Exercise 1.4.1.4 Find an example of a preordered set $\langle X, \leq\rangle$ an a subset $S \subseteq X$ with two distinct suprema.

R It is because of counter-examples like these that we shall primarily concern ourselves with partially-ordered sets in this section.

We similarly have a notion of infimum, which is just the same concept with inequalities reversed.

Definition 1.4.1.5 — Infimum Let $\langle X, \leq\rangle$ be a preordered set, let $S \subseteq X$, and let $s \in X$. Then, x is a ***infimum*** of S iff

(i). x is a lower-bound of S, and
(ii). if x' is any other lower-bound of S, then $x' \leq x$.

R A you might have guessed, infima are also called ***greatest lower-bounds***.

R Similarly, if S has an infimum x, then we write $x := \inf(S)$.

We extend the definition of infimum as follows.

$$\inf(S) := \begin{cases} -\infty & \text{if } S \text{ is not bounded below} \\ \infty & \text{if } S = \emptyset. \end{cases} \tag{1.4.1.6}$$

So that's what suprema (and infima) are, but why should you care? At least one reason is the following. Consider the set

$$S := \left\{ \left(1 + \tfrac{1}{n}\right)^n : n \in \mathbb{Z}^+ \right\} \subseteq \mathbb{Q}. \tag{1.4.1.7}$$

This set has two key properties (i) it is bounded above, and (ii) for every $x \in S$, there is some $x' \in S$ with $x < x'$. Draw yourself a picture of what this must look like. Though we don't know what a limit is yet, from the picture we see that pretty much any reasonable definition of a limit should have the property that

$$\lim_n \left[1 + \tfrac{1}{n}\right]^n = \sup(S). \tag{1.4.1.8}$$

Though we don't even have the definition to make sense of it yet, you'll recall that the answer to the left-hand side *should* be $\mathrm{e} := \exp(1)$, which of course is not rational.[6] Thus, despite the fact that $S \subseteq \mathbb{Q}$, $\sup(S) \notin \mathbb{Q}$, and in fact, for us, $\sup(S)$ just doesn't make sense (yet). Ultimately, because we want to do calculus, we want to be able to take limits. In particular, because of things like (1.4.1.8), we had better be able to take suprema as well. It turns out that throwing in all least upper-bounds gives us all the limits we were missing. This is really the motivation for demanding the existence of least upper-bounds: we want to be able to take limits.

Thus, the desired property which $\mathbb{Q}$ lacks is the following.

[6]Of course, we haven't proven that e is not rational, but right now, as we are only concerned with motivation, simply knowing that it is not rational is enough to justify the desire to have suprema.

Definition 1.4.1.9 — Least upper-bound property A preordered set has the ***least upper-bound property*** iff every nonempty subset that is bounded above has a least upper-bound.

> **R** You'll recall from (1.4.1.3) that the empty-set has supremum $-\infty$ and that sets which are not bounded above have supremum $+\infty$. Thus, the reason for the conditions "nonempty" and "bounded above" is that we don't want to necessarily require the existence of $\pm\infty$. Besides these two extremes, however, we would like everything else to have a supremum.

Of course, we have the inequality-reversed notion as well.

Definition 1.4.1.10 — Greatest lower-bound property A preordered set has the ***greatest lower-bound property*** iff every nonempty subset that is bounded below has a greatest lower-bound.

It turns out that it doesn't actually matter which property we require:

Proposition 1.4.1.11 Let X be a preordered set. Then, X has the least upper-bound property iff it has the greatest lower-bound property.

Proof. ($\Rightarrow$) Suppose that X has the least upper-bound property. Let $S \subseteq X$ be nonempty and bounded-below. We wish to show that S has an infimum.

Define

$$T := \{x \in X : x \leq x' \text{ for all } x' \in S.\} . \tag{1.4.1.12}$$

As S is nonempty, there is some $x_0 \in S$. From the definition of T, it follows that x_0 is an upper-bound of T, so that T is bounded above. T is the set of all lower-bounds of S, and so as S is bounded below, T is nonempty. Thus, because X has the least upper-bound property, T has some supremum $t \in X$.[a]

We wish to show that t is an infimum of S. We must show two things: (i) that it is a lower bound of S and (ii) that it is at least as large as every other lower-bound.

To show the first, let $x \in S$. By definition, $x' \leq x$ for all $x' \in T$, and so x is an upper-bound for T. As t is a *least* upper-bound, we then have that $t \leq x$, so that t is a lower-bound of S.

To show the second, let $t' \in X$ be some other lower-bound of S. Then, by definition, $t' \in T$, and so as t is an upper-bound of T, we have that $t' \leq t$, as desired.

($\Leftarrow$) This proof is the inequality-reversed version of the ($\Rightarrow$) proof. ■

[a] We refrain from writing $t = \sup(T)$ as X is not necessarily partially-ordered—see the remark in the definition of Suprema (Definition 1.4.1.1).

R Sometimes preordered sets which possess both of these properties are called ***Dedekind-complete***. The term *Dedekind*-complete is to contrast with the term "*Cauchy*-complete", which we will meet in Chapter 4 Uniform spaces—see Definition 4.4.7. Indeed, you should probably try to avoid saying the word "complete" by itself because (i) this can lead to confusion with Cauchy-completeness and (ii) in order theory the term "complete" (I believe?) usually implies the existence of suprema and infima of *all* subsets (not necessarily nonempty and bounded). That said, I can guarantee you I'm going to be sloppy about this myself.

The real numbers will turn out to be a Dedekind-complete *totally*-ordered set, and so it will be useful to have the following equivalent characterization of suprema and infima in totally-ordered sets.

Proposition 1.4.1.13 Let X be a totally-ordered set, let $S \subseteq X$ nonempty and bounded above, and let $x \in X$ be an upper-bound for S. Then, $x = \sup(S)$ iff for every $x' \in X$ with $x' < x$, there is some $x'' \in S$ with $x' < x'' \leq x$.

W Warning: This is *not* true if X is only partially-ordered—see Example 1.4.1.15.

R You should think of this as saying that, in particular, S contains elements 'arbitrarily close' to its supremum.

R This result is *incredibly* important, and you should definitely take note of it. In the real numbers, this will be our primary method for proving things using suprema and infima.

Proof. ($\Rightarrow$) Suppose that $x = \sup(S)$. Let $x' \in X$ be such that $x' < x$. We proceed by contradiction: suppose that there is no $x'' \in S$ such that $x' < x'' \leq x$. x being an upper-bound of S, every $x'' \in S$ is automatically less than or equal to S, so really this is just the same as saying that there is no $x'' \in S$ with $x' < x''$. *By totality*, it thus follows that we must have $x'' \leq x'$ for all $x'' \in S$, in which case x' is an upper-bound for S. But $x' < x$, which contradicts the fact that x is the *least* upper-bound for S. Thus, there must be some $x'' \in S$ such that $x < x'' \leq x$.

($\Leftarrow$) Suppose that for every $x' \in S$ with $x' < x$, there is some $x'' \in S$ with $x' < x'' \leq x$. Let $x' \in X$ be any other upper-bound for S. We would like to show that $x \leq x'$. We proceed by contradiction: suppose that it is not the case that $x \leq x'$. By totality, this is equivalent to $x' < x$. Then, by hypothesis, there must be some $x'' \in S$ such that $x' < x'' \leq x$, which contradicts the fact that x' is an upper-bound of S. Thus, it must be the case that $x = \sup(S)$. ■

Exercise 1.4.1.14 Write down and prove the analogous version of the previous proposition for the infimum.

▪ **Example 1.4.1.15 — A counter-example to Proposition 1.4.1.13 in the absence of totality** Define $X := \mathbb{Q} \sqcup \{A\}$ and declare that $A \leq q$ if $q > 0$ and $A \geq q$ if $q < 0$ for $q \in \mathbb{Q}$.[a] Then, $0 \in X$ is *not* a least upper-bound for $(-\infty, 0)$ because it is not less-than-or-equal-to A, which itself is another upper-bound of $(-\infty, 0)$ (in fact, it's just not comparable to A). On the other hand, if $q < 0$, then $q < \frac{q}{2} \leq 0$, and so $0 \in X$ does satisfy the desired property even though it is not a least upper-bound

[a] You might picture A as being 'right next to' and incomparable with 0.

R It is not uncommon to see others using facts like "$\sqrt{2}$ is not rational." as motivation for the introduction of the real numbers. This is stupid. If all we really cared about were numbers like $\sqrt{2}$, then we shouldn't be going from $\mathbb{Q}$ to $\mathbb{R}$, but rather from $\mathbb{Q}$ to $\mathbb{A}$, the *algebraic numbers* (see Definition 2.1.13). The point of $\mathbb{R}$ is not to be able to take square-roots; the point is to be able to take limits.

1.4.2 Dedekind-cuts and the real numbers

Everybody reading these notes probably already has some intuition about the real numbers, most likely gained from some sort of calculus course. Let us suppose for a moment that we know what the real numbers are and that they make sense. Given a real number $x_0 \in \mathbb{R}$, how would you encode x_0 using only $\mathbb{Q}$? The trick we use is to look at the set

$$D_{x_0} := \{x \in \mathbb{Q} : x \leq x_0\} . \tag{1.4.2.1}$$

This subset of $\mathbb{Q}$ uniquely determines x_0 because $\sup(D_{x_0}) = x_0$. The idea then is to use sets of the form (1.4.2.1) to define the real numbers. The only thing we have to do for this to make sense in $\mathbb{Q}$ alone is to get rid of the reference to x_0. We do that as follows.

Definition 1.4.2.2 — Dedekind-cut Let $\langle X, \leq\rangle$ be a preordered set and let $D \subseteq X$. Then, D is a ***Dedekind-cut*** in X iff

(i). $D \neq \emptyset$,
(ii). $D \neq X$, and

(iii). the set of all lower-bounds of the set of all upper-bounds of D is equal to D itself.

R To ease notation a bit, for $D \subseteq X$, we shall write

$$\begin{aligned} D^{\mathrm{U}} &:= \{u \in X : u \text{ is an upper-bound of } D.\} \\ D^{\mathrm{L}} &:= \{l \in X : u \text{ is a lower-bound of } D.\} . \end{aligned} \tag{1.4.2.3}$$

In this notation, ((iii)) may be written as

$$D = (D^{\mathrm{U}})^{\mathrm{L}}. \tag{1.4.2.4}$$

R The word *cut* is used because, for example, D_{x_0} of (1.4.2.1) is sort of thought as 'cutting' $\mathbb{Q}$ at the point x_0.

R Note that this almost, but doesn't quite, agree with everyone's convention. For example, with our convention, $(-\infty, 2]$ is a cut (because $(-\infty, 2]^{\mathrm{U}} = [2, \infty)$ and $[2, \infty)^{\mathrm{L}} = (-\infty, 2]$) whereas $(-\infty, 2)$ is not (for essentially the same reason). On the other hand, others sometimes use an 'open' convention in which $(-\infty, 2]$ is not a cut but $(-\infty, 2)$ is. The reason for preferring the convention we do simply comes down to the fact that $\leq$ is always playing the primary role for us, not $<$.

You'll note that D_{x_0} of (1.4.2.1) is a Dedekind-cut. Indeed,

Proposition 1.4.2.5 Let $\langle X, \leq \rangle$ be a Dedekind-complete totally-ordered set and let $D \subseteq X$ be a Dedekind-cut. Then,

$$D = \{x \in X : x \leq \sup(D)\} . \tag{1.4.2.6}$$

Proof. Let us write $D' := \{x \in X : x \leq \sup(D)\}$. As $\sup(D)$ is in particular an upper-bound of D, we immediately have the inclusion $D \subseteq D'$. On the other hand, suppose that $x \leq \sup(D)$. We wish to show that $x \in D$. As D is a Dedekind-cut, this is the same as showing that x is less than or equal to every upper-bound of D. So, let $u \in X$ be an upper-bound of D. We now wish to show that $x \leq u$. We proceed by contradiction: suppose that $u < x$ (this uses totality). However, this of course contradicts the fact that u is an upper-bound for S. Thus, we must have that $x \leq u$, so that $D' \subseteq D$, so that $D = D'$. ■

Ultimately we will be constructing the real numbers as the set of all Dedekind-cuts in $\mathbb{Q}$.[7] Thus, we will in particular want to know how to do things like add cuts, multiply cuts, etc.. Some of the time, however, the naive definition doesn't 'quite' give us a new Dedekind-cut, and so we have to 'force' it to be a cut. The $(-^{\mathrm{U}})^{\mathrm{L}}$ construction will help us do that.

Proposition 1.4.2.7 Let $\langle X, \leq \rangle$ be a preordered set and let $S, T \subseteq X$. Then,

(i). if X has no minimum, then $(\emptyset^{\mathrm{U}})^{\mathrm{L}} = \emptyset$;
(ii). $S \subseteq (S^{\mathrm{U}})^{\mathrm{L}}$;
(iii). $\left(\left((S^{\mathrm{U}})^{\mathrm{L}}\right)^{\mathrm{U}}\right)^{\mathrm{L}} = (S^{\mathrm{U}})^{\mathrm{L}}$;
(iv). $(S^{\mathrm{U}})^{\mathrm{L}} \cup (T^{\mathrm{U}})^{\mathrm{L}} \subseteq \left((S \cup T)^{\mathrm{U}}\right)^{\mathrm{L}}$, with equality if X is totally-ordered;
(v). if $S \subseteq T$, then $(S^{\mathrm{U}})^{\mathrm{L}} \subseteq (T^{\mathrm{U}})^{\mathrm{L}}$; and
(vi). if D is a Dedekind-cut and $S \subseteq D$, then $(S^{\mathrm{U}})^{\mathrm{L}} \subseteq D$.

[7] There is another construction of the reals that is commonly taught, namely, the "Cauchy sequence construction" in which a real number is defined to be an equivalence class of Cauchy sequences. While this works, I find this more appropriate if one is thinking of the real numbers as a uniform space (in this case, a metric space), whereas we are currently thinking of everything as algebraic structures *with order*. Because of this, I find it more natural to complete the underlying partially-ordered set instead of the underlying uniform space (for one thing, we haven't actually put a uniform structure on $\mathbb{Q}$ yet).

R In particular, regardless of what S is, excluding the stupid case of the empty-set or the entire set, $(S^{\mathrm{U}})^{\mathrm{L}}$ is a Dedekind-cut ((iii)) that contains S ((ii)), and furthermore is the smallest such cut ((vi)).

R The first four are somehow on a different footing than the latter two—see Theorem 3.4.1.1.

Proof. (i) Note that, vacuously, every element of X is an upper-bound of $\emptyset$, that is, $\emptyset^{\mathrm{U}} = X$. X^{L} is the set of all elements less-than-or-equal to every element of X, that is, the minima of X. Thus, of course, if we assume there are no minima, then this is empty.[a]

(ii) Let $s \in S$. We wish to show that $s \in (S^{\mathrm{U}})^{\mathrm{L}}$. So, let $u \in S^{\mathrm{U}}$. Then, $s \leq u$ because u is an upper-bound of S and $s \in S$, and so indeed s is less-than-or-equal to every element of S^{U}, that is, $s \in (S^{\mathrm{U}})^{\mathrm{L}}$.

(iii) The $\supseteq$ inclusion follows from (ii). As for the other inclusion, let $x \in \left(\left((S^{\mathrm{U}})^{\mathrm{L}}\right)^{\mathrm{U}}\right)^{\mathrm{L}}$. We wish to show that $s \in (S^{\mathrm{U}})^{\mathrm{L}}$. So, let $u \in S^{\mathrm{U}}$. We wish to show that $x \leq u$. As $x \in \left(\left((S^{\mathrm{U}})^{\mathrm{L}}\right)^{\mathrm{U}}\right)^{\mathrm{L}}$, it thus suffices to show that $u \in \left((S^{\mathrm{U}})^{\mathrm{L}}\right)^{\mathrm{U}}$. So, let $l \in (S^{\mathrm{U}})^{\mathrm{L}}$. We wish to show that $l \leq u$. However, this is true because $u \in S^{\mathrm{U}}$ and $l \in (S^{\mathrm{U}})^{\mathrm{L}}$.

(iv) Let $x \in (S^{\mathrm{U}})^{\mathrm{L}} \cup (T^{\mathrm{U}})^{\mathrm{L}}$. Without loss of generality, suppose that $x \in (S^{\mathrm{U}})^{\mathrm{L}}$. We wish to show that $x \in \left((S \cup T)^{\mathrm{U}}\right)^{\mathrm{L}}$. So, let $u \in (S \cup T)^{\mathrm{U}}$. We wish to show that $x \leq u$. As $x \in (S^{\mathrm{U}})^{\mathrm{L}}$, it suffices to show that $x \in S^{\mathrm{U}}$. However, $x \geq y$ for all $y \in S \cup T$, and so in particular $x \geq s$ for all $s \in S$, so that indeed $u \in S^{\mathrm{U}}$, as desired.

Now suppose that X is totally-ordered. Let $x \in ((S \cup T)^{\mathrm{U}})^{\mathrm{L}}$. We wish to show that $x \in S^{\mathrm{U}})^{\mathrm{L}} \cup (T^{\mathrm{U}})^{\mathrm{L}}$. We proceed by contradiction: suppose that $x \notin (S^{\mathrm{U}})^{\mathrm{L}}$ and $x \notin (T^{\mathrm{U}})^{\mathrm{L}}$. This means that there is some $u \in S^{\mathrm{U}}$ and some $v \in T^{\mathrm{U}}$ such that $x \not\leq u$ and $x \not\leq v$. By totality, without loss of generality assume that $u \geq v$, so that u is likewise an upper-bound of T. Then, $u \in (S \cup T)^{\mathrm{U}}$, and hence $x \leq u$ as $x \in \left((S \cup T)^{\mathrm{U}}\right)^{\mathrm{L}}$: a contradiction. Therefore, $x \in ((S \cup T)^{\mathrm{U}})^{\mathrm{L}}$, as desired.

(iii) Suppose that $S \subseteq T$. Let $l \in X$ be a lower-bound of S^{U}. We wish to show that l is a lower-bound of T^{U}. So, let $u \in X$ be an upper-bound of T. We wish to show that $l \leq u$. As $S \subseteq T$, u is likewise an upper-bound of S, that is, $u \in S^{\mathrm{U}}$. Then, as l is a lower-bound of S^{U}, it follows that $l \leq u$, as desired.

(vi) Let $D \subseteq X$ be a Dedekind-cut such that $S \subseteq D$. By (iii), we have that $(S^{\mathrm{U}})^{\mathrm{L}} \subseteq (D^{\mathrm{U}})^{\mathrm{L}} =$ [b], as desired. ∎

[a] In case you're wondering why we're saying "minima" and not "minimum" it's because minima are only unique in a *partially*-ordered set, but not necessarily in only a preordered-set.

[b] Because D is a dedekind-cut.

Exercise 1.4.2.8 Let $\langle X, \leq\rangle$ be a partially-ordered set and let $S, T \subseteq X$. Is it necessarily the case that $\left((S \cup T)^{\mathrm{U}}\right)^{\mathrm{L}} = (S^{\mathrm{U}})^{\mathrm{L}} \cup (T^{\mathrm{U}})^{\mathrm{L}}$? If so, prove it; if not, find a counter-example.

R In case you're wondering why we're asking, check out Kuratowski's Closure Theorem (Theorem 3.4.1.1) again.

And now we finally turn to the real numbers themselves.

Theorem 1.4.2.9 — Real numbers. There exists a unique Dedekind-complete totally-ordered field $\mathbb{R}$, the ***real numbers***, such that

(i). $\mathbb{Q} \subseteq \mathbb{R}$; and (ii). if R is any other Dedekind-complete totally-ordered field such that $\mathbb{Q} \subseteq R$, then $\mathbb{R} \subseteq R$.

Furthermore, $\mathbb{R}$ is additionally the unique Dedekind-complete totally-ordered set such that

(i). $\mathbb{Q} \subseteq \mathbb{R}$; and
(ii). if R is any other field such that $\mathbb{Q} \subseteq R$, then $\mathbb{R} \subseteq R$.

R Yet again, you will find in the proof a 'concrete' construction of $\mathbb{R}$ from $\mathbb{Q}$ (using Dedekind-cuts of course), and yet again, after the proof is over, you should never use that construction of $\mathbb{R}$ again.[a]

R In fact, $\mathbb{R}$ isn't just the smallest Dedekind-complete totally-ordered field that contains $\mathbb{Q}$: it's the smallest Dedekind-complete totally-ordered field *period*. This alone, however, is perhaps not so surprising, as we had similar results for the integers and rationals (Theorems 1.2.27 and 1.3.7 respectively). For the reals, however, we have something more: the reals aren't just the smallest Dedekind-complete totally-ordered field, they are the *only* (nonzero) Dedekind-complete totally-ordered field—see Theorem 1.4.2.49.

R You'll note how in this case, in contrast with the analogous theorems for $\mathbb{Z}$ and $\mathbb{Q}$ (Theorems 1.2.1 and 1.3.4 respectively), here it is *order* that plays the primary role and the algebraic operations that "come along for the ride".

R As you might have guessed, this is also a special case of a more general construction, which takes partially-ordered sets to *Dedekind-complete* partially-ordered sets, known as the *Dedekind-MacNeille completion*. Be careful, however: in general the arithmetic operations do not extend to the Dedekind-MacNeille completion—see Example 2.3.17.

[a] Or, if you're a masochist, feel free to try to prove something about the set of Dedekind-cuts whose elements consist of equivalence classes of ordered pairs of equivalence classes of ordered pairs of equivalence classes of sets which do not biject onto a proper subset with respect to the equivalence relation of equinumerosity. Alternatively, you can do as I say and use the properties that uniquely define the real numbers ;-)

Proof. Step 1: Prove a useful lemma

Lemma 1.4.2.10 Let $D \subseteq \mathbb{R}$ be a Dedekind-cut. Then, for all $\varepsilon > 0$, there is some $d \in D$ such that $d + \varepsilon \notin D$.

Proof. Let $\varepsilon > 0$. We wish to show that there is some $d \in D$ so that $d + \varepsilon \notin D$. We proceed by contradiction: suppose that $d + \varepsilon \in D$ for all $d \in D$. Then, for $d_0 \in D$ fixed, we have $d_0 + \varepsilon \in D$, and so $d_0 + 2\varepsilon = (d_0 + \varepsilon) + \varepsilon \in D$, and so $d_0 + 3\varepsilon = (d_0 + 2\varepsilon) + \varepsilon \in D$, etc.. As D is bounded (by any element in D^{C}), this is a contradiction. Thus, there is some $d \in D$ such that $d + \varepsilon \notin D$. ■

Step 2: Define $\mathbb{R}$ as a set
Define

$$\mathbb{R} := \left\{ D \in 2^{\mathbb{Q}} : D \text{ is a dedekind-cut.} \right\} . \tag{1.4.2.11}$$

Step 3: Define a preorder on $\mathbb{R}$
We define

$$D \leq E \text{ iff } D \subseteq E. \tag{1.4.2.12}$$

Step 4: Show that $\leq$ is a total-order
$\leq$ is automatically a partial-order because set-inclusion is always a partial-order. To show totality, let $D, E \in \mathbb{R}$. If $D \leq E$, we are done, so suppose this is not the case. We would like to show that $E \leq D$, i.e., that $E \subseteq D$, so let $e \in E$. As D is a Dedekind-cut, it suffices to show that e is a lower-bound of every upper-bound of D. So, let $u \in \mathbb{Q}$ be an upper-bound of D. We wish to show that $e \leq u$. We proceed by

contradiction: suppose that $u < e$. Now, D is not a subset of E (by hypothesis), there must be some $d \in D$ with $d \notin E$. If we can show that $e \leq d$, then we will have $u < d$ (because $u < e$), a contradiction. To show this itself (that $e \leq d$), we proceed by contradiction: suppose that $d < e$. Then, in particular, d is less than every upper-bound of E, and so, as E is a cut, we have $d \in E$: a contradiction (recall that we have taken $d \notin E$). Thus, we must that $e \leq d$, which completes the proof of totality.

Step 5: Show that $\leq$ is Dedekind-complete

As $\leq$ is a total-order, it suffices simply to show that $\leq$ has the least upper-bound property (by Proposition 1.4.1.11). To show this, let $\mathscr{S} \subseteq \mathbb{R}$ be nonempty and bounded above. Define

$$S := \left(\left(\bigcup_{D \in \mathscr{S}} D\right)^{\mathrm{U}}\right)^{\mathrm{L}}. \tag{1.4.2.13}$$

Exercise 1.4.2.14 Check that S is in fact a Dedekind-cut.

> **R** Note that we cannot leave out the $^{\mathrm{U}}$ and $^{\mathrm{L}}$ here. For example, while each of $(-\infty, -\frac{1}{n}]$ is a cut for $n \in \mathbb{Z}^+$, $\bigcup_{n \in \mathbb{Z}^+}(-\infty, -\frac{1}{n}] = (-\infty, 0)$ is not—see one of the remarks in the definition Definition 1.4.2.2 for clarification. Thus, while 'morally' all we want to do is just to take the union of the sets in $\mathscr{S}$, we first have to 'close' the set, so to speak, and that is precisely what the $(-^{\mathrm{U}})^{\mathrm{L}}$ does.

We wish to show that $S = \sup(\mathscr{S})$. As S is a superset of every element of $\mathscr{S}$, we certainly have that S is an upper-bound for $\mathscr{S}$. To show that it is a least upper-bound, let S' be some other upper-bound of $\mathscr{S}$. We wish to show that $S \leq S'$. We proceed by contradiction: suppose that $S \nleq S'$. Then, there is some $x \in S$ with $x \notin S'$. As $x \in S$, there must be some $D \in \mathscr{S}$ with $x \in D$ (by the definition of S). However, as S' is an upper-bound of $\mathscr{S}$, we have that $D \subseteq S'$, which implies that $x \in S'$: a contradiction. Thus, we must have that $S \leq S'$, so that $S = \sup(\mathscr{S})$.

STEP 6: DEFINE ADDITION

Addition of elements of $\mathbb{R}$ is just set addition:[a]

$$D + E := \{d + e : d \in D, e \in E\}. \tag{1.4.2.15}$$

Exercise 1.4.2.16 Check that $D + E$ is in fact a Dedekind-cut.

STEP 7: DEFINE THE ADDITIVE IDENTITY AND INVERSES

The cut

$$0 := \{x \in \mathbb{Q} : x \leq 0\} \tag{1.4.2.17}$$

functions as an additive identity.[b]

We define the additive inverse

$$-D := \left(\{x - y : x \leq 0 \text{ and } y \notin D\}^{\mathrm{U}}\right)^{\mathrm{L}}. \tag{1.4.2.18}$$

Exercise 1.4.2.19 Check that $-D$ is in fact a Dedekind-cut.

STEP 8: SHOW THAT $\langle \mathbb{R}, +, 0, - \rangle$ IS A COMMUTATIVE GROUP

Associativity and commutativity follow from the fact that set addition is associative and commutative.

Exercise 1.4.2.20 Check that 0 is an additive identity.

Note that we have

$$D+(-D)=\{d+(x-y):d\in D,\ x\leq 0,\ y\notin D\}\,.$$

As $y\notin D$, we have that $d<y$, and so $d-y<0$, and so $d+(x-y)<0$, and so $d+(x-y)\in 0$. In the other direction, let $x\leq 0$. Then,

$$x=d+((x+(y-d))-y), \tag{1.4.2.21}$$

and so it suffices to show that we can choose $d\in D$ and $y\in D^{\mathsf{C}}$ so that $x+(y-d)\leq 0$. By the lemma of Step 1, for every $\varepsilon>0$, there is some $d\in D$ such that $d+\varepsilon\notin D$. Then, taking $\varepsilon=-x$, we have that $d-x\notin D$, and so we may take $y:=d-x$, so that $x+(y-d)=0\leq 0$ as desired.

STEP 9: SHOW THAT $\langle\mathbb{R},+,0,\leq\rangle$ IS A TOTALLY-ORDERED COMMUTATIVE GROUP

Exercise 1.4.2.22 Check that $D\leq E$ implies $D+F\leq E+F$.

STEP 10: DEFINE MULTIPLICATION

Multiplication is more complicated. For example, the product $0\cdot 0$ *should* be $0\cdot 0=0$; however, the set product, $00:=\{de:d\in 0,e\in 0\}$, isn't even bounded above. We have to break down the definition into cases. To simplify things, let us temporarily use the notation

$$D_0^+:=\{d\in D:0\leq d\}\,. \tag{1.4.2.23}$$

Here, $\cdot$ will denote multiplication in $\mathbb{R}$ and juxtaposition will denote set multiplication. We define

$$D \cdot E := \begin{cases} D_0^+ E_0^+ \cup (-\infty, 0] & \text{if } 0 \leq D, E \\ -((-D) \cdot E) & \text{if } D \leq 0, 0 \leq E \\ -(D \cdot (-E)) & \text{if } 0 \leq D, E \leq 0 \\ (-D) \cdot (-E) & \text{if } D, E \leq 0. \end{cases} \tag{1.4.2.24}$$

Exercise 1.4.2.25 Check that $D \cdot E$ is in fact a Dedekind-cut.

From the definition (1.4.2.24), it suffices to show associativity and commutativity for the case $0 \leq D, E$. Then,

$$\begin{aligned} D \cdot (E \cdot F) = D \cdot \left(E_0^+ F_0^+ \cup 0\right) &= D_0^+ E_0^+ F_0^+ \cup 0 \\ &= (D \cdot E) \cdot F, \end{aligned} \tag{1.4.2.26}$$

and similarly for commutativity.

STEP 11: DEFINE THE MULTIPLICATIVE IDENTITY

The cut

$$1 := \{x \in \mathbb{Q} : x \leq 1\} \tag{1.4.2.27}$$

functions as a multiplicative identity.

Exercise 1.4.2.28 Check that 1 is a multiplicative identity.

STEP 12: SHOW THAT THE ADDITIVE INVERSE DISTRIBUTES

We wish to show that

$$-(D + E) = (-D) + (-E). \tag{1.4.2.29}$$

On one hand we have

$$-(D+E) \coloneqq \{x-y : x \leq 0 \text{ and } y \notin D+E\} . \quad (1.4.2.30)$$

On the other hand,

$$\begin{aligned}(-D)+(-E) &\coloneqq \{a+b : a \in -D, b \in -E\} \\ &\coloneqq \{(x_1-y_1)+(x_2-y_2) : \\ &\qquad x_1, x_2 \leq 0;\ y_1 \notin D; y_2 \notin E\} \\ &= \{(x_1+x_2)-(y_1+y_2) : \qquad (1.4.2.31) \\ &\qquad x_1, x_2 \leq 0;\ y_1 \notin D;\ y_2 \notin E\} \\ &= \{x-(y_1+y_2) : \\ &\qquad x \leq 0\,, y_1 \notin D,\ y_2 \notin E\} .\end{aligned}$$

Comparing this with (1.4.2.30) above, we see that it suffices to show that $y \notin D+E$ iff $y = y_1+y_2$ for $y_1 \notin D$ and $y_2 \notin E$.

To show this, let $y_1 \in D^{\mathrm{C}}, y_2 \in E^{\mathrm{C}}$ and suppose that $y_1+y_2 \in D+E$, so that $y_1+y_2 = d+e$ for $d \in D, e \in E$. Then, $y_2 = e+(d-y_1) < e$, which implies that $y_2 \in E$: a contradiction. Conversely, let $y \notin D+E$. Let $\varepsilon > 0$ and choose $d \in D$ such that $d+\varepsilon \notin D$. Let $M \geq 2\varepsilon$ be such that $y-M \in D+E$ but $y-(M-\varepsilon) \notin D+E$. Write $y-M = d'+e'$ for $d' \in D, e' \in E$. Without loss of generality, assume that $d' \leq d$.[c] Then,

$$y-M = d'+e' = d+(e'-(d-d')) . \quad (1.4.2.32)$$

As $d-d' \geq 0$, $e \coloneqq e'-(d-d') \in E$. Of course,

$$y-(M-\varepsilon) = d+(e+\varepsilon), \quad (1.4.2.33)$$

and so, as $y-(M-\varepsilon) \notin D+E$, it must be the case that $e+\varepsilon \notin E$ (or else we would have that $y-(M-\varepsilon)$ is the sum of an element of D and an element of E). Then,

$$y = (d+(M-\varepsilon))+(e+\varepsilon). \quad (1.4.2.34)$$

As $d + (M - \varepsilon) \geq d + \varepsilon \notin D$, it follows that $d + (M - \varepsilon) \notin D$, so that indeed $y = y_1 + y_2$ for $y_1 \notin D$ and $y_2 \notin E$.

Step 13: Show that $[E + F]_0^+ = E_0^+ + F_0^+$ for $0 \leq E, F$
If $e \in E, e \geq 0$ and $f \in F, f \geq 0$, then of course $e + f \in E + F, e + f \geq 0$. Conversely, let $x \in E + F, x \geq 0$ and write $x = e + f$ for $e \in E, f \in F$. As $x \geq 0$, we must have that either $e \geq 0$ or $f \geq 0$. Without loss of generality, assume the former. If $f \geq 0$, we are done, so instead suppose that $f < 0$. Then, $x = (e + f) + 0$. As $e + f < e, e + f \in E$, and of course, as $x \geq 0, e + f \geq 0$, so that indeed $x \in E_0^+ + \{0\} \subseteq E_0^+ + F_0^+$.

Step 14: Show that $\langle \mathbb{R}, +, 0, -, \cdot, 1 \rangle$ is a cring
All that remains to be shown is distributivity. Let $D, E, F \in \mathbb{R}$ and consider

$$D \cdot (E + F). \tag{1.4.2.35}$$

Let us first do the case with $0 \leq D, E, F$. Then, by the previous step,

$$\begin{aligned} D \cdot (E + F) &:= \left(D_0^+ [E + F]_0^+\right) \cup 0 \\ &= \left(D_0^+ (E_0^+ + F_0^+)\right) \cup 0 \\ &= \left(D_0^+ E_0^+ + D_0^+ F_0^+\right) \cup 0 \\ &= D_0^+ E_0^+ \cup 0 + D_0^+ F_0^+ \cup 0 \\ &=: D \cdot E + D \cdot F. \end{aligned} \tag{1.4.2.36}$$

Because additive inverses distribute and by the definition of multiplication, we may without loss of generality assume that $0 \leq D, E + F$. Thus, either $0 \leq E$ or $0 \leq F$. Without loss of generality assume the former. We have already done the case

$0 \leq F$, so let us instead assume that $F < 0$. Then,

$$\begin{aligned} &D \cdot E + D \cdot F \\ &\qquad = D \cdot ((E + F) + (-F)) + D \cdot F \\ &\qquad = D \cdot (E + F) + D \cdot (-F) + D \cdot F \qquad (1.4.2.37) \\ &\qquad =: D \cdot (E + F) + D \cdot F + (-(D \cdot F)) \\ &\qquad = D \cdot (E + F), \end{aligned}$$

where we have used distributivity of additive inverse in $D \cdot (-F) = D \cdot (-F + 0) = -D \cdot F$.

Step 15: Show that $\langle \mathbb{R}, +, 0, -, \cdot, 1 \rangle$ is a field

All that remains to be shown is the existence of multiplicative inverses. For $D \in \mathbb{R}$ not 0, we define

$$D^{-1} := \begin{cases} \left(\left\{ y^{-1} : y \in D^{\mathrm{C}} \right\}^{\mathrm{U}} \right)^{\mathrm{L}} & \text{if } D > 0 \\ -\left((-D)^{-1} \right) & \text{if } D < 0. \end{cases} \qquad (1.4.2.38)$$

Exercise 1.4.2.39 Check that D^{-1} is in fact a Dedekind-cut.

To finish this step, it suffices to prove that $D \cdot D^{-1} = 1$ for $D > 0$.

$$D \cdot D^{-1} = D_0^+ [D^{-1}]_0^+ \cup 0. \qquad (1.4.2.40)$$

We would like to show that this is equal to $1 := \{x \leq 1 : x \in \mathbb{Q}\}$. Let $\varepsilon > 0$ and choose $d \in D$, $d > 0$ such that $d + \varepsilon \notin D$. Thus, D is bounded above by $d + \varepsilon$ and D^{C} is bounded below by d. It follows that D^{-1} is bounded above by d^{-1}, so that $D_0^+ [D^{-1}]_0^+$ is bounded above by $(d + \varepsilon) d^{-1} = 1 + \varepsilon d^{-1}$. As ε is arbitrary (and d gets smaller as ε gets smaller), it follows that in fact $D_0^+ [D^{-1}]_0^+$ is bounded above by 1, which shows that $D \cdot D^{-1} \subseteq 1$. For the other inclusion, let $x \leq 1$. Let $\varepsilon > 0$

and choose $y \notin D$ so that $y - \varepsilon \in D$ and $y - \varepsilon > 0$. As $x \leq 1$, $d := x(y - \varepsilon) \in D$. Thus,

$$D \cdot D^{-1} \ni (x(y - \varepsilon))\, y^{-1} = x - \varepsilon x y^{-1}. \tag{1.4.2.41}$$

As this is true for all $\varepsilon > 0$, we must have that $x \in D \cdot D^{-1}$, which completes the proof of this step.

STEP 16: SHOW THAT $\langle \mathbb{R}, +, 0, -, \cdot, 1 \rangle$ IS A DEDEKIND-COMPLETE TOTALLY-ORDERED FIELD
All that remains to be shown is that $0 \leq D, E$ implies $0 \leq D \cdot E$. Of course, if $d \in D$, $d \geq 0$ and $e \in E$, $e \geq 0$, then $de \in D \cdot E$, so that $0 \leq D \cdot E$.

STEP 17: SHOW THAT $\mathbb{Q} \subseteq \mathbb{R}$
That $\mathbb{Q} \subseteq \mathbb{R}$ follows immediately from Theorem 1.3.7.

STEP 18: SHOW THAT EVERY NONZERO DEDEKIND-COMPLETE TOTALLY-ORDERED FIELD WHICH CONTAINS $\mathbb{Q}$ CONTAINS A UNIQUE COPY OF $\mathbb{R}$
Let R be a nonzero Dedekind-complete totally-ordered field which contains $\mathbb{Q}$. By Theorem 1.3.7, R contains a copy of $\mathbb{Q} \subseteq R$ (as well as $\mathbb{R}$ itself, $\mathbb{Q} \subseteq \mathbb{R}$.). Let $x \in X$. Thus, because both R and $\mathbb{R}$ contain $\mathbb{Q}$, abusing notation, we may consider

$$D_x := \{q \in \mathbb{Q} : q \leq x\} \tag{1.4.2.42}$$

both as a subset of R and $\mathbb{R}$.[d] With this abuse, we define $\phi \colon R \to \mathbb{R}$ by

$$\phi(x) := \sup(D_x), \tag{1.4.2.43}$$

where on the right-hand side, D_x is regarded as a subset of $\mathbb{R}$ and the supremum is taken in $\mathbb{R}$.

Exercise 1.4.2.44 Show that ϕ is an isomorphism of preordered fields.

STEP 19: SHOW THAT $\langle \mathbb{R}, +, 0, -, \cdot, 1\rangle$ IS UNIQUE UP TO UNIQUE ISOMORPHISM OF PREORDERED FIELDS

We leave this as an exercise.

Exercise 1.4.2.45 Prove this yourself.

STEP 20: SHOW THAT $\mathbb{R}$ IS THE 'SMALLEST' DEDEKIND-COMPLETE PARTIALLY-ORDERED SET WHICH CONTAINS $\mathbb{Q}$

Here, we are referring to the "Furthermore,..." part of the statement of the theorem. Of course, we have already shown that $\mathbb{R}$ is a Dedekind-complete partially-ordered set which contains $\mathbb{Q}$. Furthermore, if we can show (ii), then the same argument as before will show uniqueness up to unique isomorphism of preordered sets. So, let R be some other Dedekind-complete partially-ordered set such that $\mathbb{Q} \subseteq R$, that is, for which there is a unique embedding $\iota\colon \mathbb{Q} \to R$ of preordered sets (i.e. an injective nondecreasing map such that $\iota(q_1) \leq \iota(q_2)$ iff $q_1 \leq q_2$—see Exercise B.2.18). Similarly as before, let us identity the unique copy of $\mathbb{Q}$ in $\mathbb{R}$ and the unique copy of $\mathbb{Q}$ in R, so that we may define $i\colon \mathbb{R} \to R$ by

$$i(x) := \sup\left\{q \in \mathbb{Q} \subseteq R : q \leq x \in \mathbb{R}\right\}. \qquad (1.4.2.46)$$

Exercise 1.4.2.47 Show that indeed i is indeed an embedding of preordered sets.

This shows that R 'contains a copy' of $\mathbb{R}$ (as a preordered set, as opposed to as a preordered field as before).

Exercise 1.4.2.48 Show that $i \colon \mathbb{R} \to R$ is the 'unique copy' of $\mathbb{R}$ contained in R, using the proof of Step 18 as guidance.

■

[a]If ever you're wondering how to come up with these definitions, just think of what the answer should be for sets of the form (1.4.2.1).

[b]Of course this is abuse of notation. It should not cause any confusion as one is a subset of $\mathbb{Q}$ and the other is an element of $\mathbb{Q}$.

[c]While it's perhaps not quite so clear a priori why this has no loss of generality, if you look at the following computation, you will see that you can swap their roles if instead we had $d \leq d'$.

[d]If you want to be pedantic about things, there is a subfield $Q \subseteq R$ together with an isomorphism of preordered fields $\psi \colon Q \to \mathbb{Q}$, where $\mathbb{Q} \subseteq \mathbb{R}$.

As was mentioned in a remark of the previous theorem, in fact, $\mathbb{R}$ is the *only*[8] nonzero Dedekind-complete totally-ordered field.

Theorem 1.4.2.49. $\mathbb{R}$ is the unique nonzero Dedekind-complete totally-ordered field.

R You really should say *Dedekind*-complete here, instead of just "complete", as is quite common. The reason is that there is (at least) one other notion of completeness, namely Cauchy-completeness (Definition 4.4.7), and they are most definitely not the same thing. In particular, this statement is *false* if you replace "Dedekind-complete" with "Cauchy-complete"—see Example 4.4.2.20.

Proof. Let R be another nonzero Dedekind-complete totally-ordered field. By Theorem 1.3.7, R contains a copy of $\mathbb{Q} \subseteq R$. Let $x \in X$. Thus, because both R and $\mathbb{R}$ contain $\mathbb{Q}$, abusing

[8]Once again, not just the *smallest*, like with the integers and rationals, but the *only*.

notation, we may consider

$$D_x := \{q \in \mathbb{Q} : q \leq x\} \tag{1.4.2.50}$$

both as a subset of R and $\mathbb{R}$.[a] With this abuse, we define $\phi\colon R \to \mathbb{R}$ by

$$\phi(x) := \sup(D_x), \tag{1.4.2.51}$$

where on the right-hand side, D_x is regarded as a subset of $\mathbb{R}$ and the supremum is taken in $\mathbb{R}$.

Exercise 1.4.2.52 Show that ϕ is an isomorphism of preordered fields.

■

[a]If you want to be pedantic about things, there is a subfield $Q \subseteq R$ together with an isomorphism of totally-ordered fields $\psi : Q \to \mathbb{Q}$, where $\mathbb{Q} \subseteq \mathbb{R}$.

Once again, as with the the natural numbers (Proposition 1.1.4.10), the integers (Proposition 1.2.23), and the rationals Proposition 1.3.8, we would like to know that this agrees with our naive idea of what the real numbers are. For better or for worse,[9] your naive idea of a real number is probably in terms of decimal expansions. We don't yet have the tools to prove the analogous result for the reals, however, and so we postpone this until Decimal expansions in Subsection 2.4.4 Series.

Definition 1.4.2.53 — Irrational numbers An element $x \in \mathbb{R}$ is ***irrational*** iff $x \notin \mathbb{Q}$.

R In fact, it will be a little while before we can show that $\mathbb{Q}^{\mathsf{C}}$ is even nonempty. For example, it is easy to show that there is no rational number whose square is 2—but can you show that there *is* a *real* number whose

[9]No, definitely for worse.

square is 2? (See the subsubsection Square-roots in Subsection 2.4.3 Cauchyness and completeness.)

Exercise 1.4.2.54 Let $A, B \subseteq \mathbb{R}$ be nonempty and bounded above.

(i). Show that $\sup(A + B) = \sup(A) + \sup(B)$.
(ii). Is it necessarily the case that $\sup(AB) = \sup(A)\sup(B)$?
(iii). Show that $-\sup(A) = \inf(-A)$

R To clarify,

$$A+B := \{a + b : a \in A,\ b \in B\} \text{ and } AB := \{ab : a \in A,\ b \in B\} .$$

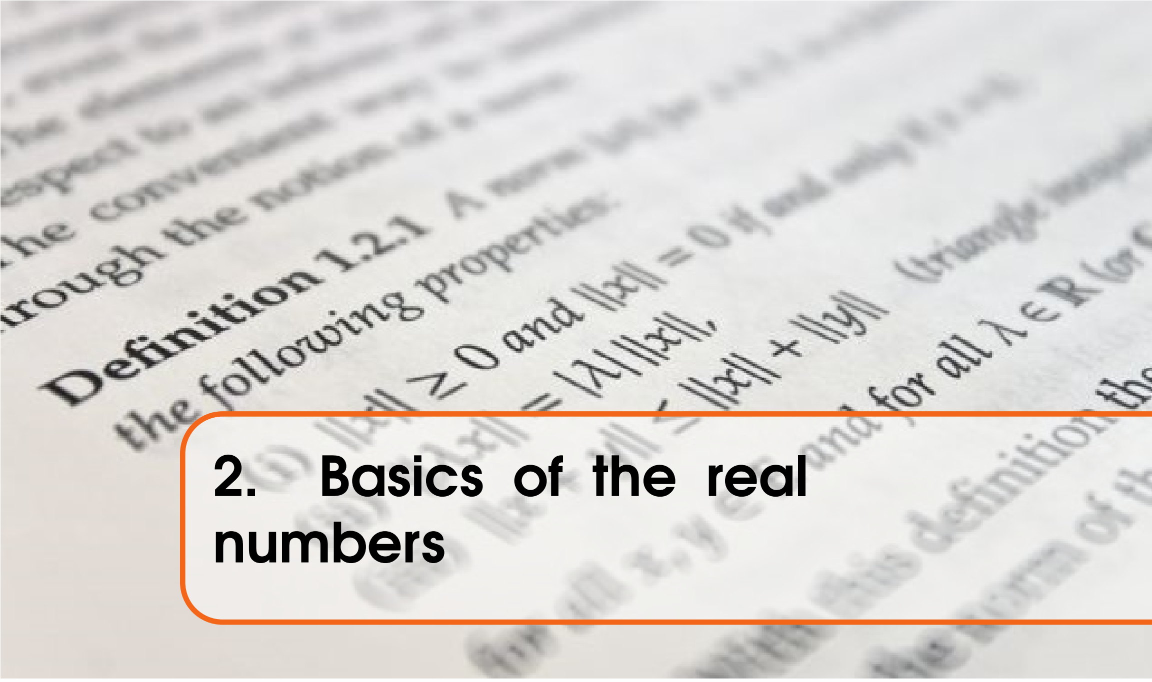
1.5 Concluding remarks

There are a couple of themes that we started to see in this chapter that you should pay particular attention to.

The first theme is that, if ever we want to have a certain object that doesn't exist in the context in which we are working, simply just enlarge the context in which you are working. For example, when we wanted additive inverses but didn't have them, we went from $\mathbb{N}$ to $\mathbb{Z}$; when we wanted multiplicative inverses but didn't have them, we went from $\mathbb{Z}$ to $\mathbb{Q}$; and when we wanted limits but didn't have them, we went from $\mathbb{Q}$ to $\mathbb{R}$.

The second theme you should take note of is the fact that we almost always defined an object by the properties that uniquely specify it. The point is that it doesn't really matter at the end of the day whether $\mathbb{R}$ is a set of Dedekind-cuts or equivalence classes of Cauchy sequences; all that matters is that it is a nonzero Dedekind-complete totally-ordered field.

The third and final theme you should take note of (which is perhaps not quite as manifest as the other two) is that the morphisms matter just as much (if not more) than the objects themselves. For example, in light of the second theme, we cannot even make sense of the idea of an object being unique without first talking about the morphisms. More significantly is that if you keep an underlying set fixed, but change the relevant morphisms, you can get completely different objects. For example, if we consider $\mathbb{Q}$ and $\mathbb{Z}$ as crings, $\langle \mathbb{Q}, +, 0, \cdot, 1 \rangle$ and $\langle \mathbb{Z}, +, 0, \cdot, 1 \rangle$, $\mathbb{Q}$ and $\mathbb{Z}$ are totally different; on the other hand, if we forget the extra structure (or, to put it another way, change our morphisms from homomorphisms of rings to just ordinary functions) then $\mathbb{Q}$ and $\mathbb{Z}$ *become the same object*, that is, $\mathbb{Q} \ncong_{\mathbf{Ring}} \mathbb{Z}$ *but* $\mathbb{Q} \cong_{\mathbf{Set}} \mathbb{Z}$. The former is obvious (for example, 2 has an inverse in $\mathbb{Q}$ but not in $\mathbb{Z}$). As for the latter, we recommend to continue reading the following chapter. . .



2. Basics of the real numbers

First of all, let us recall our definition of the real numbers.

> There exists a nonzero Dedekind-complete totally-ordered field which is unique up to isomorphism of totally-ordered fields. This field is the field of real numbers. (2.1)

Recall that we also showed (Theorem 1.3.7) that $\mathbb{R}$ must contain $\mathbb{Q}$, and hence in turn $\mathbb{Z}$ and $\mathbb{N}$.

2.1 Cardinality and countability

This section is a bit of an aside—while not on the real numbers per se, a knowledge of cardinality and countability is absolutely essential to an understanding of the real numbers.

In the first chapter, we briefly discussed the notion of cardinality and its relationship to the natural numbers. In fact, the natural numbers themselves are, as a set, precisely the finite cardinals.

The first fact we point out is that the cardinality of the natural numbers is the smallest infinite cardinal.

Proposition 2.1.1 Let κ be an infinite cardinal. Then, $|\mathbb{N}| \leq \kappa$.

R Phrased differently ,note that the contrapositive easily implies the following.

> If κ is a cardinal with $\kappa \leq |\mathbb{N}|$, then either $\kappa = |\mathbb{N}|$ or κ is finite.

Proof. Let K be any set such that $|K| = \kappa$. Recall that (Definition 1.1.3.1) to show that $|\mathbb{N}| \leq \kappa$ requires that we produce an injection from $\mathbb{N}$ into K. We construct an injection $f: \mathbb{N} \to K$ inductively. Let $k_0 \in K$ be arbitrary and let us define $f(0) := k_0$. If $K \setminus \{k_0\}$ were empty, then K would not be infinite, therefore there must be some $k_1 \in K$ distinct from k_0, so that we may define $f(1) := k_1$. Continuing this process, suppose we have defined f on $0, \ldots, n \in \mathbb{N}$, and wish to define $f(n+1)$. If $K \setminus \{f(0), \ldots, f(n)\}$ were empty, then K would be finite. Thus, there must be some $k_{n+1} \in K$ distinct from $f(0), \ldots, f(n)$. We may then define $f(n+1) := k_{n+1}$. The function produced must be injective because, by construction, $f(m)$ is distinct from $f(n)$ for all $n < m$. Hence, $|\mathbb{N}| \leq \kappa$. ■

Thus, the cardinality of the natural numbers is the smallest infinite cardinality. We give a name to this cardinality.

Definition 2.1.2 — Countability Let X be a set. Then, X is ***countably-infinite*** iff $|X| = |\mathbb{N}|$. X is ***countable*** iff either X is countably-infinite or X is finite. We write $\aleph_0 := |\mathbb{N}|$.

R It is not uncommon for people to use the term "countable" to mean what we call "countably-infinite". They would would then just say "countable or finite" in cases that we would say "countable".

Our first order of business is to decide what other sets besides the natural numbers are countably-infinite.

Proposition 2.1.3 The even natural numbers, $2\mathbb{N}$, are countably-infinite.

Proof. On one hand, $2\mathbb{N} \subseteq \mathbb{N}$, so that $|2\mathbb{N}| \leq \aleph_0$. On the other hand, $2\mathbb{N}$ is infinite, and as we just showed that $\aleph_0$ is the smallest infinite cardinal, we must have that $\aleph_0 \leq |2\mathbb{N}|$. Therefore, by antisymmetry (Bernstein-Cantor-Schröder Theorem, Theorem 1.1.3.5) of $\leq$, we have that $|2\mathbb{N}| = \aleph_0$. ■

Exercise 2.1.4 Construct an explicit bijection from $\mathbb{N}$ to $2\mathbb{N}$.

This is the first explicit example we have seen of some perhaps not-so-intuitive behavior of cardinality. On one hand, our intuition might tell us that there are half as many even natural numbers as there are natural numbers, yet, on the other hand, we have just proven (in two different ways, if you did the exercise) that $2\mathbb{N}$ and $\mathbb{N}$ have the same number of elements! This of course is not the only example of this sort of weird behavior. The next exercise shows that this is actually quite general.

Exercise 2.1.5 Let X and Y be countably-infinite sets. Show that $X \sqcup Y$ is countably-infinite.

R Note that this generalizes—see Exercise 2.1.7.

Thus, it is literally the case that $2\aleph_0 = \aleph_0$. A simple corollary of this is that $\mathbb{Z}$ is countably-infinite.

Exercise 2.1.6 Show that $|\mathbb{Z}| = \aleph_0$.

You (hopefully) just showed that $2\aleph_0 = \aleph_0$, but what about $\aleph_0^2$?

Exercise 2.1.7 Let $\mathscr{X}$ be a countable indexed collection of countable sets. Show that

$$\bigsqcup_{X \in \mathscr{X}} X \tag{2.1.8}$$

is countable.

Proposition 2.1.9 $\aleph_0^2 = \aleph_0$.

Proof. For $m \in \mathbb{N}$, define

$$X_m := \{\langle i, j\rangle \in \mathbb{N} \times \mathbb{N} : i + j = m\} \tag{2.1.10}$$

Note that each X_m is finite and also that

$$\mathbb{N} \times \mathbb{N} = \bigsqcup_{m \in \mathbb{N}} X_m. \tag{2.1.11}$$

Therefore, by the previous exercise, $|\mathbb{N} \times \mathbb{N}| =: \aleph_0^2$ is countable, i.e., $\aleph_0^2 \leq \aleph_0$. As $\aleph_0$ is not finite, we must thus have that $\aleph_0^2 = \aleph_0$ (Proposition 2.1.1). ■

Exercise 2.1.12 Use Bernstein-Cantor-Schröder and the previous proposition to show that $|\mathbb{Q}| = \aleph_0$.

This result might seem a bit crazy at first. I mean, just ‘look’ at the number line, right? There’s like bajillions more rationals than naturals. Surely it can’t be the case there there are no more rationals than natural numbers, can it? Well, yes, in fact that can be, and in fact is, precisely the case—despite what your silly intuition might be telling you, there are no more rational numbers than there are natural numbers.

We mentioned briefly before the algebraic numbers in the remark right before Subsection 1.4.2 Dedekind-cuts and the real numbers. We go ahead and define them now because they provide a good exercise in cardinality.

Definition 2.1.13 — Real algebraic numbers A real number $\alpha \in \mathbb{R}$ is ***algebraic*** iff there exists a nonzero polynomial with integer coefficients p such that $p(\alpha) = 0$. We write $\mathbb{A}_{\mathbb{R}}$ for the set of real algebraic numbers.

R Every rational number is real algebraic: if $q = \frac{m}{n} \in \mathbb{Q}$, then q is a root of the polynomial $nx - m$. On the other hand, there are real algebraic numbers that are not rational, for example, $\sqrt{2} \in \mathbb{R}$ is a root of $x^2 - 2$.[a]

R The algebraic numbers (denoted simply by $\mathbb{A}$) are those elements in $\mathbb{C}$ which are roots of a nonzero polynomial with integer coefficients. The real algebraic numbers are then those elements of $\mathbb{C}$ which are both real numbers and algebraic numbers. We haven't actually defined $\mathbb{C}$, which is why we only define the real algebraic numbers.

R In fact, both $\mathbb{A}$ and $\mathbb{A}_{\mathbb{R}}$ are fields, though this is not so easy to see. For example, can you even show that the sum of two algebraic numbers is again algebraic?

[a] I know technically we have not yet defined $\sqrt{2}$, though as I only intend you to convince you (as opposed to prove it for you) that there are algebraic numbers which are not rational, this is not an issue.

Exercise 2.1.14 Show that $\mathbb{A}_{\mathbb{R}}$ is countable.

So, we've now done both $\mathbb{Z}$ and $\mathbb{Q}$, but what about $\mathbb{R}$? At first, you might have declared it obvious that there are more real numbers than natural numbers, but perhaps the result about $\mathbb{Q}$ has now given you some doubt. In fact, it *does* turn out that there are more real numbers than there are natural numbers.

Theorem 2.1.15 — Cantor's Cardinality Theorem. Let X be a set. Then, $|X| < |2^X|$.

R There is a good chance you may have heard of the term ***Cantor's Diagonal Argument***. The argument here is a generalization of that (it's also 'cleaner'), and so we don't present the Diagonal Argument itself.

Proof. We must show two things: (i) $|X| \leq |2^X|$ and (ii) $|X| \neq |2^X|$.

The first, by definition, requires that we construct an injection from X to 2^X. This, however, is quite easy. We may define a function $X \to 2^X$ by $x \mapsto \{x\}$. This is of course an injection.

The harder part is showing that $|X| \neq |2^X|$. To show this, we must show that there is *no* surjection from X to 2^X. So, let $f: X \to 2^X$ be a function. We show that f cannot be surjective. To do this, we construct a subset of X that cannot be in the image of f.

We define

$$S := \{x \in X : x \notin f(x)\} . \tag{2.1.16}$$

We would like to show that S is not in the image of f. We proceed by contradiction: suppose that $S = f(x_0)$ for some $x_0 \in X$. Now, we must have that either $x_0 \in S$ or $x_0 \notin S$. If the former were true, then we would have that $x_0 \notin f(x_0) = S$: a contradiction. On the other hand, in the latter case, we would have $x \in f(x_0) = S$: a contradiction. Thus, as neither of these possibilities can be true, there cannot be any $x_0 \in X$ such that $f(x_0) = S$. Thus, S is not in the image of f, and so f is not surjective. ■

Now, to show that $|\mathbb{R}| > \aleph_0$, we show that $|\mathbb{R}| = 2^{\aleph_0}$. Before we do this, however, we must know a little more about the real numbers, and so we shall return to this later in the chapter—see The uncountability of the real numbers.

2.2 The absolute value

$\mathbb{R}$ already has a decent amount of structure: both an order and a field structure. We now equip it with yet another structure which will make arguments easier to work with and more intuitive (though as the entire definition involves only the order and algebraic structure, in principle we could do without it).

Definition 2.2.1 — Absolute value The ***absolute value*** of $x \in \mathbb{R}$, denoted by $|x|$, is defined by

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0. \end{cases} \tag{2.2.2}$$

For $\varepsilon > 0$ and $x_0 \in \mathbb{R}$, we write

$$\begin{aligned} B_\varepsilon(x_0) &:= \{x \in \mathbb{R} : |x - x_0| < \varepsilon\} \\ D_\varepsilon(x_0) &:= \{x \in \mathbb{R} : |x - x_0| \leq \varepsilon\}\,. \end{aligned} \tag{2.2.3}$$

R $B_\varepsilon(x_0)$ is the ***ball*** of radius ε centered at x_0 and $D_\varepsilon(x_0)$ is the ***disk*** of radius ε centered at x_0.

Exercise 2.2.4 Let $x_1, x_2 \in \mathbb{R}$. Show that the following statements are true.

(i). (Nonnegativity) $|x_1| \geq 0$.
(ii). (Definiteness) $|x_1| = 0$ iff $x_1 = 0$.
(iii). (Homogeneity) $|x_1 x_2| = |x_1||x_2|$.
(iv). (Triangle Inequality) $|x_1 + x_2| \leq |x_1| + |x_2|$.
(v). (Reverse Triangle Inequality) $||x_1| - |x_2|| \leq |x_1 - x_2|$.

R The reason that (iv) is called the *triangle inequality* is the following. First of all, by (iii), the triangle inequality can instead be written as $|x_1 - x_2| \leq |x_1| + |x_2|$. Then, if you pretend that x_1 and x_2 are vectors representing the sides of the triangle, $x_1 - x_2$ is a vector representing the third side of the triangle. The triangle inequality then states that the length of a side of a triangle is at most the sum of the lengths of the other two sides (being equal iff the angle between those two sides is precisely π). Your solution should help explain why the reverse triangle inequality is called what it is.

Intuitively, of course, $|x|$ is supposed to be the distance x is from 0. Then, $|x - y|$ is supposed to be the distance between x and y.

A simple result we have is that, if the distance between two integers is less than 1, then they are the same integer.

Proposition 2.2.5 Let $0 < \varepsilon < 1$ and let $m, n \in \mathbb{Z}$. Then, if $|m - n| < \varepsilon$, then $m = n$.

Proof. Suppose that $|m - n| < \varepsilon$. Without loss of generality, suppose that $m \leq n$, so that we can write $n = m + k$ for $k \in \mathbb{Z}_0^+$. Then, $|m - n| = k < \varepsilon < 1$. It follows from Exercise 1.2.25 that $k = 0$, so that $m = n$. ∎

2.3 The Archimedean Property

The first property of the real numbers we come to is called the Archimedean Property. The Archimedean Property essentially says that the natural numbers are unbounded.

Definition 2.3.1 — The Archimedean Property Let F be a nonzero totally-ordered field so that $\mathbb{Q} \subseteq F$ (see Theorem 1.3.7). Then, we say that F is ***Archimedean*** iff for all $x \in F$ there is some $m \in \mathbb{N} \subseteq F$ such that $x < m$.

R Note the use of *Archimedean field* in contrast with *Dedekind-complete*: one is capitalized and hte other is not. To be honest, while this convention doesn't make the most sense to me, it seems most standard to write people's names, even if not being used to refer to that person, with an upper-case letter, but to *not* use upper-case for terms derived from people's names.

Exercise 2.3.2 Let F be an Archimedean totally-ordered field and let $\varepsilon > 0$. Show that there is some $m \in \mathbb{N} \subseteq F$ such that $\frac{1}{m} < \varepsilon$.

Theorem 2.3.3 — Archimedean property of the real numbers. $\mathbb{R}$ is Archimedean.

Proof. If $x \leq 0$, we may take $m = 1$. Otherwise, $x > 0$, and so the set

$$S := \{m \in \mathbb{N} : m < x\} \tag{2.3.4}$$

is nonempty (it contains 0). On the other hand, it is also bounded-above (by x), and so by the least upper-bound property, it has a supremum: write $m_0 := \sup(S)$. We first show that $m_0 \in \mathbb{N}$, and then we will show that $x < m_0 + 2$.

By Proposition 1.4.1.13, there must be some $m_1 \in S$ such that

$$m_0 - \tfrac{1}{2} < m_1 \leq m_0. \tag{2.3.5}$$

If $m_1 = m_0$, we are done showing that $m_0 \in \mathbb{N}$, so suppose that $m_1 < m_0$. Then, we may use this same proposition again to obtain an $m_2 \in S$ such that

$$m_1 < m_2 \leq m_0. \tag{2.3.6}$$

But then $|m_1 - m_2| < \frac{1}{2}$ (because they are both strictly between $m_0 - \frac{1}{2}$ and m_0), and so $m_1 = m_2$ (by Proposition 2.2.5): a contradiction (of the fact that $m_1 < m_2$). hence, it must have been the case that $m_0 = m_1 \in \mathbb{N}$.

We cannot have that $m_0 + 1 \in S$ because then otherwise m_0 would not be an upper-bound of S. Therefore, $m_0 + 1 \in \mathbb{N} \setminus S = \{m \in \mathbb{N} : x \leq m\}$, and so $x \leq m_0 + 1$, and so $x < m_0 + 2$. ∎

Our first application of the Archimedean Property is that it allows us to define the *floor* and *ceiling* functions.

Proposition 2.3.7 — Floor and ceiling Let $x \in \mathbb{R}$.

- There is a unique integer $\lfloor x \rfloor$, the ***floor*** of x, such that

(i). $\lfloor x \rfloor \leq x$; and
(ii). if $\mathbb{Z} \ni m \leq x$, then $m \leq \lfloor x \rfloor$.

- There is a unique integer $\lceil x \rceil$, , the ***ceiling*** of x, such that

(i). $\lceil x \rceil \geq x$; and
(ii). if $\mathbb{Z} \ni m \geq x$, then $m \geq \lceil x \rceil$.

Proof. The set $\{m \in \mathbb{Z} : m \geq x\}$ is nonempty by the Archimedean Property. Then, as subsets of $\mathbb{Z}$ bounded below are well-ordered, it has a smallest element m_0. Define $\lfloor x \rfloor := m_0 - 1$.

Exercise 2.3.8 Show that $\lfloor x \rfloor$ actually has the properties claimed.

Exercise 2.3.9 Show that $\lfloor x \rfloor$ is the unique integer with these properties.

Exercise 2.3.10 Prove the analogous result for $\lceil x \rceil$.

■

It might seem like the Archimedean Property is obviously true, but it is in fact not true of all totally-ordered fields. To present such an example, we have to take a bit of an aside and discuss polynomial crings and fields of rational functions. Such crings and fields are important for many other reasons than to present an example of a totally-ordered field that does not have the Archimedean Property, so it is worthwhile to understand the material anyways, though you might want to come back to it later if you don't care about the counter-example at the time being.

Totally-ordered field which is not Archimedean

Definition 2.3.11 — Polynomial cring Let R be a totally-ordered cring, denote by $R[x]$ the set of all polynomials with coefficients in R, let $+$ and $\cdot$ on $R[x]$ be addition and multiplication of polynomials, and define $p > 0$ iff the leading coefficient of p is greater than 0 in R.[a]

Exercise 2.3.12 Show that $\langle R[x], +, 0, -, \cdot, 1, \leq \rangle$ is a totally-ordered cring. What is $0 \in R[x]$? What is $1 \in R[x]$?

$R[x]$ is the ***polynomial cring*** with coefficients in R. The ***degree*** of a polynomial p, denoted by $\deg(p)$, is the highest power of x that appears in p with nonzero coefficient.

[a] Recall that this is enough to define a total-order by Exercise 1.1.4.4.

Exercise 2.3.13 Show that the following statements are true.

(i). $\deg(p+q) \leq \max\{\deg(p), \deg(q)\}$.
(ii). $\deg(pq) \leq \deg(p) + \deg(q)$.
(iii). If R is integral, then $\deg(pq) = \deg(p) + \deg(q)$.

Exercise 2.3.14 Show that R is integral iff $R[x]$ is.

Theorem 2.3.15 — Field of rational functions. Let F be a field, so that $F[x]$ is a totally-ordered integral cring. Then, there exists unique a totally-ordered field $F(x)$, the field of ***rational functions*** with coefficients in F, such that

(i). $F[x] \subseteq F(x)$; and
(ii). if F' is any other totally-ordered field such that $F[x] \subseteq F$, then $F(x) \subseteq F'$.

R Compare this to our definition of the rational numbers in Theorem 1.3.4. $F(x)$ is to $F[x]$ as $\mathbb{Q}$ is to $\mathbb{Z}$. Indeed, we mentioned there that the passage from $\mathbb{Z}$ to $\mathbb{Q}$

was an example of the more general construction of the *fraction field* of an integral cring. This is another example of this construction. In fact, the proof of this theorem is exactly the same as the proof of Theorem 1.3.4, and so we refrain from presenting it again (it was an exercise anyways).

Of course, we have a result from $F(x)$ which is completely analogous to the result Proposition 1.3.8 for $\mathbb{Q}$ (which just says that we can write any rational function uniquely as the quotient of two relatively-prime polynomials with the denominator positive).

▪ **Example 2.3.16 — A non-Archimedean totally-ordered field** $\mathbb{R}(x)$ is a totally-ordered field, but on the other hand, $m < x$ for all $m \in \mathbb{N}$, and so $\mathbb{R}(x)$ is not Archimedean.

We mentioned in a remark when defining the real numbers (Theorem 1.4.2.9) that, in general, the arithmetic operations on a partially-ordered field do not extend to its Dedekind-MacNeille completion in a compatible way. That $\mathbb{R}(x)$ is not Archimedean immediately tells us that this field is also a counter-example to this statement.

▪ **Example 2.3.17 — A totally-ordered field whose Dedekind-MacNeille completion cannot be given the structure of a totally-ordered field** If the arithmetic operations on $\mathbb{R}(x)$ extended to give the structure of a totally-ordered field on its Dedekind-MacNeille completion, then, by uniqueness, its Dedekind-MacNeille completion would have to be $\mathbb{R}$ itself. In particular, $\mathbb{R}(x)$ would have to embed into $\mathbb{R}$, which would imply that $\mathbb{R}(x)$ is Archimedean: a contradiction.

R We noted before in a remark of Theorem 1.4.2.49 that, while they real numbers are the unique nonzero *Dedekind*-complete totally-ordered field, there are other nonzero *Cauchy*-complete totally-ordered fields.[a] $\mathbb{R}(x)$ (or rather, it's Cauchy-completion) will serve as a counter-example for this as well—see Example 4.4.2.20.

[a] You aren't supposed to know what this means yet of course. This doesn't come until Definition 4.4.7.

The second property of the real numbers we come to is what is sometimes called the density of the rationals in the reals. This is not quite literally true, though we will see why people refer to this as density when we discuss basic topology in the next chapter—see Exercise 3.2.1.2.

Theorem 2.3.18 — 'Density' of $\mathbb{Q}$ in $\mathbb{R}$. Let $a, b \in \mathbb{R}$. Then, if $a < b$, then there exists $c \in \mathbb{Q}$ such that $c \in (a, b)$.

> **R** It turns out that this is also the case for the irrational numbers, $\mathbb{Q}^{\mathrm{C}}$, but at the moment, we don't even know how to construct a single irrational number, so we will have to return to this at a later date (Theorem 2.4.3.64).[a]

[a] You might be able to show that there is no positive rational number whose square is 2, but can you show that there *is* some positive *real* number whose square is 2?

Proof. The intuitive idea is to take a positive integer m at least as large as the length of the interval $b - a$ and break-up the real line into intervals of length $\frac{1}{m}$. Then, one of the end-points of these intervals must lie in the interval (a, b). This will be our desired point.

Define $\varepsilon := b - a > 0$ and choose $m_0 \in \mathbb{N}$ with $\frac{1}{m_0} < \varepsilon$ (see Exercise 2.3.2). Define

$$S := \left\{ m \in \mathbb{N} : \frac{m}{m_0} \geq b \right\}. \tag{2.3.19}$$

By the Archimedean Property, S is nonempty. Therefore, because $\mathbb{N}$ is well-ordered, it has a least element, say $\frac{n_0}{m_0}$. We claim that $\frac{n_0 - 1}{m_0} \in (a, b)$.

First of all, we know that $n_0 - 1 \notin S$, and so $\frac{n_0-1}{m_0} < b$. On the other hand,

$$\frac{n_0-1}{m_0} = \frac{n_0}{m_0} - \frac{1}{m_0} \geq b - \frac{1}{m_0} > b - \varepsilon = b - (b-a) = a, \quad (2.3.20)$$

and so $\frac{n_0-1}{m_0} \in (a,b)$. ■

2.4 Nets, sequences, and limits

2.4.1 Nets and sequences

You probably recall sequences from calculus.

$$\text{A (real-valued) } \textit{sequence} \text{ is a function from } \mathbb{N} \text{ to } \mathbb{R}. \quad (2.4.1.1)$$

(This is not our 'official' definition of a sequence. We will present another definition later. This is why the above has no 'definition bar'.)

A net is a generalization of a sequence. It is incredibly uncommon to introduce nets in a first course on analysis (or even in second course), but we have a couple reasons for doing so. One point of introducing them to put emphasis on the *structure* $\mathbb{N}$ is equipped with in this context. In particular, the $\mathbb{N}$ in the definition of a sequence is *not* to be thought of as a crig; for the purposes of sequences, the algebraic structure does not matter. Instead, in the context of sequences, $\mathbb{N}$ should be thought of as a directed set. Another motivation, and almost certainly the more important one, for working with nets is that, when you go to generalize to examples more exotic than the real numbers, some of the results that are true in the reals would fail to generalize if we restricted ourselves to only work with sequences—see the remarks in Proposition 2.5.2.12 and Theorem 2.5.3.10. There is no getting around this: in general topology (the subject of the next chapter), you *need* to use nets.[1]

[1] I imagine that there will be quite a few readers who have already learned at least some general topology and have never encountered nets before. I would further imagine that at least some of these readers would be skeptical of my claim that you "need' nets. There are several things I could point to you to convince you of this, but one particularly important phenomenon is as follows: there are nonhomeomorphic

Definition 2.4.1.2 — Directed set A ***directed set*** is a nonempty partially-ordered set $\langle \Lambda, \leq \rangle$ such that, for every $x_1, x_2 \in X$, there is some $x_3 \in X$ with $x_1, x_2 \leq x_3$.

R In words, a partially-ordered set is directed iff you can find an upper-bound of any two elements.

R This property is sometimes called ***upward-directed***. Compare, for example, the terminology in Definitions 3.3.1 and 4.1.2.1.

Exercise 2.4.1.3 Show that totally-ordered sets are directed.

R In particular, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ all all directed, as well sets derived from these, e.g. $\mathbb{R}^+$.

Exercise 2.4.1.4 Let X be a set. Show that $\langle 2^X, \subseteq \rangle$ is directed.

▪ **Example 2.4.1.5 — A partially-ordered set that is not directed** Define $X := \{A, B\}$ and equip X with the discrete-order (Example A.3.3.7).

> **Exercise 2.4.1.6** You showed (hopefully anyways) in Example A.3.3.7 that X is a partially-ordered set. Check now that it is not directed.

Definition 2.4.1.7 — Net A (real-valued) ***net*** is a function from a directed set $\langle \Lambda, \leq \rangle$ into $\mathbb{R}$.

R Similarly as with sequences, if $x \colon \Lambda \to \mathbb{R}$ is a net, it is customary to write $x_\lambda := x(\lambda)$.

---

spaces with the same notion of sequential convergence—see Example 3.2.48. The significance of this is that, if you want to *define* a topology by saying what it means to converge (and you almost certainly will at some point if you plan to become a mathematician), you *cannot* use sequences—see the remark in Theorem 3.4.2.1.

R Single values of the sequence x_λ are referred to as ***elements*** or ***terms*** of the net $\lambda \mapsto x_\lambda$.

R It is very common that the specific directed set that is the domain is not so important and that the result works for any directed set. As a result, in an attempt to simplify notation slightly, we frequently omit the actual domain of nets. For example, we may just write $\lambda \mapsto x_\lambda \in \mathbb{R}$, with no mention of Λ. You should usually be able to infer the domain on the basis of the index we happen to be using.

R Note that nets are allowed to have only finitely many terms, that is, it is permitted that Λ itself be finite. This is not particularly interesting, however, as the directed set axiom implies that Λ has a maximum element, say λ_∞. In this case, the limit of any net $\Lambda \ni \lambda \mapsto x_\lambda$ will be x_{λ_∞}, and this case is not very useful—see Exercise 2.4.2.9.

R In general, nets will take their values in topological spaces. Of course, we haven't defined what a topological space is yet (we could, but it would probably seem like I just pulled some definition out of my ass), and so for the time being nets will take their values in $\mathbb{R}$.

R For those of you born and raised in Sequence Land: Nets are a bit like the iPhone—you don't think you need them, and then you find out they're a thing.

And now we give our 'official' definition of a sequence.

Definition 2.4.1.8 — Sequence A (real-valued) ***sequence*** is a net whose directed set is order-isomorphic (i.e. isomorphic in **Pre**) to $\langle \mathbb{N}, \leq \rangle$.

R Typically people take a sequence to be a function from $\mathbb{N}$ into $\mathbb{R}$. This is essentially fine, but then, technically speaking, a function from $\mathbb{Z}^+$ to $\mathbb{R}$ is not a sequence, and you have to reindex everything by 1 to make it a sequence. It is easier if we just ignore this by only requiring that the domain of the net be *isomorphic to* (in **Pre**) instead of *equal to* $\langle \mathbb{N}, \leq \rangle$.

R If the name of the sequence is not important, we may simply denote it by a list of its values, e.g. $\langle 7, -\frac{2}{3}, 2, 0, \ldots \rangle$.

Example 2.4.1.9 — A net which is not a sequence Recall that (Exercise A.3.3.11) $2^{\mathbb{R}}$, the power-set of the reals, is a partially-ordered set with the order relation being inclusion, and is directed by Exercise 2.4.1.4.

Now let $f\colon \mathbb{R} \to \mathbb{R}$ be any *bounded* function. That f is bounded means that

$$a_S := \sup\{|f(x)| : x \in S\} := \sup_{x \in S}\{|f(x)|\} \quad (2.4.1.10)$$

exists for all subsets $S \in 2^{\mathbb{R}}$. Therefore, the map $S \mapsto a_S$ is a net. Without knowing yet the precise definition of convergence, do you know what the limit should be?

2.4.2 Limits

We now define what it means to be the *limit* of a net. To make the definition as clean and concise as possible, we introduce the concept of nets *eventually* doing something.

Eventuality

We often use the word *eventually* in the context of nets and sequences. For example, we might say "The net $\lambda \mapsto x_\lambda$ is eventually XYZ.", "XYZ" be some sort of property. This means that there is some λ_0 such that if $\lambda \geq \lambda_0$ it follows that $\lambda \mapsto x_\lambda$ is XYZ. For example, a fact that will come in handy is that convergent nets (and in fact, Cauchy nets) are eventually bounded (Proposition 2.4.3.9). We make this formal.

Meta-definition 2.4.2.1 — Eventually XYZ Let $\Lambda \ni \lambda \mapsto x_\lambda \in \mathbb{R}$ be a net. Then, $\lambda \mapsto x_\lambda$ is ***eventually XYZ*** iff there is some λ_0 such that $\{\lambda \in \Lambda : \lambda \geq \lambda_0\} \ni \lambda \mapsto x_\lambda$ is XYZ.

R In other words, once you 'go past' λ_0, you have a net which is *actually* XYZ.

R For example, the sequence $\langle -2, -1, 0, 1, 2, \ldots \rangle$ is *eventually* positive, but of course not always positive.

R We don't have this language yet, but nets of this form are called *cofinal subnets* of the original net—see Definition 2.4.5.7.

If a net is eventually XYZ, it can be convenient to essentially just 'throw away' the terms at the beginning which are not XYZ so as to obtain a net which is not just eventually XYZ, but is XYZ *itself*. In the example above, $\langle -2, -1, 0, 1, 2, \ldots \rangle$, if you 'chop off' the beginning of the sequence, you obtain $\langle 1, 2, \ldots \rangle$, which of course itself is always positive.

For all intents and purposes, you should think of the 'first elements' of a net as not mattering; only what *eventually* happens is what matters. For example, consider the sequence $m \mapsto x_m := 1$, that is, the constant sequence $1 \in \mathbb{R}$. Obviously this should converge to 1. The point to note here is that, you can do *whatever you like* to any finite number of elements of this sequence, and you will have no effect upon the fact that it converges to 1. For example, $\langle -5, -2, 10, \frac{2}{3}, 1, 1, 1, \ldots \rangle$ should obviously still converge to 1. We can essentially 'throw away' any finite amount of the sequence and nothing will change. This idea can actually be quite important in proofs where we might know nothing about the first couple of elements.

There is another concept that is in a sense (Meta-proposition 2.4.2.3) 'complementary' to the concept of eventuality, namely, the concept of *frequently*.

Meta-definition 2.4.2.2 — Frequently XYZ Let $\Lambda \ni \lambda \mapsto x_\lambda \in \mathbb{R}$ be a net. Then, $\lambda \mapsto x_\lambda$ is ***frequently*** iff for every $\lambda \in \Lambda$ there is some $\lambda' \geq \lambda$ such that $x_{\lambda'}$ is XYZ.

R In other words, no matter 'how far you go', there will always still be at least one more term of the net that is XYZ.

R In particular, note that if a net is eventually XYZ, then it is frequently XYZ.

R Note that the properties applicable here in place of XYZ are *not* the same as they are for eventuality. The reason for this is that the definition of eventuality is reduced to the case of a certain '*subnet*'[a] having the property XYZ, whereas here the definition of frequently is reduced to the case of a certain *term* in the net having the property XYZ. Thus, for example, it doesn't make sense to say that a net is frequently bounded—the term "bounded" is really meant to be applied to *nets* (or subsets of $\mathbb{R}$), not *terms*.[b]

R Of the two, "eventually XYZ" is *by far* the more important. Indeed, one of our motivations for introducing the terminology "frequently XYZ" is that it appears in other sources, and so will be convenient for you to know when consulting other references. We ourselves make use of the term quite minimally. That said, there are at least two uses worth mentioning: being "not eventually XYZ" can be understood in terms of being "frequently XYZ" (Meta-proposition 2.4.2.3); and if a net is "frequently XYZ", then the terms which are actually XYZ define a *cofinal subnet* (Definition 2.4.5.7 and Meta-proposition 2.4.5.14).

[a]Whatever that mean (Definition 2.4.5.1).

[b]I suppose you could make sense of this, but then, as $\{x_\lambda\}$ is always bounded (it's a singleton), you would have that every net is frequently bounded, which is a bit silly.

The key relationship between these two concepts is given by the following result.

Meta-proposition 2.4.2.3 Let XYZ be a property that is such that a net $\lambda \mapsto x_\lambda$ is XYZ iff each x_λ is XYZ. Then, a net $\lambda \mapsto x_\lambda \in \mathbb{R}$ be a net is frequently not XYZ iff it is not eventually XYZ.

R In particular, for $S \subseteq \mathbb{R}$, a net is frequently contained in S iff it is not eventually contained in S^{C}.

Proof. $\Lambda \ni \lambda \mapsto x_\lambda$ is frequently not XYZ iff for every λ there is some $\lambda' \geq \lambda$ such that $x_{\lambda'}$ is not XYZ iff for every λ it is not the case that for every $\lambda' \geq \lambda$ that $x_{\lambda'}$ is XYZ iff for every λ it is not the case that $\{\lambda' \in \Lambda : \lambda' \geq \lambda\} \ni \lambda' \mapsto x_{\lambda'}$ is XYZ iff it is not the case that there is some λ such that $\{\lambda' \in \Lambda : \lambda' \geq \lambda\} \ni \lambda' \mapsto x_{\lambda'}$ is XYZ iff it is not the case that $\lambda \mapsto x_\lambda$ is eventually XYZ. ■

Now with the concept of eventuality in hand, we present the definition of convergence.

Definition 2.4.2.4 — Limit (of a net) Let $\lambda \mapsto x_\lambda$ be a net and let $x_\infty \in \mathbb{R}$. Then, x_∞ is the ***limit*** of $\lambda \mapsto x_\lambda$ iff for every $\varepsilon > 0$, $\lambda \mapsto x_\lambda$ is eventually contained in $B_\varepsilon(x_\infty)$. If a net has a limit, then we say that it ***converges***.[a]

R Note the use of the term *eventually*. Explicitly, this is equivalent to

x_∞ is the limit of $\lambda \mapsto x_\lambda$ iff for every $\varepsilon > 0$ there is some λ_0 such that, whenever $\lambda \geq \lambda_0$, it follows that $x_\lambda \in B_\varepsilon(x_\infty)$. (2.4.2.5)

Of course, when actually doing proofs, you will often need to make use of the explicit definition, but I think it's fair to say that the definition that uses the term "eventually" is both more intuitive and concise.

Exercise 2.4.2.6 — Limits are unique (in $\mathbb{R}$) Let $x_\infty, x'_\infty \in \mathbb{R}$ be limits of the net $\lambda \mapsto x_\lambda$. Show that $x_\infty = x'_\infty$.

R In general topological spaces (though not for most 'reasonable' ones), limits need not be unique (hence the reason for adding "in $\mathbb{R}$"). In fact, limits are unique iff the space is T_2—see Proposition 3.6.2.20.

R If x_∞ is the limit of $\lambda \mapsto x_\lambda$, then we write $\lim_\lambda x_\lambda = x_\infty$, or sometimes even just $\lim x_\lambda = x_\infty$. Note that this is unambiguous by the previous exercise.

R There are at least three directed sets that are commonly the domains of nets in calculus: $\mathbb{N}$, $\langle \mathbb{R}^+, \leq \rangle$, and $\langle \mathbb{R}^+, \geq \rangle$, that is, the natural numbers with the usual ordering, the positive reals with the usual ordering, and the positive reals with the *reverse* of the usual ordering. We shall denote limits of nets with these respective domains as

$$\lim_m x_m, \qquad \lim_{t\to\infty} x_t, \qquad \lim_{t\to 0^+} x_t. \tag{2.4.2.7}$$

[a]Divergence is not the same as nonconvergence. We will define divergence momentarily.

Proposition 2.4.2.8 Let $\lambda \mapsto x_\lambda$ be a net and let $x_\infty \in \mathbb{R}$. Then, it is *not* the case that $\lambda \mapsto x_\lambda$ converges to x_∞ iff there is some $\varepsilon_0 > 0$ such that $\lambda \mapsto x_\lambda$ is frequently contained in $B_{\varepsilon_0}(x_\infty)^{\mathsf{C}}$.

Proof. By Meta-proposition 2.4.2.3 and the definition of convergence, it is not the case that $\lambda \mapsto x_\lambda$ converges to x_∞ iff it is not the case that for every $\varepsilon > 0$, $\lambda \mapsto x_\lambda$ is eventually contained in $B_\varepsilon(x_\infty)$ iff there is some $\varepsilon_0 > 0$ such that it is not the case that $\lambda \mapsto x_\lambda$ is eventually contained in $B_{\varepsilon_0}(x_\infty)$ iff there is some $\varepsilon_0 > 0$ such that $\lambda \mapsto x_\lambda$ is frequently contained in $B_{\varepsilon_0}(x_\infty)$. ■

Exercise 2.4.2.9 Let $\Lambda \ni \lambda \mapsto x_\lambda \in \mathbb{R}$. Show that if Λ is finite, then (i) Λ has a maximum element λ_∞ and (ii) that $\lim_\lambda x_\lambda = x_{\lambda_\infty}$.

Exercise 2.4.2.10 Let $x_0 \in \mathbb{R}$ and define $\lambda \mapsto x_\lambda := x_0$. Show that $\lim_\lambda x_\lambda = x_0$.

R That is, constant nets converge to that constant. While trivial, it is significant in that it becomes an axiom of the convergence definition of a topological space—see Theorem 3.4.2.1.

Exercise 2.4.2.11 Show that $\lim_m \frac{1}{m} = 0$.

Proposition 2.4.2.12 Let $|a| < 1$. Then, $\lim_m a^m = 0$.

R For the first time in these notes, we encounter (in the proof) what is known as *sigma notation*. You likely already know what this is, but it should probably be included somewhere for the sake of completeness, and perhaps more importantly, it makes sense to introduce alongside sigma notation something which is less commonly known: *pi notation*.

In a general rg $\langle R, +, 0, \cdot \rangle$ (Definition A.4.12), if $\{x_m, \ldots, x_n\} \subseteq R$ is a finite subset, then we write

$$\begin{aligned} \sum_{\{x_m,\ldots,x_n\}} x_k &:= \sum_{k=m}^{n} x_k \\ &:= x_m + x_{m+1} + \cdots + x_{n-1} + x_n . \end{aligned} \tag{2.4.2.13}$$

Similarly, we write

$$\begin{aligned} \prod_{\{x_m,\ldots,x_n\}} x_k &:= \prod_{k=m}^{n} x_k \\ &:= x_m \cdot x_{m+1} \cdot \cdots \cdot x_{n-1} \cdot x_n .^{a} \end{aligned} \tag{2.4.2.14}$$

The empty sum ($n < m$) is defined to be 0, and for rigs the empty product ($n < m$) is defined to be 1.

The definition in (2.4.2.13) is ***sigma notation*** and the definition in (2.4.2.14) is ***pi notation***.[b]

[a]Note that these definitions implicitly make use of associativity of the binary operations. (Otherwise, does $\prod_{k=1}^{3} x_k$ mean $(x_1 \cdot x_2) \cdot x_3$ or $x_1 \cdot (x_2 \cdot x_3)$?).

[b]"Σ" is for "sum" and "Π" is for "product".

Proof. Define $b := \frac{1}{|a|}$. Let $M > 0$. We show that $m \mapsto b^m$ is eventually larger than M. It will then follows that $m \mapsto |a|^m$ is eventually smaller than $\frac{1}{M}$, or in other words, $m \mapsto |a|^m$ is eventually contained in $B_{\frac{1}{M}}(0)$. As $\frac{1}{M}$ is just as arbitrary as $\varepsilon > 0$, this will show that $\lim_m a^m = 0$.

We have

$$b^m = (1 + (b-1))^m = \sum_{k=0}^{m} \binom{m}{k} (b-1)^k \geq 1 + m(b-1).^{a}$$

Because $b - 1 > 0$, it follows from the Archimedean Property (Theorem 2.3.3)[b] that $m \mapsto b^m$ is eventually larger than M, which completes the proof. ∎

R Note the phrase "just as arbitrary". You will see arguments, and ones like it, quite frequently, so let us take the time at least once to spell out exactly what's happening.

In this case, to show that $\lim_m a^m = 0$, we want to show that, for every $\varepsilon > 0$, $m \mapsto |a|^m$ is eventually less than $\varepsilon > 0$. Instead, what I do, is I prove that for every $M > 0$, $m \mapsto |a|^m$ is eventually less than $\frac{1}{M}$. So, suppose I have done this, and let $\varepsilon > 0$. Define $M := \frac{1}{\varepsilon}$. Then, $m \mapsto |a|^m$ is eventually less than $\frac{1}{M} = \varepsilon$. That is, if I have proven the statement involving M, then I have proven the statement involving ε because $\frac{1}{M}$ is "just as arbitrary" as ε.

I could quite possibly be over-explaining this to the point of being unhelpful. Thus, if you thought you understood before, but this remark confused you, don't worry about it—you can just move on.

Finally, I explain a similar argument once again at the end of the proof of Theorem 2.4.3.40, so feel free to glance there now if you like.

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[a] $\binom{m}{k}$ is the $\langle m,k\rangle$ ***binomial coefficient*** and is defined by $\binom{m}{k} := \frac{m(m-1)(m-2)\cdots(m-(k+1))}{\prod_{j=1}^{k} j}$—see Definition 6.4.5.70 for elaboration if you need it.

[b] Choose $m \in \mathbb{N}$ to be strictly larger than $\frac{M-1}{b-1}$, so that $1 + m(b-1) > M$.

Definition 2.4.2.15 — Divergence Let $\lambda \mapsto x_\lambda$ be a net. Then, $\lambda \mapsto x_\lambda$ ***diverges to*** $+\infty$ iff for every $M > 0$, $\lambda \mapsto x_\lambda$ is eventually larger than M.

Similarly for $-\infty$.[a]

$\lambda \mapsto x_\lambda$ ***diverges*** iff $\lambda \mapsto |x_\lambda|$ diverges.

R Of course, the intuition is just that x_λ grows arbitrarily large.

R Note the use of the term *eventually*. Explicitly, this is equivalent to

$$\lambda \mapsto x_\lambda \text{ diverges to } +\infty \text{ iff for every } M > 0 \text{ there is some } \lambda_0 \text{ such that, whenever } \lambda \geq \lambda_0, \text{ it follows that } x_\lambda \geq M. \tag{2.4.2.16}$$

R Note that this terminology is slightly nonstandard. It is common for authors to take "diverge" to be "doesn't converge"; however, I prefer having the more refined terminology "converges", "diverges to $\pm\infty$", "diverges", and "doesn't converge".

[a] You should consider writing this definition out explicitly yourself.

Exercise 2.4.2.17

(i). What is an example of a net which neither converges nor diverges?

(ii). What is an example of a net which diverges, but doesn't diverge to either $\pm\infty$.

The biggest problem with proving that a net converges is that we first need to know what the limit is before hand. For example, consider

$$\lim_m \sum_{k=0}^{m} \frac{1}{k!}.^{2} \tag{2.4.2.18}$$

You probably recall that this *should* converge to e, but what the hell is e? It has not yet been defined. In fact, we will later define

$$\mathrm{e} := \lim_m \sum_{k=0}^{m} \frac{1}{k!},^{5} \tag{2.4.2.19}$$

[4] In case you're wondering what this crazy new symbol "!" means, for $n \in \mathbb{N}$, $n!$, the ***factorial*** of n, is defined inductively by $0! := 1$ and $(n+1)! := (n+1) \cdot n!$.

but we cannot even make this definition if we don't know a priori that the limit of (2.4.2.18) exists! Thus, we need to have a way of showing that (2.4.2.18) exists without making explicit reference to e. The concepts of *Cauchyness* and *completeness* (in the sense of uniform spaces, as opposed to in the sense of preordered sets) allow us to do this.

2.4.3 Cauchyness and completeness

Like with the concept of convergence and limits, we will first define what it means to be Cauchy, and then explain the intuition behind the definition.

Definition 2.4.3.1 — Cauchyness Let $\lambda \mapsto x_\lambda$ be a net. Then, $\lambda \mapsto x_\lambda$ is ***Cauchy*** iff for every $\varepsilon > 0$, $\lambda \mapsto x_\lambda$ is eventually contained in some ε-ball.

R Note the use of the term *eventually*. Similarly as before, explicitly, this means that

$$\lambda \mapsto x_\lambda \text{ is Cauchy iff there is some } \lambda_0 \text{ and some } \varepsilon\text{-ball } B \text{ such that, whenever } \lambda \geq \lambda_0, \text{ it follows that } x_\lambda \in B. \tag{2.4.3.2}$$

You should compare this with the definition of a limit, Definition 2.4.2.4. The only essential difference between the definition of convergence and the definition of Cauchy is that, in the former case, there is a *single* center of the ε-ball that works for all ε, whereas, in the case of Cauchyness, the center of the ball can vary with the choice of ε. Thus, we immediately have the implication:

Exercise 2.4.3.3 Let $\lambda \mapsto x_\lambda$ be a convergent net. Show that $\lambda \mapsto x_\lambda$ is Cauchy.

[7]Actually we will define e := exp(1), but this will of course turn out to be exactly this limit.

As a short aside, we note that this is not how the definition of Cauchyness is usually stated. Instead, the following equivalent condition is given as the definition.

Exercise 2.4.3.4 Let $\lambda \mapsto x_\lambda$ be a net. Show that $\lambda \mapsto x_\lambda$ is Cauchy iff for all $\varepsilon > 0$ there is some λ_0 such that whenever $\lambda_1, \lambda_2 \geq \lambda_0$ it follows that $\left|x_{\lambda_1} - x_{\lambda_2}\right| < \varepsilon$.

We choose to use the Definition 2.4.3.1 because (i) it more closely resembles the definition of convergence and (ii) it more closely resembles the definition in the more general case of uniform spaces (see Definition 4.4.1). That being said, the equivalent condition in this exercise (Exercise 2.4.3.4) is frequently easier to check. For example:

Example 2.4.3.5 In this example, we check that the sequence

$$m \mapsto s_m := \sum_{k=0}^{m} \frac{1}{k!} \tag{2.4.3.6}$$

is Cauchy. (Note that $s_m \in \mathbb{Q}$, so in fact, this sequence is Cauchy in $\mathbb{Q}$ as well as in $\mathbb{R}$.)

$$\begin{aligned} \left|s_{m_1} - s_{m_2}\right| &= \left|\sum_{k=0}^{m_1} \frac{1}{k!} - \sum_{k=0}^{m_2} \frac{1}{k!}\right| = \sum_{k=m_1+1}^{m_2} \frac{1}{k!} \\ &\leq \sum_{k=m_1+1}^{m_2} \frac{1}{2^k} = \frac{2^{m_2} - 2^{m_1}}{2^{m_1+m_2}} \leq 2^{-m_1} \end{aligned} \tag{2.4.3.7}$$

where we have without loss of generality assumed that $3 \leq m_1 \leq m_2$ (because $2^k \leq k!$ for $k \geq 4$).

We use this to check the condition of Exercise 2.4.3.4. So, let $\varepsilon > 0$. By Proposition 2.4.2.12, $\lim_m 2^{-m} = 0$, and so there is some $m_0 \in \mathbb{N}$ such that, whenever $m \geq m_0$, it follows that $2^{-m} = |2^{-m} - 0| < \varepsilon$. Hence, whenever $m_1, m_2 \geq m_0$, it follows that

$$\left|s_{m_1} - s_{m_2}\right| \leq 2^{-m_1} < \varepsilon. \tag{2.4.3.8}$$

One way to immediately tell if a net is not Cauchy is if it is eventually unbounded.

Proposition 2.4.3.9 Let $\lambda \mapsto x_\lambda$ be a Cauchy net. Then, $\lambda \mapsto x_\lambda$ is eventually bounded. In particular, if this net is a sequence, the set $\{x_m : m \in \mathbb{N}\}$ is bounded.

Proof. Applying the definition to $\varepsilon := 1$, we deduce that there must be some λ_0 and some open ball B of radius 1 such that, whenever $\lambda \geq \lambda_0$, it follows that $x_\lambda \in B$. Thus, $\lambda \mapsto x_\lambda$ is eventually in B, a bounded subset of $\mathbb{R}$, and hence itself is eventually bounded.

Now assume that $m \mapsto x_m$ is a sequence so that $m \in \mathbb{N}$. Let us also write $m_0 := \lambda_0$. The key to note here is that, unlike in the general case, the set $\{m \in \mathbb{N} : m < m_0\}$ is *finite*. This enables us to make the definition

$$r_0 := \max\left\{|x_0||, \ldots, \left|x_{m_0-1}\right|, 1 + |x_0|\right\}, \tag{2.4.3.10}$$

where x_0 is the center of the ball B. The reason for the $1 + |x_0|$ is as follows. For $m \geq m_0$, we have that

$$\begin{aligned} |x_m| = |(x_m - x_0) + x_0| &\leq |x_m - x_0| + |x_0| \\ &\leq 1 + |x_0|. \end{aligned} \tag{2.4.3.11}$$

From this, it follows that $\{x_m : m \in \mathbb{N}]\} \subseteq B_{r_0}(x_\infty)$, so that $\{x_m : m \in \mathbb{N}\}$ is bounded. ∎

Algebraic and Order Limit Theorems

Ideally, the content in this subsubsection would have been discussed very soon after defining a limit itself, however, we will use the fact that convergent nets are eventually bounded. We could have proved this, then done the Algebraic and Order Limit Theorems, but when we would have wound up essentially proving the same theorem twice (because Cauchy nets are eventually bounded as well). We decided to just postpone these results to a slightly less-than-maximally-coherent location in the notes instead.

Proposition 2.4.3.12 — Algebraic Limit Theorems Let $\lambda \mapsto x_\lambda$ and $\lambda \mapsto y_\lambda$ be two convergent nets.[a] Then,

(i). $\lim(x_\lambda + y_\lambda) = \lim_\lambda x_\lambda + \lim_\lambda y_\lambda$;
(ii). $\lim(x_\lambda y_\lambda) = (\lim_\lambda x_\lambda)(\lim_\lambda y_\lambda)$;
(iii). $\lim \frac{1}{x_\lambda} = \frac{1}{\lim_\lambda x_\lambda}$ if $\lim_\lambda x_\lambda \neq 0$; and
(iv). $\lim(\alpha x_\lambda) = \alpha \lim_\lambda x_\lambda$ for $\alpha \in \mathbb{R}$.

R Later, we will see how all of this follow automatically from continuity considerations (because $+$ is continuous, for example).

[a]The common index is supposed to indicate that the two nets have the same domain.

Proof. We leave (i), (ii), and (iv) as exercises. Should you need guidance, check out our proof of (iii).

Exercise 2.4.3.13 Prove (i).

Exercise 2.4.3.14 Prove (ii).

We prove (iii). It is likely the most difficult, and if you can follow this, you should be able to do the others on your own.

Define $x_\infty := \lim_\lambda x_\lambda$. Suppose that $x_\infty \neq 0$. Without loss of generality, assume that $x_\infty > 0$. As $\lambda \mapsto x_\lambda$ converge to $x_\infty > 0$, eventually it must be the case that $\lambda \mapsto x_\lambda$ is positive. We now prove this claim in more detail. Define $\varepsilon := \frac{1}{2}|x_\infty| = \frac{1}{2}x_\infty$. Then, there is some λ_0 such that, whenever $\lambda \geq \lambda_0$, it follows that

$$x_\infty - x_\lambda \leq |x_\lambda - x_\infty| < \varepsilon := \frac{1}{2}x_\infty. \tag{2.4.3.15}$$

It follows that, for $\lambda \geq \lambda_0$,

$$x_\lambda > x_\infty - \tfrac{1}{2}x_\infty = \tfrac{1}{2}x_\infty. \tag{2.4.3.16}$$

As $\lambda \mapsto x_\lambda$ is eventually positive, we may without loss of generality assume that there there is some $M > 0$ such that $x_\lambda, x_\infty \geq M$ *for all* λ.[a]

Now, let $\varepsilon > 0$ and, redefining notation,[b] choose λ_0 such that, whenever $\lambda \geq \lambda_0$, it follows that $|x_\lambda - x_\infty| < \varepsilon$. Then, for $\lambda \geq \lambda_0$,

$$\left|x_\lambda^{-1} - x_\infty^{-1}\right| = \frac{|x_\infty - x_\lambda|}{|x_\lambda x_\infty|} < \tfrac{1}{M^2}|x_\lambda - x_\infty| < \tfrac{1}{M^2}\varepsilon. \tag{2.4.3.17}$$

As $\frac{1}{M^2}\varepsilon$ is just as arbitrary as ε (this uses the fact that M does not depend on ε), this completes the proof.

Exercise 2.4.3.18 Prove (iv).

■

[a] Here we are making use of the trick we mentioned before about 'throwing away' the beginning terms of a net—see the end of Eventuality.

[b] That is, forget our previous definition of λ_0.

Exercise 2.4.3.19 — Order Limit Theorem Let $\lambda \mapsto x_\lambda$ and $\lambda \mapsto b_\lambda$ be two convergent nets. Show that if it is eventually the case that $x_\lambda \leq y_\lambda$, then $\lim_\lambda x_\lambda \leq \lim_\lambda y_\lambda$.

R Of course, it is also true if the limits are $\pm\infty$.

R Sometimes this (or really, I suppose, an immediate corollary of this) is called the ***Squeeze Theorem***.

Exercise 2.4.3.20 — Squeeze Theorem Let $\lambda \mapsto x_\lambda$, $\lambda \mapsto y_\lambda$, and $\lambda \mapsto z_\lambda$ be nets. Show that if (i) it is eventually the case that $x_\lambda \leq y_\lambda \leq z_\lambda$ and (ii) $\lim_\lambda x_\lambda = \lim_\lambda z_\lambda$, then $\lambda \mapsto y_\lambda$ is convergent, and $\lim_\lambda x_\lambda = \lim_\lambda y_\lambda = \lim_\lambda z_\lambda$.

R The Order Limit Theorem tells us that, *if* $\lim_\lambda y_\lambda$ exists, then it must be equal to the common value $\lim_\lambda x_\lambda = \lim_\lambda z_\lambda$. The Squeeze Theorem tells us furthermore that this limit does in fact exist.

You (hopefully) just showed in Exercise 2.4.3.3, convergent sequences are always Cauchy. However, it is in general not the case that every Cauchy sequence converges. For example, once we show that the definition of $\mathrm{e} := \lim_m \sum_{k=0}^{m} \frac{1}{k!}$ makes sense (we're about to) and that e is irrational[8], then this will serve as an example of a sequence that is Cauchy in $\mathbb{Q}$ (as we just showed) but does not converge in $\mathbb{Q}$ (because e is irrational). However, it *is* true that every Cauchy sequence does converge in $\mathbb{R}$, and in fact, you might say this is the real reason we care about $\mathbb{R}$ at all and that the motivation for requiring the existence of least upper-bounds was so that we could prove this. Before we actually prove this, however, we first need to discuss limit superiors and limit inferiors.

Limit superiors and limit inferiors

The Monotone Convergence Theorem is the tool that will allow us to define lim sup and lim inf.

Proposition 2.4.3.21 — Monotone Convergence Theorem Let $\lambda \mapsto x_\lambda$ be a net. Then,

(i). if $\lambda \mapsto x_\lambda$ is nondecreasing, then $\lim_\lambda x_\lambda = \sup\{x_\lambda : \lambda\}$; and

(ii). if $\lambda \mapsto x_\lambda$ is nonincreasing, then $\lim_\lambda x_\lambda = \inf\{x_\lambda : \lambda\}$.

R Note that we allow here the limits to be $\pm\infty$. Thus, nondecreasing and nonincreasing alone is not enough to guarantee convergence, however, from this it follows that, in the nondecreasing case, being bounded

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[8]Unfortunately this won't come for awhile—see Theorem 6.4.5.79.

above, and in the nonincreasing case, being bounded below, is sufficient to guarantee convergence.

R If you add "eventually" in front of "nondecreasing" and "bounded above", this is still sufficient to guarantee convergence, but you no longer have equality of the limit with the supremum (and similarly for the nonincreasing case).

Proof. We just do the case where $\lambda \mapsto x_\lambda$ is nondecreasing.

First, suppose that the net is not bounded above, so that $\sup_\lambda\{x_\lambda\} = +\infty$. We wish to show that $\lim_\lambda x_\lambda = +\infty$. So, let $M > 0$. As the net is not bounded above, there is some λ_0 such that $x_{\lambda_0} \geq M$. But then, whenever $\lambda \geq \lambda_0$, as the net is nondecreasing, we have $M \leq x_{\lambda_0} \leq x_\lambda$. Thus, by definition, we have that $\lim_\lambda x_\lambda = +\infty$.

Now suppose that the net is bounded above. We may then define

$$x_\infty := \sup_\lambda\{x_\lambda\} \in \mathbb{R}. \tag{2.4.3.22}$$

We want to show that $\lim_\lambda x_\lambda = x_\infty$.

So, let $\varepsilon > 0$. By Proposition 1.4.1.13, there is some x_{λ_0} such that

$$x_\infty - \varepsilon < x_{\lambda_0} \leq x_\infty. \tag{2.4.3.23}$$

Then, by monotonicity, whenever $\lambda \geq \lambda_0$, we have that

$$x_\infty - \varepsilon < x_{\lambda_0} \leq x_\lambda \leq x_\infty. \tag{2.4.3.24}$$

This, however, implies that $x_\lambda \in B_\varepsilon(x_\infty)$, so that, by definition, $\lim_\lambda x_\lambda = x_\infty$. ■

Definition 2.4.3.25 — Limit superior and limit inferior Let $x\colon \Lambda \to \mathbb{R}$ be a net and for each $\lambda_0 \in \Lambda$ define

$$u_{\lambda_0} := \sup_{\lambda \geq \lambda_0} \{x_\lambda\} \text{ and } l_{\lambda_0} := \inf_{\lambda \geq \lambda_0} \{x_\lambda\}. \tag{2.4.3.26}$$

Exercise 2.4.3.27 Check that $\lambda \mapsto u_\lambda$ is nonincreasing and that $\lambda \mapsto l_\lambda$ is nondecreasing.

Then, the ***limit superior*** and the ***limit inferior*** of $\lambda \mapsto x_\lambda$, $\limsup_\lambda x_\lambda$ and $\liminf_\lambda x_\lambda$ respectively, are defined by

$$\limsup x_\lambda := \lim_\lambda u_\lambda \tag{2.4.3.28a}$$

$$\liminf x_\lambda := \lim_\lambda l_\lambda. \tag{2.4.3.28b}$$

R Note that it is the Monotone Convergence Theorem which guarantees that this definition makes sense. In particular, unlike limits themselves, lim sup and lim inf *always* make sense (though they may be $\pm\infty$).

R The intuition is that $\lambda \mapsto u_\lambda$ is the net of 'eventual upper bounds'. $\limsup_\lambda x_\lambda$ is then the limit of this net (similarly for lim inf).

Exercise 2.4.3.29 Let $\lambda \mapsto x_\lambda$ be a net. Show one of the following two statements.

(i). $\limsup_\lambda x_\lambda$ is finite iff $\lambda \mapsto x_\lambda$ is eventually bounded above.

(ii). $\liminf_\lambda x_\lambda$ is finite iff $\lambda \mapsto x_\lambda$ is eventually bounded below.

R Both statements are true of course, but the proofs will be so similar that there is not really much point in asking you to write both down.

Exercise 2.4.3.30 Let $\lambda \mapsto x_\lambda$ be a net. Show that $\liminf_\lambda x_\lambda \leq \limsup_\lambda x_\lambda$.

R Note that if we have equality at a finite value, then, by the Squeeze Theorem (Exercise 2.4.3.20), the net converges to the common value. In fact, the converse is also true—see the following proposition (Proposition 2.4.3.31).

Proposition 2.4.3.31 Let $\lambda \mapsto x_\lambda$ be a net and let $x_\infty \in [-\infty, \infty]$. Then, $\lim_\lambda x_\lambda = x_\infty$ iff $\limsup_\lambda x_\lambda = x_\infty = \liminf_\lambda x_\lambda$.

Proof. ($\Rightarrow$) Suppose that $\lambda \mapsto x_\lambda = x_\infty$ converges. First suppose that x_∞ is finite. Then, it is eventually bounded, and so both $\limsup_\lambda x_\lambda$ and $\liminf_\lambda x_\lambda$ are finite. Define $u := \limsup_\lambda x_\lambda$ and $l := \liminf_\lambda x_\lambda$. Let $\varepsilon > 0$ and choose λ_0 such that, whenever $\lambda \geq \lambda_0$, it follows that

$$\sup_{\mu \geq \lambda}\{x_\mu\} - u < \varepsilon \text{ and } l - \inf_{\mu \geq \lambda}\{x_\mu\} < \varepsilon. \qquad (2.4.3.32)$$

Adding these two inequality, we find that, for $\lambda \geq \lambda_0$,

$$l - u < 2\varepsilon + \left(\inf_{\mu \geq \lambda}\{x_\mu\} - \sup_{\mu \geq \lambda}\{x_\mu\}\right). \qquad (2.4.3.33)$$

Now define $x_\infty := \lim_\lambda x_\lambda$ and choose λ_0' such that, whenever $\lambda \geq \lambda_0'$, we have that $|x_\lambda - x_\infty| < \varepsilon$. In other words,

$$x_\lambda - x_\infty < \varepsilon \text{ and } x_\infty - x_\lambda < \varepsilon. \qquad (2.4.3.34)$$

Taking the inf of this inequality (and using the fact that $\inf(-S) = -\sup(S)$—see Exercise 1.4.2.54), we find

$$\inf_{\mu \geq \lambda}\{x_\mu\} < \varepsilon + x_\infty \text{ and } -\sup_{\mu \geq \lambda}\{x_\mu\} < \varepsilon - x_\infty. \qquad (2.4.3.35)$$

Pick something larger than both λ_0 and λ_0' and change notation so that this new larger thing is called λ_0. Thus, now, for $\lambda \geq \lambda_0$, both sets of inequalities hold, and so

$$l - u < 2\varepsilon + ((\varepsilon + x_\infty) + (\varepsilon - x_\infty)) = 4\varepsilon. \qquad (2.4.3.36)$$

As ε was arbitrary, we have $l = u$.

Now consider the case $x_\infty = \pm\infty$. As the two cases are essentially the same, we prove this only in the case $x_\infty = +\infty$. As $\liminf_\lambda x_\lambda \leq \limsup_\lambda x_\lambda$, it suffices to show that $\liminf_\lambda x_\lambda = \infty$. Let $M > 0$. Then, there is some λ_0 such that, whenever $\lambda \geq \lambda_0$, it follows that $x_\lambda \geq M$. Hence, $\inf_{\mu \geq \lambda_0}\{x_\mu\} \geq M$. Hence, whenever $\lambda \geq \lambda_0$, it follows that

$$\inf_{\mu \geq \lambda}\{x_\mu\} \geq \inf_{\mu \geq \lambda_0}\{x_\mu\} \geq M. \qquad (2.4.3.37)$$

It follows that, by definition (Definition 2.4.2.15), $\liminf_\lambda x_\lambda = \infty$.

($\Leftarrow$) Suppose that $\limsup_\lambda x_\lambda = x_\infty = \liminf_\lambda x_\lambda$. Then, as

$$\inf_{\mu \geq \lambda}\{x_\mu\} \leq x_\lambda \leq \sup_{\mu \geq \lambda}\{x_\mu\}, \qquad (2.4.3.38)$$

the Squeeze Theorem implies that $\lim_\lambda x_\lambda = x_\infty$. ■

▪ **Example 2.4.3.39** Note that we can have equality at an *infinite* value. For example, consider the sequence $m \mapsto x_m := m$. This is obviously not bounded above, and so by definition $\limsup_m x_m = \infty$. On the other hand, $\inf_{m \geq m_0}\{x_m\} = m_0$, and so $\liminf_m x_m = \infty$ as well.

Now we can finally return to our current goal of proving that Cauchy nets converge in $\mathbb{R}$.

Theorem 2.4.3.40 — Completeness of $\mathbb{R}$. Let $\lambda \mapsto x_\lambda$ be a Cauchy net. Then, $\lambda \mapsto x_\lambda$ converges.

> R This property of having all Cauchy nets converge is known as *Cauchy-completeness*. Being Cauchy-complete is a property that a uniform space may or may not have,[a] and we will see that, when we equip $\mathbb{R}$ with a uniform structure, this result is equivalent to completeness in the sense of uniform spaces. You should be careful not to confuse Cauchy-completeness with Dedekind-completeness. For example, while $\mathbb{R}$ is the unique (up to unique isomorphism) nonzero Dedekind-complete totally-ordered field, there are Cauchy-complete totally-ordered field distinct from $\mathbb{R}$. In brief, the example will turn out to be the Cauchy-completion of $\mathbb{R}(x)$ (which cannot be Dedekind-complete because otherwise $\mathbb{R}(x)$ would embed into $\mathbb{R}$—see Example 2.3.17), but we will have to wait until the next chapter to be more precise about this.

[a] No, you are not expected to know what a uniform space is yet (though see Definition 4.1.2.1 if you can't wait to find out).

Proof. To show that $\lambda \mapsto x_\lambda$ converges, we first have to find some number $x_\infty \in \mathbb{R}$ which we think is going to be the limit of the net. Our guess, of course, is going to be $x_\infty := \limsup_\lambda x_\lambda$. We know that the net $\lambda \mapsto x_\lambda$ is eventually bounded (Proposition 2.4.3.9), so that $x_\infty \in \mathbb{R}$ is finite.

We wish to show that $\lim_\lambda x_\lambda = x_\infty$. So, let $\varepsilon > 0$. First of all, choose λ_0 so that there is some ε-ball B_ε such that

$$\{x_\lambda : \lambda \geq \lambda_0\} \subseteq B_\varepsilon.^a \tag{2.4.3.41}$$

We will use this later.

Now recall the definition of lim sup:

$$x_\infty := \lim_\lambda u_\lambda := \lim_{\lambda_0}\left(\sup_{\lambda\geq\lambda_0}\{x_\lambda\}\right), \tag{2.4.3.42}$$

so there is some λ_0' such that, whenever $\lambda \geq \lambda_0'$,

$$u_\lambda - x_\infty = |u_\lambda - x_\infty| < \varepsilon. \tag{2.4.3.43}$$

Redefine λ_0 to be the maximum of λ_0 and λ_0'. This way, both (2.4.3.41) and (2.4.3.43) will hold for $\lambda \geq \lambda_0$.

On the other hand, for every $\lambda \geq \lambda_0$, by Proposition 1.4.1.13, there is some $\lambda' \geq \lambda \geq \lambda_0$ such that

$$u_\lambda - \varepsilon < x_{\lambda'} \leq u_\lambda. \tag{2.4.3.44}$$

In particular,

$$u_\lambda - x_{\lambda'} < \varepsilon. \tag{2.4.3.45}$$

Hence,

$$\begin{aligned} |x_{\lambda'} - x_\infty| &= |(x_{\lambda'} - u_\lambda) + (u_\lambda - x_\infty)| \\ &\leq |x_{\lambda'} - u_\lambda| + |u_\lambda - x_\infty| \\ &< \varepsilon + \varepsilon = 2\varepsilon. \end{aligned} \tag{2.4.3.46}$$

We're almost done. The only problem with this is that λ' is a specific index. We need this inequality to hold for all λ sufficiently large, not just a single λ'. Fortunately, the Cauchyness can do this for us: it follows from (2.4.3.41) that, whenever $\lambda \geq \lambda_0$, $|x_\lambda - x_{\lambda'}| < 2\varepsilon$. Hence, for $\lambda \geq \lambda_0$,

$$\begin{aligned} |x_\lambda - x_\infty| &= |(x_\lambda - x_{\lambda'}) + (x_{\lambda'} - x_\infty)| \\ &\leq |x_\lambda - x_{\lambda'}| + |x_{\lambda'} - x_\infty| \\ &< 2\varepsilon + 2\varepsilon = 4\varepsilon. \end{aligned} \tag{2.4.3.47}$$

■

R You'll note that the final inequality the proof ended with was $|x_\lambda - x_\infty| < 4\varepsilon$, in contrast to the inequality (implicitly) in the definition of convergence (Definition 2.4.2.4), $|x_\lambda - x_\infty| < \varepsilon$. This of course makes no difference because $\frac{\varepsilon}{4} > 0$ is just as arbitrary as $\varepsilon > 0$. Some people actually go to the trouble of doing the proof and then going back and changing all the εs to $\frac{\varepsilon}{n}$s just so that the final inequality has an ε in it. This is completely unnecessary. Don't waste your time.

[a] This is the definition of Cauchy.

We showed above in Example 2.4.3.5 that the sequence $m \mapsto s_m := \sum_{k=0}^{m} \frac{1}{k!}$ is Cauchy. The completeness of $\mathbb{R}$ that we just established then tells us that this sequence converges, and so now we can define

$$\mathrm{e} := \lim_m \sum_{k=0}^{m} \frac{1}{k!}. \tag{2.4.3.48}$$

But let's not. Given what we currently know, it would seem like we would just be pulling this definition out of our ass. This series is not why e matters. In fact, I would argue that e itself doesn't matter—it is the function exp that arises naturally in calculus (being the unique (up to scalar multiples) function equal to its own derivative). Thus, we refrain from 'officially' defining e until we have defined the exponential, which of course we won't be able to do until we know about differentiation—see Definition 6.4.5.28.

Nevertheless, it would be nice to 'officially' know that there is at least some real number that is not rational.

Square-roots

I mentioned back right before Subsection 1.4.2 Dedekind-cuts and the real numbers that sometimes people introduce the real numbers so that we can take square-roots of numbers, but that this logic is a bit silly, because if that were the objective, then we should really be extending our number system from $\mathbb{Q}$ to $\mathbb{A}$, not from $\mathbb{Q}$ to $\mathbb{R}$. That being said, even though our justification for the introduction of the reals was really so that we can do calculus (take limits), it does turn out that we do get square-roots of all nonnegative reals. This should be viewed more as "icing on the cake" instead of the raison d'être.

Proposition 2.4.3.49 Let $x \geq 0$, define $x_0 := 1$ and

$$x_{m+1} := \frac{1}{2}\left(x_m + \frac{x}{x_m}\right) \tag{2.4.3.50}$$

for $m \geq 0$. Then, $m \mapsto x_m$ converges and $(\lim_m x_m)^2 = x$.

R In fact, the same sort of trick can be used to prove the existence and uniqueness of all m^{th} roots, $m \in \mathbb{Z}^+$. The statement is Proposition 2.4.3.56, though we leave the 'meat' of the proof as an exercise (Exercise 2.4.3.55).

R Not only does this proposition show the existence of a number whose square is x, but it tells you how to compute it as well.

Proof. Let us first assume that $m \mapsto x_m$ *does* converge, say to $x_\infty := \lim_m x_m$. Then, taking the limit of both sides of the equation (2.4.3.50), we find that it must be the case that

$$x_\infty = \frac{1}{2}\left(x_\infty + \frac{x}{x_\infty}\right), \tag{2.4.3.51}$$

and hence that

$$x_\infty^2 = x. \tag{2.4.3.52}$$

Thus, if the limit exists, it converges to a nonnegative number whose square is x (nonnegative because each x_m is nonnegative). Thus, we now check that in fact it converges.

We apply the Monotone Convergence Theorem. The sequence is bounded below by 0, so it suffices to show that it is nonincreasing. We thus look at

$$x_{m+1} - x_m = \frac{1}{2}\left(x_m + \frac{x}{x_m}\right) - x_m = \frac{x - x_m^2}{2x_m}. \tag{2.4.3.53}$$

We want this to be ≤ 0, so it suffices to show that eventually $x_m^2 \geq x$. However, for $m \geq 1$ (so that x_{m-1} makes sense),

$$\begin{aligned} x_m^2 &= \frac{1}{4}\left(x_{m-1} + \frac{x}{x_{m-1}}\right)^2 \\ &= x + \left[\frac{1}{4}\left(x_{m-1}^2 + 2x + \frac{x^2}{x_{m-1}^2}\right) - x\right] \\ &= x + \frac{1}{4}\left(x_{m-1} - \frac{x}{x_{m-1}}\right)^2 \geq x. \end{aligned} \tag{2.4.3.54}$$

■

Exercise 2.4.3.55 Let $m \in \mathbb{Z}^+$. Develop an algorithm which computes m^{th} roots.

Proposition 2.4.3.56 Let $m \in \mathbb{Z}^+$ and let $x \geq 0$. Then, there is a unique nonnegative real number, $\sqrt[m]{x}$, whose m^{th} power is x.

R Of course this is not true if you remove the word "nonnegative": $(-1)^2 = 1 = 1^2$.

R As I'm sure you're aware, if $m = 2$, it is common to write $\sqrt{x} := \sqrt[2]{x}$.

R If $x < 0$, then we consider the symbol $\sqrt[m]{x}$ to be undefined.

Proof. We have just established existence.

Exercise 2.4.3.57 Show that the map $x \mapsto x^m$ is strictly increasing on $\mathbb{R}_0^+$.

As strictly increasing (and decreasing) functions are injective, it follows that there is at most one nonnegative real number whose m^{th} power is x. ■

We would have been able to show almost from the very beginning that, if $m \in \mathbb{Z}^+$ is not a perfect square, then there is no element $x \in \mathbb{Q}$ such that $x^2 = m$. What we haven't been able to show until just now, however, is that there is a *real* number $x \in \mathbb{R}$ such that $x^2 = m$. We now finally check that indeed there is no rational number whose square is m. Thus, we will have finally established the existence of real numbers which are not rational.

Proposition 2.4.3.58 Let $m \in \mathbb{Z}^+$. Then, if m is not a perfect square, then there is no $x \in \mathbb{Q}$ such that $x^2 = m$.

R $m \in \mathbb{Z}^+$ is a ***perfect-square*** iff $m = n^2$ for some $n \in \mathbb{Z}$. In other words, an integer is a perfect square iff it is the square of another integer.

R Thus, in particular, Proposition 2.4.3.58 gives an example of a sequence that is Cauchy in $\mathbb{Q}$ but does *not* converge in $\mathbb{Q}$. In other words, $\mathbb{Q}$ is *not* Cauchy-complete.

Proof. Suppose that m is not a perfect square. Then, we can write $m = k^2 n$ where $n > 1$ is square-free[a]. It thus suffices to show that there is no rational number whose square is n. We proceed by contradiction: suppose there is some $x \in \mathbb{Q}$ such that $x^2 = n$. Write $x = \frac{a}{b}$ with $b > 0$ and $\gcd(a, b) = 1$. Then,

$$a^2 = nb^2, \tag{2.4.3.59}$$

and so every prime factor of n divides a^2, and hence a (because the factor is prime—see Definition C.5). Let p be some prime factor of n and write $n = pn'$ and $a = pa'$. This gives us

$$p^2(a')^2 = pn'b^2, \tag{2.4.3.60}$$

and hence

$$p(a')^2 = n'b^2 \tag{2.4.3.61}$$

As n is square-free, $\gcd(p, n') = 1$,[b] and hence this equation implies that p divides b^2 (by Euclid's Lemma), and hence divides b. But then $\gcd(a, b) \geq p > 1$: a contradiction. ∎

[a] $m \in \mathbb{Z}$ is ***square-free*** iff the only perfect-square which divides m is 1.

[b] $n = pn'$, and so if $p \mid n'$, then we would have $n = p^2 \cdot$ integer, that is, $p^2 \mid n$.

We now show that all intervals in $\mathbb{R}$ are exactly what you think they are (confer Definition A.3.3.3). We could have done this awhile ago now, but we wanted to wait until we could definitively provide an example of an interval in $\mathbb{Q}$ that is *not* of the form you would expect.[9]

Proposition 2.4.3.62 Let $I \subseteq \mathbb{R}$. Then, I is an interval iff either

(i). $I = [a,b]$ for $-\infty < a \leq b < \infty$,
(ii). $I = (a,b)$ for $-\infty \leq a \leq b \leq \infty$,
(iii). $I = [a,b)$ for $-\infty < a \leq b \leq \infty$, or
(iv). $I = (a,b]$ for $-\infty \leq a \leq b < \infty$,

where $a = \inf(I)$ and $b = \sup(I)$.

R Note that when the side of the interval is open, we allow $\pm\infty$ (depending on which side it is).

R In the future, we shall simply write $I = [(a,b)]$ to indicate that each end may be either open or closed—it is a pain to keep writing out all the four cases separately.

Proof. ($\Rightarrow$) Suppose that I is an interval. If I is empty, then we have that $I = (0,0)$, and so is of the form (ii). Otherwise, we may define $b := \sup(I)$ and $a := \inf(I)$.[a] For each a and b there are two possibilities: either I contains a or it does not (and similarly for b). We do one of the four cases: suppose that $a, b \in I$. Then, because I is an interval, we must have immediately that $[a,b] \subseteq I$. To show the other inclusion, let $x \in I$. We proceed by contradiction: suppose that either $x < a$ or $x > b$. Both cases are similar, so let us just assume that $x > b$. Then, b is no longer an upper-bound for I: a contradiction. Therefore, $I \subseteq [a,b]$, and so $I = [a,b]$.

($\Leftarrow$) Suppose that I is of the form (i)–(iv). Recall that the definition of an interval (Definition A.3.3.3) is that, for any $x_1, x_2 \in I$ with $x_1 \leq x_2$, then $x_1 \leq x \leq x_2$ implies that

[9]By "what you would expect", we mean something like $[a,b]$, or $[a,b)$, etc.

$x \in I$. As each of these forms satisfies this property (for trivial reasons—feel free to do the case-work if you're not convinced), I is an interval. ■

[a] You might be used to checking that a set is bounded above (resp. below) before being able to take its supremum (resp. infimum). This is essentially correct—this case is exceptional in that we don't mind if $\sup(I) = \infty$ or $\inf(I) = -\infty$.

Of course, we mentioned above, this is *not* true in $\mathbb{Q}$.

Example 2.4.3.63 — An interval in $\mathbb{Q}$ not of the form Proposition 2.4.3.62((i)–(iv)) Define $I := [0, \sqrt{2}] \cap \mathbb{Q} \subseteq \mathbb{Q}$. It follows from the fact that $[0, \sqrt{2}]$ is an interval in $\mathbb{R}$ that I is an interval in $\mathbb{Q}$ (because it satisfies the defining condition of Definition A.3.3.3). If it were of the form $[(a, b)]$, then we would have to have $I = [0, b]$ or $I = [0, b)$. From the definition of I, however, it follows that we must have $b^2 = 2$, and so as there is no such b (in $\mathbb{Q}$), I cannot be of this form.

'Density' of $\mathbb{Q}^{\mathrm{C}}$ in $\mathbb{R}$

We mentioned back when we showed the 'density' of $\mathbb{Q}$ in $\mathbb{R}$ (Theorem 2.3.18) that it was also true that $\mathbb{Q}^{\mathrm{C}}$ was 'dense' in $\mathbb{R}$. At the time, however, we could not even construct a single real number that we could prove was irrational. Now, however, we have done so, and so we can return to the issue of the 'density' of $\mathbb{Q}^{\mathrm{C}}$ in $\mathbb{R}$.

Theorem 2.4.3.64 — 'Density' of $\mathbb{Q}^{\mathrm{C}}$ in $\mathbb{R}$. Let $a, b \in \mathbb{R}$. Then, if $a < b$, then there exists $c \in \mathbb{Q}^{\mathrm{C}}$ such that $c \in (a, b)$.

Proof. We have just shown that $\mathbb{Q}^{\mathrm{C}}$ is nonempty, so let $x \in \mathbb{Q}^{\mathrm{C}}$. By 'density' of $\mathbb{Q}$, we may take $a' \in \mathbb{Q} \cap (a - x, b - x)$. Then, $a' + x \in (a, b)$, and furthermore, $a' + x =: q$ must be irrational, because if it were rational, then $x = q - a'$ would be rational: a contradiction. ■

Counter-examples

In this small subsubsection, we present a couple of counter-examples, which might seem quite surprising if you've never thought too much about these things before.

■ **Example 2.4.3.65** For $m, n \in \mathbb{Z}^+$, define[a]

$$x_{m,n} := \frac{\frac{1}{m}\frac{1}{n}}{\frac{1}{m^2} + \frac{1}{n^2}}. \tag{2.4.3.66}$$

Then, for *fixed* $n \in \mathbb{Z}^+$,

$$\lim_m x_{m,n} = \frac{0}{0 + \frac{1}{n^2}} = 0. \tag{2.4.3.67}$$

Similarly, for fixed $m \in \mathbb{Z}^+$,

$$\lim_n x_{m,n} = 0. \tag{2.4.3.68}$$

On the other hand, if we set $m = n$ and *then* take a limit, we get

$$\lim_m x_{m,m} = \lim_m \left(\frac{\frac{1}{m^2}}{\frac{1}{m^2} + \frac{1}{m^2}} \right) = \lim_m (\tfrac{1}{2}) = \tfrac{1}{2}. \tag{2.4.3.69}$$

Thus:

> Even if there is some number x_∞ such that (i) *for all* λ, $\lim_\mu x_{\lambda,\mu} = x_\infty$, and (ii) *for all* μ, $\lim_\lambda x_{\lambda,\mu} = x_\infty$, it need *not* be the case that $\lim_\lambda x_{\lambda,\lambda} = x_\infty$.

This can sort of be fixed, however—see Proposition 2.4.5.22.

[a] This is an example of a case where we would have to reindex by 1, so that we don't divide by 0, were we forced to take $m, n \in \mathbb{N}$—see the first remark in our definition of a sequence, Definition 2.4.1.8.

Example 2.4.3.70 — Iterated limits need not agree For $m, n \in \mathbb{Z}^+$, define

$$x_{m,n} := \frac{\frac{1}{m}}{\frac{1}{m} + \frac{1}{n}}. \tag{2.4.3.71}$$

Then,

$$\lim_m x_{m,n} = \frac{0}{0 + \frac{1}{n}} = 0, \tag{2.4.3.72}$$

and hence

$$\lim_n \left(\lim_m x_{m,n} \right) = 0. \tag{2.4.3.73}$$

On the other hand,

$$\lim_n x_{m,n} = \frac{\frac{1}{m}}{\frac{1}{m} + 0} = 1, \tag{2.4.3.74}$$

and hence

$$\lim_m \left(\lim_n x_{m,n} \right) = 1. \tag{2.4.3.75}$$

Thus:

> It is possible for both iterated limits, $\lim_\lambda \left(\lim_\mu x_{\lambda,\mu} \right)$ and $\lim_\mu \left(\lim_\lambda x_{\mu,\lambda} \right)$, to exist and *not* agree.

R This is actually incredibly important because it is often really tempting to interchange limits, especially if they might be hidden implicitly in something like a derivative, but this is in general *just plain wrong*.

2.4.4 Series

As you probably know from calculus, a series is an 'infinite sum'. In other words, it is a limit of a finite sum.

Definition 2.4.4.1 — Series Let $m \mapsto a_m$ be a sequence and define $s_m := \sum_{k=0}^{m} a_k$. Then,

$$\sum_{k \in \mathbb{N}} a_k := \sum_{k=0}^{\infty} a_k := \lim_{m} \sum_{k=0}^{m} a_k \tag{2.4.4.2}$$

is a ***series*** and the s_ms are the ***partial sums*** of this series.[a] The series is said to ***converge*** iff the sequence $m \mapsto s_m$ converges and similarly for ***diverge to*** $\pm\infty$ and ***diverge***. The series ***converges absolutely*** iff $\sum_{k=0}^{m} |a_k|$ converges and similarly for ***diverges absolutely***[b]. The series ***converges conditionally*** iff it converges but not absolutely, and similarly for ***diverges conditionally***.

> R: Of course, absolute convergence implies convergence, and similarly divergence to $\pm\infty$ implies divergence implies absolute divergence.

> R: Note that conditional convergence implies conditional divergence,[c] but that the converse is false—see Example 2.4.4.4.

[a] Of course this limit need not exist. In that case, the notation $\sum_{k \in \mathbb{N}} a_k$ is meaningless, or at least, doesn't represent any real number.

[b] Of course, there is no "diverges to $\pm\infty$ absolutely" because this can only diverge to $+\infty$. Also note that "diverges absolutely" is the same as "doesn't converge absolutely".

[c] Why?

Example 2.4.4.3 While we don't have the tools to prove these statements yet, we present a couple of examples to help illustrate the differences between these types of convergence.

(i). $\sum_{m \in \mathbb{Z}^+} \frac{1}{m^2}$ converges absolutely.

(ii). $\sum_{m\in\mathbb{Z}^+}\frac{(-1)^{m+1}}{m}$ converges, but not absolutely (that is, is conditionally convergent).

(iii). $\sum_{m\in\mathbb{Z}^+} m$ diverges to $+\infty$.

(iv). $\sum_{m\in\mathbb{Z}^+}(-2)^{m+1}$ diverges, but not to $\pm\infty$.

(v). $\sum_{m\in\mathbb{Z}^+}\frac{(-1)^{m+1}}{m}$ diverges absolutely, but doesn't diverge (that is, is conditionally divergent).

Example 2.4.4.4 — A series which is conditionally divergent but not conditionally convergent $\sum_{m\in\mathbb{Z}^+}(-1)^m$ is conditionally divergent but not conditionally convergent. Of course, $\sum_{m\in\mathbb{Z}^+}|(-1)^m|$ diverges, but on the other hand the original series does not diverge (it is only nonconvergent). On the other hand, the original series doesn't converge, and so certainly can't be conditionally convergent.

Exercise 2.4.4.5 — Geometric series Let $|a|<1$. Show that $\sum_{m\in\mathbb{N}} a^m = \frac{1}{1-a}$.

You'll probably recall several tests from calculus that allow us to determine whether or not a given series converges. One of the primary goals of this section is to prove several of these tests.

Perhaps the first thing you should check is whether or not the terms of the series go to 0, if only because it (often) takes no time to check at all.

Proposition 2.4.4.6 Let $m\mapsto a_m$ be a sequence. Then, if $\sum_{k\in\mathbb{N}} a_k$ converges, then $\lim_m a_m = 0$.

R As just mentioned, this is often used in the contrapositive: if $\lim_m a_m \neq 0$, then $\sum_{k\in\mathbb{N}} a_k$ doesn't converge.

Proof. Suppose that $\sum_{k\in\mathbb{N}} a_k$ converges. Let $\varepsilon>0$. Then, as the sequence of partial sums is Cauchy, there is some $m_0\in\mathbb{N}$

such that, whenever $m_0 \leq m \leq n$, it

$$\left|\sum_{k=m+1}^{n} a_k\right| = \left|\sum_{k=0}^{n} a_k - \sum_{k=0}^{m} a_k\right| < \varepsilon. \tag{2.4.4.7}$$

As this hold *for all* $m \leq n$, we may take $n := m + 1$, in which case this inequality reduces to

$$|a_{m+1} - 0| = |a_{m+1}| < \varepsilon. \tag{2.4.4.8}$$

Hence, by definition, $\lim_m a_m = 0$. ■

Thus, if the terms do not go to 0, you can immediately conclude that the series does not converge. There is a similar result about the entire 'tail-end' of a series, though it is probably more useful in proofs than as a test for convergence.

Exercise 2.4.4.9 Let $m \mapsto a_m$ be a sequence. Show that if $\sum_{k\in\mathbb{N}} a_k$ converges, then $\lim_m \sum_{k=m}^{\infty} a_k = 0$.

Proposition 2.4.4.10 — Absolute convergence implies convergence Let $m \mapsto a_m$ be a sequence and suppose that $\sum_{k\in\mathbb{N}} |a_k|$ exists. Then, $\sum_{k\in\mathbb{N}} a_k$ exists.

Proof. To show that $\sum_{k\in\mathbb{N}} a_k$ exists, we show that the sequences of partials sums $m \mapsto s_m := \sum_{k=0}^{m} a_k$ is Cauchy. Take $m \leq n$. Then,

$$\begin{aligned}\left|\sum_{k=0}^{n} a_k - \sum_{k=0}^{m} a_k\right| &= \left|\sum_{k=m+1}^{n} a_k\right| \leq \sum_{k=m+1}^{n} |a_k| \\ &= \left|\sum_{k=0}^{n}|a_k| - \sum_{k=0}^{m}|a_k|\right|.\end{aligned} \tag{2.4.4.11}$$

Now we are essentially done. Do you see why? This is important as you eventually you want to get to the point where the above is all you need to convince yourself that the statement is true.[a]

Let $\varepsilon > 0$ and choose m_0 so that whenever $m_0 \leq m \leq n$ it follows that

$$\left|\sum_{k=0}^{n}|a_k| - \sum_{k=0}^{m}|a_k|\right| < \varepsilon. \tag{2.4.4.12}$$

Then, it follows from (2.4.4.11) that whenever $m_0 \leq m \leq n$ that

$$\left|\sum_{k=0}^{n}a_k - \sum_{k=0}^{m}a_k\right| < \varepsilon, \tag{2.4.4.13}$$

which shows that the sequence of partial sums is Cauchy, and hence the series converges. ∎

[a] Don't 'cheat' though—if you don't yet know how to fill in the details, keep filling them in: eventually, after writing down the details sufficiently many times, you will be able to look at the above and confidently declare the rest as "obvious".

The next test we present is the ***Alternating Series Test***.

Definition 2.4.4.14 — Alternating series A series $\sum_{k\in\mathbb{N}} a_k$ is ***alternating*** iff $\operatorname{sgn}(a_{k+1}) = -\operatorname{sgn}(a_k)$ for all $k \in \mathbb{N}$.

R In this case, we may write $a_k = \pm(-1)^k b_k$ where $b_k \geq 0$ and the $\pm$ is determined depending on the sign of the first term.

Proposition 2.4.4.15 — Alternating Series Test Let $m \mapsto a_m$ be a nonincreasing sequence that converges to 0. Then, $\sum_{m\in\mathbb{N}}(-1)^m a_m$ converges.

Proof. The first thing to notice is that

$$\begin{aligned}\sum_{k=m}^{n}(-1)^{k-m}a_k &= a_m-(a_{m+1}-a_{m+2})-(a_{m+3}-a_{m+4})-\cdots \\ &\quad -(a_{n-1}-a_n) \\ &\leq a_m\end{aligned}$$

because each $a_k - a_{k+1} \geq 0$.[a]

Using this, we see that

$$\begin{aligned}\left|\sum_{k=0}^{n}(-1)^k a_k - \sum_{k=0}^{m}(-1)^k a_k\right| &= \left|\sum_{k=m+1}^{n}(-1)a_k\right| \qquad (2.4.4.16)\\ &\leq a_{m+1}.\end{aligned}$$

Now that the partial sums are Cauchy follows from the fact that $\lim_m a_m = 0$. ■

[a] Depending on whether $m-n$ is even or odd, the last term might have instead just be $-a_n$ instead of $-(a_{n-1}-a_n)$. Either way, the same inequality holds.

Proposition 2.4.4.17 — Comparison Test Let $m \mapsto a_m$ and $m \mapsto b_m$ be sequences such that eventually $|a_m| \leq |b_m|$. Then,

(i). if $\sum_{m\in\mathbb{N}} a_m$ diverges absolutely, then $\sum_{m\in\mathbb{N}} b_m$ diverges absolutely; and

(ii). if $\sum_{m\in\mathbb{N}} b_m$ converges absolutely, then $\sum_{m\in\mathbb{N}} a_m$ converges absolutely.

W Warning: (i) will fail if $\sum_{m\in\mathbb{N}} b_m$ is only conditionally convergent—see the following exercise (Exercise 2.4.4.19).

W Warning: (ii) will fail if $\sum_{m\in\mathbb{N}} b_m$ only *conditionally* converges—see the following exercise (Exercise 2.4.4.19).

Proof. (i) follows from the fact that the operation of taking limits (in this case, applied to the partial sums) is nondecreasing—see Exercise 2.4.3.19.

Exercise 2.4.4.18 Prove (ii).

■

Exercise 2.4.4.19

(i). Find an example of sequences $m \mapsto a_m$ and $m \mapsto b_m$ with (i) $|a_m| \leq |b_m|$, (ii) $\sum_{m\in\mathbb{N}} a_m$ divergent, but (iii) $\sum_{m\in\mathbb{N}} b_m$ not divergent.

(ii). Find an example of sequences $m \mapsto a_m$ and $m \mapsto b_m$ with (i) $|a_m| \leq |b_m|$, (ii) $\sum_{m\in\mathbb{N}} b_m$ convergent, but (iii) $\sum_{m\in\mathbb{N}} a_m$ not convergent.

Proposition 2.4.4.20 — Limit Comparison Test Let $m \mapsto a_m$ and $m \mapsto b_m$ be eventually nonnegative sequences such that $\lim_m \left|\frac{a_m}{b_m}\right|$ converges to a nonzero number. Then, $\sum_{m\in\mathbb{N}} a_m$ converges iff $\sum_{m\in\mathbb{N}} b_m$ converges.

Proof. Define $L := \lim_m \left|\frac{a_m}{b_m}\right| > 0$. Let $\varepsilon > 0$ be less than L, and choose $m_0 \in \mathbb{N}$ such that, whenever $m \geq m_0$, it follows that[a]

$$\left|\frac{a_m}{b_m} - L\right| < \varepsilon, \tag{2.4.4.21}$$

that is,

$$-\varepsilon < \frac{a_m}{b_m} - L < \varepsilon, \tag{2.4.4.22}$$

or rather

$$b_m(L-\varepsilon) < a_m < b_m(L+\varepsilon). \tag{2.4.4.23}$$

Note that $L-\varepsilon > 0$. It follows from the Comparison Test applied to the first inequality that, if $\sum_{m\in\mathbb{N}} a_m$ converges, then $\sum_{m\in\mathbb{N}} b_m$ converges. Likewise, it follows from the second inequality that, if $\sum_{m\in\mathbb{N}} b_m$ converges, then $\sum_{m\in\mathbb{N}} a_m$ converges. ■

[a]The absolute values signs went away because the sequences are eventually *nonnegative*—we have implicitly chosen $m_0 \in \mathbb{N}$ large enough so that $a_m, b_m \geq 0$ for all $m \geq m_0$.

Proposition 2.4.4.24 — Ratio Test Let $m \mapsto a_m$ be a sequence. Then, if $\lim_m \left|\frac{a_{m+1}}{a_m}\right| < 1$,[a] then $\sum_{m\in\mathbb{N}} a_m$ converges absolutely; if $\lim_m \left|\frac{a_{m+1}}{a_m}\right| > 1$, then $\sum_{m\in\mathbb{N}} a_m$ diverges.

[a]It is implicit in this assumption that we also assume that $m \mapsto a_m$ is eventually nonzero, so that this hypothesis actually makes sense.

Proof. Suppose that $r := \lim_m \left|\frac{a_{m+1}}{a_m}\right| < 1$. Let $\varepsilon > 0$ be less than $1-r$. Then, there is some m_0 such that, whenever $m \geq m_0$, it follows that

$$\left|\frac{a_{m+1}}{a_m} - r\right| \leq \left|\left|\frac{a_{m+1}}{a_m}\right| - r\right| < \varepsilon, \tag{2.4.4.25}$$

so that $\left|\frac{a_{m+1}}{a_m}\right| < r + \varepsilon =: C < 1$, so that $|a_{m+1}| < C|a_m|$, so that $\left|a_{m_0+l}\right| < C^l\left|a_{m_0}\right|$ for $l \geq 0$. As $|C| < 1$, it follows from the Comparison Test (Proposition 2.4.4.17) that $\sum_{k=m_0}^{\infty} |a_k|$ converges, and hence that $\sum_{m\in\mathbb{N}} a_m$ converges absolutely.

Exercise 2.4.4.26 Prove the case where $\lim_m \left|\frac{a_{m+1}}{a_m}\right| > 1$.

■

Proposition 2.4.4.27 — Root Test Let $m \mapsto a_m$ be a sequence. Then, if $\limsup_m |a_m|^{\frac{1}{m}} < 1$, then $\sum_{m\in\mathbb{N}} a_m$ converges absolutely; if $\limsup_m |a_m|^{\frac{1}{m}} > 1$, then $\sum_{m\in\mathbb{N}} a_m$ doesn't converge.

R My impression is that this test is generally less frequently taught in calculus classes than most of the other ones we discuss. Despite this, I think it's relatively important, certainly more so than AP Calculus might have one believe. For one thing, it's the only convergence test we present that gives a sufficient condition for convergence for general series that doesn't make reference to other series—no clever 'test' series required—just 'mindlessly' compute the limit superior, and unless you get 'unlucky' with a result of 1, you have your answer. Another reason for its importance is the Proposition (Proposition 6.4.5.4), which gives a formula for the radius of convergence of a power series—this theorem is a nearly immediate corollary of the Root Test.

W Warning: The $\limsup_m |a_m|^{\frac{1}{m}}$ case fails if you replace "doesn't converge" with "diverges". For example, take the sequence of terms

$$\langle 1, -1, 2, -2, 4, -4, 8, -8, \ldots \rangle. \tag{2.4.4.28}$$

The partial sums alternate between 0 and a power of 2, and so, while this doesn't converge, it also doesn't diverge.

Proof. Suppose that $u := \limsup_m |a_m|^{\frac{1}{m}} < 1$. Let $\varepsilon > 0$ be less than $1 - u > 0$. Then, there is some m_0 such that, whenever $m \geq m_0$, it follows that

$$\sup_{n\geq m}\{|a_n|^{\frac{1}{n}}\} - u < \varepsilon, \tag{2.4.4.29}$$

so that $\sup_{n\geq m}\{|a_n|^{\frac{1}{n}}\} < \varepsilon + u < 1$, so that $|a_n|^{\frac{1}{n}} < r := \varepsilon + u < 1$ for all $n \geq m_0$, so that $|a_n| < r^n$ for all $n \geq m_0$. It follows that $\sum_{m\in\mathbb{N}} a_m$ converges absolutely by the comparison test.

Exercise 2.4.4.30 Prove the case where

$$\limsup_m |a_m|^{\frac{1}{m}} > 1. \tag{2.4.4.31}$$

■

■ **Example 2.4.4.32 — Harmonic Series** Consider the series $\sum_{m\in\mathbb{Z}^+}(-1)^{m+1}\frac{1}{m}$ with terms $a_m := (-1)^{m+1}\frac{1}{m}$. This is called the ***Alternating Harmonic Series***, whereas the series of absolute values $\sum_{m\in\mathbb{Z}^+}\frac{1}{m}$ is the ***Harmonic Series***. We see immediately from the Alternating Series Test that the alternating harmonic series converges. In fact, it converges to $\ln(2)$,[a] though we don't know that yet (we don't even know what $\ln(2)$ is!). The harmonic series itself though diverges (so that the alternating harmonic series is conditionally convergent). To see this, we apply the Comparison Test:

$$\begin{aligned}
\sum_{m\in\mathbb{Z}^+}\frac{1}{m} &= 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}+\cdots \\
&\geq 1+\tfrac{1}{2}+\left(\tfrac{1}{4}+\tfrac{1}{4}\right)+\left(\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}\right) \\
&= 1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\cdots = \infty.
\end{aligned}$$

R If you take a string and fix its endpoints, then there are only countably many "fundamental" frequencies that the string can vibrate at (fundamental in the sense that any way in which the string might vibrate can be written as a sum of the fundamental frequency solutions). These are called the *harmonics* and the wavelength of every harmonic is of the form $\frac{1}{m}2L$, where L is the length of the string. This is where the term "harmonic" comes from (or so it seems).

[a] See Exercise 6.4.5.40.

In fact, there is a test that will tell us that the Harmonic Series diverges immediately, called the *p-series Test*. It says that $\sum_{n\in\mathbb{Z}^+}\frac{1}{n^p}$ converges (absolutely) iff $p > 1$. While I suppose it is possible to prove it at the moment,[10] I think the most natural proof makes use of the integral, and so we postpone the proof—see Proposition 6.4.5.76. More specifically, this proof will make use of the *Integral Test*, which, obviously, first requires we introduce the integral, and so we likewise postpone this as well—see Proposition 5.2.3.15.

Decimal expansions

We now return to the issue left unfinished from Chapter 1 having to do with your "naive" idea of the real numbers, namely, as decimal expansions. In mathematics, it is almost never a good idea to prove something about the real numbers using decimal expansions. On the other hand, I suppose it's nice to know that this new object does in fact correspond to what you are familiar with.

In any case, without further ado, here is the precise statement of what is meant by "decimal expansions".

Theorem 2.4.4.33 — Decimal expansions. Let $r \in \mathbb{Z}^+$ be at least 2 and let $x \in \mathbb{R}$. Then, there is a $x_0 \in \mathbb{N}$ and $x_k \in \{0,\ldots,r-1\}$ for $k \in \mathbb{Z}^+$ such that

(i).
$$x = \operatorname{sgn}(x)\left(x_0 + \sum_{k=1}^{\infty}\frac{x_k}{r^k}\right); \tag{2.4.4.34}$$
and

(ii). $k \mapsto x_k$ is not eventually the constant $r-1$.

R It is then customary to write $x = \operatorname{sgn}(x)x_0.x_1x_2x_3\ldots$, possibly using separate set of symbols to represent the numbers in $\{0,1,\ldots,r-1\}$ (for example, in hexadecimal, it is customary to use the symbol "A" to represent the number $10 \in \mathbb{Z}^+$, the symbol "B" to represent the number $11 \in \mathbb{Z}^+$, and so on).

[10] See [YL05], for example.

R r is the ***radix*** of the number system, and it correspond to the number of distinct symbols one uses to represent a number in a 'decimal expansion' (though, strictly speaking I suppose, "decimal expansion" really refers to the case $r = 10$). Of course, you cannot distinguish between numbers using a single symbol alone (at least certainly not all real numbers), and so we need at least $r \geq 2$. $r = 2$ corresponds to ***binary***, $r = 3$ corresponds to ***ternary***, $r = 8$ correspond to ***octal***, $r = 10$ corresponds to ***decimal***, $r = 16$ corresponds to ***hexadecimal***, and those are the only names I am aware of.

R The condition (ii) is needed to ensure uniqueness. For example, we have $1 = 1.00\ldots$ as well as $1 = 0.99\ldots$ for $r = 10$.

R The exact same statement without the "$\operatorname{sgn}(x)$" is still true, but it's not the standard way of writing numbers. For example, in this case, we would be writing $-3.6 = -4 + 0.4$, whereas the 'usual' decimal expansion corresponds to writing $-3.6 = -3 - 0.6 = -(3 + 0.6)$.

Proof. STEP 1: REPLACE x WITH $|x|$
We always have that $x = \operatorname{sgn}(x)|x|$. Thus, what we actually want to do is find a unique $x_0 \in \mathbb{N}$ and $x_k \in \{0, \ldots, r-1\}$ for $k \in \mathbb{Z}^+$ such that

(i).

$$|x| = x_0 + \sum_{k=1}^{\infty} \frac{x_k}{r^k}; \tag{2.4.4.35}$$

and

(ii). $k \mapsto x_k$ is not eventually the constant $r - 1$.

Thus, it suffices to prove the result in the case $x \geq 0$. We assume this throughout.

Step 2: Define x_0
First of all, let $x_0 \in \mathbb{N}$ be the largest integer less-than-or-equal to x.[a] Define $y_0 := x - x_0 \in [0, 1)$.

Step 3: Define x_1
Note that $\frac{d}{r} \in [0, 1)$ for $d \in \{0, \ldots, r-1\}$. As the set $\left\{d \in \{0, \ldots, r-1\} : \frac{d}{r} \leq y_0\right\}$ is nonempty and finite, it has a largest element. Call that element x_1 and define $y_1 := y_0 - \frac{x_1}{r} \in [0, \frac{1}{r})$.

Exercise 2.4.4.36 We have claimed that $y_1 \in [0, \frac{1}{r})$. Prove this.

Step 4: Define x_k for $k \in \mathbb{Z}^+$ inductively
Note again that $\frac{d}{r^2} \in [0, \frac{1}{r})$ for $d \in \{0, \ldots, r-1\}$, and so, similarly as before, there is a largest $x_2 \in \{0, \ldots, r-1\}$ such that $\frac{x_2}{r^2} \leq y_1$.

Continue this process inductively defining x_k for all $k \in \mathbb{Z}^+$.

Exercise 2.4.4.37 Show that in fact

$$x = x_0 + \sum_{k=1}^{\infty} \frac{x_k}{r^k}. \tag{2.4.4.38}$$

Step 5: Verify (i) and (ii)
We now check that $k \mapsto x_k$ is not eventually the constant $r-1$. To see this, note that

$$\sum_{k=k_0}^{\infty} \frac{r-1}{r^k} = \frac{1}{r^{k_0-1}} \tag{2.4.4.39}$$

for $l \in \mathbb{Z}^+$. Thus, if we had $x_k = r-1$ for $k \geq k_0$ but $x_{k_0-1} < r-1$, then $x_{k_0-1} \in \{0,\ldots,r-1\}$ would no longer be the larges element such that $\frac{x_{k_0-1}}{r^{k_0-1}} \leq y_{k_0-2}$—$x_{k_0-1}+1 \in \{0,\ldots,r-1\}$ would be a larger element which satisfies the same inequality: a contradiction. Therefore, it cannot be the case that $k \mapsto x_k$ is eventually the constant $r-1$.

Step 6: Show that $x_0 = x_0'$ in uniqueness

It remains to check uniqueness. So, let $x_0' \in \mathbb{Z}$ and $x_k' \in \{0,\ldots,r-1\}$ be such that

$$x = x_0' + \sum_{k=1}^{\infty} \frac{x_k'}{r^k} \tag{2.4.4.40}$$

and $k \mapsto x_k'$ is not eventually the constant $r-1$.

As both these 'decimal expansions' are equal to x, we have

$$x_0 - x_0' = \sum_{k=1}^{\infty} \frac{x_k' - x_k}{r^k}, \tag{2.4.4.41}$$

and hence

$$\left|x_0 - x_0'\right| \leq \sum_{k=1}^{\infty} \frac{\left|x_k' - x_k\right|}{r^k}. \tag{2.4.4.42}$$

Now, we certainly have that $\left|x_k' - x_k\right| \leq r-1$, the 'worst case scenario' being when one is 0 and the other is $r-1$. We wish to show that this can't *always* be the case, that is, for at least some k we have $\left|x_k' - x_k\right|$ is *strictly* less than $r-1$. If we can show that, then we will have

$$\left|x_0 - x_0'\right| < \sum_{k=1}^{\infty} \frac{r-1}{r^k} = 1, \tag{2.4.4.43}$$

forcing $x_0 = x_0'$ (by a corollary of Exercise 1.2.25). So, we now show that there is at least some k for which $\left|x_k' - x_k\right| < r-1$.

We proceed by contradiction: suppose that $\left|x'_k - x_k\right| = r - 1$ for all $k \in \mathbb{Z}^+$. This means that, for each k, either $x'_k = r - 1$ and $x_k = 0$, or vice-versa. Furthermore, note that regardless, the inequality (2.4.4.43) is nonstrict, which still forces $\left|x_0 k - x'_0\right| = 0$ or $\left|x_0, x'_0\right| = 1$. In the former case we are done, so we can assume without loss of generality that $x'_0 = x_0 + 1$. This implies that we must have

$$\sum_{k=1}^{\infty} \frac{x_k}{r^k} = 1 + \sum_{k=1}^{\infty} \frac{x'_k}{r^k}. \tag{2.4.4.44}$$

We wish to show that this forces $x_k = r - 1$ for all $k \in \mathbb{Z}^+$, which will contradict our hypothesis. We prove that inductively. We already know that $x_1 = r - 1$ and $x'_1 = 0$, or vice-versa. If the "vice-versa" were true, then the above equation gives

$$1 + \frac{1}{r} = \sum_{k=2}^{\infty} \frac{x_k - x'_k}{r^k} \leq \sum_{k=2}^{\infty} \frac{r-1}{r^k} = \frac{1}{r}: \tag{2.4.4.45}$$

a contradiction. Thus, we must have that $x_1 = r - 1$ and $x'_1 = 0$.

Exercise 2.4.4.46 Complete the induction argument, using our proof for the $k = 1$ case as guidance.

Step 7: Show that $x_1 = x'_1$ in uniqueness

Now that we know $x_0 = x'_0$,

$$x_0 + \sum_{k=1}^{\infty} \frac{x_k}{r^k} x = x'_0 + \sum_{k=1}^{\infty} \frac{x'_k}{r^k} \tag{2.4.4.47}$$

simplifies to

$$x_1 - x'_1 = \sum_{k=2}^{\infty} \frac{x'_k - x_k}{r^{k-1}} \tag{2.4.4.48}$$

This is exactly the same setup as in the previous step, and so the previous step gives us $x_1 = x_1'$.

STEP 8: SHOW THAT $x_k = x_k'$ FOR $k \in \mathbb{Z}^+$ IN UNIQUENESS

Applying this argument inductively, we find that $x_k = x_k'$ for all $k \in \mathbb{Z}^+$, finishing the proof of uniqueness. ■

[a] By the Archimedean Property, there is some integer larger than x. As $\mathbb{N}$ is well-ordered, there is a smallest integer strictly larger than x. That integer minus one will be the largest integer less-than-or-equal to x.

The uncountability of the real numbers

We mentioned at the end of Section 2.1 Cardinality and countability that $|\mathbb{R}| = 2^{\aleph_0}$ but at the time we did not know enough to prove it. We now return to this.

Theorem 2.4.4.49. $|\mathbb{R}| = 2^{\aleph_0}$.

Proof. To prove this, we apply the Bernstein-Cantor-Schröder Theorem (Theorem 1.1.3.5). Thus, we wish to construct in injection from $2^{\mathbb{N}}$ to $\mathbb{R}$ and an injection from $\mathbb{R}$ to $2^{\mathbb{N}}$. By Exercise A.3.31, we may replace $2^{\mathbb{N}}$ with $\{0,1\}^{\mathbb{N}}$ in this statement, and hence in return with $\{0,2\}^{\mathbb{N}}$ (you will see why we do this momentarily).

The set $\{0,2\}^{\mathbb{N}}$ is just the collection of sequences $m \mapsto a_m$ with $a_m \in \{0,2\}$. We thus define $\phi\colon \{0,2\}^{\mathbb{N}} \to \mathbb{R}$ by

$$\phi(m \mapsto x_m) := \sum_{m \in \mathbb{N}} \tfrac{a_m}{3^m}. \tag{2.4.4.50}$$

Intuitively, the sequence $m \mapsto a_m$ is thought of as the ternary expansion of a real number. Switching from $\{0,1\}$ to $\{0,2\}$ was tantamount to changing from binary to ternary and not including any numbers with 1 in their ternary expansion. The reason for this is because, in binary, we have

$$1 = .\bar{1} := .111\cdots, \tag{2.4.4.51}$$

and so the resulting function would not be injective. Of course, in ternary, we still have things like

$$1 = .\bar{2} = .222\cdots, \tag{2.4.4.52}$$

but 1 corresponds to the sequence $\langle 1, 0, 0, 0, \ldots \rangle$, which is not an element of $\{0, 2\}^{\mathbb{N}}$, and so injectivity works out.

Exercise 2.4.4.53 Show that ϕ is injective.

It follows that $2^{\aleph_0} \leq |\mathbb{R}|$.

As $|\mathbb{Q}| = \aleph_0$, it suffices to replace $2^{\mathbb{N}}$ by $2^{\mathbb{Q}}$ above, and so it suffices to produce an injection from $\mathbb{R}$ to $2^{\mathbb{Q}}$, the power-set of $\mathbb{Q}$. Define $\psi \colon \mathbb{R} \to 2^{\mathbb{Q}}$ by

$$\psi(x) := \{q \in \mathbb{Q} : q \leq x\}. \tag{2.4.4.54}$$

Exercise 2.4.4.55 Show that ψ is injective.

It follows that $|\mathbb{R}| \leq 2^{\aleph_0}$, and hence that $|\mathbb{R}| = 2^{\aleph_0}$. ∎

Addition of infinitely many terms is 'noncommutative'

We will make the title of this subsubsection precise in a moment, but for the moment we will settle for something imprecise: there exists two convergent series $\sum_{m\in\mathbb{N}} a_m$ and $\sum_{m\in\mathbb{N}} b_m$ which converge to different values, but yet $\{a_m : m \in \mathbb{N}\} = \{b_m : m \in \mathbb{N}\}$. That is, the terms themselves are the same (though in different order of course), but yet the series converge to different values! If this is your first time studying 'rigorous' mathematics, this is probably one of these "WTF!? moments" when you realize that sometimes mathematics can be so counter-intuitive so as to demand rigor—if we didn't require a proof, it would be very easy to dismiss 'commutativity' of infinite series as "obvious". In fact, it's not usually a good idea to use the word "obvious" in proofs at all—at best it's lazy, and at worst, it's just plain wrong.

Despite the fact that this crazy sort of 'noncommutativity' can happen, it never happens for *absolutely* convergent series.

Theorem 2.4.4.56. Let $m \mapsto a_m$ be a sequence such that $\sum_{m \in \mathbb{N}} a_m$ converges absolutely and let $\phi \colon \mathbb{N} \to \mathbb{N}$ be a bijection. Then, $\sum_{m \in \mathbb{N}} a_{\phi(m)}$ converges absolutely and $\sum_{m \in \mathbb{N}} a_m = \sum_{m \in \mathbb{N}} a_{\phi(m)}$.

R That ϕ is a bijection is the way we make precise the idea that $b_m \coloneqq a_{\phi(m)}$ is just a 'rearrangement' of the original terms. Thus, this theorem says that the rearrangement of any absolutely convergent series converges to the same value.

Proof. Define $S \coloneqq \sum_{m \in \mathbb{N}} a_m$. Let $\varepsilon > 0$ and choose m_0 such that, whenever $m \geq m_0$, it follows that $\left|\sum_{k=0}^{m} a_k - S\right| < \varepsilon$. Choose m_1 such that, whenever $m \geq m_1$, it follows that $\sum_{k=m+1}^{\infty} |a_k| < \varepsilon$ (we may do this because of the absolute convergence—see Exercise 2.4.4.9). Replace m_0 by $\max\{m_0, m_1\}$, so that, whenever $m \geq m_0$, it follows that both of these inequalities hold. Define

$$n_0 \coloneqq \max\left(\phi^{-1}\left(\{m \in \mathbb{N} : m \leq m_0\}\right)\right). \tag{2.4.4.57}$$

The set $\{m \in \mathbb{N} : m \leq n_0\}$ is finite, and so ϕ of that set is finite, and so the definition of n_0 makes sense. This definition guarantees that

$$\{0, 1, \ldots, m_0 - 1, m_0\} \subseteq \phi\left(\{0, 1, \ldots, n_0 - 1, n_0\}\right). \tag{2.4.4.58}$$

Suppose that $n \geq n_0$. Then,

$$\begin{aligned} \left|\sum_{k=0}^{n} a_{\phi(k)} - S\right| &\leq \left|\sum_{k=0}^{n} a_{\phi(k)} - \sum_{k=0}^{m_0} a_k\right| + \left|\sum_{k=0}^{m_0} a_k - S\right| \\ &<^{a} \sum_{k=m_0+1}^{\infty} |a_k| + \varepsilon < 2\varepsilon. \end{aligned} \tag{2.4.4.59}$$

■

[a] In the first term, because of our choice of n_0, every a_k for $0 \leq k \leq a_{m_0}$ appears somewhere in $a_{\phi(k)}$ for $0 \leq k \leq n$. Thus, the difference only contains terms with index at least $m_0 + 1$. We then apply the triangle inequality again.

And now we turn to the precise statement of the idea that "addition of infinitely many real numbers is 'noncommutative'.".

Theorem 2.4.4.60. Let $m \mapsto a_m$ be a sequence and define

$$a_m^+ := \tfrac{1}{2}(|a_m| + a_m) \text{ and } a_m^- := \tfrac{1}{2}(|a_m| - a_m). \quad (2.4.4.61)$$

Then,

(i). if $\sum_{m\in\mathbb{N}} a_m$ converges conditionally, then

$$\sum_{m\in\mathbb{N}} a_m^+ = \infty = \sum_{m\in\mathbb{N}} a_m^-; \quad (2.4.4.62)$$

and

(ii). if

$$\sum_{m\in\mathbb{N}} a_m^+ = \infty = \sum_{m\in\mathbb{N}} a_m^-, \quad (2.4.4.63)$$

then for every $x, y \in [-\infty, \infty]$ with $x \leq y$, there exists a bijection $\phi\colon \mathbb{N} \to \mathbb{N}$ such that $\liminf_m \sum_{k=0}^m a_{\phi(k)} = x$ and $\limsup_m \sum_{k=0}^m a_{\phi(k)} = y$.

In particular, taking $x = y$, there is a bijection $\phi\colon \mathbb{N} \to \mathbb{N}$ such that $\sum_{m\in\mathbb{N}} a_{\phi(m)} = x$.

R Note that $a_m = a_m^+ - a_m^-$ (and $|a_m| = a_m^+ + a_m^-$) with $a_m^\pm \geq 0$. Furthermore, $a_m^+ > 0$ iff $a_m > 0$ and $a_m^- > 0$ iff $a_m < 0$ (and are 0 otherwise). This trick of defining the 'nonnegative part' and 'nonpositive' part of a function is not that uncommon,[a] and it would do you well to remember it.

R The 'point' of this theorem, at least for us now, is the case in which $\sum_{m\in\mathbb{N}} a_m$ is conditionally convergent. However, it holds more generally, and so we may as well state it more generally (especially as this will eventually be of use to us—see the proof of Theorem 5.2.2.36).

R This theorem says something really quite surprising: Given a series that converges conditionally, you can rearrange the terms to obtain a series which converges to *any real number you choose whatsoever.*[b]

---

[a]For example, see Definition 5.2.1.40.

[b]And in fact it says *even more* than this—what we are referring to in this remark is just the special case.

Proof. [a] (i) Suppose that $\sum_{m\in\mathbb{N}} a_m$ converges conditionally. As $|a_m| = a_m^+ + a_m^-$, we must have that

$$\infty = \sum_{m\in\mathbb{N}} |a_m| = \sum_{m\in\mathbb{N}} [a_m^+ + a_m^-], \tag{2.4.4.64}$$

so that at least one of $\sum_{m\in\mathbb{N}} a_m^+$ and $\sum_{m\in\mathbb{N}} a_m^-$ must diverge. On the other hand, as $a_m = a_m^+ - a_m^-$,

$$\infty > \sum_{m\in\mathbb{N}} a_m = \sum_{m\in\mathbb{N}} [a_m^+ - a_m^-], \tag{2.4.4.65}$$

and so we cannot just have one of $\sum_{m\in\mathbb{N}} a_m^+$ and $\sum_{m\in\mathbb{N}} a_m^-$ diverge. Thus, we must have that

$$\sum_{m\in\mathbb{N}} a_m^+ = \infty = \sum_{m\in\mathbb{N}} a_m^-. \tag{2.4.4.66}$$

(ii) Suppose that

$$\sum_{m\in\mathbb{N}} a_m^+ = \infty = \sum_{m\in\mathbb{N}} a_m^-. \tag{2.4.4.67}$$

Let $x, y \in [-\infty, \infty]$ with $x \leq y$.

Note that $a_m^+ = a_m$ and $a_m^- = 0$ if $a_m \geq 0$, and $a_m^+ = 0$ and $a_m^- = -a_m$ if $a_m \leq 0$. Thus, modulo the existence of zero terms which make no difference, every term in both of the series of (2.4.4.66) is a term in the original series $\sum_{m\in\mathbb{N}} a_m$ (up to a minus sign in the latter case). We will build up a rearrangement of the original series $\sum_{m\in\mathbb{N}} a_m$ using a_m^+ and $-a_m^-$.

The intuition is this: because both of the series of (2.4.4.66) diverge, I can move as 'far to the right' as I like by choosing terms of the form a_m^+, and likewise, I can move as 'far to the left' as I like by choosing terms of the form $-a_m^-$. We make this precise as follows.

First, suppose that x, y are finite.[b]

Let m_0 be the smallest natural number such that

$$\sum_{k=0}^{m_0} a_k^+ \geq y. \tag{2.4.4.68}$$

Such an m_0 exists because the first series in (2.4.4.66) diverges. Note that, because m_0 is the *smallest* such number, we must have that

$$\sum_{k=0}^{m_0-1} a_k^+ < y, \tag{2.4.4.69}$$

so that

$$0 \leq \sum_{k=0}^{m_0} a_k^+ - y < a_{m_0}^+. \tag{2.4.4.70}$$

Then, let n_0 be the smallest natural number such that

$$\sum_{k=0}^{m_0} a_k^+ - \sum_{k=0}^{n_0} a_k^- \leq x. \tag{2.4.4.71}$$

Similarly as before, we now have that

$$0 \leq x - \left(\sum_{k=0}^{m_0} a_k^+ - \sum_{k=0}^{n_0} a_k^- \right) < a_{n_0}^-. \tag{2.4.4.72}$$

Do this again: let m_1 and n_1 be the smallest natural numbers such that

$$\sum_{k=0}^{m_0} a_k^+ - \sum_{k=0}^{n_0} a_k^- + \sum_{k=m_0+1}^{m_1} a_k^+ \geq y \tag{2.4.4.73}$$

and

$$\sum_{k=0}^{m_0} a_k^+ - \sum_{k=0}^{n_0} a_k^- + \sum_{k=m_0+1}^{m_1} a_k^+ - \sum_{k=n_0+1}^{n_1} a_k^- \leq x \tag{2.4.4.74}$$

respectively. (Of course, inequalities analogous to (2.4.4.70) and (2.4.4.72) hold here as well.) Continue this process inductively. The series

$$\begin{split} \sum_{k=0}^{m_0} a_k^+ - \sum_{k=0}^{n_0} a_k^- + \sum_{k=m_0+1}^{m_1} a_k^+ - \sum_{k=n_0+1}^{n_1} a_k^- \\ + \sum_{k=m_1+1}^{m_2} a_k^+ - \sum_{k=n_1+1}^{n_2} a_k^- + \cdots \end{split} \tag{2.4.4.75}$$

is a rearrangement of the original series. Denote the partial sums of this series by S_m. Thus, the inequalities (2.4.4.70) and (2.4.4.72), in terms of S_m, look like[c]

$$0 \leq S_m - y < a_{i_m}^+ \text{ and } 0 \leq x - S_m < a_{j_m}^-, \tag{2.4.4.76}$$

where $i, j : \mathbb{N} \to \mathbb{N}$ are strictly increasing functions of m (these are the m_k and n_ks). Hence,

$$0 \leq \sup_{m \geq m_0} \{S_m\} - y < \sup_{m \geq m_0} \{a_{i_m}^+\} \tag{2.4.4.77a}$$

$$0 \leq x - \inf_{m \geq m_0} \{S_m\} < \inf_{m \geq m_0} \{a_{j_m}^-\} \tag{2.4.4.77b}$$

Recall however that $\sum_{m\in\mathbb{N}} a_m$ converges. It follows that $\lim_m a_m = 0$, and so in turn $\lim_m a_m^+ = 0 = \lim_m a_m^-$. Thus, taking the limit of (2.4.4.77) with respect to m_0, we obtain $\liminf_m S_m = x$ and $\limsup_m S_m = y$.

Exercise 2.4.4.78 Modify this argument to prove the case where at least one of x or y is not finite.

■

[a] Adapted from [Rud76].

[b] Note in the statement that we are allowed to take $x = -\infty$ and $y = +\infty$!

[c] Note that I can replace m_0 with $m \leq m_0$ on the *left-hand* side of (2.4.4.70) and the inequality still remains valid. Similarly, I can replace n_0 with $m \leq n_0$ on the *left-hand* side of (2.4.4.72) and the inequality still remains valid.

2.4.5 Subnets and subsequences

The concept of a subnet is *almost* what you think it should be. To help understand the concept before we go to the precise definition, let's think of what subnets of *sequences* should be.

Let $x\colon \mathbb{N} \to \mathbb{R}$ be a sequence and let $S \subseteq \mathbb{N}$. When should $x|_S$ be a *subsequence* of the original $m \mapsto x_m$? Well certainly if S is finite, we should not consider $x|_S$ to be a subsequence—we need the indices to get arbitrarily large. Moreover, whatever our definition of subsequence is, it should have the property that, if $m \mapsto x_m$ converges to x_∞, then every subsequence of $m \mapsto x_m$ should converge to x_∞ as well. If S is allowed to be finite, then of course this will not be the case. For example, if we allowed this, (0) would be a subsequence of $\langle 0, 1, 1, 1, \dots\rangle$. This is just silly. Thus, a key requirement is that elements of S have to be able to become arbitrarily large.

Now let Λ be a general directed set and let $\Lambda' \subseteq \Lambda$ be a subset whose elements are arbitrarily large. (Precisely, this means that, for all $\lambda \in \Lambda$, there is some $\mu \in \Lambda'$ such that $\mu \geq \lambda$.) One way to see we need to make this requirement is because, without this requirement, Λ' would not itself be a directed set in general. However, if we do require the elements of S to be arbitrarily large, then Λ' will be a directed set, and so $a|_S$ will indeed be a net, and certainly it will turn-out that $a|_{\Lambda'}$ is a subnet of a. However, it is *not* the case that every subnet of $\lambda \mapsto x_\lambda$ is of this form. The reason for this is ultimately because, if we don't allow for more general subnets, then theorems we want to be true will fail to be true (see, for example, the proofs of Theorems 2.5.3.10 and 3.4.2.1).

Definition 2.4.5.1 — Subnet Let $x\colon \Lambda \to \mathbb{R}$ be a be a net. Then, a ***subnet*** of a is a net $b : \Lambda' \to \mathbb{R}$ such that

(i). for all $\mu \in \Lambda'$, $y_\mu = x_{\lambda_\mu}$ for some $\lambda_\mu \in \Lambda$; and
(ii). whenever $U \subseteq \mathbb{R}$ eventually contains x, it eventually contains y.

A ***subsequence*** is a subnet that is a sequence.

R In words, a subnet of a net is a net whose terms are all terms from the original net and is eventually contained in every set that eventually contains the original net.

R See the following proposition (Proposition 2.4.5.2) for an equivalent way to state this, that could possibly be more intuitive for you.

R Recall (see the paragraphs preceding this definition) that our definition should have the property that, if $\lambda \mapsto x_\lambda$ converges to x_∞, then all subnets should likewise converge to x_∞. (ii) is precisely the condition that guarantees this. See Proposition 2.4.5.16 for the precise statement and proof.

R There are at least two definitions in the literature that are distinct from this one. Our definition is strictly weaker than both of them (see Propositions 2.4.5.9 and 2.4.5.11, Example 2.4.5.10, and Exercise 2.4.5.12). These definitions are not so good because they do not correspond precisely to the notion of filterings and filters (see Section 3.3 Filter bases).[a] This definition also makes a few proofs easier (see, for example, the proofs of Theorems 2.5.3.10 and 3.4.2.1). I also personally find this definition easier to understand than either of the ones given in Proposition 2.4.5.9 or Proposition 2.4.5.11. As nets seem not to be taught very often at least in part because of the fact that the definition of a subnet is a bit tricky,[b] I think it is quite important to make this definition in particular as clean as possible.

R If $y\colon \Lambda' \to \mathbb{R}$ is a subnet of a net $x\colon \Lambda \to \mathbb{R}$, then, by (i), it follows that there is some function $\iota : \Lambda' \to \Lambda$ such that $y = x \circ \iota$. However, *in general there will not be a unique such function*. This almost never matters, and it is customary to write $y_\mu = x_{\iota(\mu)} \coloneqq x_{\lambda_\mu}$ for some noncanonically chosen ι.

R Notwithstanding the fact that our definition of subnet is nonstandard, our definition of *subsequence* would still be slightly different than most authors. The primary reason for this is because typically people do not introduce nets in a first analysis course, in which case the 'naive' definition of subsequence, i.e., a subnet of the form $x|_S$, works. For them, subsequences are what we would call *cofinal subnets* of a sequence (see Definition 2.4.5.7). In particular, we allow for 'repeats' whereas most authors will not, e.g., we consider $\langle 1, 1, 1, 2, 3, \ldots \rangle$ to be a subsequence of $\langle 1, 2, 3, 4, 5, \ldots \rangle$.

[a] You are neither supposed to know what these are yet nor why this is significant.

[b] Or at least, this is the impression that I have gotten.

Proposition 2.4.5.2 Let $\lambda \mapsto x_\lambda \in \mathbb{R}$ be a net. Then, $\mu \mapsto x_{\lambda_\mu}$ is a subnet of $\lambda \mapsto x_\lambda$ iff for all λ_0 there is some μ_0 such that

$$\left\{x_{\lambda_\mu} : \mu \geq \mu_0\right\} \subseteq \{x_\lambda : \lambda \geq \lambda_0\}\,. \tag{2.4.5.3}$$

Proof. ($\Rightarrow$) Suppose that $\mu \mapsto x_{\lambda_\mu}$ is a subnet of $\lambda \mapsto x_{\lambda_\mu}$. Let λ_0 be arbitrary. Of course, $\{x_\lambda : \lambda \geq \lambda_0\}$ eventually contains $\lambda \mapsto x_\lambda$, and so by the definition of a subnet, it eventually contains $\mu \mapsto x_{\lambda_\mu}$. That is, there is some μ_0 such that, whenever $\mu \geq \mu_0$, it follows that $x_\mu \in \{x_\lambda : \lambda \geq \lambda_0\}$. In other words,

$$\left\{x_{\lambda_\mu} : \mu \geq \mu_0\right\} \subseteq \{x_\lambda : \lambda \geq \lambda_0\}\,. \tag{2.4.5.4}$$

Suppose that for all λ_0 there is some μ_0 such that

$$\left\{x_{\lambda_\mu} : \mu \geq \mu_0\right\} \subseteq \{x_\lambda : \lambda \geq \lambda_0\}\,. \tag{2.4.5.5}$$

Let $U \subseteq \mathbb{R}$ eventually contain $\lambda \mapsto x_\lambda$. This means that there is some λ_0 such that, whenever $\lambda \geq \lambda_0$, it follows that $x_\lambda \in U$. In other words, $\{x_\lambda : \lambda \geq \lambda_0\} \subseteq U$. By hypothesis, there is some μ_0 such that $\left\{x_{\lambda_\mu} : \mu \geq \mu_0\right\} \subseteq \{x_\lambda : \lambda \geq \lambda_0\}$. Hence, $\left\{x_{\lambda_\mu} : \mu \geq \mu_0\right\} \subseteq U$, that is, whenever $\mu \geq \mu_0$, it follows that $x_{\lambda_\mu} \in U$. In other words, $\mu \mapsto x_{\lambda_\mu}$ is eventually contained in U. By definition then, $\mu \mapsto x_{\lambda_\mu}$ is a subnet of λx_λ. ■

As subnets of the more 'naive type' are still quite important, we do given them a special name.

Definition 2.4.5.6 — Cofinal subset Let S be a subset of a preordered set X. Then, S is ***cofinal*** iff for every $x \in X$ there is some $s \in S$ such that $s \geq x$.

> **R** Of course, saying that a subset is cofinal is just our fancy-schmancy way of saying that the elements are arbitrarily large.

One way to see that we need elements to grow arbitrarily large is because, in a directed set, the subset will itself be directed.

Definition 2.4.5.7 — Cofinal subnet A ***cofinal subnet*** of a net $x\colon \Lambda \to \mathbb{R}$ is a subnet of the form $a|_{\Lambda'} : \Lambda' \to \mathbb{R}$ for $\Lambda' \subseteq \Lambda$ cofinal.

Exercise 2.4.5.8 Let $a\colon \Lambda \to \mathbb{R}$ be a net and let $\Lambda' \subseteq \mathbb{R}$. Show that if Λ' is cofinal, then $a|_{\Lambda'}\colon \Lambda' \to \mathbb{R}$ is a subnet of $a\colon \Lambda \to \mathbb{R}$.

R Thus, our definition does in fact make sense.

R If $\Lambda' \subseteq \Lambda$ is cofinal, we say that Λ' *defines* the cofinal subnet $x|_{\Lambda'}\colon \Lambda' \to \mathbb{R}$.

R One key difference between the definition of a subnet and the more 'naive' definition (that is, if one were only to allow *cofinal* subnets) is that you are allowed to repeat elements in a subnet, for example, $\langle 0, 0, 1, 2, 3, \ldots \rangle$ is a subnet of $\langle 0, 1, 2, 3, \ldots \rangle$, but *not* a cofinal subnet.

We mentioned in the definition of a subnet, Definition 2.4.5.1, that there are at least two definitions of subnets in the literature that are distinct from ours. Our definition is strictly weaker than both of these, as we now show.

Proposition 2.4.5.9 Let $a\colon \Lambda \to \mathbb{R}$ and $b\colon \Lambda' \to \mathbb{R}$ be nets. Then, if there is a function $\iota\colon \Lambda' \to \Lambda$ such that (i) $b = a \circ \iota$ and (ii) for all $\lambda \in \Lambda$ there is some $\mu_0 \in \Lambda'$ such that, whenever $\mu \geq \mu_0$, it follows that $\iota(\mu) \geq \lambda$, then b is a subnet of a.

R Thus, in different notation, a subnet of $\lambda \mapsto x_\lambda$ is a net of the form $\mu \mapsto x_{\lambda_\mu}$, where the function $\mu \mapsto \lambda_\mu$ has the property that, for all λ_0, there is some μ_0 such that, whenever $\mu \geq \mu_0$, it follows that $\lambda_\mu \geq \lambda_0$. Note that, of course, in this case, there *is* a canonically chosen $\iota : \Lambda' \to \Lambda$ (confer the remarks in the definition of a subnet, Definition 2.4.5.1).

R This is sometimes taken as the definition of a subnet (for example, see [Kel55, pg. 70]). Our definition is strictly weaker than this one as the following example shows. This definition is more or less perfectly okay for almost all purposes. The reason we have chosen the definition we have over this one (aside from the fact that it makes some proofs slightly easier), is that it is more natural in the sense that our definition is the one that corresponds to the analogous notion with filters (see Section 3.3 Filter bases).

Proof. Suppose that there is a function $\iota : \Lambda' \to \Lambda$ such that (i) $b = a \circ \iota$ and (ii) for all $\lambda \in \Lambda$ there is some $\mu_0 \in \Lambda'$ such that, whenever $\mu \geq \mu_0$, it follows that $\iota(\mu) \geq \lambda$. Let $U \subseteq \mathbb{R}$ eventually contain a. Then, there is some $\lambda_0 \in \Lambda$ such that, whenever $\lambda \geq \lambda_0$, it follows that $x_\lambda \in U$. Let $\mu_0 \in \Lambda'$ be such that, whenever $\mu \geq \mu_0$, it follows that $\iota(\mu) =: \lambda_\mu \geq \lambda_0$. Thus, whenever $\mu \geq \mu_0$, it follows that $x_{\lambda_\mu} \in U$, so that $\mu \mapsto x_{\lambda_\mu}$ is eventually contained in U, and hence is a subnet of $\lambda \mapsto x_\lambda$. ■

Example 2.4.5.10 — A subnet that would not be a subnet in the sense of Proposition 2.4.5.9 Consider the constant sequence $m \mapsto x_m := 0$ and define $\iota : \mathbb{N} \to \mathbb{N}$ by $\iota(n) := 0$. Then, $n \mapsto x_{\iota(n)} = 0$, and so is certainly eventually contained in every set which eventually contains a, and so is a subnet. On the other hand, ι definitely does not satisfy (ii) of Proposition 2.4.5.9. Thus, this is an example of a subnet which would not be considered a subnet if we had taken the conditions in Proposition 2.4.5.9 as our definition of a subnet.

And now we come to the second definition that is sometimes in the literature.

Proposition 2.4.5.11 Let $a \colon \Lambda \to \mathbb{R}$ and $b \colon \Lambda' \to \mathbb{R}$ be nets. Then, if there is a function $\iota \colon \Lambda' \to \Lambda$ such that (i) $b = a \circ \iota$, (ii) is nondecreasing, and (iii) has cofinal image, then b is a subnet of a.

R This is sometimes taken as the definition of a subnet (for example, this is currently[a] the definition given on Wikipedia). The definition that is sometimes given as described in Proposition 2.4.5.9 is strictly weaker than this definition (see the next exercise), and so in turn our definition is strictly weaker than this definition. This definition also fails to exactly correspond to the analogous notion with filters, but, unlike the 'definition' of Proposition 2.4.5.9, using this definition can actually make things quite a bit more difficult.[b] The problem is that we often construct a subnet of the form in Proposition 2.4.5.9, and then showing that ι is nondecreasing is an extra step at best and requires modification of the subnet at worst. As far as I am aware, there is just no good reason to use as a definition.

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[a] 6 July 2015

[b] Or maybe even impossible? I don't know because I don't use this definition.

Proof. We apply the previous proposition. Let $\lambda \in \Lambda$ be arbitrary. Then, because ι has cofinal image, there is some $\mu_0 \in \Lambda'$ such that $\iota(\mu_0) \geq \lambda$. Because ι is nondecreasing, it follows that, if $\mu \geq \mu_0$, then $\iota(\mu) \geq \iota(\mu_0) \geq \lambda$. ∎

Exercise 2.4.5.12 — A subnet that would not be a subnet in the sense of Proposition 2.4.5.11 Find a net $x\colon \Lambda \to \mathbb{R}$, a directed set Λ' with function $\iota : \Lambda' \to \Lambda$ that has the property that, for all $\lambda \in \Lambda$, there is some $\mu_0 \in \Lambda'$ such that, whenever $\mu \geq \mu_0$, it follows that $\iota(\mu) \geq \lambda$, but yet either (i) is not nondecreasing or (ii) does not have cofinal image.

(R) That is to say, the conditions of Proposition 2.4.5.9 are strictly weaker than the conditions of Proposition 2.4.5.11.

Example 2.4.5.13

(i). $\langle 0, 0, 0, \ldots \rangle$ is a subsequence of $\langle 0, 1, 0, 1, 0, 1, \ldots \rangle$.

(ii). $\langle 0, 2, 4, , \ldots \rangle$ is a subsequence of $\langle 0, 1, 2, \ldots \rangle$.
(iii). $\langle 1, 1, 1, \ldots \rangle$ is *not* a subsequence of $\langle 0, 1, 2, \ldots \rangle$.
(iv). $\langle 0, 1, 4, 9, 16, \ldots \rangle$ is a subsequence of $\langle 0, 1, 2, \ldots \rangle$.

We mentioned when we defined the terminology "frequently XYZ" (Meta-definition 2.4.2.2) that one of its uses is that in a net which is frequently XYZ, the terms which are actually XYZ define a cofinal subnet. Now that we know what a cofinal subnet is, we can return to this.

Meta-proposition 2.4.5.14 Let $\Lambda \ni \lambda \mapsto x_\lambda$. Then, if $\lambda \mapsto x_\lambda$ is frequently XYZ, then

$$\{\lambda \in \Lambda : x_\lambda \text{ is XYZ.}\} \tag{2.4.5.15}$$

defines a cofinal subnet of $\lambda \mapsto x_\lambda$.

Proof. We must show that $\Lambda' := \{\lambda \in \Lambda : x_\lambda \text{ is XYZ.}\}$ is cofinal in Λ. So, let $\lambda_0 \in \Lambda$. By the definition of frequently (Meta-definition 2.4.2.2), this means that there is some x_λ that is XYZ, for $\lambda \geq \lambda_0$. That is, $\lambda \in \Lambda'$, as desired. ■

The following two results, while neither particularly difficult nor deep, are important because they become an axioms of the convergence definition of topological spaces—see Kelley's Convergence Theorem (Theorem 3.4.2.1).

Proposition 2.4.5.16 Let $\mu \mapsto x_{\lambda_\mu}$ be a subnet of a net $\lambda \mapsto x_\lambda$. Then, if $\lim_\lambda x_\lambda = x_\infty$, then $\lim_\mu x_{\lambda_\mu} = x_\infty$.

Proof. Suppose that $\lim_\lambda x_\lambda = x_\infty$. Let $\varepsilon > 0$. Then, as $\lim_\lambda x_\lambda = x_\infty$, $B_\varepsilon(x_\infty)$ eventually contains $\lambda \mapsto x_\lambda$. By the definition of a subnet, $B_\varepsilon(x_\infty)$ then eventually contains $\mu \mapsto x_{\lambda_\mu}$. By definition, this means that $\lim_\mu x_{\lambda_\mu} = x_\infty$. ■

Proposition 2.4.5.17 Let $\lambda \mapsto x_\lambda$ be a net. Then, if every cofinal subnet $\mu \mapsto x_{\lambda_\mu}$ has in turn a subnet itself $\nu \mapsto x_{\lambda_{\mu_\nu}}$ such that $\lim_\nu x_{\lambda_{\mu_\nu}} = x_\infty$, then $\lim_\lambda x_\lambda = x_\infty$.

R The converse is true too of course by the previous result.

Proof. Suppose that every cofinal subnet $\mu \mapsto x_{\lambda_\mu}$ has in turn a subnet itself $\nu \mapsto x_{\lambda_{\mu_\nu}}$ such that $\lim_\nu x_{\lambda_{\mu_\nu}} = x_\infty$. We proceed by contradiction: suppose that it is not the case that $\lim_\lambda x_\lambda = x_\infty$. Then,

$$\text{there is some } \varepsilon_0 > 0 \text{ such that for all } \lambda \text{ there is some } \mu_\lambda \geq \lambda \text{ such that } \left|x_{\mu_\lambda} - x_\infty\right| \geq \varepsilon_0. \tag{2.4.5.18}$$

Define $S := \{\mu_\lambda : \lambda\}$ and denote by $\lambda \mapsto x_{\mu_\lambda}$ the corresponding cofinal subnet, so that $\left|x_{\mu_\lambda} - x_\infty\right| \geq \varepsilon_0$ for all λ.[a]

We wish to show that $\lambda \mapsto x_{\mu_\lambda}$ has no subnet which converges to x_∞. This will be a contradiction, thereby proving the result. To show this itself, we again proceed by contradiction: suppose there is some subnet $\nu \mapsto x_{\mu_{\lambda_\nu}}$ of $\lambda \mapsto x_{\mu_\lambda}$ that converges to x_∞. By (2.4.5.18), we have that $\left|x_{\mu_{\lambda_\nu}} - x_\infty\right| \geq \varepsilon_0$, or rather, $x_{\mu_{\lambda_\nu}} \in B_{\varepsilon_0}(x_\infty)^\complement$, for all ν. But then $\nu \mapsto x_{\mu_{\lambda_\nu}}$ is certainly not eventually contained in $B_{\varepsilon_0}(x_\infty)$: a contradiction. ■

[a] Note that the index here μ_λ is not the usual. Of course, it makes no difference—λ and μ are just letters—we point this out simply because this can be an easy-to-miss detail if you're reading fast.

The following result, together with Propositions 2.4.5.16 and 2.4.5.17 (and the fact that constant nets converge to that constant) turn out to be sufficient to characterize the topology on a topological space. Before we present it, however, we must first define an order on a product of preordered sets.

Definition 2.4.5.19 — Product order Let $\mathscr{P}$ be a collection of preordered sets and let $x, y \in \prod_{P \in \mathcal{P}} P$. Then, we define $x \leq_{\mathcal{P}} y$ iff $x_P \leq_P y_P$ for all $P \in \mathcal{P}$.

Exercise 2.4.5.20 Show that $\leq_{\mathcal{P}}$ is a preorder.

R In any category, there is a notion of a *product*. It turns out that this is in fact the product in the category of preordered sets **Pre**.

Exercise 2.4.5.21 If $\mathscr{P}$ is a collection of directed sets, show that $\langle \prod_{P \in \mathscr{P}} P, \leq_{\mathscr{P}} \rangle$ is a directed set.

Proposition 2.4.5.22 Let I be a directed set[a] and for each $i \in I$ let $x^i : \Lambda^i \to \mathbb{R}$ be a convergent net. Then, if $(x^\infty)_\infty := \lim_i \lim_\lambda (x^i)_\lambda$ exists, then $I \times \prod_{i \in I} \Lambda^i \ni \langle i, \lambda \rangle \mapsto (x^i)_{\lambda^i}$ converges to $(x^\infty)_\infty$.

R In other words, if you have a 'net's worth of nets' such that the iterated limit converges, then you can write this limit as a limit of a single net (which itself is formed from the 'net's worth of nets').

R Note that if you insist upon working with only sequences, you haven't a chance in hell to make something like this work.[b] In fact, recall our counter-example (Example 2.4.3.65), in which we had $\lim_m (a^n)_m = 0$ for all $n \in \mathbb{N}$, and so of course we had that $\lim_n (\lim_m (x^n)_m)$ existed (and was equal to 0). In fact, the same was true with m and n reversed. We then hoped that $\lim_m (x^m)_m = 0$, but found that this was not in fact the case. This result tells us that you can indeed form a net from a nets worth of convergent nets that converges to the thing you would like to, the catch being that the answer is not as nice as one might have liked.

R As messy as this answer might seem, in some sense, it's the best we could do. Can you write down any other net at all formed from just the $(x^\lambda)_{\lambda^\mu}$s? As all the directed sets Λ^i are in general distinct, this is essentially the simplest thing we can write down.

---

[a] "I" is for "index".

[b] Okay, perhaps I'm not being completely fair to the sequence-loyalists out there. In general, I imagine you *should* be able to conjure up a single sequence from the $(a^n)_m$s which converges to $(x^\infty)_\infty$; however, there's no way you're going to get an *explicit formula* for that sequence that's going to work all the time—this result is nice in that it gives a very explicit expression for *any* 'net's worth of nets' you might come up with.

Proof. Suppose that $(x^\infty)_\infty := \lim_i \lim_\lambda (x^i)_\lambda$ exists. Let $\varepsilon > 0$. Let $i_0 \in I$ be such that, whenever $i \geq i_0$, it follows that

$$\left|\lim_\lambda (x^i)_\lambda - (x^\infty)_\infty\right| < \varepsilon. \tag{2.4.5.23}$$

Define

$$(x^i)_\infty := \lim_\lambda (x^i)_\lambda. \tag{2.4.5.24}$$

Let $\lambda_0^i \in \Lambda^i$ be such that whenever $\lambda^i \geq \lambda_0^i$ it follows that

$$\left|(x^i)_{\lambda^i} - (x^i)_\infty\right| < \varepsilon. \tag{2.4.5.25}$$

Then, whenever $\langle i, \lambda\rangle \geq \langle i_0, \lambda_0\rangle$, by definition of the product order, $i \geq i_0$ and $\lambda^i \geq \lambda_0^i$ for all $i \in I$, and so

$$\begin{aligned}\left|(x^i)_{\lambda^i} - (x^\infty)_\infty\right| &\leq \left|(x^i)_{\lambda^i} - (x^i)_\infty\right| \\ &\quad + \left|(x^i)_\infty - (x^\infty)_\infty\right| \\ &< \varepsilon + \varepsilon = 2\varepsilon.\end{aligned} \tag{2.4.5.26}$$

■

We end this section with a result on the relationship of limit superiors and limit inferiors of nets to their subnets.

Proposition 2.4.5.27 Let $\mu \mapsto x_{\lambda_\mu}$ be a subnet of a net $\lambda \mapsto x_\lambda$. Then,

$$\liminf_\lambda x_\lambda \leq \liminf_\mu x_{\lambda_\mu} \leq \limsup_\mu x_{\lambda_\mu} \leq \limsup_\lambda x_\lambda .$$

Proof. We already know the middle inequality holds from Exercise 2.4.3.30. We prove the lim sup inequality holds; the other is similar.

Let λ_0 be an arbitrary index. Then, by Proposition 2.4.5.2, there is some μ_0 such that

$$\left\{x_{\lambda_\mu} : \mu \geq \mu_0\right\} \subseteq \{x_\lambda : \lambda \geq \lambda_0\} . \tag{2.4.5.28}$$

Hence

$$\sup_{\mu \geq \mu_0} \{x_{\lambda_\mu}\} \leq \sup_{\lambda \geq \lambda_0} \{x_\lambda\} . \tag{2.4.5.29}$$

It follows from the definition of the limit superior (2.4.3.28) and the Order Limit Theorem (Exercise 2.4.3.19) that $\limsup_\mu x_{\lambda_\mu} \leq \lim_\lambda x_\lambda$. ∎

2.5 Basic topology of the Euclidean space

Though we have not defined it yet (and will not do so until the next chapter), a *topological space* is the most general context in which one can make precise the notion of *continuity*.[11] The point is, if continuity is something you care about, then topology in turn is something you should also care about.

In this last section, we work in not $\mathbb{R}$ itself, but rather $\mathbb{R}^d := \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{d}$, ***Euclidean space***. The reason we do so is because (i) basically everything works verbatim if you just replace $\mathbb{R}$ with $\mathbb{R}^d$ and (ii) I think picturing these concepts in $\mathbb{R}^2$ or $\mathbb{R}^3$ is more

[11]This is arguably a slight lie, but in any case, I think it's fair to say that continuity is really the point of working with topological spaces.

enlightening than picturing them in $\mathbb{R}$. There are a couple of places in which we will need to reference old concepts that we have not technically defined in $\mathbb{R}^d$ (e.g. limits of nets). However, the correct definition in all such cases, unless otherwise stated, is given simply by replacing "$\mathbb{R}$" with "$\mathbb{R}^d$". If you're still bothered by this, just pretend $d = 1$.

Most of the structure of $\mathbb{R}$ generalizes to $\mathbb{R}^d$ in a straightforward manner.[12]. For example, for $x, y \in \mathbb{R}^d$,

$$x + y := \langle x_1 + y_1, \ldots, x_d + y_d \rangle \in \mathbb{R}^d, \tag{2.5.1}$$

where x_k and y_k are the *coordinates* of x and y respectively—see Definition A.3.1.6. You can define multiplication similarly if you like, but this is not common. Upon doing so, we obtain a cring that is *not even integral* (e.g. $\langle 1, 0\rangle \cdot \langle 0, 1\rangle = \langle 0, 0\rangle$), much less a field. We can also equip $\mathbb{R}^d$ with the product order, but this will only given us a *partially-ordered* cring, not a *totally-ordered* cring (e.g. $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ are not comparable).

There is also the issue of the Absolute value (Definition 2.2.1). For $x \in \mathbb{R}^d$, we define

$$|x| := \sqrt{|x_1|^2 + \cdots + |x_d|^2}.\text{[13][14]} \tag{2.5.2}$$

The motivation for such a definition comes of course from the Pythagorean Theorem, though it is certainly not the only choice we could have made—see, for example, the L^p-norm (Definition 5.3.1). The definitions of $B_\varepsilon(x_0)$ and $D_\varepsilon(x_0)$ (given in the remark of Definition 2.2.1) generalize accordingly.

2.5.1 Continuity

Roughly speaking, a continuous function is a function which 'preserves limits', that is, if you take the limit as x goes to a, then $f(x)$ goes to $f(a)$. Thus, to formulate the definition of continuity, we must first say what we mean by limit of a function (so far, we have only said what it means to take the limit of a *net*). In turn, the notion of a *limit point* will make this definition slightly easier to formulate (it's also quite an important concept in its own right).

[12]Though upon generalization we lose a lot of the nice properties we had in $\mathbb{R}$!

Definition 2.5.1.1 — Limit point Let $S \subseteq \mathbb{R}^d$ and let $x_0 \in \mathbb{R}^d$. Then, x_0 is a ***limit point*** of S iff there exists a net $\lambda \mapsto x_\lambda \in S$ with $x_\lambda \neq x_0$ such that $\lim_\lambda x_\lambda = x_0$.

R The reason for the requirement $x_\lambda \neq x_0$ is two-fold: first of all, we need to make this same requirement in the definition of a limit of a function for reasons explained there, and second of all, we need this condition for the notion of a limit point to agree with the notion of an accumulation point—see Proposition 2.5.2.12. (See the respective definitions, Definitions 2.5.1.2 and 2.5.2.9, for an explanation of why we make the corresponding conditions in these definitions.)

Definition 2.5.1.2 — Limit (of a function) Let $D \subseteq \mathbb{R}^d$,[a] let $x_0 \in \mathbb{R}^d$ be a limit point of D, let $f\colon D \to \mathbb{R}^e$, and let $L \in \mathbb{R}^e$. Then, y is a ***limit*** of f at x_0 iff for every net $\lambda \mapsto x_\lambda \in D$ such that (i) $x_\lambda \neq x_0$ and (ii) $\lim_\lambda x_\lambda = x_0$ we have $\lim_\lambda f(x_\lambda) = y$.

Exercise 2.5.1.3 Let $y, y' \in \mathbb{R}^e$ be limits of the function $f\colon D \to \mathbb{R}^e$ at $x_0 \in \mathbb{R}^d$. Show that $y = y'$.

R If y is the limit of f at x_0, then we write $\lim_{x\to x_0} f(x) = y$. Note that this is unambiguous by the previous exercise.

R That $x_0 \in \mathbb{R}^d$ is a limit point of D guarantees that there is at least one such net. This rules out the possibility of the condition being fulfilled vacuously. For example, in the stupid case in which the domain S is a point, if we didn't require that a be a limit point,[b] then (vacuously) every $y \in \mathbb{R}^e$ would be a limit of f at x_0, which in particular would destroy uniqueness.

R Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases} \tag{2.5.1.4}$$

Then we *would like* to say that $\lim_{x \to 0} f(x) = 0$. This is the motivation for imposing the constraint $x_\lambda \neq x_0$. Because, for example, the constant net $\lambda \mapsto x_\lambda := x_0$ does *not* satisfy $\lim_\lambda f(x_\lambda) = 0$.

R There are several equivalent ways to state the definition of a limit of a function (see the following exercise, for example), and it's quite possible that this is not the one you've seen taken as a definition before. Our motivation for taking this as our definition is that it makes it quite easy to carry over our knowledge about limits of nets to limits of functions.

[a] "D" is for "domain".

[b] Note that, for $D = \{x_0\}$, a is *not* a limit point of D.

The following equivalent condition is also commonly taken as the definition of a limit, the so-called "ε-δ" definition of a limit.

Exercise 2.5.1.5 Let $D \subseteq \mathbb{R}^d$, let $x_0 \in \mathbb{R}^d$ be a limit point of D, let $f: D \to \mathbb{R}^e$, and let $y \in \mathbb{R}^e$. Show that $\lim_{x \to x_0} f(x) = y$ iff for every $\varepsilon > 0$ there is some $\delta > 0$ such that, whenever $0 < |x - x_0| < \delta$, $x \in D$, it follows that $|f(x) - y| < \varepsilon$.

R The intuition is as follows: No matter how close we want to make $f(x)$ to y, we can do so by making x sufficiently close to x_0.

R The motivation for the condition $0 < |x - x_0|$ is the same as the motivation for the condition $x_\lambda \neq x_0$ in the previous definition.

Of course, a lot of the results we had for limits of nets have analogues for limits of functions. In particular, there are versions of the Algebraic Limit Theorems and the Order Limit Theorem for limits of functions.

Proposition 2.5.1.6 — Algebraic Limit Theorems (for functions) Let $D \subseteq \mathbb{R}^d$, let $x_0 \in \mathbb{R}^d$ be a limit point of D, and let $f, g\colon D \to \mathbb{R}^e$ both have limits at x_0. Then,

(i). $\lim_{x\to x_0}[f(x)+g(x)] = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$;
(ii). $\lim_{x\to x_0}[f(x)g(x)] = \left(\lim_{x\to x_0} f(x)\right)\left(\lim_{x\to x_0} g(x)\right)$;
(iii). $\lim_{x\to x_0}\frac{1}{f(x)} = \frac{1}{\lim_{x\to x_0} f(x)}$ if $\lim_{x\to x_0} f(x) \in [\mathbb{R}^e]^{\times}$[a]; and
(iv). $\lim_{x\to x_0}[\alpha f(x)] = \alpha \lim_{x\to x_0} f(x)$ for $\alpha \in \mathbb{R}$.

R The product is defined component-wise.

[a] Recall that ((A.4.17)) the notation $[\mathbb{R}^e]^{\times}$ means the set of invertible elements in $\mathbb{R}^e$. Concretely, this means that every component of $\lim_{x\to x_0} f(x)$ is nonzero.

Proof. We leave this as an exercise.

Exercise 2.5.1.7 Prove this yourself, using the Algebraic Limit Theorems (for nets).

■

Exercise 2.5.1.8 — Order Limit Theorem (for functions) Let $D \subseteq \mathbb{R}^d$, let $x_0 \in \mathbb{R}^d$ be a limit point of D, and let $f, g\colon D \to \mathbb{R}^e$ both have limits at x_0. Then, if $f(x) \leq g(x)$ for all $x \in D$, then $\lim_{x\to x_0} f(x) \leq \lim_{x\to x_0} g(x)$.

R $\mathbb{R}^e$ is equipped with the Product order.

Exercise 2.5.1.9 — Squeeze Theorem (for functions) Let $D \subseteq \mathbb{R}^d$, let $x_0 \in \mathbb{R}^d$ be a limit point of D, and let $f, g, h\colon D \to \mathbb{R}^e$. Show that if (i) $f(x) = h(x)$ in an ε-ball centered at x_0 and (ii) $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x)$, then $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x)$.

R As before, the content of this theorem is the the limit $\lim_{x\to x_0} g(x)$ *exists*. We already knew that (by the Order Limit Theorem), if it existed, it would have to be equal to the common value $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x)$.

Before we finally turn to continuity, we first define limit superiors and limit inferiors of *functions*.

Definition 2.5.1.10 — Limit superior and limit inferior (of a function) Let $D \subseteq \mathbb{R}^d$, let $x_0 \in \mathbb{R}^d$ be a limit point of D, and let $f\colon D \to \mathbb{R}^e$. Then, the ***limit superior*** and ***limit inferior*** of f at x_0, $\limsup_{x\to x_0} f(x)$ and $\liminf_{x\to x_0} f(x)$ respectively, are defined by

$$\limsup_{x\to x_0} f(x) := \lim_{\varepsilon\to 0^+} \sup\{f(x) : x \in B_\varepsilon(x_0)\}$$
$$\liminf_{x\to x_0} f(x) := \lim_{\varepsilon\to 0^+} \inf\{f(x) : x \in B_\varepsilon(x_0)\}.$$

R Note that, the same as before, the Monotone Convergence Theorem (Proposition 2.4.3.21) guarantees that these limits always exist, though they may be $\pm\infty$.

R Also note that, as suprema and infima are quite special to $\mathbb{R}$ (as opposed to $\mathbb{R}^d$ for $d \geq 2$), this is quite specific for functions with values in $\mathbb{R}$.[a] On the other hand, we can replace the domain with any topological space—see Definition 3.2.18.

[a] I suppose for functions with values in $\mathbb{R}^d$ for $d \geq 2$, you could take the lim sup and lim inf 'componentwise', but that I think is a bit awkward.

Having defined limits of functions, we can present the definition of continuity itself.

Definition 2.5.1.12 — Continuous (real) function Let $D \subseteq \mathbb{R}^d$, let $x_0 \in \mathbb{R}^d$ be a limit point of D, and let $f\colon D \to \mathbb{R}^e$. Then, f is ***continuous*** at x_0 iff $\lim_{x \to x_0} f(x) = f(x_0)$. f is ***continuous*** iff it is continuous at x_0 for all $x_0 \in D$.

R Note how this can be written instead as $\lim_{x \to x_0} f(x) = f(\lim_{x \to x_0} x)$. Thus, you might say that continuous functions are precisely the functions which 'commute' with, or 'preserve', limits.

Example 2.5.1.13 — Dirichlet Function Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q}^{\mathsf{C}}. \end{cases} \tag{2.5.1.14}$$

This is the ***Dirichlet Function***.

Exercise 2.5.1.15 Where is the Dirichlet Function continuous?

Example 2.5.1.16 — Thomae Function Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \text{ with } \gcd(m,n) = 1,\ n > 0 \\ 0 & \text{if } x \in \mathbb{Q}^{\mathsf{C}}. \end{cases}$$

This is the ***Thomae Function***.

Exercise 2.5.1.17 Show that the Thomae Function is continuous at $x \in \mathbb{R}$ iff $x \in \mathbb{Q}^{\mathsf{C}}$. Hint: For a fixed $n \in \mathbb{Z}^+$, how many rational numbers are there in the interval $[0,1]$ with denominator smaller than n?

Exercise 2.5.1.18 Let $f\colon \mathbb{R}^d \to \mathbb{R}^e$ be a function and let $x_0 \in \mathbb{R}^d$. Show that the following are equivalent.

(i). f is continuous at x_0.
(ii). For every net $\lambda \mapsto x_\lambda$ such that (i) $x_\lambda \neq x_0$ and (ii) $\lim_\lambda x_\lambda = x_0$ we have $\lim_\lambda f(x_\lambda) = f(x_0)$.
(iii). For every $\varepsilon > 0$ there is some $\delta > 0$ such that, whenever $0 < |x - x_0| < \delta$, it follows that $|f(x) - f(x_0)| < \varepsilon$.[a]
(iv). For every $\varepsilon > 0$ there is some $\delta > 0$ such that $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$.
(v). For every $\varepsilon > 0$ there is some $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0)))$.

[a]Note that as $|f(x_0) - f(x_0)| < \varepsilon$ for all $\varepsilon > 0$, the "$<$" in "$0 < |x - x_0| < \delta$" is not strictly necessary anymore.

Exercise 2.5.1.19 Let $D \subseteq \mathbb{R}^d$, let $x_0 \in \mathbb{R}^d$ be a limit point of D, and let $f, g\colon D \to \mathbb{R}^e$ be continuous at $x_0 \in \mathbb{R}^d$.

(i). Show that $f + g$ is continuous at $x_0 \in \mathbb{R}^d$.
(ii). Show that fg is continuous at $x_0 \in \mathbb{R}^d$.
(iii). Show that $\frac{1}{f}$ is continuous at $x_0 \in \mathbb{R}^d$ if $f(x_0) \in [\mathbb{R}^e]^\times$.
(iv). Show that αf is continuous at $x_0 \in \mathbb{R}^d$ for $\alpha \in \mathbb{R}$.

The equivalences of Exercise 2.5.1.18 are nice, but they all somehow have the 'disadvantage' that they make reference to the points of $\mathbb{R}^d$.[15] There is, however, a way to characterize continuity without making reference to points at all. This is done by the introduction of *open sets*. Before we get there however, we take an aside to define something that ideally we could have done a long time ago, but first needed continuity to discuss.

Exponentials

Despite all that we've done, there is still something you thought you knew how to do with numbers since grade school, but we have yet

[15]Admittedly, at this point, it should probably not be clear as to why this would be a disadvantage at all.

to define: exponentials.[16] Now that we know what it means to be continuous, we can finally define exponentials.

Theorem 2.5.1.20 — Exponentials. Let $a \in \mathbb{R}^+$. Then, there is a unique continuous function, $\mathbb{R} \ni x \mapsto a^x \in \mathbb{R}$, such that

(i). $a^{x+y} = a^x a^y$; and
(ii). $a^1 = a$.

Functions of this form are called ***exponential functions***. a is the ***base*** and x is the ***exponent***.

R We also define: $0^a := 0$ for $a > 0$, and $0^0 := 1$—see Exercise 5.1.5.69.[a] 0^a is undefined for $a < 0$.

[a] The reason for this definition is *nontrivial*, and we will have to wait for *Lebesgue measure* to justify it. That being said, you can see intuitively why we make this definition simply by graphing the function $\langle x, y\rangle \mapsto x^y$ on $\mathbb{R}^+ \times \mathbb{R}^+$. Despite our reasons, I should probably note that I imagine most simply leave 0^0 undefined.

Proof. STEP 1: DEFINE a^m FOR $m \in \mathbb{N}$
For $m \in \mathbb{N}$, a^m is defined by

$$a^m := \underbrace{a \cdot \cdots \cdot a}_{m}. \tag{2.5.1.21}$$

STEP 2: DEFINE $a^{1/n}$ FOR $n \in \mathbb{Z}^+$
For $n \in \mathbb{Z}^+$, $a^{\frac{1}{n}}$ is defined to be the unique positive real number that has the property that $(a^{\frac{1}{n}})^n = a$—see Proposition 2.4.3.56.

STEP 3: DEFINE a^r FOR $r \in \mathbb{Q}$
For $\frac{p}{q} \in \mathbb{Q}$ with $p \neq 0$, $q \in \mathbb{Z}^+$, and $\gcd(p, q) = 1$, $a^{\frac{p}{q}}$ is defined by

$$a^{\frac{p}{q}} := (a^p)^{\frac{1}{q}}. \tag{2.5.1.22}$$

[16] Did you really think you knew what something like $\sqrt{2}^{\sqrt{3}}$ was? What are you going to do? Multiply $\sqrt{2}$ by itself $\sqrt{3}$ times? Good luck with that.

STEP 4: DEFINE a^x FOR $x \in \mathbb{R}$

For $x \in \mathbb{R}$, let $\lambda \mapsto x_\lambda \in \mathbb{Q}$ be some net converging to x.[a] Then, a^x is defined by

$$a^x := \lim_\lambda a^{x_\lambda}. \tag{2.5.1.23}$$

Exercise 2.5.1.24 Show that this is well-defined in the sense that if $\mu \mapsto b_\mu \in \mathbb{Q}$ also converges to a, then $\lim_\lambda x^{a_\lambda} = \lim_\mu x^{b_\mu}$.

STEP 5: CHECK PROPERTIES

$a^1 = a$ by definition.

Exercise 2.5.1.25 Show that $a^{x+y} = a^x a^y$ for all $x, y \in \mathbb{R}$.

STEP 6: SHOW UNIQUENESS

Exercise 2.5.1.26 Show that this is the unique continuous function that is a at 1 and takes sums to products.

■

[a]Note that some such net exists by 'Density' of $\mathbb{Q}$ in $\mathbb{R}$.

Exercise 2.5.1.27 Show that $(a^x)^y = a^{xy}$.

2.5.2 Open and closed sets

Definition 2.5.2.1 — Open set Let $U \subseteq \mathbb{R}^d$. Then, U is ***open*** iff for every $x \in U$ there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

> **R** The intuition is that there is 'wiggle room' around every point.

Exercise 2.5.2.2 Explain why $\emptyset$ is open.

> **R** When we generalize to topological spaces, we will require that the empty-set is open.

Exercise 2.5.2.3 Show that $\mathbb{R}^d$ is open.

> **R** Likewise, when we generalize to topological spaces, we will also require that the entire set be open.

Exercise 2.5.2.4 Let $x \in \mathbb{R}^d$ and $\varepsilon > 0$. Show that $B_\varepsilon(x)$ is open.

The following result is incredibly important. It is neither particularly deep nor particularly difficult, but it is what is taken as the *definition* of continuity when we generalize to topological spaces,[17] even though it might not be particularly intuitive at first.

Theorem 2.5.2.5. Let $f\colon \mathbb{R}^d \to \mathbb{R}^d$. Then, f is continuous iff the preimage of every open set is open (i.e. iff $U \subseteq \mathbb{R}^d$ open implies $f^{-1}(U) \subseteq \mathbb{R}^d$ is open).

[17] Okay, so this is *almost* true, but not quite—see Definition 3.1.3.1 and Exercise 3.1.3.3.

Proof. ($\Rightarrow$) Suppose that f is continuous. Let $U \subseteq \mathbb{R}^d$ be open and let $x \in f^{-1}(U)$. Then, $f(x) \in U$, so because U is open, there is some $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subseteq U$. Then, by Exercise 2.5.1.18.(iv), there is some $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \subseteq U$. It follows that $B_\delta(x) \subseteq f^{-1}(U)$, and so $f^{-1}(U)$ is open.

($\Leftarrow$) Suppose that the preimage of every open set is open. Let $x \in \mathbb{R}^d$ and $\varepsilon > 0$. $B_\varepsilon(f(x))$ is open by Exercise 2.5.2.4, and so $f^{-1}(B_\varepsilon(f(x)))$ is open. As this set contains x, there is some $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$, and so $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. Thus, f is continuous by Exercise 2.5.1.18.(iv). ■

'Dual' (but not *opposite*!)[18] to the notion of an open set is that of a *closed* set.

Definition 2.5.2.6 — Closed set Let $C \subseteq \mathbb{R}^d$. Then, C is ***closed*** iff C^{C} is open.

■ **Example 2.5.2.7 — $\mathbb{R}^d$ and the empty-set** Hopefully you saw in Exercises 2.5.2.2 and 2.5.2.3 that both $\emptyset$ and $\mathbb{R}^d$ are open. As $\emptyset^{\mathsf{C}} = \mathbb{R}^d$ and $[\mathbb{R}^d]^{\mathsf{C}} = \emptyset$, it follows that both $\emptyset$ and $\mathbb{R}^d$ are closed *as well*. In particular, it is possible for sets to be *both open and closed*—sometimes such sets are referred to as ***clopen***. This is what we meant when we said that openness and closedness are "dual" but not "opposite".

Exercise 2.5.2.8 Let $f\colon \mathbb{R}^d \to \mathbb{R}^e$. Show that f is continuous iff the preimage of every closed set is closed.

Showing that C^{C} is open to show that C is closed can in fact be a very efficient way of doing so. Nevertheless, it would be nice to have

[18]Don't make the mistake Hitler did—see https://www.youtube.com/watch?v=SyD4p8_y8Kw.

a direct way of checking that C is closed, which is why we introduce the notion of *accumulation point*.

Definition 2.5.2.9 — Accumulation point Let $S \subseteq \mathbb{R}^d$ and let $x_0 \in \mathbb{R}^d$. Then, x_0 is an ***accumulation point*** of S iff for every $\varepsilon > 0$, $B_\varepsilon(x_0)$ intersects S at a point *distinct from* x_0.

R You might think of the accumulation points of S as being 'infinitely close' to S in some sense.

R The reason we require that it intersect at a point *distinct* from x_0 is because some results would just fail to be true without it (for example, Proposition 2.5.2.23). A similar problem which would arise is that the Bolzano-Weierstrass Theorem (Corollary 2.5.3.14) would be trivial (and hence have no content) without this extra condition. This condition should be seen as exactly analogous to the condition in the definition of a limit point (Definition 2.5.1.1) that $x_\lambda \neq x_0$. Indeed, we will see momentarily that accumulation points are the same as limit points (Proposition 2.5.2.12), and the proof will make plain that these conditions correspond to one another.

The condition "distinct from x_0" is very important as, as mentioned in the remark, some results are just not true without it. That said, there are cases in which it can be slightly annoying to worry about, which leads us to the following definition.

Definition 2.5.2.10 — Adherent point Let $S \subseteq \mathbb{R}^d$ and let $x_0 \in \mathbb{R}^d$. Then, x_0 is an ***adherent point*** of S iff for every $\varepsilon > 0$, $B_\varepsilon(x_0)$ intersects S.

R Note of course that this is exactly the same as the definition of accumulation point without the condition "distinct from x_0".

R Note that automatically every point of S is an adherent point of S, which is one convenience adherent points have over accumulation points. In contrast, many nonempty sets have no accumulation points at all.[a]

R While certainly not useless, I think it is unquestionably fair to say, that of the two, the concept of an accumulation point is quite a bit more useful (certainly, anyways, it is the concept that we will make primary use of).

[a]Example?

Exercise 2.5.2.11 Let $S \subseteq \mathbb{R}^d$ and let $x_0 \in \mathbb{R}^d$. Show that x_0 is an accumulation point of S iff for every $\varepsilon > 0$, $B_\varepsilon(x_0)$ intersects S at *infinitely many points*.

W Warning: Many of the results we prove in this section generalize perfectly to the case of a general topological space. This is not one of them! For example, obviously this cannot be true in a topological space which only has finitely many points.

W Warning: This fails, even in $\mathbb{R}^d$, if you replace "accumulation point" with "adherent point". Counterexample?

Proposition 2.5.2.12 Let $S \subseteq \mathbb{R}^d$ and let $x_0 \in \mathbb{R}^d$. Then, x_0 is an accumulation point of S iff it is a limit point of S.

R If you replace "net" with "sequence" in the definition of a limit point, then this result will be *false* in general! Sequences are just fine if we restrict ourselves to $\mathbb{R}^d$, but when we generalize, this result would fail to hold if we constrained ourselves to only work with sequences.

Proof. ($\Rightarrow$) Suppose that x_0 is an accumulation point of S. Let $\varepsilon > 0$. Then, $B_\varepsilon(x_0) \cap S$ contains some element x_ε distinct from x_0. Note that the positive-real numbers $\langle \mathbb{R}^+, \preceq \rangle$ equipped with the *reverse* of the usual ordering $\preceq$ (i.e.,$x \preceq y$ is defined to be true iff $y \leq x$) is a directed set, so that $\varepsilon \mapsto x_\varepsilon$ is a net with $x_\varepsilon \neq x_0$. We claim that $\lim_\varepsilon x_\varepsilon = x_0$, so that x_0 will then be a limit point of S. Let $\varepsilon > 0$. Suppose that $\delta \succeq \varepsilon$ (we are taking our 'λ_0' in the definition of the limit of a net, Definition 2.4.2.4, to be ε itself). Then, $x_\delta \in B_\delta(x_0)$, and so $|x_\delta - x_0| < \delta$. As $\delta \succeq \varepsilon$, we have $\delta \leq \varepsilon$, and so $|x_\delta - x_0| < \varepsilon$, which shows that $\lim_\varepsilon x_\varepsilon = x_0$, and so x_0 is a limit point of S.

($\Leftarrow$) Suppose that x_0 is a limit point of S. Then, there is some net $\lambda \mapsto x_\lambda \in S$ with $x_\lambda \neq x_0$ such that $\lim_\lambda x_\lambda = x_0$. Let $\varepsilon > 0$. Then, there is some $x_{\lambda_0} \neq x_0$ such that $\left|x_{\lambda_0} - x_0\right| < \varepsilon$. In other words, $x_{\lambda_0} \in B_\varepsilon(x_0) \cap S$, so that x_0 is an accumulation point of S. ■

There is of course an analogous result for adherent points.

Proposition 2.5.2.13 Let $S \subseteq \mathbb{R}^d$ and let $x_0 \in \mathbb{R}^d$. Then, x_0 is an adherent point of S iff there is a net $\lambda \mapsto x_\lambda \in S$ such that $x_0 = \lim_\lambda x_\lambda$.

R Note that the condition "there is a net $\lambda \mapsto x_\lambda \in S$ such that $x_0 = \lim_\lambda x_\lambda$" is *almost* the definition of a limit point, the difference being that there is no requirement that $x_\lambda \neq x_0$. To the best of my knowledge, there is no term for this condition. I suppose to be completely parallel to the accumulation point-limit point equivalence, it would make sense to have one, but there is no standard term, and as it winds up being equivalent to adherent point anyways, there isn't that much benefit in creating one.

Proof. We leave this as an exercise.

Exercise 2.5.2.14 Prove this yourself, using the proof of the previous result as guidance.

■

Certainly, every accumulation point is an adherent point. Additionally, there is a name (and characterization) of those accumulation points which are not adherent points, namely, "*isolated point*".

Definition 2.5.2.15 — Isolated point Let $S \subseteq \mathbb{R}^d$ and let $x_0 \in \mathbb{R}^d$. Then, x_0 is an ***isolated point*** iff there is some $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(x_0) \cap S = \{x_0\}$.

R Note that it follows straight from the definition that $x_0 \in S$.

R The name is a fitting one: the intuition is that x_0 is "isolated" from the rest of S by a distance of at least ε_0.

Proposition 2.5.2.16 Let $S \subseteq \mathbb{R}^d$ and let $x_0 \in \mathbb{R}^d$. Then, x_0 is an adherent point of S iff either x_0 is an accumulation point of S or x_0 is an isolated point of S.

R Note that this or is *exclusive*.

R In particular, as every element of S is an adherent point, every element of S is either an accumulation point of S or an isolated point of S.

Proof. ($\Rightarrow$) Suppose that x_0 is an adherent point of S. If x_0 is an accumulation point, we're done, so suppose this is not the case. This means that there is some $\varepsilon_0 > 0$ such that it is *not* the case that $B_{\varepsilon_0}(x_0)$ intersects S at a point distinct from x_0. This can fail for two reasons: either it doesn't intersect S at all or it intersects S only at x_0. However, the former cannot happen as x_0 is an adherent point, and so it must be the case that $B_{\varepsilon_0}(x_0) \cap S = \{x_0\}$. Thus, x_0 is an isolated point of S. This shows that at least one of the statements is true. It remains to show that at most one of them is true.

So, suppose that x_0 is an accumulation point of S. This means that for every $\varepsilon > 0$, $B_\varepsilon(x_0) \cap S$ contains some point besides x_0, and in particular, this intersection is not equal to $\{x_0\}$. Thus, x_0 is not an isolated point of S.

Now suppose that x_0 is an isolated point of S. Then, there is some $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(x_0) \cap S = \{x_0\}$, and so in particular, this ε_0-ball does not intersect S at a point distinct from x_0. Thus, x_0 is not an accumulation point.

($\Leftarrow$) Suppose that either x_0 is an accumulation point of S or x_0 is an isolated point of S. If x_0 is an accumulation point, then x_0 is certainly an adherent point (from the definition). On the other hand, if x_0 is an isolated point, then in fact $x_0 \in S$, and so certainly $B_\varepsilon(x_0)$ intersects S for all $\varepsilon > 0$ (namely at x_0), and so is likewise an adherent point. ■

The following result is an equivalent characterization of being a closed set and was our motivation for introducing accumulation points at all.

Theorem 2.5.2.17. Let $C \subseteq \mathbb{R}^d$. Then, the following are equivalent.

(i). C is closed.
(ii). C contains all its accumulation points.
(iii). C contains all its limit points.
(iv). C contains all its adherent points.
(v). C contains all points $x_0 \in \mathbb{R}^d$ for which there is a net $\lambda \mapsto x_\lambda \in C$ such that $x_0 = \lim_\lambda x_\lambda$.
(vi). C is equal to its set of adherent points.
(vii). C is equal to the set of all points $x_0 \in \mathbb{R}^d$ for which there is a net $\lambda \mapsto x_\lambda \in C$ such that $x_0 = \lim_\lambda x_\lambda$.

R In practice, you use this to to show a set is closed as follows: given an arbitrary convergent net $\lambda \mapsto x_\lambda \in C$, you must show that $\lim_\lambda x_\lambda \in C$. This is really an application of (v), as you don't have to worry about x_λ being equal to the limit or not, which is arguably one reason why adherent points can be useful (being equivalent to this characterization by Proposition 2.5.2.13).

R I am quite confident this is not in fact the correct etymology of the term, but you might think of closed sets as being 'closed under the operation of taking limits'.

Proof. ((*ii*) $\Leftrightarrow$ (*iii*)) This follows from Proposition 2.5.2.12.

((*iv*) $\Leftrightarrow$ (*v*)) This follows from Proposition 2.5.2.13.

((*vi*) $\Leftrightarrow$ (*vii*)) This follows from Proposition 2.5.2.13.

((*i*) $\Rightarrow$ (*ii*)) Suppose that C is closed. Let $x \in \mathbb{R}^d$ be an accumulation point of C. We proceed by contradiction: suppose that $x \in C^{\mathsf{C}}$. Then, because C^{C} is open, there is some $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(x) \subseteq C^{\mathsf{C}}$. But then, $B_{\varepsilon_0}(x) \cap C$ is empty, a contradiction of the fact that x is an accumulation point of C. Thus, we must have had that $x \in C$.

((*ii*) $\Rightarrow$ (*i*)) Suppose that C contains all its accumulation points. Let $x \in C^{\mathsf{C}}$. Then, x is not an accumulation point of C, and so there must be some ε_0 such that $B_\varepsilon(x) \cap C$ is empty (it cannot even contain x because $x \notin C$). In other words, it must be the case that $B_\varepsilon(x) \subseteq C^{\mathsf{C}}$, so that C^{C} is open.

((*iv*) $\Rightarrow$ (*ii*)) This is immediate as every accumulation point is an adherent point.

((*ii*) $\Rightarrow$ (*iv*)) Suppose that C contains all its accumulation points. Let $x_0 \in \mathbb{R}^d$ be an adherent point of C. By the previous result, either x_0 is an accumulation point of C or an isolated point of C. In the former case, $x_0 \in C$ by hypothesis, and in the latter case, $x_0 \in C$ because isolated points of a set are always contained in that set. Either way, $x_0 \in C$, and so C contains all its adherent points.

($(iv) \Rightarrow (vi)$) Suppose that C contains all its adherent points. Then, as every set is contained in its set of adherent points,[a], it follows that C is equal to its set of adherent points.

($(vi) \Rightarrow (iv)$) Immediate ■

[a] See the remark in the definition Definition 2.5.2.10.

Thus, while C is closed iff it is equal to its set of adherent points, this is in general not true if you replace "adherent" with "accumulation". In fact, this condition even has a name.

Definition 2.5.2.18 — Perfect set Let $C \subseteq \mathbb{R}^d$. Then, C is ***perfect*** iff it is equal to its set of accumulation points.

R In particular, it contains its set of accumulation points, and perfect sets are necessarily closed.

There are several equivalent ways to state this condition.

Proposition 2.5.2.19 Let $C \subseteq \mathbb{R}^d$. Then, the following are equivalent.

(i). C is perfect.
(ii). C is closed and every element of C is an accumulation point of C.
(iii). C is closed and has no isolated points.

Proof. ($(i) \Rightarrow (ii)$) Suppose that C is perfect. Then, by definition, C is equal to its set of accumulation points, and so in particular it contains all its accumulation points, and so is closed by Theorem 2.5.2.17. Likewise, every element of C is an accumulation point of C.

(((*ii*) $\Rightarrow$ (*iii*)) Suppose that C is closed and every element of C is an accumulation point of C. By Theorem 2.5.2.17, C contains its accumulation points, and hence is equal to its set of accumulation points. By Theorem 2.5.2.17 again, it follows that its set of accumulation points is equal to its set of adherent points. By Proposition 2.5.2.16, an adherent point is either an accumulation point or an isolated point, whence it follows that C has no isolated points.

(((*iii*) $\Rightarrow$ (*i*)) Suppose that C is closed and has no isolated points. By Theorem 2.5.2.17, C is equal to its set of adherent points, which, by Proposition 2.5.2.16, is simply its set of accumulation points. ■

Perfect sets will almost be of no use to use at all. However, there is at least one interesting fact about perfect sets that we will make use of later on (when discussing generalized Cantor sets—see Example 5.1.5.20).

Proposition 2.5.2.20 Let $C \subseteq \mathbb{R}^d$. Then, if C is perfect, then either C is empty or uncountable.

W Warning: This doesn't generalize to topological spaces.[a]

[a]This will be obvious as soon as you know what a topological space is—certainly subsets of finite topological spaces cannot be uncountable.

Proof. We leave this as an exercise.

Exercise 2.5.2.21 Prove this yourself.

R Hint: See [Abb02, Theorem 3.4.3].

■

Exercise 2.5.2.22 Let $S \subseteq \mathbb{R}$ be closed and bounded above. Show that $\sup(S) \in S$.

R Similarly, of course, if S is closed and bounded below, then $\inf(S) \in S$.

Proposition 2.5.2.23 Let $m \mapsto x_m$ be a sequence that is not eventually constant and let $x \in \mathbb{R}^d$. Then, x is an accumulation point of $\{x_m : m \in \mathbb{Z}^+\}$ iff there is a subsequence of $m \mapsto x_m$ which converges to x.

R Note that this result would be *false* if we did not require that $B_\varepsilon(x_0)$ intersect the set at a point *distinct* from x_0 in the definition of an accumulation point (Definition 2.5.2.9). For example, if we did not require this, $1 \in \mathbb{R}$ would be an accumulation point of $\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \rangle$, but of course no subsequence converges to 1.

W Warning: This is *not* true if you replace "sequence" with "net"—see the following counter-example.

W Warning: This is *not* true in general topological spaces—see Example 3.2.31.

Proof. $(\Rightarrow)$ Suppose that x is an accumulation point of $\{x_m : m \in \mathbb{Z}^+\}$. We construct a subsequence $n \mapsto x_{m_n}$ inductively that has the property that $x_{m_n} \in B_{\frac{1}{n}}(x)$ with $x_{m_n} \neq x$. If we can do so, then $n \mapsto x_{m_n}$ will converge to x by the Archimedean Property (that is, because numbers of the form $\frac{1}{n}$ can be chosen to be arbitrarily small). Because x is an accumulation point, we have that $B_1(x) \cap \{x_m : m \in \mathbb{Z}^+\}$ contains some point distinct from x, and so we can take x_{m_0} to be any such element. Suppose now that we have constructed $x_{m_0}, \ldots, x_{m_n}$ and we wish to construct $x_{m_{n+1}}$. By Exercise 2.5.2.11, not only is $B_{\frac{1}{n+1}}(x) \cap \{x_m : m \in \mathbb{Z}\}$ nonempty, but it is infinite. Therefore,

$B_{\frac{1}{n+1}} \cap \{x_m : m > m_n\}$ is nonempty,[a] and so we can choose $x_{m_{n+1}}$ to be any element in this set distinct from x.

R Note that this direction of the proof fails for nets in general. We are implicitly using the fact that infinite subsets of $\mathbb{N}$ are cofinal in $\mathbb{N}$ (and hence give us a (strict) subsequence), and this is not true for general directed sets.

($\Leftarrow$) Suppose that there is a subsequence of $m \mapsto x_m$ which converges to x. Denote this subsequence by $n \mapsto x_{m_n}$. Let $\varepsilon > 0$. Then, there is some n_0 such that, whenever $n \geq n_0$, it follows that $x_{m_n} \in B_\varepsilon(x)$. In particular, $B_\varepsilon(x) \cap \{x_m : m \in \mathbb{Z}^+\}$ contains an element distinct from x (because the sequence $m \mapsto x_m$ is not eventually constant), and so x is an accumulation point of $\{x_m : m \in \mathbb{Z}^+\}$. ■

[a]Because this set is obtained from the *infinite* set $B_{\frac{1}{n+1}}(x) \cap \{x_m : m \in \mathbb{Z}\}$ by removing *finitely* many points.

■ Example 2.5.2.24 — An accumulation point of a net to which no subnet converges [a] Define $\mathbb{R}^+ \ni \lambda \mapsto x_\lambda := \frac{1}{\lambda}$. Then,

$$\{x_\lambda : \lambda \in \mathbb{R}^+\} = (0, \infty). \tag{2.5.2.25}$$

Thus, for example, $1 \in (0, \infty)$ is an accumulation point of the net. However, as $\lim_\lambda x_\lambda = 0$, it follows that no subnet can converge to 1.[b]

[a]This example was inspired by a similar example showed to me by a student.

[b]Of course, there is nothing special about 1—any element of $(0, \infty)$ would work just as well.

There is a ‘dual’ (well, sort of) notion of accumulation point, though it is perhaps not quite as useful.

Definition 2.5.2.26 — Interior point Let $S \subseteq \mathbb{R}^d$ and let $x_0 \in \mathbb{R}^d$. Then, x_0 is an ***interior point*** of S iff there is some ε_0 such that $B_{\varepsilon_0}(x_0) \subseteq S$.

The result dual to Theorem 2.5.2.17 is in the following easy exercise.

Exercise 2.5.2.27 Let $U \subseteq \mathbb{R}^d$. Show that U is open iff every point in U is an interior point.

The next couple of results are incredibly important for the same reason that Theorem 2.5.2.5 (the characterization of continuity in terms of open sets) was: they are neither deep nor difficult (in fact, they're quite trivial), but they will be defining requirements of open sets in the more general setting of topological spaces.

Theorem 2.5.2.28. Let $\mathscr{U}$ be a collection of open subsets of $\mathbb{R}^d$. Then,

$$\bigcup_{U \in \mathcal{U}} U \tag{2.5.2.29}$$

is open.

> R In other words, an *arbitrary* union of open sets is open.

Proof. Let $x \in \bigcup_{U \in \mathcal{U}} U$. Then, $x \in U$ for some $U \in \mathcal{U}$. Because U is open, there is some $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(x) \subseteq U \subseteq \bigcup_{U \in \mathcal{U}} U$, and so $\bigcup_{U \in \mathcal{U}} U$ is open. ∎

Theorem 2.5.2.30. Let $U_1, \ldots, U_m \subseteq \mathbb{R}^d$ be open. Then,

$$\bigcap_{k=1}^{m} U_k \tag{2.5.2.31}$$

is open.

R In other words, the *finite* intersection of open sets is open.

Proof. Let $x \in \bigcap_{k=1}^{m} U_k$, so that $x \in U_k$ for $1 \leq k \leq m$. Let $\varepsilon_k > 0$ be such that $B_{\varepsilon_k}(x) \subseteq U_k$. Define $\varepsilon_0 := \min\{\varepsilon_1, \ldots, \varepsilon_m\} > 0$. Then, $B_{\varepsilon_0}(x) \subseteq B_{\varepsilon_k}(x) \subseteq U_k$ for all k, and so $B_{\varepsilon_0}(x) \subseteq \bigcap_{k=1}^{m} U_k$, so that $\bigcap_{k=1}^{m} U_k$ is open. ■

Exercise 2.5.2.32 Find an infinite collection of open sets whose intersection is *not* open.

Exercise 2.5.2.33 Let $\mathscr{C}$ be a collection of closed subsets of $\mathbb{R}^d$. Show that

$$\bigcap_{C \in \mathscr{C}} C \tag{2.5.2.34}$$

is closed.

Exercise 2.5.2.35 Let $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ be closed. Show that

$$\bigcup_{k=1}^{m} C_k \tag{2.5.2.36}$$

is closed.

The closure of a set is the 'smallest' closed set which contains it. Likewise, the interior of a set is the 'largest' open set which it contains. The sense in which these are respectively "smallest" and "largest" is made precise by the following results.

Proposition 2.5.2.37 — Closure Let $S \subseteq \mathbb{R}^d$. Then, there exists a unique set $\mathrm{Cls}(S) \subseteq \mathbb{R}^d$, the ***closure*** of S, that satisfies

(i). $\mathrm{Cls}(S)$ is closed;

(ii). $S \subseteq \mathrm{Cls}(S)$; and

(iii). if C is any other closed set which contains S, then $\mathrm{Cls}(S) \subseteq C$.

Furthermore, explicitly, we have

$$\mathrm{Cls}(S) = \bigcap_{\substack{C \subseteq \mathbb{R}^d \text{ closed} \\ S \subseteq C}} C. \tag{2.5.2.38}$$

R Compare this with the definition of the integers, rationals, and reals (Theorems 1.2.1, 1.3.4 and 1.4.2.9).

R See Proposition 2.5.2.45 for another description of the closure, as the union of the set with its accumulation points.

R Sometimes people denote the closure by $\bar{S}$. We prefer the notation $\mathrm{Cls}(S)$ because (i) it is less ambiguous (the over-bar is used to denote many things in mathematics) and (ii) the notation $\mathrm{Cls}(S)$ is just slightly more descriptive.

Proof. Define

$$\mathrm{Cls}(S) := \bigcap_{\substack{C \subseteq \mathbb{R} \text{ closed} \\ S \subseteq C}} C. \tag{2.5.2.39}$$

$\mathrm{Cls}(S)$ is closed because the intersection of an arbitrary collection of closed sets is closed. $S \subseteq \mathrm{Cls}(S)$ because, by definition, S is contained in every subset in the intersection (2.5.2.39). Let C be some other closed set which contains S. Then, C itself appears in the intersection of (2.5.2.39), and so $\mathrm{Cls}(S) \subseteq C$.

If C is some other subset of $\mathbb{R}^d$ which satisfies (i)–(iii), then, by (iii) applied to $\mathrm{Cls}(S)$, we have that $\mathrm{Cls}(S) \subseteq C$. On the other hand, by (iii) applied to C, we have that $C \subseteq \mathrm{Cls}(S)$. Thus, $\mathrm{Cls}(S) = C$. ■

We have a dual result for the interior.

Proposition 2.5.2.40 — Interior Let $S \subseteq \mathbb{R}^d$. Then, there exists a unique set $\operatorname{Int}(S) \subseteq \mathbb{R}^d$, the ***interior*** of S, that satisfies

(i). $\operatorname{Int}(S)$ is open;
(ii). $\operatorname{Int}(S) \subseteq S$; and
(iii). if U is any other open set which is contained in S, then $U \subseteq \operatorname{Int}(S)$.

Furthermore, explicitly, we have

$$\operatorname{Int}(S) = \bigcup_{\substack{U \subseteq \mathbb{R} \text{ open} \\ U \subseteq S}} U. \qquad (2.5.2.41)$$

R See Proposition 2.5.2.46 for another description of the interior, as the set of (surprise, surprise) interior points.

R Sometimes people denote the interior by S°. We prefer the notation $\operatorname{Int}(S)$ for essentially the same reasons as we prefer the notation $\operatorname{Cls}(S)$.

Proof. We leave this as an exercise.

Exercise 2.5.2.42 Complete this proof by yourself, using the dual proof for the closure as guidance.

∎

One might ask "What about the smallest *open* set which contains S?" (and "dually"). In general, however, there is no smallest open set that contains S.

▪ **Example 2.5.2.43 — A set for which there is not a smallest open set which contains S** Define $S := \{0\} \subseteq \mathbb{R}$. Suppose there is a smallest open set U which contains 0. As U

is open, there is some $\varepsilon > 0$ such that $B_\varepsilon(0) \subseteq U$. On the off chance we have $B_\varepsilon(0)$, use $\frac{\varepsilon}{2}$ instead: $B_{\frac{\varepsilon}{2}}(0)$ will be a *proper* subset of U. Thus, there can be no smallest open set which contains $\{0\}$.

Exercise 2.5.2.44 Find a set $S \subseteq \mathbb{R}^d$ for which there is no largest set closed set contained in S.

There is a relatively concrete description of the closure.

Proposition 2.5.2.45 Let $S \subseteq \mathbb{R}^d$. Then, $\mathrm{Cls}(S)$ is the union of S and its set of accumulation points.

Proof. Let C be the union of S and its accumulation points. We simply have to verify that it satisfies the axioms of the definition of the closure in Proposition 2.5.2.37.

To show that it is closed, we must show that it contains all its accumulation points. So, let x be an accumulation point of C. If $x \in S$, we are done, so we may as well assume that $x \notin S$. We show that x is an accumulation point of S, so that $x \in C$. Let $\varepsilon > 0$. We wish to show that $B_\varepsilon(x)$ intersects S (as $x \notin S$, we know the point of intersection must be distinct from x). As x is an accumulation point of C, we know, however, that $B_\varepsilon(x)$ contains either a point of S or an accumulation point of S distinct from x. In the former case, we are done, so let $x_\varepsilon \in B_\varepsilon(x)$ be an accumulation point of S distinct from x. Because x_ε is an accumulation point of S, it must be the case that $B_\varepsilon(x_\varepsilon)$ intersects S at a point $x_\varepsilon \in S$ distinct from x_ε. But then, by the triangle inequality, $x_\varepsilon \in B_{2\varepsilon}(x)$. Thus, x is an accumulation point of S (x_ε and x must be distinct because one is in S and the other is not), and hence an element of C. Thus, C is closed.

Because any closed set must contain all its accumulation points (Theorem 2.5.2.17), it follows that C must be contained in any closed set which contains S, and so $C = \mathrm{Cls}(S)$. ■

There is a dual concrete description of the interior.

Proposition 2.5.2.46 Let $S \subseteq \mathbb{R}^d$. Then, $\mathrm{Int}(S)$ is the set of interior points of S.

Proof. Because of the dual result to Theorem 2.5.2.17, namely Exercise 2.5.2.27 (a set is open iff all of its points are interior points), just as in the dual proof above, it suffices to show that the set of interior points of S is open.

So, let $x \in \mathbb{R}^d$ be an interior point of S. Then, there is some $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(x) \subseteq S$. To show that the set of interior points is open, we need to show that in fact every point of $B_{\varepsilon_0}(S)$ is an interior point of S. This however follows from the fact that balls are open (Exercise 2.5.2.4). ■

Exercise 2.5.2.47 Let $S \subseteq \mathbb{R}^d$.

(i). Show that S is closed iff $S = \mathrm{Cls}(S)$.
(ii). Show that S is open iff $S = \mathrm{Int}(S)$.

In the next chapter, we will define a topological space as a set equipped with a choice of open sets. The choice of open sets will be called a *topology*. Of course, it turns out that there are many equivalent ways to specify a topology, and one way to do this is by defining what the closure (or interior) of each set is. The following result is important because, when we go to generalize, it will play the role of axioms which a closure (or interior) 'operator' must satisfy.

Theorem 2.5.2.48 — Kuratowski Closure Axioms. Let $S, T \subseteq \mathbb{R}^d$. Then,

(i). $\mathrm{Cls}(\emptyset) = \emptyset$;
(ii). $S \subseteq \mathrm{Cls}(S)$;
(iii). $\mathrm{Cls}(S) = \mathrm{Cls}(\mathrm{Cls}(S))$; and
(iv). $\mathrm{Cls}(S \cup T) = \mathrm{Cls}(S) \cup \mathrm{Cls}(T)$.

R Careful: The closure of a *finite* union is the union of the closures, but this does not hold in general—see Exercise 2.5.2.51 below.

Proof. The empty-set is closed, contains the empty-set, and is contained in every closed set which contains the empty-set, and hence, by definition, $\mathrm{Cls}(\emptyset) = \emptyset$.

(ii) follows from the definition.

(iii) follows from the fact that the closure of a set is closed (by definition) and the fact that the closure of a closed set is itself (Exercise 2.5.2.47).

We now prove (iv). We show that $\mathrm{Cls}(S) \cup \mathrm{Cls}(T)$ satisfies the axioms of the closure of $S \cup T$. $\mathrm{Cls}(S) \cup \mathrm{Cls}(T)$ is a closed set which contains $S \cup T$, and so it therefore contains $\mathrm{Cls}(S \cup T)$. Let C be some other closed set which contains $S \cup T$. C therefore contains S, and so it must contain $\mathrm{Cls}(S)$. Likewise, it must contain $\mathrm{Cls}(T)$, and so C must contain $\mathrm{Cls}(S) \cup \mathrm{Cls}(T)$. Therefore, by definition, $\mathrm{Cls}(S) \cup \mathrm{Cls}(T) = \mathrm{Cls}(S \cup T)$. ■

Of course, we have the dual result for interior.

Theorem 2.5.2.49 — Kuratowski Interior Axioms. Let $S, T \subseteq \mathbb{R}^d$. Then,

(i). $\mathrm{Int}(\mathbb{R}) = \mathbb{R}^d$;
(ii). $\mathrm{Int}(S) \subseteq S$;
(iii). $\mathrm{Int}(S) = \mathrm{Int}\left(\mathrm{Int}(S)\right)$; and
(iv). $\mathrm{Int}(S \cap T) = \mathrm{Int}(S) \cap \mathrm{Int}(T)$.

Proof. We leave this as an exercise.

Exercise 2.5.2.50 Complete this proof yourself, using the dual proof for the closure as guidance.

■

Exercise 2.5.2.51 Let $\mathscr{S} \subseteq 2^{\mathbb{R}^d}$ be a collection of subsets of $\mathbb{R}^d$. Show that the following are true.

(i). $\bigcap_{S\in\mathscr{S}} \mathrm{Cls}(S) \supseteq \mathrm{Cls}\left(\bigcap_{S\in\mathscr{S}} S\right)$.
(ii). $\bigcup_{S\in\mathscr{S}} \mathrm{Int}(S) \subseteq \mathrm{Int}\left(\bigcup_{S\in\mathscr{S}} S\right)$.
(iii). $\bigcup_{S\in\mathscr{S}} \mathrm{Cls}(S) \subseteq \mathrm{Cls}\left(\bigcup_{S\in\mathscr{S}} S\right)$.
(iv). $\bigcap_{S\in\mathscr{S}} \mathrm{Int}(S) \supseteq \mathrm{Int}\left(\bigcup_{S\in\mathscr{S}} S\right)$.

Find examples to show that we need not have equality in general. In fact, show that (i) and (ii) can fail even in the case where $\mathscr{S}$ is *finite*.[a]

[a]Of course, in this case, you can find counter-examples for $|\mathscr{S}| = 2$.

2.5.3 Quasicompactness

You will find with experience that closed intervals on the real line are particularly nice to work with. For example, you'll probably recall from calculus that, on a closed interval, every continuous function *attains* a maximum and minimum (the Extreme Value Theorem (Corollary 3.8.2.5)). In particular, continuous functions are bounded on closed intervals. This is not just true of all closed intervals, however, but is in fact true about any closed bounded subset of $\mathbb{R}^d$.

The objective then is to characterize closed bounded sets in such a way that will generalize to arbitrary topological spaces. The characterization we are looking for is what is called *quasicompactness*.

Definition 2.5.3.1 — Cover Let $S \subseteq \mathbb{R}^d$ and let $\mathscr{U} \subseteq 2^{\mathbb{R}^d}$. Then, $\mathscr{U}$ is a *cover* of S iff $S \subseteq \bigcup_{U\in\mathscr{U}} U$. $\mathscr{U}$ is an *open cover* iff every $U \in \mathscr{U}$ is open. A *subcover* of $\mathscr{U}$ is a subset $\mathscr{V} \subseteq \mathscr{U}$ that is still a cover of S.

Definition 2.5.3.2 — Quasicompact Let $S \subseteq \mathbb{R}^d$. Then, S is ***quasicompact*** iff every open cover of S has a finite subcover.

R For most authors, and in fact, for probably all authors of introductory analysis books, my definition of quasicompact for them will be called just *compact*. Instead, I reserve the term *compact* for spaces which are both quasicompact *and* T_2—see Definition 3.6.2.19.[a] As $\mathbb{R}^d$ is T_2, the notions of compact and quasicompact are the same for the real numbers, but in general they will differ. To the best of my knowledge,[b] the term quasicompact originated in algebraic geometry because it was felt that things that should *not* intuitively be thought of as compact nevertheless satisfied the defining condition above. Thus, it was decided that such spaces should be referred to as quasicompact and that instead the term compact should be reserved for spaces which are both quasicompact and T_2. I prefer this terminology for two reasons: (i) the terminology is more precise, that is, I have two terms to work with instead of just one; and (ii) I strongly feel that it is a bad idea for the terminology we use to be dependent on the mathematics we happen to be doing at the moment—"compact" should not mean one thing today and something else tomorrow just because I decided to work on something different. Terminology should be as consistent as possible across all of mathematics.

[a] You are not expected to know what T_2 means—see the next chapter for details (specifically, Definition 3.6.2.18), though keep in mind, the details don't matter at the moment.

[b] Actually, since first writing this remark, I found that Bourbaki [Bou66, pg. 83] uses exactly this terminology, so perhaps the root of this terminology goes back further.

And now of course we had better check that this condition properly characterizes "closed and bounded" in the real numbers, as desired.

Theorem 2.5.3.3 — Heine-Borel Theorem. Let $S \subseteq \mathbb{R}^d$. Then, S is closed and bounded iff it is quasicompact.

R This fails in general topological spaces (for one thing, "boundedness" does not make sense in arbitrary topological spaces). In fact, it will not even hold topological spaces in which the notion of boundedness *does* make sense. That said, there is something closely resembling a generalization for uniform spaces—see Theorem 4.4.10.

W Warning: A common mistake I have seen is for students to assert that, for example, $[0,1] \subseteq \mathbb{R}$ is quasicompact *for other topologies besides the usual one*. In the next chapter, we will start equipping $\mathbb{R}$ with other new topologies, and this result *fails* for other topologies. For example, $[0,1]$ is *not* quasicompact in the cocountable topology on $\mathbb{R}$—see Example 3.6.2.23.

Proof. [a] $(\Rightarrow)$ STEP 1: ASSUME HYPOTHESES
Suppose that S is closed and bounded. Let $\mathscr{U}$ be an open cover of S. We proceed by contradiction: suppose that $\mathscr{U}$ has no finite subcover.

STEP 2: CONSTRUCT A SEQUENCE OF NONINCREASING CLOSED RECTANGLES WHOSE SIDE LENGTHS GO TO 0 AND WHOSE INTERSECTION WITH S HAS NO FINITE SUBCOVER
We do this step for $d = 1$ as it makes the proof clearer. We will mention in footnotes where things must be modified for $d \geq 2$.

Let $M > 0$ be such that $S \subseteq [-M, M]$.[b] Then, at least one of $[-M, 0] \cap S$ and $[0, M] \cap S$ must not have a finite subcover of $\mathcal{U}$, because if they both did, then the cover of $[-M, 0] \cap S$ together with the cover of $[0, M] \cap S$ would comprise a finite subcover of S. Without loss of generality, assume that $[0, M] \cap S$ has no finite subcover.[c] Then, by exactly the same logic, either $\left[0, \frac{M}{2}\right] \cap S$ or $\left[\frac{M}{2}, M\right] \cap S$ has no finite subcover. Proceeding inductively, we obtain a nonincreasing sequence of closed rectangles $R_0 \supseteq R_1 \supseteq R_2 \supseteq \cdots$ such that (i) $I_k \cap S$ has no finite subcover and (ii) the lengths of the intervals converges to 0.

STEP 3: CONSTRUCT AN ELEMENT IN $S \cap \bigcap_{k \in \mathbb{N}} R_k$

Note that, as $R_k \cap S$ has no finite subcover, in particular, it cannot be empty (otherwise any subset of $\mathcal{U}$ would cover it). So, let $x_k \in R_k \cap S$. Because the side lengths of the rectangles go to 0, $m \mapsto x_m$ is a Cauchy sequence, and hence converges to some $x_\infty \in \mathbb{R}^d$. As S is closed, we have in addition that $x_\infty \in S$. We wish to show that in addition $x_\infty \in I_k$ for all k. Write $R_k = [a_{k,1}, b_{k,1}] \times \cdots \times [a_{k,d}, b_{k,d}]$ for $a_{k,l} \leq b_{k,l}$. We wish to show that $a_{k,l} \leq x_\infty \leq b_{k,l}$. We show just $[x_\infty]_l \leq b_{k,l}$ (the other inequality is similar). We proceed by contradiction: suppose that there is some b_{k_0,l_0} such that $[x_\infty]_{l_0} > b_{k_0,l_0}$. Then, there is some m_0 such that, whenever $m \geq m_0$, it follows that $[x_m]_{l_0} > b_{k_0,l_0}$. However, for m at least as large as k_0, we need $x_m \in R_m \subseteq I_{k_0}$, so that $x_{m,l_0} \leq b_{k_0,l_0}$: a contradiction. Therefore, $x_\infty \leq b_k$ for all k, and so $x_\infty \in R_k$ for all k.

Step 4: Deduce the contradiction
As $x_\infty \in S$, there is some $U \in \mathscr{U}$ such that $x_\infty \in U$. Then, because the side lengths of the rectangles converge to 0 and U is open, there must be some I_{m_0} such that $x_\infty \in I_{m_0} \subseteq U$. But then, $\{U\}$ is a finite open cover of $I_{m_0} \cap S$: a contradiction. Therefore, S is quasicompact.

$(\Leftarrow)$

Step 5: Assume hypotheses
Suppose that S is quasicompact.

Step 6: Show that S is bounded
The cover

$$\{B_M(0) : M > 0\} \tag{2.5.3.4}$$

covers all of $\mathbb{R}^d$, and so certainly covers S. Therefore, there is a finite subcover

$$\left\{B_{M_1}(0), \ldots, B_{M_m}(0) : M_1, \ldots, M_m > 0\right\}. \tag{2.5.3.5}$$

Define $M := \max\{M_1, \ldots, M_m\}$. Then, $B_M(0)$ contains each $B_{M_k}(0)$, and so contains S. Therefore, S is bounded.

STEP 7: SHOW THAT S IS CLOSED
Let $\lambda \mapsto x_\lambda \in S$ be a net converging to $x_\infty \in \mathbb{R}^d$. We must show that $x_\infty \in S$—see Theorem 2.5.2.17. We proceed by contradiction: suppose that $x_\infty \notin S$. Then, for each $s \in S$, because $s \neq x_\infty$, there is some $\varepsilon_s > 0$ such that $x_\infty \notin B_{\varepsilon_s}(s)$.[d] The collection $\left\{B_{\varepsilon_s}(s) : s \in S\right\}$ is certainly an open cover of S, and so there is some finite subcover $\{B_{\varepsilon_{s_1}}(s_1), \ldots, B_{\varepsilon_{sm}}(s_m)\}$. Define $\varepsilon_0 := \min_{1 \leq k \leq m}\{|s_k - x_\infty| - \varepsilon_k\}$. As $\lim_\lambda x_\lambda = x_\infty$, there is some $x_{\lambda_0} \in B_{\varepsilon_0}(x_\infty)$. However, by the Reverse Triangle Inequality (Exercise 2.2.4.(v)),

$$\begin{aligned} \left|x_{\lambda_0} - s_k\right| &= \left|(x_{\lambda_0} - x_\infty) + (x_\infty - s_k)\right| \\ &\geq \left|\left|x_{\lambda_0} - x_\infty\right| - |x_\infty - s_k|\right| \\ &\geq |x_\infty - s_k| - \left|x_{\lambda_0} - x_\infty\right| \\ &> |x_\infty - s_k| - \varepsilon_0 \\ &\geq |x_\infty - s_k| - (|s_k - x_\infty| - \varepsilon_k) \\ &= \varepsilon_k, \end{aligned} \tag{2.5.3.6}$$

and so, $x_{\lambda_0} \notin B_{\varepsilon_{s_k}}(s_k)$, a contradiction of the fact that $\{B_{\varepsilon_{s_1}}(s_1), \ldots, B_{\varepsilon_{sm}}(s_m)\}$ is a cover of S. Therefore, $\lim_\lambda x_\lambda = x_\infty \in S$, and we are done. ∎

[a] Proof adapted from [Abb02, pg. 87–89].
[b] In general, we want $S \subseteq [-M, M] \times \cdots \times [-M, M]$.
[c] In higher dimensions, $[-M, M] \times \cdots \times [-M, M]$ breaks up into not 2 cases, but 2^d, all with side length M. The same logic gives us that S intersected with at least one of these ‘quadrants’ has no finite subcover.
[d] For what it’s worth, this step does not work in general.

Equivalent formulations of quasicompactness

If for some reason you find the definition of quasicompactness in terms of open covers off-putting, there are a couple of other equivalent formulations of the concept that we present in this section. In contrast to the Heine-Borel Theorem, these characterizations of quasicompactness do generalize to arbitrary topological spaces.

The first characterization we come to is essentially the characterization you get by 'taking complements' in the definition of quasicompactness, and rewording things in terms of closed sets. To phrase this characterization, it will be convenient to have a term that makes the statement less verbose.

Definition 2.5.3.7 — Finite-intersection property Let X be a set, let $S \subseteq X$, let $\mathscr{C} \subseteq 2^X$ be a collection of subsets of X. Then, $\mathscr{C}$ has the ***finite-intersection property*** with S iff every finite subset $\{C_1, \ldots, C_m\} \subseteq \mathscr{C}$ intersects S: $(C_1 \cap \cdots \cap C_m) \cap S \neq \emptyset$. For $S = X$, we simply say that $\mathscr{C}$ has the ***finite-intersection property***.[a]

[a] That is, in this case we omit the phrase "with X".

Proposition 2.5.3.8 Let $K \subseteq \mathbb{R}^d$ and let $\mathscr{C}$ be a collection of closed subsets of $\mathbb{R}^d$. Then, K is quasicompact iff whenever $\mathscr{C}$ has the finite-intersection property with K, the entire intersection $\bigcap_{C \in \mathscr{C}} C$ also intersects K.

Proof. ($\Rightarrow$) Suppose that K is quasicompact. Let $\mathscr{C}$ be a collection of closed subsets of $\mathbb{R}^d$ that has the property that the intersection of any finite number of elements of $\mathscr{C}$ intersects K. We proceed by contradiction: suppose that $\bigcap_{C \in \mathscr{C}} C$ does not intersect K. Then, $\left(\bigcap_{C \in \mathscr{C}} C\right)^{\mathsf{C}} = \bigcup_{C \in \mathscr{C}} C^{\mathsf{C}}$ contains K,[a] and therefore the collection $\mathscr{U} := \left\{C^{\mathsf{C}} : C \in \mathscr{C}\right\}$ constitutes an open cover of K. Because K is quasicompact, it follows that there is some finite subcover $C_1^{\mathsf{C}} \cup \cdots \cup C_m^{\mathsf{C}} \supseteq K$. But then $C_1 \cap \cdots \cap C_m$ does not intersect K: a contradiction. Therefore, $\bigcap_{C \in \mathscr{C}} C$ intersects K.

($\Leftarrow$) Suppose that $\mathscr{C}$ has the finite-intersection property with K.

Exercise 2.5.3.9 Prove the converse.

■

[a] Recall De Morgan's Laws (Exercise A.2.3.12).

Theorem 2.5.3.10. Let $K \subseteq \mathbb{R}^d$. Then, K is quasicompact iff every net $\lambda \mapsto x_\lambda \in K$ has a subnet that converges to a limit in K.

> R This is yet another result that will not hold in general if you replace the word "net" with "sequence" (though it will hold in $\mathbb{R}^d$).

Proof. ($\Rightarrow$) Suppose that K is quasicompact. Let $\Lambda \ni \lambda \mapsto x_\lambda \in K$ be a net. Define

$$C_\lambda := \mathrm{Cls}\left(\left\{x_\mu : \mu \geq \lambda\right\}\right) \tag{2.5.3.11}$$

and $\mathscr{C} := \{C_\lambda : \lambda \in \Lambda\}$.

Exercise 2.5.3.12 Show that $\mathscr{C}$ has the finite-intersection property with K.

By the previous characterization of quasicompactness, it follows that $\bigcap_{\lambda \in \Lambda} C_\lambda$ intersects K, so let $x \in K$ be in this intersection.

Then,

$$\text{for every } \varepsilon > 0 \text{ and for every } \mu \text{, there is some } x_{\lambda_{\varepsilon,\mu}} \in B_\varepsilon(x) \text{ with } \lambda_{\varepsilon,\mu} \geq \mu.^b \tag{2.5.3.13}$$

Define

$$\Lambda' := \left\{\langle \varepsilon, \lambda\rangle : \varepsilon \in \mathbb{R}^+, \lambda \in \Lambda \text{ such that } x_\lambda \in B_\varepsilon(x)\right\}.$$

We order $\mathbb{R}^+$ with the reverse of the usual ordering. Then, $\mathbb{R}^+ \times \Lambda$ is directed. We verify that Λ' is likewise directed.

Let $\varepsilon_1, \varepsilon_2 > 0$ and let $\lambda_1, \lambda_2 \in \Lambda$ be such that $x_{\lambda_k} \in B_{\varepsilon_k}(x)$. Take $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ and let λ_3 be at least as large as λ_1 and λ_2. By (2.5.3.13), there is some $\lambda \geq \lambda_3$ such that $x_\lambda \in B_\varepsilon(x)$. Then, $\langle \varepsilon, \lambda \rangle \in \Lambda'$ and $\langle \varepsilon, \lambda \rangle \geq \langle \varepsilon_k, \lambda_k \rangle$, so that Λ' is directed.

Now, for $\langle \varepsilon, \mu \rangle \in \Lambda'$, pick some $\lambda_{\varepsilon,\mu}$ such that (i) $x_{\lambda_{\varepsilon,\mu}} \in B_\varepsilon(x)$ and (ii) $\lambda_{\varepsilon,\mu} \geq \mu$. Of course $\langle \varepsilon, \mu \rangle \mapsto x_{\varepsilon,\mu}$ converges to x, but we still need to check that it is a subnet.

Let λ_0 be an arbitrary index. Then, if $\langle \varepsilon, \mu \rangle \geq (1, \lambda_0)$, it follows that $\lambda_{\varepsilon,\mu} \geq \mu \geq \lambda_0$, and so, by Proposition 2.4.5.9, this is indeed a subnet.

($\Leftarrow$) Suppose that every net $\lambda \mapsto x_\lambda$ has a subnet that converges to a limit in K. Let $\mathscr{C}$ be a collection of closed sets such that the intersection of finitely many elements of $\mathscr{C}$ intersects K. Let $\tilde{\mathscr{C}}$ be the collection of all finite subsets of $\mathscr{C}$ ordered by inclusion, so that it is indeed a directed set. For each element $\tilde{C} \in \tilde{\mathscr{C}}$, let $x_{\tilde{C}} \in \left(\bigcap_{C \in \tilde{C}} C\right) \cap K$, which by assumption is nonempty. Then, $\tilde{\mathscr{C}} \ni \tilde{C} \mapsto x_{\tilde{C}} \in K$ is a net, and so by hypothesis, this has a subnet $\mu \mapsto x_{\tilde{C}_\mu}$ that converges to $x \in K$. We wish to show that $x \in (\bigcap_{C \in \mathscr{C}} C) \cap K$. We already know that $x \in K$, so to show this, it suffices to show that $x \in C$ for all $c \in \mathscr{C}$. To show this, it suffices to show that $\mu \mapsto x_{\tilde{C}_\mu}$ is eventually in each $C \in \mathscr{C}$ (we will then have that $x \in C$ because each C is closed). By the definition of a subnet, it suffices to show that each C eventually contains $\tilde{C} \mapsto x_{\tilde{C}}$. However, for $C_0 \in \mathscr{C}$, $\{C_0\} \in \tilde{C}$, and so whenever $\tilde{C} \geq \{C_0\}$, i.e. $C_0 \in \tilde{C}$, certainly $x_{\tilde{C}} \in C_0$ as in fact $x_{\tilde{C}} \in \left(\bigcap_{C \in \tilde{C}} C\right) \cap K$. ■

[a] This is the statement that x is an accumulation point of $\{x_\lambda : \lambda \geq \mu\}$ for each μ.

[b] This is the statement that x is an accumulation point of $\{x_\lambda : \lambda \geq \mu\}$ for each μ.

Corollary 2.5.3.14 — Bolzano-Weierstrass Theorem Every eventually bounded net in $\mathbb{R}^d$ has a convergent subnet.

Proof. Every eventually bounded net is eventually contained in some closed rectangle interval $[-M,M]\times\cdots\times[-M,M]$, and so, every eventually bounded net has a subnet which is contained (not *eventually* contained) in $[-M,M]\times\cdots\times[-M,M]$. $[-M,M]\times\cdots\times[-M,M]$ is quasicompact by the Heine-Borel Theorem (Theorem 2.5.3.3). By the subnet characterization of quasicompactness (the previous theorem), it follows that this net has a convergent subnet. ■

Corollary 2.5.3.15 Bounded infinite subsets of $\mathbb{R}^d$ have accumulation points.

R This is sometimes also called the Bolzano-Weierstrass Theorem.

Proof. Any infinite subset of $\mathbb{R}^d$ will have a sequence of distinct points contained in it. If the set is bounded, this sequence will be bounded, and so by the Bolzano-Weierstrass Theorem has a convergent subnet. The limit of this subnet is an accumulation point of the set (Proposition 2.5.2.23). ■

2.6 Summary

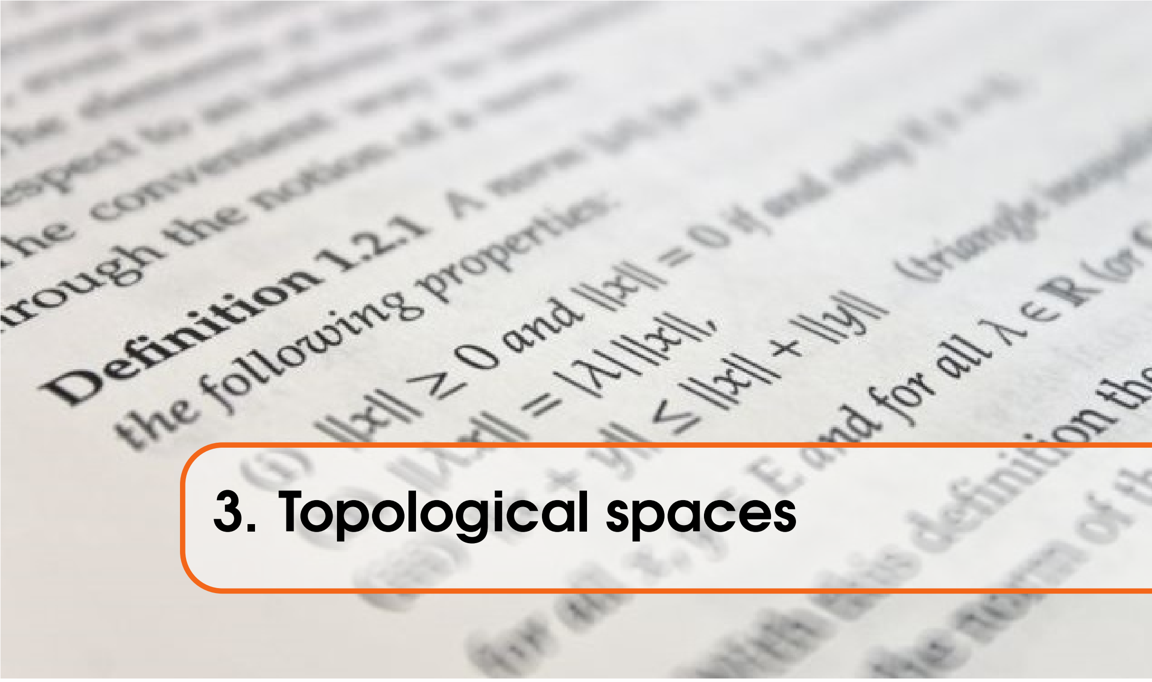
This has been a rather long chapter and we have covered many different properties of the real numbers. For convenience, we summarize here some of the main points we have covered.

(i). The real numbers are the unique (up to isomorphism of totally-ordered fields) nonzero Dedekind-complete totally-ordered field.
(ii). The real numbers are Cauchy-complete (Theorem 2.4.3.40).
(iii). The real numbers have cardinality $2^{\aleph_0} > \aleph_0$ (Theorems 2.1.15 and 2.4.4.49).
(iv). The real numbers are Archimedean (the natural numbers are unbounded)—see Theorem 2.3.3.

(v). Nondecreasing/nonincreasing nets bounded above/bounded below converge (Monotone Convergence Theorem).

(vi). A subset of $\mathbb{R}$ is closed and bounded iff it is quasicompact (Heine-Borel Theorem).

(vii). Every bounded net in $\mathbb{R}$ has a convergent subnet (Bolzano-Weierstrass Theorem).



3. Topological spaces

We have mentioned topological spaces several times throughout the notes already, and finally we turn to studying general topological spaces themselves. I said before that I think it is fair to say that the purpose of topological spaces is to introduce the most general context as possible in which one can talk about continuity. The motivating result in this regard is Theorem 2.5.2.5, which says that a function is continuous iff the preimage of every open set is open. The idea is then to axiomatize the notion of open set: a topological space will be a set X equipped with a collection of subsets $\mathscr{U} \subseteq 2^X$, called the *open sets*. Of course, if we want to be able to say anything of interest, we can't just take any old collection of subsets—we must require the collection to satisfy some conditions. The conditions we require are those that come from Theorems 2.5.2.28 and 2.5.2.30, namely, that an arbitrary union and finite intersection of open sets is open. So, without further ado, I present to you, the definition of a topological space.

3.1 The definition of a topological space

Definition 3.1.1 — Topological space A ***topological space*** is a set X equipped with a collection of subsets $\mathscr{U} \subseteq 2^X$, the ***topology*** on X, such that

(i). $\emptyset, X \in \mathscr{U}$;
(ii). $\bigcup_{U \in \mathscr{V}} U \in \mathscr{U}$ if $\mathscr{V} \subseteq \mathscr{U}$; and
(iii). $\bigcap_{k=1}^{m} U_k \in \mathscr{U}$ if $U_k \in \mathscr{U}$ for $1 \leq k \leq m$.

R The elements of $\mathscr{U}$ are the ***open sets***.

R In other words, a topological space is a set X equipped with a collection of subsets (containing at least $\emptyset$ and X) closed under arbitrary union and finite intersection.

R A subset $C \subseteq X$ is ***closed*** iff C^{C} is open. (Recall that this is precisely the definition we gave in $\mathbb{R}$—see Definition 2.5.2.6.)

W Warning: A common mistake beginners make is to think "More open sets means fewer closed sets.". This is *wrong*. There are always the same number of open sets as closed sets: $U \mapsto U^{\mathsf{C}}$ gives a bijection between them.

R In order to exclude stupid things like the empty topology, we require that the empty-set and the entire set are open. (Recall that these were open in $\mathbb{R}$—see Exercises 2.5.2.2 and 2.5.2.3.)

Of course, you can specify a topology just as well by saying what the closed sets are (the open sets are then just the complements of these sets).

Exercise 3.1.2 Let X be a set and let $\mathscr{C} \subseteq 2^X$ be a collection of subsets of X such that

(i). $\emptyset, X \in \mathscr{C}$;
(ii). $\bigcap_{C \in \mathscr{D}} C$ if $\mathscr{D} \subseteq \mathscr{C}$; and
(iii). $\bigcup_{k=1}^{m} C_k \in \mathscr{C}$ if $C_k \in \mathscr{C}$ for $1 \leq k \leq m$.

Show that there is a unique topology on X whose collection of closed sets is precisely $\mathscr{C}$.

R In other words, your collection of closed sets must be nontrivial, closed under arbitrary intersection, and closed under finite union, just as was the case in $\mathbb{R}$—see Exercises 2.5.2.33 and 2.5.2.35.

Definition 3.1.3 — G_δ and F_σ sets Let $S \subseteq X$ be a subset of a topological space. Then, S is a G_δ ***set*** iff S is the countable intersection of open sets. S is an F_σ ***set*** iff S i the countable union of closed sets.

R For us, these concepts will not come up very often, so it is not imperative that you remember these terms—indeed, you can probably just skip this definition for now. They do come up, however, so it would be incomplete to not include them, and if they're going to be included somewhere, this is not a bad place to do so.

3.1.1 (Neighborhood) bases and generating collections

It is often not necessary to define every single open set explicitly, but rather, only a special class of open sets that determine all the others. For example, in the real numbers, to determine whether an arbitrary set was open, we made use of the 'special' open sets $B_\varepsilon(x)$: you could determine whether an arbitrary set was open simply by knowing the ε balls—see Definition 2.5.2.1. The idea that generalizes this notion is that of a *base* for a topology.

Definition 3.1.1.1 — Base Let X be a topological space and let $\mathscr{B}$ be a collection of open sets of X. Then, $\mathscr{B}$ is a ***base*** for the topology of X iff the statement that a subset U of X is open is equivalent to the statement that, for every $x \in U$, there is some $B \in \mathscr{B}$ such that $x \in B \subseteq U$.

▪ **Example 3.1.1.2** The collection $\left\{B_\varepsilon(x) : x \in \mathbb{R}^d,\ \varepsilon > 0\right\}$ is a base for the topology of $\mathbb{R}^d$.

The real reason bases are important is because they allow us to *define* topologies, and so it is important to know when a collection of subsets of a set form a base for some topology.

Proposition 3.1.1.3 Let X be a set and let $\mathscr{B}$ be a collection of subsets of X. Then, there exists a unique topology for which $\mathscr{B}$ is a base iff

(i). $\mathscr{B}$ covers X; and
(ii). for every $x \in X$ and $B_1, B_2 \in \mathscr{B}$ with $x \in B_1, B_2$, there is some $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

R If X does not a priori come with a topology, we will still refer to any collection of sets that satisfy (i)–(ii) as a *base*.

Proof. ($\Rightarrow$) Suppose that there is a unique topology $\mathscr{U}$ for which $\mathscr{B}$ is a base. We first show that $\mathscr{B}$ covers X. We proceed by contradiction: suppose there is some $x \in X$ which is not contained in any $B \in \mathscr{B}$. Then, as $\mathscr{B}$ is a base for the topology, X would not be open: a contradiction. Therefore, $\mathscr{B}$ covers X.

Now for the second property: let $x \in X$ and let $B_1, B_2 \in \mathscr{B}$ be such that $x \in B_1, B_2$. By the definition of a base, we have that $\mathscr{B} \subseteq \mathscr{U}$. In particular, $B_1 \cap B_2 \in \mathscr{U}$, and so because $\mathscr{B}$ is a base for $\mathscr{U}$, there must be some $x \in B_3 \in \mathscr{B}$ such that $B_3 \subseteq B_1 \cap B_2$.

($\Leftarrow$) Suppose that (i) $\mathscr{B}$ covers X, and that (ii) for every for $x \in X$ and $B_1, B_2 \in \mathscr{B}$ with $x \in B_1 \cap B_2$, there is some

$B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. We declare $U \subseteq X$ to be open iff for every $x \in U$ there is some $B \in \mathscr{B}$ with $x \in B \subseteq U$. By the definition of bases, this was the only possibility. We need only check that this is in fact a topology. The empty-set is vacuously open. X is open because $\mathscr{B}$ covers X. Let $\mathscr{V}$ be a collection of open sets and let $x \in \bigcup_{U \in \mathscr{V}} U$. Then, $x \in U$ for some $U \in \mathscr{V}$, and so there is some $B \in \mathscr{B}$ such that $x \in B \subseteq U \subseteq \bigcup_{U \in \mathscr{V}} U$. Thus, $\bigcup_{U \in \mathscr{V}} U$ is open. Let $U_1, \ldots, U_m$ be open and let $x \in \bigcap_{k=1}^m U_k$. Then, there is some $B_k \in \mathscr{B}$ such that $x \in B_k \subseteq U_k$. By (ii), there is some $B \in \mathscr{B}$ with $x \in B \subseteq B_1 \cap \cdots \cap B_m \subseteq \bigcap_{k=1}^m U_k$, and so $\bigcap_{k=1}^m U_k$ is open. ■

Exercise 3.1.1.4 Let X be a topological space and let $\mathscr{B}$ be a collection of subsets of X. Show that $\mathscr{B}$ is a base for the topology of X iff it has the property that $U \subseteq X$ open is equivalent to U being a union of elements of $\mathscr{B}$.

There is a similar way of defining a topology. Instead of specifying a base for all open sets, you specify a base for all the open sets at a point (see the following definitions) for every point.

Definition 3.1.1.5 — Neighborhood Let X be a topological space and let $S \subseteq N \subseteq X$. Then, N is a ***neighborhood*** of S iff there is some open set $U \subseteq X$ such that $S \subseteq U \subseteq N$. An ***open neighborhood*** of S is just an open set which contains S. A(n open) neighborhood of a point x is a(n open) neighborhood of $\{x\}$.

R The intuition is that neighborhoods have the "wiggle room" that is characteristic of open sets, but are not necessarily open themselves. For example, in $\mathbb{R}$, $D_\varepsilon(x_0)$ is a neighborhood of x_0, but isn't actually open.[a]

[a]Recall that $D_\varepsilon(x_0) := \{x \in \mathbb{R} : |x - x_0| \leq \varepsilon\}$—see Definition 2.2.1.

Definition 3.1.1.6 — Neighborhood base Let X be a topological space, let $x \in X$, and let $\mathscr{B}_x$ be a collection of neighborhoods[a] of x. Then, $\mathscr{B}_x$ is a ***neighborhood base*** of x iff for every neighborhood $N \subseteq X$ of x there is some $B_x \in \mathscr{B}_x$ such that $B_x \subseteq N$. If $\mathscr{B}_x$ is a neighborhood base of x for all $x \in X$, then $\{\mathscr{B}_x : x \in X\}$ is a ***neighborhood base*** of the topology.

[a]Not necessarily open!

▪ **Example 3.1.1.7 — A neighborhood base with no open sets** Take $X := \mathbb{R}$, and for $x \in X$ define $\mathscr{B}_x := \{D_\varepsilon(x) : \varepsilon > 0\}$. This is a neighborhood base but yet no $D_\varepsilon(x)$ is open.

There are two important related facts about neighborhood bases—the first that they can be used to determine exactly which sets are open (Proposition 3.1.1.8) and the second that they can be used to define topologies (Proposition 3.1.1.9).

Proposition 3.1.1.8 Let X be a topological space and for each $x \in X$ let $\mathscr{B}_x$ be a collection of neighborhoods of x. Then, $\{\mathscr{B}_x : x \in X\}$ is a neighborhood base of the topology iff the statement that $U \subseteq X$ is open is equivalent to the statement that for every $x \in U$ there is some $B_x \in \mathscr{B}_x$ such that $B_x \subseteq U$.

Proof. ($\Rightarrow$) Suppose that $\mathscr{B}_x$ is a neighborhood base of x for all $x \in X$. Let $U \subseteq X$. First suppose that U is open. Let $x \in U$. Then, U is a neighborhood of x, and so there is some $B_x \in \mathscr{B}_x$ such that $B_x \subseteq U$. Conversely, suppose that for every $x \in U$ there is some $B_x \in \mathscr{B}_x$ such that $B_x \subseteq U$. As B_x is a neighborhood of x, there is some open $U_x \subseteq B_x$ with $x \in U_x$. Then, $U = \bigcup_{x \in U} U_x$, and so U is open.

($\Leftarrow$) Suppose that the statement that $U \subseteq X$ is open is equivalent to the statement that for every $x \in U$ there is some

$B_x \in \mathscr{B}_x$ such that $B_x \subseteq U$. Let $x \in X$ and let $N \subseteq X$ be a neighborhood of x. Then, there is some open set $U \subseteq N$ with $x \in U$. By the hypothesis, there is them some $B_x \in \mathscr{B}_x$ such that $B_x \subseteq U \subseteq N$. Thus, by definition, $\mathscr{B}_x$ is a neighborhood base at x. ■

Once again, neighborhood bases allow us to *define* topologies.

Proposition 3.1.1.9 Let X be a set and for each $x \in X$ let $\mathscr{B}_x$ be a nonempty collection of subsets of X which contain x. Then, there exists a unique topology for which $\{\mathscr{B}_x : x \in X\}$ is a neighborhood base iff for every $x \in X$ and $B_1, B_2 \in \mathscr{B}_x$, there is a subset $U \subseteq B_1 \cap B_2$ such that (i) $x \in U$ and (ii) for every $y \in U$ there is some $B_y \in \mathscr{B}_y$ such that $B_y \subseteq U$.

Furthermore, if every element of each $\mathscr{B}_x$ is open, then $\bigcup_{x\in X} \mathscr{B}_x$ is a base for this topology.

W Warning: The elements of $\mathscr{B}_x$ *need not be open*—see Example 3.1.1.7. In particular, $\bigcup_{x\in X} \mathscr{B}_x$ will not be a base in general.

R This remark contains an explanation of my first failed attempt to formulate this result. As such, it is not particularly important, and may be skipped (along with the following example, Example 3.1.1.11).

I first mistakenly thought that we could instead get away with the easier property that each $\mathscr{B}_x$ is a filter base.[a]

For convenience of language, let us say that a collection $\{\mathscr{B}_x : x \in X\}$ where each $\mathscr{B}_x$ is a collection of sets that contain x 'creates' a topology iff the statement that U is open in this topology is equivalent to the statement that for every $x \in U$ there is some $B \in \mathscr{B}_x$ such that $B \subseteq U$.[b]

Using this language, I initially thought that $\{\mathscr{B}_x : x \in X\}$ would be a neighborhood base for a unique topology iff each $\mathscr{B}_x$ were a filter base. If you only assume that each $\mathscr{B}_x$ is a filter base, however, then you will be unable to prove that elements of $\mathscr{B}_x$ are neighborhoods of x (despite the fact that it would still "create" a unique topology)—see Example 3.1.1.11.

Instead, you might then try to relax the requirement that elements of $\mathscr{B}_x$ be neighborhoods of x, but if you do this, then you become unable to do the other direction, that is, you can have a topology "created" by $\{\mathscr{B}_x : x \in X\}$ for which not every $\mathscr{B}_x$ is a filter base—see Example 3.1.1.11 again.

R The previous remark notwithstanding, it is still true that if $\{\mathscr{B}_x : x \in X\}$ is a neighborhood base for a unique topology, then for every $x \in X$ and $B_1, B_2 \in \mathscr{B}_x$, there is some $B_3 \in \mathscr{B}_x$ such that $B_3 \subseteq B_1 \cap B_2$—it's just that this is not *equivalent* to being a neighborhood base for a unique topology.

---

[a] $\mathscr{B}_x$ would be a "filter base" iff for every $B_1, B_2 \in \mathscr{B}_x$, there is some $B_3 \in \mathscr{B}_x$ such that $B_3 \subseteq B_1 \cap B_2$—see Definition 3.3.1.

[b] This of course is just the definition of a neighborhood base, except I no longer want to require that elements of $\mathscr{B}_x$ be neighborhoods of x. You should also note that this is ad hoc 'throw away' terminology—besides the related Example 3.1.1.11, we shall never use this term again.

Proof. ($\Rightarrow$) Suppose that there exists a unique topology for which $\mathscr{B}_x$ is a neighborhood base at x. Let $x \in X$ and let $B_1, B_2 \in \mathscr{B}_x$. Then, B_1 and B_2 are neighborhoods of x, and so there are open sets $U_1 \subseteq U_2 \subseteq B_2$ containing x. Then, $U_1 \cap U_2$ contains x and is open. As $\{\mathscr{B}_x : x \in X\}$ is a neighborhood base for the topology, this means that for every $y \in U_1 \cap U_2$, there is some $B_y \in \mathscr{B}_y$ such that $B_y \subseteq U_1 \cap U_2$.

($\Leftarrow$) Suppose that for every $x \in X$ and $B_1, B_2 \in \mathscr{B}_x$, there is some $U \subseteq B_1 \cap B_2$ such that (i) $x \in U$ and (ii) for every $y \in U$ there is some $B_y \in \mathscr{B}_y$ such that $B_y \subseteq U$.

Claim 3.1.1.10 For every $x \in X$ and $B_1, \ldots, B_m \in \mathscr{B}_x$, there is some $U \subseteq B_1 \cap \cdots \cap P_m$ such that (i) $x \in U$ and (ii) for every $y \in U$ there is some $B_y \in \mathscr{B}_y$ such that $B_y \subseteq U$.

Proof. By hypothesis, there is some $U_{12} \subseteq B_1 \cap B_2$ such that (i) $x \in U$ and (ii) for every $y \in U$ there is some $B_y \in \mathscr{B}_y$ such that $B_y \subseteq U_{12}$ As of course $x \in U_{12}$, there is some $B_{12} \in \mathscr{B}_x$ such that $B_{12} \subseteq U_{12}$. Applying the hypothesis again, there is some $U_{123} \subseteq B_{12} \cap B_3$ such that (i) $x \in U_{123}$ and (ii) for every $y \in U_{123}$ there is some $B_y \in \mathscr{B}_y$ such that $B_y \subseteq U_{123}$. We have that $U_{123} \subseteq B_{12} \cap B_3 \subseteq B_1 \cap B_2 \cap B_3$. By repeating this process inductively, we find a subset $U_{1\ldots m} \subseteq B_1 \cap \cdots \cap B_m$ such that (i) $x \in U_{1\ldots m}$ and (ii) for every $y \in U_{1\cdots m}$ there is some $B_y \in \mathscr{B}_y$ such that $B_y \subseteq U_{1\ldots m}$, as desired. ■

We declare $U \subseteq X$ to be open iff for every $x \in U$ there is some $B_x \in \mathscr{B}_x$ with $B \subseteq U$. By definition of neighborhood bases, this was the only possibility. We need to check that this is in fact a topology, and that each element of every $\mathscr{B}_x$ is in fact a neighborhood of x in this topology. The empty-set is vacuously open. X is open because each $\mathscr{B}_x$ is nonempty. Let $\mathscr{V}$ be a collection of open sets and let $x \in \bigcup_{U \in \mathscr{V}} U$. Then, $x \in U$ for some $U \in \mathscr{V}$, and so there is some $B_x \in \mathscr{B}_x$ such that $B \subseteq U \subseteq \bigcup_{U \in \mathscr{V}} U$. Thus, $\bigcup_{U \in \mathscr{V}} U$ is open. Let $U_1, \ldots, U_m$ be open and let $x \in \bigcap_{k=1}^m U_k$. Then, $x \in U_k$ for each k, and so there is some $B_k \in \mathscr{B}_x$ such that $x \in B_k \subseteq U_k$. By the claim, there is a subset $U \subseteq B_1 \cap \cdots \cap B_m$ such that (i) $x \in U$ and (ii) for every $y \in U$ there is some $B_y \in \mathscr{B}_y$ such that $B_y \subseteq U$. As in particular $x \in U$, there is some $B_x \in \mathscr{B}_x$ such that $B_x \subseteq U \subseteq U_1 \cap \cdots \cap U_m$, and so $U_1 \cap \cdots \cap U_m$ is open.

It remains to check that each $B \in \mathscr{B}_x$ is a neighborhood of x. Taking $B_1 := B =: B_2$ in the hypothesis, we see that there is some subset $U \subseteq B$ such that (i) $x \in U$ and (ii) for every $y \in U$ there is some $B_y \in \mathscr{B}_y$ such that $B_y \subseteq U$. The second condition is just the statement that U is open. Thus, as $x \in U \subseteq B$, B is a neighborhood of x.

Now suppose that $\mathscr{B}_x$ consists purely of open sets. Define $\mathscr{B} := \bigcup_{x \in X} \mathscr{B}_x$. We wish to show that $\mathscr{B}$ is a base for the topology. To do that, we must show that $U \subseteq X$ is open iff for all $x \in U$ there is some $B \in \mathscr{B}$ such that $x \in B \subseteq U$. One direction is easy: if U is open and $x \in X$, then in fact there is some $B \in \mathscr{B}_x$ such that $x \in B \subseteq U$.

Conversely, suppose that for all $x \in U$ there is some $B \in \mathscr{B}$ such that $x \in B \subseteq U$. We wish to show that U is open. So, let $x \in U$. We wish to find some $B \in \mathscr{B}_x$ such that $x \in B \subseteq U$. By hypothesis and the definition of $\mathscr{B}$, we know that there is some $B_y \in \mathscr{B}_y$ such that $x \in B_y \subseteq U$ for $y \in X$. However, we're assuming that every element in $\mathscr{B}_y$ is open, and so in fact there is some $B \in \mathscr{B}_x$ such that $x \in B \subseteq B_y \subseteq U$, as desired. ■

▪ **Example 3.1.1.11 — Counter-examples to a naive formulation of Proposition 3.1.1.9** We first give an example of a collection $\{\mathscr{B}_x : x \in X\}$ where each $\mathscr{B}_x$ is a filter base containing x for which the elements of $\mathscr{B}_x$ are not neighborhoods of x in the unique topology "created"[a] by $\{\mathscr{B}_x : x \in X\}$.

Define $X := \{1, 2, 3\}$, $\mathscr{B}_1 := \{\{1, 2\}\}$, $\mathscr{B}_2 := \{\{2, 3\}\}$, and $\mathscr{B}_3 := \{\{3, 1\}\}$. Then, for every $x \in X$, we immediately have the property that for every $B_1, B_2 \in \mathscr{B}_x$, there is some $B_3 \in \mathscr{B}_x$ such that $B_3 \subseteq B_1 \cap B_2$. On the other hand, if $\{\mathscr{B}_x : x \in X\}$ were a neighborhood base for a topology on X, then this topology would have to be the indiscrete topology, in which case $\{1, 2\}$, for example, would not be a neighborhood of 1.

We now check that this does in fact "create" the indiscrete topology. So, let $U \subseteq X$ be nonempty. Then, without loss of generality, we would have $1 \in U$, and hence $\{1,2\} \subseteq U$ as B_1 is (by assumption) a neighborhood base of $1 \in X$. But then $2 \in U$, and hence $\{2,3\} \subseteq U$ as B_2 is (by assumption) a neighborhood base of $2 \in X$. But then $3 \in U$, and so $U = X$.

We now give an example in which $\{\mathscr{B}_x : x \in X\}$ "creates" a unique topology but for which not every $\mathscr{B}_x$ is a filter base (in which case we necessarily must have that elements of $\mathscr{B}_x$ are *not* all neighborhoods of x—otherwise, Proposition 3.1.1.9 implies they would have to be a filter base).

Let all the definitions be the same as before, except redefine

$$\mathscr{B}_1 := \{\{1,2\},\{1,3\}\} \text{ and } \mathscr{B}_3 := \{\{1,2,3\}\}. \quad (3.1.1.12)$$

Again, this "creates" the indiscrete topology, but $\mathscr{B}_1$ is not a filter base because $\{1,2\} \cap \{1,3\} = \{1\}$ does not contain as a subset any element of $\mathscr{B}_1$.

[a] In the sense described in the remark of Proposition 3.1.1.9.

Sometimes we have a collection of sets that we would like to be open, but they do not necessarily form a base, and so in this case we cannot just invoke Proposition 3.1.1.3. However, what we can do is take the 'smallest' topology which contains these sets.

Proposition 3.1.1.13 — Generating collection (of a topology) Let X be a set and let $\mathscr{S} \subseteq 2^X$. Then, there exists a unique topology $\mathscr{U}$ on X, the topology ***generated*** by $\mathscr{S}$, such that

(i). $\mathscr{S} \subseteq \mathscr{U}$; and
(ii). if $\mathscr{U}'$ is any other topology on X containing $\mathscr{S}$, it follows that $\mathscr{U} \subseteq \mathscr{U}'$.

Furthermore, the collection of all finite intersections of elements of $\mathscr{S} \cup \{X\}$ is a base for this topology.[a]. $\mathscr{S}$ is a ***generating collection***.

R Compare this with the definition of the integers, rationals, reals, closure, and interior (Theorems 1.2.1, 1.3.4 and 1.4.2.9 and Propositions 3.2.35 and 3.2.39).

[a]We throw X in in case $\mathscr{S}$ did not cover X (recall that bases (Definition 3.1.1.1) need to cover the space).

Proof. Let $\mathscr{B}$ be the collection of all finite intersections of elements of $\mathscr{S} \cup \{X\}$. This definitely covers X as $X \in \mathscr{B}$. Furthermore, the intersection of any two elements of $\mathscr{B}$ is also an element of $\mathscr{B}$, by definition. Therefore, there is a unique topology $\mathscr{U}$ on X for which $\mathscr{B}$ is a base (Proposition 3.1.1.3).

By construction, $\mathscr{S} \subseteq \mathscr{B} \subseteq \mathscr{U}$. On the other hand, if $\mathscr{U}'$ is any other topology for which every element of $\mathscr{S}$ is open, then, because topologies are closed under finite intersection, $\mathscr{U}'$ must contain $\mathscr{B}$, and hence it must contain $\mathscr{U}$ (because $\mathscr{U}'$ is closed under arbitrary union and every element of $\mathscr{U}$ is a union of elements of $\mathscr{B}$ (Exercise 3.1.1.4)).

The uniqueness proof is one that is hopefully familiar by now: if $\mathscr{V}$ also satisfies both these properties, then on one hand we already know that $\mathscr{U} \subseteq \mathscr{V}$, but on the other hand, (ii) applied to $\mathscr{V}$ gives $\mathscr{V} \subseteq \mathscr{V}$, and hence $\mathscr{U} = \mathscr{V}$. ■

Generating collections are actually quite nice because several things can be checked just by looking at generating collections, which are generally significantly 'smaller'[1] than the entire topology—see Exercises 3.1.3.4 and 3.2.1.4, and the Alexander Subbase Theorem (Theorem 3.2.1.5).

3.1.2 Some basic examples

We can use the notion of a base to define the *order topology*.

[1]Though not literally in the sense of cardinality.

Definition 3.1.2.1 — Order topology Let X be a totally-ordered set, and let $\mathscr{B}$ be the collection of sets of the form $(a, b) := \{x \in X : a < x < b\}$ for $a, b \in X$ with $a < b$. (We also allow $a = -\infty$ and $b = +\infty$, in which case $(a, +\infty) := \{x \in X : x > a\}$ and similarly for $(-\infty, b)$ and $(-\infty, +\infty)$.)

Exercise 3.1.2.2 Use Proposition 3.1.1.3 to show that $\mathscr{B}$ is a base for a topology.

The topology defined by $\mathscr{B}$ is the ***order topology*** on X.

Example 3.1.2.3 — A partially-ordered set whose open intervals do not form a base Define

$$X := \{x, y_1, y_2, y_3, z_1, z_2\}, \tag{3.1.2.4}$$

and declare that

$$x \leq y_k \text{ and } z_1 \geq y_1, y_2 \text{ and } z_2 \geq y_2, y_3 \tag{3.1.2.5}$$

for all k.[a] Then,

$$(x, z_1) = \{y_1, y_2\} \text{ and } (x, z_2) = \{y_2, y_3\}. \tag{3.1.2.6}$$

Thus, if the open[b] intervals formed a base for a topology, then

$$(x, z_1) \cap (x, z_2) = \{y_2\} \tag{3.1.2.7}$$

would have to be open. As this is a singleton, the only way this could happen is if $\{y_2\}$ is an open interval itself. However,, $\{y_2\}$ is not an interval because if we had $\{y_2\} = (a, b)$ for $a < y_2$ and $b > y_2$, we would necessarily have $a = x$ or $a = -\infty$, and $b = z_1, z_2, +\infty$. In all of these 6 cases, (a, b) must contain at least either y_1 or y_3 as well, and so in particular, it would not be the case that $\{y_2\} = (a, b)$.

R This is why we only define the order topology for *totally*-ordered sets.

[a] Of course, we also have all additional relations needed to make this reflexive and transitive.

[b] Warning: Here, "open" is referring to the strict inequalities in the definition of (a, b), and *not* whether or not it is open in the relevant topology.

▪ **Example 3.1.2.8 — $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$** $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are all totally-ordered and so we may (and do) equip them with the order topology.

The order topology on $\mathbb{N}$ and $\mathbb{Z}$ are examples of a very special topology, the 'finest' one possible, namely the *discrete topology*.

Definition 3.1.2.9 — Discrete topology Let X be any set. The ***discrete topology*** is the topology in which *every* subset is open.

R Note that every set is also *closed*.

R In other words, the topology is just the power-set of X. This is a topology for tautological reasons.

R Using the "wiggle room" intuition again, in the discrete topology, every singleton $\{x\}$ is open, and so you might interpret this as saying that there is no "wiggle room" around x at all. Certainly, I think it's fair to say that you should imagine points of a discrete space as being, well, discrete.

Proposition 3.1.2.10 The order topologies on $\mathbb{N}$ and $\mathbb{Z}$ are both discrete.

Proof. We prove that the order topology on $\mathbb{Z}$ is discrete and leave the case of $\mathbb{N}$ as an exercise (a very similar argument will work). We wish to show that every subset of $\mathbb{Z}$ is open. To show this, it suffices to show that each singleton set is open because an arbitrary set is going to be a union of singletons.

So, let $m \in \mathbb{Z}$. To show that $\{m\}$ is open, it suffices to find $a, b \in \mathbb{Z}$ such that $(a, b) = \{m\}$ (because every element in the base of a topology is automatically open). Take $a := m - 1$ and $b := m + 1$. Recall (Exercise 1.2.25) that there is no integer between 0 and 1. It follows that there is no integer between k and $k + 1$ for all $k \in \mathbb{Z}$, and so indeed, $(m - 1, m + 1) = \{m\}$.

Exercise 3.1.2.11 Show that the order topology on $\mathbb{N}$ is discrete.

■

Exercise 3.1.2.12 Why is the topology on $\mathbb{Q}$ not discrete?

The discrete topology is the 'finest' topology you can have (finer means more open sets). The 'coarsest' topology you can have is the *indiscrete topology*.

Definition 3.1.2.13 — Indiscrete topology Let X be any set. The ***indiscrete topology*** is just $\{\emptyset, X\}$.

R The definition of a topology requires that at least the empty-set and the entire set are open. The indiscrete topology is when nothing else is open. It is not particularly useful, but it can be an easy source of some counter-examples (e.g. Example 3.2.7).

3.1.3 Continuity

We now finally vastly generalize our definition of continuity.

Definition 3.1.3.1 — Continuous function Let $f : X \to Y$ be a function between topological spaces and let $x \in X$. Then, f is ***continuous*** at x iff the preimage of every neighborhood of $f(x)$ is a neighborhood of x. f is ***continuous*** iff f is continuous at x for all $x \in X$.

W Warning: Here, you *cannot* replace "neighborhood" with "open neighborhood" For example, consider the function $f\colon \mathbb{R}\to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}. \tag{3.1.3.2}$$

Then, this *should* be continuous at $x = 1$, but yet it is *not* the case the preimage of every open neighborhood of $f(1) = 1$ is open is an open neighborhood of 1: $f^{-1}(B_{\frac{1}{2}}(1)) = [0,\infty)$. While this is a neighborhood of $x = 1$, it is not an *open* neighborhood of $x = 1$.

Exercise 3.1.3.3 Show that $f\colon X\to Y$ is continuous iff the preimage of every open set is open.

In fact, we can do a bit better than this.

Exercise 3.1.3.4 Let X and Y be topological spaces and let $\mathscr{S}$ be generate the topology of Y. Show that f is continuous iff $f^{-1}(S)$ is open for all $S\in\mathscr{S}$.

R In particular, this is true for $\mathscr{S}$ a *base* for the topology of Y

Exercise 3.1.3.5 Let $f\colon X\to Y$ be any function between two topological spaces.

(i). Show that, if X has the discrete topology, then f is continuous.
(ii). Show that, if Y has the indiscrete topology, then f is continuous.

R In other words, every function *on* a discrete space is continuous. Similarly, every function *into* an indiscrete space is continuous.

Example 3.1.3.6 — The category of topological spaces The category of topological spaces is the category **Top**

(i). whose collection of objects $\mathrm{Obj}(\mathbf{Top})$ is the collection of all topological spaces;
(ii). with morphism set $\mathrm{Mor}_{\mathbf{Top}}(X,Y)$ precisely the set of all continuous functions from X to Y;
(iii). whose composition is given by ordinary function composition; and
(iv). whose identities are given by the identity functions.

Exercise 3.1.3.7 Show that the composition of two continuous functions is continuous.

R Note that this is something you need to check in order for **Top** to actually form a category. You also need to verify that the identity function is continuous, but this is trivial (the preimage of a set is itself, so...).

Definition 3.1.3.8 — Homeomorphism Let $f\colon X\to Y$ be a function between topological spaces. Then, f is a ***homeomorphism*** iff f is an isomorphism in **Top**.

Exercise 3.1.3.9 Show that a function is a homeomorphism iff (i) it is bijective, (ii) it is continuous, and (iii) its inverse is continuous.

Exercise 3.1.3.10 Find an example of a function that is (i) bijective, (ii) continuous, but (iii) is not a homeomorphism.

R In other words, find a bijective continuous function whose inverse is not continuous.

R Contrast this with, for example, isomorphisms in **Ring**. It follows immediately from the definition that a function between groups is an isomorphism in **Ring** iff (i) it is bijective, (ii) it is a homomorphism, and (iii) its inverse is a homomorphism. However, by Exercise B.2.7, if the original function is a homomorphism, then we get that its inverse is a homomorphism for free, so we only need to actually check (i) and (ii). The point is, with rings (and other algebraic objects), the isomorphisms are just bijective homomorphisms, but this is *not* the case in **Top**.

Exercise 3.1.3.11 — Embedding Let $f\colon X \to Y$ be a function between topological spaces. Show that f is an embedding in **Top**[a] iff f is a homeomorphism onto its image.

[a] See Definition B.2.16.

■ **Example 3.1.3.12 — An injective continuous function that is not an embedding** Let X be any set. Then, $\mathrm{id}_X\colon \langle X, \text{discrete}\rangle \to \langle X, \text{indiscrete}\rangle$ is an injective (in fact, bijective) function that is continuous, but will not be an embedding as long as X has at least two points.

R While a trivial example, it has significance in that it tells us that the naive definition of an embedding in a concrete category as "injective morphism" is the *wrong* definition of embedding—see Definition B.2.16.

Exercise 3.1.3.13 Let X be a set, and let $\mathscr{U}$ and $\mathscr{V}$ be two topologies on X. Then, if $\langle X, \mathscr{U}\rangle$ is homeomorphic to $\langle X, \mathscr{V}\rangle$, must we have that $\mathscr{U} = \mathscr{V}$?

There is a useful result that is applicable in general whenever you want to check that a "piece-wise" function is continuous.

Proposition 3.1.3.14 — Pasting Lemma Let X and Y be topological spaces, let $C_1, C_2 \subseteq X$ be closed, and let f_1 :

$C_1 \to Y$ and $f_2 : C_2 \to Y$ be continuous. Then, if $f_1|_{C_1 \cap C_2} = f_2|_{C_1 \cap C_2}$, then the function $f : C_1 \cup C_2 \to Y$ defined by

$$f(x) := \begin{cases} f_1(x) & \text{if } x \in C_1 \\ f_2(x) & \text{if } x \in C_2 \end{cases} \tag{3.1.3.15}$$

is well-defined and continuous.

Proof. We saw before in the reals (Exercise 2.5.2.8) that a function is continuous iff the preimage of every closed set is closed. This is still true (Proposition 3.2.21) with essentially the same proof as before. We use this to prove the result.

First of all, we needed to assume that $f_1|_{C_1 \cap C_2} = f_2|_{C_1 \cap C_2}$ in order for the definition of f to actually be a function. We now check that f is indeed continuous.

Let $D \subseteq Y$ be closed. Define $D_1 := f^{-1}(D) \cap C_1$ and $D_2 := f^{-1}(D) \cap C_2$, so that $f^{-1}(D) = D_1 \cup D_2$. It thus suffices to show that each D_k is closed. By $1 \leftrightarrow 2$ symmetry, it suffice to show that just D_1 is closed. However, this is the case because

$$D_1 := f^{-1}(D) \cap C_1 = f_1^{-1}(D) \cap C_1 \tag{3.1.3.16}$$

and f_1 is continuous. ∎

3.1.4 Some motivation

At this point, it is reasonable for one to ask "Why do we care about such generality in an introductory *real* analysis course? Shouldn't we only be concerned with the real numbers for now?". I would argue that, even in the case where you really are only interested in the real numbers, abstraction can shed light onto such a specific example. The real numbers are many things: a field, a totally-ordered set, a totally-ordered field, a metric space, a uniform space, a topological space, a manifold, etc.. In mathematics, we are not just concerned about *what* is true, but *why* things are true. Because the real numbers are so special, having so much structure, there are many things true about them. But by the same token, because there is so much structure, unless you go through the proofs yourself in detail, it can be difficult to recall *why* things are true—does property XYZ hold for the real numbers because they are a totally-ordered field or because they are a metric space or because they are a topological vector space or because. . . ? Instead, however, if we step back and only study $\mathbb{R}$ as a topological space, it becomes clearer why certain things are true. This allows us to tell easily what is true about the real numbers because they are a topological space, as opposed to what is true about the real numbers because they are a field, etc..

For example, consider the following.

■ **Example 3.1.4.1** Though we have not technically defined it yet, hopefully you are familiar with the function arctan from calculus (see Definition 6.4.5.58). $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is a homeomorphism (see Exercise 6.4.5.60).

Exercise 3.1.4.2 Find a homeomorphism from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $(0, 1)$.

Thus, from the point of view of general topology, there is no distinction between $\mathbb{R}$ and $(0, 1)$. This is completely analogous to the sense in which there is no difference between $\mathbb{R}$ and $2^{\mathbb{N}}$ at the level of sets. ($\mathbb{R}$ and $(0, 1)$ are isomorphic in **Top**, and $\mathbb{R}$ and $2^{\mathbb{N}}$ are isomorphic in **Set**.) Recall from the very end of Chapter 1 What is a number?: morphisms matter.

Perhaps a good example of the more general context making it clearer *why* things are true is the Intermediate Value Theorem. Here is the 'classical' statement you are probably familiar with from calculus.[2]

$$\text{Let } f\colon [a, b] \to \mathbb{R} \text{ be continuous. Then, for all } y \text{ between } f(a) \text{ and } f(b) \text{ (inclusive) there is some } x \in [a, b] \text{ such that } f(x) = y. \tag{3.1.4.3}$$

[2]See Corollary 3.8.1.31 for the proof.

Compare that with the more general statement.[3]

$$\text{The continuous image of a connected set is connected.} \tag{3.1.4.4}$$

I think it's fair to say that the latter is much more elegant, and perhaps even easier to understand.[4]

3.2 A review

Quite a many things that we did with the real numbers in the last chapter carry over to general topological spaces no problem. For convenience, we present here all definitions and theorems which carry over nearly verbatim to topological spaces. We will try to point out when things do *not* carry over identically (for example, limits no longer need be unique (Example 3.2.7)).

One important thing to keep in mind is: *Open neighborhoods of a point in a general topological space play the same role that ε-balls did in $\mathbb{R}$.* Not only do we use this to determine appropriate generalizations, but if you are feeling a bit uncomfortable with topological spaces, this should help your intuition.

For the entirety of this section, X and Y will be general topological spaces. We recommend that this section be used mainly as a reference—the pedagogy of the concepts is contained in the last chapter.

Definition 3.2.1 — Net A ***net*** in X is a function from a nonempty directed set $\langle \Lambda, \leq \rangle$ into X.

Definition 3.2.2 — Sequence A ***sequence*** is a net whose directed set is order-isomorphic (i.e. isomorphic in **Pre**) to $\langle \mathbb{N}, \leq \rangle$.

[3] See Theorem 3.8.1.28 for the proof.

[4] Admittedly, the proof to go from the general statement to the 'classical' statement is not completely trivial (a subset of $\mathbb{R}$ is connected iff it is an interval—see Theorem 3.8.1.20), but the two results are still 'morally' the same.

Meta-definition 3.2.3 — Eventually XYZ Let $\Lambda \ni \lambda \mapsto x_\lambda \in X$ be a net. Then, $\lambda \mapsto a_\lambda$ is ***eventually XYZ*** iff there is some λ_0 such that $\{\lambda \in \Lambda : \lambda \geq \lambda_0\} \ni \lambda \mapsto x_\lambda$ is XYZ.

Meta-definition 3.2.4 — Frequently XYZ Let $\Lambda \ni \lambda \mapsto x_\lambda \in X$ be a net. Then, $\lambda \mapsto x_\lambda$ is ***frequently*** iff for every $\lambda \in \Lambda$ there is some $\lambda' \geq \lambda$ such that $x_{\lambda'}$ is XYZ.

Meta-proposition 3.2.5 Let XYZ be a property that is such that a net $\lambda \mapsto x_\lambda$ is XYZ iff each x_λ is XYZ. Then, a net $\lambda \mapsto x_\lambda \in \mathbb{R}$ be a net is frequently not XYZ iff it is not eventually XYZ.

R In particular, for $S \subseteq X$, a net is frequently contained in S iff it is not eventually contained in S^{C}.

Definition 3.2.6 — Limit (of a net) Let $\lambda \mapsto x_\lambda$ be a net and let $x_\infty \in X$. Then, x_∞ is a ***limit*** of $\lambda \mapsto x_\lambda$ iff for every open neighborhood U of x_∞, $\lambda \mapsto x_\lambda$ is eventually contained in U. If a net has a limit, then we say that it ***converges***.

R Note that we no longer have the concept of *diverge*, as this required to at least have a notion of "unbounded", a concept which doesn't make sense for general topological spaces.

W Warning: Note that limits no longer need be unique—see the following example. In particular, you have to be careful about using the notation $\lim_\lambda x_\lambda$—if there is more than one limit, to which element should the notation $\lim_\lambda x_\lambda$ be referring to? That said, in the case limits are unique (e.g. in T_2 spaces—see Proposition 3.6.2.20), we will use the notation $\lim_\lambda x_\lambda$ to denote *the* limit of $\lambda \mapsto x_\lambda$.

■ **Example 3.2.7 — Limits need not be unique** Define $X := \{0, 1\}$ and equip X with the indiscrete topology (Definition 3.1.2.13). Then, the constant net $\lambda \mapsto x_\lambda := 0$ converges to both 0 and 1 (the only open neighborhood of both of these points is X itself, and of course the net is eventually contained in X).

Proposition 3.2.8 Let $\lambda \mapsto x_\lambda$ be a net and let $x_\infty \in X$. Then, it is *not* the case that $\lambda \mapsto x_\lambda$ converges to x_∞ iff there is an open neighborhood U of x such that $\lambda \mapsto x_\lambda$ is frequently contained in U^{C}.

Definition 3.2.9 — Subnet Let $x : \Lambda \to X$ be a be a net. Then, a ***subnet*** of x is a net $y : \Lambda' \to X$ such that

(i). for all $\mu \in \Lambda'$, $y_\mu = x_{\lambda_\mu}$ for some $\lambda_\mu \in \Lambda$; and
(ii). whenever $U \subseteq X$ eventually contains x, it eventually contains y.

A ***cofinal subnet*** of x is a net of the form $x|_{\Lambda'}$ for $\Lambda' \subseteq \Lambda$ cofinal. A ***subsequence*** is a subnet that is a sequence.

Proposition 3.2.10 Let $\lambda \mapsto x_\lambda \in X$ be a net. Then, $\mu \mapsto x_{\lambda_\mu}$ is a subnet of $\lambda \mapsto x_{\lambda_\mu}$ iff for all λ_0 there is some μ_0 such that

$$\left\{x_{\lambda_\mu} : \mu \geq \mu_0\right\} \subseteq \{x_\lambda : \lambda \geq \lambda_0\}\,. \tag{3.2.11}$$

Proposition 3.2.12 Let $x\colon \Lambda \to X$ and $y\colon \Lambda' \to X$ be nets. Then, if there is a function $\iota\colon \Lambda' \to \Lambda$ such that (i) $y = x \circ \iota$ and (ii) for all $\lambda \in \Lambda$ there is some $\mu_0 \in \Lambda'$ such that, whenever $\mu \geq \mu_0$, it follows that $\iota(\mu) \geq \lambda$, then y is a subnet of x.

Proposition 3.2.13 Let $x\colon \Lambda \to X$ and $y\colon \Lambda' \to X$ be nets. Then, if there is a function $\iota\colon \Lambda' \to \Lambda$ such that (i) $y = x \circ \iota$,

(ii) is nondecreasing, and (iii) has cofinal image, then y is a subnet of a.

Theorem 3.2.14 — Kelley's Convergence Axioms.

(i). Constant nets converge to the constant.

(ii). Let $\mu \mapsto x_{\lambda_\mu}$ be a subnet of a net $\lambda \mapsto x_\lambda$. Then, if $\lim_\lambda x_\lambda = a_\infty$, then $\lim_\mu x_{\lambda_\mu} = a_\infty$.

(iii). Let $\lambda \mapsto x_\lambda$ be a net. Then, if every cofinal subnet $\mu \mapsto x_{\lambda_\mu}$ has in turn a subnet itself $\nu \mapsto x_{\lambda_{\mu_\nu}}$ such that $\lim_\nu x_{\lambda_{\mu_\nu}} = x_\infty$, then $\lim_\lambda x_\lambda = x_\infty$.

(iv). Let I be a directed set and for each $i \in I$ let $x^i : \Lambda^i \to X$ be a convergent net. Then, if $(x^\infty)_\infty := \lim_i \lim_\lambda (x^i)_\lambda$ exists, then $I \times \prod_{i \in I} \Lambda^i \ni \langle i, \lambda \rangle \mapsto (x^i)_{\lambda^i}$ converges to $(x^\infty)_\infty$.

R We will see later (Theorem 3.4.2.1) that these three properties can be used to define a topology (hence, the reason we refer to them as *axioms*).

R As far as Kelley's Convergence Theorem (Theorem 3.4.2.1) is concerned, we can[a] replace (ii) and (iii) here with just the single axiom

$$\lim_\lambda x_\lambda = x_\infty \text{ iff every subnet } \mu \mapsto x_{\lambda_\mu} \text{ has in turn a subnet itself } \nu \mapsto x_{\lambda_{\mu_\nu}} \text{ such that } \lim_\nu x_{\lambda_{\mu_\nu}} = x_\infty. \tag{3.2.15}$$

This is a bit cleaner, I suppose, if only because it cuts four axioms down to three. That said, notice that the term "cofinal" has been dropped here. This is because (iii) without "cofinal" will imply (ii) (so that, without the condition "cofinal" appearing in (iii), we needn't separately list (ii)), but this should no longer be the case if you include the condition "cofinal". As we will want to make use of the version with the "cofinal" appearing (and not just for the proof of Kelley's Convergence Theorem), we list the four-axiom version of Kelley's Convergence Axioms.

[a]The proof of this fact is given in Proposition 3.4.2.2.

Definition 3.2.16 — Limit point Let $S \subseteq X$ and let $x_0 \in X$. Then, x_0 is a ***limit point*** of S iff there exists a net $\lambda \mapsto x_\lambda \in S$ with $x_\lambda \neq x_0$ that converges to x_0.

R Note how in $\mathbb{R}^d$ (Definition 2.5.1.1) we wrote "such that $\lim_\lambda x_\lambda = x$" here. We can no longer do this as limits might not be unique—see the remark in Definition 3.2.6. Similar tweaks apply to statements throughout this section.

Definition 3.2.17 — Limit (of a function) Let $f : X \to Y$ be a function between topological spaces, and let $x_0 \in X$ and $y \in Y$. Then, y is a ***limit*** of f at x_0 iff for every net $\lambda \mapsto x_\lambda$ with $x_\lambda \neq x_0$ that converges to x_0, $\lambda \mapsto f(x_\lambda)$ converges to y.

R Similarly as for limits of nets, we don't have uniqueness in general, in which case the notion $\lim_{x\to a} f(x)$ is ambiguous, though we will still make use of this notation when limits are unique.

W Warning: While in the real numbers this might be true if you replace the word "net" with the word "sequence", however, in general topological spaces, this fails—see Example 3.2.50.

Definition 3.2.18 — Limit superior and limit inferior (of a function) Let $D \subseteq \mathbb{R}^d$, let $x_0 \in \mathbb{R}^d$ be a limit point of D, and let $f : D \to \mathbb{R}^e$. Then, the ***limit superior*** and ***limit inferior*** of f at x_0, $\limsup_{x\to x_0} f(x)$ and $\liminf_{x\to x_0} f(x)$ respectively, are defined by

$$\limsup_{x\to x_0} f(x) := \lim_{U\in\mathscr{U}_{x_0}} \sup\{f(x) : x \in U\} \tag{3.2.19a}$$

$$\liminf_{x\to x_0} f(x) := \lim_{U\in\mathscr{U}_{x_0}} \inf\{f(x) : x \in U\}, \tag{3.2.19b}$$

where $\mathscr{U}_{x_0}$ is the collection of open sets containing x_0 regarded as a directed set with reverse inclusion.

Proposition 3.2.20 Let $f\colon X \to Y$ be a function and let $x_0 \in X$. Then, f is continuous at x_0 iff $f(x_0)$ is a limit of f at x_0. f is continuous iff it is continuous at x_0 for all $x_0 \in X$.

> **R** If limits were unique, we could write this more transparently as $\lim_{x\to x_0} f(x) = f(x_0)$.

Proposition 3.2.21 Let $f\colon X \to Y$ be a function. Then, f is continuous iff the preimage of every closed set is closed.

Definition 3.2.22 — Accumulation point Let $S \subseteq X$ and let $x \in X$. Then, x is an ***accumulation point*** of S iff every open neighborhood of x intersects S at a point distinct from x.

Definition 3.2.23 — Adherent point Let $S \subseteq X$ and let $x_0 \in X$. Then, x_0 is an ***adherent point*** of X iff every open neighborhood of x intersects S.

Proposition 3.2.24 Let $S \subseteq X$ and let $x_0 \in X$. Then, x_0 is an accumulation point of S iff it is a limit point of S.

> **R** If you replace "net" with "sequence" in the definition of a limit point, then this result will be *false* in general—see Example 3.2.45

Proposition 3.2.25 Let $S \subseteq X$ and let $x_0 \in X$. Then, x_0 is an adherent point of S iff there is a net $\lambda \mapsto x_\lambda \in S$ that converges to x_0.

Definition 3.2.26 — Isolated point Let $S \subseteq X$ and let $x \in X$. Then, x is an ***isolated point*** iff there is an open neighborhood U of x such that $U \cap S = \{x\}$.

Proposition 3.2.27 Let $S \subseteq X$ and let $x_0 \in X$. Then, if x_0 is an adherent point of S, then either x_0 is an accumulation point of S or x_0 is an isolated point of S.

R Note that this or is *exclusive*.

Theorem 3.2.28. Let $C \subseteq X$. Then, the following are equivalent.

(i). C is closed.
(ii). C contains all its accumulation points.
(iii). C contains all its limit points.
(iv). C contains all its adherent points.
(v). C contains all points $x_0 \in X$ for which there is a net $\lambda \mapsto x_\lambda \in C$ converging to x_0.
(vi). C is equal to its set of adherent points.
(vii). C is equal to the set of all points $x_0 \in X$ for which there is a net $\lambda \mapsto x_\lambda \in C$ converging to x_0.

Definition 3.2.29 — Perfect set Let $C \subseteq X$. Then, C is ***perfect*** iff it is equal to its set of accumulation points.

There are several equivalent ways to state this condition.

Proposition 3.2.30 Let $C \subseteq X$. Then, the following are equivalent.

(i). C is perfect.
(ii). C is closed and every element of C is an accumulation point of C.
(iii). C is closed and has no isolated points.

We proved that (Proposition 2.5.2.23) in $\mathbb{R}$ that $x \in \mathbb{R}$ is an accumulation point of a sequence iff there was a subsequence that converged to x. We also gave an example (Example 2.5.2.24) of how this fails (even in $\mathbb{R}$) for general nets. Not only this, but this also fails to hold (for sequences) in general topological spaces.

▪ **Example 3.2.31 — An accumulation point of a sequence to which no subsequence converges** Define $X := \{x_1, x_2, x_3\}$ and

$$\mathscr{U} := \{\emptyset, X, \{x_1, x_2\}\} . \tag{3.2.32}$$

Consider the sequence $m \mapsto a_m$ defined by

$$a_m := \begin{cases} x_2 & \text{if } m = 0 \\ x_3 & \text{otherwise.} \end{cases} \tag{3.2.33}$$

Then, it is true that every open neighborhood of $x_1 \in X$ contains an element of the set $\{a_m : m \in \mathbb{N}\}$, namely $a_0 := x_2$, distinct from x_1. On the other hand, no subsequence of $m \mapsto a_m$ converges to x_1 as it is eventually outside an open neighborhood of x_1 (namely the open neighborhood $\{x_1, x_2\}$).

Definition 3.2.34 — Interior point Let $S \subseteq X$ and let $x_0 \in X$. Then, x_0 is an ***interior point*** of S iff there is some open neighborhood U of x_0 such that $U \subseteq S$.

Proposition 3.2.35 — Closure Let $S \subseteq X$. Then, there exists a unique set $\mathrm{Cls}(S) \subseteq X$, the ***closure*** of S, that satisfies

(i). $\mathrm{Cls}(S)$ is closed;
(ii). $S \subseteq \mathrm{Cls}(S)$; and
(iii). if C is any other closed set which contains S, then $\mathrm{Cls}(S) \subseteq C$.

Furthermore, explicitly, we have

$$\mathrm{Cls}(S) = \bigcap_{\substack{C \subseteq X \text{ closed} \\ S \subseteq C}} C . \tag{3.2.36}$$

Proposition 3.2.37 Let $S \subseteq X$. Then, $\mathrm{Cls}(S)$ is the union of S and its set of accumulation points.

Theorem 3.2.38 — Kuratowski Closure Axioms. Let $S, T \subseteq X$. Then,

(i). $\mathrm{Cls}(\emptyset) = \emptyset$;
(ii). $S \subseteq \mathrm{Cls}(S)$;
(iii). $\mathrm{Cls}(S) = \mathrm{Cls}\left(\mathrm{Cls}(S)\right)$; and
(iv). $\mathrm{Cls}(S \cup T) = \mathrm{Cls}(S) \cup \mathrm{Cls}(T)$.

Proposition 3.2.39 — Interior Let $S \subseteq X$. Then, there exists a unique set $\mathrm{Int}(S) \subseteq \mathbb{R}$, the ***interior*** of S, that satisfies

(i). $\mathrm{Int}(S)$ is open;
(ii). $\mathrm{Int}(S) \subseteq S$; and
(iii). if U is any other open set which is contained in S, then $U \subseteq \mathrm{Int}(U)$.

Furthermore, explicitly, we have

$$\mathrm{Int}(S) = \bigcup_{\substack{U \subseteq X \text{ open} \\ U \subseteq S}} U. \tag{3.2.40}$$

Proposition 3.2.41 Let $S \subseteq X$. Then, $\mathrm{Int}(S)$ is the set of interior points of S.

Theorem 3.2.42 — Kuratowski Interior Axioms. Let $S, T \subseteq X$. Then,

(i). $\mathrm{Int}(X) = X$;
(ii). $\mathrm{Int}(S) \subseteq S$;
(iii). $\mathrm{Int}(S) = \mathrm{Int}\left(\mathrm{Int}(S)\right)$; and
(iv). $\mathrm{Int}(S \cap T) = \mathrm{Int}(S) \cap \mathrm{Int}(T)$.

Proposition 3.2.43 Let $S \subseteq X$. Then, S is closed iff $S = \mathrm{Cls}(S)$.

Proposition 3.2.44 Let $S \subseteq X$. Then, S is open iff $S = \operatorname{Int}(S)$.

We postponed the following counter-examples because we technically had not introduced the notion of closure in a general topological space.

▪ **Example 3.2.45 — A limit point that is not a sequential limit point** Define $X := \mathbb{R}$. We equip $\mathbb{R}$ with a nonstandard topology, the so-called ***cocountable topology***.[a] Let $C \subseteq X$ and declare that

$$C \text{ is closed iff either (i) } C = X \text{ or (ii) } C \text{ is countable.} \tag{3.2.46}$$

Because the finite union of countable sets is countable and an arbitrary intersection of countable sets is countable (obviously), it follows that this defines a topology on X.

We first show that every convergent sequence is eventually constant. So, let $m \mapsto x_m \in X$ be sequence that converges to $x_\infty \in X$. We proceed by contradiction: suppose that it is not eventually constant. Then, the set

$$\{m \in \mathbb{N} : x_m \neq x_\infty\} \tag{3.2.47}$$

is cofinal, and hence defines a subsequence $n \mapsto x_{m_n}$ that (i) converges to x_∞ but (ii) is never equal to x_∞. Hence, the set $C := \{x_{m_n} : n \in \mathbb{N}\}$ does not contain x_∞, and so C^{C} is an open neighborhood of x_∞. But of course $n \mapsto x_{m_n}$ is not eventually contained in C^{C}—no term of this subsequence is contained in C^{C}. Therefore, $n \mapsto x_{m_n}$ cannot converge to x_∞: a contradiction. Therefore, $m \mapsto x_m$ must be eventually constant.

Define $U := \mathbb{Q}^{\mathrm{C}}$. As $U^{\mathrm{C}} = \mathbb{Q}$ is countable, U is open. We note that the closure of U is all of $\mathbb{R}$: no countable set can contain U, and so the only closed set which contains U is $\mathbb{R}$ itself. It thus follows that,[b] in particular, 0 is a limit point of U. On the other hand, as every convergent sequence is eventually constant, no sequence in $U := \mathbb{Q}^{\mathrm{C}}$ can converge to 0.

[a] ***Cocountable*** means that the complement is countable. The name of the topology here derives from the fact that the *open* sets have countable complement (except the empty-set of course).

[b] Because the closure of a set is that set union its accumulation points (Proposition 3.2.37), and accumulation points are the same as limit points (Proposition 3.2.24).

Example 3.2.48 — Two distinct topologies with the same notion of sequential convergence Let X be as in Example 3.2.45 and denote the topology on X given there (the cocountable topology) by $\mathscr{U}$. Denote by $\mathscr{D}$ the discrete topology on X. We saw in Example 3.2.45 that sequences converge iff they are eventually constant. However, we also have the following result.

> **Exercise 3.2.49** Let X be a discrete space and let $\lambda \mapsto x_\lambda \in X$ be net. Show that $\lambda \mapsto x_\lambda$ iff it is eventually constant.

Thus, a given sequence $m \mapsto x_m \in X$ converges with respect to $\mathscr{U}$ iff it converges with respect to $\mathscr{D}$, and in this case, they converge to the same limit. On the other hand, the set $\{0\}$ is *not* open with respect to $\mathscr{U}$[a] but it is open with respect to $\mathscr{D}$.

Even though the topologies are distinct, perhaps it is the case that they are *homeomorphic*? We show that this cannot happen. In fact, we show that no bijective function from $\phi : \langle X, \mathscr{U} \rangle \to \langle X, \mathscr{D} \rangle$ is continuous. If ϕ were such a function, then $\phi^{-1}(\mathbb{Q}^\mathsf{C})$ would be uncountable and proper, and hence would not be closed, a contradiction of the fact that ϕ is continuous and $\mathbb{Q}^\mathsf{C}$ is closed with respect to $\mathscr{D}$.

[a] This uses the fact that the real numbers are uncountable!

Example 3.2.50 — A sequentially-continuous function that is not continuous Let X be as in Example 3.2.45 ($\mathbb{R}$

with the cocountable topology) and define $f\colon X \to X$ by

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{Q}^{\mathsf{C}}. \end{cases} \tag{3.2.51}$$

(Note that this is just the Dirichlet Function (Example 2.5.1.13)!)

This function is certainly not continuous because, for example, the preimage of $\{-1\}$ is not closed. On the other hand, because every convergent sequence is eventually constant and the condition that $x_\lambda \neq x$ in the definition of a limit (Definition 3.2.17), it follows that f vacuously satisfies the continuity condition for sequences.

Definition 3.2.52 — Cover Let $S \subseteq X$ and let $\mathscr{U} \subseteq 2^X$. Then, $\mathscr{U}$ is a ***cover*** of S iff $S \subseteq \bigcup_{U \in \mathscr{U}} U$. $\mathscr{U}$ is an ***open cover*** iff every $U \in \mathscr{U}$ is open. A ***subcover*** of $\mathscr{U}$ is a subset $\mathscr{V} \subseteq \mathscr{U}$ that is still a cover of S.

Definition 3.2.53 — Quasicompact Let $S \subseteq X$. Then, S is ***quasicompact*** iff every open cover of S has a finite subcover.

Exercise 3.2.54 Show that finite spaces are quasicompact.

Exercise 3.2.55 Show that closed subsets of quasicompact spaces are quasicompact.

Definition 3.2.56 — Finite-intersection property Let $S \subseteq X$ and let $\mathscr{C} \subseteq 2^X$ be a collection of subsets of X. Then, $\mathscr{C}$ has the ***finite-intersection property*** with S iff every finite subset $\{C_1, \ldots, C_m\} \subseteq \mathscr{C}$ intersects S: $(C_1 \cap \cdots \cap C_m) \cap S \neq \emptyset$. For $S = X$, we simply say that $\mathscr{C}$ has the ***finite-intersection property***

Proposition 3.2.57 Let $K \subseteq X$ and let $\mathscr{C}$ be a collection of closed subsets of X. Then, K is quasicompact iff whenever $\mathscr{C}$ has the finite-intersection property with K, the entire intersection $\bigcap_{C \in \mathscr{C}} C$ also intersects K.

Proposition 3.2.58 Let $K \subseteq X$. Then, K is quasicompact iff every net $\lambda \mapsto a_\lambda \in K$ has a subnet that converges to a limit in K.

Note that the Heine-Borel and Bolzano-Weierstrass Theorems (Theorem 2.5.3.3 and Corollary 2.5.3.14) do not hold in general, but we will wait until after having studied integration before writing down counter-examples—see Examples 5.3.26 and 5.3.31.

3.2.1 A couple new things

We present in this subsection a couple of facts that, while not technically review per se, make more sense to place here before we begin study of new topological concepts in earnest.

Definition 3.2.1.1 — Dense Let X be a topological space and let $S \subseteq X$. Then, S is ***dense*** in X iff $\mathrm{Cls}(S) = X$.

R In other words, every element of X is either an accumulation point of S or an element of S (or both). Recall (the remark in Definition 2.5.2.9) our intuition for accumulation points are points which are "infinitely close" to the set. Thus, intuitively, the statement "S is dense in X." can be understood as saying that every point of X is "infinitely close" to S.

Exercise 3.2.1.2 Show that $\mathbb{Q}$ and $\mathbb{Q}^{\mathrm{C}}$ are both dense in $\mathbb{R}$.

R We mentioned way back when we discussed 'density' of $\mathbb{Q}$ and $\mathbb{Q}^{\mathrm{C}}$ in $\mathbb{R}$ (Theorems 2.3.18 and 2.4.3.64) that these results aren't literally the statements that $\mathbb{Q}$ and $\mathbb{Q}^{\mathrm{C}}$ are dense in $\mathbb{R}$. When we say that $\mathbb{Q}$ is dense in $\mathbb{R}$, what we mean of course is that $\mathrm{Cls}(\mathbb{Q}) = \mathbb{R}$ (and similarly for $\mathbb{Q}^{\mathrm{C}}$). People 'abuse' language and refer to the properties of Theorems 2.3.18 and 2.4.3.64 as "density" because density in this sense (that is, in the sense of the definition above) follows as an easy corollary of these theorems.

The next couple of facts have to do with neighborhood bases, generating collections, and their relations to concepts reviewed in the previous subsection.

Exercise 3.2.1.3 Let X be a topological space, let $\{\mathscr{B}_x : x \in X\}$ be a neighborhood base for X, let $\lambda \mapsto x_\lambda \in X$ be a net, and let $x_\infty \in X$. Show that $\lambda \mapsto x_\infty$ converges to x_∞ iff $\lambda \mapsto x_\lambda$ is eventually contained in every element of $\mathscr{B}_{x_\infty}$.

Exercise 3.2.1.4 Let X be a topological space, let $\mathscr{S}$ generate the topology of X, let $\lambda \mapsto x_\lambda \in X$ be a net, and let $x_\infty \in X$. Show that $\lambda \mapsto x_\lambda$ converges to x_∞ iff $\lambda \mapsto x_\lambda$ is eventually contained in every element of $\mathscr{S}$ which contains x_∞.

R In other words, for the purposes of convergence, it suffices to look only at a generating collection of the topology.

There is another characterization of quasicompactness that we did not introduce in the previous chapter because it involves the concept of generating collections, something that we hadn't defined in that context.

Theorem 3.2.1.5 — Alexander Subbase Theorem. Let X be a topological space and let $\mathscr{S}$ be a generating collection for the topology on X. Then, X is quasicompact iff every cover by elements of $\mathscr{S}$ has a finite subcover.

R This is of course just the defining property of quasicompactness, the only change being that we need only check covers whose elements come from the generating collection $\mathscr{S}$.

R In particular, if $\mathscr{S}$ does not cover X, then X is quasicompact.

R The term ***subbase*** is sometimes used for generating collections which cover the space. In fact, people usually make the requirement that the generating collection cover the space, but there is no need.

Proof. [a] ($\Rightarrow$) Of course, if X is quasicompact, then *every* open cover has a finite subcover, and so certainly open covers that come from $\mathscr{S}$ will have finite subcovers.

($\Leftarrow$) Step 1: Make hypotheses
Suppose that every cover by elements of $\mathscr{S}$ has a finite subcover. To show that X is quasicompact, we prove the contrapositive of the defining condition of quasicompactness. That is, we show that every collection of open sets that has the property that no finite subset covers X, also does not cover X. So, let $\mathscr{U}$ be a collection of open sets that has the property that no finite subset covers X. We show that $\mathscr{U}$ itself does not cover X.

Step 2: Enlarge $\mathscr{U}$ to a maximal collection
Let $\tilde{\mathscr{U}}$ be the collection of all collections of open sets which (i) contain $\mathscr{U}$ (ii) also have the property that no finite subset covers X. This is a set that is partially-ordered by inclusion, and so we intend to apply Zorn's Lemma (Theorem A.3.5.9) to extract a maximal such element. So, let $\tilde{\mathscr{W}}$ be a well-ordered subset of $\tilde{\mathscr{U}}$ and define

$$\mathscr{W}_0 := \bigcup_{\mathscr{W} \in \tilde{\mathscr{W}}} \mathscr{W}. \tag{3.2.1.6}$$

Certainly $\mathscr{W}_0$ is a collection of open sets, a collection which contains $\mathscr{U}$. In order to be an upper-bound for $\tilde{\mathscr{W}}$, however, we need to check that no finite subset covers X. So, let $W_1, \ldots, W_m \in \mathscr{W}_0$. Then, each $W_k \in \mathscr{W}_k$ for some $\mathscr{W}_k \in \tilde{\mathscr{W}}$.

Because $\tilde{\mathscr{W}}$ is in particular totally-ordered, one of $\mathscr{W}_1, \dots, \mathscr{W}_m$ must contain all the others, and in particular, all the W_ks are contained in a single $\mathscr{W}_k$. It follows that $\{W_1, \dots, W_m\}$ cannot cover X.

Therefore, by Zorn's Lemma, there is a maximal collection of open sets $\mathscr{U}_0$ that (i) contains $\mathscr{U}$ and (ii) has the property that no finite subset covers X. To show that $\mathscr{U}$ does not cover X, it suffices to show that $\mathscr{U}_0$ does not cover X.

STEP 3: SHOW THAT IF AN ELEMENT OF $\mathscr{U}_0$ CONTAINS AN INTERSECTION OF OPEN SETS, $\mathscr{U}_0$ MUST CONTAIN ONE OF THOSE SETS

Let $U \in \mathscr{U}_0$ and suppose that $U_1 \cap \cdots \cap U_m \subseteq U$ for $U_1, \dots, U_m$ open. We proceed by contradiction: suppose that $U_k \notin \mathscr{U}_0$ for all k. This means that, by maximality, $\mathscr{U}_0 \cup \{U_k\}$ must have the property that some finite subset covers X, and so, for each U_k, there are finitely many $U_k^1, \dots, U_k^{n_k}$ such that $X = U_k \cup U_k^1 \cup \cdots \cup U_k^{n_k}$. But then

$$\begin{aligned} U \cup \bigcup_{k=1}^m \bigcup_{l=1}^{n_k} U_k^l &\supseteq \left(\bigcap_{k=1}^m U_k \right) \cup \left(\bigcup_{k=1}^m \bigcup_{l=1}^{n_k} U_k^l \right) \\ &\supseteq \bigcap_{k=1}^m \left(U_k \cup U_k^1 \cup \cdots \cup U_k^{n_k} \right) \\ &= X, \end{aligned} \tag{3.2.1.7}$$

so that

$$U \cup \bigcup_{k=1}^m \bigcup_{l=1}^{n_k} U_k^l = X, \tag{3.2.1.8}$$

a contradiction of that fact that $\mathscr{U}_0$ contains no finite subset which covers X. Therefore, some $U_k \in X$.

STEP 4: DEDUCE THAT $\mathscr{U}_0$ DOES NOT COVER X

Define

$$\mathscr{V}_0 := \mathscr{U}_0 \cap \mathscr{S}. \tag{3.2.1.9}$$

First of all, as no finite subset of $\mathscr{U}_0$ covers X, in particular, $X \notin \mathscr{U}_0$, so that

$$\mathscr{V}_0 = \mathscr{U}_0 \cap (\mathscr{S} \cup \{X\}). \tag{3.2.1.10}$$

As every element of $\mathscr{V}_0$ comes from $\mathscr{U}_0$, it follows that no finite subset of $\mathscr{V}_0$ covers X. On the other hand, every element of $\mathscr{V}_0$ comes from $\mathscr{S}$, so that, by hypothesis, it in turn follows that $\mathscr{V}_0$ does not cover X.[b] Thus, we will be done if we can show that

$$\bigcup_{V \in \mathscr{V}_0} V = \bigcup_{U \in \mathscr{U}_0} U. \tag{3.2.1.11}$$

The $\subseteq$ inclusion is obvious because $\mathscr{V}_0 \subseteq \mathscr{U}_0$. For the other inclusion, let $x \in \bigcup_{U \in \mathscr{U}_0} U$. Then, $x \in U$ for some $U \in \mathscr{U}_0$. Because $\mathscr{S}$ is a generating collection, the collection of all finite intersections of elements of $\mathscr{S} \cup \{X\}$ is a base (Proposition 3.1.1.13), and so there are $U_1, \ldots, U_m \in \mathscr{S}$ such that $x \in U_1 \cap \cdots \cap U_m \subseteq U$. But then, by the previous step, $U_k \in \mathscr{U}_0$ for some U_k. Of course, U_k came from $\mathscr{S} \cup \{X\}$, and so in fact $U_k \in \mathscr{U}_0 \cap (\mathscr{S} \cup \{X\}) = \mathscr{V}_0$, so that indeed $x \in \bigcup_{V \in \mathscr{V}_0} V$. ■

[a] Proof adapted from [Kel55, pg. 139].

[b] If it did cover X, then there would have to be a finite subset which still covers X.

3.3 Filter bases

This section is a bit of an aside and can probably be skipped without too much trouble. Our motivation for covering filter bases is (i) it will help us demonstrate our definition of subnet is the 'correct' one, as opposed to, for example, the one currently[5] given on Wikipedia, and (ii) it is something that is important enough that you should probably at least be aware of its existence and will almost certainly eventually encounter it if you decide to become a mathematician.

Filter bases are actually an alternative to nets. In principle, one could do the entirety of topology never speaking of nets and instead using only filter bases. This would be one motivation for introducing them (though we have decided to primarily stick to nets).

Definition 3.3.1 — Filter base Let $\langle X, \leq \rangle$ be a partially-ordered set and let $F \subset X$ be nonempty and not containing a minimum.[a] Then, F is a ***filter base*** of X iff for $x_1, x_2 \in F$, there is some $x_3 \in F$ such that $x_3 \leq x_1, x_2$.[b]

R This is the definition of an abstract filter base in any partially-ordered set. For us, we will essentially only be interested in filter bases of the partially-ordered set $\langle 2^X, \subseteq \rangle$ for X a topological space: if F is a filter base of $\langle 2^X, \subseteq \rangle$, X a topological space, then we shall say that F is a ***filter base in*** X. (Note that the elements of a filter base do not have to be open sets.)

R This condition is exactly analogous to the condition for directed sets Λ, that for $x_1, x_2 \in \Lambda$, there is some $x_3 \in \Lambda$ with $x_3 \geq x_1, x_2$. In fact, you might even say that the definition is what it is because *a filter base ordered by reverse-inclusion is a directed set*.

[a] As mentioned in the remark, the primary case of interest is when the partially-ordered set is the power set 2^X of some other set X, in which case the minimum is the $\emptyset$. Thus, the condition of disallowing the minimum (if one exists) is put in place so as to disallow $\emptyset \in \mathcal{F}$.

[b] This property is called being ***downward-directed***.

The relation between nets and filter bases is given by the following.

[5] 5 July 2015

Definition 3.3.2 — Derived filter base Let X be a topological space, let $\lambda \mapsto x_\lambda \in X$ be a net, and define

$$\mathcal{F}_{\lambda \mapsto x_\lambda} := \{F \subseteq X : F \text{ eventually contains } \lambda \mapsto x_\lambda\} .$$

Exercise 3.3.3 Show that $\mathcal{F}$ is a filter base.

$\mathcal{F}_{\lambda \mapsto x_\lambda}$ is the ***derived filter*** of $\lambda \mapsto x_\lambda$.

R Given a net, we just defined a canonically associated filter. Of course, there will be many nets which give us this filter. For example, I can change a single term of a net without affecting its derived filter.[a]

R Because of this, one might argue that filter bases are more fundamental than nets. Nets somehow contain extra information that is completely irrelevant to topology. An example of this is how one can always throw away finitely many terms of a sequence without affecting anything of importance. Because the derived filter base of a net only contains information about what *eventually* happens with the net, this extraneous information is lost when passing from the net to its derived filter.

R I personally find nets much more intuitive than filter bases (probably because they are a much more straightforward generalization of sequences than filter bases are), and thinking of how filter bases come from nets helps me understand some of the intuition of filter bases themselves.

[a] In general at least. In stupid cases, of course, e.g. if the domain of the net is a single point, then this will change the derived filter.

At the bare minimum, in order for filter bases and nets to be effectively equivalent for the purposes of topology, we must at least (i) define convergence of filter bases and (ii) show that convergence of a net agrees with convergence of its derived filter.

Definition 3.3.4 — Limit (of a filter base) Let X be a topological space, let $\mathcal{F}$ be a filter base in X, and let $x_\infty \in X$. Then, x_∞ is a ***limit*** of $\mathcal{F}$ iff for every open neighborhood U of x_∞ there is some $F \in \mathcal{F}$ such that $F \subseteq U$. If a filter base has a limit, then we say that it ***converges***.

Proposition 3.3.5 Let X be a topological space, let $\lambda \mapsto x_\lambda \in X$ be a net, and let $x_\infty \in X$. Then, $\lambda \mapsto x_\lambda$ converges to x_∞ iff $\mathcal{F}_{\lambda \mapsto x_\lambda}$ converges to x_∞.

Proof. ($\Rightarrow$) Suppose that $\lambda \mapsto x_\lambda$ converges to x_∞. Let U be an open neighborhood of x_∞. Then, $\lambda \mapsto x_\lambda$ is eventually contained in U. Therefore, $\mathcal{F}_{\lambda \mapsto x_\lambda} \ni U \subseteq U$, and so $\mathcal{F}_{\lambda \mapsto x_\lambda}$ converges to x_∞.

($\Leftarrow$) Suppose that $\mathcal{F}_{\lambda \mapsto x_\lambda}$ converges to x_∞. Let U be an open neighborhood of x_∞. Then, there is some $F \in \mathcal{F}_{\lambda \mapsto x_\lambda}$ such that $F \subseteq U$. By definition of derived filter bases, it follows that $\lambda \mapsto x_\lambda$ is eventually contained in F, and hence eventually contained in U. Therefore, $\lambda \mapsto x_\lambda$ converges to x_∞. ■

Now we turn to filterings and subnets. A filtering is to a filter base as a subnet is to a net. Recall that one of our motivations for talking about filter bases at all was to argue that our definition of subnet was the 'correct' one. One nice thing about filter bases is that there is no confusion about what the definition of a filtering should be.[6] We will then show that our definition of subnet coincides with the notion of a filtering.

Definition 3.3.6 — Filtering Let $\mathcal{F}$ be a filter base on X. Then, a ***filtering*** of $\mathcal{F}$ is a filter base $\mathcal{G}$ that has the property that, for every $F \in \mathcal{F}$, there is some $G \in \mathcal{G}$ such that $G \subseteq F$.

[6] Though evidently there is some confusion as to what they should be called—see the remark in the definition below.

R Note that in some places[a] this is called a *refinement*. This is poor terminology because it disagrees with the usual definition of refinements of covers—see Definition 4.1.1.29. For comparison, we reproduce that definition here.

$$\mathcal{G} \text{ is a } \textbf{\textit{refinement}} \text{ of } \mathcal{F} \text{ iff for every } G \in \mathcal{G} \text{ there is some } F \in \mathcal{F} \text{ such that } G \subseteq F. \tag{3.3.7}$$

[a] *cough*—Wikipedia—*cough*

Exercise 3.3.8 Let $\mathcal{F}$ and $\mathcal{G}$ be filter bases. Show that if $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{G}$ is a filtering of $\mathcal{F}$.

In fact, for derived filter bases, every filtering is of this form.

Proposition 3.3.9 Let X be a topological space, let $\Lambda \ni \lambda \mapsto x_\lambda \in X$ be a net, and let $\Lambda' \ni \mu \mapsto \lambda_\mu \in \Lambda$. Then the following are equivalent.

(i). $\mu \mapsto x_{\lambda_\mu}$ is a subnet of $\lambda \mapsto x_\lambda$.
(ii). $\mathcal{F}_{\lambda \mapsto x_\lambda} \subseteq \mathcal{F}_{\mu \mapsto x_{\lambda_\mu}}$.
(iii). $\mathcal{F}_{\mu \mapsto x_{\lambda_\mu}}$ is a filtering of $\mathcal{F}_{\lambda \mapsto x_\lambda}$.

R In particular, as the definitions of subnet given in [Kel55] (Proposition 2.4.5.9) and on Wikipedia[a] (Proposition 2.4.5.11) are *not* equivalent (Example 2.4.5.10 and Exercise 2.4.5.12) to our definition of subnet (Definition 3.2.9), they are in turn not equivalent to filterings of filter bases!

R Of course, in general there are filterings not of this form (Exercise 3.3.10), but for *derived* filter bases, filtering is equivalent to containing (as sets).

[a] As of 16 July 2015

Proof. ($(i) \Rightarrow (ii)$) Suppose that $\mu \mapsto x_{\lambda_\mu}$ is a subnet of $\lambda \mapsto x_\lambda$. Let $F \in \mathcal{F}_{\lambda\mapsto x_\lambda}$. Then, $\lambda \mapsto x_\lambda$ is eventually in F, and so[a] $\mu \mapsto x_{\lambda_\mu}$ is eventually in F, and so $F \in \mathcal{F}_{\mu\mapsto x_{\lambda_\mu}}$.

($(ii) \Rightarrow (iii)$) Exercise 3.3.8

($(iii) \Rightarrow (i)$) Suppose that $\mathcal{F}_{\mu\mapsto x_{\lambda_\mu}}$ is a filtering of $\mathcal{F}_{\lambda\mapsto x_\lambda}$. Let $F \subseteq X$ be such that F eventually contains $\lambda \mapsto x_\lambda$. Then, $F \in \mathcal{F}_{\lambda\mapsto x_\lambda}$, and so there is some $F' \in \mathcal{F}_{\mu\mapsto x_{\lambda_\mu}}$ such that $F' \subseteq F$. As $\mu \mapsto x_{\lambda_\mu}$ is eventually contained in F', it is eventually contained in F, and so[b] $\mu \mapsto x_{\lambda_\mu}$ is a subnet of $\lambda \mapsto x_\lambda$. ■

[a] See the definition of subnet, Definition 3.2.9.
[b] Once again, by the definition of subnets, Definition 3.2.9.

Exercise 3.3.10 Find an example of a filter base $\mathcal{F}$ and a filtering $\mathcal{G}$ of $\mathcal{F}$, but for which $\mathcal{G} \not\subseteq \mathcal{F}$.

In the next section, we present several new ways of defining topologies. One of these ways will be by defining what it means for filters to converge, and so present here as theorems the results that will be used as axioms.

Proposition 3.3.11 — Kelley's Filter Convergence Axioms Let X be a topological space. Then,

(i). $\mathcal{P}_x$ converges to x, where $\mathcal{P}_x := \{U \subseteq X : x \in U\}$;[a]
(ii). Let $\mathcal{G}$ be a filtering of the filter base $\mathcal{F}$. Then, if $\mathcal{F}$ converges to x, then $\mathcal{G}$ converges to x.
(iii). Let $\mathcal{F}$ be a filter. Then, if every filtering $\mathcal{G} \supseteq \mathcal{F}$ has in turn a filtering $\mathcal{H}$ such that $\mathcal{H}$ converges to x, then $\mathcal{F}$ converges to x; and

(iv). for all directed sets I and filters $\mathcal{F}^i$ converging to $x^i \in X$, for $i \in I$, if $\mathcal{F}_{i \mapsto x^i}$ converges to x^∞, then

$$\begin{aligned}\mathcal{F}_\infty := \{U \subseteq X : \text{there exists } i_U \in I \text{ such that,}\\ \text{whenever } i \geq i_U,\, U \supseteq F^i \text{ for some } F^i \in \mathcal{F}^i.\}\end{aligned}$$

also converges to x^∞.

R In other words, $\mathcal{F}_\infty$ consists of those sets that eventually contain some element of $\mathcal{F}^i$.

R Note that these are completely analogous to Kelley's (Net) Convergence Axioms (Theorem 3.2.14), with one itsy-bitsy exception. You'll recall that in the third axiom for nets, there is mention of *cofinal* subnets. I don't believe there is any concept for filterings that is analogous to cofinal subnets, and so there is no "strict" here. That said, you'll notice I have written "$\mathcal{G} \supseteq \mathcal{F}$"—this is not a big deal, but I write this instead of just "$\mathcal{F}$ is a filtering of $\mathcal{G}$" to make the axioms just a *little* bit more parallel, even though the condition $\mathcal{G} \supseteq \mathcal{F}$ is *not* the analogue of being strict.[b]

R Perhaps this could be made into an argument that somehow nets are more fundamental—to define a topology by convergence of filters, you have to make use of nets (in this case, $i \mapsto x^i$).[c]

---

[a]"$\mathcal{P}$" is for *principal*, the etymology being from the use of the word "principal" in the context of ideals in ring theory (to the best of my knowledge anyways).

[b]Indeed, for derived filter bases, this is equivalent to being an (ordinary) subnet—see Proposition 3.3.9.

[c]Or rather, I am not aware of a way to state the axioms without at least implicitly using nets.

Proof. (i) We leave this as an exercise.

Exercise 3.3.12 Show (i).

(ii) We leave this as an exercise.

Exercise 3.3.13 Show (ii)($\Rightarrow$).

(iii) Suppose that for every filtering $\mathcal{G} \supseteq \mathcal{F}$ there is a filtering $\mathcal{H}$ of $\mathcal{G}$ that converges to x. We proceed by contradiction: suppose that $\mathcal{F}$ doesn't converge to x. Then, there is some open neighborhood $x \in U \subseteq X$ such that $F \not\subseteq U$ for all $F \in \mathcal{F}$. For each $F \in \mathcal{F}$, let $x_F \in U^{\mathrm{C}} \cap F$, so that $\langle \mathcal{F}, \supseteq \rangle \ni F \mapsto x_F$ is a net. We claim that $\mathcal{F}_{F \mapsto x_F}$ is a filtering of $\mathcal{F}$.

So, let $F_0 \in \mathcal{F}$. We wish to show that that in fact F_0 itself eventually contains $F \mapsto x_F$. So, suppose that $F \geq F_0$, that is, $F \subseteq F_0$. Then, $x_F \in F \subseteq F_0$. Thus, whenever $F \geq F_0$, it follows that $x_F \in F_0$, so that indeed F_0 eventually contains $F \mapsto x_F$.

We wish to show that $\mathcal{F}_{F \mapsto x_F}$ itself has no filtering which converges to x. This will be a contradiction, thereby proving the result. To show this itself, we again proceed by contradiction: suppose that there is some filtering $\mathcal{H}$ of $\mathcal{F}_{F \mapsto x_F}$ that converges to x. As $\mathcal{H}$ converges to x, there must be some $H \in \mathcal{H}$ such that $x \in H \subseteq U$. On the other hand, as $\mathcal{H}$ is a filtering of $\mathcal{F}_{F \mapsto x_F}$, there must be some $G \subseteq H$ which eventually contains $F \mapsto x_F$. As $H \subseteq U$, of course $G \subseteq U$, which implies that at least one $x_F \in G \subseteq U$, a contradiction of the fact that $x_F \notin U$ (by construction).

(iv) Let I be a directed set, for each $i \in I$ let $\mathcal{F}^i$ be a filter converging to $x^i \in X$, and suppose that $\mathcal{F}_{i \mapsto x^i}$ converges to x^∞. By Proposition 3.3.5, $i \mapsto x^i$ converges to x^∞. To show that $\mathcal{F}_\infty$ converges to x^∞, it suffices to show that every open neighborhood of x^∞ is an element of $\mathcal{F}_\infty$. So, let U be an open neighborhood of x^∞. Then, as $\mathcal{F}_{i \mapsto x^i}$ converges to x^∞, it follows that there is some $V \in \mathcal{F}_{i \mapsto x^i}$ such that $V \subseteq U$. As

$i \mapsto x^i$ is eventually contained in V, there is some $i_0 \in I$ such that, whenever $i \geq i_0$, it follows that $x^i \in V \subseteq U$. Thus, for i sufficiently large, U is an open neighborhood of x^i. But then, because $\mathcal{F}^i$ converges to x^i, it follows that there is some $F^i \in \mathcal{F}^i$ such that $F^i \subseteq U$, and so $U \in \mathcal{F}_\infty$ as desired. ∎

3.4 Equivalent definitions of topological spaces

There is more than one way to define a topology. By definition, the specification of a topology is just the specification of what sets are open. Sometimes, however, what the open sets should be is not nearly as obvious as, for example, what convergence of nets should mean. In this section, we present a couple of other ways you may define a topological space. Which one is most useful, of course, will depend on the particular problem at hand.

To summarize, we have already shown that we may define a topology in the following ways. We can define a topology

(i). by specifying the open sets (Definition 3.1.1);
(ii). by specifying the closed sets (Exercise 3.1.2);
(iii). by specifying a base for the topology (Proposition 3.1.1.3);
(iv). by specifying a neighborhood base for the topology (Proposition 3.1.1.9); or
(v). by specifying a generating collection for the topology (Proposition 3.1.1.13).

Of course, there are many other ways to specify a topology as well, and it is these methods that are the topic of this section.

3.4.1 Definition by specification of closures or interiors

We have mentioned the Kuratowski Closure (Interior) Axioms as well as Kelley's (Filter) Convergence Axioms. The former allow us to define a topology in the following way.

Theorem 3.4.1.1 — Kuratowski's Closure Theorem. Let X be a set and let $\mathrm{C} : 2^X \to 2^X$ be a function on the power-set of X. Then, if

(i). $\mathrm{C}(\emptyset) = \emptyset$;
(ii). $S \subseteq \mathrm{C}(S)$;
(iii). $\mathrm{C}(S) = \mathrm{C}(\mathrm{C}(S))$; and
(iv). $\mathrm{C}(S \cup T) = \mathrm{C}(S) \cup \mathrm{C}(T)$,

then there exists a unique topology on X such that $\mathrm{Cls}(S) = \mathrm{C}(S)$.

Proof. Step 1: Make hypotheses
Suppose that (i) $\mathrm{C}(\emptyset) = \emptyset$, (ii) $S \subseteq \mathrm{C}(S)$, (iii) $\mathrm{C}(S) = \mathrm{C}(\mathrm{C}(S))$, and (iv) $\mathrm{C}(S \cup T) = \mathrm{C}(S) \cup \mathrm{C}(T)$.

Step 2: Show that $S \subseteq T$ implies $\mathrm{C}(S) \subseteq \mathrm{C}(T)$
Suppose that $S \subseteq T$, so that $T = S \cup (T \setminus S)$, and hence $\mathrm{C}(T) = \mathrm{C}(S) \cup \mathrm{C}(T \setminus S)$, and so $\mathrm{C}(S) \subseteq \mathrm{C}(T)$.

Step 3: Define what should be the closed sets
The idea of the proof is that the closed sets should be precisely the sets that are equal to their closure (Proposition 3.2.43). We thus make the definition

$$\mathscr{C} := \left\{ C \in 2^X : C = \mathrm{C}(C) \right\}. \tag{3.4.1.2}$$

Step 4: Verify that this defines a topology
By (i), the empty-set is closed (by which we mean it is an element of $\mathscr{C}$). By (ii), we have

$$X \subseteq \mathrm{C}(X) \subseteq X, \tag{3.4.1.3}$$

so that X is closed as well. Let $\mathscr{D} \subseteq \mathscr{C}$ and define

$$B := \bigcap_{C \in \mathscr{D}} C \tag{3.4.1.4}$$

Then, of course, $B \subseteq C$ for all $C \in \mathscr{D}$, and so by Step 2, we have that $\mathrm{C}(B) \subseteq \mathrm{C}(C)$ for all $C \in \mathscr{D}$, and so

$$\mathrm{C}(B) \subseteq \bigcap_{C \in \mathscr{D}} \mathrm{C}(C) = \bigcap_{C \in \mathscr{D}} C =: B. \tag{3.4.1.5}$$

As $B \subseteq \mathrm{C}(B)$ by (ii), we have that $B = \mathrm{C}(B)$, so that $\mathscr{C}$ is closed under arbitrary intersections. We now check that it is closed under finite unions, so let $C, D \in \mathscr{C}$. Then,

$$\mathrm{C}(C \cup D) = \mathrm{C}(C) \cup \mathrm{C}(D) = C \cup D, \tag{3.4.1.6}$$

and so $C \cup D \in \mathscr{C}$. Thus, $\mathscr{C}$ is closed under finite unions, and hence defines a topology by Exercise 3.1.2.[a]

Step 5: Show that $\mathrm{Cls}(S) = \mathrm{C}(S)$

Let $S \subseteq X$. By (iii), $\mathrm{C}(S)$ is closed, and by (ii), it contains S. Therefore, $\mathrm{Cls}(S) \subseteq \mathrm{C}(S)$. It follows that $\mathrm{C}\left(\mathrm{Cls}(S)\right) \subseteq \mathrm{C}(S)$. On the other hand, because $S \subseteq \mathrm{Cls}(S)$, it follows that $\mathrm{C}(S) \subseteq \mathrm{C}\left(\mathrm{Cls}(S)\right) =^{b} \mathrm{Cls}(S)$, and so indeed $\mathrm{Cls}(S) = \mathrm{C}(S)$.

Step 6: Demonstrate uniqueness

If another topology $\mathscr{V}$ satisfies this property, that is, $\mathrm{Cls}_{\mathscr{V}}(S) = \mathrm{C}(S)$, then we have that $\mathrm{Cls}_{\mathscr{V}}(S) = \mathrm{Cls}(S)$ (no subscript indicates the closure in the topology defined by $\mathscr{C}$). It follows that a set S is closed with respect to $\mathscr{V}$ iff it is equal to $\mathrm{Cls}_{\mathscr{V}}(S)$ iff it is equal to $\mathrm{Cls}(S)$ iff it is closed in the topology defined by $\mathscr{C}$. Thus, the two topologies have the same closed sets, and hence are the same. ■

[a] This is the exercise that tells us we can define a topology by specifying the closed sets.

[b] Because $\mathrm{Cls}(S) \in \mathscr{C}$, as it must be because $\mathrm{Cls}(S)$ is closed.

Similarly, we have a ‘dual’ interior theorem.

Theorem 3.4.1.7 — Kuratowski's Interior Theorem. Let X be a set and let $\mathrm{I} : 2^X \to 2^X$ be a function on the power-set of X. Then, if

(i). $\mathrm{I}(X) = X$;
(ii). $\mathrm{I}(S) \subseteq S$;
(iii). $\mathrm{I}(S) = \mathrm{I}(\mathrm{I}(S))$; and
(iv). $\mathrm{I}(S \cap T) = \mathrm{I}(S) \cap \mathrm{I}(T)$,

then there exists a unique topology on X such that $\mathrm{Int}(S) = \mathrm{I}(S)$.

R We omit the proof as it is completely 'dual' to the corresponding closure proof.

3.4.2 Definition by specification of convergence

The previous two results showed that we can define a topology by defining what the closure or interior of every set should be. The next result says that we can define a topology by defining what it means for nets to converge.

Theorem 3.4.2.1 — Kelley's Convergence Theorem. Let X be a set, denote by $\mathscr{N}$ the collection of all nets in X, and let $\to$ be a relation on $\mathscr{N} \times X$. Then, if

(i). $(\lambda \mapsto x_\infty) \to x_\infty$;
(ii). if $(\lambda \mapsto x_\lambda) \to x_\infty$, then $(\mu \mapsto x_{\lambda_\mu}) \to x_\infty$ for every subnet $\mu \mapsto x_{\lambda_\mu}$ of $\lambda \mapsto x_\lambda$;
(iii). if every cofinal subnet $\mu \mapsto x_{\lambda_\mu}$ has in turn a subnet $\nu \mapsto x_{\lambda_{\mu_\nu}}$ such that $(\nu \mapsto x_{\lambda_{\mu_\nu}}) \to x_\infty$, then $(\lambda \mapsto x_\lambda) \to x_\infty$; and
(iv). for all directed sets I and nets $x^i : \Lambda^i \to X$, if $x^i \to (x^i)_\infty$ and $(i \mapsto (x^i)_\infty) \to (x^\infty)_\infty$, then $\left(I \times \prod_{i \in I} \Lambda^i \ni \langle i, \lambda \rangle \mapsto (x^i)_{\lambda^i}\right) \to (x^\infty)_\infty$,

then there is a unique topology on X such that $\lambda \mapsto x_\lambda$ converges to x_∞ iff $(\lambda \mapsto x_\lambda) \to x_\infty$.

R People sometimes attempt to define a topology by defining what it means for *sequences* to converge. This is nonsensical. For example, (iv) doesn't even make sense in this context. Moreover, because of examples like Example 3.2.48 (the cocountable topology and discrete topology on $\mathbb{R}$ have the same notion of sequential convergence), trying to rectify this in general is hopeless. My best guess is that this is because people tend to shy away from the usage of nets for some reason, and so they tend to define topologies using convergence of sequences. In any case, I don't recall ever seeing a case where such a definition was given that *didn't* make sense after you replaced the word "sequence" with the word "net'. That is to say, even though they are technically wrong, there is usually a very easy fix.

R We will see later that a topological space is what is called "T_2" iff limits are unique (Proposition 3.6.2.20). Thus, defining a topology in this way is often a super easy way to guarantee at least this separation axiom.

Proof. STEP 1: MAKE HYPOTHESES
Suppose that (i) $(\lambda \mapsto x_\infty) \to x_\infty$; (ii) if $(\lambda \mapsto x_\lambda) \to x_\infty$, then $(\mu \mapsto x_{\lambda_\mu}) \to x_\infty$ for every subnet $\mu \mapsto x_{\lambda_\mu}$ of $\lambda \mapsto x_\lambda$; (iii) if every cofinal subnet $\mu \mapsto x_{\lambda_\mu}$ has in turn a subnet $\nu \mapsto x_{\lambda_{\mu_\nu}}$ such that $(\nu \mapsto x_{\lambda_{\mu_\nu}}) \to x_\infty$, then $(\lambda \mapsto x_\lambda) \to x_\infty$; and (iv) for all directed sets I and nets $x^i : \Lambda^i \to X$, if $x^i \to (x^i)_\infty$ and $(i \mapsto (x^i)_\infty) \to (x^\infty)_\infty$, then $\left(I \times \prod_{i\in I} \Lambda^i \ni \langle i, \lambda\rangle \mapsto (x^i)_{\lambda^i}\right) \to (x^\infty)_\infty$.

STEP 2: DEFINE THE NOTION OF AN *ADHERENT POINT*
For a subset $S \subseteq X$ and $x \in X$, we say that x is an *adherent point*[a] of S iff there is some net $\lambda \mapsto x_\lambda \in S$ such that $(\lambda \mapsto x_\lambda) \to x$.

STEP 3: DEFINE WHAT SHOULD BE THE CLOSURE

For a subset $S \subseteq X$, we define $\mathrm{C}(S)$ to be the set of adherent points of S.

STEP 4: SHOW THAT C SATISFIES KURATOWSKI'S CLOSURE AXIOMS

The empty-set has no adherent points, and so trivially $\mathrm{C}(\emptyset) = \emptyset$.

It follows from (i) that $S \subseteq \mathrm{C}(S)$.

We wish to show that $\mathrm{C}(\mathrm{C}(S)) \subseteq \mathrm{C}(S)$. So, let $(x^\infty)_\infty \in \mathrm{C}(\mathrm{C}(S))$. Then, there is a net $I \ni i \mapsto x^i \in \mathrm{C}(S)$ such that $(i \mapsto x^i) \to (x^\infty)_\infty$. As each $x^i \in \mathrm{C}(S)$, there is a net $\Lambda^i \ni \lambda^i \mapsto (x^i)_{\lambda^i} \in S$ such that $\left(\lambda^i \mapsto (x^i)_{\lambda^i}\right) \to (x^i)_\infty$. By (iv), it then follows that $\left(I \times \prod_{i \in I} \Lambda^i \ni \langle i, \lambda \rangle \mapsto (x^i)_{\lambda^i}\right) \to (x^\infty)_\infty$. As each $(x^i)_{\lambda^i} \in S$, it follows that $(x^\infty)_\infty$ is an adherent point of S, that is, $(x^\infty)_\infty \in \mathrm{C}(S)$, and hence $\mathrm{C}(\mathrm{C}(S)) = \mathrm{C}(S)$ (that $\mathrm{C}(S) \subseteq \mathrm{C}(\mathrm{C}(S))$ follows from the fact that $S \subseteq \mathrm{C}(S)$).

We now show that $\mathrm{C}(S \cup T) = \mathrm{C}(S) \cup \mathrm{C}(T)$. We have that $S \subseteq \mathrm{C}(S \cup T)$, and so $\mathrm{C}(S) \subseteq \mathrm{C}(S \cup T)$. Similarly for T, and so we have $\mathrm{C}(S) \cup \mathrm{C}(T) \subseteq \mathrm{C}(S \cup T)$.

We now show that $\mathrm{C}(S \cup T) \subseteq \mathrm{C}(S) \cup \mathrm{C}(T)$. So, let $x_\infty \in \mathrm{C}(S \cup T)$, so that there is a net $\Lambda \ni \lambda \mapsto x_\lambda \in S \cup T$ such that $(\lambda \mapsto x_\lambda) \to x_\infty$. Define

$$\Lambda_S := \{\lambda \in \Lambda : x_\lambda \in S\} \text{ and } \Lambda_T := \{\lambda \in \Lambda : x_\lambda \in T\} .$$

As $\Lambda = \Lambda_S \cup \Lambda_T$, either Λ_S or Λ_T is cofinal in Λ. Without loss of generality, suppose that Λ_S is cofinal in Λ, so that $x|_{\Lambda_S}$ is a cofinal subnet of $\lambda \mapsto x_\lambda$. By hypothesis, this in turn must have a subnet $\mu \mapsto x_{\lambda_\mu}$, $\lambda_\mu \in \Lambda_S$, such that $(\mu \mapsto x_{\lambda_\mu}) \to x_\infty$. Thus, $x_\infty \in \mathrm{C}(S)$, and so $\mathrm{C}(S \cup T) \subseteq \mathrm{C}(S) \cup \mathrm{C}(T)$.

It follows by Kuratowski's Closure Theorem (Theorem 3.4.1.1) that there is a unique topology on X such that $\mathrm{Cls}(S) = \mathrm{C}(S)$.

STEP 5: SHOW THAT $\lambda \mapsto x_\lambda$ CONVERGES TO x_∞ IFF $(\lambda \mapsto x_\lambda) \to x_\infty$

Suppose that $\Lambda \ni \lambda \mapsto x_\lambda$ converges to x_∞. We proceed by contradiction: suppose that it is not the case that $(\lambda \mapsto x_\lambda) \to x_\infty$. Then, there must be some cofinal subnet $I \ni \mu \mapsto x_{\lambda_\mu}$ that has no subnet $\nu \mapsto x_{\lambda_{\mu_\nu}}$ such that $(\nu \mapsto x_{\lambda_{\mu_\nu}}) \to x_\infty$. To obtain a contradiction, we construct such a subnet.

For each μ_0, define $S_{\mu_0} := \{x_{\lambda_\mu} : \mu \geq \mu_0\}$. As $\lambda \mapsto x_\lambda$ converges to x_∞, so does $\mu \mapsto x_{\lambda_\mu}$, and so $x_\infty \in \mathrm{Cls}(S_{\mu_0})$ for all μ_0. Hence, by the definition of our closure, for each $\mu \in I$ there is a net $\Lambda^\mu \ni \nu^\mu \mapsto x_{\lambda_{\mu_{\nu^\mu}}} \in S_\mu$, so that $\mu_{\nu^\mu} \geq \mu$, with $\left(\nu^\mu \mapsto x_{\lambda_{\mu_{\nu^\mu}}}\right) \to x_\infty$.[b] From (iv), it follows that $\left(I \times \prod_{\mu \in I} \Lambda^\mu \ni \langle \mu, \nu \rangle \mapsto x_{\lambda_{\mu_{\nu^\mu}}}\right) \to x_\infty$. Thus, if this is in fact a subnet of $\mu \mapsto x_{\lambda_\mu}$, we will have our contradiction. To show this, we apply Proposition 2.4.5.9. So, let $\mu_0 \in I$ be arbitrary. We must find an index in $I \times \prod_{\mu \in I} \Lambda^\mu$ that has the property that, whenever $\langle \mu, \nu \rangle$ is at least as large as that index, it follows that $\mu_{\nu^\mu} \geq \mu_0$. However, we know that $\mu_{\nu^\mu} \geq \mu$ for all μ, and so we can take $\langle \mu_0, \nu_0 \rangle$ as our index for any choice of ν_0.

Now suppose that $(\Lambda \ni \lambda \mapsto x_\lambda) \to x_\infty$. We proceed by contradiction: suppose that $\lambda \mapsto x_\lambda$ does not converge to x_∞. Then, there is a cofinal subnet $\mu \mapsto x_{\lambda_\mu}$ which has no subnet converging to x_∞. In particular, $\mu \mapsto x_{\lambda_\mu}$ itself does not converge to x_∞, and so there is an open neighborhood U of x_∞ which does not eventually contain $\mu \mapsto x_{\lambda_\mu}$. It follows that $I := \{\lambda_\mu : x_{\lambda_\mu} \notin U\}$ is cofinal, so that $I \ni \mu \mapsto x_{\lambda_\mu}$ is a subnet contained in U^{C}. On the other hand, we know that $(I \ni \mu \mapsto x_{\lambda_\mu}) \to x_\infty$[c], so that $x_\infty \in \mathrm{Cls}(\{x_{\lambda_\mu} : \mu \in I\})$, but this is a contradiction of the fact that x_∞ has an open neighborhood (U) which contains no point of the set $\{x_{\lambda_\mu} : \mu \in I\}$.

STEP 6: DEMONSTRATE UNIQUENESS

Recall that sets are closed iff they contain all their limit points. If we have two topologies with the same notion of convergence, then the set of limit points of a given set in each topology are the same, and consequently, a set is closed in one iff it is closed in the other. ■

[a]This of course will turn out to agree with the notion of adherent point for the topology.

[b]The superscript μ in ν^μ is just to help us keep track of which directed set ν^μ is contained in.

[c]By (ii), because it is a subnet of $\lambda \mapsto x_\lambda$ and $(\lambda \mapsto x_\lambda) \to x_\infty$.

We mentioned in a remark of Kelley's Convergence Axioms (Theorem 3.2.14) that the second and third axioms are in fact equivalent to just a single axiom. We now prove this.

Proposition 3.4.2.2 Let X be a set, denote by $\mathcal{N}$ the collection of all nets in X, and let $\to$ be a relation on $\mathcal{N} \times X$. Then, the following are equivalent.

(i). $(\lambda \mapsto x_\lambda) \to x_\infty$ iff every subnet $\mu \mapsto x_{\lambda_\mu}$ has in turn a subnet $\nu \mapsto x_{\lambda_{\mu_\nu}}$ such that $(\nu \mapsto x_{\lambda_{\mu_\nu}}) \to x_\infty$.

(ii). (a). If $(\lambda \mapsto x_\lambda) \to x_\infty$ and $\mu \mapsto x_{\lambda_\mu}$ is a subnet of $\lambda \mapsto x_\lambda$, then $(\mu \mapsto x_{\lambda_\mu}) \to x_\infty$; and

(b). If every cofinal subnet $\mu \mapsto x_{\lambda_\mu}$ of $\lambda \mapsto x_\lambda$ has in turn a subnet $\nu \mapsto x_{\lambda_{\mu_\nu}}$ such that $(\nu \mapsto x_{\lambda_{\mu_\nu}}) \to x_\infty$, then $(\lambda \mapsto x_\lambda) \to x_\infty$.

Proof. ($\Rightarrow$) Suppose that $(\lambda \mapsto x_\lambda) \to x_\infty$ iff every subnet $\mu \mapsto x_{\lambda_\mu}$ has in turn a subnet $\nu \mapsto x_{\lambda_{\mu_\nu}}$ such that $(\nu \mapsto x_{\lambda_{\mu_\nu}}) \to x_\infty$.

We first check (ii)(a). So, let $\lambda \mapsto x_\lambda$ be a net such that $(\lambda \mapsto x_\lambda) \to x_\infty$ and let $\mu \mapsto x_{\lambda_\mu}$ be a subnet. To show that $(\mu \mapsto x_{\lambda_\mu}) \to x_\infty$, we show that every subnet $\nu \mapsto x_{\lambda_{\mu_\nu}}$ has in turn a subnet $\xi \mapsto x_{\lambda_{\mu_{\nu_\xi}}}$ such that $(\xi \mapsto x_{\lambda_{\mu_{\nu_\xi}}}) \to x_\infty$. So, let $\nu \mapsto x_{\lambda_{\mu_\nu}}$ be a subnet of $\mu \mapsto x_{\lambda_\mu}$. This itself is also a subnet of $\lambda \mapsto x_\lambda$, and so by hypothesis, it has a subnet $\xi \mapsto x_{\lambda_{\mu_{\nu_\xi}}}$ such that $(\xi \mapsto x_{\lambda_{\mu_{\nu_\xi}}}) \to x_\infty$.

Now, suppose that every cofinal subnet $\mu \mapsto x_{\lambda_\mu}$ of $\lambda \mapsto x_\lambda$ has in turn a subnet $\nu \mapsto x_{\lambda_{\mu_\nu}}$ such that $(\nu \mapsto x_{\lambda_{\mu_\nu}}) \to x_\infty$.

We wish to show that $(\lambda \mapsto x_\lambda) \to x_\infty$. To do that, we show that *every* subnet (not just every cofinal subnet) $\mu \mapsto x_{\lambda_\mu}$ has in turn a subnet $\nu \mapsto x_{\lambda_{\mu_\nu}}$ such that $(\nu \mapsto x_{\lambda_{\mu_\nu}}) \to x_\infty$. So, let $M \ni \mu \mapsto x_{\lambda_\mu}$ be a subnet of $\Lambda \ni \lambda \mapsto x_\lambda$. By Proposition 3.2.10, for every $\lambda \in \Lambda$, there is some $\mu_\lambda \in M$ such that

$$\left\{x_{\lambda_\mu} : \mu \geq \mu_\lambda\right\} \subseteq \{x_{\lambda'} : \lambda' \geq \lambda\}\,. \tag{3.4.2.3}$$

Let $U \subseteq X$ eventually contain $\mu \mapsto x_{\lambda_\mu}$. Then, there is some $\mu_U \in M$ such that, whenever $\mu \geq \mu_U$, it follows that $x_{\lambda_\mu} \in U$. For $U \subseteq X$ that eventually contains $\mu \mapsto x_{\lambda_\mu}$ and $\lambda \in \Lambda$, let $\mu_{U,\lambda} \in M$ be at least as large as μ_U and μ_λ. In particular, $x_{\lambda_{\mu_{U,\lambda}}} \in \{x_{\lambda'} : \lambda' \geq \lambda\}$, and so without loss of generality, assume that $\lambda_{\mu_{U,\lambda}} \geq \lambda$. It follows that

$$\left\{\lambda_{\mu_{U,\lambda}} : U \subseteq X \text{ eventually contains } \mu \mapsto x_{\lambda_\mu},\ \lambda \in \Lambda\right\}$$

is cofinal in Λ, and hence defines a corresponding cofinal subnet. By hypothesis, this has in turn a subnet $N \ni \nu \mapsto x_{\lambda_{\mu_{U_\nu,\lambda_\nu}}}$ such that $(\nu \mapsto x_{\lambda_{\mu_{U_\nu,\lambda_\nu}}}) \to x_\infty$.

Now, for $\mu' \in M$ and $\nu' \in N$, let $\mu_{\mu',\nu'} \in M$ be at least as large as both μ' and μ_{U_ν,λ_ν}. It follows that

$$M \times N \ni \langle \mu', \nu' \rangle \mapsto x_{\lambda_{\mu_{\mu',\nu'}}} \tag{3.4.2.4}$$

is a subnet of both $\mu \mapsto x_{\lambda_\mu}$ and $\nu \mapsto x_{\lambda_{\mu_{U_\nu,\lambda_\nu}}}$. The latter fact together with (ii)(b) imply that $(\langle \mu', \nu' \rangle \mapsto x_{\lambda_{\mu_{\mu',\nu'}}}) \to x_\infty$. Thus, we have constructed a subnet of $\mu \mapsto x_{\lambda_\mu}$ that is related by x_∞ via $\to$, and hence, by hypothesis, $(\mu \mapsto x_{\lambda_\mu}) \to x_\infty$, as desired.

($\Leftarrow$) Suppose that (a) if $(\lambda \mapsto x_\lambda) \to x_\infty$ and $\mu \mapsto x_{\lambda_\mu}$ is a subnet of $\lambda \mapsto x_\lambda$, then $(\mu \mapsto x_{\lambda_\mu}) \to x_\infty$; and (b) if every cofinal subnet $\mu \mapsto x_{\lambda_\mu}$ of $\lambda \mapsto x_\lambda$ has in turn a subnet $\nu \mapsto x_{\lambda_{\mu_\nu}}$ such that $(\nu \mapsto x_{\lambda_{\mu_\nu}}) \to x_\infty$, then $(\lambda \mapsto x_\lambda) \to x_\infty$. The ($\Leftarrow$) direction of (i) is true by hypothesis, so it suffices to show the ($\Rightarrow$) direction of (i). So, suppose that $(\lambda \mapsto x_\lambda) \to x_\infty$ and let $\mu \mapsto x_{\lambda_\mu}$ be a subnet. Then, by hypothesis, we also have that $(\mu \mapsto x_{\lambda_\mu}) \to x_\infty$, and so this subnet $\mu \mapsto x_\mu$ has in turn a subnet (namely itself) that is related to x_∞ by $\to$. ■

And of course, there is an analogous result for filters.

Theorem 3.4.2.5 — Kelley's Filter Convergence Theorem. Let X be a set, denoted by $\tilde{\mathcal{F}}$ the collection of all filter bases in X, and let $\to$ be a relation on $\tilde{\mathcal{F}} \times X$. Then, if

(i). $\mathcal{P}_x \to x$, where $\mathcal{P}_x := \{U \subseteq X : x \in U\}$;[a]

(ii). if $\mathcal{F} \to x$, then $\mathcal{G} \to x$ for every filtering $\mathcal{G}$ of $\mathcal{F}$;

(iii). every filtering $\mathcal{G} \supseteq \mathcal{F}$ has in turn a filtering $\mathcal{H}$ such that $\mathcal{H} \to x$, then $\mathcal{F} \to x$; and

(iv). for all directed sets I and convergent filters $\mathcal{F}^i \to x^i \in X$, for $i \in I$, if $\mathcal{F}_{i \mapsto x^i} \to x^\infty$, then

$$\begin{aligned}\mathcal{F}_\infty := \{U \subseteq X : {} & \text{there exists } i_U \in I \text{ such that,} \\ & \text{whenever } i \geq i_U,\ U \supseteq F^i \text{ for some} \\ & F^i \in \mathcal{F}^i.\} \to x^\infty,\end{aligned} \tag{3.4.2.6}$$

then there is a unique topology on X such that $\mathcal{F}$ converges to $x_\infty \in X$ iff $\mathcal{F} \to x_\infty$.

[a] "$\mathcal{P}$" is for *principal*, the etymology being from the use of the word "principal" in the context of ideals in ring theory (to the best of my knowledge anyways).

Proof. Step 1: Make hypotheses
Suppose that (i)$\mathcal{P}_x \to x$, where $\mathcal{P}_x := \{U \subseteq X : x \in U\}$; (ii) if $\mathcal{F} \to x$, then $\mathcal{G} \to x$ for every filtering $\mathcal{G}$ of $\mathcal{F}$; (iii); if every filtering $\mathcal{G} \supseteq \mathcal{F}$ has in turn a filtering $\mathcal{H}$ such that $\mathcal{H} \to x$, then $\mathcal{F} \to x$; and (iv) for all directed sets I and

convergent filters $\mathcal{F}^i \to x^i \in X$, for $i \in I$, if $\mathcal{F}_{i\mapsto x^i} \to x^\infty$, then $\mathcal{F}_\infty \to x^\infty$.

STEP 2: DEFINE THE NOTION OF AN *ADHERENT POINT*
For a subset $S \subseteq X$ and $x \in X$, we say that x is an *adherent point*[a] iff there is a filter base $\mathcal{F}$ such that $F \subseteq S$ for all $F \in \mathcal{F}$ with $\mathcal{F} \to x_\infty$.

STEP 3: DEFINE WHAT SHOULD BE THE CLOSURE
For a subset $S \subseteq X$, we define $\mathrm{C}(S)$ to be the set of adherent points of S.

STEP 4: SHOW THAT IF $S \subseteq T$, THEN $\mathrm{C}(S) \subseteq \mathrm{C}(T)$
Suppose that $S \subseteq T$. Let $x \in \mathrm{C}(S)$. Then, there is a filter base $\mathcal{F}$ such that $F \subseteq S$ for all $F \in \mathcal{F}$ and such that $\mathcal{F} \to x$. As $S \subseteq T$, every $F \in \mathcal{F}$ also satisfies $F \subseteq T$, and so of course we have that $x \in \mathrm{C}(T)$ as well.

STEP 5: SHOW THAT C SATISFIES KURATOWSKI'S CLOSURE AXIOMS
The empty-set has no adherent points, and so trivially $\mathrm{C}(\emptyset) = \emptyset$.

For $x \in S$, $\mathcal{P}_x \to x$. Furthermore, $\{\{x\}\}$ is a filtering of $\mathcal{P}_x$, and so, $\{\{x\}\} \to x$, and hence x is an adherent point of S, that is, $x \in \mathrm{C}(S)$, so that $S \subseteq \mathrm{C}(S)$.

We wish to show that $\mathrm{C}(\mathrm{C}(S)) \subseteq \mathrm{C}(S)$. So, let $x^\infty \in \mathrm{C}(\mathrm{C}(S))$. Then, there is a filter base $\mathcal{F}$ such that $F \subseteq \mathrm{C}(S)$ for all $F \in \mathcal{F}_\infty$ with $\mathcal{F} \to x^\infty$. For each $F \in \mathcal{F}$, let $x_F \in F$, so that $\mathcal{F} \ni F \mapsto x_F$ is a net. Note that $\mathcal{F}_{F\mapsto x_F} \supseteq \mathcal{F}$, so that $\mathcal{F}_{F\mapsto x_F} \to x^\infty$. As each $x_F \in \mathrm{C}(S)$, for each $F \in \mathcal{F}$, there is a filter base $\mathcal{G}^F$ such that $G \subseteq S$ for all $G \in \mathcal{G}^F$ with $\mathcal{G}^F \to x_F$. By (iv), we then have that

$$\begin{aligned}\mathcal{F}_\infty := \{U \subseteq X :\ & \text{there exists } F_U \in \mathcal{F} \text{ such that},\\ & \text{whenever } F \subseteq F_u,\ U \supseteq G^F \text{ for some} \\ & G^F \in \mathcal{G}^F.\} \to x^\infty.\end{aligned} \tag{3.4.2.7}$$

Note that, as each $G^F \subseteq U$, if $U \in \mathcal{F}_\infty$, then so is $U \cap S$. It follows that

$$\mathcal{G}_\infty := \{U \cap S : U \in \mathcal{F}_\infty\} \tag{3.4.2.8}$$

is a filter base that is a filtering of $\mathcal{F}_\infty$. As $\mathcal{F}_\infty \to x_\infty$, it follows that $\mathcal{G}_\infty \to x_\infty$. As $G \subseteq S$ for all $G \in \mathcal{G}_\infty$, it follows that $x_\infty \in \mathrm{C}(S)$, as desired.

We now show that $\mathrm{C}(S \cup T) = \mathrm{C}(S) \cup \mathrm{C}(T)$. By Step 4, we have that $\mathrm{C}(S) \subseteq \mathrm{C}(S \cup T)$. Likewise for T, and so $\mathrm{C}(S) \cup \mathrm{C}(T) \subseteq \mathrm{C}(S \cup T)$.

We now show that $\mathrm{C}(S \cup T) \subseteq \mathrm{C}(S) \cup \mathrm{C}(T)$. So, let $x \in \mathrm{C}(S \cup T)$, so that there is a filter base $\mathcal{F}$ such that $F \subseteq S \cup T$ for all $F \in \mathcal{F}$ with $\mathcal{F} \to x_\infty$. If $F \cap S = \emptyset$ for all $F \in \mathcal{F}$, then $F \subseteq T$ for all $F \in \mathcal{F}$, in which case we will have that $x \in \mathrm{C}(T)$, and we are done. Thus, we may as well assume that $F \cap S$ is nonempty for all $F \in \mathcal{F}$. It follows that $\mathcal{F}_S := \{S \cap F : F \in \mathcal{F}\}$ is a filtering of $\mathcal{F}$. We have that $\mathcal{F}_S \to x$, and so $x \in \mathrm{C}(S)$, as desired.

It follows by Kuratowski's Closure Theorem (Theorem 3.4.1.1) that there is a unique topology on X such that $\mathrm{Cls}(S) = \mathrm{C}(S)$.

Step 6: Show that $\mathcal{F}$ converges to x_∞ iff $\mathcal{F} \to x_\infty$

Suppose that $\mathcal{F}$ converges to x_∞. We wish to show that $\mathcal{F} \to x_\infty$. We proceed by contradiction: suppose that it is not the case that $\mathcal{F} \to x_\infty$. Then, there must be some filtering $\mathcal{G}$ of $\mathcal{F}$ that has no filtering $\mathcal{H}$ such that $\mathcal{H} \to x_\infty$. To obtain a contradiction, we construct such a filtering. As $\mathcal{F}$ converges to x_∞, so does $\mathcal{G}$. For every $G \in \mathcal{G}$, let $x_G \in G$, so that $\mathcal{G} \ni G \mapsto x_G \in X$ is a net, and furthermore, $\mathcal{F}_{G \mapsto x_G}$ is a filtering of $\mathcal{G}$, and so likewise converges to x_∞. Hence, $G \mapsto x_G$ converges to x_∞. For each $G_0 \in G$, define $S_{G_0} := \{x_G : G \supseteq G_0\}$. As $G \mapsto x_G$ converges to x_∞, it follows that $x_\infty \in \mathrm{Cls}(S_{G_0})$ for all $G_0 \in \mathcal{G}$. Hence, by the definition of our closure, for each

$G \in \mathcal{G}$, there is a filter base $\mathcal{H}_G$ such that $H \subseteq S_G$ for all $H \in \mathcal{H}_G$ and $\mathcal{H}_G \to x_\infty$. From (iv), it follows that

$$\begin{aligned}\mathcal{H}_\infty := \{U \subseteq X : \text{ there exists } G_U \in \mathcal{G} \text{ such that },\\ \text{whenever } G \subseteq G_U,\ U \supseteq H_G \text{ for some}\\ H_G \in \mathcal{H}_G.\} \to x_\infty.\end{aligned} \tag{3.4.2.9}$$

Note that, as $H_G \subseteq S_G \subseteq G$, if $U \in \mathcal{H}_\infty$, then $U \cap G \in \mathcal{H}_\infty$ for all $G \in \mathcal{G}$. Hence,

$$\mathcal{K}_\infty := \{U \cap G : G \in \mathcal{G}\} \tag{3.4.2.10}$$

is a filtering of $\mathcal{H}_\infty$, and hence $\mathcal{K}_\infty \to x_\infty$. However, it is likewise a filtering of $\mathcal{G}$, and so we have our contradiction.

Now suppose that $\mathcal{F} \to x$. We proceed by contradiction: suppose that $\mathcal{F}$ does not converge to x. Then, there is a filtering $\mathcal{G}$ of $\mathcal{F}$ which has no filtering converging to x_∞. In particular, $\mathcal{G}$ itself does not converge to x_∞, and so there is an open neighborhood U of x_∞ such that $G \nsubseteq U$ for all $G \in \mathcal{G}$. So, for each $G \in \mathcal{G}$, let $x_G \in G \setminus U$. $G \mapsto x_G$ is then a net contained in U^{C}. On the other hand, $\mathcal{H} := \left\{F \cap U^{\mathsf{C}} : F \in \mathcal{F}_{G \mapsto x_G}\right\}$ is a filtering of $\mathcal{G}$, and hence of $\mathcal{F}$, and so $\mathcal{H} \to x$. As $H \subseteq U^{\mathsf{C}}$ by construction for all $H \in \mathcal{H}$, it follows that $x \in \mathrm{Cls}(U^{\mathsf{C}}) = U^{\mathsf{C}}$: a contradiction.

Step 7: Demonstrate uniqueness

Note that $x \in X$ is an adherent point of $S \subseteq X$ iff there is a filter base $\mathcal{F}$ such that $F \subseteq S$ for all $F \in \mathcal{F}$ with $\mathcal{F}$ converging to x. Recall that sets are closed iff they contain all their adherent points.[b] If we have two topologies with the same notion of filter convergence, then the adherent points of a given set in each topology are the same, and consequently, a set is closed in one iff it is closed in the other. ■

[a] This of course will turn out to agree with the notion of adherent point for the topology.

[b] Because an adherent point of S is either an element of S of an accumulation point of S.

And of course, there is an analogous result to Proposition 3.4.2.2 which gives an axiom equivalent to the second and third.

Proposition 3.4.2.11 Let X be a set, denoted by $\tilde{\mathcal{F}}$ the collection of all filter bases in X, and let $\to$ be a relation on $\tilde{\mathcal{F}} \times X$. Then, the following are equivalent.

(i). $\mathcal{F} \to x$ iff for every filtering $\mathcal{G}$ of $\mathcal{F}$, there is some filtering $\mathcal{H}$ of $\mathcal{G}$ such that $\mathcal{H} \to x$.

(ii). (a). If $\mathcal{F} \to x$ and $\mathcal{G}$ is a filtering of $\mathcal{F}$, then $\mathcal{G} \to x$; and

(b). If every filtering $\mathcal{G} \supseteq \mathcal{F}$ has in turn a filtering $\mathcal{H}$ such that $\mathcal{H} \to x$, then $\mathcal{F} \to x$.

Proof. ($(i) \Rightarrow (ii)$) Suppose that $\mathcal{F} \to x$ iff for every filtering $\mathcal{G}$ of $\mathcal{F}$, there is some filtering $\mathcal{H}$ of $\mathcal{G}$ such that $\mathcal{H} \to x$.

We first check (ii)(a). So, suppose that $\mathcal{F} \to x$ and let $\mathcal{G}$ be a filtering of $\mathcal{F}$. We wish to show that $\mathcal{G} \to x$. To show this, let $\mathcal{H}$ be a filtering of $\mathcal{G}$. This is likewise a filtering of $\mathcal{F}$, and so, by hypothesis, this has a filtering $\mathcal{I}$ such that $\mathcal{I} \to x$. Thus, we have shown that every filtering of $\mathcal{G}$ has in turn a filtering $\mathcal{I}$ such that $\mathcal{I} \to x$. Thus, by hypothesis, $\mathcal{G} \to x$. This shows (ii)(a).

We now check (ii)(b). So, suppose that every filtering $\mathcal{G} \supseteq \mathcal{F}$ has in turn a filtering $\mathcal{H}$ such that $\mathcal{H} \to x$. We wish to show that $\mathcal{F} \to x$. To do that, we show that *every* filtering $\mathcal{G}$ of $\mathcal{F}$ (not just those filter bases which are supersets of $\mathcal{F}$) have a filtering $\mathcal{H}$ such that $\mathcal{H} \to x$.

So, let $\mathcal{G}$ be a filtering of $\mathcal{F}$. Define $\mathcal{H} := \mathcal{G} \cup \mathcal{F}$. We first check that this is in fact a filter base. So, let $U_1, U_2 \in \mathcal{H}$. If both U_1 and U_2 lie in either $\mathcal{G}$ or $\mathcal{F}$, then certainly there will be some $U_3 \in \mathcal{H}$ with $U_3 \subseteq U_1, U_2$ (simply because $\mathcal{G}$ and $\mathcal{F}$ are filter bases). So, without loss of generality suppose that $U_1 \in \mathcal{G}$ and $U_2 \in \mathcal{F}$. As $\mathcal{G}$ is a filtering of $\mathcal{F}$, there is some $V \in \mathcal{G}$ such that $V \subseteq U_2$. As $\mathcal{G}$ is a filter base, there is some $U_3 \in \mathcal{G}$ such that $U_3 \subseteq U_1, V$, and hence $U_3 \subseteq U_1, U_2$.

Thus, $\mathcal{H}$ is indeed a filter base. Furthermore, $\mathcal{H} \supseteq \mathcal{F}$, and so, by hypothesis, has a filter $\mathcal{I}$ such that $\mathcal{I} \to x$. However, $\mathcal{H} \supseteq \mathcal{G}$, and so certainly is a filtering of $\mathcal{G}$, so that in turn $\mathcal{I}$ is a filtering of $\mathcal{G}$. Thus, we have shown that every filtering $\mathcal{G}$ of $\mathcal{F}$ has in turn a filtering $\mathcal{I}$ such that $\mathcal{I} \to x$, and hence, by hypothesis, $\mathcal{F} \to x$, as desired.

((*ii*) $\Rightarrow$ (*i*)) Suppose that (a) if $\mathcal{F} \to x$ and $\mathcal{G}$ is a filtering of $\mathcal{F}$, then $\mathcal{G} \to x$; and (b) if every filtering $\mathcal{G} \supseteq \mathcal{F}$ has in turn a filtering $\mathcal{H}$ of such that $\mathcal{H} \to x$, then $\mathcal{F} \to x$. The ($\Leftarrow$) direction (i) is true by hypothesis, so it suffices to show the ($\Rightarrow$) direction of (i). So, suppose that $\mathcal{F} \to x$ and let $\mathcal{G}$ be a filtering. Then, by hypothesis, we also have that $\mathcal{G} \to x$, and so this filtering has in turn a filtering (namely itself) that is related to x by $\to$. ∎

3.4.3 Definition by specification of continuous functions

The following two results define a topology on a set by simply 'declaring' that a collection of functions be continuous. This is similar in nature to how we defined a topology with a generating collection (Proposition 3.1.1.13)—in that case, we started with a collection of subsets, and simply 'declared' them to be open—here, we start with a collection of *functions*, and simply 'declare' them to be continuous.

Proposition 3.4.3.1 — Initial topology Let X be a set, let $\mathscr{Y}$ be an indexed collection of topological spaces, and for each $Y \in \mathscr{Y}$ let $f_Y : X \to Y$ be a function. Then, there exists a unique topology $\mathscr{U}$ on X, the ***initial topology*** with respect to $\{f_Y : Y \in \mathscr{Y}\}$, such that

(i). $f_Y : X \to Y$ is continuous with respect to $\mathscr{U}$ for all $Y \in \mathscr{Y}$; and

(ii). if $\mathscr{U}'$ is another topology for which each $f_Y : X \to Y$ is continuous, then $\mathscr{U} \subseteq \mathscr{U}'$.

Furthermore, if $\mathscr{S}_Y$ generates the topology on Y, then the collection

$$\{f_Y^{-1}(U) : Y \in \mathscr{Y},\ U \in \mathscr{S}_Y\} \tag{3.4.3.2}$$

generates $\mathscr{U}$.

R In other words, the initial topology is the smallest topology for which each f_Y is continuous.

R But what about the largest such topology? Well, the largest such topology is always going to be the discrete topology, which is not very interesting. This is how you remember whether the initial topology is the smallest or largest—it can't be the largest because the discrete topology always works.

R In particular,

$$\{f_Y^{-1}(V) : Y \in \mathscr{Y},\ V \subseteq Y \text{ open}\} \tag{3.4.3.3}$$

generates the initial topology.

W Warning: $\{f_Y^{-1}(U) : Y \in \mathscr{Y},\ U \in \mathscr{S}_Y\}$ need not be a base for the topology even if each $\mathscr{S}_Y$ is (indeed, or even if each $\mathscr{S}_Y$ is the topology of Y itself)—see, for example, the Product topology (Proposition 3.5.3.2).

R Compare this with the definition of the integers, rationals, reals, closure, interior, and generating collections (Theorems 1.2.1, 1.3.4 and 1.4.2.9 and Propositions 3.1.1.13, 3.2.35 and 3.2.39).

Proof. For $Y \in \mathscr{Y}$, let $\mathscr{S}_Y$ be any generating collection, define

$$\mathscr{S} := \{f_Y^{-1}(V) : Y \in \mathscr{Y},\ V \in \mathscr{S}_Y\}, \tag{3.4.3.4}$$

and let $\mathscr{U}$ be the topology generated by $\mathscr{S}$ (Proposition 3.1.1.13).

As every element of $\mathscr{S}$ is open, it is certainly the case that each f_Y is continuous by Exercise 3.1.3.4 (it suffices to check continuity on a generating collection). On the other hand, if $\mathscr{U}'$ is another topology for each each f_Y is continuous, then it must certainly contain $\mathscr{S}$, in which case $\mathscr{U}'$ contains $\mathscr{U}$ by the definition of a generating collection.

Exercise 3.4.3.5 Show that the initial topology is unique.

■

There is a nice characterization of convergence in initial topologies.

Proposition 3.4.3.6 Let X have the initial topology with respect to the collection $\{f_Y : Y \in \mathscr{Y}\}$, let $\lambda \mapsto x_\lambda \in X$ be a net, and let $x_\infty \in X$. Then, $\lambda \mapsto x_\lambda$ converges to x_∞ in X iff $\lambda \mapsto f_Y(x_\lambda)$ converges to $f_Y(x_\infty)$ in Y for all $Y \in \mathscr{Y}$. Furthermore, the initial topology is the unique topology that has this property.

Proof. ($\Rightarrow$) Suppose that $\lambda \mapsto x_\lambda$ converges to x_∞ in X. Then, because each f_Y is continuous, it follows that $\lambda \mapsto f_Y(x_\lambda)$ converges to $f_Y(x_\infty)$ in Y for all $Y \in \mathscr{Y}$.

($\Leftarrow$) Suppose that $\lambda \mapsto f_Y(x_\lambda)$ converges to $f_Y(x_\infty)$ in Y for all $Y \in \mathscr{Y}$. To show that $\lambda \mapsto x_\lambda$ converges to x_∞ in X, we apply Exercise 3.2.1.4 (it suffices to check convergence on a generating collections). We know that

$$\{f_Y^{-1}(U) : Y \in \mathscr{Y},\ U \subseteq Y \text{ open}\} \tag{3.4.3.7}$$

generates the initial topology, and so we wish to show that $\lambda \mapsto x_\lambda$ is eventually contained in $f_Y^{-1}(U)$. However, as

$f_Y^{-1}(U)$ is an open neighborhood of x_∞, then U is an open neighborhood of $f_Y(x_\infty)$,[a] and so $\lambda \mapsto f_Y(x_\lambda)$ is eventually contained in U because $\lambda \mapsto f_Y(x_\lambda)$ converges to $f_Y(x_\infty)$. But then $\lambda \mapsto x_\lambda$ is eventually contained in $f_Y^{-1}(U)$, and we are done.

Uniqueness follows from the uniqueness stated in Kelley's Convergence Theorem. ■

[a] Of course we already knew that U was open—the point is that $x_\infty \in U$.

Initial topologies have the nice property that you can determine whether functions into X are continuous or not by looking at their composition with each f_Y.

Proposition 3.4.3.8 Let X have the initial topology with respect to the collection $\{f_Y : Y \in \mathscr{Y}\}$, let Z be a topological space, and let $f : Z \to X$ be a function. Then, f is continuous iff $f_Y \circ f$ is continuous for all $Y \in \mathscr{Y}$. Furthermore, the initial topology is the unique topology with this property.

Proof. ($\Rightarrow$) Suppose that f is continuous. Then, because each f_Y is itself continuous and compositions of continuous functions are continuous, it follows that $f_Y \circ f$ is continuous for all $Y \in \mathscr{Y}$.

($\Leftarrow$) Suppose that $f_Y \circ f$ is continuous for all $Y \in \mathscr{Y}$. Let $\lambda \mapsto x_\lambda$ converge to $x_\infty \in X$. To show that f is continuous, it suffices to show that $\lambda \mapsto f(x_\lambda)$ converges to $f(x_\infty)$. However, because each $f_Y \circ f$ is continuous, $\lambda \mapsto f_Y(f(x_\lambda))$ converges to $f_Y(f(x_\infty))$. Therefore, by the previous result, we do indeed have that $\lambda \mapsto f(x_\lambda)$ converges to $f(x_\infty)$.

To show uniqueness, we apply the previous proposition. We wish to show that if a topology on X has the property that f is continuous iff each $f_Y \circ f$ is continuous, then it also has the property that $\lambda \mapsto x_\lambda$ converges to x_∞ iff each $\lambda \mapsto f(x_\lambda)$ converges to $f(x_\infty)$. As the identity function id_X is continuous, each $f_Y = f_Y \circ \mathrm{id}_X$ is continuous, and so if $\lambda \mapsto x_\lambda$ converges to x_∞, then certainly each $\lambda \mapsto f_Y(x_\lambda)$ converges to $f_Y(x_\infty)$.

Conversely, suppose that each $\lambda \mapsto f_Y(x_\lambda)$ converges to $f_Y(x_\infty)$. Denote the index set of $\lambda \mapsto x_\lambda$ by Λ and define $\Lambda_\infty := \Lambda \sqcup \{\infty\}$ and extend the order on Λ to Λ_∞ be declaring that $\lambda \leq \infty$ for all $\lambda \in \Lambda$. Λ_∞ is then a directed set. Define $\mathcal{B} := \{(\lambda, \infty] \subseteq \Lambda_\infty : \lambda \in \Lambda\} \cup \{\{\lambda\} \subseteq \Lambda_\infty : \lambda \in \Lambda\}$. As Λ is directed, this is a base, and so defines a unique topology on Λ_∞ (Proposition 3.1.1.3). Furthermore, the topology is defined in such a way that $\Lambda \ni \lambda \mapsto \lambda \in \Lambda_\infty$ converges to $\infty \in \Lambda_\infty$.

Extend $x\colon \Lambda \to X$ to a function on Λ_∞ be defining $x(\infty) := x_\infty$. As $\lambda \mapsto \lambda \in \Lambda_\infty$ converges to infinity, if we can show that $x\colon \Lambda_\infty \to X$ is continuous, we will have that $\lim x_\lambda := \lim_\lambda x(\lambda) = x(\infty) := x_\infty$, as desired. By hypothesis, it suffices to show that $f_Y \circ x$ is continuous for all $Y \in \mathcal{Y}$. The hypothesis that $\lambda \mapsto f_Y(x_\lambda)$ converges to $f_Y(x_\infty)$ for all $Y \in \mathcal{Y}$ tells us that $f_Y \circ x\colon \Lambda_\infty \to Y$ is continuous at $\infty \in \Lambda_\infty$. On the other hand, for $\lambda \in \Lambda$, as $\{\lambda\}$ is an open neighborhood of λ, every net in Λ_∞ converging to λ must be eventually constant, and so $f_Y \circ x$ is vacuously continuous at every other point. Hence, $f_Y \circ x$ is continuous, and we are done. ■

There is a 'dual' version of the initial topology, in which the functions map *into* the set on which we would like to define a topology.

Proposition 3.4.3.9 — Final topology Let X be a set, let $\mathcal{Y}$ be an indexed collection of topological spaces, and for each $Y \in \mathcal{Y}$ let $f_Y : Y \to X$ be a function. Then, there exists a unique topology $\mathcal{U}$ on X, the ***final topology*** with respect to $\{f_Y : Y \in \mathcal{Y}\}$, such that

(i). $f_Y : Y \to X$ is continuous with respect to $\mathcal{U}$ for all $Y \in \mathcal{Y}$; and (ii). if $\mathscr{U}'$ is another topology for which each f_Y is continuous, then $\mathscr{U} \supseteq \mathscr{U}'$.

Furthermore,

$$\mathscr{U} = \{U \in 2^X : f_Y^{-1}(U) \text{ is open for all } Y \in \mathscr{Y}.\}. \quad (3.4.3.10)$$

R In other words, the final topology is the largest topology for which each f_Y is continuous.

R But what about the smallest such topology? Well, the smallest such topology is always going to be the indiscrete topology, which is not very interesting. This is how you remember whether the final topology is the smallest or largest—it can't be the smallest because the indiscrete topology always works.

Proof. Define

$$\mathscr{U} := \{U \in 2^X : f_Y^{-1}(U) \text{ is open for all } Y \in \mathscr{Y}.\}. \quad (3.4.3.11)$$

Exercise 3.4.3.12 Check that $\mathscr{U}$ is actually a topology.

R We didn't need to do any such checking in the construction of the initial topology because there we just took the topology *generated* by the collection.

Of course every f_Y is continuous with respect to $\mathscr{U}$ as, by definition, the preimage of every element of $\mathscr{U}$ is open. Furthermore, anything larger than $\mathscr{U}$ would necessarily have to contain some set for which the preimage under some f_Y would not be open, and so that f_Y would not be continuous.

Exercise 3.4.3.13 Show that the final topology is unique.

■

There is likewise a 'dual' result to Proposition 3.4.3.8 which tells us when continuous functions *on* a space equipped with a final topology are continuous.[7]

Proposition 3.4.3.14 Let X have the final topology with respect to the collection $\{f_Y : Y \in \mathscr{Y}\}$, let Z be a topological space, and let $f \colon X \to Z$ be a function. Then, f is continuous iff $f \circ f_Y$ is continuous for all $Y \in \mathscr{Y}$. Furthermore, the final topology is the unique topology with this property.

Proof. ($\Rightarrow$) Suppose that f is continuous. Then, because each f_Y is itself continuous and compositions of continuous functions are continuous, it follows that $f \circ f_Y$ is continuous for all $Y \in \mathscr{Y}$.

($\Leftarrow$) Suppose that $f \circ f_Y$ is continuous for all $Y \in \mathscr{Y}$. Let $U \subseteq Z$ be open. We must show that $f^{-1}(U)$ is open. However, from (3.4.3.10), we know that $f^{-1}(U)$ will be open iff $f_Y^{-1}\left(f^{-1}(U)\right)$ will be open for all $Y \in \mathscr{Y}$. However, $f_Y^{-1}\left(f^{-1}(U)\right) = [f \circ f_Y]^{-1}(U)$ is open because $f \circ f_Y$ is continuous.

Let $\mathscr{U}$ be another topology that has the property that $f \colon X \to Z$ is continuous iff $f \circ f_Y$ is continuous for all $Y \in \mathscr{Y}$. To show that $\mathscr{U}$ is the final topology, by the definition (Proposition 3.4.3.9), it suffices to show that each $f_Y : Y \to X$ is continuous with respect to $\mathscr{U}$ and that any other such topology is contained in $\mathscr{U}$. As $\mathrm{id}_X : \langle X, \mathscr{U}\rangle \to \langle X, \mathscr{U}\rangle$ is continuous, by hypothesis, it follows that $\mathrm{id}_X \circ f_Y = f_Y$ is

[7] I am not aware of a 'dual' to Proposition 3.4.3.6 that characterizes convergence in final topologies.

continuous with respect to $\mathscr{U}$. Let $\mathscr{U}'$ be another topology such that f_Y is continuous with respect to $\mathscr{U}'$ for all $Y \in \mathscr{Y}$. We must show that $\mathscr{U}' \subseteq \mathscr{U}$. To show this, it suffices to show that $\mathrm{id}_X : \langle X, \mathscr{U} \rangle \to \langle X, \mathscr{U}' \rangle$ is continuous. By the defining property of $\mathscr{U}$, to show this, it suffices to show that the composition of this with each f_Y is continuous. In other words, it suffices to show that each f_Y is continuous with respect to $\mathscr{U}'$, but this is true by hypothesis. ■

3.4.4 Summary

We now quickly recap all the ways in which we know how to specify a topology on a set.

(i). We can specify the open sets (Definition 3.1.1)..
(ii). We can specify the closed sets (Exercise 3.1.2).
(iii). We can specify a base (Definition 3.1.1.1).
(iv). We can specify a neighborhood base (Definition 3.1.1.6).
(v). We can generate a topology (Proposition 3.1.1.13).
(vi). We can define the closure of each set (Theorem 3.4.1.1).
(vii). We can define the interior of each set (Theorem 3.4.1.7).
(viii). We can define convergence of nets (Theorem 3.4.2.1).
(ix). We can define convergence of filters (Theorem 3.4.2.5).
(x). We can declare that functions on the space are continuous (the initial topology—see Proposition 3.4.3.1).
(xi). We can declare that functions into the space are continuous (the final topology—see Proposition 3.4.3.9.

3.5 New topologies from old

The purpose of this section is to present several ways of constructing new topologies spaces from old. In brief, the subspace topology will be the topology we put on subsets of topological spaces, the quotient topology will be the topology we put on quotients of topological spaces[8], the product topology is the topology we put on Cartesian-products, the disjoint-union topology (surprise, surprise) is

[8]Quotients in the sense of Definition A.3.2.16.

the topology we put on disjoint-unions of topological spaces. They key to defining all of these topologies are the initial (for the subspace and product topologies) and final topologies (for the quotient and disjoint-union topologies) (Propositions 3.4.3.1 and 3.4.3.9).

3.5.1 The subspace topology

All *subsets* of topological spaces have a canonically associated topology, called the *subspace topology*. Note that this is not completely immediate—for example, it is not the case that every subset of a ring is a ring. There is definitely something to define and something to check (that the subspace topology is in fact a topology).

Proposition 3.5.1.1 — Subspace topology Let X be a topological space and let $S \subseteq X$. Then, there exists a unique topology $\mathscr{U}$ on S, the ***subspace topology***, that has the property that a function into S is continuous iff it is continuous regarded as a function into X. Furthermore, the subspace topology is the initial topology with respect to the inclusion $\iota : S \hookrightarrow X$. In particular,

$$\mathscr{U} = \{U \cap S : U \subseteq X \text{ is open.}\}. \tag{3.5.1.2}$$

R One fact to take note of is that, if $U \subseteq S$ is open in the subspace topology, then there is some open $U' \subseteq X$ such that $U = U' \cap S$. Similarly, if $C \subseteq S$ is closed in the subspace topology, then there is some closed $C' \subseteq X$ such that $C = C' \cap S$.

W Warning: Just because $U \subseteq S$ is open in S, does *not* mean that U is open in X—see Example 3.5.1.4.

R Unless otherwise stated, subsets of topological spaces are always equipped with the subspace topology.

Proof. All of this follows from the definition of the initial topology and its characterization of continuity of functions into initial topologies (Propositions 3.4.3.1 and 3.4.3.8), with exception of the fact that $\mathscr{U} = \{U \cap S : U \subseteq X \text{ is open.}\}$. Proposition 3.4.3.1 tells us that this generates the initial topology, but it does not tell us that it is a topology itself. Of course, however, if a generating collection is itself a topology, then the topology it generates is just itself. Therefore, it suffices just to check that $\{U \cap S : U \subseteq X \text{ is open.}\}$ is in fact a topology.

Exercise 3.5.1.3 Check that $\{U \cap S : U \subseteq X \text{ is open.}\}$ is in fact a topology.

■

Example 3.5.1.4 Consider the subspace topology on $[0,1] \subseteq \mathbb{R}$. Note that $(\frac{1}{2},1] = (\frac{1}{2},\infty) \cap [0,1]$ is *open in* $[0,1]$, despite the fact that it obviously not open in $\mathbb{R}$.

Example 3.5.1.5 Note that the order topology on $\mathbb{Q}$ is the space as the subspace topology inherited from $\mathbb{R}$. This is of course because they are both equipped with the order topology with respect to the same order. Likewise, for $\mathbb{N} \subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq \mathbb{Q}$.

Here is a neat little application of the subspace topology that a priori doesn't seem like it would have to make use of the subspace topology at all.[9]

Proposition 3.5.1.6 Let X be a topological space, let $K \subseteq X$ be quasicompact, and let $C \subseteq X$ be closed. Then, $K \cap C$ is quasicompact and closed.

[9] And indeed, you can probably quite easily find a proof that doesn't use it.

Proof. $K \cap C$ is closed in the subspace topology of K, and hence is quasicompact in K by Exercise 3.2.55.

Now, let $\mathscr{U}$ be an open[a] cover of $K \cap C$. Then, $\{U \cap K : U \in \mathscr{U}\}$ is an open cover of $K \cap C$ in K, and so as $K \cap C$ is quasicompact in K, there are finitely many $U_1, \ldots, U_m \in \mathscr{U}$ such that

$$K \cap C \subseteq (U_1 \cap K) \cup \cdots \cup \cdots (U_m \cap K). \qquad (3.5.1.7)$$

But then certainly

$$K \cap C \subseteq U_1 \cup \cdots \cup U_m, \qquad (3.5.1.8)$$

that is, $\{U_1, \ldots, U_m\}$ is an open subcover of $\mathscr{U}$. ■

[a]Open in X, that is.

3.5.2 The quotient topology

Whenever we have a surjective function from a topological space X onto a set Y, we can use this function and the topology on X to place a topology on Y. Recall (Exercise A.3.2.18) that every surjective function can be viewed as a quotient function—this of course is the etymology of the term "quotient topology".

Proposition 3.5.2.1 Let X be a topological space, let Y be a set, and let $\mathrm{q} : X \to Y$ be surjective. Then, there exists a unique topology $\mathscr{U}$ on Y, the ***quotient topology***, that has the property that a function on Y is continuous iff its composition with q is continuous. Furthermore, the quotient topology is the final topology with respect to $\mathrm{q} : X \to Y$. In particular,

$$\mathscr{U} = \{U \subseteq Y : \mathrm{q}^{-1}(U) \text{ is open.}\}. \qquad (3.5.2.2)$$

R Unless otherwise stated, quotients of topological spaces are always equipped with the quotient topology.

R If you ever hear mathematicians talking about some sort of "gluing" construction, what they're actually doing is constructing a topological space by first defining an equivalence relation $\sim$ on a topological space X (points that are "glued" together are defined to be equivalent), and then equipping the quotient, $X/\sim$,[a] with the quotient topology $X \to X/\sim$. For example, if you take a square[b] and "glue" two opposite sides together,[c] you get a cylinder.

[a]Recall that (Definition A.3.2.16) $X/\sim$ is the set of equivalence classes.

[b]Say, $[0,1] \times [0,1]$ endowed with the product topology (Proposition 3.5.3.2).

[c]Precisely, you would say that two distinct points are equivalent to one another iff they have x-coordinate 0 and x-coordinate 1 respectively and the *same* y-coordinate.

Proof. All of this follows from the definition of the final topology and its characterization in terms of continuity of functions defined on final topologies (Propositions 3.4.3.9 and 3.4.3.14). ■

3.5.3 The product topology

The product topology is the canonical topology we put on a Cartesian-product of topological spaces $X \times Y$. While we do technically use this in places, we use it in ways where we could have gotten-away with not speaking of the product topology per se (for example, the product topology on $\mathbb{R} \times \mathbb{R}$ is the same as the topology defined by ε-balls). One significant reason we go to the trouble of talking about the product topology explicitly is for the proof of producing a *counter-example* to the statement

> A space is quasicompact iff every net has a convergent *cofinal* subnet.[a]
>
> [a]In case you're skimming and didn't read the context, this statement is *false*.

We mentioned when we defined subnets (Definition 3.2.9) that the notion of a cofinal subnet was the more obvious "naive" notion (that is, you just take terms from the original net, subject to the only condition that your indices get arbitrarily large), but that this "naive" notion was insufficient because it didn't allow us to prove certain theorems. They key result, that spaces are quasicompact iff every net has a convergent subnet (Proposition 3.2.58), was precisely the example of a theorem I had in mind.

In brief, the counter-example will be

$$X := \prod_{2^{\mathbb{N}}} \{0,1\}, \tag{3.5.3.1}$$

that is, an uncountable product (precisely, a product over the power set of $\mathbb{N}$) of the two-element set $\{0,1\}$. Obviously $\{0,1\}$ is quasicompact (all finite spaces are), and then a theorem called Tychonoff's Theorem (Theorem 3.5.3.14), which says that *arbitrary* products of quasicompact spaces are quasicompact, will tell us that X is quasicompact.

So before we do anything else then, we must first define the product topology.

Proposition 3.5.3.2 — Product topology Let $\mathscr{X}$ be an indexed collection of topological spaces. Then, there exists a unique topology, the ***product topology***, on $\prod_{X\in\mathscr{X}} X$, that has the property that a function into $\prod_{X\in\mathscr{X}} X$ is continuous iff each component of the function is continuous. Furthermore, the product topology is the initial topology with respect to $\{\pi_X : X \in \mathscr{X}\}$.[a] In particular, if $\mathscr{S}_X$ is a generating collection for the topology on $X \in \mathscr{X}$, then the collection

$$\{\pi_X^{-1}(U) : U \in \mathscr{S}_X\} \tag{3.5.3.3}$$

generates the product topology, so that

$$\mathscr{B} := \left\{\prod_{X\in\mathscr{X}} S_X : S_X \in \mathscr{S}_X \text{ and } \right. \\ \left. \text{all but finitely many } S_X = X\right\} \tag{3.5.3.4}$$

is a base for the product topology.

R The "all but finitely many" phrase is *crucial*. For example, in the space

$$\prod_{\mathbb{N}} \mathbb{R}, \tag{3.5.3.5}$$

that is, a countably-infinite product of $\mathbb{R}$, the set

$$(0,1) \times (0,1) \times (0,1) \times \cdots \tag{3.5.3.6}$$

is *not* even open in the product topology on $\prod_{\mathbb{N}} \mathbb{R}$ (much less an element of any base). The topology in which all sets of the form $U_0 \times U_1 \times U_2 \times \cdots$ with each U_k open (and not necessarily equal to all of $\mathbb{R}$)[b] might be your more naive guess, but it is 'wrong' in the sense that, if we allow things like this, then we lose the property that the continuity of a function is determined by the continuity of the components of the function—see Example 3.5.3.7 below.

R Unless otherwise stated, products of topological spaces are always equipped with the product topology.

[a]Recall (Definition A.3.1.6) that $\pi_X : \prod_{X \in \mathscr{X}} X \to X$ is just the projection.

[b]This is called the ***box topology***).

Proof. All of this follows from the definition of the initial topology, its characterization in terms of continuity of functions into initial topologies, and the defining result of generating collections (Propositions 3.1.1.13, 3.4.3.1 and 3.4.3.8). ■

We mentioned in the remarks of this theorem that if you take as a base sets of the form $\prod_{X \in \mathscr{X}} S_X$ (with $S_X \neq X$ *for all* $X \in \mathscr{X}$ permissible), then you will lose the property that a function into the product is continuous iff each of its components is. We now present a counter-example.

■ **Example 3.5.3.7 — A discontinuous function into the box topology with each component continuous** Define *as a set*

$$X := \prod_{2^{\mathbb{N}}} \mathbb{R} \tag{3.5.3.8}$$

and consider the function

$$\mathrm{id}_X : \langle X, \text{product topology} \rangle \to \langle X, \text{box topology} \rangle. \tag{3.5.3.9}$$

Then, each component of this function is continuous because the component of the identity is just the projection from X onto the corresponding copy of $\mathbb{R}$ (the preimage of $U \subseteq \mathbb{R}$ under this projection is open in the product topology because in fact, according to the theorem, such an element is in a generating collection of the product topology). On the other hand,

$$\prod_{2^{\mathbb{N}}} (0, 1) \tag{3.5.3.10}$$

is open in the box topology (by definition), but not open in the product topology by the theorem above.

There is a relatively useful corollary of the definition of the product topology that characterizes convergence.

Corollary 3.5.3.11 Let $\mathscr{X}$ be an indexed collection of topological spaces, let $\lambda \mapsto x_\lambda \in \prod_{X \in \mathscr{X}} X$ be a net, and let $x_\infty \in \prod_{X \in \mathscr{X}} X$. Then, $\lambda \mapsto x_\lambda$ converges to x_∞ iff $\lambda \mapsto [x_\lambda]_X$ converges to $[x_\infty]_X$ in X for all $X \in \mathscr{X}$.[a]

R In other words, nets converge to an element in a product iff every component converges to the corresponding component of that element.

[a] To clarify, $[x_\infty]_X$ is the X-component of $x_\infty \in \prod_{X \in \mathscr{X}} X$.

Proof. As the product topology on $\prod_{X\in\mathscr{X}} X$ is the initial topology with respect to the projections, $\{\pi_X : X \in \mathscr{X}\}$, this result follows from Proposition 3.4.3.6. ■

Exercise 3.5.3.12 — Closure of a product is the product of the closures Let $\mathscr{X}$ be an indexed collection of topological spaces, and for each $X \in \mathscr{X}$ let $S_X \subseteq X$. Show that $\mathrm{Cls}\left(\prod_{X\in\mathscr{X}} S_X\right) = \prod_{X\in\mathscr{X}} \mathrm{Cls}(S_X)$.

Exercise 3.5.3.13 — Projections are open Let $\mathscr{X}$ be an indexed collection of topological spaces, let $X \in \mathscr{X}$, and let $U \subseteq \prod_{X\in\mathscr{X}} X$ be open. Show that $\pi_X(U) \subseteq X$ is open.

R Functions that have the property that the image of open sets are open are ***open functions***.

Now that we have defined the product topology, we prove a relatively difficult[10] result concerning quasicompactness of products which will allow us to produce the desired counter-example.

Theorem 3.5.3.14 — Tychonoff's Theorem. Let $\mathscr{X}$ be a collection of quasicompact spaces. Then, $\prod_{X\in\mathscr{X}} X$ is quasicompact.

Proof. [a] Recall that the product topology on $\prod_{X\in\mathscr{X}} X$ has a generating collection of the form $\pi_X^{-1}(U_X)$ for $U_X \subseteq X$ open, where $\pi_X : \prod_{X\in\mathscr{X}} X \to X$ is the projection. We apply the Alexander Subbase Theorem (Theorem 3.2.1.5) to this generating collection.

So, let $\mathscr{U}$ be an open cover of $\prod_{X\in\mathscr{X}} X$ of subsets of the form $\pi_X^{-1}(U_X)$. It suffices to show that if no finite subset of $\mathscr{U}$ covers $\prod_{X\in\mathscr{X}} X$, then $\mathscr{U}$ itself does not cover $\prod_{X\in\mathscr{X}} X$.

[10] At least compared to what we've been doing, though really the 'meat' is contained in the Alexander Subbase Theorem (Theorem 3.2.1.5).

Define

$$\mathscr{U}_X := \left\{ U \subseteq X \text{ open} : \pi_X^{-1}(U) \in \mathscr{U} \right\} . \tag{3.5.3.15}$$

If a finite subset of $\mathscr{U}_X$ covered X, then its preimage would cover $\prod_{X \in \mathscr{X}} X$, and so by quasicompactness of X, it follows that $\mathscr{U}_X$ does not cover X, so choose $x_X \in X$ not contained in $\bigcup_{U \in \mathscr{U}_X} U$. Then, the element $x \in \prod_{X \in \mathscr{X}} X$ whose coordinate at $X \in \mathscr{X}$ is x_X is not contained in $\bigcup_{U \in \mathscr{U}} U$, so that $\mathscr{U}$ in fact does not cover $\prod_{X \in \mathscr{X}} X$. ■

[a] Proof adapted from [Kel55, pg. 143].

And finally we are able to present our counter-example.

▪ Example 3.5.3.16 — A quasicompact space with a net that has no convergent *cofinal* subnet [a] Define

$$X := \prod_{S \subseteq \mathbb{N}} \{0, 1\} = \{0, 1\}^{2^{\mathbb{N}}} \tag{3.5.3.17}$$

that is, a product of $2^{\aleph_0}$ copies of the two element set $\{0, 1\}$. In other words, it is the set of all functions from $2^{\mathbb{N}}$ into $\{0, 1\}$—in fact, for the most of this example, we shall think of this space as the collection of functions. This is quasicompact by Tychonoff's Theorem (and because finite sets are quasicompact—see Exercise 3.2.54). On the other hand, we may define a sequence $m \mapsto x_m \in X$ as follows.

$$x_m(S) := \begin{cases} 1 & \text{if } m \in S \\ 0 & \text{if } m \notin S, \end{cases} \tag{3.5.3.18}$$

where $S \subseteq \mathbb{N}$. (x_m is a function from $2^{\mathbb{N}}$ into $\{0, 1\}$, and so $x_m(S)$ is the value of this function at the element $S \in 2^{\mathbb{N}}$.)

We now show that this sequence has no convergent cofinal subnet (necessarily also a sequence). We proceed by contradiction: suppose that there were a cofinal subset $\Lambda' \subseteq \mathbb{N}$,

$\Lambda' = \{m_0, m_1, m_2, \ldots\}$ (with $m_n \leq m_{n+1}$), such that $n \mapsto x_{m_n}$ converges. Then, by Corollary 3.5.3.11 (nets in products converge iff each component does), for each $S \subseteq \mathbb{N}$, the sequence $n \mapsto x_{m_n}(S) \in \{0, 1\}$ would have to converge. Thus, for each $S \subseteq \mathbb{N}$, the sequence $n \mapsto x_{m_n}(S)$ would have to be either eventually 0 or eventually 1. In other words, either (i) for all but finitely many $m_n \in \Lambda'$, $m_n \notin S$; or (ii) for all but finitely many $m_n \in \Lambda'$, $m_n \in S$. In other words, for all $S \subseteq \mathbb{N}$, either (i) there is a cofinite[b] subset of Λ' that is contained in S^{C} or (ii) there is a cofinite subset of Λ' that is contained in S.[c]

So, take $S := \{m_0, m_2, m_4, \ldots\} \subset \Lambda'$. Then, there is some cofinite subset $\Lambda'' \subseteq \Lambda'$ such that either $\Lambda'' \subseteq S$ or $\Lambda'' \subseteq S^{\mathsf{C}}$. In the former case, we have that

$$\text{finite set } = \Lambda' \setminus \Lambda'' \supseteq \Lambda' \setminus S = \{m_1, m_3, m_5, \ldots\} :$$

a contradiction. Thus, we must have that $\Lambda'' \subseteq S^{\mathsf{C}}$, and so

$$\text{finite set } = \Lambda' \setminus \Lambda'' \supseteq \Lambda' \setminus S^{\mathsf{C}} = \{m_0, m_2, m_4, \ldots\} :$$

a contradiction. As both possibilities resulted in a contradiction, this itself is a contradiction, and so our assumption that there was a convergent cofinal subnet must have been incorrect. Therefore, $m \mapsto x_m$ contains no convergent cofinal subnet, despite the fact that X is quasicompact.

[a] This example was shown to me by Eric Wofsey on mathoverflow.net.

[b] Just as cocountable means that the complement is countable, ***cofinite*** means that the complement is finite.

[c] Cofinite in Λ', that is.

3.5.4 The disjoint-union topology

Our inclusion of the disjoint-union topology is mostly because of its obvious duality to the product topology—it feels incomplete not to include it. On the other hand, while in principle, being completely dual to the product topology, it should be no more or less difficult work with, in practice it seems that it is *much* easier to get a handle on, and so probably doesn't deserve as in-depth a treatment.

Proposition 3.5.4.1 — Disjoint-union topology Let $\mathscr{X}$ be an indexed collection of topological spaces. Then, there exists a unique topology, the ***disjoint-union topology***, on $\coprod_{X\in\mathscr{X}} X$, that has the property that a function defined on $\coprod_{X\in\mathscr{X}} X$ is continuous iff its restriction to each component is continuous. Furthermore, the disjoint-union topology is the final topology with respect to $\{\iota_X : X \in \mathscr{X}\}$.[a] In particular, a set $U \subseteq \coprod_{X\in\mathscr{X}} X$ is open iff $U \cap \iota_X(X)$ is open for all $X \in \mathscr{X}$.

R Unless otherwise stated, disjoint-unions of topological spaces are always equipped with the disjoint-union topology.

W Warning: Just because $X = S \cup T$ and $S \cap T = \emptyset$, does *not* mean that X has the disjoint-union topology. For example, by that logic, as every space is the disjoint-union of its points, every space would be discrete. When I point this out to you now, it could very well appear that this deduction is so obviously incorrect as to not be worth mentioning, but I have definitely seen students make what is essentially this very same mistake in contexts where it is not as blatant that this is the (wrong) implication one is using.

[a] Recall (Definition A.3.1.1) that $\iota_X : X \to \coprod_{X\in\mathscr{X}} X$ is just the inclusion.

Proof. All of this follows from the definition of the final topology and its characterization in terms of continuity of functions defined on final topologies (Propositions 3.4.3.9 and 3.4.3.14). ∎

3.6 Separation Axioms

We have mentioned the term "T_2" a couple of times now (see, for example, the definition of quasicompactness (Definition 3.2.53) and the Heine-Borel Theorem (Theorem 2.5.3.3). One of the purposes of this section is to explain what we meant by this. The term "T_2" is a separation axiom, and you should know the other separation axioms as well if you plan to become a mathematician, but this is admittedly not a priority for this course (a study of them in detail is better suited for a course on general topology itself).[11]

3.6.1 Separation of subsets

In this subsection, we will define various levels of "separation" of *subsets* of a topological space. In the next section, we will then define corresponding levels of separation for *spaces*.

Throughout this subsection, let $S_1, S_2 \subseteq X$ be *disjoint* subsets of a topological space X. In the various definitions will follow, we will say things like "S_1 and S_2 are XYZ.". If $S_1 = \{x_1\}$ and $S_2 = \{x_2\}$ are singletons, then instead we will say that "x_1 and x_2 are XYZ.". In fact, this[12] is probably the case of most interest (though certainly not the only case).

Definition 3.6.1.1 — Topologically-distinguishable S_1 and S_2 are ***topologically-distinguishable*** iff there is a neighborhood of S_1 not intersecting S_2 *or* there is a neighborhood of S_2 not intersecting S_1.

W Warning: The term "distinguishable" comes from the fact that, in the case $S_1 = \{x_1\}$ and $S_2 = \{x_2\}$, then x_1 and x_2 being topologically-indistinguishable (i.e. not topologically-distinguishable) means that x_1 and x_2 are contained in precisely the same open sets. Thus, in this sense, the topology cannot tell the difference between x_1 and x_2, they are "indistinguishable". However, if S_1 and S_2 are not singletons, then this

[11]Says the person who decided to include the most detailed account I'm aware of (except for probably [SJ70]) ....

[12]That is, the case when the sets are singletons.

need not be true, that is, the open sets containing S_1 and S_2 need not coincide—see Example 3.6.1.4.

R For example, in an indiscrete space, no two points are topologically-distinguishable. In $\mathbb{R}$ on the other hand (and almost every space you work with that is not expressly cooked-up for the sole purpose of being a counter-example to something), any two points are topologically-distinguishable.

Example 3.6.1.2 — Two distinct points which are not topologically-distinguishable We just mentioned this in the remark above, but decided to place it in an example of its own to make it easier to spot if skimming. Take $X := \{x_1, x_2\}$ and equip X with the indiscrete topology, that is,

$$\mathcal{U} := \{\emptyset, X\}. \tag{3.6.1.3}$$

Then, $x_1 \neq x_2$ but x_1 and x_2 are contained in precisely the same open sets.

Example 3.6.1.4 — Two sets which are topologically-indistinguishable but don't have the same open neighborhoods Define $X := \{x_1, x_2, x_3\}$ and

$$\mathcal{U} := \{\emptyset, X, \{x_1, x_2\}\}. \tag{3.6.1.5}$$

Define $S_1 := \{x_1\}$ and $S_2 := \{x_2, x_3\}$. Then, every neighborhood of S_1 (there are only two, $\{x_1, x_2\}$ and X) intersects S_2. Similarly, every neighborhood of S_2 (there is only one, X) intersects S_1. Thus, S_1 and S_2 are topologically-indistinguishable. On the other hand, $\{x_1, x_2\}$ is an open set which contains S_1 but not S_2.

Definition 3.6.1.6 — Separated S_1 and S_2 are ***separated*** iff there is a neighborhood U_1 of S_1 not intersecting S_2 *and* a neighborhood U_2 of S_2 not intersecting S_1.

R The difference between this and topological-distinguishability is that this has to happen to *both* S_1 and S_2, whereas, in the case of topological-distinguishability, we only require that at least one of them has an open neighborhood that does not intersect the other.

▪ **Example 3.6.1.7 — Two points which are topologically-distinguishable but not separated** Define $X := \{x_1, x_2\}$ and

$$\mathscr{U} := \{\emptyset, X, \{x_1\}\}\,. \tag{3.6.1.8}$$

Then, x_1 and x_2 are topologically-distinguishable as $\{x_1\}$ is an open neighborhood of x_1 that does not contain x_2. On the other hand, every neighborhood of x_2 contains x_1.

R This is the ***Sierpinski Space***.

Definition 3.6.1.9 — Separated by neighborhoods S_1 and S_2 are ***separated by neighborhoods*** iff there is a neighborhood U_1 of S_1 and a neighborhood U_2 of S_2 with U_1 and U_2 disjoint.

R Equivalently, we may replace U_1 and U_2 with open neighborhoods.

R This is just like being separated, except that we may put U_1 around S_1 and U_2 around S_2 '*simultaneously*' and have no intersection, whereas in the separated case, the U_1 and U_2 that 'work' will in general intersect.

■ **Example 3.6.1.10 — Two points which are separated but not separated by neighborhoods** Define $X := \{x_1, x_2, x_3\}$ and

$$\mathcal{U} := \{\emptyset, X, \{x_1, x_3\}, \{x_2, x_3\}, \{x_3\}\}. \tag{3.6.1.11}$$

Then, $\{x_1, x_3\}$ is an open neighborhood of x_1 which does not contain x_2 and $\{x_2, x_3\}$ is an open neighborhood of x_2 which does not contain x_1. On the other hand, every open neighborhood of x_1 intersects every open neighborhood of x_2 at x_3.

Definition 3.6.1.12 — Separated by closed neighborhoods S_1 and S_2 are ***separated by closed neighborhoods*** iff there is a closed neighborhood C_1 of S_1 and a closed neighborhood C_2 of S_2 with C_1 and C_2 disjoint.

R This is the same as being separated by neighborhoods, except that we further require that the neighborhoods are closed.[a]

[a]Recall that neighborhoods do not have to be open—see Definition 3.1.1.5.

■ **Example 3.6.1.13 — Two points which are separated by neighborhoods but not by closed neighborhoods** Define $X := \{x_1, x_2, x_3\}$ and

$$\mathcal{U} := \{\emptyset, X, \{x_1\}, \{x_3\}, \{x_1, x_3\}\}. \tag{3.6.1.14}$$

Then, $\{x_1\}$ is a neighborhood of x_1, $\{x_3\}$ is a neighborhood of x_3, and these two neighborhoods are disjoint, so that x_1 and x_3 are separated by neighborhoods. On the other hand, x_1 only has four neighborhoods: $\{x_1\}$, $\{x_1, x_3\}$, X, and $\{x_1, x_2\}$.[a] Of these, only two are closed, X and $\{x_1, x_2\}$. X certainly intersects every closed neighborhood of x_3, and so, if x_1 and x_3 are to be separated by closed neighborhoods, the closed neighborhood of x_1 must be $\{x_1, x_2\}$. By $1 \leftrightarrow 3$ symmetry, the only closed neighborhood of x_3 that might 'work' is $\{x_2, x_3\}$, however, these two closed neighborhoods intersect, namely at x_2. Therefore, we cannot separate x_1 and x_3 with *closed* neighborhoods.

[a] Note that $\{x_1, x_2\}$ is a neighborhood of x_1, but *not* an open neighborhood.

There is an equivalent, alternative way to think about being separated by closed neighborhoods that you may find useful.

Proposition 3.6.1.15 S_1 and S_2 are separated by closed neighborhoods iff they are separated by open neighborhoods with disjoint closure.

Proof. ($\Rightarrow$) Suppose that S_1 and S_2 are separated by closed neighborhoods C_1 and C_2 respectively. By the definition of a neighborhood (Definition 3.1.1.5), there are then open neighborhoods U_1 and U_2 with $S_1 \subseteq U_1 \subseteq C_1$ and $S_2 \subseteq U_2 \subseteq C_2$. Then, $\mathrm{Cls}(U_1) \subseteq C_1$ and $\mathrm{Cls}(U_2) \subseteq C_2$, and so U_1 and U_2 constitute open neighborhoods of S_1 and S_2 with disjoint closures.

($\Leftarrow$) Suppose that S_1 and S_2 are separated by open neighborhoods with disjoint closure. Then, these closures constitute closed neighborhoods which separate S_1 and S_2. ∎

Definition 3.6.1.16 — Completely-separated S_1 and S_2 are ***completely-separated*** or ***separated by (continuous) functions*** iff there is a continuous function $f\colon X \to [0, 1]$ such that $f|_{S_1} = 0$ and $f|_{S_2} = 1$.

R Why does being completely-separated imply being separated by closed neighborhoods?[a]

[a] All the other implications are true too (i.e. separated implies topologically-distinguishable, separated by neighborhoods implies sep-

arated, etc.), this is just the first one that is not completely obvious, which is why it is the only one we have asked about.

Example 3.6.1.17 — Two points which are separated by closed neighborhoods but not completely-separated [a] Define

$$\begin{aligned} S &:= [(0,1)\times(0,1)]\cap[\mathbb{Q}\times\mathbb{Q}] \\ &= \{\langle x,y\rangle \in (0,1)\times(0,1) : x,y\in\mathbb{Q}\}, \end{aligned} \tag{3.6.1.18}$$

$$\begin{aligned} T &:= \{\tfrac{1}{2}\}\times\{r\sqrt{2}\in(0,1) : r\in\mathbb{Q}\} \\ &= \{\langle\tfrac{1}{2}, r\sqrt{2}\rangle : r\in\mathbb{Q},\ r\sqrt{2}\in(0,1)\} \end{aligned} \tag{3.6.1.19}$$

and

$$X := S\cup T\cup\{\langle 0,0\rangle\}\cup\{\langle 1,0\rangle\}. \tag{3.6.1.20}$$

We define a topology on X by defining a neighborhood base at each point (see Proposition 3.1.1.9). For $\langle x,y\rangle\in X$, there are four cases: (i) $\langle x,y\rangle = \langle 0,0\rangle$, (ii) $\langle x,y\rangle = \langle 1,0\rangle$, (iii) $\langle x,y\rangle\in T$, and (iv) $\langle x,y\rangle\in S$. We define

$$\mathscr{B}_{\langle x,y\rangle} := \begin{cases} \{U\subseteq S : U \text{ is open in } S.^{b}\} & \text{if } \langle x,y\rangle\in S \\ \left\{U^m_{\langle x,y\rangle} : m\in\mathbb{Z}^+\right\} & \text{if } \langle x,y\rangle\in T \\ & \quad\text{or } x = \langle 0,0\rangle; \langle 1,0\rangle, \end{cases} \tag{3.6.1.21}$$

where

$$\begin{aligned} U^m_{\langle 0,0\rangle} &:= \{\langle 0,0\rangle\}\cup\left\{\langle x,y\rangle\in(0,\tfrac{1}{4})\times(0,\tfrac{1}{m}) : x,y\in\mathbb{Q}\right\} \\ U^m_{\langle 1,0\rangle} &:= \{\langle 1,0\rangle\}\cup\left\{\langle x,y\rangle\in(\tfrac{3}{4},1)\times(0,\tfrac{1}{m}) : x,y\in\mathbb{Q}\right\} \\ U^m_{\langle\frac{1}{2},r\sqrt{2}\rangle} &:= \left\{\langle x,y\rangle\in(\tfrac{1}{4},\tfrac{3}{4})\times(r\sqrt{2}-\tfrac{1}{m}, r\sqrt{2}+\tfrac{1}{m}) :\right. \\ &\qquad \left. x,y\in\mathbb{Q}\right\} \text{ for } m\in\mathbb{Z}^+ \text{ with} \\ &\qquad \left(r\sqrt{2}-\tfrac{1}{m}, r\sqrt{2}+\tfrac{1}{m}\right)\subseteq(0,1). \end{aligned} \tag{3.6.1.22}$$

By Proposition 3.1.1.9, there is a unique topology for which $\mathscr{B}_{\langle x,y\rangle}$ is a neighborhood base of $\langle x,y\rangle\in X$.

The closures of $\{\langle 0,0\rangle\} \cup (0,\frac{1}{4}) \times (0,\frac{1}{n})$ and $\{\langle 1,0\rangle\} \cup (\frac{3}{4},1) \times (0,\frac{1}{n})$ in X must be disjoint as, in particular, any point in the former cannot have x-coordinate exceeding $\frac{1}{4}$ and any point in the latter cannot have x-coordinate strictly less than $\frac{3}{4}$.

On the other hand, $\langle 0,0\rangle$ and $\langle 1,0\rangle$ cannot be separated by a function. To see this, suppose that $f\colon X \to \mathbb{R}$ were a continuous function such that $f(\langle 0,0\rangle) = 0$ and $f(\langle 1,0\rangle) = 1$. Then, $f^{-1}([0,\frac{1}{4}))$ would be an open neighborhood of $\langle 0,0\rangle$, and so must contain $U^m_{\langle 0,0\rangle}$ for some $m \in \mathbb{Z}^+$. Similarly, $f^{-1}((\frac{3}{4},1])$ must contain $U^n_{\langle 1,0\rangle}$ for some $n \in \mathbb{Z}^+$. Let $r \in \mathbb{Q}$ be such that $r\sqrt{2} < \min\{\frac{1}{m},\frac{1}{n}\}$. Obviously, $f(\langle \frac{1}{2}, r\sqrt{2}\rangle)$ cannot be in both $[0,\frac{1}{4})$ and $(\frac{3}{4},1]$ as these sets are disjoint, so without loss of generality assume that it is not contained in $[0,\frac{1}{4})$, so let $U \subseteq [0,1]$ be an open neighborhood of $f(\langle \frac{1}{2}, r\sqrt{2}\rangle)$ with $\mathrm{Cls}(U)$ disjoint from $\mathrm{Cls}\left([0,\frac{1}{4})\right)$, so that the preimages of $\mathrm{Cls}(U)$ and $\mathrm{Cls}\left([0,\frac{1}{4})\right)$ are disjoint closed neighborhoods of $\langle \frac{1}{2}, r\sqrt{2}\rangle$ and $\langle 0,0\rangle$ respectively. On the other hand, a closed neighborhood of $\langle \frac{1}{2}, r\sqrt{2}\rangle$ must contain $U^o_{\langle \frac{1}{2}, r\sqrt{2}\rangle}$ for $o \in \mathbb{Z}^+$ with $r\sqrt{2} - \frac{1}{o} > 0$ (and $r\sqrt{2} + \frac{1}{o} < 1$). As $r < \frac{1}{\sqrt{2}m} < \frac{1}{m}$, we have that $\frac{1}{o} < \frac{\sqrt{2}}{m}$, and hence $r\sqrt{2} - \frac{1}{o} < \frac{1}{m} - \frac{1}{o} < \frac{1}{m}$, and so there is some rational $q \in \mathbb{Q}$ with $r\sqrt{2} - \frac{1}{o} < q < \frac{1}{m}$. But then, $\mathrm{Cls}\left(U^m_{\langle 0,0\rangle}\right)$ and $\mathrm{Cls}\left(U^o_{\langle \frac{1}{2}, r\sqrt{2}\rangle}\right)$ must intersect at $\langle \frac{1}{4}, q\rangle$, a contradiction of disjointedness.

R This is the ***Arens Square***.

[a] This is significantly more nontrivial than the preceding counterexamples and comes from [SJ70, pg. 98].

[b] Open in the usual topology (the subspace topology inherited from $(0,1) \times (0,1)$).

Definition 3.6.1.23 — Perfectly-separated S_1 and S_2 are ***perfectly-separated*** or ***precisely-separated*** iff there is a continuous function $f\colon X \to [0,1]$ such that $S_1 = f^{-1}(0)$ and $S_2 = f^{-1}(1)$.

R Being completely-separated means that f is 0 on S_1 and 1 on S_2. Being perfectly-separated means that, furthermore, f is 0 *nowhere else* except on S_1 and f is 1 *nowhere else* except on S_2.

▪ **Example 3.6.1.24 — Two points which are completely-separated but not perfectly-separated** [a] This example is fairly similar to the cocountable topology example—see Example 3.2.45.

Define $X := \mathbb{R}$. Let $C \subseteq X$ and declare that

$$X \text{ is closed iff either (i) } C \text{ contains } 0 \text{ or (ii) } C \text{ is finite.} \tag{3.6.1.25}$$

You can check for yourself that this satisfies the defining conditions for a topology in terms of closed sets (Exercise 3.1.2).

We wish to show that $0, 1 \in X$ are completely-separated, but not perfectly-separated. We first check that they are completely-separated by producing a continuous function $f\colon X \to [0,1]$ such that $f(0) = 0$ and $f(1) = 1$. Define

$$f(x) := \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.6.1.26}$$

We first check that f is continuous. Let $C \subseteq [0,1]$ be closed. If C contains $0 \in [0,1]$, then $f^{-1}(C)$ contains $0 \in X$, and so is closed. If it does not contain 0, then $f^{-1}(C)$ is finite—either C contained 1 in which case $f^{-1}(C) = \{1\}$ or it did not in which case $f^{-1}(C) = \emptyset$.

Now we show that $0, 1 \in X$ are not *perfectly*-separated. Suppose that there exists a continuous function $f\colon X \to [0,1]$

such that $\{0\} = f^{-1}(0)$ and $\{1\} = f^{-1}(1)$. Then,

$$\begin{aligned}\{0\} = f^{-1}(0) &= f^{-1}\left(\bigcap_{m\in\mathbb{Z}^+}[0,\tfrac{1}{m})\right) \\ &=^{b} \bigcap_{m\in\mathbb{Z}^+} f^{-1}\left([0,\tfrac{1}{m})\right).\end{aligned} \tag{3.6.1.27}$$

That is, $\{0\}$ is a G_δ set.[c] However, by definition, a set is open iff it does not contain $0 \in X$ or its complement is finite. Of course, all the sets appearing in (3.6.1.27) must be of the latter kind (because $0 \in f^{-1}\left([0,\tfrac{1}{m})\right)$). However, taking the complement of this equation, we find

$$\mathbb{R} \setminus \{0\}^{d} = \bigcup_{m\in\mathbb{Z}^+} f^{-1}\left([0,\tfrac{1}{m})\right)^{\mathsf{C}}, \tag{3.6.1.28}$$

so that $\mathbb{R} \setminus \{0\}$ is a countably-infinite union of finite sets: a contradiction.

R This is the ***Uncountable Fort Space***.

[a] This comes from [SJ70, pg. 52].
[b] Exercise A.3.27.(ii)
[c] Recall that this is just a fancy-shmancy term for a set which is a countable intersection of open sets—see Definition 3.1.3.
[d] De Morgan's Laws—see Exercise A.2.3.12

Note that we obviously have the implications

perfectly-separated ⇒ completely-separated ⇒ separated by closed neighborhoods ⇒ separated by neighborhoods ⇒ separated ⇒ topologically-distinguishable ⇒ distinct. (3.6.1.29)

Here, "distinct" literally means that S_1 and S_2 are not the same thing, i.e. $S_1 \neq S_2$. Furthermore, we have presented counter-examples after each definition to show that each implication is strict.

3.6.2 Separation axioms of spaces

In the previous subsection, we defined several levels of "separation" between two disjoints subsets of a topological space. We now use these definitions to put conditions on topological spaces themselves.

Throughout this section, X will be a topological space.

This is the first time in these notes that a result is stated 'officially' that refers to material we have not yet covered. For example, we are going to introduce a bunch of properties that a space may or may not possess, and for each of them, we are going to ask the question "Is this preserved under subspaces, quotients, products, and disjoint unions?". In some cases, to answer the questions, we needed to refer to material coming as late as Chapter 6, for example, to refer to the natural log. However, the logical placement of such material is certainly in this section, in parallel with all the other analogous results, and so I didn't feel as if it made pedagogical sense to postpone such results by several chapters just because we don't 'officially' know what ln is yet—the reality is that, if you're reading this, you almost certainly know[a] what ln is, and in any case, the development isn't actually circular, and you can flip to the definition (Definition 6.4.5.37) and come back to the result when you feel comfortable.

[a]Or think you know anyways.

Definition 3.6.2.1 — T_0 X is T_0 iff any two distinct points are topologically-distinguishable.

R Sometimes this condition is called ***Kolmogorov***. You will find that a lot (if not all) of the separation axioms of spaces have other names. We have chosen the names we have because (i) other terminology is less consistent and (ii) it carries less information (the subscript 0 in T_0 has some significance).

R This is an insanely reasonable condition. I might even argue that if you have a space you're trying to study that is not T_0, you're doing something wrong. If the topology cannot distinguish between two points, either (i) you may as well identify those two points (see Proposition 3.6.2.3) or (ii) you should probably consider adding more structure to your space that does distinguish them.

R Apparently the "T" in all these separation axioms is for the German "Trennungsaxiom", or so Munkres ([Mun00, pg. 211]) informs me.

■ **Example 3.6.2.2 — A space which is not T_0** Any indiscrete space with at least two points. The example above in Example 3.6.1.2 worked.

Proposition 3.6.2.3 — T_0 quotient Let X be a topological space. Then, there exists a unique[a] topological space $\mathrm{T0}(X)$, the T_0 ***quotient*** of X, and a surjective map $\mathrm{q} : X \to \mathrm{T0}(X)$ such that

(i). $\mathrm{T0}(X)$ is T_0; and
(ii). if Y is another T_0 space with a continuous map $\phi : X \to Y$, then there is a unique continuous map $\phi' : \mathrm{T0}(X) \to Y$ such that $\phi = \phi' \circ \mathrm{q}$.

R As T_0 is sometimes called Kolmogorov, so to this is sometimes called the ***Kolmogorov quotient***.

R Compare this with the definitions of the integers, rationals, reals, closure, interior, generating collections, initial topology, and final topology (Theorems 1.2.1, 1.3.4 and 1.4.2.9 and Propositions 3.2.35, 3.2.39, 3.4.3.1 and 3.4.3.9). Note how this is a bit different—the key difference here is that the map from X to $\mathrm{T0}(X)$ is now *surjective* (i.e. a quotient map) instead of in all the previous cases where it was *injective* (i.e. an inclusion).

[a] Up to homeomorphism.

Proof. Define $x_1 \sim x_2$ iff the open sets which contain x_1 are precisely the same as the open sets which contain x_2.

Exercise 3.6.2.4 Show that $\sim$ is an equivalence relation.

Define $\mathrm{T0}(X) := X/\sim$ and let $\mathrm{q} : X \to \mathrm{T0}(X)$ be the quotient map (Definition A.3.2.16).

Exercise 3.6.2.5 Show that $\mathrm{T0}(X)$ satisfies (i) and (ii).

Exercise 3.6.2.6 Show that $\mathrm{T0}(X)$ is unique.

■

Exercise 3.6.2.7 Show that a subspace of a T_0 space is T_0.

W Warning: While this might seem like it should be obvious, it's not true for other separation axioms (Example 3.6.2.99), and so there is indeed something to check.

Exercise 3.6.2.8 Show that an arbitrary product of T_0 spaces is T_0.

You'll notice that we didn't ask about quotients or disjoint unions. This is because separation axioms are essentially never preserved under quotients[13] and are essentially always preserved under disjoint union.

[13] For example, $\mathbb{R}$ is perfectly-T_4 (see Definition 3.6.2.115), but it has a quotient which is not even T_0—see Example 3.6.2.9.

■ **Example 3.6.2.9 — A quotient of $\mathbb{R}$ that is not T_0** Consider the partition $\{(-\infty,0]\cup(1,\infty),(0,1]\}$ of $\mathbb{R}$ and let $\sim$ denote the corresponding equivalence relation.[a] We claim that $\mathbb{R}/\sim$ is not T_0.

Recall (Proposition 3.5.2.1) that a set is open in the quotient topology iff its preimage under the quotient map is open. In this case, the quotient has just two points, one represented by $(-\infty,0]\cup(1,\infty)$ and the other represented by $(0,1]$. The preimage of both of these points are just the respective sets, neither of which are open, and hence neither of the points in the quotient is open. As these are the only two nonempty proper subsets, there are no nonempty proper open sets in the quotient, that is, the quotient has the indiscrete topology, which is never T_0.[b]

[a]Recall that partitions define equivalence relations—see Exercise A.3.2.13.

[b]Except on a one point set and the empty-set.

Definition 3.6.2.10 — T_1 X is T_1 iff any two distinct points are separated.

R Sometimes this condition is called ***accessible*** of ***Fréchet***. I also prefer T_1 over "accessible" because, not only does it carry slightly more information, but it's also a lot more common. I would *definitely* recommend not to use the term "Fréchet" to describe this, as the term "Fréchet space" is usually meant to describe something else entirely.

W Warning: This is not the same as "any two topologically-distinguishable points are separated". That condition is called R_0 and is rarely, if ever, used.[a]

R In contrast to the T_0 condition, there are *many* important examples of spaces that are not T_1.

[a]There is no difference between R_0 and T_1 for spaces which are T_0.

■ **Example 3.6.2.11 — A space that is T_0 but not T_1** The Sierpinski Space (Example 3.6.1.7) worked (briefly, it was a two point set with the only nontrivial open set being a singleton).

While the above is probably the best way to state T_1 as a definition because it is phrased in a way that is analogous to other separation conditions, it is often best to think of T_1 spaces are spaces in which points are closed.

Proposition 3.6.2.12 Let X be a topological space. Then, X is T_1 iff $\{x\}$ is closed for all $x \in X$.

Proof. ($\Rightarrow$) Suppose that X is T_1. Let $x \in X$. For all $y \in X$ distinct from x, let U_y be an open neighborhood of y that does not contain x. Then,

$$\bigcup_{y \neq x} U_y = X \setminus \{x\} \tag{3.6.2.13}$$

is open, and so

$$\bigcap_{y \neq x} U_y^{\mathsf{C}} = \{x\} \tag{3.6.2.14}$$

is closed.

($\Leftarrow$) Suppose that $\{x\}$ is closed for all $x \in X$. Let $x_1, x_2 \in X$. Then, $\{x_1\}^{\mathsf{C}}$ is an open neighborhood of x_2 that does not contain x_1 and $\{x_2\}^{\mathsf{C}}$ is an open neighborhood of x_1 that does not contain x_2. ■

Corollary 3.6.2.15 Every finite T_1 topological space is discrete.

R In particular, it is a waste of time to look for counterexamples in finite spaces if your space is T_1.

Proof. If the space is T_1, then every point is closed. If the space is finite, then every subset is a union of finitely many points, that is, a finite union of closed sets, and hence closed. ■

Exercise 3.6.2.16 Show that a subspace of a T_1 space is T_1.

Exercise 3.6.2.17 Show that an arbitrary product of T_1 spaces is T_1.

Definition 3.6.2.18 — T_2 X is T_2 iff any two distinct points are separated by neighborhoods.

> R This is quite often referred to as ***Hausdorff***. In contrast to most of the alternative terminologies, this actually might be more common than the term "T_2".

Definition 3.6.2.19 — Compact X is ***compact*** iff it is quasi-compact and T_2.

Proposition 3.6.2.20 Let X be a topological space. Then, X is T_2 iff limits are unique.

> R Because of lack of uniqueness in general, we have been hesitant to write $\lim_\lambda x_\lambda$—if limits are not unique, then what limit does this symbol refer to? Hereafter, however, in T_2 spaces, we will not hesitate to use this notation.

Proof. ($\Rightarrow$) Suppose that X is T_2. Let $\lambda \mapsto x_\lambda \in X$ be a net and let $x_\infty, x'_\infty \in X$ be limits of X. We proceed by contradiction: suppose that $x_\infty \neq x'_\infty$. Then, there exist disjoint open neighborhoods U of x_∞ and U' of x'_∞. As $\lambda \mapsto x_\lambda$ converges to x_∞, it is eventually contained in U. But then, as U and U' are disjoint, $\lambda \mapsto x_\lambda$ is not eventually contained in U', a contradiction of the fact that $\lambda \mapsto x_\lambda$ converges to x'_∞.

($\Leftarrow$) Suppose that limits are unique. Let $x_\infty, x'_\infty \in X$ be distinct. We wish to show that there exist disjoint open neighborhoods of x_∞ and x'_∞. We proceed by contradiction: suppose that every open neighborhood of x_∞ intersects every open neighborhood of x'_∞. Let $\mathcal{N}_{x_\infty}$ and $\mathcal{N}_{x'_\infty}$ denote the collection of all open neighborhoods of x_∞ and x'_∞ respectively. Order them with respect to reverse-inclusion so as to form directed sets, and equip $\mathcal{N}_{x_\infty} \times \mathcal{N}_{x'_\infty}$ with the product order (Definition 2.4.5.19). By hypothesis, for all $\langle U, U'\rangle \in \mathcal{N}_{x_\infty} \times \mathcal{N}_{x'_\infty}$, there is some $x_{\langle U,U'\rangle} \in U \cap U'$. Then, $\langle U, U'\rangle \mapsto x_{U,U'}$ converges to both x_∞ and x'_∞, and hence, as limits are unique, $x_\infty = x'_\infty$: a contradiction. ■

Exercise 3.6.2.21 Show that quasicompact subsets (which are in fact compact by Exercise 3.6.2.27) of a T_2 space can be separated by neighborhoods.

Exercise 3.6.2.22 Show that quasicompact subsets of a T_2 space are closed.

W Warning: The converse is false[a]—see the following counter-example.

[a]This is in contrast to the T_1 situation in which T_1 *is* equivalent to points being closed.

▪ Example 3.6.2.23 — A space for which every quasi-compact subset is closed that is not T_2 Let X be as in Example 3.2.45, that is the real numbers with the cocountable topology.

Any open neighborhood of 0, must intersect any open neighborhood of 1, because both of these neighborhoods have countable complements and so, by the uncountability of $\mathbb{R}$, must intersect. Therefore, X is not T_2.

We claim that a subset of X is quasicompact iff it is finite. Finite subsets are always quasicompact. On the other hand, take $K \subseteq X$ infinite. Then, there is in particular a countably-infinite subset $\{x_0, x_1, x_2, \ldots\} \subseteq K$. Define $C_m := \{x_k : k \geq m\}$ and $\mathscr{C} := \{C_m : m \in \mathbb{N}\}$. Then, $\mathscr{C}$ is a collection of closed subsets of X. Furthermore, the intersection of any finitely many of them intersects K. On the other hand, the intersection of all of them is empty. Therefore, K is not quasicompact. That is, infinite subsets of X are never quasicompact, and so in fact subsets of X are quasicompact iff they are finite.

Now, if a subset of X is quasicompact, it is finite, and hence closed (by the definition of the topology).

On the other hand, the condition "quasicompact subsets are closed" is strictly stronger than being T_1 as well (this condition implies T_1 by Proposition 3.6.2.12).[14]

▪ **Example 3.6.2.24 — A T_1 space for which not every quasicompact subset is closed** Define $X := \mathbb{N}$. We equip X with a nonstandard topology, the so-called ***cofinite topology***. Let $C \subseteq X$ and declare that

$$C \text{ is closed iff either (i) } C = X \text{ or (ii) } C \text{ is finite.} \tag{3.6.2.25}$$

Because the finite union and arbitrary intersection of finite sets is finite (obviously), it follows that this defines a topology on X. X is T_1 (Proposition 3.6.2.12 again) because points are finite, hence closed.

[14]To remember where the condition "quasicompact subsets are closed" fits into the series of implications (that is, strictly between T_1 and T_2), I secretly remember this condition as "$T_{1\frac{3}{4}}$ ($T_{1\frac{1}{2}}$ is inappropriate because the $\frac{1}{2}$ should be reserved for some sort of "separation by closed neighborhoods"). I hesitate to make this an 'official' term because, besides the fact that no one else uses it, I don't see how this property is really a *separation* axiom.

We claim that $\mathbb{Z}^+ \subseteq \mathbb{N}$ is quasicompact (of course this is not closed because it is not finite). So, let $\mathscr{U}$ be an open cover of $\mathbb{Z}^+$. Let $U \in \mathscr{U}$ be an open set containing $1 \in \mathbb{Z}^+$. From the definition of the topology, $\mathbb{N} \setminus U$ is finite, and so $\mathbb{Z}^+ \setminus U \subseteq \mathbb{N} \setminus U$ is finite, so let us write $\mathbb{Z}^+ \setminus U = \{m_1, \ldots, m_n\}$. As $\mathscr{U}$ covers $\mathbb{Z}^+$, there must be some $U_k \in \mathscr{U}$ with $m_k \in U_k$. Then, $\{U, U_1, \ldots, U_n\}$ is a finite subcover of $\mathscr{U}$.

■ **Example 3.6.2.26 — A space that is T_1 but not T_2** We showed that the real numbers with the cocountable topology is not T_2 in Example 3.6.2.23. On the other hand, by definition, countable sets are closed, and so certainly points are closed, and so X is T_1.

Exercise 3.6.2.27 Show that a subspace of a T_2 space is T_2.

Exercise 3.6.2.28 Show that an arbitrary product of T_2 spaces is T_2.

R Thus, by Tychonoff's Theorem (Theorem 3.5.3.14) an arbitrary product of compact spaces is compact.

Here is where the consistency of the terminology breaks-down. If you guessed that the term T_3 refers to spaces in which any two distinct points are separated by closed neighborhoods, you'd be wrong. Unfortunately, it seems that the term T_3 was already taken (we'll see what it means in a bit) when someone went to write down the following definition, and so it is called $T_{2\frac{1}{2}}$.

Definition 3.6.2.29 — $T_{2\frac{1}{2}}$ X is $T_{2\frac{1}{2}}$ iff any two distinct points are separated by closed neighborhoods.

R The alternate terminology for this seems to be ***Urysohn***.

Example 3.6.2.30 — A space that is T_2 but not $T_{2\frac{1}{2}}$ [a] This example is very similar to the Arens Square—see Example 3.6.1.17.

Define

$$\begin{aligned} S :=& ((0,1)\times(0,1))\cap(\mathbb{Q}\times\mathbb{Q}) \\ =& \{\langle x,y\rangle \in (0,1)\times(0,1) : x,y\in\mathbb{Q}\} \end{aligned} \tag{3.6.2.31}$$

and

$$X := S\cup\{\langle 0,0\rangle\}\cup\{\langle 1,0\rangle\}. \tag{3.6.2.32}$$

(This is the rationals in the open unit square together with the bottom-left and bottom-right corners.) We define a topology on X by defining a neighborhood base at each point—see Proposition 3.1.1.9. For $(x,y)\in X$, there are three cases: (i) $\langle x,y\rangle = \langle 0,0\rangle$, (ii) $\langle x,y\rangle = \langle 1,0\rangle$, and (iii) $\langle x,y\rangle\in S$. We define

$$\mathscr{B}_{\langle x,y\rangle} := \begin{cases} \left\{U\subseteq S : U\text{ is open in } S.^{b}\right\} & \text{if } \langle x,y\rangle\in S \\ \left\{U^m_{\langle x,y\rangle} : m\in\mathbb{Z}^+\right\} & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} U^m_{\langle 0,0\rangle} &:= \{\langle 0,0\rangle\}\cup\left\{\langle x,y\rangle\in(0,\tfrac{1}{2})\times(0,\tfrac{1}{m}) : x,y\in\mathbb{Q}\right\} \\ U^m_{\langle 1,0\rangle} &:= \{\langle 1,0\rangle\}\cup\left\{\langle x,y\rangle\in(\tfrac{1}{2},1)\times(0,\tfrac{1}{m}) : x,y\in\mathbb{Q}\right\}. \end{aligned}$$

By Proposition 3.1.1.9, there is a unique topology for which $\mathscr{B}_{\langle x,y\rangle}$ is a neighborhood base of $\langle x,y\rangle\in X$.

The closure of any open neighborhood of $\langle 0,0\rangle$ contains points with x-coordinate $\frac{1}{2}$ and y-coordinate arbitrarily small, and the same goes for any open neighborhood of $\langle 1,0\rangle$. Thus, these two points are not separated by closed neighborhoods.[c]

On the other hand, the x-coordinate of every point in every open neighborhood of $\langle 0,0\rangle$ is strictly less than $\frac{1}{2}$, and similarly the x-coordinate of every point in every open neighborhood of $\langle 1,0\rangle$ is strictly greater than $\frac{1}{2}$. In particular, these two points are separated by neighborhoods.

For any point $\langle x, y\rangle$ distinct from $\langle 0, 0\rangle$ and $\langle 1, 0\rangle$, we can separate $\langle x, y\rangle$ from $\langle 0, 0\rangle$ with neighborhoods by taking $U^m_{\langle 0,0\rangle} \ni \langle 0, 0\rangle$ for $m \in \mathbb{Z}^+$ with $\frac{1}{m} < y$ and any ε-ball about $\langle x, y\rangle$ of with radius less than $y - \frac{1}{m}$. Similarly for $\langle x, y\rangle$ and $\langle 1, 0\rangle$. Of course, any two points distinct from both $\langle 0, 0\rangle$ and $\langle 1, 0\rangle$ are separated by neighborhoods because they can be separated by neighborhoods in S. Thus, X is indeed T_2.

R This is the ***Simplified Arens Square***.

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[a] This comes from [SJ70, pg. 100].

[b] Open in the usual topology (the subspace topology inherited from $(0, 1) \times (0, 1)$).

[c] Recall that two points are separated by closed neighborhoods iff they are separated by open neighborhoods with disjoint closure—see Proposition 3.6.1.15.

Exercise 3.6.2.33 Show that a subspace of a $T_{2\frac{1}{2}}$ space is $T_{2\frac{1}{2}}$.

Exercise 3.6.2.34 Show that an arbitrary product of $T_{2\frac{1}{2}}$ spaces is $T_{2\frac{1}{2}}$.

You might be thinking to yourself “Well, if T_3 wasn't separated by closed neighborhoods, then it must be completely-separated, right?”. Sorry. Wrong again.

Definition 3.6.2.35 — Completely-T_2 X is ***completely-T_2*** iff any two distinct points are completely-separated.

R Naturally, this is sometimes called ***completely-Hausdorff***.

R Sometimes people will also say that ***continuous functions separate points***.

■ **Example 3.6.2.36 — A space that is $T_{2\frac{1}{2}}$ but not completely-T_2** The Arens Square from Example 3.6.1.17 will do the trick. There, we provided an example of two points which are separated by closed neighborhoods, but not completely-separated. This is of course already enough to show that the Arens Square is not completely-T_2, but we still need to check that *any* two points can be separated by closed neighborhoods.

Denote the Arens Square by X and let $\langle \frac{1}{2}, r\sqrt{2}\rangle \in X$ for $r \in \mathbb{Q}$. Take $m \in \mathbb{Z}^+$ such that $r\sqrt{2} - \frac{1}{m} > 0$ and take $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < r\sqrt{2} - \frac{1}{m}$. Then, using the definition (3.6.1.22), we see that the closures of $U^m_{(1,0)}$ and $U_{\langle \frac{1}{2}, r\sqrt{2}\rangle}$ are disjoint. Similarly, a point of this form can be separated from $(1,0)$ by closed neighborhoods.

For $\langle x, y\rangle \in S$, just take $m \in \mathbb{Z}^+$ with $\frac{1}{m} < y$. Then, the closure of $U^m_{\langle 0,0\rangle}$ and any ε-ball around $\langle x, y\rangle$ with radius less than $y - \frac{1}{m}$ will be disjoint. A similar trick works for $\langle 1, 0\rangle$ of course.

For $r, q \in \mathbb{Q}$ with $r < q$, choose $m, n \in \mathbb{Z}^+$ so that $r\sqrt{2} + \frac{1}{m} < q\sqrt{2} - \frac{1}{n}$ (and so that $r\sqrt{2} - \frac{1}{m} > 0$ and $q\sqrt{2} + \frac{1}{n} < 1$ so that the open neighborhoods are actually contained in X). Then, the closures of $U^m_{\langle \frac{1}{2}, r\sqrt{2}\rangle}$ and $U^n_{\langle \frac{1}{2}, q\sqrt{2}\rangle}$ are disjoint.

For $\langle x, y\rangle \in S$ and $\langle \frac{1}{2}, r\sqrt{2}\rangle \in T$, we must have that $y \neq r\sqrt{2}$, and so, without loss of generality, that $y < r\sqrt{2}$. Then, choose $m \in \mathbb{Z}^+$ large enough so that $r\sqrt{2} - \frac{1}{m} > y$ (and so that $U^m_{\langle \frac{1}{2}, \sqrt{2}\rangle}$ is contained in X), and take a ε-ball about $\langle x, y\rangle$ with radius less than $(r\sqrt{2} - \frac{1}{m}) - y$. Then, the closure of these two neighborhoods will be disjoint.

Finally, can separate two points in S by closed neighborhoods because we could do so in $(0,1) \times (0,1)$ (recall that S just has the subspace topology).

Exercise 3.6.2.37 Show that a subspace of a completely-T_2 space is completely-T_2.

Exercise 3.6.2.38 Show that an arbitrary product of completely-T_2 spaces is completely-T_2.

And of course, as now you probably saw coming, T_3 does not mean distinct points can be perfectly-separated.

Definition 3.6.2.39 — Perfectly-T_2 X is ***perfectly-T_2*** iff any two distinct points can be perfectly-separated.

R Disclaimer: I have never seen this term before. Then again, I've never seen *any* term to describe such spaces. But honestly, if completely-T_2 means you can completely-separate points, then a space in which you can perfectly-separate points is going to be called . . .

■ Example 3.6.2.40 — A space that is completely-T_2 but not perfectly-T_2 The Uncountable Fort Space from Example 3.6.1.24 will do the trick. There, we provided an example of two points which are completely-separated, but not perfectly-separated. This is already enough to show that the Uncountable Fort Space is not perfectly-T_2, but we still need to check that *any* two points can be completely-separated.

Recall that the Uncountable Fort Space was defined to be $X := \mathbb{R}$ with the closed sets being precisely the finite sets and also the sets which contained 0. We showed above in Example 3.6.1.24 that $1 \in X$ can be completely-separated from $0 \in X$. Of course, there was nothing special about 1, and so all we need to do is to show that we can completely-separate any two nonzero points in X.

So, let $x_1, x_2 \in X$ be nonzero and distinct, and define $f\colon X \to [0,1]$ by

$$f(x) := \begin{cases} 0 & \text{if } x = x_1 \\ 1 & \text{if } x = x_2 \\ \frac{1}{2} & \text{otherwise.} \end{cases} \tag{3.6.2.41}$$

Then, if $C \subseteq [0,1]$ closed contains $\frac{1}{2}$, $f^{-1}(C)$ contains $0 \in X$, and is hence closed. Otherwise, it is finite, as $f^{-1}\left([0,1] \setminus \{\frac{1}{2}\}\right) = \{x_1, x_2\}$, and hence closed. Thus, f is continuous, and so completely-separates x_1 and x_2.

Exercise 3.6.2.42 Show that a subspace of a perfectly-T_2 space is perfectly-T_2.

Now should be the point where we say "Exercise: Show that an arbitrary product of perfectly-T_2 spaces is perfectly-T_2." Unfortunately, however, that is false.

■ Example 3.6.2.43 — A product of perfectly-T_2 spaces that is not perfectly-T_2 The counter-example will be an uncountable product of the real numbers.[a]

We first check that real numbers are perfectly-T_2.[b] Let $x_1, x_2 \in \mathbb{R}$ be distinct. Without loss of generality, suppose that $x_1 < x_2$. Define $f\colon X \to Y$ by

$$f(x) := \begin{cases} \frac{1}{2} & \text{if } x \in (-\infty, x_1 - \frac{1}{2}] \\ 1 + (x - x_1) & \text{if } x \in [x_1 - \frac{1}{2}, x_1] \\ -\frac{x - x_1}{x_2 - x_1} + 1 & \text{if } x \in [x_1, x_2] \\ x - x_2 & \text{if } x \in [x_2, x_2 + \frac{1}{2}] \\ \frac{1}{2} & \text{if } x \in [x_2, \infty). \end{cases} \tag{3.6.2.44}$$

[c]

This is continuous by the Pasting Lemma, and furthermore satisfies $f^{-1}(1) = \{x_1\}$ and $f^{-1}(0) = \{x_2\}$, and so $\mathbb{R}$ is indeed perfectly-T_2.

Now, let I be an *uncountable* index set[d] and define $X :=\prod_I \mathbb{R}$. We wish to show that X is not perfectly-T_2. It suffices to show that there is no continuous function $f\colon X\to[0,1]$ such that $\{\langle 0,0,0,\ldots\rangle\}=f^{-1}(0)$.[e] We proceed by contradiction: let $f\colon X\to[0,1]$ be such a function.

For each $n\in\mathbb{Z}^+$, $f^{-1}([0,\frac{1}{n}))$ is an open neighborhood of $\langle 0,0,\ldots\rangle$, and so there are finitely many indices $i^n_1,\ldots,i^n_{m_n}\in I$ and open sets $U^n_{i^n_1},\ldots,U^n_{i^n_{m_n}}\subseteq\mathbb{R}$ containing 0 with

$$\langle 0,0,\ldots\rangle\in U^n_{i^n_1}\times\cdots\times U^n_{i^n_{m_n}}\times\prod_{\substack{i\in I\\ i\neq i^n_1,\ldots,i^n_{m_n}}}\mathbb{R}\subseteq f^{-1}([0,\tfrac{1}{n})).$$

Taking the intersection over all $n\in\mathbb{Z}^+$, we find that (because $\{\langle 0,0,\ldots\rangle\}=\bigcap_{n\in\mathbb{Z}^+}f^{-1}([0,\frac{1}{n}))$).

$$\bigcap_{n\in\mathbb{Z}^+}\left(U^n_{i^n_1}\times\cdots\times U^n_{i^n_{m_n}}\times\prod_{\substack{i\in I\\ i\neq i^n_1,\ldots,i^n_{m_n}}}\mathbb{R}\right)=\{\langle 0,0,\ldots\rangle\}.$$

However, for each $n\in\mathbb{Z}^+$, we only have finitely many indices $i^n_1,\ldots,i^n_{m_n}$, and so in total we have only countably many indices appearing at all. This means that there are still uncountably many coordinates in the above intersection which are equal to $\mathbb{R}$: a contradiction.

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[a]The conclusion is actually true for *countable* products—see Proposition 3.6.2.45.

[b]In fact, they're uniformly-perfectly-T_4—see Definition 4.3.1.17 and Proposition 4.3.1.18.

[c]The picture is that it starts at $\frac{1}{2}$, climbs up linearly to 1 at x_1, decreases linearly down to 0 at x_2, and then increases linearly back up to $\frac{1}{2}$, and stays there.

[d]"Index set" is not a technical term, that is, it's not a special sort of set—it just means that it's some random set that we happen to be using for 'indexing'.

[e]Because then we definitely cannot perfectly separate $\langle 0,0,0,\ldots\rangle$ from any other point. Also, note that, while my notation might suggest that this is a countable collection of coordinates, this is *not* the case—I just mean that this is the element in X all of whose coordinates are 0.

Despite this, it *is* true for countably-infinite products.

Proposition 3.6.2.45 A countable product of perfectly-T_2 spaces is perfectly-T_2.

W Warning: Recall that this fails for arbitrary (not-necessarily-countable) products—see Example 3.6.2.43.

Proof. W Warning: This references material not yet covered.

Let $\{X_m : m \in \mathbb{N}\}$ be a countable collection of perfectly-T_2 spaces and let $x, y \in \prod_{m\in\mathbb{N}} X_m$ be distinct. Define $S := \{m \in \mathbb{N} : x_m = y_m\}$, so that S^{C} is nonempty.

First suppose that S^{C} is finite. For $m \in S$, let $g_m \colon X_m \to [0,1]$ be continuous and such that $g_m^{-1}(1) = \{x_m\} = \{y_m\}$. For $m \in S^{\mathrm{C}}$, let $f_m \colon X_m \to [-1,1]$ be such that $f_m^{-1}(1) = \{x_m\}$ and $f_m^{-1}(-1) = \{y_m\}$. Write $S^{\mathrm{C}} = \{i_1, \ldots, i_{m_0}\}$. Finally, define $f\colon \prod_{m\in\mathbb{N}} X_m \to [-1,1]$ by

$$f := \left(\prod_{k\in S} g_k\right) \cdot \left(\frac{f_{i_1} + \cdots + f_{i_{m_0}}}{m_0}\right).^{ab} \tag{3.6.2.46}$$

We claim that $f(z) = +1$ iff $z = x$ and $f(z) = -1$ iff $z = y$. As the proofs are similar, we only prove the x case. One direction is easy: if $z = x$, then each $g_k(z) = 1$ and each $f_{i_k} = 1$, and so $f(z) = 1$. Conversely, if $f = 1$, then as the absolute value of the 'average' term is bounded by 1, if any $g_k(z) < 1$, then we would have $|f(z)| < 1$, and so, we must have that each $g_k(z) = 1$, which forces $z_k = x_k$ for all $k \in S$, as well as $\frac{f_{i_1}(z)+\cdots+f_{i_{m_0}}(z)}{m_0} = 1$. As $f_{i_k}(z) \in [-1,1]$, this forces each $f_{i_k}(z) = 1$, which in turn forces $z_k = x_k$ for all $k \in S^{\mathrm{C}}$, and so $z = x$.

Now suppose that S^{C} is infinite and enumerate S^{C} as $S^{\mathrm{C}} = \{i_1, i_2, \ldots\}$. Let g_m be as before, but now, for $i_k \in S^{\mathrm{C}}$, let

$f_{i_k}: X_{i_k} \to [0, \frac{1}{2^k}]$ be such that $f_{i_k}^{-1}(\frac{1}{2^k}) = \{x_{i_k}\}$ and $f_{i_k}^{-1}(0) = \{y_{i_k}\}$. Now define $f\colon \prod_{m\in\mathbb{N}} X_m \to [-1,1]$ by

$$f := \left(\prod_{k\in S} g_k\right)\cdot\left(2\sum_{k=1}^{\infty} f_{i_k} - 1\right). \tag{3.6.2.47}$$

I claim once again that $f(z) = +1$ iff $z = x$ and $f(z) = -1$ iff $z = y$. This time the cases are a bit different and so we do them each separately. That $z = x$ implies $f(z) = +1$ and $z = y$ implies $f(z) = -1$ are easy. Conversely, suppose that $f(z) = 1$. Similarly as before, this forces each $g_k(z) = 1$, so that $z_k = x_k$ for all $k \in S$, and hence $2\sum_{k=1}^{\infty} f_{i_k}(z) - 1 = 1$, and hence $\sum_{k=1}^{\infty} f_{i_k}(z) = 1$. As $f_{i_k}(z) \leq \frac{1}{2^k}$, this forces $f_{i_k}(z) = \frac{1}{2^k}$ for all $k \in S^{\mathrm{c}}$, and hence $z_k = x_k$ for all $k \in S^{\mathrm{C}}$, and hence $z = x$, as desired. Now suppose that $f(z) = -1$. As the absolute value of the second term is bounded by 1, we cannot have that any $g_k(z) < 1$, and so once again we have each $g_k(z) = 1$, so that $z_k = y_k$ for all $k \in S$, and hence $2\sum_{k=1}^{\infty} f_{i_k}(z) - 1 = -1$, and hence $\sum_{k=1}^{\infty} f_{i_k}(z) = 0$. As all of these terms are nonnegative, this forces each $f_{i_k}(z) = 0$, so that $z_k = y_k$ for all $y \in S^{\mathrm{C}}$, and hence $z = y$, as desired.

There is one important detail we have yet to check: that our definition of f in each case is actually *continuous*. To do this, we must show that $\prod_{k\in S} g_k$ and, in the second case, $\sum_{k=1}^{\infty} f_{i_k}$, actually converge to a continuous functions. To do so, we make use of *completeness* of the set of all bounded continuous functions on X (Exercise 4.4.1.26). Thus, we want to show that the partial sums of this series are Cauchy. So, enumerate $S = \{j_1, j_2, \ldots\}$ and take the codomain of g_{j_k} to be $[2^{2^{-k}}, 1]$. This ensures that the codomain of $\ln(g_{j_k})^c$ will be $[-2^{-k}, 1]$, which is enough to ensure that the partial sums of $\sum_{k=1}^{\infty} \ln(g_{j_k})$ Cauchy, as desired. As the codomain of f_{i_k} is already $[0, \frac{1}{2^k}]$, we needn't modify the codomain in order that the partial sums be Cauchy. ∎

[a] We have to work a bit harder to ensure that the product converges to a continuous function, but as that argument complicates the proof and is not part of the core idea, we postpone this correction until the end.

[b] Of course, each g_k is a function on X_k but yet f is a function on X. Thus, in order for this to make sense, when we write g_k we don't mean $g_k(z)$ for $z \in X$, because that is nonsensical, but rather $g_k(z_k)$. Similar comments apply throughout this example.

[c] See Definition 6.4.5.37 for the definition of ln.

But now we've gone through all the separation properties, right? How could T_3 be a thing? Well, now we enter a collection of separation axioms of a different nature—now we will focus on separating *closed sets*. One unfortunate fact, however, is that, if points are not closed, then these new separation axioms are *not* strictly stronger than the separation axioms we just presented. There are thus two versions of the following separation axioms: ones in which the points are closed and ones in which they aren't.

Definition 3.6.2.48 — Regular X is ***regular*** iff any closed set and a point not contained in it can be separated by neighborhoods.

R You'll note that we skipped right over being topologically-distinguishable and (just) separated. This is because a point is automatically separated from a closed set—the complement of the closed set is an open neighborhood of the point which does not intersect the closed set.

R Unfortunately, the terminology of separation axioms is so fucked that there's really no way of being consistent with all of the literature. There are sources which reverse my conventions of regular and T_3. We explain the motivation of our choice of convention in the definition of T_3 (Definition 3.6.2.50).

■ **Example 3.6.2.49 — A space that is regular but not T_0**
The indiscrete topology on any set with at least two points

is (almost) vacuously regular but not T_0: there are only two closed sets, $\emptyset$ and X, and X doesn't not contain any points, and for $\emptyset$ and $x \in X$, they are separated by the neighborhoods $\emptyset \subseteq \emptyset$ and $x \in X$.

Stupid examples like this is why we almost always care about the case when we in addition impose the condition of being T_1. This finally brings us to the definition of T_3.

Definition 3.6.2.50 — T_3 X is T_3 iff it is T_1 and regular.

R The T_1 condition (points are closed) is added so that this is a strict specialization of being T_2. In fact, by the following proposition, we could have equally well said T_0.

R We choose to call this T_3 instead of regular because then we have $T_3 \to T_2$. If the terms were reversed, then there would be T_3 spaces (like indiscrete spaces) that were not T_2—ew.

W Warning: Up until now, all of the T_k axioms (including things like $T_{2\frac{1}{2}}$, completely-T_2, and perfectly-T_2) have been strictly comparable, that is, T_1 strictly implies T_0, T_2 strictly implies T_1, etc.. With the introduction of T_3, this is no longer the case. It turns out that T_3 implies $T_{2\frac{1}{2}}$, but that T_3 is not comparable with either completely-T_2 or perfectly-T_2.

Proposition 3.6.2.51 If X is T_0 and regular, then it is T_2.

Proof. Suppose that X is T_0 and regular. Let $x_1, x_2 \in X$. Because X is T_0, without loss of generality there is some open neighborhood U of x_1 which does not contain x_2. Then, U^{C} is closed and does not contain x_1, so because X is regular, there are disjoint open neighborhoods V_1 and V_2 of x_1 and U^{C}

respectively. Then, as $x_2 \in U^{\mathrm{C}}$, this implies that x_1 and x_2 are separated by neighborhoods. ■

Before having defined T_3, we have a perfect chain of strict implications: perfectly-T_2 implies completely-T_2 implies $T_{2\frac{1}{2}}$ implies T_2 implies T_1 implies T_0. We now have a new condition, T_3, and unfortunately it does not fit into this chain.[15] T_3 does not imply perfectly-T_3. In fact, it doesn't even imply completely-T_2 (Example 3.6.2.52). On the other hand, it *does* imply $T_{2\frac{1}{2}}$—this is because T_3 is equivalent to $T_{3\frac{1}{2}}$ (Proposition 3.6.2.69) which certainly implies $T_{2\frac{1}{2}}$. The other direction isn't true either—perfectly-T_2 does not imply T_3 (Example 3.6.2.63).

■ **Example 3.6.2.52 — A space that is T_3 but not completely-T_2** [a] For $m \in 2\mathbb{Z}$, define

$$L_m := \{m\} \times [0, \tfrac{1}{2}). \tag{3.6.2.53}$$

For $n \in 1+2\mathbb{Z}$ and $k \in \mathbb{Z}$ with $k \geq 2$, let $T_{n,k}$ be the *legs* of an isosceles triangle in $\mathbb{R}^2 \times \mathbb{R}^2$ with apex at $p_{n,k} := \langle n, 1-\frac{1}{k}\rangle$ and base $[n-(1-\frac{1}{k}), n+(1-\frac{1}{k})] \times \{0\}$ (including the end-points of the legs on the base but not the apex $p_{n,k}$), precisely

$$T_{n,k} := \left\{\langle n \pm t, 1-t-\tfrac{1}{k}\rangle : 0 < t \leq 1-\tfrac{1}{k}\right\}. \tag{3.6.2.54}$$

Then, define

$$\begin{aligned} Y_0 &:= \bigcup_{m\in 2\mathbb{Z}} L_m \\ Y_1 &:= \bigcup_{\substack{n\in 1+2\mathbb{Z} \\ k\in\mathbb{Z}, k\geq 2}} \{p_{n,k}\} \\ Y_2 &:= \bigcup_{\substack{n\in 1+2\mathbb{Z} \\ k\in\mathbb{Z}, k\geq 2}} T_{n,k} \\ Y &:= Y_0 \cup Y_1 \cup Y_2. \end{aligned} \tag{3.6.2.55}$$

[15] See (3.6.3.1) for the final diagram of implications.

We define a topology on Y by defining a neighborhood base at each point—see Proposition 3.1.1.9. For $\langle x, y\rangle \in Y_2$, we declare a neighborhood base to be simply $\{\langle x, y\rangle\}$. For $\langle x, y\rangle = p_{n,k}$, we declare a neighborhood base of $\langle x, y\rangle$ to consist of subsets of the form $\{p_{n,k}\} \cup S$ for $S \subseteq T_{n,k}$ cofinite in $T_{n,k}$. For $\langle x, y\rangle \in L_m$ for $m \in 2\mathbb{Z}$, we declare a neighborhood base of $\langle x, y\rangle$ to consist of subsets of the form $\{\langle m, y\rangle\} \cup S$ for $S \subseteq Y \cap ((m-1, m+1) \times \{y\})$ cofinite in $Y \cap ((m-1, m+1) \times \{y\})$.

Finally define

$$X := Y \sqcup \{p_1, p_2\} \tag{3.6.2.56}$$

for distinct new points p_1, p_2. To define a topology on X, we once again use Proposition 3.1.1.9. A neighborhood base at $\langle x, y\rangle \in Y$ consists of precisely the same sets as it did before. We furthermore declare that

$$\begin{aligned} \mathscr{B}_{p_1} &:= \left\{U^\alpha_{p_1} : \alpha \in \mathbb{R}\right\} \\ \mathscr{B}_{p_2} &:= \left\{U^\alpha_{p_2} : \alpha \in \mathbb{R}\right\} \end{aligned} \tag{3.6.2.57}$$

to be neighborhood bases of p_1 and p_2 respectively, where

$$\begin{aligned} U^\alpha_{p_1} &:= \{\langle x, y\rangle \in Y : x < \alpha\} \cup \{p_1\} \\ U^\alpha_{p_2} &:= \{\langle x, y\rangle \in Y : x > \alpha\} \cup \{p_2\}. \end{aligned} \tag{3.6.2.58}$$

Note that in all cases, every element of $\mathscr{B}_p$ is *open*,[b] which does not follow automatically from the definition of a neighborhood base—see the remark in Proposition 3.1.1.9.

We first check that X is not completely-T_2 by showing that no continuous function can separate p_1 from p_2. So, let $f : X \to [0, 1]$ be continuous.

Write $\alpha := f(p_{n,k})$. Then,

$$\begin{aligned} f^{-1}(\alpha) \cap T_{n,k} &= f^{-1}\left(\bigcap_{m\in\mathbb{Z}^+} (\alpha - \tfrac{1}{m}, \alpha + \tfrac{1}{m})\right) \cap T_{n,k} \\ &= \bigcap_{m\in\mathbb{Z}^+} S_m, \end{aligned}$$

where

$$S_m := f^{-1}((\alpha - \tfrac{1}{m}, \alpha + \tfrac{1}{m})) \cap T_{n,k} \tag{3.6.2.59}$$

is cofinite in $T_{n,k}$. Hence,

$$T_{n,k} \setminus \left(f^{-1}(\alpha) \cap T_{n,k}\right) = \bigcup_{m \in \mathbb{Z}^+} T_{n,k} \setminus S_m. \tag{3.6.2.60}$$

In other words, the number of points in $T_{n,k}$ that map to a different value other than $f(p_{n,k})$ is at most countable. Less precisely, f is constant on $T_{n,k}$ modulo a countable set.

Let $C_{n,k}$ denote the set of y-values in $[0, \frac{1}{k})$ for which there is some point in $T_{n,k}$ with that y-coordinate and that maps to $f(p_{n,k})$. We just showed that this set is cocountable in $[0, 1-\frac{1}{k})$, and hence it is certainly cocountable in $[0, \frac{1}{2}) \subseteq [0, 1-\frac{1}{k})$. Then, the intersection of them $\bigcap_{k \in \mathbb{Z}, k \geq 2} C_{n,k}$ must in turn then be cocountable in $[0, \frac{1}{2})$, and in particular, be nonempty. So, let $y_m \in L_m$ be such that for every $k \geq 2$ there is some $q^-_{m,k} \in T_{m-1,k}$ and some $q^+_{m,k} \in T_{m+1,k}$, each with the same y-coordinate a y_m, with $f(q^-_{m,k}) = f(p_{m-1,k})$ and $f(q^+_{m,k}) = f(p_{m+1,k})$. Because the open neighborhoods of y_m are cofinite in $((m-1, m+1) \times \{y\}) \cap Y$, the sequence $k \mapsto q^-_{m,k}$ must eventually be in every neighborhood of y_m, and so we have $\lim_k q_{m-1,k} = y_m = \lim_k q_{m+1,k}$. It follows that

$$\begin{aligned} f(y_m) &= \lim_k f(q^+_{m,k}) = \lim_k f(p_{m+1,k}) \\ &= \lim_k f(q^-_{m+2,k}) = f(y_{m+2}). \end{aligned} \tag{3.6.2.61}$$

The crux: for each $m \in 2\mathbb{Z}$, there exists points $y_m \in L_m$ and $y_{m+2} \in L_{m+2}$ with $f(y_m) = f(y_{m+2})$. By the definition of the open neighborhoods of p_1 and p_2, it follows that $f(p_1) = \lim_{m \to -\infty} f(y_m) = \lim_{m \to +\infty} f(y_m) = f(p_2)$, so that p_1 and p_2 are not completely-separated, and so X is not completely-T_2.

We now check that X is T_3. We first check that it is T_1. p_1 is closed because

$$\{p_1\} = \bigcap_{\alpha \in \mathbb{R}} (U_{p_2}^{\alpha})^{\mathsf{C}}. \tag{3.6.2.62}$$

Similarly for p_2. Each point $\langle m, y\rangle \in L_m$ is closed because the union of all open sets not contained in $\mathscr{B}_{\langle m,y\rangle}$, $\mathscr{B}_{p_1}$, or $\mathscr{B}_{p_2}$ is precisely the complement of $\langle m, y\rangle$. Similarly for the $p_{n,k}$s. For $\langle x, y\rangle \in T_{n,k}$, the intersection over $\{\langle x', y'\rangle\}^{\mathsf{C}}$ for $\langle x', y'\rangle \neq \langle x, y\rangle$ in $T_{n,k}{}^{c}$ is $\{\langle x, y\rangle\}$ along with at least one neighborhood of every other point that doesn't contain $\langle x, y\rangle$[d] and so is closed.

Now we just need to show that we can separate closed sets from points with open neighborhoods. One thing to note is that we need only separate closed sets that are complements of some element of some $\mathscr{B}_p$ from points, because every closed set is an intersection of sets of this form.[e] So, let $p \in X$ and let $C \subseteq X$ be a complement of some element in the neighborhood base not containing p. There is nothing to do but just break it down into cases.

First suppose that $p = p_1$ and $C \subseteq Y \cup \{p_2\}$. If C contained points of arbitrarily small x-coordinate, then because it is closed, it would have to contain p_1. Thus, there is some $x_0 \in \mathbb{R}$ such that the x-coordinate of every point (besides p_2 of course, which doesn't actually have an x-coordinate) is at least x_0. Then we can take $U_{p_1}^{\frac{x_0}{8}}$ as our open neighborhood of p_1 and $U_{p_2}^{\frac{x_0}{4}}$ as our open neighborhood of C. Similarly if our point is p_2.

We can easily separate points of $T_{n,k}$ from closed sets as these points themselves are open.

Now take $p = \langle m, y\rangle \in L_m$ and $C \subseteq X$ not containing p. If C intersects $((m-1, m+1) \times \{y\}) \cap Y$ at more than finitely many points, then y would be an accumulation point of C, and hence would be contained in C. Thus, it must intersect $((m-1, m+1) \times \{y\}) \cap Y$ at at most finitely many points, in which case we can simply remove them to obtain an open neighborhood of $\langle m, y\rangle$. These finitely many points are elements of $T_{m-1,k}$ or $T_{m+1,k}$ for some k, and so are in particular open. Thus, the union of these finitely many points together with $U_{p_1}^{(m-1)-(1-\frac{1}{k})}$ and $U_{p_2}^{(m+1)+(1+\frac{1}{k})}$ must contain C (because these two neighborhoods of p_1 and p_2 contain all $T_{n,l}$ for $n \neq m-1, m+1$).

Finally if one of the points is some $p_{n,k}$, then using very similar logic as in the previous case, C must intersect $T_{n,k}$ at at most finitely many points. Removing these points from $T_{n,k}$ gives us an open neighborhood which is disjoint from the neighborhood formed from the union of these finitely many points (which are open) and the complement of $T_{n,k}$.

R I do not know of a name for this space, but in order to have something to refer to, I shall call it the ***Thomas Tent Space***.[f]

[a] Evidently, this example originally comes from [Tho69], though I first heard about it from Brian M. Scott on math.stackexchange.

[b] This does require a quick check.

[c] Recall that points of $T_{n,k}$ are open.

[d] The only points which have neighborhoods that contain $\langle x, y\rangle$ are $p_{n,k}$ and possibly elements of L_{m-1} and L_{m+1}, but as in these cases all elements of the neighborhood case are cofinite, we can just pick a neighborhood that does not contain $\langle x, y\rangle$.

[e] This uses the fact that every element in every $\mathscr{B}_p$ is open, so that all of these sets together actually form a base for the topology—see Proposition 3.1.1.9 and Exercise 3.1.1.4.

[f] The $T_{n,k}$s are like "tents", and indeed, that is the word Scott used to describe them.

Example 3.6.2.63 — A space that is perfectly-T_2 but not T_3

W Warning: This references material not yet covered.

Define $X := \mathbb{R}$. We equip $\mathbb{R}$ with the so-called ***cocountable extension topology***. Let $C \subseteq X$ and declare that

$$C \text{ is closed iff it is the union of a countable set and a set closed in the usual topology on } \mathbb{R}. \tag{3.6.2.64}$$

Now, let $x_1, x_2 \in X$ be distinct, and let $\phi : \mathbb{R} \to (0,1)$ be any homeomorphism (in the usual topology).[a] Now define $f : X \to [0,1]$ by

$$f(x) := \begin{cases} 0 & \text{if } x = x_1 \\ 1 & \text{if } x = x_2 \\ \phi(x) & \text{otherwise.} \end{cases} \tag{3.6.2.65}$$

The preimage of any set in $[0,1]$ which contains neither 0 nor 1 will be closed because ϕ is a homeomorphism. If it does contain 0 or 1, then the preimage will be the union of a closed set in the usual topology on $\mathbb{R}$ and a finite set, and hence will be closed. Thus, f is continuous, and so X is perfectly-T_2. We know check that X is not T_3.

$\mathbb{Q} \subseteq X$ is closed because it is countable. Any open neighborhood of $\mathbb{Q}$ must be of the form of a usual open neighborhood of $\mathbb{Q}$ with countably many points removed. Of course, the only usual open neighborhood of $\mathbb{Q}$ is all of $\mathbb{R}$, and so every open neighborhood of $\mathbb{Q}$ is just $\mathbb{R}$ with countably many points removed. On the other hand, $\sqrt{2} \notin \mathbb{Q}$ and any open neighborhood of $\sqrt{2}$ is a usual open neighborhood with countably many points removed. Thus, by uncountability of $\mathbb{R}$, these two must intersection, and so you cannot separate $\mathbb{Q}$ and $\sqrt{2}$ with neighborhoods. Thus, X is not T_3.

[a] Such a homeomorphism exists by Definition 6.4.5.58 (hopefully you can construct a homeomorphism between $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $(0,1)$).

Thus, once you hit $T_{2\frac{1}{2}}$, you can increase your separation in one of two noncomparable ways: you can make your space completely-T_2

or you can make your space T_3. At the end of the section, we will summarize how all the separation axioms relate to each other.

Exercise 3.6.2.66 Show that a subspace of a T_3 space is T_3.

Exercise 3.6.2.67 Show that an arbitrary product of T_3 spaces is T_3.

Definition 3.6.2.68 — $T_{3\frac{1}{2}}$ X is $T_{3\frac{1}{2}}$ iff it is T_1 and any closed set can be separated by closed neighborhoods from a point it does not contain.

> **R** This is not universally accepted terminology. Often people use the term $T_{3\frac{1}{2}}$ to refer to what I call completely-T_3. I imagine this is the case because it turns out that my $T_{3\frac{1}{2}}$ is equivalent to T_3 (see the next proposition).

> **R** You will notice a pattern with the terminology. If T_k means you can separate XYZ from ABC with neighborhoods, then $T_{k\frac{1}{2}}$ means you can separate XYZ from ABC with *closed* neighborhoods; completely-T_k means you can completely-separate XYZ from ABC; perfectly-T_k means you can perfectly-separate XYZ from ABC. This admittedly creates conflict with some people's terminology, but the terminology of separation axioms is so varied from source to source that it was impossible to come up with a naming system that didn't conflict with something. I chose what I did because it is the most systematic that does not completely depart from the established nomenclature.

Proposition 3.6.2.69 X is T_3 iff X is $T_{3\frac{1}{2}}$.

Proof. ($\Rightarrow$) Suppose that X is T_3. Let $C \subseteq X$ be closed and let $x \in C^{\mathrm{C}}$. As X is T_3, there is an open neighborhood U of C

and an open neighborhood V of x which are disjoint. Then, V^{C} is a closed set not containing x, and so there is an open neighborhood W_1 of V^{C} and an open neighborhood W_2 of x which are disjoint. Then,

$$C \subseteq U \subseteq \operatorname{Cls}(U) \subseteq V^{\mathsf{C}} \subseteq W_1 \tag{3.6.2.70}$$

and

$$x \in W_2 \subseteq \operatorname{Cls}(W_2) \subseteq W_1^{\mathsf{C}}, \tag{3.6.2.71}$$

and so U and W_2 are open neighborhoods of C and x respectively with disjoint closures, so that X is $T_{3\frac{1}{2}}$ by Proposition 3.6.1.15.

($\Leftarrow$) This is immediate—if sets are separated by closed neighborhoods, then they are certainly separated by neighborhoods. ■

Now would be the time to ask about subspaces and products of $T_{3\frac{1}{2}}$ spaces, but as $T_{3\frac{1}{2}}$ is equivalent to T_3, we[16] have already addressed these questions in Exercises 3.6.2.66 and 3.6.2.67.

Definition 3.6.2.72 — Completely-T_3 X is ***completely-T_3*** iff it is T_1 and any closed set can be completely-separated from a point it does not contain.

> R As noted above, sometimes people use the term $T_{3\frac{1}{2}}$ for this property. This is also sometimes called ***Tychonoff*** (probably because of the Tychonoff Embedding Theorem (Theorem 3.6.2.113).

■ **Example 3.6.2.73 — A space that is $T_{3\frac{1}{2}}$ but not completely-T_3** The Thomas Tent Space X from Example 3.6.2.52 will do the trick. We showed there that it is

[16] And by "we" I mean "you".

T_3, but not completely-T_2. From Proposition 3.6.2.69, it follows that X is $T_{3\frac{1}{2}}$. However, it if were completely-T_3, then it would also be completely-T_2—a contradiction. Therefore, X is likewise not completely-T_3.

Exercise 3.6.2.74 Show that a subspace of a completely-T_3 space is completely-T_3.

Exercise 3.6.2.75 Show that an arbitrary product of completely-T_3 spaces is completely-T_3.

One characterization of completely-T_3 spaces that can be quite useful is that they are precisely the topological spaces which appear as (up to homeomorphism) subspaces of products of the closed interval $[0, 1]$ (this is the Tychonoff Embedding Theorem (Theorem 3.6.2.113), though we will have to wait for Urysohn's Lemma to prove it.

Definition 3.6.2.76 — Perfectly-T_3 X is ***perfectly-T_3*** iff it is T_1 and any closed set can be perfectly-separated from a point it does not contain.

■ Example 3.6.2.77 — A space that is completely-T_3 but not perfectly-T_3 The Uncountable Fort Space from Example 3.6.1.24 will do the trick. In Example 3.6.2.40, we showed that this was completely-T_2 but not perfectly-T_2. As it is not perfectly-T_2, it is certainly not perfectly-T_3, though we still need to check that it is completely-T_3.

Recall that the Uncountable Fort Space was defined to be $X := \mathbb{R}$ with the closed sets being precisely the finite sets and also the sets which contained 0.

So, let $C \subseteq X$ be closed and let $x_0 \in C^{\mathsf{C}}$. First let us do the case where $x_0 = 0$. Then, we may define $f: X \to [0, 1]$ by

$$f(x) := \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{otherwise.} \end{cases} \tag{3.6.2.78}$$

Then, $f^{-1}(1) = C$ is closed and $f^{-1}(0)$ contains $0 \in X$, and so is closed. Thus, f is continuous. Now consider the case where $0 \in C$. Then, we may define $f\colon X \to [0,1]$ by

$$f(x) := \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.6.2.79}$$

$f^{-1}(1)$ is just a point and so is closed and $f^{-1}(0)$ contains $0 \in X$ and so is closed. Finally, let us consider the case where $x_0 \neq 0$ and $0 \notin C$. Then, we may define $f\colon X \to [0,1]$ by

$$f(x) := \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x = x_0 \\ \frac{1}{2} & \text{otherwise.} \end{cases} \tag{3.6.2.80}$$

Then, $f^{-1}(1) = C$ is closed, $f^{-1}(0) = \{x_0\}$ is finite and hence closed, and $f^{-1}(\frac{1}{2})$ contains $0 \in X$ and is hence closed. Thus, indeed, X is completely-T_3.

Exercise 3.6.2.81 Show that a subspace of a perfectly-T_3 space is perfectly-T_3.

Note that the same example (Example 3.6.2.43) that showed that an arbitrary product of perfectly-T_2 spaces need not be perfectly-T_2 also shows that an arbitrary product of perfectly-T_3 spaces need not be even perfectly-T_2, much less perfectly-T_3. However, in fact, it's even worse: not even a finite product of perfectly-T_3 spaces need be perfectly-T_3. And now we return to the issue of the product of two perfectly-T_3 spaces.

▪ **Example 3.6.2.82 — A product of two perfectly-T_3 spaces that is not perfectly-T_3** [a]

W Warning: This references material not yet covered.

Define $X := [0,1] \times [0,1]$ but do *not* equip X with the product topology. Instead, we will equip it with the order topology

(Definition 3.1.2.1) with respect to a particular total-order, the *lexicographic ordering*.[b] For $\langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in X$, define

$$\langle x_1, y_1\rangle \leq \langle x_2, y_2\rangle \text{ iff } x_1 < x_2 \text{ or } (x_1 = x_2 \text{ and } y_1 < y_2).$$

Now define $A := \{\langle x, y\rangle \in X : y = 0 \text{ or } y = 1.\}$. That is, A is the union of the top and bottom edges of the unit square equipped with lexicographic ordering. A is our example of a perfectly-T_3 spaces such that $A \times A$ is not perfectly-T_3.

Note that the interior of the 'top edge' of A and the 'bottom edge' of A are homeomorphic copies of the Sorgenfrey line (Example 3.6.2.125). In that example, we showed that the Sorgenfrey Line is perfectly-T_4.

Exercise 3.6.2.83 Use the fact that the Sorgenfrey Line is perfectly-T_4 to show that A is perfectly-T_4.

As A contains a copy (in fact, it contains two) of the Sorgenfrey Line, $A \times A$ contains a copy of the Sorgenfrey Plane, which we showed was not T_4 in Example 3.6.2.125. Thus, as $A \times A$ has a subspace that is not T_4, $A \times A$ cannot be homeomorphic to a metric space.[cd]

Exercise 3.6.2.84 Show that A is compact.

Lemma 3.6.2.85 Let X be a topological space. Then, if X is compact and $\{\langle x, x\rangle \in X \times X : x \in X\}$ is a G_δ set, then X is homeomorphic to a metric space.

Proof. We leave this as an exercise.

Exercise 3.6.2.86 Prove this yourself.

∎

Define

$$\Delta := \{\langle a, a\rangle \in A \times A : a \in A\}. \tag{3.6.2.87}$$

Exercise 3.6.2.88 Show that Δ is closed.

As A is compact, if Δ were a G_δ set, then A would be homeomorphic to a metric space by the lemma, and so $A\times A$ would be homeomorphic to a metric space: a contradiction. Therefore, Δ is not a G_δ set. However, if $A\times A$ were perfectly-T_3, then all closed subsets of $A\times A$ should be G_δs. As Δ is closed but not a G_δ, it follows that $A\times A$ cannot be perfectly-T_3.

R A is the ***Double Arrow Space***.

[a]This example comes from (again) Brian M. Scott's math.stackexchange answer.

[b]We saw this briefly once before in Example 1.2.29.

[c]Because subspaces of metric spaces are of course metric spaces, and metric spaces are uniformly-perfectly-T_4 (Proposition 4.3.1.18).

[d]Topological spaces that are homeomorphic to metric spaces are called ***metrizable***.

The separation axioms T_0 through perfectly-T_2 all had to do with separation of points. All the T_3 separation axioms had to do with separating closed sets from points. Finally, the T_4 axioms have to do with separating closed sets from closed sets.

Definition 3.6.2.89 — Normal X is ***normal*** iff any two disjoint closed subsets can be separated by neighborhoods.

R Similarly as with the definition of regular (Definition 3.6.2.48), we do not need to consider topological-distinguishability and mere separatedness.

R Similarly as with the term "regular", some authors reverse my conventions of normal and T_4. The motivation of our choice of convention is the same as it was for T_3.

▪ Example 3.6.2.90 — A space that is normal but not T_0 The indiscrete topology on any set with at least two points is (almost) vacuously normal but not T_0: there are only two closed sets $\emptyset$ and X, and these can certainly be separated from each other by neighborhoods.

Stupid examples like this is why we almost always care about the case when we in addition impose the condition of being T_1.

Definition 3.6.2.91 — T_4 X is T_4 iff it is T_1 and normal.

R Just as before, the condition of T_1 is imposed so that this is a strict specialization of being T_3.

There are at least two relatively large families of spaces that are T_4: metric spaces (which are in fact perfectly-T_4—see Proposition 4.3.1.18) and *compact* spaces.

Proposition 3.6.2.92 If X is compact, then it is T_4.

Proof. Suppose that X is compact. Then, closed subsets are quasicompact (Exercise 3.2.55), and hence can be separated by neighborhoods by Exercise 3.6.2.21. ■

Recall that (Example 3.6.2.52) there are T_3 spaces that are *not* completely-T_2. This does not happen with T_4 and completely-T_3: it turns out that every T_4 space is completely-T_3, as we shall see in a moment, in fact, they are what is called completely-T_4—see Theorem 3.6.2.106. In particular, we shouldn't try to look for a space that is T_4 but not completely-T_3. On the other hand, there are spaces that are perfectly-T_3 but not T_4.

▪ Example 3.6.2.93 — A space that is perfectly-T_3 but not T_4 [a]

W Warning: This references material not yet covered.

Define $X := \mathbb{R} \times \mathbb{R}_0^+$, the upper-half plane, and define a base for a topology on X by

$$\begin{aligned}\mathscr{B} := &\left\{B_\varepsilon(\langle x, y\rangle) \subseteq \mathbb{R} \times \mathbb{R}^+ : \varepsilon > 0,\ \langle x, y\rangle \in \mathbb{R} \times \mathbb{R}^+\right\} \\ &\cup \{B_\varepsilon(\langle x, \varepsilon\rangle) \cup \{\langle x, 0\rangle\} : x \in \mathbb{R},\ \varepsilon > 0\}\,.\end{aligned}$$

That is, we take all ε-balls contained in the (strict) upper half-plane together with all ε-balls in the upper half-plane which are 'tangent' to the x-axis (together with the point of tangency).

We first check that X is perfectly-T_3. So, let $C \subseteq X$ be closed and let $\langle x_0, y_0\rangle \in C^{\mathsf{C}}$. The subspace topology of $\mathbb{R}^+ \times \mathbb{R}^+ \subseteq X$ is just the usual topology, which, being a metric space, is perfectly-T_4. Thus, we only need to check the case where $y_0 = 0$. Let $\varepsilon > 0$ be such that $B_\varepsilon(\langle x_0, \varepsilon\rangle)$ is disjoint from C. Then let $\delta > 0$ be sufficiently small and less than 1 so that $\mathrm{dist}_C^{-1}([0, \delta)) \subseteq B_\varepsilon(\langle x_0, \varepsilon\rangle)^{\mathsf{C}}$. Then define $f\colon X \to [0, 1]$ by

$$f(\langle x, y\rangle) := \begin{cases} 0 & \text{if } \langle x, y\rangle = \langle x_0, 0\rangle \\ \delta \cdot \left(\frac{(x - x_0)^2 + y^2}{2\varepsilon y}\right) & \text{if } \langle x, y\rangle \in D_\varepsilon(\langle x_0, \varepsilon\rangle) \setminus \{\langle x_0, 0\rangle\} \\ \delta & \text{if } \langle x, y\rangle \in B_\varepsilon(\langle x_0, \varepsilon\rangle) \cap \\ & \quad \mathrm{dist}_C^{-1}([0, \delta))^{\mathsf{C}} \cap (\mathbb{R} \times \mathbb{R}^+) \\ \mathrm{dist}_C(\langle x, y\rangle) & \text{if } \langle x, y\rangle \in \mathrm{dist}_C^{-1}([0, \delta]). \end{cases}$$

This is continuous by the Pasting Lemma (Proposition 3.1.3.14).

We now check that X is not T_4. To do this, we show that $\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q} \subseteq \mathbb{R} \times \mathbb{R}_0^+$ (as subsets of the x-axis) are closed and cannot be separated by neighborhoods.[b] In fact, we check that every subset $S \subseteq \mathbb{R} \times \{0\}$ is closed. To show that, we show that S^{C} is open. For $\langle x, y\rangle \in S^{\mathsf{C}}$, if $y > 0$, then certainly we can put an ε-ball around it that does not intersect the x-axis, and hence does not intersection S. On the other hand, for $\langle x, 0\rangle \in S^{\mathsf{C}}$, $B_\varepsilon(\langle x, \varepsilon\rangle) \cup \{\langle x, 0\rangle\}$ intersects the x-axis only at $\langle x, 0\rangle$, and so does not intersect S. Therefore, S^{C} is open, and so S is closed.

We now check that $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ cannot be separated by neighborhoods. Let U be an open neighborhood of $\mathbb{R} \setminus \mathbb{Q}$. We show that there is some point $q_0 \in \mathbb{Q}$ every neighborhood of which intersects U.

For $x \in \mathbb{R} \setminus \mathbb{Q}$, let $\varepsilon_x > 0$ be such that

$$\mathbb{R} \setminus \mathbb{Q} \ni x \in B_{\varepsilon_x}(\langle x, \varepsilon_x \rangle) \cup \{x\} \subseteq U. \tag{3.6.2.94}$$

For $m \in \mathbb{Z}^+$, define

$$S_m := \{x \in \mathbb{R} \setminus \mathbb{Q} : \varepsilon_x > \tfrac{1}{m}\}. \tag{3.6.2.95}$$

Then,

$$\mathbb{R} = \bigcup_{m \in \mathbb{Z}^+} S_m \cup \bigcup_{x \in \mathbb{Q}} \{x\}, \tag{3.6.2.96}$$

and so

$$\mathbb{R} = \bigcup_{m \in \mathbb{Z}^+} \mathrm{Cls}_{\mathbb{R}}(S_m) \cup \bigcup_{x \in \mathbb{Q}} \{x\}.^c \tag{3.6.2.97}$$

As $\mathbb{R}$ is a complete metric space, by the Baire Category Theorem (Theorem 4.4.3.1), there must be some $m_0 \in \mathbb{Z}^+$ such that $\mathrm{Cls}_{\mathbb{R}}(S_{m_0})$ does *not* have empty interior. In particular, its interior must contains some rational point, so let $q_0 \in \mathrm{Cls}_{\mathbb{R}}(S_{m_0}) \cap \mathbb{Q}$. Then, for every $\varepsilon > 0$, $(q_0 - \varepsilon, q_0 + \varepsilon)$ intersects S_{m_0}, say at x_ε. That $x_\varepsilon \in S_{m_0}$ means that (from (3.6.2.94) and (3.6.2.95))

$$B_{1/m_0}(\langle x_\varepsilon, \tfrac{1}{m_0} \rangle) \subseteq {}^d B_{\varepsilon_{x_\varepsilon}}(\langle x_\varepsilon, \varepsilon_{x_\varepsilon} \rangle) \subseteq U. \tag{3.6.2.98}$$

But then, for all ε sufficiently small,

$$\langle x_\varepsilon, \varepsilon \rangle \in B_\varepsilon(\langle q_0, \varepsilon \rangle) \cap B_{1/m_0}(\langle x_\varepsilon, \tfrac{1}{m_0} \rangle) \subseteq B_\varepsilon(\langle q_0, \varepsilon \rangle) \cap U.$$

Thus, every neighborhood of $\langle q_0, 0 \rangle \in \mathbb{Q}$ intersects U.

R This is ***Niemytzki's Tangent Disk Topology***.

[a]This example comes from [SJ70, pg. 100].

[b]We write $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ instead of $\mathbb{Q}\times\{0\}$ and $(\mathbb{R}\setminus\mathbb{Q})\times\{0\}$. Note that we cannot write $\mathbb{Q}^{\mathrm{C}}$ to denote the irrationals as usual because, in this context, $\mathbb{Q}^{\mathrm{C}}$ means $(\mathbb{R}\times\mathbb{R}^+_0)\setminus\mathbb{Q}$

[c]The subscript $\mathbb{R}$ here is to indicate that we are taking the closure with respect to the usual topology on $\mathbb{R}$.

[d]As $\frac{1}{m_0}<\varepsilon_{x_\varepsilon}$, the Triangle Inequality shows that if you are within $\frac{1}{m_0}$ of $\langle x_\varepsilon,\frac{1}{m_0}\rangle$ then you must be within $\varepsilon_{x_\varepsilon}$ of $\langle x_\varepsilon,\varepsilon_{x_\varepsilon}\rangle$.

It is at this point that we (should) turn to the issue of subspaces and products of T_4 spaces. Unfortunately, probably a bit surprisingly, it is neither the case that a subspace of a T_4 space is necessarily T_4 nor the case that the product of two T_4 spaces is necessarily T_4. We now finally return to the issue of subspaces and products of T_4 spaces.

▪ **Example 3.6.2.99 — A subspace of a T_4 space that is not T_4** By Tychonoff's Theorem (Theorem 3.5.3.14), and the fact that compact spaces are T_4 (Proposition 3.6.2.92), arbitrary products of $[0,1]$ are T_4. Furthermore, by the Tychonoff Embedding Theorem (Theorem 3.6.2.113), every completely-T_3 space is homeomorphic to a subspace of a product of copies of $[0,1]$, that is, a T_4 space. Thus, it suffices to exhibit a single space that is completely-T_3 but not T_4. Example 3.6.2.93 (the previous example) is an example of such a space.

We mentioned that the product of two T_4 spaces need not be T_4. In fact, it's worse than this: the product of two *perfectly-T_4* spaces need not even be T_4, and so we wait for this counter-example until having defined "perfectly-T_4"—see Example 3.6.2.125.

Definition 3.6.2.100 — $T_{4\frac{1}{2}}$ X is $T_{4\frac{1}{2}}$ iff it is T_1 and any two disjoint closed subsets can be separated by closed neighborhoods.

R Just as with $T_{3\frac{1}{2}}$ (Definition 3.6.2.68), this terminology is not standard, presumably because it is actually just equivalent to T_4.

R It turns out that T_4 is equivalent to $T_{4\frac{1}{2}}$. We don't say any more about this now, because in fact T_4 is equivalent to completely-T_4 (and as usual, the implication that completely-T_4 implies $T_{4\frac{1}{2}}$ is relatively easy).

Definition 3.6.2.101 — Completely-T_4 X is ***completely-T_4*** iff it is T_1 and any two disjoint closed subsets can be completely-separated.

This is in fact *equivalent* to being T_4, though this is relatively nontrivial, and the statement even has a name associated to it. Before we prove that, however, we first present a useful 'lemma'.

Proposition 3.6.2.102 Let X be a T_1 topological space. Then,

(i). X is T_3 iff whenever U is an open neighborhood of $x \in X$, there is an open neighborhood V of x such that $x \in V \subseteq \mathrm{Cls}(V) \subseteq U$; and

(ii). X is T_4 iff whenever U is an open neighborhood of a closed subset $C \subseteq X$, there is an open neighborhood V of C such that $C \subseteq V \subseteq \mathrm{Cls}(V) \subseteq U$.

Proof. We first prove (i).

($\Rightarrow$) Suppose that X is T_3. Let U be an open neighborhood of $x \in X$. Then, U^{C} is a closed set that does not contain x, and so there are disjoint open neighborhoods V and W of x and U^{C} respectively. Then,

$$x \in V \subseteq \mathrm{Cls}(V) \subseteq W^{\mathrm{C}} \subseteq U. \tag{3.6.2.103}$$

($\Leftarrow$) Suppose that whenever U is an open neighborhood of $x \in X$, there is an open neighborhood V of x such that $x \in V \subseteq \mathrm{Cls}(V) \subseteq U$. Let $C \subseteq X$ be closed and let $x \in C^{\mathrm{C}}$. Then, C^{C} is an open neighborhood of x, and so there is an open neighborhood V of x such that

$$x \in V \subseteq \mathrm{Cls}(V) \subseteq C^{\mathsf{C}}, \tag{3.6.2.104}$$

and so V and $\mathrm{Cls}(V)^{\mathsf{C}}$ are disjoint open neighborhoods of x and C respectively, so that X is T_3.

(ii)

Exercise 3.6.2.105 Prove (ii).

■

Theorem 3.6.2.106 — Urysohn's Lemma. [a] If X is T_4, then X is completely-T_4.

R This is often stated as "If $C_1, C_2 \subseteq X$ are disjoint closed subsets of a T_4 space X, then there is a continuous function $f : X \longrightarrow [0,1]$ that is 0 on C_1 and 1 on C_2.".

[a] Proof adapted from [Mun00, pg. 207].

Proof. Suppose that X is T_4. Let $C_1, C_2 \subseteq X$ be closed.

Let $\{r_m : m \in \mathbb{N}\}$ be an enumeration of the rationals in $[0,1]$ with $r_0 = 1$ and $r_1 = 0$. We define a collection of open sets $\{U_{r_m} : m \in \mathbb{N}\}$ inductively. During this process, we will apply Proposition 3.6.2.102 repeatedly.

First of all, define $U_1 := C_2^{\mathsf{C}}$. Then, U_1 is an open neighborhood of C_1, and so there is some other open neighborhood U_0 of C_1 such that $C_1 \subseteq U_0 \subseteq \mathrm{Cls}(U_0) \subseteq U_1$.

Suppose that we have defined $U_{r_0}, \ldots U_{r_m}$ such that $\mathrm{Cls}(U_p) \subseteq U_q$ if $r < q$ for $r, q \in \{r_0, \ldots, r_m\}$. We wish to define U_{m+1} so that this property still remains to be true. As there are only finitely many rational numbers in $\{r_0, \ldots, r_m\}$ there is a largest $p_0 \in \{r_0, \ldots, r_m\}$ with $p_0 < r_{m+1}$ and similarly there is a smallest $q_0 \in \{r_0, \ldots, r_m\}$ with $r_{m+1} < q_0$.

Take $U_{r_{m+1}}$ so that

$$\mathrm{Cls}(U_{p_0}) \subseteq U_{r_{m+1}} \subseteq \mathrm{Cls}(U_{r_{m+1}}) \subseteq U_{q_0}.^a \tag{3.6.2.107}$$

Proceeding inductively, this allows us to define U_{r_m} for all $m \in \mathbb{N}$, and hence, we have defined U_r for all $r \in \mathbb{Q} \cap [0,1]$. By construction, it follows that

$$\mathrm{Cls}(U_p) \subseteq U_q \tag{3.6.2.108}$$

for $p, q \in \mathbb{Q} \cap [0,1]$ for $p < q$.

For $x \in X$, define

$$Q_x \coloneqq \{r \in \mathbb{Q} \cap [0,1] : x \in U_r\}. \tag{3.6.2.109}$$

Note that Q_x is empty iff $x \in C_2$. We then in turn define

$$f(x) \coloneqq \begin{cases} 1 & \text{if } x \in C_2 \\ \inf(Q_x) & \text{otherwise.} \end{cases} \tag{3.6.2.110}$$

Of course $f(C_2) = \{1\}$. Furthermore, if $x \in C_1$, then $x \in U_0$, and so indeed $f(x) = 0$. Thus, we need only check that f is continuous.

So, let $x_0 \in X$. First suppose that $f(x_0) \neq 0, 1$. The other two cases are similar. Let $\varepsilon > 0$ be such that $B_\varepsilon(f(x_0)) \subseteq [0,1]$ (if $f(x_0) = 0$, for example, then you will instead use $[0, \varepsilon)$ in place of $B_\varepsilon(f(x_0))$). Let $p, q \in \mathbb{Q}$ be such that

$$f(x_0) - \varepsilon < p < f(x_0) < q < f(x_0) + \varepsilon. \tag{3.6.2.111}$$

Then, $U \coloneqq U_q \setminus \mathrm{Cls}(U_p)$ is open in X. We claim that $f(U) \subseteq B_\varepsilon(x_0)$. This will show that f is continuous at x_0, and hence continuous as x_0 was arbitrary.

So, let $x \in U_q \setminus \mathrm{Cls}(U_p)$. Then, $q \in Q_x$, and so $f(x) \coloneqq \inf(Q_x) \leq q$. On the other hand, if $r \in Q$, so that $x \in U_r$, we cannot have that $U_r \subseteq U_p$ ($x \notin U_p$), and so we cannot have $r \leq p$, so that we must have $p < r$. Therefore, p is a lower-bound for Q_x, and so $f(x) = \inf(Q_x) \geq p$. Hence,

$$f(x_0) - \varepsilon < p \leq f(x) \leq q < f(x_0) + \varepsilon, \tag{3.6.2.112}$$

so that $f(x) \in B_\varepsilon(x_0)$, so that $f(U) \subseteq B_\varepsilon(x_0)$, as desired. ∎

[a]Here we are applying Proposition 3.6.2.102.(ii).

Thus, as T_4 is equivalent to $T_{4\frac{1}{2}}$ is equivalent to completely-T_4, we needn't ask additional questions about subspaces or products of $T_{4\frac{1}{2}}$ or completely-T_4 spaces, and, as already mentioned, these issues are addressed in Examples 3.6.2.99 and 3.6.2.125.

We mentioned shortly after the definition of completely-T_3 that completely-T_3 spaces are precisely those spaces which embed into products of $[0,1]$. Having proven Urysohn's Lemma, we can now return to this result.

Theorem 3.6.2.113 — Tychonoff Embedding Theorem. Let X a topological space. Then, X is completely-T_3 iff it is homeomorphic to a subspace of $\prod_S [0,1]$ for some set S.

Proof. [a] ($\Rightarrow$) Suppose that X is completely-T_3. Define $\iota\colon X \to \prod_{\mathrm{Mor}_{\mathbf{Top}}(X,[0,1])}[0,1]$ by $\iota(x)_f := f(x)$, that is, the f-component of $\iota(x)$ (for f an element of the index set $\mathrm{Mor}_{\mathbf{Top}}(X,[0,1])$) is $f(x) \in [0,1]$. We claim that ι is a homeomorphism onto its image.

We first show that ι is actually continuous. So, let $\lambda \mapsto x_\lambda \in X$ converge to $x_\infty \in X$. We wish to show that $\lambda \mapsto \iota(x_\lambda)$ converges to $\iota(x_\infty)$. However, by our characterization of convergence in the product topology (Corollary 3.5.3.11), this is the case iff $\lambda \mapsto \iota(x_\lambda)_f := f(x_\lambda)$ converges to $\iota(x_\infty)_f := f(x_\infty)$ for all $f \in \mathrm{Mor}_{\mathbf{Top}}(X,[0,1])$, however, this is the case because each such f is continuous.

We next check that ι is injective. Let $x_1, x_2 \in X$ be distinct. Then, as X is in particular completely-T_2, there is a continuous function $f\colon X \to [0,1]$ such that $f(x_1) = 0$ and $f(x_2) = 1$. Thus, $\iota(x_1)_f \neq \iota(x_2)_f$, and so $\iota(x_1) \neq \iota(x_2)$. Thus, ι is injective.

As ι is injective, it has an inverse $\iota(X) \to X$. To show that ι is a homeomorphism onto its image, we wish to show that this inverse is continuous. To show that, we want to show that the preimage of every open set under this inverse is closed.

However, the preimage of a set $U \subseteq X$ under this inverse is just $\iota(U)$, so it suffices to show that $\iota(U) \subseteq \prod_{f\in \mathrm{Mor}_{\mathbf{Top}}(X,[0,1])}[0,1]$ is open for every open $U \subseteq X$.

So, let $U \subseteq X$ be open. Let $y_0 \in \iota(U)$. We find an open subset V of $\iota(X)$ with $y_0 \in V \subseteq \iota(U)$.

Write $y_0 = \iota(x_0)$ for $x_0 \in U$. As X is completely-T_3 there is a continuous function $f_0 \colon X \to [0,1]$ such that $f_0(U^{\mathsf{C}}) = 0$ and $f_0(x_0) = 1$. Define $V := \pi_{f_0}^{-1}((0,1]) \cap \iota(X)$, where $\pi_{f_0} \colon \prod_{f\in \mathrm{Mor}_{\mathbf{Top}}(X,[0,1])}[0,1] \to [0,1]$ is the projection. This is certainly open in $\iota(X)$, and so all that remains to be shown is that (i) $y_0 = \iota(x_0) \in V$ and that (ii) $V \subseteq \iota(U)$.

However, as

$$\pi_{f_0}(y_0) := \pi_{f_0}(\iota(x_0)) := f_0(x_0) = 1 \in (0,1], \quad (3.6.2.114)$$

it follows that $y_0 \in V$. For the other condition, let $y \in V \subseteq \iota(X)$, and write $y = \iota(x)$ for some $x \in X$. As $\iota(x) = y \in V \subseteq \pi_{f_0}^{-1}((0,1]))$, $f_0(x) > 0$, and so $x \notin U^{\mathsf{C}}$, and so $x \in U$, and so $y \in \iota(U)$. Thus, $V \subseteq \iota(U)$, as desired.

($\Leftarrow$) Suppose that X is homeomorphic to a subspace of $\prod_S[0,1]$ for some set S. $\prod_S[0,1]$ is quasicompact by Tychonoff's Theorem (Theorem 3.5.3.14) and T_2 by Exercise 3.6.2.28, that is, compact, hence T_4 by Proposition 3.6.2.92, hence completely-T_4 by Urysohn's Lemma (Theorem 3.6.2.106), hence completely-T_3. As subspaces of completely-T_3 spaces are completely-T_3 (Exercise 3.6.2.74), it follows that X is completely-T_3. ∎

[a] Proof adapted from [Mun00, Theorem 34.2].

Definition 3.6.2.115 X is ***perfectly-T_4*** iff it is T_1 and any two disjoint closed can be perfectly-separated.

R For some reason, this is sometimes called T_6. What is T_5 you ask? Evidently T_5 means T_1 and every subspace is T_4.

There are other conditions equivalent to perfectly-T_4 that are sometimes easier to check.

Proposition 3.6.2.116 The following are equivalent.

(i). X is perfectly-T_4.
(ii). X is T_1 and for every closed $C \subseteq X$, there is a continuous function $f\colon X \to [0,1]$ such that $C = f^{-1}(0)$.
(iii). X is T_4 and every closed subset of X is a G_δ.

Proof. [a]

W Warning: This references material not yet covered.

($(i) \Rightarrow (ii)$) Suppose that X is perfectly-T_4. X is certainly T_1. Let $C \subseteq X$ be closed. If $X \setminus C = \emptyset$, take f to be the function the constant function $x \mapsto 0$. Otherwise, let $x_0 \in C^{\mathsf{C}}$. As $\{x_0\}$ is closed, there is a continuous $f\colon X \to [0,1]$ such that $C = f^{-1}(0)$ and $\{x_0\} = f^{-1}(1)$.

($(ii) \Rightarrow (ii)$) Suppose that X is T_1 and for every closed $C \subseteq X$, there is a continuous function $f\colon X \to [0,1]$ such that $C = f^{-1}(0)$. Let $C_1, C_2 \subseteq X$ be closed and disjoint. Let $f_1, f_2\colon X \to [0,1]$ be continuous and such that $C_1 = f_1^{-1}(0)$ and $C_2 = f_2^{-1}(0)$. Now, define $g\colon X \to [0,1]$ by

$$g(x) := \frac{f_1(x)}{f_1(x) + f_2(x)}. \tag{3.6.2.117}$$

Notice that the denominator never vanishes as the zero sets of f_1 and f_2 are disjoint. This is therefore a continuous function on X taking values in $[0,1]$. It is certainly 0 on C_1 and 1 on C_2. Conversely, if $g(x) = 0$, then $f_1(x) = 0$, and so $x \in C_1$. Thus, indeed, $g^{-1}(0) = C_1$. Similarly, if $g(x) = 1$, then $f_1(x) = f_1(x) + f_2(x)$, and so $f_2(x) = 0$, and so $x \in C_2$. Thus, $g^{-1}(1) = C_2$. Thus, X is perfectly-T_4

($(i) \Rightarrow (iii)$) Suppose that X is perfectly-T_4. X is certainly T_4. On the other hand, let $C \subseteq X$ be closed and let $f\colon X \to [0,1]$

be continuous and such that $C = f^{-1}(0)$. Then,

$$C = \bigcap_{m \in \mathbb{Z}^+} f^{-1}\left([0, \tfrac{1}{m})\right), \qquad (3.6.2.118)$$

and so C is a G_δ.

($(iii) \Rightarrow (ii)$) Suppose that X is T_4 and every closed subset of X is a G_δ. X is certainly T_1. Let $C \subseteq X$ be closed. Then, C is a G_δ, and so we can write $C = \bigcap_{m \in \mathbb{N}} U_m$, for $U_m \subseteq X$ open. Without loss of generality, assume that $U_m \supseteq U_{m+1}$.[b] As X is T_4, by Urysohn's Lemma (Theorem 3.6.2.106), there is a continuous function $f_m \colon X \to [0, 1]$ such that $f_m(C) = 0$ and $f_m(U_m^{\complement}) = 1$. Define $f \colon X \to [0, 1]$ by

$$f(x) := \sum_{m \in \mathbb{N}} \frac{f_m(x)}{2^{m+1}}. \qquad (3.6.2.119)$$

This defines a continuous function on X by Exercise 4.4.1.26 (completeness of bounded continuous functions) with values in $[0, 1]$. It is certainly 0 on C. On the other hand, if $x \notin C$, then $x \in U_m^{\complement}$ for some $m \in \mathbb{N}$, in which case $f(x) \geq \frac{f_m(x)}{2^{m+1}} = \frac{1}{2^{m+1}} > 0$. Thus, indeed, taking the contrapositive gives us $f(x) = 0$ implies $x \in C$, so that $C = f^{-1}(0)$, as desired. ■

[a] In case you're wondering "Is there a good reason he didn't do this in the usual "$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$' order?", the answer is "No, not really.". I just happened to think of the proofs in this order, and it's not clear to me that rewriting things in the 'usual' order would necessarily make things clearer.

[b] If necessary, replace U_m with $\bigcap_{k=0}^{m} U_k$.

■ **Example 3.6.2.120 — A space that is completely-T_4 but not perfectly-T_4** The Uncountable Fort Space from Example 3.6.1.24 will once again do the trick. In Example 3.6.2.40, we show that it is not perfectly-T_2, and so it is certainly not going to be perfectly-T_4. We still need to check that it is completely-T_4.

Recall that the Uncountable Fort Space was defined to be $X := \mathbb{R}$ with the closed sets being precisely the finite sets and also the sets which contained 0.

So, let $C_1, C_2 \subseteq X$ be disjoint closed sets. Let us first do the case where neither C_1 nor C_2 contains $0 \in X$. Then, we may define $f\colon X \to [0,1]$ by

$$f(x) := \begin{cases} 0 & \text{if } x \in C_1 \\ 1 & \text{if } x \in C_2 \\ \frac{1}{2} & \text{otherwise.} \end{cases} \tag{3.6.2.121}$$

The preimage of 0 is C_1 is closed, the preimage of 1 is C_2 is closed, and the preimage of $\frac{1}{2}$ contains $0 \in X$ and so is closed. Thus, this function is continuous. Now suppose that $0 \in C_1$. Then, we may define $f\colon X \to [0,1]$ by

$$f(x) := \begin{cases} 0 & \text{if } x \in C_1 \\ 1 & \text{if } x \in C_2. \end{cases} \tag{3.6.2.122}$$

The preimage of 0 is C_1 is closed and the preimage of 1 is C_2 is closed. Thus, this function is continuous.

And now we turn to the usual issues of subspaces and products of perfectly-T_4 spaces.

Proposition 3.6.2.123 Let X be a perfectly-T_4 space and let $S \subseteq X$. Then, S is perfectly-T_4.

Proof. [a] Let $C, D \subseteq Y$ be closed and disjoint. Write $C = C' \cap S$ and $D = D' \cap S$ for $C, D \subseteq Y$ closed.[b] As X is perfectly-T_4, there is a continuous function $f\colon X \to [0,1]$ such that $C' = f^{-1}(0)$ and a continuous function $g\colon X \to [0,1]$ such that

$D' = g^{-1}(0)$. Define $h\colon S \to [0,1]$ by

$$h(s) := \begin{cases} 1 & \text{if } s \in D \\ \frac{f(s)}{f(s)+g(s)} & \text{otherwise.} \end{cases} \tag{3.6.2.124}$$

Note that $f(x) + g(x) = 0$ iff $f(x) = 0 = g(x)$ iff $s \in C' \cap D'$. However, as $s \in S$, this would force $s \in C' \cap D' \cap S = C \cap D = \emptyset$. Thus, the denominator is never 0, and so defines a continuous function on S.

For $s \in D$, $g(s) = 0$, in which case we have $h(s) = 1$. For $s \in C$, we have $f(s) = 0$, in which case we have $h(s) = 0$. Conversely, if $h(s) = 0$, then $f(s) = 0$, and so $s \in C$. Similarly, if $h(s) = 1$, then $f(s) = f(s) + g(s)$, and so $g(s) = 0$, and so $s \in D$.

Thus, h perfectly separates C and D in S.

Finally, as subspaces of T_1 spaces are T_1 (Exercise 3.6.2.16), this shows that S is perfectly-T_4. ■

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[a] Proof adapted from math.stackexchange.

[b] See the defining result of the subspace topology Proposition 3.5.1.1.

Unfortunately, products of perfectly-T_4 spaces need not even be T_4, much less perfectly-T_4.

▪ **Example 3.6.2.125 — A product of two perfectly-T_4 spaces that is not T_4**

W Warning: This references material not yet covered.

Define $S := \mathbb{R}$[a] and equip S with the topology defined by the base

$$\{[a,b) : a,b \in \mathbb{R} \cup \{\pm\infty\},\ a < b\}\,. \tag{3.6.2.126}$$

S is called the ***Sorgenfrey Line***.[b] Note that

$$(a,b) = \bigcup_{b-a>\varepsilon>0} [a-\varepsilon, b) \tag{3.6.2.127}$$

is open. Thus, every set that is open in the usual topology is also open in the Sorgenfrey Line. In particular, points are closed because they are closed in the usual topology, and so S is T_1 (Proposition 3.6.2.12).

We first prove that, if $C \subseteq S$ is closed and $x \in S$ is an accumulation point of C *in the usual topology*, then $x \in C$ or $x - \varepsilon \in C$ for every $\varepsilon > 0$. If $x \in C$, we're done. Otherwise, $x \in C^{\mathsf{C}}$, and so there is some $[x, a) \subseteq C^{\mathsf{C}}$ for $a > x$. On the other hand, as x is an accumulation point of C in the usual topology, we have that for every $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon)$ intersects C. Choosing ε sufficiently small so that $x + \varepsilon < a$, as $[x, x + \varepsilon) \subseteq [x, a) \subseteq C^{\mathsf{C}}$, we in fact must have that $(x - \varepsilon, x)$ intersects C, as desired.

We now show that S is perfectly-T_4. So, let $C, D \subseteq S$ be closed and disjoint. Define $g_C, g_D \colon S \to [0, 1]$ respectively by

$$g_C(x) := \max\{\operatorname{dist}_C(x), \tfrac{1}{2}\} \tag{3.6.2.128}$$

and

$$g_D(x) := \min\{1 - \operatorname{dist}_D(x), \tfrac{1}{2}\}. \tag{3.6.2.129}$$

As they are continuous with respect to the usual topology, they are continuous with respect to the Sorgenfrey Topology.

Define

$$P := \left\{p \in (C \cup D)^{\mathsf{C}} : \text{for every } \varepsilon > 0,\ p - \varepsilon \in C \cup D.\right\}.$$

For $p \in P$, define $b_p := \inf\{x \in C \cup D : x \geq p\}$, and let us say that "$C$ comes next" iff b_p is an accumulation point of C in the usual topology, similarly for "D comes next", and "nothing comes next" if this set is empty. Finally pick a_p such that $p < a_p < b_p$.[c]

Now, let $f \colon S \to [0, 1]$ be the function that is defined to be 0, on C, 1 on D, $\frac{1}{2}$ on $[p, \infty)$ if "nothing comes next", given by g_C or g_D on $(-\infty, \min\{\inf(C), \inf(D)\})$ according to whether

$\inf(C) \leq \inf(D)$ or vice-versa, change linearly from $\frac{1}{2}$ at p to $g_C(a_p)$ at a_p over $[p, a_p)$ if "C comes next" and then is given by g_C on $[a_p, b_p)$, and changes linearly from $\frac{1}{2}$ at p to $g_D(a_p)$ at a_p over $[p, a_p)$ if "D comes next" and then is given by g_D on $[a_p, b_p)$. First note that this actually gives a well-defined function on all of S. As all of domains of this 'piece-wise' definition are closed, by the Pasting Lemma, this defines a continuous function.

f is obviously 0 on C and 1 on D. Conversely, the only way we might have $f(x) = 0$ with $x \notin C$ is if $g_C(x) = 0$ for $x \in [a_p, b_p)$ for some $p \in P$. However, that $g_C(x) = 0$ implies that $\mathrm{dist}_C(x) = 0$, so that x is an accumulation point of C in the original topology. The only time this would not automatically give that $x \in C$ is if $x - \varepsilon \in C$ for every $\varepsilon > 0$, but then we would have $x \in P$, in which case $f(x) = \frac{1}{2}$. Thus, $f(x) = 0$ in fact implies that $x \in C$. Similarly, $f(x) = 1$ implies that $x \in D$. Thus, S is perfectly-T_4.

We now check that $S \times S$ is not T_4.[d]

Define $L := \{\langle x, -x\rangle \in S \times S : x \in \mathbb{R}\}$. Note that, as the topology of $S \times S$ is finer than that of $\mathbb{R}^2$, L is automatically closed. Now define

$$Q := \{\langle x, -x\rangle \in S \times S : x \in \mathbb{Q}\} \tag{3.6.2.130}$$

and

$$I := \left\{\langle x, -x\rangle \in S \times S : x \in \mathbb{Q}^{\mathrm{C}}\right\}. \tag{3.6.2.131}$$

We first check that Q and I are closed. We will then check that they cannot be separated by neighborhoods, thereby demonstrating that $S \times S$ it not T_4.[e]

Note that $([x, x+1) \times [-x, -x+1)) \cap L = \{\langle x, -x\rangle\}$, so that $\{\langle x, -x\rangle\}$ is open in L, so that the subspace topology of L is the discrete topology.

Also note that as L is closed in the usual topology, and the Sorgenfrey Topology is finer than the usual topology, it is closed in the Sorgenfrey Topology as well. Thus, if $\lambda \mapsto x_\lambda \in Q$ is a net converging to $x_\infty \in S \times S$, in fact we must have that $x_\infty \in L$. As L has the discrete topology, this implies that the net $\lambda \mapsto x_\lambda \in Q$ must be eventually constant, which in turn implies that $x_\infty \in Q$. Thus, Q is closed. The same exact argument works with I in place of Q, and so I is likewise closed.

We now check that Q and I cannot be separated by neighborhoods. Let U be an open neighborhood of I. We show that there is some point $q_0 \in Q$ every neighborhood of which intersects U.

For $\langle x, -x\rangle \in I$, let $\varepsilon_x > 0$ be such that

$$I \ni \langle x, -x\rangle \in [x, x+\varepsilon_x) \times [-x, -x+\varepsilon_x) \subseteq U. \tag{3.6.2.132}$$

For $m \in \mathbb{Z}^+$, define

$$S_m := \left\{x \in \mathbb{Q}^{\mathrm{C}} : \varepsilon_x > \tfrac{1}{m}\right\}.^{f} \tag{3.6.2.133}$$

Then,

$$\mathbb{R} = \bigcup_{m \in \mathbb{Z}^+} S_m \cup \bigcup_{x \in \mathbb{Q}} \{x\}, \tag{3.6.2.134}$$

and so

$$\mathbb{R} = \bigcup_{m \in \mathbb{Z}^+} \mathrm{Cls}(S_m) \cup \bigcup_{x \in \mathbb{Q}} \{x\}. \tag{3.6.2.135}$$

By the Baire Category Theorem (Theorem 4.4.3.1), there must be some $m_0 \in \mathbb{Z}^+$ such that $\mathrm{Cls}(S_{m_0})$ does *not* have empty interior. So, let $(a, b) \subseteq \mathrm{Cls}(S_m)$ and let $\mathbb{Q} \ni q_0 \in (a, b)$. Then, for every $\varepsilon > 0$, $(q_0 - \varepsilon, q_0 + \varepsilon)$ intersects S_m, say at x_ε. That $x_\varepsilon \in S_{m_0}$ means that (from (3.6.2.132) and (3.6.2.133))

$$\begin{aligned}[x_\varepsilon, x_\varepsilon + \tfrac{1}{m_0}) \times [-x_\varepsilon, -x_\varepsilon + \tfrac{1}{m_0}) \subseteq \\ [x_\varepsilon, x_\varepsilon + \varepsilon_{x_\varepsilon}) \times [-x_\varepsilon, -x_\varepsilon + \varepsilon_{x_\varepsilon}) \subseteq U.\end{aligned} \tag{3.6.2.136}$$

But then, for all ε sufficiently small, if $q_0 \leq x_\varepsilon$,

$$\begin{aligned}\langle x_\varepsilon, -q_0\rangle \in ([q_0, q_0+\varepsilon) \times [-q_0, -q_0+\varepsilon)) \\ \cap \left([x_\varepsilon, x_\varepsilon + \tfrac{1}{m_0}) \times [-x_\varepsilon, -x_\varepsilon + \tfrac{1}{m_0})\right) \\ \subseteq ([q_0, q_0+\varepsilon) \times [-q_0, -q_0+\varepsilon)) \cap U,\end{aligned}$$

and similarly, if $x_\varepsilon \leq q_0$, we have that $[q_0, q_0+\varepsilon) \times [-q_0, -q_0+\varepsilon)$ intersects U at $\langle q_0, -x_\varepsilon\rangle$. Thus, every neighborhood of $\langle q_0, 0\rangle \in \mathbb{Q}$ intersects U, and so $S \times S$ is in fact not even T_4.

---

[a]"S" is for *Sorgenfrey*, some guy's name.

[b]It is also sometimes called the ***lower-limit topology*** because it has the property that a net $\lambda \mapsto x_\lambda$ converges to $x_\infty \in S$ iff it is eventually contained in $[x_\infty, x_\infty + \varepsilon)$ for every $\varepsilon > 0$, that is, iff it converges to x_∞ "from the right". I actually prefer this term for S itself, as it is more descriptive, but as $S \times S$ is called the Sorgenfrey Plane (and no one calls this the "lower-limit plane" or something of the like), I have decided to just use "Sorgenfrey" for both.

[c]Such an a_p exists because there must be some $[p, a_p) \subseteq (C \cup D)^{\mathsf{C}}$. If we happen to have $a_p = b_p$, instead replace a_p with some element of (p, a_p).

[d]$S \times S$ is the ***Sorgenfrey Plane***.

[e]Note that the remainder of the argument is very similar to the end of the argument given in Example 3.6.2.93, that is, the argument that show that Niemytzki's Tangent Disk Topology is not T_4.

[f]Note that this is a subset of the real numbers with the *usual* topology.

3.6.3 Summary

We summarize what we have covered so far in this section.

First of all, there are several levels of separation between two different objects in a space

Distinct $\Leftarrow^{a}$ Topologically-distinguishable $\Leftarrow^{b}$ Separated
$\Leftarrow^{c}$ Separated by neighborhoods $\Leftarrow^{d}$ Separated by closed neighborhoods $\Leftarrow^{e}$ Completely-separated $\Leftarrow^{f}$ Perfectly-separated

[a] A two-point space with the indiscrete topology—see Example 3.6.1.2.

[b] The Sierpinski Space—see Example 3.6.1.7.

[c] A certain three-point space—see Example 3.6.1.10.

[d] Another three-point space—see Example 3.6.1.13.

[e] The Arens Square—see Example 3.6.1.17.

[f] The Uncountable Fort Space—see Example 3.6.1.24.

The arrows indicated implication of course, and all these implications are strict, as indicated by the examples referenced in the footnotes.

There are three 'families' of separation axioms of spaces: (i) separation of pairs of points, (ii) separation of closed sets from points, and (iii) and separation of disjoint closed sets.

In the first family:

(i). Points being topologically-distinguishable is T_0 (Definition 3.6.2.1).
(ii). Points being separated is T_1 (Definition 3.6.2.10).
(iii). Points being separated by neighborhoods is T_2 (Definition 3.6.2.18).
(iv). Points being separated by closed neighborhoods is $T_{2\frac{1}{2}}$ (Definition 3.6.2.29).
(v). Points being completely-separated is completely-T_2 (Definition 3.6.2.35).
(vi). Points being perfectly-separated is perfectly-T_2 (Definition 3.6.2.39).

In general, appending "$\frac{1}{2}$", "completely", or "perfectly" to a separation axiom in this way changes whatever the separation axiom was now to "separated by closed neighborhoods", "completely-separated", and "perfectly-separated" respectively.

Closed sets and points, as well as closed sets and closed sets, are automatically separated, and so there are no separation axioms analogous to T_0 and T_1 for families (ii) and (iii).

For the other two "families", in order for them to be directly comparable with the first, we require that points be closed (that is, we explicitly require spaces in the second two "families" to be T_1—see Proposition 3.6.2.12). Without this extra assumption, the separation axioms are called "regular" and "normal" respectively.

Thus, for the second family:[17]

(i). Closed sets and points being separated by neighborhoods is T_3 (Definition 3.6.2.50).
(ii). Closed sets and points being separated by closed neighborhoods is $T_{3\frac{1}{2}}$ (Definition 3.6.2.68).
(iii). Closed sets and points being completely-separated is completely-T_3 (Definition 3.6.2.72).
(iv). Closed sets and points being perfectly-separated is perfectly-T_3 (Definition 3.6.2.76).

And similarly for the third family: [18]

(i). Closed sets being separated by neighborhoods is T_4 (Definition 3.6.2.50).
(ii). Closed sets being separated by closed neighborhoods is $T_{4\frac{1}{2}}$ (Definition 3.6.2.100).
(iii). Closed sets being completely-separated is completely-T_4 (Definition 3.6.2.101).
(iv). Closed set being perfectly-separated is perfectly-T_4 (Definition 3.6.2.115).

We now illustrate how all these axioms are related to each other. All implications are strict (unless otherwise indicated by a $\Leftrightarrow$), in which case the offending counter-example is given in the indicated footnote. Perhaps the only real surprise is the equivalence of $T_{4\frac{1}{2}}$ and

[17]Remember that we require the spaces to be T_1 in addition to these properties.
[18]Remember that we require the spaces to be T_1 in addition to these properties.

completely-T_4: Urysohn's Lemma (Theorem 3.6.2.106).[19]

$$
\begin{tikzcd}
T_0 \\
T_1 \arrow[u, Rightarrow, "20"] \\
T_2 \arrow[u, Rightarrow, "21"] \\
T_{2\frac{1}{2}} \arrow[u, Rightarrow, "22"] & T_3 \arrow[l, Rightarrow, "23"'] \arrow[r, Leftrightarrow, "24"'] & T_{3\frac{1}{2}} & T_4 \arrow[l, Rightarrow, "25"'] \arrow[r, Leftrightarrow, "26"'] & T_{4\frac{1}{2}} \\
\text{Completely-}T_2 \arrow[u, Rightarrow, "27"] & & \text{Completely-}T_3 \arrow[ll, Rightarrow, "28"'] \arrow[u, Rightarrow, "29"] & & \text{Completely-}T_4 \arrow[ll, Rightarrow, "30"'] \arrow[u, Leftrightarrow, "31"] \\
\text{Perfectly-}T_2 \arrow[u, Rightarrow, "32"] & & \text{Perfectly-}T_3 \arrow[ll, Rightarrow, "33"'] \arrow[u, Rightarrow, "34"] & & \text{Perfectly-}T_4 \arrow[ll, Rightarrow, "35"'] \arrow[u, Rightarrow, "36"]
\end{tikzcd} \quad . \qquad (3.6.3.1)
$$

Finally, we review which of these separation properties are preserved under subspaces and products.[37]

[19]Though perhaps it's worth nothing that $T_{k\frac{1}{2}}$ is equivalent to completely-T_k for $k = 3, 4$ but *not* $k = 2$.

[20]The Sierpinsi Space—see Example 3.6.1.7.

[21]$\mathbb{R}$ with the cocountable topology—see Example 3.2.45.

[22]The Simplified Arens Square—see Example 3.6.2.30.

[23]$\mathbb{R}$ with the cocountable extension topology—see Example 3.6.2.63.

[24]See Proposition 3.6.2.69.

[25]Niemytzki's Tangent Disk Topology—see Example 3.6.2.93.

[26]See Urysohn's Lemma (Theorem 3.6.2.106).

[27]The Arens Square—see Example 3.6.1.17.

[28]$\mathbb{R}$ with the cocountable extension topology—see Example 3.6.2.63.

[29]The Thomas Tent Space—see Example 3.6.2.52.

[30]Niemytzki's Tangent Disk Topology—see Example 3.6.2.93.

[31]Urysohn's Lemma, Theorem 3.6.2.106.

[32]The Uncountable Fort Space–see Example 3.6.1.24.

[33]$\mathbb{R}$ wit the cocountable extension topology—see Example 3.6.2.63.

[34]The Uncountable Fort Space—see Example 3.6.1.24.

[35]Niemytzki's Tangent Disk Topology—see Example 3.6.2.93.

[36]The Uncountable Fort Space—see Example 3.6.1.24.

[37]Recall that (Example 3.6.2.9) none of them should be preserved under quotienting and all of them should be preserved under disjoint union.

Every separation axiom we have seen is preserved under taking subspaces (Exercises 3.6.2.7, 3.6.2.16, 3.6.2.27, 3.6.2.33, 3.6.2.37, 3.6.2.42, 3.6.2.66, 3.6.2.74 and 3.6.2.81 and Proposition 3.6.2.123) except for T_4 (which is of course equivalent to $T_{4\frac{1}{2}}$ and completely-T_4 by Urysohn's Lemma (Theorem 3.6.2.106) (Example 3.6.2.99).

The following separation axioms are preserved under products, and the products allowed are arbitrary unless otherwise stated.

(i). T_0 (Exercise 3.6.2.8)
(ii). T_1 (Exercise 3.8.2.8)
(iii). T_2 (Exercise 3.6.2.28).
(iv). $T_{2\frac{1}{2}}$ (Exercise 3.6.2.34)
(v). Completely-T_2 (Exercise 3.6.2.38)
(vi). Perfectly-T_2 (countably-infinite product works (Proposition 3.6.2.45), arbitrary products don't (Example 3.6.2.43))
(vii). $T_3 \Leftrightarrow T_{3\frac{1}{2}}$ (Exercise 3.6.2.67)
(viii). Completely-T_3 (Exercise 3.6.2.75)

On the other hand, products of perfectly-T_3 spaces, $T_4 \Leftrightarrow T_{4\frac{1}{2}} \Leftrightarrow$ completely-T_4, and perfectly-T_4 are in general never preserved under products, not even finite ones (Examples 3.6.2.82 and 3.6.2.125). Thus, the 'rule' is that a separation axiom is preserved under products iff it is not a "perfectly" axiom, the sole exception being that T_4 (and it's equivalents) is not.[38][39]

3.7 Local properties

For most topological properties, there is a "local" version.

> **Meta-definition 3.7.1 — Locally XYZ** A topological space is ***locally XYZ*** iff each point has a neighborhood base consisting of sets that are XYZ.

The following result is probably the most useful for checking whether or not a space is actually locally XYZ.

[38] This should be easy to remember because T_4 is also the only separation axiom not preserved under subspaces.

[39] And in addition to this, perfectly-T_2 spaces happen to be preserved under countably-infinite products, but still not arbitrary ones.

Meta-proposition 3.7.2 Let X be a topological space. Then, X is locally XYZ iff for every $x \in X$ and open set $U \subseteq X$ containing x, there is a neighborhood $N \subseteq U$ of x that is XYZ.

Proof. ($\Rightarrow$) Suppose that X is locally XYZ. Let $x \in X$ and let $U \subseteq X$ be an open set containing X. By hypothesis, x has a neighborhood base consisting of sets that are XYZ, and so by the definition of neighborhood base (Definition 3.1.1.6), there is some neighborhood $N \subseteq U$ of x that is XYZ.

($\Leftarrow$) Suppose that for every $x \in X$ and open set $U \subseteq X$ containing x, there is a neighborhood $N \subseteq U$ of x that is XYZ. Define

$$\mathscr{B}_x := \{N \subseteq X : N \text{ a neighborhood of } x \text{ that is XYZ.}\} .$$

We claim that this is a neighborhood base for the topology. To show this, we must show that $U \subseteq X$ is open iff for every $x \in U$ there is some element $B \in \mathscr{B}_x$ with $x \in B \subseteq U$.

So, suppose that $U \subseteq X$ is open. By hypothesis, there is a neighborhood $N \subseteq U$ of x that is XYZ, that is, there is some $N \in \mathscr{B}_x$ with $x \in N \subseteq U$. Conversely, suppose that for every $x \in U$ there is some element $B \in \mathscr{B}_x$ with $x \in B \subseteq U$. Then, in particular, every point in U has a neighborhood contained in U, and so U is open. ∎

Of particular importance are the notions of local connectedness, local quasicompactness, and locally (completely/perfectly)-T_k.

Exercise 3.7.3 Show that a space is T_1 iff it is locally T_1.

Exercise 3.7.4 Show that if a space is T_2 then it is locally T_2. Find a counter-example to show that converse is false.

Exercise 3.7.5 Show that if a space is locally quasicompact and T_2, then it is locally compact. Find a counter-example to show the converse is false.

The following is the result related to local quasicompactness that will be used in the proof of the Haar-Howes Theorem.

Proposition 3.7.6 Let X be a locally compact space, let $K \subseteq X$ be quasicompact, and let $U \subseteq X$ contain K. Then, there is an open set $V \subseteq X$ with compact closure such that

$$K \subseteq V \subseteq \mathrm{Cls}(V) \subseteq U. \tag{3.7.7}$$

Proof. For each $x \in X$, let $B_x \subseteq X$ be an open neighborhood of X with compact closure, and for $x \in U$ choose B_x sufficiently small so that $B_x \subseteq U$. Define $K_x := K \cap B_x$ and $U_x := U \cap B_x$. As closed subsets of compact sets are compact, K_x and $B_x \setminus U_x$ are compact in B_x, and, as compact spaces are T_4, we can find disjoint open (in B_x) neighborhoods $V_x \supseteq K_x$ and $W_x \supseteq B_x \setminus U_x$. Write $V_x = V'_x \cap B_x$ and $W_x = W'_x \cap B_x$ for $V'_x, W'_x \subseteq X$ open. Note that

$$V_x \subseteq W_x^{\complement} \subseteq B_x^{\complement} \cup U_x \tag{3.7.8}$$

and hence

$$\begin{split} V_x = V_x \cap \mathrm{Cls}(B_x) \subseteq W_x^{\complement} \cap \mathrm{Cls}(B_x) \subseteq \\ U_x \cap \mathrm{Cls}(B_x) = U_x, \end{split} \tag{3.7.9}$$

so that $\mathrm{Cls}(U_x) \subseteq W_x^{\complement} \cap \mathrm{Cls}(B_x) \subseteq U_x$.

By quasicompactness, there are $x_1, \ldots, x_m \in X$ such that $K \subseteq V_{x_1} \cup \cdots \cup V_{x_m}$. Furthermore,

$$\begin{split} \mathrm{Cls}(V_{x_1}) \cup \cdots \cup V_{x_m}) &= \mathrm{Cls}(V_{x_1} \cup \cdots \cup \mathrm{Cls}(V_{x_m}) \\ &\subseteq (\mathrm{Cls}(B_{x_1}) \setminus W_{x_1}) \cup \cdots \cup (\mathrm{Cls}(B_{x_m}) \setminus W_{x_m}) \\ &\subseteq U_{x_1} \cup \cdots \cup U_{x_m} \subseteq U. \end{split} \tag{3.7.10}$$

Thus, $V_{x_1} \cup \cdots \cup V_{x_m}$ works. ∎

Function spaces topologies

Given topological spaces X and Y, we haven't yet discussed much possible topologies one may put on $\mathrm{Mor}_{\mathbf{Top}}(X,Y)$. An important one you are almost certain to encounter in your mathematical career is the *quasicompact-open topology*, which has the following natural generalization.

Meta-definition 3.7.11 — XYZ-open topology Let X and Y be topological spaces. Then, the ***XYZ-open topology*** on $\mathrm{Mor}_{\mathbf{Top}}(X,Y)$ is the topology generated by the collection

$$\begin{aligned}\{\{f\in \mathrm{Mor}_{\mathbf{Top}}(X,Y) : f(N)\subseteq V\} :\\ N\subseteq X \text{ is XYZ and } V\subseteq Y \text{ is open.}\}\,.\end{aligned} \tag{3.7.12}$$

Exercise 3.7.13 Let X and Y be topological spaces, and let $\mathscr{T}$ be a generation collection for the topology of Y. Show that

$$\begin{aligned}\{\{f\in \mathrm{Mor}_{\mathbf{Top}}(X,Y) : f(N)\subseteq V\} :\\ N\subseteq X \text{ is XYZ and } V\in\mathscr{T}\}\end{aligned} \tag{3.7.14}$$

generates the XYZ-open topology on $\mathrm{Mor}_{\mathbf{Top}}(X,Y)$.

R In other words, in the definition Meta-definition 3.7.11, we may as well just look at elements in a generating collection, instead of all open subsets of Y.

The relevance to local properties is given by the following result, which itself seems to be [Are46] the original motivation for the introduction of the quasicompact-open topology.

Meta-theorem 3.7.15. Let X. Then, X is locally-XYZ,

$$\mathrm{Mor}_{\mathbf{Top}}(X,Y)\times X\ni\langle f,x\rangle\mapsto f(x)\in Y \tag{3.7.16}$$

is continuous for all topologies spaces Y, where $\mathrm{Mor}_{\mathbf{Top}}(X,Y)$ is equipped with the XYZ topology.

Proof. Suppose that X is locally-XYZ. Let Y be a topological space, let $\langle f, x\rangle \in \mathrm{Mor}_{\mathbf{Top}}(X, Y) \times X$, and let $V \subseteq Y$ be an open neighborhood of $f(x) \in Y$. Then, $f^{-1}(V)$ is an open neighborhood of $x \in X$, and hence, as X is locally-XYZ, there is a neighborhood $N \subseteq f^{-1}(V)$ of $x \in X$ that is XYZ. Write

$$B_{N,V} := \left\{\check{f} \in \mathrm{Mor}_{\mathbf{Top}}(X, Y) : \check{f}(N) \subseteq V\right\}. \tag{3.7.17}$$

As $N \subseteq f^{-1}(V)$, $f \in B_{N,V}$, and so $B_{N,V} \times N \subseteq \mathrm{Mor}_{\mathbf{Top}}(X, Y) \times X$ is a neighborhood of $\langle f, x\rangle \in \mathrm{Mor}_{\mathbf{Top}}(X, Y) \times X$. Furthermore, if $\langle \check{f}, \check{x}\rangle \in B_{N,V} \times N$, then $\check{f}(\check{x}) \in V$, and so $B_{N,V} \times N$ is indeed a neighborhood of $\langle f, x\rangle$ that is mapped into V via the evaluation map $\mathrm{Mor}_{\mathbf{Top}}(X, Y) \times X \to Y$, as desired. ■

Exercise 3.7.18 Let X be a topological space. If

$$\mathrm{Mor}_{\mathbf{Top}}(X, Y) \times X \ni \langle f, x\rangle \mapsto f(x) \in Y \tag{3.7.19}$$

is continuous for all topologies spaces Y, where $\mathrm{Mor}_{\mathbf{Top}}(X, Y)$ is equipped with the XYZ topology, is it necessarily true that X is locally-XYZ?

R In other words, is the converse of the previous result Meta-theorem 3.7.15 true?

For an application of the quasicompact-open topology in particular, see Proposition 4.2.3.27.

3.8 The Intermediate and Extreme Value Theorems

3.8.1 Connectedness and the Intermediate Value Theorem

You'll recall from calculus that the classical statement of the Intermediate Value Theorem is

> Let $f : [a, b] \to \mathbb{R}$ be continuous. Then, for all y between $f(a)$ and $f(b)$ (inclusive), there exists $x \in [a, b]$ such that $f(x) = y$. (3.8.1.1)

We will see that the proper way to interpret this statement is that the image of f is connected, so that the image must contain everything in-between $f(a)$ and $f(b)$ as well. Of course, in order to make this precise, we have to first define what it means to be connected.

Definition 3.8.1.2 — Connected and disconnected Let X be a topological space. Then, X is ***disconnected*** iff there exist disjoint nonempty closed sets $C, D \subset X$ such that $X = C \cup D$. X is ***connected*** iff it is not disconnected. A subset S of X is connected iff it is connected in its subspace topology.

R The intuition for the definition of disconnected of course is that we 'break up' the space into two separate pieces which have no overlap.

R Another way to say this is that X is disconnected iff it has a partition into two closed sets.

R For $S \subseteq X$, note that it is *not* the case that S is disconnected iff $S = C \cup D$ for nonempty disjoint closed subsets $C, D \subseteq X$. The reason for the difference is that *subsets of S which are closed in S need not be closed in X*. For example, $(0, 1) \cup (2, 3)$ is a disconnected space, but it cannot be written as the union of two nonempty disjoint closed subsets of $\mathbb{R}$.

R This is usually phrased in terms of open sets instead of closed sets. It turns out the definitions are equivalent—see the following result. The reason we state the definition in terms of closed sets is because there is a related concept called *hyperconnected* in which they are not equivalent and the 'correct' notion is the one stated in terms of closed sets.

Proposition 3.8.1.3 Let X be a topological space. Then, X is disconnected iff there exist disjoint nonempty open sets $U, V \subset X$ such that $X = U \cup V$.

Proof. ($\Rightarrow$) Suppose that X is disconnected. Then, by definition, there are disjoint nonempty closed sets $C, D \subset X$ such that $X = C \cup D$. As C and D are disjoint, in fact $D = C^{\mathsf{C}}$, and so D is itself open. Similarly C is open. This is the desired result.

($\Leftarrow$) Suppose that there exist disjoint nonempty open sets $U, V \subset X$ such that $X = U \cup V$. Similarly as before, $V = U^{\mathsf{C}}$, and hence is closed, etc. etc.. ■

Proposition 3.8.1.4 Let X be a topological space and let $\mathscr{U}$ be a collection of connected subsets of X with nonempty intersection. Then, $\bigcup_{U \in \mathscr{U}} U$ is connected.

Proof. To simplify notation, let us write $X' := \bigcup_{U \in \mathscr{U}} U$. We proceed by contradiction: suppose that $\bigcup_{U \in \mathscr{U}} U$ is disconnected, so that we may write

$$X' = V \cup W \tag{3.8.1.5}$$

for $V, W \subseteq X'$ open, nonempty, and disjoint. Let $x_0 \in \bigcap_{U \in \mathscr{U}} U$ and without loss of generality assume that $x_0 \in V$. For each $U \in \mathscr{U}$, let us write

$$U_V := U \cap V \text{ and } U_W := U \cap W, \tag{3.8.1.6}$$

so that

$$U = U_V \cup U_W \tag{3.8.1.7}$$

for all $U \in \mathscr{U}$. As U is connected, it follows that, for each U, either U_V or U_W is empty. However, we know that $x_0 \in U_V$, and so in fact, we must have that $U_W = \emptyset$ for all $U \in \mathscr{U}$, which in turn implies that $W = \emptyset$: a contradiction. ■

Definition 3.8.1.8 — Connected component Let X be a topological space and let x_1, x_2. Then, x_1 and x_2 are ***connected*** (to each other) iff there exists a connected set $U \subseteq X$ with $x_1, x_2 \in U$.

Proposition 3.8.1.9 The relation of being connected to is an equivalence relation on X.

Proof. x is connected to itself because $\{x\}$ is connected. The relation is symmetric because the definition of the relation is symmetric. If x_1 is connected to x_2 and x_2 is connected to x_3, then there is some connected set U which contains x_1 and x_2, and there is some connected set V which contains x_2 and x_3. As U and V both contain x_2, it follows from the previous proposition that $U \cup V$ is connected, and hence x_1 is connected to x_3. ■

A ***connected component*** of X is an equivalence class of some point with respect to the relation of being connected to.

We have a pretty explicit description of connected components.

Proposition 3.8.1.10 Let X be a topological space and let $x \in X$. Then, the connected component of X is

$$\bigcup_{\substack{U \subseteq X \\ U \text{ connected} \\ x \in U}} U. \tag{3.8.1.11}$$

R In particular, by Proposition 3.8.1.4, every connected component of X is connected.

Proof. Define

$$U_x := \bigcup_{\substack{U \subseteq X \\ U \text{ connected} \\ x \in U}} U. \tag{3.8.1.12}$$

By Proposition 3.8.1.4, this is a connected set that contains x. It follows that every element of U_x is connected to x. To show that it is the connected component of x, that is, the equivalence class of x with respect to the relation of being "connected to", we must show that every other point that is connected to x is contained in U_x.

So, let $y \in X$ be connected to x. Then, there is a connected set V with $x, y \in V$. Then, V appears in the union (3.8.1.12), and so $V \subseteq U_x$, and in particular, $y \in U_x$, as desired. ∎

One particularly important property of connected components, at least in locally connected spaces, is that they are both open *and* closed. This is not a bad way of telling if a space is connected (a space is connected iff they only clopen sets are $\emptyset$ and itself).

Proposition 3.8.1.13 Let X be a topological space. Then, X is locally connected iff every connected component of every open subset is open.

R In particular, as X is the disjoint union of its connected components (by Corollary A.3.2.11), if $[x]_\sim \subseteq X$ is a connected component of X, then $[x]_\sim^{\complement}$ is the union of all the other connected components, and hence is open. Thus, $[x]_\sim$ is closed. That is, in a locally connected space, all the connected components are clopen.

Proof. ($\Rightarrow$) Suppose that X is locally connected. Let $U \subseteq X$ be open and for $x, y \in U$, let us write $x \sim y$ iff x is connected to y. Let $x \in [x_0]_\sim$. By definition, this means that there is some connected set $V \subseteq U$ such that $x, x_0 \in V$. As the space is locally connected, x has a neighborhood base consisting of connected sets. So, let N be a connected neighborhood of x. Then, V and N intersect, namely at x, and so by Proposition 3.8.1.4, $V \cup N$ is again connected. As $V \cup N$ certainly contains x_0, from the previous result, we have that $N \cup V \subseteq [x]_\sim$, and hence that $x \in N \subseteq [x]_\sim$. Thus, $[x]_\sim$ is open, as desired.

($\Leftarrow$) Suppose that every connected component of every open subset is open. To show that X is locally connected, we apply Meta-proposition 3.7.2, that is, we must show that for every $x \in X$ and $U \subseteq X$ be open containing x there is a connected neighborhood $N \subseteq U$ of x. So, let $x \in X$ and let $U \subseteq X$ be open and containing x. By hypothesis, the connected component of x in U is open, and hence constitutes a connected neighborhood of x contained in U, as desired. ■

Exercise 3.8.1.14 Is it true that a space is locally connected iff every connected component is clopen?

Definition 3.8.1.15 — Totally-disconnected Let X be a topological space. Then, X is ***totally-disconnected*** iff every connected component of X is a point.

Proposition 3.8.1.16 Let X be a discrete topological space. Then, X is totally-disconnected.

Proof. Let $U \subseteq X$ have at least two distinct points x_1 and x_2. Then,

$$U = \{x_1\} \cup (U \setminus \{x_1\}), \tag{3.8.1.17}$$

and as every subset in a discrete space is open, it follows that U is disconnected. ■

Example 3.8.1.18 — $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are totally-disconnected That $\mathbb{N}$ and $\mathbb{Z}$ are totally-disconnected follows from the fact that they are discrete.

$\mathbb{Q}$ is also totally-disconnected,[a] but this is more difficult to see. Let $U \subseteq \mathbb{Q}$ have at least two distinct points q_1 and q_2. Without loss of generality, suppose that $q_1 < q_2$. Then, by 'density' of $\mathbb{Q}^{\mathrm{C}}$ in $\mathbb{R}$ (Theorem 2.4.3.64), there is some $x \in \mathbb{Q}^{\mathrm{C}}$ with $q_1 < x < q_2$. Define

$$V := (-\infty, x) \cap U \text{ and } W := (x, \infty) \cap U. \tag{3.8.1.19}$$

Both V and W are open by the definition of the subspace topology (Proposition 3.5.1.1) of U, and both are nonempty because $q_1 < x$ and $q_2 > x$. Thus, as $U = V \cup W$, U is disconnected, and hence $\mathbb{Q}$ is totally-disconnected.

[a] In particular, there are totally-disconnected spaces which are not discrete.

We mentioned at the beginning of this section that the 'proper' way to interpret the Intermediate Value Theorem is the statement that the image of connected sets are connected. There is one other thing we first need to check though—we need to check that intervals in $\mathbb{R}$ are in fact connected.

Theorem 3.8.1.20. Let $I \subseteq \mathbb{R}$. Then, I is connected iff it is an interval.

R You might say that this result is to connectedness as the Heine-Borel Theorem is to quasicompactness—this result characterizes connectedness in $\mathbb{R}$ and the Heine-Borel Theorem characterizes quasicompactness in $\mathbb{R}$. A key difference, however, is their generalization to $\mathbb{R}^d$—there is no such thing as an interval in $\mathbb{R}^d$,[a] whereas the Heine-Borel Theorem holds verbatim in $\mathbb{R}^d$.

R In particular, $\mathbb{R}$ itself is connected, in contrast with $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$.

[a] I suppose you could equip $\mathbb{R}^d$ with the product order, but this is not totally-ordered for $d \geq 2$, and so you cannot define the order-topology.

Proof. ($\Rightarrow$) Suppose that I is connected. Let $a, b \in I$ with $a \leq b$ and let $x \in \mathbb{R}$ with $a \leq x \leq b$. We must show that $x \in I$. If either $x = a$ or $x = b$, we are done, so we may as well suppose that $a < x < b$. We proceed by contradiction: suppose that $x \notin I$. Then,

$$I = (I \cap (-\infty, x)) \cup (I \cap (x, \infty)), \tag{3.8.1.21}$$

and so as I is connected, we must have that either $I \cap (-\infty, x)$ is empty or $I \cap (x, \infty)$ is empty. But a is in the former and b is in the latter: a contradiction.

($\Leftarrow$) Suppose that I is an interval. As I is an interval, by Proposition 2.4.3.62, we have that $I = [(a, b)]$ for $a = \inf(I)$ and $b = \sup(I)$.[a] We wish to show that I is connected. If $a = b$, then either $I = \emptyset$ or $I = \{a\}$, in which case I is trivially connected. Therefore, we may assume without loss of generality that $a < b$.

We proceed by contradiction: suppose that I is disconnected. The, we have that $I = U \cup V$ for $U, V \subseteq I$ open in I disjoint and nonempty. As U and V are open in I and cover I, we must have that at least one of them contains an open neighborhood of a, so without loss of generality, suppose that $[(a, x_0) \subseteq U$ for some $x_0 \in \mathbb{R}$ with $a < x_0 \leq b$. Now define

$$S := \{x \in I : [(a, x) \subseteq U\} . \tag{3.8.1.22}$$

We just showed that this set is nonempty. It is also bounded above by b as b is in particular an upper-bound of I. Therefore, it has a supremum. We wish to show that $\sup(S) = b$.

Note that $a < \sup(S) \leq b$. If $\sup(S) = b$, we are done, otherwise $a < \sup(S) < b$, and so $sup(S) \in I = [(a, b)]$.

We show that in fact $\sup(S) \in U$. We proceed by contradiction: suppose that $\sup(S) \in V$ (here is where we use the fact that $\sup(S) \in I$). As V is open, there is a neighborhood of $\sup(S)$ completely contained in V. On the other hand, by Proposition 1.4.1.13, this neighborhood has to contain some element of S, which in turn would imply that it would have to contain some element of U. But then U intersects V: a contradiction. Therefore, $\sup(S) \in U$.

Because U is open, there is some $\varepsilon > 0$ such that $(\sup(S) - \varepsilon, \sup(S) + \varepsilon) \subseteq U$. By Proposition 1.4.1.13, there must be some $x \in S$ with $\sup(S) - \varepsilon < x \leq \sup(S)$, so that $[(a, x) \subseteq U$. But then,

$$\begin{aligned} &[(a,x)) \cup (\sup(S) - \varepsilon, \sup(S) + \varepsilon) \\ &\qquad = [(a, \sup(S) + \varepsilon) \subseteq U, \end{aligned} \tag{3.8.1.23}$$

and so, in particular, there is some $x' > \sup(S)$ such that $[(a, x') \subseteq U$, a contradiction as S is the supremum of the set of all such elements. Therefore, we must have that $\sup(S) = b$.

Now that we have finally succeeded in showing that $\sup(S) = b$, we finish the proof by coming to a contradiction of the assumption of disconnectedness. As $\sup(S) = b$, this means that for every $x \in I$ with $a < x < b$, we have that $[(a, x) \subseteq U$, and hence

$$\bigcup_{a<x<b} [(a, x) = [(a, b) \subseteq U. \tag{3.8.1.24}$$

Thus, either $V = \{b\}$ or $V = \emptyset$: a contradiction of being open in I or being nonempty respectively. ■

[a] Recall that this notation just means that the end-points can be either open or closed—see the remark in Proposition 2.4.3.62.

As an application, we can use this to 'classify' all open sets in the real numbers.

Theorem 3.8.1.25. Let $U \subseteq \mathbb{R}$ be open. Then, U is a countable disjoint union of open intervals.

R Note that this is very special to $\mathbb{R}$—I don't see how one could hope to generalize this to $\mathbb{R}^d$ for $d \geq 2$.

Proof. For $x, y \in \mathbb{R}$, let us write $x \sim y$ iff x is connected to y. Then,

$$U = \bigcup_{x \in U} [x]_\sim, \tag{3.8.1.26}$$

that is, U is the disjoint union of its connected components.[a] By the previous result, each $[x]_\sim$ is an interval. By Proposition 3.8.1.13, each $[x]_\sim$ is open, and hence an open interval. Thus, U is the disjoint union of open intervals, and so all that remains to be shown is that the union is countable.

Exercise 3.8.1.27 Show that the union in (3.8.1.26) is a countable one.

■

[a] Because every equivalence relation determines a partition—see Corollary A.3.2.11.

And now we finally get to the statement of the 'true' Intermediate Value Theorem.

Theorem 3.8.1.28 — Intermediate Value Theorem. Let $f: X \to Y$ be a continuous function and let $S \subseteq X$ be connected. Then, $f(S)$ is connected.

R Sometimes this is abbreviated ***IVT***.

R In other words, the continuous image of a connected set is connected.

R In general, if a function $f\colon X\to Y$ has the property that the image of a connected set is connected, then we say that f has the ***intermediate value property*** (also referred to as *Darboux continuous*—see Theorem 6.4.2.35). Thus, the Intermediate Value Theorem says that "Continuous functions have the intermediate value property.". Continuous functions are not the only such functions with this property, however. Darboux's Theorem (Theorem 6.4.2.35) says that any function that is the derivative of another function (from $\mathbb{R}$ to $\mathbb{R}$) has the intermediate value property. See also Exercise 3.8.1.32.

Proof. We proceed by contradiction: suppose that $f(S)$ is disconnected. Then, $f(S)=U\cup V$ for $U,V\subseteq f(S)$ open in $f(S)$ disjoint and nonempty. By continuity, we have that[a]

$$S\supseteq^{b} f|_S{}^{-1}(U)\cup f|_S{}^{-1}(V)\supseteq S, \tag{3.8.1.29}$$

and so

$$S=f|_S{}^{-1}(U)\cup f|_S{}^{-1}(V). \tag{3.8.1.30}$$

As U and V are open in S and f is continuous, $f|_S{}^{-1}(U)$ and $f|_S{}^{-1}(V)$ are open in S. They also must be disjoint, for a point which lied in their intersection would be mapped into $U\cap V=\emptyset$ via f. Therefore, because S is connected, we have that either $f|_S{}^{-1}(U)$ or $f|_S{}^{-1}(V)$ is empty, which implies respectively that either U or V is empty: a contradiction. ■

[a] Also recall that $f^{-1}(f(S))\supseteq S$—see Exercise A.3.29.(ii).

[b] This follows from the fact that we are applying the preimage of *the restriction* of f to S.

As a corollary of this (and the fact that a subnet of $\mathbb{R}$ is connected iff it is an interval—see Theorem 3.8.1.20), we have the classical statement of the Intermediate Value Theorem.

Corollary 3.8.1.31 — Classical Intermediate Value Theorem Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then, $f([a,b])$ is an interval. In particular, any element between $f(a)$ and $f(b)$ is in the image of f.

Proof. The "in particular" part follows from the definition of an interval (Definition A.3.3.3).

$[a,b]$ is connected by Theorem 3.8.1.20, and so, by the Intermediate Value Theorem, $f([a,b])$ is connected, and so by Theorem 3.8.1.20 again, is an interval. ■

Exercise 3.8.1.32 Are there functions which are not continuous but which still have the intermediate value property?

3.8.2 Quasicompactness and the Extreme Value Theorem

You'll recall from calculus that the classical statement of the Extreme Value Theorem is

> Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then, there exists $x_1, x_2 \in [a,b]$ such that $f(x_1) = \inf_{x\in[a,b]}\{f(x)\}$ and $f(x_2) = \sup_{x\in[a,b]}\{f(x)\}$. In other words, continuous functions *attain* their maximum and minimum on closed intervals. (3.8.2.1)

This is actually a special case of (or follows easily from) a *much* more general, elegant statement.

Theorem 3.8.2.2 — Extreme Value Theorem. Let $f\colon X \to Y$ be continuous and let $K \subseteq X$. Then, if K is quasicompact, then $f(K)$ is quasicompact.

R Sometimes this is abbreviated ***EVT***.

R In other words, the continuous image of a quasicompact set is quasicompact.

R In general, if a function $f\colon X \to Y$ has the property that the image of a quasicompact set is quasicompact, then we say that f has the ***extreme value property***. Thus, the Extreme Value Theorem says that "Continuous functions have the extreme value property.".

Proof. Suppose that K is quasicompact. Let $\mathscr{U}$ be an open cover of $f(K)$. Then, $f^{-1}(\mathscr{U}) := \left\{ f^{-1}(U) : U \in \mathscr{U} \right\}$ is an open cover of K, and therefore it has a finite subcover $\{f^{-1}(U_1), \ldots f^{-1}(U_m)\}$. In other words,

$$K \subseteq f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_m), \tag{3.8.2.3}$$

and hence

$$\begin{aligned} f(K) &\subseteq f\left(f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_m) \right) \\ &=^a f\left(f^{-1}(U_1) \right) \cup \cdots \cup f\left(f^{-1}(U_m) \right) \\ &\subseteq^b U_1 \cup \cdots \cup U_m, \end{aligned} \tag{3.8.2.4}$$

so that $\{U_1, \ldots, U_m\}$ is a finite subcover of $f(K)$, and hence $f(K)$ is quasicompact. ■

[a] By Exercise A.3.27.(iii).
[b] By Exercise A.3.29.(i).

And now we can present the classical version of the theorem.

Corollary 3.8.2.5 — Classical Extreme Value Theorem Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then, $f([a,b])$ is closed and bounded. In particular, it attains a maximum and minimum on $[a,b]$.

Proof. The "in particular" is a result of Exercise 2.5.2.22, the statement that closed bounded sets contain their supremum and infimum.

By the Heine-Borel Theorem (Theorem 2.5.3.3), $[a, b]$ is quasicompact. Therefore, by the Extreme Value Theorem, $f([a, b])$ is quasicompact, and therefore, closed and bounded, again by the Heine-Borel Theorem. ∎

In fact, we may as well just combine the classical statements into one.

Corollary 3.8.2.6 — Classical Intermediate-Extreme Value Theorem Let $f\colon [a, b] \to \mathbb{R}$ be continuous. Then, $f([a, b])$ is a closed, bounded, interval.

Exercise 3.8.2.7 Are there functions which are not continuous but which still have the extreme value property?

Finally, we end (the subsubsection) with a handy little result that is a corollary of the Extreme Value Theorem. Hopefully you found an example in Exercise 3.1.3.10 of a continuous bijective function that was not a homeomorphism. In certain special cases, however, you can immediately make this deduction.

Exercise 3.8.2.8 Show that continuous injective function from a quasicompact space into a T_2 space is a homeomorphism onto its image.

Characterizations of quasicompactness

What follows is a summary of all the equivalent characterizations of quasicompactness we are aware of. We have seen several characterization so far, but we have one more to go. IMHO, it is a bit more difficult to understand, and not as useful as the others, so we have procrastinated dealing with it until this subsection on quasicompactness. While the real objective of this subsection was to do the Extreme Value Theorem, it probably makes more sense to summarize these characterizations here than anywhere else.

First of all, we will need one last definition before getting to our final characterization.

Definition 3.8.2.9 — Ultranet Let X be a set and let $\lambda \mapsto x_\lambda$ be a net. Then, $\lambda \mapsto x_\lambda$ is an ***ultranet*** iff for every $S \subseteq X$, $\lambda \mapsto x_\lambda$ is eventually contained in S or eventually contained in S^{C}.

R This is sometimes also called a ***universal net***. I prefer the term "ultranet" because (i) the word "universal" here isn't really being used in the sense in which it is usually meant and (ii) "ultranet" is meant to suggest a similarity with something called an *ultra-filter*. An ***ultra-filter*** is just a fancy word for a proper maximal filter, and it turns out that[a] the derived filter of an ultranet is an ultra-filter, hence the terminology.

[a]Disclaimer: I didn't actually check this because it doesn't matter for us, so I don't guarantee that this statement is exactly true on those nose.

This allows us to present our final characterization (together with a couple of 'lemmas' that can be of use in their own right).

Proposition 3.8.2.10 Let X be a set, let $S \subseteq X$, and let $\lambda \mapsto x_\lambda \in X$ be an ultranet. Then, if $\lambda \mapsto x_\lambda$ is frequently contained in S, then $\lambda \mapsto x_\lambda$ is eventually contained in S.

R Note that the converse is *always* true (by the definitions Meta-definitions 3.2.3 and 3.2.4).

Proof. Suppose that $\lambda \mapsto x_\lambda$ is frequently contained in S. If $\lambda \mapsto x_\lambda$ is eventually contained in S, we're done, so suppose that's not the case. Then, as $\lambda \mapsto x_\lambda$ is an ultranet, it must be the case that $\lambda \mapsto x_\lambda$ is eventually contained in S^{C}. But then it is *not* the case that $\lambda \mapsto x_\lambda$ is frequently contained in S (Meta-proposition 3.2.5): a contradiction. ■

Proposition 3.8.2.11 Let X be a topological space and let $\lambda \mapsto x_\lambda \in X$ be a net. Then, there is a subnet $\mu \mapsto x_{\lambda_\mu}$ of $\lambda \mapsto x_\lambda$ that is a ultranet.

R In brief, every net has an ultra-subnet.

Proof. [a] STEP 1: DEFINE $\mathscr{S}$
Let $\widetilde{\mathcal{F}}$ be the collection of all filter bases $\mathcal{F}$ that contain $\mathcal{F}_{\lambda\mapsto x_\lambda}$ and have the property that $\lambda \mapsto x_\lambda$ is frequently contained in F for all $F \in \mathcal{F}$. Note that $\mathcal{F}_{\lambda\mapsto x_\lambda} \in \widetilde{\mathcal{F}}$ (this is the derived filter base of $\lambda \mapsto x_\lambda$ (Definition 3.3.2)).

STEP 2: OBTAIN A MAXIMAL ELEMENT $\mathcal{F}_0$ OF $\widetilde{\mathcal{F}}$
Regard $\widetilde{\mathcal{F}}$ as a partially-ordered subset with respect to inclusion. Let $\widetilde{\mathcal{W}} \subseteq \widetilde{\mathcal{W}}$ be a well-ordered subset, and define

$$\mathcal{F} := \bigcup_{\mathcal{W}\in\widetilde{\mathcal{W}}} \mathcal{W}. \tag{3.8.2.12}$$

Exercise 3.8.2.13 Show that $\mathcal{F}$ is an upper-bound of $\widetilde{\mathcal{W}}$ in $\widetilde{\mathcal{F}}$. Explicitly, show that $\mathcal{F}$ is (i) a filter base, (ii) contains $\mathcal{F}_{\lambda\mapsto x_\lambda}$, (iii) every element of $\mathcal{F}$ frequently contains $\lambda \mapsto x_\lambda$, and (iv) $\mathcal{F}$ is a superset of every element of $\widetilde{\mathcal{W}}$.

By Zorn's Lemma (Theorem A.3.5.9), $\widetilde{\mathcal{F}}$ has a maximal element. Call such a maximal element $\mathcal{F}_0$.

STEP 3: FIND A SUBNET $\mu \mapsto x_{\lambda_\mu}$ EVENTUALLY CONTAINED IN EACH ELEMENT OF $\mathcal{F}_0$
Denote the index set of $\lambda \mapsto x_\lambda$ by Λ. Order $\mathcal{F}_0$ by reverse inclusion and equip $\Lambda \times \mathcal{F}_0$ with the product order (Definition 2.4.5.19). For $\lambda_0 \in \Lambda$ and $F \in \mathcal{F}_0$, as $\lambda \mapsto x_\lambda$ is frequently contained in F, there is some $\lambda_{\lambda_0,F} \geq \lambda_0$ such that $x_{\lambda_{\lambda_0,F}} \in F$. We claim that

$$\Lambda \times \mathcal{F}_0 \ni \langle \lambda_0, F\rangle \mapsto x_{\lambda_{\lambda_0,F}} \tag{3.8.2.14}$$

is a subnet of $\lambda \mapsto x_\lambda$ eventually contained in every element of $\mathcal{F}$.

First of all, suppose that U eventually contained $\lambda \mapsto x_\lambda$. As $\mathcal{F}_0$ contains $\mathcal{F}_{\lambda\mapsto x_\lambda}$, in fact $U \in \mathcal{F}_0$. Also choose $\lambda_0 \in \Lambda$ such that, whenever $\lambda \geq \lambda_0$, it follows that $x_\lambda \in U$. Now, suppose that $\langle \lambda, F\rangle \geq \langle \lambda_0, F\rangle$. This means that $\lambda \geq \lambda_0$ and that $F \subseteq U$. As $\lambda_{\lambda_0,F} \geq \lambda_0$, it follows that $x_{\lambda_{\lambda_0,F}}$, and hence as $F \subseteq U$, that $x_{\lambda_{\lambda_0,F}} \in U$. Thus, $\langle \lambda_0, F\rangle \mapsto x_{\lambda_{\lambda_0,F}}$ is eventually contained in U, and hence this constitutes a subnet of $\lambda \mapsto x_\lambda$.

Now let $F \in \mathcal{F}_0$. Fix any index λ_0. Then, whenever $\langle \lambda, F'\rangle \geq \langle \lambda_0, F\rangle$, it follows that $x_{\lambda_{\lambda,F'}} \in F' \subseteq F$, and so indeed $\langle \lambda, F'\rangle \mapsto x_{\lambda_{\lambda,F'}}$ is eventually contained in F, as desired.

STEP 4: SHOW THAT $\mu \mapsto x_{\lambda_\mu}$ IS AN ULTRANET

Let $S \subseteq X$. We wish to show that $\mu \mapsto x_{\lambda_\mu}$ is eventually contained in S or S^{C}. If $S \in \mathcal{F}_0$, then by the previous part, $\mu \mapsto x_{\lambda_\mu}$ is eventually contained in S, and we are done, so suppose that $S \notin \mathcal{F}_0$. This means that (Meta-proposition 3.2.5), $\lambda \mapsto x_\lambda$ is frequently in S. Thus, there must be some $F \in \mathcal{F}_0$ such that $S \cap F = \emptyset$, otherwise $\mathcal{F}_0 \cup \{S\}$ would be strictly larger than $\mathcal{F}_0$ in $\widetilde{\mathcal{F}}$. $S \cap F = \emptyset$ implies that $F \subseteq S^{\mathsf{C}}$, and so $\lambda \mapsto x_\lambda$ is frequently contained in S^{C} as it is frequently contained in F. But then (Meta-proposition 3.2.5 again) $\lambda \mapsto x_\lambda$ is eventually contained in S, and we are done. ■

[a] Proof adapted from [How91, pg. 90].

Proposition 3.8.2.15 Let X be a topological space. Then, X is quasicompact iff every ultranet in X converges.

Proof. ($\Rightarrow$) Suppose that X is quasicompact. We first show that, for every net $\lambda \mapsto x_\lambda \in X$, there is some $x_\infty \in X$ that has the property that $\lambda \mapsto x_\lambda$ is frequently contained in U for every open neighborhood U of x_∞. We proceed by contradiction: suppose that there is a net $\lambda \mapsto x_\lambda \mapsto x_\lambda \in X$ that has the property that for every $x \in X$ there is some open neighborhood $U_x \subseteq X$ of x for which it is not the case that $\lambda \mapsto x_\lambda$ if frequently contained in U_x. This means that $\lambda \mapsto x_\lambda$ is eventually contained in U_x^{C} for all $x \in X$ (Metaproposition 3.2.5), and hence every subnet of $\lambda \mapsto x_\lambda$ must eventually be contained in U_x^{C} for all $x \in X$. But then $\lambda \mapsto x_\lambda$ doesn't have any convergent subnet: a contradiction. Thus, it is indeed the case that for every net $\lambda \mapsto x_\lambda \in X$, there is some $x_\infty \in X$ that has the property that $\lambda \mapsto x_\lambda$ is frequently contained in U for every open neighborhood U of x_∞.

Now take $\lambda \mapsto x_\lambda \in X$ be an ultranet. It then follows by the previous proposition (Proposition 3.8.2.10) that in fact $\lambda \mapsto x_\lambda$ is *eventually* contained in every open neighborhood U of x_∞, and hence converges to x_∞, as desired.

($\Leftarrow$) Suppose that every ultranet in X converges. Then, by the previous result (Proposition 3.8.2.11), in particular every net in X has a convergent subnet, and hence X is quasicompact. ■

And now, for convenience, we collection all characterizations together.

Theorem 3.8.2.16. Let X be a topological space. Then, the following are equivalent.

(i). X is quasicompact.[a]

(ii). Every collection of closed sets in X that has the property that every finite intersection is nonempty, has nonempty intersection.

(iii). Every net in X has a convergent subnet.

(iv). Every ultranet in X converges.

[a]That is, every open cover of X has a finite subcover.

Proof. ((i) $\Leftrightarrow$ (ii)) Proposition 3.2.57

((i) $\Leftrightarrow$ (iii)) Proposition 3.2.58

($(i) \Leftrightarrow (iv)$) Proposition 3.8.2.15 ■

4. Uniform spaces

A uniform space is the most general context in which one can talk about concepts such as uniform continuity, uniform convergence, Cauchyness, completeness, etc.. To formalize this notion, we will equip a set with a distinguished set of covers, called *uniform covers*. The example you should always keep in the back of your mind is the collection of all ε-balls for a *fixed* ε: $\mathscr{U}_\varepsilon := \{B_\varepsilon(x) : x \in \mathbb{R}\}$. The idea is that, somehow, all of the sets in the same uniform cover are of the 'same size'.

Having specified the uniform covers, we will then be able to say things like a net $\lambda \mapsto x_\lambda$ is Cauchy iff for every uniform cover $\mathscr{U}$, there is some $U \in \mathscr{U}$ such that $\lambda \mapsto x_\lambda$ is eventually contained in U. In the case that the collection of uniform covers is $\{\mathscr{U}_\varepsilon : \varepsilon > 0\}$,[1] you can check that this is precisely the definition of Cauchyness we had in $\mathbb{R}$ (Definition 2.4.3.1).

Moreover, the generalization from $\mathbb{R}$ to uniform spaces is not a needless abstraction. Indeed, I am *required* to cover metric spaces (Definition 4.2.1.6) in this course, and this is just a very special type of uniform space. Indeed, essentially every topological space we look

[1]Disclaimer: The collection of all the $\mathscr{U}_\varepsilon$ is not actually a uniformity but rather a *uniform base*—see Definition 4.1.3.1.

at in these notes—besides ones cooked up for the express purpose of producing a counter-example—has a canonical uniformity. On the other hand, it is certainly not the case that every topological space we encounter will be a metric space. For example, something as simple as all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ has no canonical metric,[2] but is trivially a uniform space (because it is a topological group—see Proposition 4.2.2.16).

4.1 Basic definitions and facts

A uniform space will wind-up being a set equipped with a special set of covers, the *uniform covers*. Of course, however, as you should expect, we cannot just take *any* collection of covers and declare them to be the uniform covers—the collection of uniform covers has to satisfy certain reasonable properties, analogous to the properties satisfied by the collections of all ε-balls. The key requirement is that the collection of uniform covers has to be *downward-directed* with respect to a relation called *star-refinement*. Thus, before getting to the definition of a uniform space itself, we must say what we mean by "star-refinement" (we will say what we mean by "downward-directed" in the definition of a uniform space itself).

4.1.1 Star-refinements

Definition 4.1.1.1 — Star Let X be a set, let $S \subseteq X$, and let $\mathscr{U}$ be a collection of subsets of X. Then, the ***star*** of S with respect to $\mathscr{U}$, $\mathrm{Star}_{\mathscr{U}}(S)$, is defined by

$$\mathrm{Star}_{\mathscr{U}}(S) := \bigcup_{\substack{U \in \mathscr{U} \\ U \cap S \neq \emptyset}} U. \tag{4.1.1.2}$$

The star of a point is the star of its singleton and denoted $\mathrm{Star}_{\mathscr{U}}(x)$.

[2] For what it's worth, I believe the topology is metrizable (homeomorphic to a metric space), but certainly not with any metric you would like to work with, much less a canonical one.

R In other words, the star of a set with respect to a cover is the union of all elements of the cover which intersect the set.

To help guide your intuition, stars play similar roles as ε-balls did in $\mathbb{R}^d$. For example, if $\mathscr{U}_\varepsilon := \{B_\varepsilon(x) : x \in \mathbb{R}^d\}$ is the cover of $\mathbb{R}$ ball all ε-balls, then the star 'centered' at a point $x \in \mathbb{R}^d$ is

$$\operatorname{Star}_{\mathscr{U}_\varepsilon}(x) = B_{2\varepsilon}(x). \tag{4.1.1.3}$$

We will certainly be wanting to take the 'image' and 'preimage' of a cover.

Definition 4.1.1.4 — Image and preimage of a cover Let $f\colon X \to Y$ be a function, let $\mathscr{U}$ be a cover of X, and $\mathscr{V}$ be a cover of Y. Then, the ***image*** of $\mathscr{U}$, $f(\mathscr{U})$, is defined by

$$f(\mathscr{U}) := \{f(U) : U \in \mathscr{U}\}. \tag{4.1.1.5}$$

The ***preimage*** of $\mathscr{V}$, $f^{-1}(\mathscr{V})$, is defined by

$$f^{-1}(\mathscr{V}) := \{f^{-1}(V) : V \in \mathscr{V}\}. \tag{4.1.1.6}$$

R We needed to make these definitions because, technically speaking, we only defined the image an preimage of *subsets* of X and Y respectively. As $\mathscr{U}$ and $\mathscr{V}$ are subsets of 2^X and 2^Y respectively, not X and Y, to talk about their 'usual' image and preimage, we would need to have a function from 2^X to 2^Y.[a] In particular, note that the definition of the preimage of a cover is *not* $\{U \in 2^X : f(U) \in \mathscr{V}\}$.

[a] Actually, given a function $f\colon X \to Y$, we obtain a function from 2^X to 2^Y *and* a function from 2^Y to 2^X: $f\colon 2^X \to 2^Y$ (the function that sends a set to its image) and $f^{-1} : 2^Y \to 2^X$ (the function that sends a set to its preimage) respectively. The preimage of a cover is the image of the cover with respect to the preimage function $f^{-1} : 2^Y \to 2^X$. Likewise, the image of a cover is the image of the cover with respect to the image function $f\colon 2^X \to 2^Y$.

▪ **Example 4.1.1.7 — The star of the preimage is not the preimage of the star** One might hope for the preimage of the star to be the star of the preimage, that is, for $f\colon X\to Y$, $V\subseteq Y$, and $\mathscr{V}$ a cover of Y, that

$$f^{-1}\left(\mathrm{Star}_{\mathscr{V}}(V)\right)=\mathrm{Star}_{f^{-1}(\mathscr{V})}(f^{-1}(V)). \tag{4.1.1.8}$$

Unfortunately, this is not necessarily the case. For example, take f to be the inclusion $\mathbb{R}\hookrightarrow\mathbb{R}^2$ (with image the x-axis), take V to be any subset of $\mathbb{R}^2$ which does not intersect the x-axis (e.g. $V=\{\langle x,y\rangle\in\mathbb{R}^2:y=1\}$), and take $\mathscr{V}:=\{\mathbb{R}^2\}$, namely the cover of $\mathbb{R}^2$ consisting of only $\mathbb{R}^2$ itself. Then,

$$\mathrm{Star}_{\mathscr{V}}(V)=\mathbb{R}^2, \tag{4.1.1.9}$$

and so

$$f^{-1}\left(\mathrm{Star}_{\mathscr{V}}(V)\right)=\mathbb{R}. \tag{4.1.1.10}$$

On the other hand,

$$\mathrm{Star}_{f^{-1}(\mathscr{V})}(f^{-1}(V))=\emptyset \tag{4.1.1.11}$$

simply because $f^{-1}(V)=\emptyset$.

As the image tends to be even more poorly behaved,[3] you might guess that we have a similar problem with the image. You would be correct.

▪ **Example 4.1.1.12 — The star of the image is not the image of the star** Take $f\colon\mathbb{R}^2\to\mathbb{R}$ to be the projection onto the x-axis, define

$$\mathscr{U}:=\{[m+y,m+1+y]\times\{y\}:m\in\mathbb{Z},\ y\in\mathbb{R}\}, \tag{4.1.1.13}$$

[3]For example, preimages preserves intersections but images in general do not—see Exercise A.3.27.

and $U_0 := (0,1) \times \{0\}$. Then,

$$\mathrm{Star}_{\mathscr{U}}(U_0) = [0,1] \times \{0\},^{a} \tag{4.1.1.14}$$

and so

$$f\left(\mathrm{Star}_{\mathscr{U}}(U_0)\right) = [0,1]. \tag{4.1.1.15}$$

On the other hand,

$$f(\mathscr{U}) = \{[m+y, m+1+y] : m \in \mathbb{Z},\ y \in \mathbb{R}\} \tag{4.1.1.16}$$

and $f(U_0) = (0,1)$, and so $f(U_0)$ intersects $[m+y, m+1+y]$ for $m = 0$ and $-1 < y < 1$, and so

$$\mathrm{Star}_{f(\mathscr{U})} f(U_0) \supseteq \bigcup_{-1<y<1} [y, y+1] = (-1,2), \tag{4.1.1.17}$$

which is strictly larger than $f\left(\mathrm{Star}_{\mathscr{U}}(U_0)\right)$.

[a] The only elements of $\mathscr{U}$ which have y-coordinate 0 are of the form $[m, m+1] \times \{0\}$ for $m \in \mathbb{Z}$, and this intersects $(0,1)$ only for $m = 0$.

On the other hand, we do have the following. Despite this, we always have one inclusion.

Proposition 4.1.1.18 Let $f : X \to Y$ be a function, let $S \subseteq X$, let $T \subseteq Y$, let $\mathscr{U}$ be a cover of X, and let $\mathscr{V}$ be a cover of Y. Then, we have the following.

(i).

$$\mathrm{Star}_{f^{-1}(\mathscr{V})}(f^{-1}(T)) \subseteq f^{-1}\left(\mathrm{Star}_{\mathscr{V}}(T)\right). \tag{4.1.1.19}$$

Furthermore, if f is surjective, then we have equality.

(ii).

$$\mathrm{Star}_{f(\mathscr{S})}(f(S)) \supseteq f(\mathrm{Star}_{\mathscr{U}}(U)). \tag{4.1.1.20}$$

Furthermore, if f is injective, then we have equality.

(iii).

$$\mathrm{Star}_{f^{-1}(\mathscr{V})}(S) = f^{-1}\left(\mathrm{Star}_{\mathscr{V}}(f(S))\right). \tag{4.1.1.21}$$

(iv).

$$\mathrm{Star}_{f(\mathscr{U})}(T) = f\left(\mathrm{Star}_{\mathscr{U}}\, f^{-1}(T)\right). \tag{4.1.1.22}$$

Proof. (i) Note that we have that (Exercise A.3.27.(ii) and Exercise A.3.29.(i)) $V \cap T \supseteq f(f^{-1}(V) \cap f^{-1}(T))$. Therefore, if $f^{-1}(V)$ intersects $f^{-1}(T)$, it must be the case that V intersects T. Furthermore, f is surjective, so that $f(f^{-1}(T')) = T'$ for all $T' \subseteq Y$ (Exercise A.3.29), then we would in fact that that $V \cap T = f(f^{-1}(V) \cap f^{-1}(T))$, so that in this case V intersects T iff $f^{-1}(V)$ intersects $f^{-1}(T)$.

$$\begin{aligned}\mathrm{Star}_{f^{-1}(\mathscr{V})}(f^{-1}(T)) &:= \bigcup_{\substack{V \in \mathscr{V} \\ f^{-1}(V) \cap f^{-1}(T) \neq \emptyset}} f^{-1}(V) \\ &=^{a} f^{-1}\left(\bigcup_{\substack{V \in \mathscr{V} \\ f^{-1}(V) \cap f^{-1}(T) \neq \emptyset}} V\right) \\ &\subseteq^{b} f^{-1}\left(\bigcup_{\substack{V \in \mathscr{V} \\ V \cap T \neq \emptyset}} V\right) \\ &=: f^{-1}\left(\mathrm{Star}_{\mathscr{V}}(T)\right).\end{aligned} \tag{4.1.1.23}$$

(ii) Note that we have that (Exercise A.3.27.(iv) and Exercise A.3.29.(ii)) $U \cap S \subseteq f^{-1}(f(U) \cap f(S))$. Therefore, if U intersects S, it must be the case that $f(U)$ intersects $f(S)$. Furthermore, note that as f is injective, $f(S_1 \cap S_2) = f(S_1) \cap f(S_2)$ (Exercise A.3.27) and $f^{-1}(f(S')) = S'$ (Exercise A.3.29) for all subsets $S_1, S_2, S' \subseteq X$. Thus, if f is injective,

$$\begin{aligned} U \cap S &= f^{-1}(f(U)) \cap f^{-1}(f(S)) \\ &= f^{-1}(f(U) \cap f(S)), \end{aligned} \tag{4.1.1.24}$$

and hence

$$f(U \cap S) = f(U) \cap f(S), \tag{4.1.1.25}$$

so in this case U intersects S iff $f(U)$ intersects $f(S)$. Hence,

$$\begin{aligned} f(\mathrm{Star}_{\mathscr{U}}(S)) &:= f\left(\bigcup_{\substack{U \in \mathscr{U} \\ U \cap S \neq \emptyset}} U \right) \\ &=^c \bigcup_{\substack{U \in \mathscr{U} \\ U \cap S \neq \emptyset}} f(U) \\ &\subseteq^d \bigcup_{\substack{U \in \mathscr{U} \\ f(U) \cap f(S) \neq \emptyset}} f(U) \\ &=: \mathrm{Star}_{f(\mathscr{U})} f(S). \end{aligned} \tag{4.1.1.26}$$

(iii) Note that $f^{-1}(T)$ intersects S iff T intersects $f(S)$. Using this, we have

$$\begin{aligned} \mathrm{Star}_{f^{-1}(\mathscr{V})}(S) &:= \bigcup_{\substack{V \in \mathscr{V} \\ f^{-1}(V) \cap S \neq \emptyset}} f^{-1}(V) \\ &= f^{-1}\left(\bigcup_{\substack{V \in \mathscr{V} \\ V \cap f(S) \neq \emptyset}} V \right) \\ &=: f^{-1}\left(\mathrm{Star}_{\mathscr{V}}(f(S))\right). \end{aligned} \tag{4.1.1.27}$$

(iv) Similarly as in (iii),

$$\operatorname{Star}_{f(\mathscr{U})}(T) := \bigcup_{\substack{U\in\mathscr{U} \\ f(U)\cap T\neq\emptyset}} f(U) = f\left(\bigcup_{\substack{U\in\mathscr{U} \\ U\cap f^{-1}(T)\neq\emptyset}} U\right) \qquad (4.1.1.28)$$
$$=: f\left(\operatorname{Star}_{\mathscr{U}}(f^{-1}(T))\right).$$

■

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[a]Exercise A.3.27.(i)

[b]Here we are using the fact that $f^{-1}(V)$ intersects $f^{-1}(T)$ implies that V intersects T. Also note that we have equality here if f is surjective.

[c]Exercise A.3.27.(iii)

[d]Here we are using the fact that U intersects S implies that $f(U)$ intersects $f(S)$. Also note that we have equality here if f is injective.

Definition 4.1.1.29 — Refinement and star-refinement Let X be a set, and let $\mathscr{U}$ and $\mathscr{V}$ be covers on X.

(i). $\mathscr{U}$ is a ***refinement*** of $\mathscr{V}$, written $\mathscr{U} \preceq \mathscr{V}$ iff for every $U \in \mathscr{U}$ there is some $V \in \mathscr{V}$ such that $U \subseteq V$.

(ii). $\mathscr{U}$ is a ***star-refinement*** of $\mathscr{V}$, written $\mathscr{U} \lll \mathscr{V}$ iff for every $U \in \mathscr{U}$ there is a $V \in \mathscr{V}$ such that $\operatorname{Star}_{\mathscr{U}}(U) \subseteq V$.

R The intuition is that every element of $\mathscr{U}$ is small enough to be contained in some element of $\mathscr{V}$.

R In other words, $\mathscr{U}$ is a star-refinement of $\mathscr{V}$ iff for all $U \in \mathscr{U}$, there is some $V \in \mathscr{V}$ such that, whenever $U' \in \mathscr{U}$ intersects U, it follows that $U' \subseteq V$. The intuition for star-refinements is the same as for refinements, except that a star-refinement is *much* finer than a mere refinement.

Proposition 4.1.1.30 Let X be a set, let $\mathscr{U}$ and $\mathscr{V}$ be covers of X, and let $S \subseteq X$. Then, if $\mathscr{U}$ refines $\mathscr{V}$, then

$$\mathrm{Star}_{\mathscr{U}}(S) \subseteq \mathrm{Star}_{\mathscr{V}}(S). \tag{4.1.1.31}$$

R In particular, (4.1.1.31) holds if $\mathscr{U}$ *star*-refines $\mathscr{V}$.

Proof. Let $U \in \mathscr{U}$ intersect S. From the definition of refinement, there is some $V \in \mathscr{V}$ such that $U \subseteq V$. As U intersects S, V certainly intersects S, and so $V \subseteq \mathrm{Star}_{\mathscr{V}}(S)$, and hence $U \subseteq \mathrm{Star}_{\mathscr{U}}(S)$. Taking the union over all elements of $\mathscr{U}$ which intersect S, we find $\mathrm{Star}_{\mathscr{U}}(S) \subseteq \mathrm{Star}_{\mathscr{V}}(S)$. ■

Exercise 4.1.1.32 Show that $\preceq$ is a preorder, but not a partial-order.

Exercise 4.1.1.33 Show that $\lll$ is transitive, but not even reflexive.

Any two covers always have a common refinement. In fact, they have a canonical (but not unique!) largest one.

Definition 4.1.1.34 — Meet of covers Let X be a set, and let $\mathscr{U}$ and $\mathscr{V}$ be covers of X. Then, the ***meet*** of $\mathscr{U}$ and $\mathscr{V}$, $\mathscr{U} \wedge \mathscr{V}$, is defined by

$$\mathscr{U} \wedge \mathscr{V} := \{U \cap V : U \in \mathscr{U} \text{ and } V \in \mathscr{V}\}. \tag{4.1.1.35}$$

R The term "meet" and notation "$\mathscr{U} \wedge \mathscr{V}$" is notation taken from the theory of partially-ordered sets where $x \wedge y$ (the *meet*) of x and y is defined to be $\inf\{x, y\}$. In our case, however, this is abuse of notation and terminology as $\preceq$ is not a partial-order (and so infima need not be unique—see Exercise 1.4.1.2).

Exercise 4.1.1.36 Show that (i) $\mathscr{U} \wedge \mathscr{V} \preceq \mathscr{U}, \mathscr{V}$; and (ii) if $\mathscr{W}$ refines both $\mathscr{U}$ and $\mathscr{V}$, then it refines $\mathscr{U} \wedge \mathscr{V}$. Find an example to show that it is *not* the unique such cover with these two properties.

Exercise 4.1.1.37 Let X be a set and let $\mathscr{U}$ be a cover of X. Show that there does not necessarily exist a cover $\mathscr{V}$ of X such that (i) $\mathscr{V}$ star-refines $\mathscr{U}$; and (ii) if $\mathscr{W}$ star-refines $\mathscr{U}$, then it star-refines $\mathscr{V}$?

R In particular, there can't be any construction that does for star-refinements what the meet does for refinements.

Proposition 4.1.1.38 Let X be a set, let $x \in X$, and let $\mathscr{U}$ and $\mathscr{V}$ be covers of X. Then,

$$\mathrm{Star}_{\mathscr{U}}(x) \cap \mathrm{Star}_{\mathscr{V}}(x) = \mathrm{Star}_{\mathscr{U} \wedge \mathscr{V}}(x). \tag{4.1.1.39}$$

W Warning: This fails if you replace x with a general set S—see the following counter-example (though we should always have the $\supseteq$ inclusion).

Proof. We simply 'compute'.

$$\begin{aligned}\mathrm{Star}_{\mathscr{U}}(x) \cap \mathrm{Star}_{\mathscr{V}}(x) &:= \left(\bigcup_{\substack{U \in \mathscr{U} \\ x \in U}} U\right) \cap \left(\bigcup_{\substack{V \in \mathscr{V} \\ x \in V}} V\right) \\ &= \bigcup_{\substack{U \in \mathscr{U} \text{ s.t. } x \in U \\ V \in \mathscr{V} \text{ s.t. } x \in V}} U \cap V \\ &= \bigcup_{\substack{U \in \mathscr{U}, V \in \mathscr{V} \\ x \in U \cap V}} U \cap V =: \mathrm{Star}_{\mathscr{U} \wedge \mathscr{V}}(x).\end{aligned} \tag{4.1.1.40}$$

∎

▪ **Example 4.1.1.41 —** $\mathrm{Star}_{\mathscr{U}}(S) \cap \mathrm{Star}_{\mathscr{V}}(S) \neq \mathrm{Star}_{\mathscr{U} \wedge \mathscr{V}}(S)$ Define $X := \{0, 1, 2, 3\}$, and

$$\mathscr{U} := \{\{0, 1\}, \{2\}, \{3\}\} \text{ and } \mathscr{V} := \{\{0, 2\}, \{1, 3\}\} .$$

Then,

$$\mathscr{U} \wedge \mathscr{V} = \{\emptyset, \{0\}, \{1\}, \{2\}, \{3\}\} . \tag{4.1.1.42}$$

Define $S := \{1, 2\}$. Then, $\mathrm{Star}_{\mathscr{U}}(S) = \{0, 1\} \cup \{2\} = \{0, 1, 2\}$ and $\mathrm{Star}_{\mathscr{V}}(S) = \{0, 2\} \cup \{1, 3\} = \{0, 1, 2, 3\}$, and so

$$\mathrm{Star}_{\mathscr{U}}(S) \cap \mathrm{Star}_{\mathscr{V}}(S) = \{0, 1, 2\}. \tag{4.1.1.43}$$

On the other hand,

$$\mathrm{Star}_{\mathscr{U} \wedge \mathscr{V}}(S) = \{1, 2\}. \tag{4.1.1.44}$$

Exercise 4.1.1.45 Show that if $\mathscr{U}_1 \preceq \mathscr{U}_2$ (resp.$\mathscr{U}_1 \lll \mathscr{U}_2$) and $\mathscr{V}_1 \preceq \mathscr{V}_2$ (resp. $\mathscr{V}_1 \lll \mathscr{V}_2$), then $\mathscr{U}_1 \wedge \mathscr{V}_1 \preceq \mathscr{U}_2 \wedge \mathscr{V}_2$ (resp. $\mathscr{U}_1 \wedge \mathscr{V}_1 \lll \mathscr{U}_2 \wedge \mathscr{V}_2$).

Proposition 4.1.1.46 Let $f\colon X \to Y$ be a function and let $\mathscr{U}$ and $\mathscr{V}$ be covers of Y such that $\mathscr{U} \preceq \mathscr{V}$ (resp. $\mathscr{U} \lll \mathscr{V}$). Then, $f^{-1}(\mathscr{U}) \preceq f^{-1}(\mathscr{V})$ (resp. $f^{-1}(\mathscr{U}) \lll f^{-1}(\mathscr{V})$).

Proof. We first do the case with $\mathscr{U} \preceq \mathscr{V}$. Let $f^{-1}(U) \in f^{-1}(\mathscr{U})$ for $U \in \mathscr{U}$. Then, as $\mathscr{U} \preceq \mathscr{V}$, there is some $V \in \mathscr{V}$ such that $U \subseteq V$. Then, $f^{-1}(U) \subseteq f^{-1}(V)$, and so $f^{-1}(\mathscr{U}) \preceq f^{-1}(\mathscr{V})$.

Now we do the case $\mathscr{U} \lll \mathscr{V}$. Let $f^{-1}(U) \in f^{-1}(\mathscr{U})$ for $U \in \mathscr{U}$. Then, as $\mathscr{U} \lll \mathscr{V}$, there is some $V \in \mathscr{V}$ such that

$\mathrm{Star}_{\mathscr{U}}(U) \subseteq V$. Hence,

$$\mathrm{Star}_{f^{-1}(\mathscr{U})}(f^{-1}(U)) \subseteq f^{-1}(\mathrm{Star}_{\mathscr{U}}(U)) \subseteq f^{-1}(V), \quad (4.1.1.47)$$

where we have applied Proposition 4.1.1.18, and so $f^{-1}(\mathscr{U}) \lll f^{-1}(\mathscr{V})$. ■

Exercise 4.1.1.48 Show that if $\mathscr{U} \leq \mathscr{V}$, then $f(\mathscr{U}) \leq f(\mathscr{V})$.

Proposition 4.1.1.49 Let $f\colon X \to Y$ be a function and let $\mathscr{U}$ and $\mathscr{V}$ be covers of Y. Then, $f^{-1}(\mathscr{U} \wedge \mathscr{V}) = f^{-1}(\mathscr{U}) \wedge f^{-1}(\mathscr{V})$.

Proof. We simply 'compute'.

$$\begin{aligned} & f^{-1}(\mathscr{U}) \wedge f^{-1}(\mathscr{V}) \\ & \qquad := \left\{ f^{-1}(U) \cap f^{-1}(V) : U \in \mathscr{U}, V \in \mathscr{V} \right\} \\ & \qquad = \left\{ f^{-1}(U \cap V) : U \in \mathscr{U}, V \in \mathscr{V} \right\} \\ & \qquad =: f^{-1}(\mathscr{U} \wedge \mathscr{V}). \end{aligned} \quad (4.1.1.50)$$

■

Unfortunately, however, in general, it will not be the case that the image preserves star-refinements.

Exercise 4.1.1.51 Find an example of covers $\mathscr{U}$ and $\mathscr{V}$ with $\mathscr{U} \lll \mathscr{V}$, but $f(\mathscr{U})$ not a star-refinement of $f(\mathscr{V})$.

However, in special cases, it will.

Proposition 4.1.1.52 Let $f\colon X \to Y$ be a function and let $\mathscr{U}$ and $\mathscr{V}$ be covers of X such that $\mathscr{U} \lll \mathscr{V}$. Then, if f is surjective and $f^{-1}(f(U)) = U$ for all $U \in \mathscr{U}$, then $f(\mathscr{U}) \lll f(\mathscr{V})$.

Proof. Suppose that $f^{-1}(f(U)) = U$ for all $U \in \mathscr{U}$. Let $f(U) \in f(\mathscr{U})$. Then, there is some $V \in \mathscr{V}$ such that $\mathrm{Star}_{\mathscr{U}}(U) \subseteq V$. As $f^{-1}(f(U)) = U$ for all $U \in \mathscr{U}$, we have

$$\mathrm{Star}_{\mathscr{U}}(U) = \mathrm{Star}_{f^{-1}(f(\mathscr{U}))}(U). \tag{4.1.1.53}$$

By Proposition 4.1.1.18, we have

$$\mathrm{Star}_{f^{-1}(f(\mathscr{U}))}(U) = f^{-1}\left(\mathrm{Star}_{f(\mathscr{U})}(f(U))\right). \tag{4.1.1.54}$$

Putting this together, we get

$$f^{-1}\left(\mathrm{Star}_{f(\mathscr{U})}(f(U))\right) = \mathrm{Star}_{\mathscr{U}}(U) \subseteq V, \tag{4.1.1.55}$$

and hence, as f is surjective

$$\mathrm{Star}_{f(\mathscr{U})}(f(U)) = f\left(f^{-1}\left(\mathrm{Star}_{f(\mathscr{U})}(f(U))\right)\right) \subseteq f(V)$$

Thus, $f(\mathscr{U}) \lll f(\mathscr{V})$. ■

4.1.2 Uniform spaces

Definition 4.1.2.1 — Uniform space A ***uniform space*** is a set X equipped with a nonempty collection $\widetilde{\mathscr{U}}$ of covers, the ***uniformity***, such that

(i). (Upward-closed) if $\mathscr{U} \in \widetilde{\mathscr{U}}$ and $\mathscr{U} \lll \mathscr{V}$, then $\mathscr{V} \in \widetilde{\mathscr{U}}$; and

(ii). (Downward-directed) if $\mathscr{U}, \mathscr{V} \in \widetilde{\mathscr{U}}$, then there is some $\mathscr{W} \in \widetilde{\mathscr{U}}$ such that $\mathscr{W} \lll \mathscr{U}$ and $\mathscr{W} \lll \mathscr{V}$.

R The elements of $\widetilde{\mathscr{U}}$ are ***uniform covers***.

R The intuition is that, in a given uniform cover $\mathscr{U}$, every element of $\mathscr{U}$ is 'of the same size' (think $\mathscr{U} := \{B_\varepsilon(x) : x \in \mathbb{R}\}$ for a *fixed* $\varepsilon > 0$).

R Note that, by taking $\mathscr{U} = \mathscr{V}$ in (ii), we see that, in particular, every uniform cover is star-refined by some other uniform cover.

R Note that the cover $\{X\}$ is an element of every uniformity. This follows from the fact that any collection of uniform covers is required to be nonempty and the fact that collections of uniform covers are upward-closed with respect to star-refinement. (We mention this because sometimes that $\{X\}$ is a uniform cover is taken as an axiom, in place of the requirement that the collection of uniform covers simply be nonempty.)

R Another common way to define uniform spaces are to make use of what are called *entourages*. An *entourage* is a collection of relations on X that has to satisfy a list of axioms. For example, in the case of $\mathbb{R}^d$, for every $\varepsilon > 0$, you would have the relation $x \sim_\varepsilon y$ iff $|x - y| < \varepsilon$. The translation between that definition and this one is as follows: given a relation $\sim$ in an entourage, you obtain a corresponding uniform cover $\{B_x : x \in X\}$, where $B_x := \{y \in X : y \sim x\}$; and given a uniform cover $\mathscr{U}$, you obtain a relation $\sim$ defined by $x \sim y$ iff there is some $U \in \mathscr{U}$ with $x, y \in U$. We will not use this definition, but as it is pretty common, it is worth knowing how to understand it when you encounter it.

Of incredible importance is that uniformities *define* a canonical topology. Thus, we can think of uniform spaces as topological spaces with *extra structure*.

Proposition 4.1.2.2 — Uniform topology Let $\langle X, \widetilde{\mathscr{U}}\rangle$ be a uniform space. Then, for $x \in X$,

$$\mathscr{B}_x := \left\{\mathrm{Star}_{\mathscr{U}}(x) : \mathscr{U} \in \widetilde{\mathscr{U}}\right\} \tag{4.1.2.3}$$

is a neighborhood base at x. The topology defined by this neighborhood base is the ***uniform topology*** on X with respect to $\widetilde{\mathscr{U}}$.

R Explicitly, this means that $U \subseteq X$ is open iff for every $x \in U$ there is some uniform cover $\mathscr{U}$ such that $\mathrm{Star}_{\mathscr{U}}(x) \subseteq U$. Note how this is precisely the same as our definition in $\mathbb{R}^d$ when you replace the start with $B_\varepsilon(x)$—see Definition 2.5.2.1.

R Unless otherwise stated, uniform spaces are *always* equipped with the uniform topology.

Proof. Let $\mathrm{Star}_{\mathscr{U}_1}(x), \mathrm{Star}_{\mathscr{U}_2}(x) \in \mathscr{B}_x$. Let $\mathscr{U}_3$ be a common star-refinement of both $\mathscr{U}_1$ and $\mathscr{U}_2$. Define

$$\begin{aligned} U := \Big\{ y \in \mathrm{Star}_{\mathscr{U}_3}(x) : \text{there is some } \mathscr{V} \in \widetilde{\mathscr{U}} \\ \text{such that } \mathrm{Star}_{\mathscr{V}}(y) \subseteq \mathrm{Star}_{\mathscr{U}_3}(x) \Big\} . \end{aligned} \tag{4.1.2.4}$$

We wish to show that $U \subseteq \mathrm{Star}_{\mathscr{U}_1}(x) \cap \mathrm{Star}_{\mathscr{U}_2}(x)$ such that (i) $x \in U$ and (ii) for every $y \in U$ there is some $\mathscr{V} \in \widetilde{\mathscr{U}}$ such that $\mathrm{Star}_{\mathscr{V}}(y) \subseteq U$—see Proposition 3.1.1.9 (the result which tells us how to define a topology by defining a neighborhood base).

To show that $U \subseteq \mathrm{Star}_{\mathscr{U}_1}(x) \cap \mathrm{Star}_{\mathscr{U}_2}(x)$, we show that

$$\mathrm{Star}_{\mathscr{U}_3}(x) \subseteq \mathrm{Star}_{\mathscr{U}_1}(x) \cap \mathrm{Star}_{\mathscr{U}_2}(x). \tag{4.1.2.5}$$

By $1 \leftrightarrow 2$ symmetry, it suffices to just prove that $\mathrm{Star}_{\mathscr{U}_3} \subseteq \mathrm{Star}_{\mathscr{U}_1}$. By definition, we have

$$\mathrm{Star}_{\mathscr{U}_3}(x) := \bigcup_{\substack{U \in \mathscr{U}_3 \\ x \in U}} U. \tag{4.1.2.6}$$

So, let $U \in \mathscr{U}_3$ contain x. Because $\mathscr{U}_3$ star-refines $\mathscr{U}_1$, there is some $V \in \mathscr{U}_1$ such that

$$\mathrm{Star}_{\mathscr{U}_3}(U) \subseteq V. \tag{4.1.2.7}$$

In particular, $U \subseteq V$. Then, V contains x, and so $V \subseteq \mathrm{Star}_{\mathscr{U}_1}(x)$, and so $U \subseteq \mathrm{Star}_{\mathscr{U}_1}(x)$. It follows that

$$\mathrm{Star}_{\mathscr{U}_3}(x) \subseteq \mathrm{Star}_{\mathscr{U}_1}(x), \tag{4.1.2.8}$$

as desired.

$x \in U$ as tautologically $\mathrm{Star}_{\mathscr{U}_3}(x) \subseteq \mathrm{Star}_{\mathscr{U}_3}(x)$.

Now, let $y \in U$. We wish to find a cover $\mathscr{V} \in \widetilde{\mathscr{U}}$ such that $\mathrm{Star}_{\mathscr{V}}(y) \subseteq U$. By definition, there is a cover $\mathscr{W} \in \widetilde{\mathscr{U}}$ such that $\mathrm{Star}_{\mathscr{W}}(y) \subseteq \mathrm{Star}_{\mathscr{U}_3}(x)$. Let $\mathscr{V} \in \widetilde{\mathscr{U}}$ be a star-refinement of $\mathscr{W}$. We wish to show that $\mathrm{Star}_{\mathscr{V}}(y) \subseteq U$. So, let $z \in \mathrm{Star}_{\mathscr{V}}(y)$. To show that $z \in U$, it suffices to show that $\mathrm{Star}_{\mathscr{V}}(z) \subseteq \mathrm{Star}_{\mathscr{U}_3}(x)$. As $\mathrm{Star}_{\mathscr{W}}(y) \subseteq \mathrm{Star}_{\mathscr{U}_3}(x)$, it in turn suffices to show that $\mathrm{Star}_{\mathscr{V}}(z) \subseteq \mathrm{Star}_{\mathscr{W}}(y)$. So, let $V \in \mathscr{V}$ be such that $z \in V$. We wish to show that $V \subseteq \mathrm{Star}_{\mathscr{W}}(y)$. As $\mathscr{V} \lll \mathscr{W}$, there is some $W \in \mathscr{W}$ such that $\mathrm{Star}_{\mathscr{V}}(V) \subseteq W$. As $V \subseteq \mathrm{Star}_{\mathscr{V}}(V)$, it suffices to show that $W \subseteq \mathrm{Star}_{\mathscr{W}}(y)$, that is, it suffices to show that $y \in W$. As $\mathrm{Star}_{\mathscr{V}}(V) \subseteq W$, it suffices to show that $y \in \mathrm{Star}_{\mathscr{V}}(V)$. However, as $z \in \mathrm{Star}_{\mathscr{V}}(y)$, $y \in \mathrm{Star}_{\mathscr{V}}(z) \subseteq \mathrm{Star}_{\mathscr{V}}(V)$, as desired.

R Note that this proof did not make use of the upward-closed axiom. Thus, in fact, uniform bases (see below in Definition 4.1.3.1) suffice to define the uniform topology as well.

■

■ **Example 4.1.2.9 — Discrete and indiscrete uniform spaces** Just as with topological spaces, we can always put the largest and the smallest uniformity on a set X. The former case, in which every cover of X is a uniform cover, is the ***discrete uniformity***, and the latter, in which the only uniform cover is $\{X\}$, is the ***indiscrete uniformity***.

Exercise 4.1.2.10 Show that the uniform topology with respect to the discrete uniformity is the discrete topology and that the uniform topology with respect to the indiscrete uniformity is the indiscrete topology.

Just as we have continuous maps between topological spaces, we have *uniformly*-continuous maps between uniform spaces.

Definition 4.1.2.11 — Uniformly-continuous function Let $f\colon X \to Y$ be a function between uniform spaces. Then, f is ***uniformly-continuous*** iff the preimage of every uniform cover is a uniform cover.

Example 4.1.2.12 — The category of uniform spaces The category of uniform spaces is the category **Uni**

(i). whose collection of objects $\mathrm{Obj}(\mathbf{Uni})$ is the collection of all uniform spaces;
(ii). with morphism set $\mathrm{Mor}_{\mathbf{Uni}}(X,Y)$ precisely the set of all uniformly-continuous functions from X to Y;
(iii). whose composition is given by ordinary function composition; and
(iv). whose identities are given by the identity functions.

Exercise 4.1.2.13 Show that the composition of two uniformly-continuous functions is uniformly-continuous.

R Note that this is something you need to check in order for **Uni** to actually form a category ($\mathrm{Mor}_{\mathbf{Uni}}(X,Y)$ needs to be closed under composition). You also need to verify the identity function is uniformly-continuous, but this is trivial (the preimage of a cover is itself, so. . .).

Definition 4.1.2.14 — Uniform-homeomorphism
Let $f\colon X \to Y$ be a function between uniform spaces. Then, f is a ***uniform-homeomorphism*** iff it is an isomorphism in **Uni**.

Exercise 4.1.2.15 Show that a function is a uniform-homeomorphism iff (i) it is bijective, (ii) it is uniformly-continuous, and (iii) its inverse is uniformly-continuous.

Exercise 4.1.2.16 Show that if a function is uniformly-continuous, then it is continuous.

Exercise 4.1.2.17 Find an example of a function that is bijective and uniformly-continuous, but not a uniform-homeomorphism.

4.1.3 Uniform bases, and the initial and final uniformities

It is usually convenient to not specify every uniform cover explicitly, but rather, to specify a certain collection of uniform covers analogous to specifying a base for a topology.[4]

Definition 4.1.3.1 — Uniform base Let X be a uniform space and let $\widetilde{\mathscr{B}}$ be a collection of uniform covers of X. Then, $\widetilde{\mathscr{B}}$ is a ***uniform base*** for the uniformity on X iff the statement that a cover $\mathscr{U}$ is a uniform cover is equivalent to the statement that there is some $\mathscr{B} \in \widetilde{\mathscr{B}}$ such that $\mathscr{B} \lll \mathscr{U}$.

R You should compare this to the definition of a base for a topology (Definition 3.1.1.1).

And just like with bases, the real reason uniform bases are important is because they allow us to *define* uniformities. Thus, same

[4]Unfortunately, the way in which one might like to define a uniformity that is analogous to generating collections for topologies doesn't work—see Example 4.1.3.11.

as before, it is important to know when a collection of covers of a set form a uniform base for some uniformity.

Proposition 4.1.3.2 Let X be a set and let $\widetilde{\mathscr{B}}$ be a nonempty collection of covers of X. Then, there exists a unique uniformity for which $\widetilde{\mathscr{B}}$ is a uniform base iff $\widetilde{\mathscr{B}}$ is downward-directed with respect to $\preccurlyeq$.

R Just as we did for bases, if a set X does not a priori come with a uniformity, we will still refer to any collection of covers that is downward-directed with respect to $\preccurlyeq$ as a *uniform base*.

Proof. ($\Rightarrow$) Suppose that there exists a uniformity for which $\widetilde{\mathscr{B}}$ is a uniform base. Let $\mathscr{B}, \mathscr{C} \in \widetilde{\mathscr{B}}$. Then, there is certainly some uniform cover $\mathscr{U}$ which star-refines both $\mathscr{B}$ and $\mathscr{C}$ (recall that covers in $\widetilde{\mathscr{B}}$ are a priori taken to be uniform covers). However, because $\mathscr{U}$ is a uniform cover and $\widetilde{\mathscr{B}}$ is a uniform base, there is some $\mathscr{D} \in \widetilde{\mathscr{B}}$ such that $\mathscr{D} \preccurlyeq \mathscr{U}$. As $\mathscr{U}$ star-refines both $\mathscr{B}$ and $\mathscr{C}$, it follows that $\mathscr{D}$ does as well. Thus, $\widetilde{\mathscr{B}}$ is indeed downward-directed with respect to star-refinement.

($\Leftarrow$) Suppose that $\widetilde{\mathscr{B}}$ is downward-directed with respect to $\preccurlyeq$. Define $\widetilde{\mathscr{U}}$ to be the collection of covers that are star-refined by some element of $\widetilde{\mathscr{B}}$. By the definition of uniform bases, this was the only possibility. As $\widetilde{\mathscr{B}}$ is nonempty and downward-directed with respect to $\preccurlyeq$, it follows that $\widetilde{\mathscr{U}}$ contains $\widetilde{\mathscr{B}}$, and in particular is nonempty. $\widetilde{\mathscr{U}}$ is upward-closed with respect to $\preccurlyeq$ because if $\mathscr{U}$ is a uniform cover and star-refines $\mathscr{V}$, then there is some $\mathscr{B} \in \widetilde{\mathscr{B}}$ that star-refines $\mathscr{U}$, and hence in turn star-refines $\mathscr{V}$. We now check that it is downward-directed with respect to $\preccurlyeq$. If $\mathscr{U}$ and $\mathscr{V}$ are covers, then there are $\mathscr{B}, \mathscr{C} \in \widetilde{\mathscr{B}}$ that star-refine $\mathscr{U}$ and $\mathscr{V}$ respectively. Because $\widetilde{\mathscr{B}}$ is downward-directed, there is then some $\mathscr{D} \in \widetilde{\mathscr{B}}$ which star-refines both $\mathscr{B}$ and $\mathscr{C}$, and hence both $\mathscr{U}$ and $\mathscr{V}$. ■

As we mentioned above at the end of the proof of Proposition 4.1.2.2, uniform bases define the uniform topology just as well as the entire uniformity.

Corollary 4.1.3.3 Let $\widetilde{\mathscr{B}}$ be a uniform base for the uniform space X. Then, for $x \in X$,

$$\mathscr{B}_x := \left\{ \mathrm{Star}_{\mathscr{B}}(x) : \mathscr{B} \in \widetilde{\mathscr{B}} \right\} \tag{4.1.3.4}$$

is a neighborhood base at x for the uniform topology.

Exercise 4.1.3.5 Show that $\widetilde{\mathscr{U}} := \{\{\{x\} : x \in X\}\}$ is a uniform base for the discrete uniformity.

R That is, the collection consisting of just a *single* open cover, which itself is just the collection of all singletons, forms a uniform base. In other words, you need to check that $\mathscr{U}$ star-refines itself.[a]

R The 'dual' result for the indiscrete uniformity is trivial—the indiscrete uniformity by definition only has a single cover to begin with (namely $\{X\}$), and that single cover certainly forms a uniform base for itself.

[a] Of course, while $\lll$ in general is not reflexive, that doesn't mean we can't at least have $\mathscr{U} \lll \mathscr{U}$ *some* of the time.

We will want to check that two collections of uniform covers are the same by just looking at uniform bases. We did not present it because we did not need to make use of it, but of course there is an analogous result for bases of topological spaces.

Proposition 4.1.3.6 Let X be a set, and let $\widetilde{\mathscr{B}}$ and $\widetilde{\mathscr{C}}$ be uniform bases on X. Then, $\widetilde{\mathscr{B}}$ and $\widetilde{\mathscr{C}}$ determine the same uniformity iff for every $\mathscr{B} \in \widetilde{\mathscr{B}}$, there is some $\mathscr{C} \in \widetilde{\mathscr{C}}$ with

$\mathscr{C} \lll \mathscr{B}$; and for every $\mathscr{C} \in \widetilde{\mathscr{C}}$, there is some $\mathscr{B} \in \widetilde{\mathscr{B}}$ with $\mathscr{B} \lll \mathscr{C}$.

Proof. ($\Rightarrow$) Suppose that $\widetilde{\mathscr{B}}$ and $\widetilde{\mathscr{C}}$ determine the same uniformity. Let $\mathscr{B} \in \widetilde{\mathscr{B}}$. Then, $\mathscr{B}$ is in particular in the uniformity generated by $\widetilde{\mathscr{B}}$, and hence in the uniformity generated by $\widetilde{\mathscr{C}}$. Thus, there is some $\mathscr{C} \in \widetilde{\mathscr{C}}$ such that $\mathscr{C} \lll \mathscr{B}$. By $\widetilde{\mathscr{B}} \leftrightarrow \widetilde{\mathscr{C}}$ symmetry, the other result is true as well.

($\Leftarrow$) Suppose that for every $\mathscr{B} \in \widetilde{\mathscr{B}}$, there is some $\mathscr{C} \in \widetilde{\mathscr{C}}$ with $\mathscr{C} \lll \mathscr{B}$; and for every $\mathscr{C} \in \widetilde{\mathscr{C}}$, there is some $\mathscr{B} \in \widetilde{\mathscr{B}}$ with $\mathscr{B} \lll \mathscr{C}$. Let $\mathscr{U}$ be a uniform cover in the uniformity determined by $\widetilde{\mathscr{B}}$. Then, there is some $\mathscr{B} \in \widetilde{\mathscr{B}}$ such that $\mathscr{B} \lll \mathscr{U}$. By the hypothesis, then, there is some $\mathscr{C} \in \widetilde{\mathscr{C}}$ with $\mathscr{C} \lll \mathscr{B} \lll \mathscr{U}$. Thus, $\mathscr{U}$ is in the uniformity determined by $\widetilde{\mathscr{C}}$. By $\widetilde{\mathscr{B}} \leftrightarrow \widetilde{\mathscr{C}}$ symmetry, the reverse inclusion is also true. ■

We will also want to check whether a function is uniformly-continuous by simply looking at a uniform base.

Proposition 4.1.3.7 Let $f\colon \langle X, \widetilde{\mathscr{U}}\rangle \to \langle Y, \widetilde{\mathscr{V}}\rangle$ be a function between uniform spaces and let $\widetilde{\mathscr{C}}$ be a uniform base for $\widetilde{\mathscr{V}}$. Then, f is uniformly-continuous iff $f^{-1}(\mathscr{C}) \in \widetilde{\mathscr{U}}$ for each $\mathscr{C} \in \widetilde{\mathscr{C}}$.

Proof. ($\Rightarrow$) There is nothing to check (because every open cover in a uniform base is itself a uniform cover).

($\Leftarrow$) Suppose that $f^{-1}(\mathscr{C}) \in \widetilde{\mathscr{U}}$ for each $\mathscr{C} \in \widetilde{\mathscr{C}}$. We need to show that the preimage of *every* uniform cover is a uniform cover. So, let $\mathscr{V} \in \widetilde{\mathscr{V}}$. Then, there is some $\mathscr{C} \in \widetilde{\mathscr{C}}$ such that $\mathscr{C} \lll \mathscr{V}$. Then, by Proposition 4.1.1.46, it follows that $f^{-1}(\mathscr{C}) \lll f^{-1}(\mathscr{V})$. As $f^{-1}(\mathscr{C}) \in \widetilde{\mathscr{U}}$ and $\widetilde{\mathscr{U}}$ is upward-closed with respect to $\lll$, it follows that $f^{-1}(\mathscr{V}) \in \widetilde{\mathscr{U}}$, so that f is uniformly-continuous. ■

A uniform space does not start its life as a topological space—instead, it obtains a canonical topology from its uniformity. In particular, it does not make sense a priori to just restrict to open covers. On the other hand, once we specify the uniform covers, a topology is determined, and then, it turns out (as the following proposition shows), that it suffices to just look at *open* uniform covers, more precisely, those uniform covers obtained by taking the interior of the covers in your uniform base.

Proposition 4.1.3.8 Let $\widetilde{\mathscr{B}}$ be a uniform base on a set X and let $\mathscr{B} \in \widetilde{\mathscr{B}}$. Then, (i) $\operatorname{Int}(\mathscr{B}) := \{\operatorname{Int}(B) : B \in \mathscr{B}\}$ is (still) a cover of X and (ii) $\operatorname{Int}(\widetilde{\mathscr{B}}) := \left\{\operatorname{Int}(\mathscr{B}) : \mathscr{B} \in \widetilde{\mathscr{B}}\right\}$ is (still) a uniform base on X that generates the same uniformity as $\widetilde{\mathscr{B}}$.

Proof. Let $\mathscr{B} \in \widetilde{\mathscr{B}}$ and let $x \in X$. Let $\mathscr{C}$ be a star-refinement of $\mathscr{B}$. Let $C \in \mathscr{C}$ contain x and let $B \in \mathscr{B}$ be such that $\operatorname{Star}_{\mathscr{C}}(C) \subseteq B$. Then,

$$x \in \operatorname{Star}_{\mathscr{C}}(x) \subseteq \operatorname{Star}_{\mathscr{C}}(C) \subseteq B, \tag{4.1.3.9}$$

and so $x \in \operatorname{Int}(B)$ (because $\operatorname{Star}_{\mathscr{C}}(x)$ is a neighborhood of x), and so indeed $\operatorname{Int}(\mathscr{B})$ is a cover of X.

Let $\mathscr{B}, \mathscr{C} \in \widetilde{\mathscr{B}}$ and let $\mathscr{D}$ be a common star-refinement of $\mathscr{B}$ and $\mathscr{C}$. We show that $\operatorname{Int}(\mathscr{D})$ is a common star-refinement of $\operatorname{Int}(\mathscr{B})$ and $\operatorname{Int}(\mathscr{C})$. Because of $\mathscr{B} \leftrightarrow \mathscr{C}$ symmetry, it suffices to show that it is a star-refinement of $\operatorname{Int}(\mathscr{B})$. So, let $\operatorname{Int}(D) \in \operatorname{Int}(\mathscr{D})$. Then, there is some $B \in \mathscr{B}$ such that $\operatorname{Star}_{\mathscr{D}}(D) \subseteq B$, and so because union of the interiors is contained in the interior of the union (Exercise 2.5.2.51.(ii)), we have

$$\begin{aligned}\operatorname{Star}_{\operatorname{Int}(\mathscr{D})}(\operatorname{Int}(D)) \subseteq \operatorname{Star}_{\operatorname{Int}(\mathscr{D})}(D) &\subseteq \operatorname{Int}(\operatorname{Star}_{\mathscr{D}} D) \\ &\subseteq \operatorname{Int}(B),\end{aligned} \tag{4.1.3.10}$$

and so indeed $\operatorname{Int}(\mathscr{D})$ star-refines $\operatorname{Int}(\mathscr{B})$.

It remains to show that $\widetilde{\mathscr{B}}$ and $\mathrm{Int}(\widetilde{\mathscr{B}})$ induce the same uniform structure. To show this, we apply Proposition 4.1.3.6. Let $\mathscr{B} \in \widetilde{\mathscr{B}}$ and let $\mathrm{Int}(\mathscr{C}) \in \mathrm{Int}(\widetilde{\mathscr{B}})$. If $\mathscr{D} \in \widetilde{\mathscr{B}}$ is a star-refinement of $\mathscr{B}$, then $\mathrm{Int}(\mathscr{D}) \in \mathrm{Int}(\widetilde{\mathscr{B}})$ certainly star-refines $\mathscr{B}$ as well. In the other direction, we wish to find a cover in $\widetilde{\mathscr{B}}$ that star-refines $\mathrm{Int}(\mathscr{C})$. So, let $\mathscr{D} \in \widetilde{\mathscr{B}}$ be a star-refinement of $\mathscr{C}$ and let $\mathscr{E} \in \widetilde{\mathscr{B}}$ be in turn a star-refinement of $\mathscr{D}$. We wish to show that $\mathscr{E} \in \widetilde{\mathscr{B}}$ star-refines $\mathrm{Int}(\mathscr{C})$. So, let $E \in \mathscr{E}$. Then, there is some $D \in \mathscr{D}$ such that $\mathrm{Star}_{\mathscr{E}}(E) \subseteq D$. In turn, there is some $C \in \mathscr{C}$ such that $\mathrm{Star}_{\mathscr{D}}(D) \subseteq C$. We wish to show that $\mathrm{Star}_{\mathscr{E}}(E) \subseteq \mathrm{Int}(C) \in \mathrm{Int}(\mathscr{C})$. So, let $x \in \mathrm{Star}_{\mathscr{E}}(E)$. Then, $x \in D$, and so $\mathrm{Star}_{\mathscr{D}}(x) \subseteq \mathrm{Star}_{\mathscr{D}}(D) \subseteq C$. As $\mathrm{Star}_{\mathscr{D}}(x)$ is a neighborhood base of x, this implies that $x \in \mathrm{Int}(C)$. Thus, $\mathrm{Star}_{\mathscr{E}}(E) \subseteq \mathrm{Int}(C)$, and so $\mathscr{E}$ star-refines $\mathrm{Int}(\mathscr{C})$, as desired. ■

As the idea of a uniformity is inherently 'global' in nature, there really isn't a way to define a uniformity that is analogous to the method of defining a topology by specifying neighborhood bases.[5] Furthermore, a bit more unexpected, and quite a bit more unfortunate, is that we cannot generate a uniformity by simply declaring any collection of covers to be uniform in a way analogous to specifying a generating collection of a topology.

▪ **Example 4.1.3.11 — A collection of covers for which there is no unique minimal uniformity containing the collection** Before we begin with the actual counter-example, let us elaborate on what we are trying to do. If you look back to generating collections for topologies (Proposition 3.1.1.13), you'll see that what we would like to be true is the following:

[5] Of course one cannot hope to make a statement like this precise (What does it mean for a method of defining a uniformity to be "analogous to" a method of defining a topology?), but hopefully the intuition is clear. All elements in a uniform cover, no matter where they are in the space, are supposed to be thought of as the same size. How could one hope to encode the idea of two sets living 'far away' are of the same size by the specification of local information alone?

if $\widetilde{\mathscr{S}}$ is a nonempty collection of covers on X, then there is a unique uniformity $\widetilde{\mathscr{U}}$ such that (i) every cover in $\widetilde{\mathscr{S}}$ is uniform with respect to $\widetilde{\mathscr{U}}$, and (ii) that if $\widetilde{\mathscr{U}}'$ is any other uniformity for which this is true then $\widetilde{\mathscr{U}}' \supseteq \widetilde{\mathscr{U}}$. The objective then is to construct a collection $\widetilde{\mathscr{S}}$ for which there exists no such uniformity.

Define $X := \{1, 2, 3\}$ and

$$\mathscr{U} := \{\{1, 2\}, \{2, 3\}\}. \tag{4.1.3.12}$$

We construct incomparable uniformities $\widetilde{\mathscr{U}_1}$ and $\widetilde{\mathscr{U}_2}$ which contain $\{\mathscr{U}\}$ and for which there is *no* uniformity $\widetilde{\mathscr{U}}$ containing $\{\mathscr{U}\}$ and contained in both $\widetilde{\mathscr{U}_1}$ and $\widetilde{\mathscr{U}_2}$. This is enough, because if there were a minimal uniformity containing $\{\mathscr{U}\}$ minimality would dictate it be contained in both $\widetilde{\mathscr{U}_1}$ and $\widetilde{\mathscr{U}_2}$.

Define

$$\mathscr{U}_1 := \{\{1\}, \{2\}, \{3\}, \{1, 2\}\} \tag{4.1.3.13}$$

and

$$\mathscr{U}_2 := \{\{1\}, \{2\}, \{3\}, \{2, 3\}\}\,.^a \tag{4.1.3.14}$$

These covers both star-refine themselves, and hence both $\{\mathscr{U}_1\}$ and $\{\mathscr{U}_2\}$ form uniform bases: denote by $\widetilde{\mathscr{U}_1}$ and $\widetilde{\mathscr{U}_2}$ the uniformities generated by these respective uniform bases (so that $\widetilde{\mathscr{U}_k}$ is the collection of covers which are star-refined by $\mathscr{U}_k$). As $\mathscr{U}$ is star-refined by both $\mathscr{U}_1$ and $\mathscr{U}_2$, $\mathscr{U}$ is contained in both $\widetilde{\mathscr{U}_1}$ and $\widetilde{\mathscr{U}_2}$. It remains to check that there is no uniformity contained in both which contains $\mathscr{U}$.

So, let $\widetilde{\mathscr{U}}$ be a uniformity containing $\mathscr{U}$ and contained in both $\widetilde{\mathscr{U}_1}$ and $\widetilde{\mathscr{U}_2}$. $\widetilde{\mathscr{U}}$ must then contain a star-refinement $\mathscr{V}$ of $\mathscr{U}$. As $\mathscr{V}$ is in both $\widetilde{\mathscr{U}_1}$ and $\widetilde{\mathscr{U}_2}$, $\mathscr{V}$ must then in turn be star-refined by both $\mathscr{U}_1$ and $\mathscr{U}_2$. However, there is no cover of X which star-refines $\mathscr{U}$ and is star-refined by both $\mathscr{U}_1$ and $\mathscr{U}_2$:[b] a contradiction. Therefore, there is no such $\widetilde{\mathscr{U}}$.

[a] Draw a picture.

[b] $\mathrm{Star}_{\mathscr{U}_1}(\{2\}) = \{1, 2\}$ and $\mathrm{Star}_{\mathscr{U}_2}(\{2\}) = \{2, 3\}$, and so $\mathscr{V}$ would have to contain a set which contains $\{1, 2\}$ and a set which contains $\{2, 3\}$. This implies that the star of either of these sets with respect to $\mathscr{V}$ is $\{1, 2, 3\}$, which is not contained in any element of $\mathscr{U}$.

With topological spaces, we could define a topology by specifying the closure or interior, or defining a notion of convergence. To the best of my knowledge, there are no analogous methods for defining uniformities. As for the initial and final topologies, however, there are analogous constructions (which you might not have expected as we no longer can simply 'generate' uniformities like we could with topologies).

Proposition 4.1.3.15 — Initial uniformity Let X be a set, let $\mathscr{Y}$ be an indexed collection of uniform spaces, and for each $Y \in \mathscr{Y}$ let $f_Y : X \to Y$ be a function. Then, there exists a unique uniformity $\widetilde{\mathscr{U}}$ on X, the ***initial uniformity*** with respect to $\{f_Y : Y \in \mathscr{Y}\}$, such that

(i). $f_Y : X \to Y$ is uniformly-continuous with respect to $\widetilde{\mathscr{U}}$ for all $Y \in \mathscr{Y}$; and

(ii). if $\widetilde{\mathscr{U}}'$ is another uniformity for which each f_Y is uniformly-continuous, then $\widetilde{\mathscr{U}} \subseteq \widetilde{\mathscr{U}}'$.

Furthermore,

(i). if $\widetilde{\mathscr{C}}_Y$ is a uniform base for $\mathscr{Y}$, then the collection of all finite meets of covers of the form $f_Y^{-1}(\mathscr{V})$ with $\mathscr{V} \in \widetilde{\mathscr{C}}_Y$ is a uniform base for $\widetilde{\mathscr{U}}$; and

(ii). the uniform topology of the initial uniformity is the same as the initial topology.

R In other words, the initial uniformity is the smallest uniformity for which each f_Y is uniformly-continuous.

R But what about the largest such uniformity? Well, the largest such uniformity is always going to be the discrete uniformity, which is not very uninteresting.

This is how you remember whether the initial uniformity is the smallest or largest—it can't be the largest because the discrete uniformity always works.

Proof. Let $\widetilde{\mathscr{C}_Y}$ be any uniform base for Y and let $\widetilde{\mathscr{C}}$ be the collection of all finite meets of covers of the form $f_Y^{-1}(\mathscr{V})$ for $\mathscr{V} \in \widetilde{\mathscr{C}_Y}$ and $Y \in \mathscr{Y}$. We wish to check that this is a uniform base.

So, let $f_{Y_1}^{-1}(\mathscr{V}_1) \wedge \cdots \wedge f_{Y_m}^{-1}(\mathscr{V}_m)$ and $f_{Y_{m+1}}^{-1}(\mathscr{V}_m) \wedge \cdots \wedge f_{Y_{m+n}}^{-1}(\mathscr{V}_{m+n})$ be two elements of $\widetilde{\mathscr{C}}$. Let $\mathscr{W}_k$ be a star-refinement of $\mathscr{V}_k$ in $\widetilde{\mathscr{C}_{Y_k}}$. Then, because preimage preserves star-refinement (Proposition 4.1.1.46) and the meet is 'compatible' with the relation of star-refinement (Exercise 4.1.1.45), it follows that

$$f_{Y_1}^{-1}(\mathscr{W}_1) \wedge \cdots \wedge f_{Y_{m+n}}^{-1}(\mathscr{W}_{m+n}) \tag{4.1.3.16}$$

is a star-refinement of the two initial covers.

Thus, $\widetilde{\mathscr{B}}$ is a uniform base. Denote by $\widetilde{\mathscr{U}}$ the uniformity generated by $\widetilde{\mathscr{U}}$.

By construction, $f_Y^{-1}(\mathscr{V})$ is a uniform cover for all $\mathscr{V} \in \widetilde{\mathscr{C}_Y}$, so each f_Y is certainly uniformly-continuous. On the other hand, if $\widetilde{\mathscr{U}'}$ were a uniformity for which each f_Y were uniformly-continuous, then $\widetilde{\mathscr{U}'}$ would have to contain every cover of the form $f_Y^{-1}(\mathscr{V})$ for $\mathscr{V} \in \widetilde{\mathscr{C}_Y}$ a uniform cover of Y. As uniformities are closed under meets,[a] $\widetilde{\mathscr{U}'}$ would have to contain $\widetilde{\mathscr{C}}$ and hence $\widetilde{\mathscr{U}}$, as desired.

Exercise 4.1.3.17 Show that the initial uniformity is unique.

We finally check that the uniform topology of the initial uniformity agrees with the initial topology. First of all, each f_Y is uniformly-continuous, hence continuous, and so as the initial topology is the coarsest (i.e. smallest) topology for which each f_Y is continuous, the initial topology must be coarser than the uniform topology of the initial uniformity.

In the other direction, let $U \subseteq X$ be open with respect to the uniform topology of the initial uniformity. By the defining result of the uniform topology (Proposition 4.1.2.2), this means that for all $x \in U$, there is some $\mathscr{B} \in \widetilde{\mathscr{B}}$ such that $x \in \mathrm{Star}_{\mathscr{B}}(x) \subseteq U$. Write $\mathscr{B} = f_{Y_1}^{-1}(\mathscr{V}_1) \wedge \cdots \wedge f_{Y_m}^{-1}(\mathscr{V}_m)$. As, for points, the star with respect to a wedge is the intersection of the stars (Proposition 4.1.1.38),

$$\begin{aligned}\mathrm{Star}_{\mathscr{B}}(x) &= \mathrm{Star}_{f_{Y_1}^{-1}(\mathscr{V}_1)}(x) \cap \cdots \cap \mathrm{Star}_{f_{Y_m}^{-1}(\mathscr{V}_m)}(x) \\ &= f_{Y_1}^{-1}\left(\mathrm{Star}_{\mathscr{V}_1}(f_{Y_1}(x))\right) \cap \cdots \cap f_{Y_m}^{-1}\left(\mathrm{Star}_{\mathscr{V}_m}(f_{Y_m}(x))\right).\end{aligned}$$

where we have applied Proposition 4.1.1.18. As each $\mathrm{Star}_{\mathscr{V}_k}(f_{Y_k}(x))$ is a neighborhood of $f_{Y_k}(x) \in Y_k$, $f_{Y_k}^{-1}\left(\mathrm{Star}_{\mathscr{V}_k}(f_{Y_k}(x))\right)$ is a neighborhood of $x \in X$ for the initial topology, and so, by the above equality, $\mathrm{Star}_{\mathscr{B}}(x)$ is likewise a neighborhood of $x \in X$ for the initial topology. This shows that every point in U has a neighborhood for the initial topology contained in U, and hence U is likewise open for the initial topology, as desired. ■

[a] Why?

Of course, we have a result that is perfectly analogous to Proposition 3.4.3.8 (a function is continuous iff its composition with each f_Y is continuous).

Proposition 4.1.3.18 Let X have the initial uniformity with respect to the collection $\{f_Y : Y \in \mathscr{Y}\}$, let Z be a uniform space, and let $f : Z \to X$ be a function. Then, f is uniformly-continuous iff $f_Y \circ f$ is uniformly-continuous for all $Y \in \mathscr{Y}$. Furthermore, the initial uniformity is the unique uniformity with this property.

Proof. We leave the proof as an exercise.

Exercise 4.1.3.19 Prove this result, using the proof of Proposition 3.4.3.8 (the analogous result for the initial topology) as guidance.

■

And just as we had with topological spaces, there is a 'dual' version of the initial uniformity.

Proposition 4.1.3.20 — Final uniformity Let X be a set, let $\mathscr{Y}$ be an indexed collection of uniform spaces, and for each $Y \in \mathscr{Y}$ let $f_Y : Y \to X$ be a function. Then, there exists a unique uniformity $\widetilde{\mathscr{U}}$ on X, the ***final uniformity*** with respect to $\{f_Y : Y \in \mathscr{Y}\}$, such that

(i). $f_Y : Y \to X$ is uniformly-continuous with respect to $\widetilde{\mathscr{U}}$; and

(ii). if $\widetilde{\mathscr{U}}'$ is another uniformity for which each f_Y is uniformly-continuous, then $\widetilde{\mathscr{U}} \supseteq \widetilde{\mathscr{U}}'$.

Furthermore, the uniform topology of the final uniformity is coarser than the final topology.

R In other words, the final uniformity is the largest uniformity for which each f_Y is uniformly-continuous.

R But what about the smallest such uniformity? Well, the smallest such uniformity is always going to be the indiscrete uniformity, which is not very interest. This is how you remember whether the final uniformity is the smallest or largest—it can't be the smallest because the indiscrete uniformity always works.

R For both the initial and final topologies, as well as the initial uniformity, we had a relatively concrete description of the induced structure in terms of the structure on the elements of $\mathscr{Y}$. To the best of my knowledge, there is no such analogous description for the final uniformity. For better insight as to why that is, see the proof.

W Warning: The uniform topology of the final uniformity can be *strictly* coarser than the final topology—see the following counter-example.

Proof. [a] Let $\mathscr{Z}$ be an indexed collection containing a copy of a uniform space Z for all functions $g_Z\colon X \to Z$ that have the property that $g_Z \circ f_Y$ is uniformly-continuous for all $Y \in \mathscr{Y}$. Let $\widetilde{\mathscr{U}}$ be the initial uniformity with respect to $\{g_Z : Z \in \mathscr{Z}\}$.

By the previous result Proposition 4.1.3.18, each f_Y is uniformly-continuous $\widetilde{\mathscr{U}}$.

Let $\widetilde{\mathscr{U}'}$ be another uniformity on X for which each f_Y is uniformly-continuous. Let $\mathscr{U}' \in \widetilde{\mathscr{U}'}$. We wish to show that $\mathscr{U}' \in \widetilde{\mathscr{U}}$. To shows this, it suffices to show that $\mathrm{id}_X\colon \langle X, \widetilde{\mathscr{U}}\rangle \to \langle X, \widetilde{\mathscr{U}'}\rangle$ is uniformly-continuous. However, $\mathrm{id}_X \circ f_Y = f_Y$ is uniformly-continuous by assumption for all $Y \in \mathscr{Y}$, and so id_X is among the g_Zs, and in particular, is uniformly-continuous, as desired.

Exercise 4.1.3.21 Show that the final uniformity is unique.

We finally check that the uniform topology of the final uniformity is coarser than the final topology. First of all, each f_Y is uniformly-continuous, hence continuous, and so as the final topology is the finest topology for which each f_Y is continuous, the final topology must be finer than the uniform topology of the final uniformity. ■

[a] Proof adapted from [Pre02, Theorem 1.2.1.1].

▪ Example 4.1.3.22 — The final topology need not agree with the uniform topology of the final uniformity [a] Let $q\colon \mathbb{R} \to \{(-\infty, 0], (0, \infty)\} =: X$ denote the quotient map. To simplify notation, let us write $x_1 := (-\infty, 0]$ and $x_2 := (0, \infty)$. A subset of X is open in the final topology iff its preimage under q is open. It follows that the final topology on X is $\{\emptyset, \{x_2\}, X\}$.

Obviously, every cover of X either contains X itself or it does not. If it does not, then it must contain separately x_1 and x_2, and so must be $\{\{x_1\}, \{x_2\}\}$ (or this together with the empty-set). The preimage of this cover under q is the cover of q $\{(-\infty, 0], (0, \infty)\}$, which is never a uniform cover of $\mathbb{R}$.[b] Thus, every uniform cover of X (in the final uniformity) must contain X. It follows that the uniform topology of the final uniformity is the indiscrete topology, which is strictly coarser than the final topology.

[a] Adapted from [A05, Exercise 2.1.10].

[b] To be a uniform cover of $\mathbb{R}$, it must be star-refined by a cover of ε-balls, and the star of an ε-ball centered at $0 \in \mathbb{R}$ is contained in neither $(-\infty, 0]$ nor $(0, \infty)$.

And the result 'dual' to Proposition 4.1.3.18:

Proposition 4.1.3.23 Let X have the final uniformity with respect to the collection $\{f_Y : Y \in \mathscr{Y}\}$, let Z be a uniform space, and let $f\colon X \to Z$. Then, f is uniformly-continuous iff $f_Y \circ f$ is uniformly-continuous for all $Y \in \mathscr{Y}$. Furthermore, the final uniformity is the unique uniformity with this property.

Proof. We leave the proof as an exercise.

Exercise 4.1.3.24 Prove this result, using the proof of Proposition 3.4.3.14 (the analogous result for the final topology) as guidance

■

Of course, just as with topological spaces, a key application of the initial and final uniformities is that they provide canonical uniformities on subsets, quotients, products, and disjoint-unions. The definitions and results are completely analogous to the case of topological spaces, and so we omit stating them explicitly.

After having discussed the real numbers themselves as a uniform space, we show below (Example 4.1.3.38) that functions even as nice as polynomials are not uniformly continuous. On the other hand, when restricted to *quasicompact* sets, all continuous functions are uniformly-continuous.

Proposition 4.1.3.25 — Cantor-Heine Theorem
Let $f\colon X \to Y$ be a continuous function between uniform spaces and let $K \subseteq X$ be quasicompact. Then, $f|_K : K \to Y$ is uniformly-continuous.

R Ideally we would have presented this result shortly after giving the definition of uniformly-continuous functions, however, we do technically need the notion of the subspace uniformity to state this result.

Proof. Let $\mathscr{V}$ be a uniform cover of Y. By Propositions 4.1.3.7 and 4.1.3.8 (the the interior covers define the same uniformity and it suffices to check uniform-continuity on a uniform base), without loss of generality we can take $\mathscr{V}$ to be an open cover. We would like to show that $f|_K{}^{-1}(\mathscr{V}) = f^{-1}(\mathscr{V}) \wedge \{K\}$ is a uniform-cover of K. To do this, by upward-closedness, it suffices to find a uniform-cover of K which star-refines $f^{-1}(\mathscr{V}) \wedge \{K\}$.

$f^{-1}(\mathscr{V})$, while not necessarily a uniform cover of X, will certainly be an open cover, and in particular will be an open cover of K. So, for $x \in K$, let $V_x \in \mathscr{V}$ be such that $x \in f^{-1}(V_x)$. Then, choose an open[a] uniform cover $\mathscr{U}_x$ of X such that[b]

$$\mathrm{Star}_{\mathscr{U}_x \wedge \{K\}}(x) \subseteq f^{-1}(V_x) \cap K. \tag{4.1.3.26}$$

As

$$\left\{\mathrm{Star}_{\mathscr{U}_x \wedge \{K\}}(x) : x \in K\right\} \tag{4.1.3.27}$$

is an open cover of K, there is a finite subcover. So, let $x_1, \ldots, x_m \in K$ be such that

$$\left\{\mathrm{Star}_{\mathscr{U}_{x_k} \wedge \{K\}}(x_k) : 1 \leq k \leq m\right\} \tag{4.1.3.28}$$

is an open cover of K. Let $\mathscr{U}$ be a common star-refinement of each $\mathscr{U}_k{}^c$. Then, for $x \in K$, if $x \in \mathrm{Star}_{\mathscr{U}_k \wedge \{K\}}(x_k)$, then

$$\mathrm{Star}_{\mathscr{U}_0 \wedge \{K\}}(x) \subseteq f^{-1}(V_{x_k}) \cap K \tag{4.1.3.29}$$

for all $x \in K$.

Let $\mathscr{U}_0$ be in turn a star-refinement of $\mathscr{U}_0$. We show that $\mathscr{U} \wedge \{K\}$ is a star-refinement of $f^{-1}(\mathscr{V}) \wedge \{K\}$. So, let $U \in \mathscr{U}$. Let $U_0 \in \mathscr{U}_0$ be such that $\mathrm{Star}_{\mathscr{U}}(U) \subseteq U_0$. Let $x \in U$ and pick k so that (4.1.3.29) holds. Then,

$$\begin{aligned} \mathrm{Star}_{\mathscr{U} \wedge \{K\}}(U \cap K) \subseteq U_0 \cap K \subseteq \mathrm{Star}_{\mathscr{U}_0 \wedge \{K\}}(x) \\ \subseteq f^{-1}(V_{x_k}) \cap K. \end{aligned} \tag{4.1.3.30}$$

■

[a] Applying Proposition 4.1.3.8 again.

[b] This implicitly uses the fact that the subspace topology is the same as the uniform topology of the subspace uniformity—see Proposition 4.1.3.15.

[c] It is here that the finiteness given to us by quasicompactness is key.

■ **Example 4.1.3.31 — The real numbers** The real numbers have a canonical uniformity (and in fact, we will see below that this is just a special base of a more general construction): let $\varepsilon > 0$ and define

$$\mathscr{U}_\varepsilon := \{B_\varepsilon(x) : x \in \mathbb{R}\} \tag{4.1.3.32}$$

and

$$\widetilde{\mathscr{U}} := \{\mathscr{U}_\varepsilon : \varepsilon > 0\}\,. \tag{4.1.3.33}$$

Exercise 4.1.3.34 Show that $\widetilde{\mathscr{U}}$ is a uniform base on $\mathbb{R}$.

Exercise 4.1.3.35 Show that $f : \mathbb{R} \to \mathbb{R}$ is uniformly-continuous iff for every $\varepsilon > 0$ there is some $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$ for every $x \in \mathbb{R}$.

R Compare this with the condition for $f : \mathbb{R} \to \mathbb{R}$ being *continuous at* $a \in \mathbb{R}$ given in Exercise 2.5.1.18.(iii). We will spell-it-out here for convenience:

$$f : \mathbb{R} \to \mathbb{R} \text{ is continuous iff for every } x \in \mathbb{R} \text{ and for every } \varepsilon > 0 \text{ there is some } \delta > 0 \text{ such that } f(B_\delta(x)) \subseteq B_\varepsilon(f(x)). \tag{4.1.3.36}$$

The key difference between continuity and uniform-continuity is *the location in which the quantification "for every $x \in \mathbb{R}$" appears*. In the former (just continuous case), your choice of δ is *allowed to depend on* x, whereas to be uniformly-continuous, *a single δ has to 'work' for every $x \in \mathbb{R}$*.

R The result you just proved characterizing uniform-continuity in $\mathbb{R}$ is often taken as the definition of uniform-continuity. Had we studied uniform-continuity in the context of just the real numbers first (as opposed to in the context of uniform spaces), we would have done the same. My personal feeling, however, is that uniform continuity is not that incredibly important, at least not to the point where it is worth going out of our way to discuss it just in the context of $\mathbb{R}$. The real reason we discuss uniform spaces is for the purpose of discussing Cauchyness and completeness, not uniform continuity per se (and also of course because a huge collection of examples of topological spaces are canonically uniform spaces).

■ Example 4.1.3.37 — A uniformly-continuous function By Proposition 4.1.3.25, any continuous function restricted to a quasicompact set will be uniformly-continuous, so, for example the function $x \mapsto x^2$ is uniformly-continuous on $[0, 1]$. However, be careful: it is not uniformly-continuous on all of $\mathbb{R}$.

■ Example 4.1.3.38 — A continuous function that is not uniformly-continuous Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := x^2$. Of course f is continuous (because it is the product of continuous functions—see Exercise 2.5.1.19).

On the other hand, we show that f does not satisfy the condition given in the previous exercise. Take $\varepsilon := 1$. Then, if f were uniformly-continuous, there should be some $\delta > 0$ such that

$$\begin{aligned}\left\{x^2 : |x - x_0| < \delta\right\} &=: f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0)) \\ &:= \left\{x \in \mathbb{R} : \left|x - x_0^2\right| < 1\right\}\end{aligned} \tag{4.1.3.39}$$

for all $x_0 \in \mathbb{R}$. However, $(x_0 + \frac{1}{2}\delta)^2$ is an element of the left-hand side, but

$$\left(x_0 + \tfrac{1}{2}\delta\right)^2 - x_0^2 = \delta x_0 + \tfrac{1}{4}\delta^2 \tag{4.1.3.40}$$

is not less than 1 in general (for example, for $x_0 = \frac{1}{\delta}(1 - \frac{1}{4}\delta^2)$.

On the other hand, by Proposition 4.1.3.25, f restricted to any closed interval is uniformly-continuous.

> **R** In particular, *the product of uniformly-continuous functions is not necessarily uniformly-continuous*. This sucks, but alas, what's a mathematician to do?

4.2 Semimetric spaces and topological groups

As was previously mentioned, one big motivation for studying uniform spaces is that a huge collection of very important examples of topological spaces admit a canonical uniformity. Two such families of spaces that we will study are *semimetric spaces* and *topological groups*.

4.2.1 Semimetric spaces

Before we talk about any sort of uniformity, we had better first say what we mean by *semimetric space*.

Definition 4.2.1.1 — Semimetric and metric Let X be a set. Then, a ***semimetric*** on X is a function $|\cdot,\cdot| : X \times X \to \mathbb{R}_0^+$ such that

(i). $|x, x| = 0$;
(ii). (Symmetry) $|x, y| = |y, x|$;
(iii). (Triangle Inequality) $|x, z| \leq |x, y| + |y, z|$.

$|\cdot,\cdot|$ is a ***metric*** if furthermore (Definiteness) $|x, y| = 0$ implies $x = y$.

R Semimetrics are also sometimes called ***pseudometrics***. However, the term seminorm (something we haven't discussed yet—see Definition 4.2.3.2) is actually much more common than either of these terms, and as metrics are to norms (also something we haven't discussed yet—see Definition 4.2.3.2 again) as semimetrics/pseudometrics are to seminorms, I feel as if the terminology "semimetric" is more appropriate.

If fact, you should be warned that *some authors use a different meaning than us for "semimetric"*.

R It is much more common to denote (semi)metrics by "$d(\cdot, \cdot)$", however, this conflicts with our conventions of reserving the letter "d" for dimension.

Example 4.2.1.2 Let $X := \mathbb{R}$ and define $|x, y| := |x - y|$. Then, $|\cdot,\cdot|$ is in fact a metric.

R Of course, this is where the notation $|\cdot,\cdot|$ in general comes from.

Example 4.2.1.3 — A semimetric that is not a metric Let X be a topological space and let $K \subseteq X$ be quasicompact. For $f, g \in \mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$, we define

$$|f, g|_K := \sup_{x \in K}\{|f(x) - g(x)|\}.^{a}. \tag{4.2.1.4}$$

In general, this will not be a metric. For example, take $X := \mathbb{R}$ and $K := [0, 1]$. Then, $|f, 0|_K = 0$ iff $f|_{[0,1]} = 0$, but of course, there are many nonzero real-valued continuous functions on $\mathbb{R}$ that vanish on $[0, 1]$ (by Urysohn's Lemma (Theorem 3.6.2.106), for example, if you want to use a sledgehammer (or maybe just a hammer?) to swat a fly).

[a] We require that K be quasicompact so that $f - g$ is bounded on K (by the Extreme Value Theorem (Theorem 3.8.2.2))

Definition 4.2.1.5 — Semimetric space A ***semimetric space*** is a set X equipped with a collection $\mathscr{D}$ of semimetrics.

> R For some reason, it seems that semimetric spaces are also referred to as ***gauge spaces***. Off the top of my head, I can think of at least two other distinct ways in which the term "gauge" is used in mathematics, and so I would recommend not using this terminology

Definition 4.2.1.6 — Metric space A ***metric space*** $(X, |\cdot,\cdot|)$ is a semimetric space $\langle X, \mathscr{D}\rangle$ in which $\mathscr{D}$ is a singleton, $\mathscr{D} = \{|\cdot,\cdot|\}$, and $|\cdot,\cdot|$ is a metric.

> R The reason we take $\mathscr{D}$ to be a singleton instead of just an arbitrary collection of *metrics* is to agree with standard terminology (metric spaces are almost always taken to be sets equipped with a (*single*) metric).

■ **Example 4.2.1.7 — A semimetric space that is not a metric space** Let X be a topological space, and for $K \subseteq X$ quasicompact nonempty, let $|\cdot,\cdot|_K$ be the semimetric on $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ in (4.2.1.4), that is

$$|f, g| := \sup_{x \in K} \{|f(x) - g(x)|\}. \tag{4.2.1.8}$$

We already know from Example 4.2.1.3 that each $|\cdot,\cdot|_K$ is a semimetric on $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$.

As $\mathscr{D}$ clearly contains more than one element (at least so long as X contains more than one point), you might think that this shows that this cannot be a metric space. However, the real question is *"Is it uniformly-homeomorphic* to a metric space?",[a] and for general topological spaces the answer is *no*.

[a]Of course, this doesn't quite make sense yet as we have not put a uniformity on semimetric spaces.

We noted at the very beginning of this chapter that the star of a point with respect to $\mathscr{B}_\varepsilon$ in $\mathbb{R}$ is just the ball of 2ε. You should be careful, however, as this does not hold in general.

■ **Example 4.2.1.9 — A metric space for which** $\mathrm{Star}_{\mathscr{B}_\varepsilon}(x) \neq B_{2\varepsilon}(x)$ Define $X := \{0, 1\}$ equipped with the metric $|\cdot,\cdot|$ for which $|0, 1| = 1$, $\varepsilon := 1$, and $x_0 := 0$. Then,

$$\mathscr{B}_\varepsilon = \{\{0\}, \{1\}\}, \tag{4.2.1.10}$$

and so

$$\mathrm{Star}_{\mathscr{B}_\varepsilon}(x_0) = \{0\}. \tag{4.2.1.11}$$

On the other hand,

$$B_{2\varepsilon}(x_0) = \{0, 1\}. \tag{4.2.1.12}$$

Despite this, we always have one inclusion.

Exercise 4.2.1.13 Let X be a metric space, let $x \in X$, and let $\varepsilon > 0$. Show that $\mathrm{Star}_{\mathscr{B}_\varepsilon}(x) \subseteq B_{2\varepsilon}(x)$.

Exercise 4.2.1.14

(i). Show that $\mathrm{Star}_{\mathscr{B}_\varepsilon}(B_\varepsilon(x)) = B_{3\varepsilon}(x)$ in $\mathbb{R}^d$.
(ii). Find a metric space in which this fails.

It's worth noting that, in a metric space, for every closed subset, the distance (as defined below in (4.2.1.16)) from a point to the closed subset is a continuous function of that point.

Proposition 4.2.1.15 Let $\langle X, |\cdot,\cdot|\rangle$ be a metric space and let $C \subseteq X$. Then, the function $\mathrm{dist}_C : X \to \mathbb{R}$ defined by

$$\mathrm{dist}_C(x) := \inf_{c \in C}\{|x, c|\} \tag{4.2.1.16}$$

is uniformly-continuous and furthermore $\mathrm{dist}_C^{-1}(0) = C$.

R The statement that $\mathrm{dist}_C^{-1}(0) = C$ means that, not only is dist_C 0 on C, but in fact, it isn't 0 anywhere else as well. Contrast this with a set that is not closed, e.g. $(0, 1) \subseteq \mathbb{R}$, for which $1 \in \mathbb{R}$ is a distance of 0 from this set but not actually an element of the set.

Proof. Let $\varepsilon > 0$. Let $x_1, x_2 \in X$ lie in some ε ball. Choose some $c \in C$ such that $|x_1, c| - \mathrm{dist}_C(x_1) < \varepsilon$.[a] Then,

$$\begin{aligned}\mathrm{dist}_C(x_2) &\le |x_2, c| \le |x_2, x_1| + |x_1, c| \\ &< 2\varepsilon + (\mathrm{dist}_C(x_1) + \varepsilon) \\ &= 3\varepsilon + \mathrm{dist}_C(x_1),\end{aligned} \tag{4.2.1.17}$$

and so

$$\mathrm{dist}_C(x_2) - \mathrm{dist}_C(x_1) < 3\varepsilon. \tag{4.2.1.18}$$

By $1 \leftrightarrow 2$ symmetry, we also have that

$$\mathrm{dist}_C(x_1) - \mathrm{dist}_C(x_2) < 3\varepsilon, \tag{4.2.1.19}$$

and hence

$$\left|\mathrm{dist}_C(x_1) - \mathrm{dist}_C(x_2)\right| < 3\varepsilon. \tag{4.2.1.20}$$

This shows that dist_C is uniformly-continuous (by Proposition 4.2.1.26).

Of course $C \subseteq \mathrm{dist}_C^{-1}(0)$. On the other hand, if $x \in \mathrm{dist}_C^{-1}(0)$, then x is an accumulation point of C, and hence contained in C. Thus, $C = \mathrm{dist}_C^{-1}(0)$. ■

[a]Using Proposition 1.4.1.13.

Now that we've gotten that out of the way, we are ready to equip semimetric spaces with a topology and uniformity.

Definition 4.2.1.21 — Uniformity on a semimetric space Let $\langle X, \mathscr{D}\rangle$ be a semimetric space, and equip X with the uniformity generated by the uniform base defined by

$$\begin{split}\widetilde{\mathscr{B}}_{\mathscr{D}} := \Big\{\mathscr{B}^{|\cdot,\cdot|_1,\ldots,|\cdot,\cdot|_m}_{\varepsilon_1,\ldots,\varepsilon_m} : m \in \mathbb{Z}^+; \\ |\cdot,\cdot|_1, \ldots, |\cdot,\cdot|_m \in \mathscr{D};\ \varepsilon_1, \ldots, \varepsilon_m > 0\Big\},\end{split} \tag{4.2.1.22}$$

where

$$\mathscr{B}^{|\cdot,\cdot|_1,\ldots,|\cdot,\cdot|_m}_{\varepsilon_1,\ldots,\varepsilon_m} := \left\{B^{|\cdot,\cdot|_1,\ldots,|\cdot,\cdot|_m}_{\varepsilon_1,\ldots,\varepsilon_m}(x) : x \in X\right\} \tag{4.2.1.23}$$

and

$$\begin{split}B^{|\cdot,\cdot|_1,\ldots,|\cdot,\cdot|_m}_{\varepsilon_1,\ldots,\varepsilon_m}(x) := \{y \in X : \\ |y,x|_1 < \varepsilon_1, \ldots, |y,x|_m < \varepsilon_m\}.\end{split} \tag{4.2.1.24}$$

Exercise 4.2.1.25 Show that $\widetilde{\mathscr{B}}_{\mathscr{D}}$ is indeed a uniform base.

R So the notation admittedly makes this look a bit atrocious. Let's try to break it down. $B^{|\cdot,\cdot|_1,\ldots,|\cdot,\cdot|_m}_{\varepsilon_1,\ldots,\varepsilon_m}$

is just like the ε-balls you know and love, but now, as we have more than one semimetric, we don't just have ε-balls, but instead we have $\langle \varepsilon_1, \ldots, \varepsilon_m \rangle$-balls for all $m \in \mathbb{Z}^+$. Then, we just do the same as we did when there was just one metric: for each choice of $\langle \varepsilon_1, \ldots, \varepsilon_m \rangle$ and corresponding semimetric, there is a uniform cover consisting of $\langle \varepsilon_1, \ldots, \varepsilon_m \rangle$-balls centered at each point, and the collection of all of these uniform covers is the uniform base.

Our first order of business is to make explicit what it means to be uniformly-continuous for semimetric spaces.

Proposition 4.2.1.26 Let $f\colon \langle X, \mathscr{D} \rangle \to \langle Y, \mathscr{E} \rangle$ be a function between semimetric spaces. Then, f is uniformly continuous iff for every $|\cdot,\cdot| \in \mathscr{E}$ and $\varepsilon > 0$, there are some $|\cdot,\cdot|_1, \ldots, |\cdot,\cdot|_m \in \mathscr{D}$ and $\delta_1, \ldots, \delta_m > 0$ such that, whenever $|x_1, x_2|_k < \delta_k$ for $1 \leq k \leq m$, it follows that $|f(x_1), f(x_2)| < \varepsilon$.

R In particular, for metric spaces, f is uniformly-continuous iff for every $\varepsilon > 0$ there is some $\delta > 0$ such that, whenever $|x_1, x_2| < \delta$, it follows that $|f(x_1), f(x_2)| < \varepsilon$.

Proof. ($\Rightarrow$) Suppose that f is uniformly-continuous. Let $|\cdot,\cdot|_1 \in \mathscr{E}$ and let $\varepsilon > 0$. Then, as f is uniformly-continuous, $f^{1}\left(\mathscr{B}_\varepsilon^{|\cdot,\cdot|}\right)$ is a uniform cover, and so, by the definition of a uniform base (Definition 4.1.3.1), there are $|\cdot,\cdot|_1, \ldots, |\cdot,\cdot|_m \in \mathscr{D}$ and $\delta_1, \ldots, \delta_m > 0$ such that $\mathscr{B}_{\delta_1,\ldots,\delta_m}^{|\cdot,\cdot|_1,\ldots,|\cdot,\cdot|_m} \lll f^{-1}\left(\mathscr{B}_\varepsilon^{|\cdot,\cdot|}\right)$. Thus, for every $B_{\delta_1,\ldots,\delta_m}^{|\cdot,\cdot|_1,\ldots,|\cdot,\cdot|_m}(x) \in \mathscr{B}_{\delta_1,\ldots,\delta_m}^{|\cdot,\cdot|_1,\ldots,|\cdot,\cdot|_m}$ there is some $f^{-1}(B_\varepsilon^{|\cdot,\cdot|}(y_x)) \in f^{-1}(\mathscr{B}_\varepsilon^{|\cdot,\cdot|})$ such that

$$\operatorname{Star}_{\mathscr{B}_{\delta_1,\ldots,\delta_m}^{|\cdot,\cdot|_1,\ldots,|\cdot,\cdot|_m}}\left(B_{\delta_1,\ldots,\delta_m}^{|\cdot,\cdot|_1,\ldots,|\cdot,\cdot|_m}(x)\right) \subseteq f^{-1}(B_\varepsilon^{|\cdot,\cdot|}(y_x)).$$

So, let $x_1, x_2 \in X$ and suppose that $|x_1, x_2|_k < \delta_k$ for $1 \leq k \leq m$. This means that $x_1, x_2 \in B^{|\cdot,\cdot|_1,\ldots,|\cdot,\cdot|_m}_{\delta_1,\ldots,\delta_m}(x_2)$, and so, by the above equation, $x_1, x_2 \in f^{-1}(B^{|\cdot,\cdot|}_{\varepsilon}(y_x))$. Hence,

$$\begin{aligned} |f(x_1), f(x_2)| &\leq \left|f(x_1), y_{x_1}\right| + \left|y_{x_1}, f(x_2)\right| \\ &< \varepsilon + \varepsilon = 2\varepsilon. \end{aligned} \tag{4.2.1.27}$$

($\Leftarrow$) Suppose that for every $|\cdot,\cdot| \in \mathscr{E}$ and $\varepsilon > 0$, there are some $|\cdot,\cdot|_1, \ldots, |\cdot,\cdot|_m \in \mathscr{D}$ and $\delta_1, \ldots, \delta_m > 0$ such that, whenever $|x_1, x_2|_k < \delta_k$ for $1 \leq k \leq m$, it follows that $|f(x_1), f(x_2)| < \varepsilon$. Let $|\cdot,\cdot|_1, \ldots, |\cdot,\cdot|_m \in \mathscr{E}^a$ and let $\varepsilon_1, \ldots, \varepsilon_m > 0$. We wish to show that $f^{-1}\left(\mathscr{B}^{|\cdot,\cdot|_1,\ldots,|\cdot,\cdot|_m}_{\varepsilon_1,\ldots,\varepsilon_m}\right)$ is a uniform cover.

By hypothesis, for each k, there are finitely many $|\cdot,\cdot|_{k,1}, \ldots, |\cdot,\cdot|_{k,m_k} \in \mathscr{D}$ and $\delta_{k_1}, \ldots, \delta_{k,m_k} > 0$ such that, whenever $|x_1, x_2|_{k,l} < \delta_{k,l}$ for $1 \leq l \leq m_k$, it follows that $|f(x_1), f(x_2)|_k < \frac{1}{2}\varepsilon_k$. We claim that $\mathscr{B}^{|\cdot,\cdot|_{1,1},\ldots,|\cdot,\cdot|_{m,m}}_{\delta_{1,1},\ldots,\delta_{m,m_m}}$ star-refines $f^{-1}\left(\mathscr{B}^{|\cdot,\cdot|_1,\ldots,|\cdot,\cdot|_m}_{\varepsilon_1,\ldots,\varepsilon_m}\right)$, which will complete the proof.

So, let $B^{|\cdot,\cdot|_{1,1},\ldots,|\cdot,\cdot|_{m,m_m}}_{\delta_{1,1},\ldots,\delta_{m,m_m}}(x) \in \mathscr{B}^{|\cdot,\cdot|_{1,1},\ldots,|\cdot,\cdot|_{m,m}}_{\delta_{1,1},\ldots,\delta_{m,m_m}}$. We claim that

$$\begin{aligned} \operatorname{Star}_{\mathscr{B}^{|\cdot,\cdot|_{1,1},\ldots,|\cdot,\cdot|_{m,m}}_{\delta_{1,1},\ldots,\delta_{m,m_m}}}\left(B^{|\cdot,\cdot|_{1,1},\ldots,|\cdot,\cdot|_{m,m_m}}_{\delta_{1,1},\ldots,\delta_{m,m_m}}(x)\right) \subseteq \\ f^{-1}\left(B^{|\cdot,\cdot|_1,\ldots,|\cdot,\cdot|_m}_{\varepsilon_1,\ldots,\varepsilon_m}(f(x))\right), \end{aligned} \tag{4.2.1.28}$$

which itself will complete the proof.

So, let $x' \in \operatorname{Star}_{\mathscr{B}^{|\cdot,\cdot|_{1,1},\ldots,|\cdot,\cdot|_{m,m}}_{\delta_{1,1},\ldots,\delta_{m,m_m}}}\left(B^{|\cdot,\cdot|_{1,1},\ldots,|\cdot,\cdot|_{m,m_m}}_{\delta_{1,1},\ldots,\delta_{m,m_m}}(x)\right)$. We wish to show that $f(x') \in B^{|\cdot,\cdot|_1,\ldots,|\cdot,\cdot|_m}_{\varepsilon_1,\ldots,\varepsilon_m}(f(x))$. That $x' \in \operatorname{Star}_{\mathscr{B}^{|\cdot,\cdot|_{1,1},\ldots,|\cdot,\cdot|_{m,m}}_{\delta_{1,1},\ldots,\delta_{m,m_m}}}\left(B^{|\cdot,\cdot|_{1,1},\ldots,|\cdot,\cdot|_{m,m_m}}_{\delta_{1,1},\ldots,\delta_{m,m_m}}(x)\right)$ means that there is some $x'' \in X$ with $x'' \in B^{|\cdot,\cdot|_{1,1},\ldots,|\cdot,\cdot|_{m,m_m}}_{\delta_{1,1},\ldots,\delta_{m,m_m}}(x')$ and $x'' \in B^{|\cdot,\cdot|_{1,1},\ldots,|\cdot,\cdot|_{m,m_m}}_{\delta_{1,1},\ldots,\delta_{m,m_m}}(x)$. As $|x'', x'|_{k,l} < \delta_k$ for $1 \leq k \leq m_l$ (and similarly for $|x'', x|_{k,l}$), it follows from our hypothesis that $|f(x''), f(x')|_k < \frac{1}{2}\varepsilon_k$ and $|f(x''), f(x)|_k < \frac{1}{2}\varepsilon_k$ for $1 \leq k \leq m$. Hence,

$$\begin{aligned} |f(x'), f(x)|_k &\leq |f(x'), f(x'')|_k + |f(x''), f(x)|_k \\ &< \tfrac{1}{2}\varepsilon_k + \tfrac{1}{2}\varepsilon_k = \varepsilon_k . \end{aligned} \tag{4.2.1.29}$$

Thus, $f(x') \in B_{\varepsilon_1,\ldots,\varepsilon_m}^{|\cdot,\cdot|_1,\ldots,|\cdot,\cdot|_m}(f(x))$, as desired. ■

[a]Note that these semimetrics are elements of $\mathscr{E}$, whereas the same symbols in the previous sentence were used to represent elements of $\mathscr{D}$.

Similarly, it would be nice to have a more explicit description of the uniform topology of the uniformity of a semimetric space.

Exercise 4.2.1.30 Let $\langle X, \mathscr{D}\rangle$ be a semimetric space. Show that

$$\begin{aligned} \Big\{ & B_{\varepsilon_1,\ldots,\varepsilon_m}^{|\cdot,\cdot|_1,\ldots,|\cdot,\cdot|_m}(x) : m \in \mathbb{Z}^+; \\ & |\cdot,\cdot|_1, \ldots, |\cdot,\cdot|_m \in \mathscr{D};\ \varepsilon_1, \ldots, \varepsilon_m > 0 \Big\} \end{aligned} \tag{4.2.1.31}$$

is a neighborhood base at x consisting of open sets for the uniform topology of the uniformity associated to the semimetric space $\langle X, \mathscr{D}\rangle$ defined in Definition 4.2.1.21.

R In particular, for a metric space $\langle X, |\cdot,\cdot|\rangle$, a subset $U \subseteq X$ is open iff for every $x \in U$ there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

■ **Example 4.2.1.32 — Discrete metric** Not only does the discrete topology come from a uniformity, but so to does the discrete uniformity in turn come from a metric.

Let X be a set and for $x, y \in X$, define

$$|x, y| := \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases} \tag{4.2.1.33}$$

Exercise 4.2.1.34 Show that $|\cdot,\cdot|$ is indeed a metric on X.

Exercise 4.2.1.35 Show that the uniformity defined by $|\cdot,\cdot|$ is the discrete uniformity.

Exercise 4.2.1.36 Let $\langle X, \mathscr{D}\rangle$ be a semimetric space, let $\lambda \mapsto x_\lambda \in X$ be a net, and let $x_\infty \in X$. Show that $\lambda \mapsto x_\lambda$ converges to x_∞ iff for every $|\cdot,\cdot| \in \mathscr{D}$ and for every $\varepsilon > 0$, $\lambda \mapsto x_\lambda$ is eventually contained in $B_\varepsilon^{|\cdot,\cdot|}(x_\infty)$.

R In other words, a net converges to a point in a semimetric space iff it converges to that point with respect to each semimetric.

Exercise 4.2.1.37 Why does the indiscrete uniformity (on a set with at least two elements) not come from a metric?

But before we head onto topological groups, what about the morphisms in the category of semimetric spaces, you ask? Good question.

Definition 4.2.1.38 — Bounded map (of semimetric spaces) Let $f\colon \langle X, \mathscr{D}\rangle \to \langle Y, \mathscr{E}\rangle$ be a function between semimetric spaces. Then, f is ***bounded*** iff for every $|\cdot,\cdot| \in \mathscr{E}$, there are *finitely-many* $|\cdot,\cdot|_1, \ldots, |\cdot,\cdot|_m \in \mathscr{D}$ and constants $K_1, \ldots, K_m \geq 0$ such that

$$|f(x_1), f(x_2)| \leq K_1|x_1, x_2|_1 + \cdots + K_m|x_1, x_2|_m \tag{4.2.1.39}$$

for all $x_1, x_2 \in X$.

R If X and Y are metric spaces with metric $|\cdot,\cdot|_X$ and $|\cdot,\cdot|_Y$ respectively, this condition reads just

$$|f(x_1), f(x_2)|_Y \leq K|x_1, x_2|_X. \tag{4.2.1.40}$$

In this case, f is called ***Lipschitz-continuous***.

R Of all the categories we've come across, that the bounded maps are the 'right' notion of morphism between semimetric spaces is probably the least obvious.[a] The motivation for the definition is that, for seminormed vector spaces, this definition is equivalent to continuity—see Exercise 4.2.3.12. Perhaps a simpler explanation is that it guarantees uniform-continuity.

[a] Of course, we can declare any collection of morphisms we like. It's just that, taking the morphisms to be *all* functions when the objects are groups (for example) is not particularly useful—the category won't be able to tell that the groups are groups!

Exercise 4.2.1.41 Show that bounded maps between semimetric spaces are uniformly-continuous.

■ **Example 4.2.1.42 — The category of semimetric spaces** The category of semimetric spaces is the category **SemiMet**

(i). whose collection of objects $\mathrm{Obj}(\mathbf{SemiMet})$ is the collection of all semimetric spaces;
(ii). with morphism set $\mathrm{Mor}_{\mathbf{SemiMet}}(X,Y)$ is precisely the set of all bounded maps from X to Y;
(iii). whose composition is given by ordinary function composition; and
(iv). whose identities are given by the identity functions.

R Every semimetric space is canonically a uniform space—see Definition 4.2.1.21. By the previous exercise, every bounded map is likewise uniformly-continuous. Therefore, in fact, the category **SemiMet** *embeds*[a] in **Uni**: the thing to take note of is that *both* the objects *and* the morphisms have to be contained in **Uni**.

[a] We have not defined what precisely this means for categories, but with a little mathematical maturity, you can probably figure it out. In any case, it's okay if you don't know the precise definition.

■ Example 4.2.1.43 — A Lipschitz-continuous function The function $x \mapsto x$ from $\mathbb{R}$ to $\mathbb{R}$.

■ Example 4.2.1.44 — A uniformly-continuous function that is not Lipschitz-continuous Define $f\colon [0,1] \to \mathbb{R}$ by $f(x) := \sqrt{x}$. This function is continuous on $[0,1]$, and hence uniformly-continuous by the Cantor-Heine Theorem (Proposition 4.1.3.25) ($[0,1]$ is quasicompact by the Heine-Borel Theorem (Theorem 2.5.3.3)). On the other hand, to show that it is *not* Lipschitz-continuous, we need to show that

$$\frac{\sqrt{x}-\sqrt{y}}{x-y} \tag{4.2.1.45}$$

is *not* bounded for $x, y \in [0,1]$ distinct. However, simply take $x = 0$. Then, we need to show that

$$\frac{\sqrt{y}}{y} = \frac{1}{\sqrt{y}} \tag{4.2.1.46}$$

is not bounded on $[0,1]$. Equivalently, you can show that $\lim_{y\to 0^+} \sqrt{y} = 0$.[a]

[a] We have technically not defined one-sided limits. If this bothers you, it's not a bad exercise to try to come up with the definition yourself.

4.2.2 Topological groups

Before we talk about any sort of uniformity, we had better first define what we mean by a topological group.

Definition 4.2.2.1 — Topological group A ***topological group*** is a group $\langle G, \cdot, 1, -^{-1}\rangle$ equipped with a topology such that

(i). $\cdot : G \times G \to G$ is continuous; and
(ii). $-^{-1} : G \to G$ is continuous.

R That is to say, a topological group is a thing that is both a group and a topological space, subject to a couple of 'compatibility' axioms that demand that the two structures 'work together'. This is very analogous to our definition of preordered rgs (Definition 1.1.4.1)—a preordered rg is both a preordered set and a rg subject to a couple of ''compatibility'' conditions. This idea is not uncommon throughout all of mathematics, and, as you might have expected by this point, can be unified (to an extent anyways) with the use of categories.

R Recall that in a remark of the definition of a group (Definition A.4.8), we made a slight deal about "having inverses" not being stated as an *extra property* but rather as *extra structure*. This is one reason why. When we go to define a topological group, the operation of taking inverses should be thought of as just that—an operation, on the *same footing as the product*. If we think of the operation of taking inverses as on the same footing as the product, then we almost *have* to also assume that the inverse operation is likewise continuous, whereas if it were thought of just as an existence property, it would not make as much sense to do this.

W Warning: You do need to check both: it is possible that multiplication be continuous but not inversion, or vice-versa—see the following counter-examples.

■ **Example 4.2.2.2 — A topology on a group for which multiplication is continuous but inversion is not** [a] Define

$G := \mathbb{Q}^\times = \{q \in \mathbb{Q} : q \neq 0\}$ equipped with the ordinary multiplication in $\mathbb{Q}$.

We define a topology on G by specifying a neighborhood base for the topology. For $q \in G$, define

$$\mathscr{B}_q := \{(q + n\mathbb{Z}) \cap G : n \in \mathbb{Z},\ n \neq 0\} . \tag{4.2.2.3}$$

We claim that this satisfies the hypotheses of Proposition 3.1.1.9, so that there will be a unique topology on G for which $\{\mathscr{B}_q : q \in G\}$ is a neighborhood base. So, let $(q + n_1\mathbb{Z}) \cap G, (q + n_2\mathbb{Z}) \cap G \in \mathscr{B}_q$. Then, $(q + \mathrm{lcm}(n_1, n_2)\mathbb{Z}) \cap G \subseteq ((q + n_1\mathbb{Z}) \cap G) \cap ((q + n_2\mathbb{Z}) \cap G)$, where here lcm least common multiple (Definition C.8). Furthermore, for $r \in (q + \mathrm{lcm}(n_1, n_2)\mathbb{Z}) \cap G$, we have that $(r + \mathrm{lcm}(n_1, n_2)\mathbb{Z}) \cap G \subseteq (q + \mathrm{lcm}(n_1, n_2)\mathbb{Z}) \cap G$. Thus, there is a unique topology on G for which $\{\mathscr{B}_q : q \in G\}$ is a neighborhood base.

We now wish to show that all the elements in this neighborhood base are in fact open neighborhoods. So, fix an element in the neighborhood base $(q_0+n_0\mathbb{Z})\cap G$ and let $q \in (q_0+n_0\mathbb{Z})\cap G$. We need to show that an element of $\mathscr{B}_q$ is contained in $(q_0 + n_0\mathbb{Z}) \cap G$. We claim that in fact $q + n_0\mathbb{Z} = q_0 + n_0\mathbb{Z}$. As $q \in q_0 + n_0\mathbb{Z}$, we can write $q = q_0 + n_0k$ for some $k \in \mathbb{Z}$. Hence, $q + n_0\mathbb{Z} = q_0 + n_0k + n_0\mathbb{Z} = q_0 + n_0\mathbb{Z}$. Thus, all elements of the neighborhood base are in fact open. It follows (Proposition 3.1.1.9 again) that

$$\{q + n\mathbb{Z} : q \in G,\ n \in \mathbb{Z},\ n \neq 0\} \tag{4.2.2.4}$$

is a base for this topology.

The odd integers, $(1 + 2\mathbb{Z}) \cap G = 1 + 2\mathbb{Z}$, thus themselves form an open set in this topology (take $q = 1$ and $n = 2$), and its preimage under inversion is $\left\{\frac{1}{m} : m \in 2\mathbb{Z} + 1\right\}$. If this were open, then in particular, we would be able to find an element of $\mathscr{B}_1$ contained it in, which is impossible as the only integers in this set are ± 1.

We now show that multiplication on the other hand *is* continuous. So, let $\lambda \mapsto \langle q_\lambda, r_\lambda \rangle \in G \times G$ be a net converging to $\langle q, r \rangle \in G \times G$. We wish to show that $\lambda \mapsto q_\lambda r_\lambda$ converges to qr. To do that, it suffices to show that $\lambda \mapsto q_\lambda r_\lambda$ is eventually contained in $qr + n\mathbb{Z}$ for all nonzero $n \in \mathbb{Z}$. So, fix $n \in \mathbb{Z}$ nonzero arbitrary. That $\lambda \mapsto \langle q_\lambda, r_\lambda \rangle$ converges to $\langle q, r \rangle$ implies that $\lambda \mapsto q_\lambda$ converge to q and $\lambda \mapsto r_\lambda$ converges to r (Corollary 3.5.3.11). Thus, there is some λ_0 such that $q_\lambda \in q + n\mathbb{Z}$ and $r_\lambda \in r + n\mathbb{Z}$ whenever $\lambda \geq \lambda_0$. Of course, however, if $q_\lambda \in q + n\mathbb{Z}$ and $r_\lambda \in r + n\mathbb{Z}$, then we have that $q_\lambda r_\lambda \in qr + n\mathbb{Z}$, as desired.

[a]This is adapted from an answer on math.stackexchange.

■ **Example 4.2.2.5 — A topology on a group for which inversion is continuous but multiplication is not** Define $G := \mathbb{R}^\times$ and equip G with the cocountable topology. Inversion is bijective, and so the preimage of a countable set is countable, and hence the preimage of every closed set under inversion is closed. Thus, inversion is continuous.

We now check that multiplication is not continuous. The set $\{x \in G : x \neq 1\}$ is cocountable, and hence open. Therefore, its preimage under multiplication,

$$S := \{\langle x, y \rangle \in G \times G : xy \neq 1\}, \tag{4.2.2.6}$$

should likewise be open. Thus, as $\langle 1, 2 \rangle \in S$, if S were open, we should be able to find $U, V \subseteq G$ open (i.e. cocountable) such that $\langle 1, 2 \rangle \in U \times V \subseteq S$. If this were true, then in particular we would have that

$$\begin{aligned} \left\{\langle x, \tfrac{1}{x} \rangle : x \in \mathbb{R}^\times\right\} &= S^\mathsf{C} \subseteq (U \times V)^\mathsf{C} \\ &= (U^\mathsf{C} \times \mathbb{R}^\times) \cup (\mathbb{R}^\times \times V^\mathsf{C}). \end{aligned} \tag{4.2.2.7}$$

There are uncountably many $x \in \mathbb{R}^\times$ not contained in U^C, and as V^C is countable, at least one of them (in fact, uncountably many of them) must have the property that $\frac{1}{x}$ is likewise not in V^{C}. This, however, yields an element of S^{C} that cannot be contained in $(U \times V)^{\mathsf{C}}$: a contradiction.

Example 4.2.2.8 — The category of topological groups The category of topological groups is the category **TopGrp**

(i). whose collection of objects $\mathrm{Obj}(\mathbf{TopGrp})$ is the collection of all topological groups;
(ii). with morphism set $\mathrm{Mor}_{\mathbf{TopGrp}}(G, H)$ is precisely the set of all continuous group homomorphisms from G to H;
(iii). whose composition is given by ordinary function composition; and
(iv). whose identities are given by the identity functions.

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Example 4.2.2.9 — A topological group that is not T_0 Define $G := \mathbb{R}/\mathbb{Q}$.[a] Let $\mathrm{q} : \mathbb{R} \to G$ be the quotient map (i.e. the map that sends an element to its equivalence class) and equip $\mathbb{R}/\mathbb{Q}$ with the quotient topology.

Exercise 4.2.2.10 Show that $+ : G \times G \to G$ and $-^{-1} : G \to G$ are continuous.

We now check that the quotient topology on G is not T_0. So, let $x \in \mathbb{R}$ be irrational. We wish to show every open neighborhood of $x + \mathbb{Q}$ contains $0 + \mathbb{Q}$ and conversely. So, let $U \ni x + \mathbb{Q}$ be open. Then, by definition, $\mathrm{q}^{-1}(U)$ is an open neighborhood of x, and so by density, must contain some rational number $r \in \mathrm{q}^{-1}(U)$. But then, $0 + \mathbb{Q} = r + \mathbb{Q} \in \mathrm{q}\left(\mathrm{q}^{-1}(U)\right) \subseteq U$. On the other hand, if U is an open neighborhood of $0 + \mathbb{Q}$, $\mathrm{q}^{-1}(U)$ is an open neighborhood of 0, and so by density again, must contain some $\varepsilon > 0$ so that $x - \varepsilon$ is rational. But if $x - \varepsilon \in \mathbb{Q}$, then $x + \mathbb{Q} = \varepsilon + \mathbb{Q} \in \mathrm{q}\left(\mathrm{q}^{-1}(U)\right) \subseteq U$.

R We show in the next section that every T_0 uniform space is in fact (uniformly-)completely-T_3. Thus, this serves as an example of a uniform space which is not (uniformly-)completely-T_3.

[a] This is the quotient group construction—see Definition A.4.1.5.

A large number of examples of topological groups arise from totally-ordered rngs (or more generally, totally-ordered commutative groups).

Exercise 4.2.2.11 Let G be a totally-ordered commutative group. Show that G is a topological group with respect to the order topology.

Before we put a uniformity on topological groups, it will be useful to know at least one basic fact about them.

Proposition 4.2.2.12 Let G be a topological group and let U be a neighborhood of the identity. Then, there exists an open neighborhood V of the identity such that (i) $VV \subseteq U$ and (ii) $V^{-1} = V$.

R The notation means what you think it means: $VV := \{v_1v_2 : v_1 \in V,\ v_2 \in V\}$ (note how this is not the same as $V^2 := \{v^2 : v \in V\}$) and $V^{-1} := \{v^{-1} : v \in V\}$.

Proof. Regarding the group operation $\cdot$ as a function from $G \times G$ to G, we know that $\cdot^{-1}(U)$ is an open neighborhood of $\langle 1, 1\rangle$ in $G \times G$, and therefore we have that $V \times W \subseteq \cdot^{-1}(U)$ for some $V, W \subseteq G$ open neighborhoods of the identity (by the definition of the product topology Proposition 3.5.3.2). Replace V with $V \cap W$, another open neighborhood of the identity, so that $V \times V \subseteq \cdot^{-1}(U)$. In other words, $VV \subseteq U$. Now do this exact same construction again and find another open neighborhood of the identity W (replacing our 'old' W)

with $WW \subseteq V$. Now define

$$W' := W \cap [-^{-1}]^{-1}(W), \tag{4.2.2.13}$$

that is, the intersection of W with the preimage of W under the inverse function $-^{-1} : G \to G$. This will be yet another open neighborhood of the identity, with both $W', (W')^{-1} \subseteq W$. Finally, define

$$W'' := (W')(W')^{-1}. \tag{4.2.2.14}$$

This certainly satisfies $(W'')^{-1} = W''$, and furthermore,

$$\begin{aligned} W''W'' &:= (W')(W')^{-1}(W')(W')^{-1} \subseteq WWWW \\ &\subseteq VV \subseteq U. \end{aligned} \tag{4.2.2.15}$$

■

Proposition 4.2.2.16 — Uniformities on a topological group Let G be a topological group and define

$$\widetilde{\mathscr{B}}_{G,L} := \{\mathscr{B}_{U,L} : U \ni 1 \text{ is open.}\} \tag{4.2.2.17}$$

and

$$\widetilde{\mathscr{B}}_{G,R} := \{\mathscr{B}_{U,R} : U \ni 1 \text{ is open.}\} \tag{4.2.2.18}$$

where

$$\mathscr{B}_{U,L} := \{gU : g \in G\} \tag{4.2.2.19a}$$

$$\mathscr{B}_{U,R} := \{Ug : g \in G\}\,. \tag{4.2.2.19b}$$

Then, $\widetilde{\mathscr{B}}_{G,L}$ and $\widetilde{\mathscr{B}}_{R,L}$ are uniform bases, which generate respectively the ***left uniformity*** $\widetilde{\mathscr{U}}_{G,L}$ and ***right uniformity*** $\widetilde{\mathscr{U}}_{G,R}$.

Furthermore,

(i). for every $g \in G$, the maps $G \ni x \mapsto gx \in G$ and $G \ni x \mapsto xg \in G$ are uniformly continuous with respect to the left and right uniformities respectively;

(ii). $\langle G, \widetilde{\mathscr{U}}_{G,L}\rangle \ni x \mapsto x^{-1} \in \langle G, \widetilde{\mathscr{U}}_{G,R}\rangle$ is a uniform-homeomorphism and vice-versa;

(iii). the uniform topologies of both the left and right uniformities are the same as the original topology.

R Note that we could have equally well taken the covers $\mathscr{B}_{U,L}$ and $\mathscr{B}_{U,R}$ for only $U \in \mathscr{N}$, $\mathscr{N}$ a fixed neighborhood base of the identity (by Proposition 4.1.3.6).

R It also turns out that $\langle G, \widetilde{\mathscr{U}}_{G,L}\rangle$ is complete iff $\langle G, \widetilde{\mathscr{U}}_{G,R}\rangle$ is complete (indeed, this follows immediately from (ii))—see Definition 4.4.7 for the definition of completeness.

R On the other hand, it is *not* necessarily the case that a net is Cauchy with respect to the left uniformity iff it is Cauchy with respect to the right uniformity–see Exercise 4.4.4.

R For almost all purposes, whether we use the left or right uniformity is a matter of convention. Moreover, a lot of the important examples, especially in functional analysis, are *commutative* groups, in which case the two uniformities are the same. In any case, for the sake of definitiveness, unless otherwise stated, assume we are using the *left* uniformity.

Proof. We leave this as an exercise.

Exercise 4.2.2.20 Prove this yourself.

■

We had a potential problem here—G started its life as a topological group, and in particular, as a topological space. We then defined on it two uniform structures, from each of which it obtains the uniform topology. The question arises: “Are these topologies the same, and if not, which one should we use?”. Fortunately, as stated above, they are the same. However, we have have yet another potential problem to deal with here—$\mathbb{R}$ is both a metric space and a topological group (with respect to $+$), so we could use either the (semi)metric uniformity or the topological group uniformity.[6] Fortunately, we needn't worry about this, because the two uniformities are the same.

Exercise 4.2.2.21 Show that $\widetilde{\mathscr{B}}_{\langle \mathbb{R}, |\cdot,\cdot|\rangle}$ and $\widetilde{\mathscr{B}}_{\langle \mathbb{R}, +\rangle}$ define the same uniformity on $\mathbb{R}$.

You might say that functional analysis is the study of topological vector spaces (the term "functional" a result of the fact that many 'spaces' of functions are topological vector spaces). As vector spaces are in particular a group (just forget about the scalars), everything we say regarding the uniformities of topological groups also applies to uniformities of topological vector spaces. Thus, a knowledge of uniform spaces is very useful when studying functional analysis. In particular, the following result is used ubiquitously (to the point where it is so common that it is not really even mentioned).

Proposition 4.2.2.22 Let $f\colon G \to H$ be a group homomorphism between topological groups and let $x_0 \in G$. Then, if f is continuous at x_0, then f is uniformly-continuous.

R This shows that the category **TopGrp** *embeds* into the category **Uni**. We already knew that the *objects* 'embedded' (from the canonical uniformity on topological groups given in Proposition 4.2.2.16)—this result tells us furthermore that the morphisms 'embed' as well.

Proof. STEP 1: MAKE HYPOTHESES
Suppose that f is continuous at x_0.

STEP 2: SHOW THAT f IS CONTINUOUS.
We first show that f is continuous (as opposed to just continuous at x_0). To show that, we show that f is continuous at

[6] They both agree of course because $\mathbb{R}$ is commutative.

$x \in G$ for arbitrary x. Let V be a neighborhood of $f(x) \in H$. Then, $f(x_0)f(x)^{-1}V$ is a neighborhood of $f(x_0) \in H$. As f is continuous at x_0, it follows that $f^{-1}\left(f(x_0)f(x)^{-1}V\right)$ is a neighborhood of x_0, and so $xx_0^{-1}f^{-1}\left(f(x_0)f(x)^{-1}V\right)$ is a neighborhood of x.[a] However,

$$\begin{aligned} & f\left(xx_0^{-1}f^{-1}\left(f(x_0)f(x)^{-1}V\right)\right) \\ & \qquad = f(x)f(x_0)^{-1}f\left(f^{-1}\left(f(x_0)f(x)^{-1}V\right)\right) \\ & \qquad \subseteq f(x)f(x_0)^{-1}f(x_0)f(x)^{-1}V = V, \end{aligned} \tag{4.2.2.23}$$

so that

$$xx_0^{-1}f^{-1}\left(f(x_0)f(x)^{-1}V\right) \subseteq f^{-1}(V), \tag{4.2.2.24}$$

so that $f^{-1}(V)$ is a neighborhood of x, so that f is continuous at x.

STEP 3: SHOW THAT f IS UNIFORMLY-CONTINUOUS. To show that f is uniformly-continuous, we apply Proposition 4.1.3.7 (it suffices to check uniform-continuity on a uniform base). So, let $V \subseteq H$ be an open neighborhood of the identity and consider the uniform cover $\mathscr{U}_V := \{hV : h \in H\}$. To show that $f^{-1}(\mathscr{U}_V)$ is a uniform cover, it suffices to find an open neighborhood $U \subseteq G$ of the identity such that $\mathscr{U}_U \lll f^{-1}(\mathscr{U}_V)$. Take U' to be an open neighborhood of the identity such that $U'U' \subseteq f^{-1}(V)$, and then in turn take U to be an open neighborhood of the identity such that (i) $UU \subseteq U'$ and (ii) $U = U^{-1}$ (which we may do by Proposition 4.2.2.12). We wish to show that

$$\mathrm{Star}_{\mathscr{U}_U}(U) := \bigcup_{\substack{x \in G \\ xU \cap U \neq \emptyset}} xU \subseteq f^{-1}(V). \tag{4.2.2.25}$$

It will follow from this (see (4.2.2.26)) that $\mathscr{U}_U \lll f^{-1}(\mathscr{U}_V)$. So, let $x \in G$ be such that $xU \cap U \neq \emptyset$. Then, there are

$u_1, u_2 \in U$ such that $xu_1 = u_2$, so that $x = u_2 u_1^{-1} \in UU^{-1} = UU \subseteq U'$. Thus, $xU \subseteq U'U' \subseteq f^{-1}(V)$. (4.2.2.25) follows from this.

We then have

$$\begin{aligned}\operatorname{Star}_{\mathscr{U}_U}(x_0U) &= \bigcup_{\substack{x\in G\\ xU\cap x_0U\neq\emptyset}} xU = \bigcup_{\substack{x\in G\\ xU\cap U\neq\emptyset}} x_0xU \\ &= x_0 \operatorname{Star}_{\mathscr{U}_U}(U) \subseteq x_0f^{-1}(V) \\ &\subseteq f^{-1}(f(x_0)V) \in f^{-1}(\mathscr{U}_V),\end{aligned} \tag{4.2.2.26}$$

so that $\mathscr{U}_U \lll f^{-1}(\mathscr{U}_V)$. ■

[a]The juxtaposition here is being used to denote multiplication in the group. Be careful not to confuse preimages with inverse elements (even though the same symbol is used, the context makes the notation unambiguous).

4.2.3 Topological vector spaces and algebras

An *incredibly* family of examples of topological groups are the topological vector spaces.

Definition 4.2.3.1 — Topological vector space A ***topological vector space*** is real vector space $\langle V, +, 0, \mathbb{R}, \cdot\rangle$ such that

(i). $\langle V, +, 0\rangle$ is a topological group; and
(ii). $\cdot : \mathbb{R} \times V \to V$ is continuous.

R Of course, this definition makes sense if we were to replace $\mathbb{R}$ with any topological field[a], but for our purposes, restricting ourselves to working over the reals will be sufficient. The other case of most interest is over $\mathbb{C}$, but we have not even defined the complex numbers.

[a]What do you think the definition of a topological field should be?

One big reason why we are interested in topological vector spaces is because almost all of the examples of semimetrics we counter actually come *seminorms*.

Definition 4.2.3.2 — Seminorm and norm Let V be a real vector space. Then, a ***seminorm*** on V is a function $\|\cdot\| : V \to \mathbb{R}_0^+$ such that

(i). (Homogeneity) $\|\alpha v\| = \|\alpha\| \|v\|$ for $\alpha \in \mathbb{R}$ and $v \in V$;
(ii). (Triangle Inequality) $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$.

$\|\cdot\|$ is a ***norm*** if furthermore (Definiteness) $\|x\| = 0$ implies $x = 0$.

Definition 4.2.3.3 — Semimetric induced by a seminorm Let V be a real vector space, let $\|\cdot\|$ be a seminorm on V, and let $v_1, v_2 \in V$. Then, the ***semimetric induced by*** $\|\cdot\|$, $|\cdot,\cdot|$, is defined by

$$|v_1, v_2| := \|v_1 - v_2\|. \tag{4.2.3.4}$$

Exercise 4.2.3.5 Check that $|\cdot,\cdot|$ is indeed a semimetric. Show that if $\|\cdot\|$ is a norm then $|\cdot,\cdot|$ is a metric.

R Intuitively, the seminorm of something is like its 'size' and semimetric is like 'distance'—the 'distance' between two vectors is the 'size' of their difference.

Definition 4.2.3.6 — Seminormed vector space and normed vector space A ***seminormed vector space*** is a real vector space V together with a collection of seminorms $\mathscr{D}$ such that $\langle V, \mathscr{D}\rangle$ is a semimetric space. $\langle V, \mathscr{D}\rangle$ is a ***normed vector space*** iff it is furthermore a metric space.

As by now you should have expected, the morphisms of seminormed vector spaces are the bounded (Definition 4.2.1.38) linear maps.

Example 4.2.3.7 — The category of seminormed vector spaces The category of topological spaces is the category **SemiVect**

(i). whose collection of objects Obj(**SemiVect**) is the collection of all topological spaces;
(ii). with morphism set $\mathrm{Mor}_{\mathbf{SemiVect}}(V,W)$ precisely the set of all bounded linear maps from V to W;
(iii). whose composition is given by ordinary function composition; and
(iv). whose identities are given by the identity functions.

Exercise 4.2.3.8 Let $f\colon \langle V,\mathscr{D}\rangle \to \langle W,\mathscr{E}\rangle$ be a linear map between seminormed vector spaces. Show that f is bounded iff for every $\|\cdot\|_0 \in \mathscr{E}$ there are *finitely-many* $\|\cdot\|_1,\ldots,\|\cdot\|_m \in \mathscr{D}$ and constants $K_1,\ldots,K_m \geq 0$ such that

$$\|f(v)\|_0 \leq K_1\|v\|_1 + \cdots + K_m\|v\|_m \tag{4.2.3.9}$$

for all $v \in V$.

R Compare this with the definition of bounded maps of semimetric spaces (Definition 4.2.1.38). Essentially this boils down to the statement that, to check that f is bounded, it suffices to check for $x_2 = 0$ and $x_1 =: v$ arbitrary (in the notation of Definition 4.2.1.38).

Note that a priori a seminormed vector space is not a topological vector space. However, being a semimetric space, it is in fact a uniform space, and so in turn is equipped with its uniform topology. The question is then whether it is a topological vector space with respect to the uniform topology. Of course, the answer is in the affirmative. Once we know that the seminormed vector space V is likewise a topological vector space, we know in turn that its underlying topological group $\langle V, +, 0, -\rangle$ induces in turn yet another uniform structure, and so a new question arises as to whether or not this uniform structures agrees with the one induced from the semimetric space structure. Of course, the answer to this is likewise in the affirmative.

Exercise 4.2.3.10 Let V be a seminormed vector space. Show that V is a topological vector space with respect to the uniform topology induced by the semimetric uniformity.

Exercise 4.2.3.11 Let V be a seminormed vector space. Show that the uniformity induced by the topological group structure $\langle V, +, 0, -\rangle$ is the same as the semimetric uniformity.

As a mater of fact, the morphisms don't care whether you're thinking of things as a semimetric space or as a topological group either.

Exercise 4.2.3.12 Let $f : \langle V, \mathscr{D}\rangle \to \langle W, \mathscr{E}\rangle$ be a *linear* map between two seminormed vector spaces. Show that it is continuous iff it is bounded.

> Unless otherwise stated, seminormed vector spaces are always equipped with the uniformity induced by the semimetric space structure (or, equivalently, the uniformity induced by the topological group structure).

In fact, a lot of examples of seminormed vector spaces have *even more* structure, namely, the structure of an algebra.

Definition 4.2.3.13 — Associative-algebra
An ***associative-algebra*** is a set A equipped with the structure of a vector space over a field F $\langle A, +, 0, -, F\rangle$ and the structure of a ring $\langle A, +, 0, -, \cdot, 1\rangle$ such that

(i). $(\alpha_1\alpha_2) \cdot a = \alpha_1 \cdot (\alpha_2 \cdot a)$ for $\alpha_1, \alpha_2 \in F$ and $a \in A$; and
(ii). $\alpha \cdot (a_1 a_2) = (\alpha \cdot a_1)a_2$ for $\alpha \in F$ and $a_1, a_2 \in A$.

R That is, an associative-algebra is both a vector space and a ring subject to a couple of compatibility axioms.

R The prefix "associative" here is to distinguish this concept from other things which go by the term "algebra", e.g. "lie algebra" or as used in general algebra. For us, there is no just plain "algebra"[a]

[a]Though you could certainly define such a thing, simply by dropping the assumption that the multiplication is associative, but as we don't a name for not-necessarily-associative rgs, we would either have to create such a term or explicitly list the axioms here, both of which are messy options. As we have no need for not-necessarily-associative algebras, we simply do not bother.

Definition 4.2.3.14 — Homomorphism (of associative-algebras) Let A and B be associative-algebras and let $f\colon A \to B$ be a function. Then, f is a ***homomorphism*** iff f is both a linear map of the underlying vectors spaces and a ring homomorphism of the underlying vector spaces.

■ **Example 4.2.3.15 — The category of associative-algebras over a field F** The category of associative-algebras over a field F is the category $\_\mathbf{Alg}F$

(i). whose collection of objects $\mathrm{Obj}(\_\mathbf{Alg}F)$ is the collection of all associative-algebras;
(ii). with morphism set $\mathrm{Mor}_{\_\mathbf{Alg}F}(A, B)$ is precisely the set of all homomorphisms from A to B;
(iii). whose composition is given by ordinary function composition; and
(iv). whose identities are given by the identity functions.

Definition 4.2.3.16 — Seminormed algebra A ***seminormed algebra*** is an associative-algebra whose underlying vector space is a seminormed vector space such that $\|a_1 a_2\| \leq \|a_1\| \|a_2\|$ for $a_1, a_2 \in A$.

R Note that we don't mention the field we are working over because, by definition, seminormed vector spaces are always over $\mathbb{R}$—see Definition 4.2.3.6.

R Note that seminormed algebras are *associative*-algebras. The term "seminormed associative-algebras" is unnecessarily verbose.

Example 4.2.3.17 — The category of seminormed algebras The category of seminormed algebras over a field F is the category $\mathbf{Semi}F\textbf{-}\mathbf{Alg}$

(i). whose collection of objects $\mathrm{Obj}(\mathbf{Semi}F\textbf{-}\mathbf{Alg})$ is the collection of all seminormed algebras;
(ii). with morphism set $\mathrm{Mor}_{\mathbf{Semi}F\textbf{-}\mathbf{Alg}}(A, B)$ is precisely the set of all bounded homomorphisms from A to B;
(iii). whose composition is given by ordinary function composition; and
(iv). whose identities are given by the identity functions.

We're just about to put all these definitions to use for the purpose of introducing, among other things, the notion of *uniform convergence*. It will be important to be able to contrast this with the weaker notion of *pointwise convergence*.

Definition 4.2.3.18 — Pointwise convergence Let X be a set,let Y be a topological space, let $\lambda \mapsto f_\lambda \in \mathrm{Mor}_{\mathbf{Set}}(X, Y)$ be a net, and let $f_\infty \in \mathrm{Mor}_{\mathbf{Set}}(X, Y)$. Then, $\lambda \mapsto f_\lambda$ ***converges pointwise*** to f_∞ iff for every $x \in X$, $\lambda \mapsto f_\lambda(x)$ converges to $f_\infty(x)$.

R Thus, $\lambda \mapsto f_\lambda$ converges pointwise to f_∞ iff if you plug in x and take the limit, you get $f_\infty(x)$.

■ **Example 4.2.3.19** For $m \in \mathbb{N}$, define $f\colon [0,1] \to \mathbb{R}$ by $f(x) := x^m$. Then, $m \mapsto f_m$ converges pointwise to the function

$$f_\infty(x) := \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1. \end{cases} \tag{4.2.3.20}$$

It turns out that this does *not* converge uniformly to f_∞ because, if it did, f_∞ would need to be continuous—see Theorem 4.4.1.1.

Exercise 4.2.3.21 Let X be a set and let Y be a topological space. Show that the relation of pointwise convergence between nets in $\mathrm{Mor}_{\mathbf{Set}}(X,Y)$ and elements of $\mathrm{Mor}_{\mathbf{Set}}(X,Y)$ satisfies the axioms of Kelley's Convergence Theorem (Theorem 3.4.2.1), so as to define a unique topology on $\mathrm{Mor}_{\mathbf{Set}}(X,Y)$ for which the notion of convergence is precisely pointwise convergence, the ***topology of pointwise convergence***.

With these new definitions in hand, we now present an incredibly important example of a seminormed algebra, an example that was a large part of the motivation for introducing seminormed algebras at all.

■ **Example 4.2.3.22 — Uniform convergence on quasicompact subsets** Let X be a topological space and define

$$A := \mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R}). \tag{4.2.3.23}$$

Pointwise addition and pointwise scalar multiplication gives A the structure of a real vector space. Pointwise multiplication gives A in turn the structure of a real associative-algebra. The collection $\{\|\cdot\|_K : K \subseteq X \text{ quasicompact}\}$, where

$$\|f\|_K := \sup_{x\in K}\{|f(x)|\}, \tag{4.2.3.24}$$

the ***supremum seminorm*** on K, then gives A the structure of a seminormed algebra. It is thus canonically a uniform space (and in turn a topological space). If X itself is quasicompact, convergence in $\mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})$ is called ***uniform convergence***.[a] Thus, in the general case, people refer to convergence in the topological space $\mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})$ as ***uniform convergence on quasicompact subsets***, though, as this is never ambiguous, you can probably just say "uniform convergence" all the time for short[b]. This is important enough to spell out in detail.

> $\lambda \mapsto f_\lambda \in \mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})$ converges (uniformly) to $f_\infty \in \mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})$ iff for every quasicompact subset $K \subseteq X$ $\lambda \mapsto x_\lambda$ is eventually contained in $B_{\|\cdot\|_K}(f_\infty) := \left\{ f \in \mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R}) : \|f - f_\infty\|_K \right\}$.

Even more explicitly:

> $\lambda \mapsto f_\lambda \in \mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})$ converges (uniformly) to $f_\infty \in \mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})$ iff for every quasicompact subset $K \subseteq X$ and for every $\varepsilon > 0$ there is some λ_0 such that, whenever $\lambda \geq \lambda_0$, it follows that $|f_\lambda(x) - f_\infty(x)| < \varepsilon$ for all $x \in K$.

This is in contrast to *pointwise convergence* (Definition 4.2.3.18): here, we can choose a single λ_0 that 'works' for all $x \in K$, whereas, in the pointwise case, your λ_0 will have to depend on x. That is to say, we can choose λ_0 "uniformly" to work for *all* $x \in K$.

The reason the case X quasicompact is special is because, in this case, it is actually isomorphic (in the category of seminormed algebras) to a normed algebra.

Exercise 4.2.3.25 Let X be quasicompact. Show that

$$\begin{aligned} \mathrm{id}_X \colon \langle X, \{\|\cdot\|_X\}\rangle \to \\ \langle X, \{\|\cdot\|_K : K \subseteq X \text{ quasicompact}\}\rangle . \end{aligned} \tag{4.2.3.26}$$

is an isomorphism in the category of seminormed algebras.

R The point is that, if all we care about is the seminormed algebra structure, we may always assume without loss of generality that $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is in fact a *normed* algebra with the single norm being given by $\|f\|_X := \sup_{x\in X}\{|f(x)|\}$, the ***supremum norm***. In this case, we do not write $\|f\|_X$ but rather $\|f\|_\infty$. The reason for this notation will become clear when we study L^p spaces—see Section 5.3 L^p spaces.

Unless otherwise stated, $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is a always given the structure of a seminormed algebra, the associative-algebra structure defined pointwise and the seminorms being $\|\cdot\|_K$ for $K \subseteq X$ quasicompact.

Of *incredible* importance is that this space is in fact complete (at least for so-called *quasicompactly-generated* spaces). Of course, we need to first actually define what we mean by complete, and so we postpone this result—see Theorem 4.4.1.1.

[a] Despite the name and the context in which we're presenting it, uniform convergence actually has nothing to do with uniform spaces per se (in contrast to uniform continuity, for example). $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ has a topology, and hence a notion of convergence, which we happen to call "uniform convergence". In particular, we only needed to equip $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ with a topology to define uniform convergence. The reason we waited until the chapter on uniform spaces, of course, is because $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ obtains its

topology from a family of semimetrics (or in the case X is quasicompact, just a single metric), not because of any direct connection with uniform convergence and uniform spaces.

[b]This is further justified in the exercise below.

Another reason to believe that the topology of uniform convergence on quasicompact subsets is a 'natural' topology to use on $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is that, in many cases, it agrees with another frequently-encountered topology, namely the quasicompact-open topology (Meta-definition 3.7.11).

Proposition 4.2.3.27 The topology on $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is the same as the quasicompact-open topology iff X is completely-T_2.

R Recall that, as stated in the previous example, the topology on $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is defined by the semi-norms $\|\cdot\|_K$, for $K \subseteq X$ quasicompact.

Proof. By Exercise 4.2.1.30 and Proposition 3.1.1.9,[a] a base for usual topology on $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is given by the collection of sets of the form

$$\begin{aligned} & f + B^{K_1,\ldots,K_m}_{\varepsilon_1,\ldots,\varepsilon_n} \\ & \quad := f + \left\{ g \in \mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R}) : \|g\|_{K_k} < \varepsilon_k \right\} \end{aligned} \tag{4.2.3.28}$$

for $m \in \mathbb{Z}^+$, $K_k \subseteq X$ quasi-compact, and $\varepsilon_k > 0$, where for convenience we have abbreviated $B^{K_1,\ldots,K_m}_{\varepsilon_1,\ldots,\varepsilon_m} := B^{|\cdot,\cdot|_{K_1},\ldots,|\cdot,\cdot|_{K_m}}_{\varepsilon_1,\ldots,\varepsilon_m}$. On the other hand, a base for the quasicompact-open topology is given by

$$\begin{aligned} & O_{K_1,a_1,\varepsilon_1;\ldots;K_n,a_n,\varepsilon_n} \\ & \quad := \left\{ f \in C(X; F) : f(K_k) \subseteq B_{\varepsilon_k}(a_k) \right\} \end{aligned} \tag{4.2.3.29}$$

for $K_k \subseteq X$ quasicompact, $a_k \in \mathbb{R}$, and $\varepsilon_k > 0$.

($\Rightarrow$) Suppose that the topology on $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is the same as the quasicompact-open topology. Let $x, y \in X$. Then, for every $\varepsilon > 0$, the set $O_{\{x\},1,\varepsilon;\{y\},0,\varepsilon}$ is open in the locally convex topology, and hence contains a set of the form

$$f + B^{K_1,\ldots,K_n}_{\varepsilon_1,\ldots,\varepsilon_n}, \tag{4.2.3.30}$$

and hence, in particular, contains f. By taking ε sufficiently small, we can guarantee that $f(y) \neq f(x)$, so that

$$\frac{f - f(y)}{f(x) - f(y)} \tag{4.2.3.31}$$

is continuous, is 0 at y, and 1 at x. Thus, X is completely T_2.

($\Leftarrow$) Suppose that X is completely T_2. Let $K_1, \ldots, K_n \subseteq X$ be arbitrary quasicompact subsets, let $a_1, \ldots, a_n \in \mathbb{R}$, and let $\varepsilon_1, \ldots, \varepsilon_m > 0$. These arbitrary choices determine an arbitrary 'O-set' of the form (4.2.3.29). Let h be an element of this set. We wish to show that it contains some set of the form (4.2.3.28) that contains h. If K_k meets K_l, then, in order that this "O-set" be nonempty, it had better be the case that $B_{\varepsilon_k}(a_k)$ meet $B_{\varepsilon_l}(a_l)$, in which case the intersection contains some $B_\varepsilon(a) \subseteq B_{\varepsilon_k}(a_k) \cap B_{\varepsilon_l}(a_l)$. Now, remove K_j and K_k from this list and replace them respectively with just $\langle K_j \cup K_k, a, \varepsilon\rangle$. After making these replacements, we obtain a new 'O-set' smaller than the original, and furthermore, with the quasicompact sets disjoint. As X is completely-T_2, there exists a continuous function $f \colon X \to \mathbb{R}$ such that $f(K_k) = \{a_k\}$ for $1 \leq k \leq m$. Then, $f + B^{K_1,\ldots,K_m}_{\varepsilon_1,\ldots,\varepsilon_n}$ is contained in this new 'O-set', and hence is contained in the original 'O-set' as well. Furthermore

$$\begin{aligned} &\sup\{|h(x) - f(x)| : x \in K_k\} \\ &\qquad = \sup\{|h(x) - a_k| : x \in K_k\} < \varepsilon_k, \end{aligned}$$

and hence contains h, as desired.

For the other direction, let $f \in \mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})$, $K_1,\ldots,K_n \subseteq X$ quasicompact, and $\varepsilon_1,\ldots,\varepsilon_m > 0$ all be arbitrary. Then,

$$\begin{aligned} f + B^{K_1,\ldots,K_n}_{\varepsilon_1,\ldots,\varepsilon_n} & \\ &\coloneqq f + \left\{ g \in \mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R}) : \lVert g \rVert_{K_k} < \varepsilon_k \right\} \\ &= f + \left\{ g \in \mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R}) : g(K_k) \subseteq B_{\varepsilon_k}(0) \right\} \end{aligned} \tag{4.2.3.32}$$

is an arbitrary set of the form (4.2.3.28). Finally, let h be an arbitrary element of this set. The function $x \mapsto |h(x) - f(x)|$ achieves a maximum δ_k on K_k. By hypothesis, we have $\delta_k < \varepsilon_k$. Replace δ_k with $\frac{\delta_k+\varepsilon_k}{2}$, so that we still have $\delta_k < \varepsilon_k$, but we now also have the strict inequality $|h(x) - f(x)| < \delta_k$ for all $x \in K_k$.

For each K_i and $x \in K_k$, let $U_x \subseteq K_k$ be open and such that f is strictly within $\frac{\varepsilon_k-\delta_k}{2}$ of $f(x)$ on U_x. Then, there is a finite cover—denote the *closure* (in K_k) of each element of of this finite cover by $L_{k,1},\ldots,L_{k,n_k}$ with $x_{k,l} \in L_{k,l}$ being the x of U_x. Thus, K_k is covered by $L_{k,1},\ldots,L_{k,n_k}$ and f is strictly within $\varepsilon_k - \delta_k$ of $f(x_{k,l})$ on $L_{k,l}$ Do this for each K_k to get compact sets $L_{k,l}$ for $1 \leq k \leq n$ and $1 \leq l \leq m_n$. As h is within δ_k of f on all of K_k, it will certainly be within δ_k of $f(x_{k,l})$ for all l as $x_{k,l} \in K_k$. Thus,

$$h \in O_{L_{1,1},f(x_{1,1}),\delta_1;\ldots,L_{m,n_m},f(x_n,x_{n_m}),\delta_m}. \tag{4.2.3.33}$$

Similarly, if g is an element of this "O-set", we have that

$$\begin{aligned} \sup &\left\{ |g(x) - f(x)| : x \in L_{k,l} \right\} \\ &\leq \sup \left\{ \left| g(x) - f(x_{k,l}) \right| : x \in L_{k,l} \right\} \\ &\qquad + \sup \left\{ \left| f(x_{k,l}) - f(x) \right| : x \in L_{k,l} \right\} \\ &< \delta_k + (\varepsilon_k - \delta_k) < \varepsilon_k. \end{aligned} \tag{4.2.3.34}$$

Thus, g is contained in the set in (4.2.3.32), and hence we have

$$h \in O_{L_{1,1},f(x_{1,1}),\delta_1;\ldots,L_{m,n_m},f(x_n,x_{n_m}),\delta_m} \subseteq f + B^{K_1,\ldots,K_n}_{\varepsilon_1,\ldots,\varepsilon_n},$$

as desired. ∎

[a] These results give respectively a neighborhood base for semimetric spaces and state how neighborhood bases define topologies.

4.3 T_0 uniform spaces are uniformly-completely-T_3

Our goal in this subsection is to show that all T_0 uniform spaces are uniformly-T_3. Of course, to prove this, we had better say what uniformly-completely-T_3 means.

4.3.1 Separation axioms in uniform spaces

Throughout this subsection, let $S_1, S_2 \subseteq X$ be *disjoint* subsets of a uniform space X.

Definition 4.3.1.1 — Uniformly-distinguishable S_1 and S_2 are ***uniformly-distinguishable*** iff there is some uniform cover $\mathscr{U}$ for which $\mathrm{Star}_{\mathscr{U}}(S_1)$ does not intersect S_2 or there is a uniform cover $\mathscr{U}$ for which $\mathrm{Star}_{\mathscr{U}}(S_2)$ does not intersect S_1.

Definition 4.3.1.2 — Uniformly-separated S_1 and S_2 are ***uniformly-separated*** iff there is some uniform cover $\mathscr{U}$ for which $\mathrm{Star}_{\mathscr{U}}(S_1)$ does not intersect S_2 and there is a uniform cover $\mathscr{U}$ for which $\mathrm{Star}_{\mathscr{U}}(S_2)$ does not intersect S_1.

Definition 4.3.1.3 — Uniformly-separated S_1 and S_2 are ***uniformly-separated by neighborhoods*** iff there is some uniform cover $\mathscr{U}$ for which $\mathrm{Star}_{\mathscr{U}}(S_1)$ is disjoint from $\mathrm{Star}_{\mathscr{U}}(S_2)$.

Definition 4.3.1.4 — Uniformly-completely-separated S_1 and S_2 are ***uniformly-completely-separated*** iff there is a uniformly-continuous function $f\colon X \to [0,1]$ such that $f|_{S_1} = 0$ and $f|_{S_2} = 1$.

Exercise 4.3.1.5 Show that if S_1 and S_2 are uniformly-completely-separated, then they are uniformly-separated.

Definition 4.3.1.6 — Uniformly-perfectly-separated S_1 and S_2 are ***uniformly-perfectly-separated*** iff there is a uniformly-continuous function $f\colon X \to [0,1]$ such that $S_1 = f^{-1}(0)$ and $S_2 = f^{-1}(1)$.

Definition 4.3.1.7 — Uniformly-T_0 X is ***uniformly-T_0*** iff any two distinct points are uniformly-distinguishable.

Definition 4.3.1.8 — Uniformly-T_1 X is ***uniformly-T_1*** iff any two distinct points are uniformly-separated.

Definition 4.3.1.9 — Uniformly-T_2 X is ***uniformly-T_2*** iff any two distinct points can be uniformly-separated by neighborhoods.

Definition 4.3.1.10 — Uniformly-completely-T_2 X is ***uniformly-completely-T_2*** iff any two distinct points can be uniformly-completely-separated.

Definition 4.3.1.11 — Uniformly-perfectly-T_2 X is ***uniformly-perfectly-T_2*** iff any two distinct points can be uniformly-perfectly-separated.

Definition 4.3.1.12 — Uniformly-T_3 X is ***uniformly-T_3*** iff it is T_1 and any closed set and a point not contained in it can be uniformly-separated by neighborhoods.

Definition 4.3.1.13 — Uniformly-completely-T_3 X is ***uniformly-completely-T_3*** iff it is T_1 and any closed set and a point not contained in it can be uniformly-completely-separated.

Definition 4.3.1.14 — Uniformly-perfectly-T_3 X is ***uniformly-perfectly-*** T_3 iff it is T_1 and any closed set and a point not contained in it can be uniformly-perfectly-separated.

Definition 4.3.1.15 — Uniformly-T_4 X is ***uniformly-*** T_4 iff it is T_1 and any two disjoint closed subsets can be uniformly-separated by neighborhoods.

Definition 4.3.1.16 — Uniformly-completely-T_4 X is ***uniformly-completely-*** T_4 iff it is T_1 and any two disjoint closed subsets can be uniformly-completely-separated.

Definition 4.3.1.17 — Uniformly-perfectly-T_4 X is ***uniformly-perfectly-*** T_4 iff it is T_1 and any closed set and any two disjoint closed subsets can be can be uniformly-perfectly-separated.

The goal of this section is to prove that all of these separation axioms from uniformly-T_0 to uniformly-completely-T_3 (that is, T_0 implies uniformly-completely-T_3—see Corollary 4.3.2.15). We also present counter-examples to show that all these equivalent axioms are strictly weaker than both uniformly-T_4 (see Example 4.3.3.1) and uniformly-perfectly-T_3 (see Example 4.3.3.3). Before we begin our proof of that result, however, we present a smaller, but still quite useful result, which says that a relatively broad collection of spaces (metric spaces) satisfies the strongest separation axiom one could possibly hope for.

Proposition 4.3.1.18 Metric spaces are uniformly-perfectly-T_4.

Proof. Let X be a metric space and let $C_1, C_2 \subseteq X$ be closed and disjoint.

Exercise 4.3.1.19 Show that there is some $f_1 : X \to [0, \frac{1}{2}]$ uniformly-continuous, equal to 0 precisely on C_1, and equal to $\frac{1}{2}$ on C_2. Similarly, show that there is some $f_2 : X \to [0, \frac{1}{2}]$ uniformly-continuous, equal to $\frac{1}{2}$ precisely on C_2, and equal to 0 on C_1.

Define $f := f_1 + f_2$. Then, this is certainly 0 on C_1 and 1 on C_2. Conversely, suppose that $f(x) = 0$. Then, in particular, $f_1(x) = 0 = f_2(x) = 0$, and so in particular $x \in C_1$. On the other hand, suppose that $f(x) = 1$. Then, we must have in particular that $f_2(x) = \frac{1}{2}$, which implies that $x \in C_2$. Thus, $f^{-1}(0) = C_1$ and $f^{-1}(1) = C_2$, and hence X is uniformly-perfectly-T_4. ∎

4.3.2 The key result

We actually prove a stronger result which requires the notion of the *diameter* of a set (in a metric space).

Definition 4.3.2.1 — Diameter Let $\langle X, |\cdot,\cdot|\rangle$ be a metric space and let $S \subseteq X$. Then, the ***diameter*** of S, $\operatorname{diam}(S)$, is defined by

$$\operatorname{diam}(S) := \sup_{x,y\in S} \{|x,y|\}. \tag{4.3.2.2}$$

R Of course, it may be the case that $\operatorname{diam}(S) = \infty$.

And now we are ready to state our key result.

Theorem 4.3.2.3. Let $\mathscr{U}$ be a uniform cover of a T_0 uniform space X. Then, there exists a metric space Y and a uniformly-continuous surjective function $\mathrm{q} : X \to Y$ such that, if $\operatorname{diam}(S) < 1$ for $S \subseteq Y$, then $\mathrm{q}^{-1}(S)$ will be contained in some element of $\mathscr{U}$.

Proof. [a] To construct Y, we shall put a semimetric on X and then take the quotient set with respect to the equivalence relation of 'being infinitely close to each other'.

STEP 1: CONSTRUCT A SEQUENCE OF STAR-REFINEMENTS OF $\mathscr{U}$

Let us write $\mathscr{U}_0 := \mathscr{U}$. Then, we take a star-refinement $\mathscr{U}_1$ of $\mathscr{U}_0$, in turn another star-refinement $\mathscr{U}_2$ of $\mathscr{U}_1$, and so on.

STEP 2: DEFINE $\ell(x_1, x_2)$ FOR $x_1, x_2 \in X$

Define

$$\ell(x_1, x_2) := \begin{cases} 2 & \text{if } x_2 \notin \mathrm{Star}_{\mathscr{U}_0}(x_1) \\ 2^{1-\max\{m\in\mathbb{N}:x_2\in\mathrm{Star}_{\mathscr{U}_m}(x_1)\}} & \text{otherwise.} \end{cases} \tag{4.3.2.4}$$

Note that this in particular implies that $\ell(x_1, x_2) = 0$ if $x_2 \in \mathrm{Star}_{\mathscr{U}_m}(x_1)$ for all $m \in \mathbb{N}$. (We do need to make the other extreme case explicit as the maximum of the empty-set is $-\infty$.) Thus, the statement that X is T_0 (together with the fact that stars form a neighborhood base for the topology (Proposition 4.1.2.2)) implies that either $\ell(x_1, x_2) > 0$ or $\ell(x_2, x_1) > 0$.

STEP 3: DEFINE $\ell(\mathscr{P})$ FOR PATHS $\mathscr{P}$

For the purposes of this proof, a *path* from x_1 to x_2 will be a finite sequence of points $\langle x^\infty, x^1, \ldots, x^m\rangle$ with $x^\infty = x_1$ and $x^m = x_2$.[b] If $\mathscr{P} = \langle x^\infty, \ldots, x^m\rangle$, then we define $\ell(\mathscr{P}) := \ell(x^\infty, x^1) + \ell(x^1, x^2) + \cdots + \ell(x^{m-1}, x^m)$. We shall call this the *length* of the path.

STEP 4: DEFINE THE SEMIMETRIC

Finally, we define

$$\begin{aligned} |x_1, x_2| := \min\{1, \inf(\{\ell(\mathscr{P}) :& \\ &\mathscr{P} \text{ is a path from } x_1 \text{ to } x_2 \text{ or a path from } x_2 \text{ to } x_1.\})\}. \end{aligned}$$

STEP 5: SHOW THAT THIS IS IN FACT A SEMIMETRIC
From the definition, we have that $|\cdot,\cdot|$ is symmetric (this is the reason for putting the "or" in the definition—note that the definition of $\ell(x_1, x_2)$ is not manifestly symmetric). The triangle inequality follows from the fact that a path from x_1 to x_3 and a path from x_3 to x_2 gives us a path from x_1 to x_2, with the length of this new path being the sum of the lengths of the other two. Thus, $|\cdot,\cdot|$ is in fact a semimetric.

STEP 6: CONSTRUCT Y
Define $x_1 \sim x_2$ iff $|x_1, x_2| = 0$. That this is an equivalence relation follows from the fact that $|\cdot,\cdot|$ is a semimetric. Thus, we may define

$$Y := X/\sim . \tag{4.3.2.5}$$

STEP 7: CONSTRUCT THE METRIC ON Y
We abuse notation and write the induced metric on Y with the same symbol $|\cdot,\cdot|$ as the semimetric on X:

$$|[x_1]_\sim, [x_2]_\sim| := |x_1, x_2|. \tag{4.3.2.6}$$

Exercise 4.3.2.7 Check that $|\cdot,\cdot|$ on Y is well-defined.

STEP 8: SHOW THAT THIS IN FACT A METRIC
$|\cdot,\cdot|$ on Y is automatically symmetric and satisfies the triangle inequality because $|\cdot,\cdot|$ on X does. Furthermore, if $|[x_1]_\sim, [x_2]_\sim| = 0$, then $|x_1, x_2| = 0$, and so $x_1 \sim x_2$ by the definition of $\sim$. Thus, $|\cdot,\cdot|$ is indeed a metric on Y.

STEP 9: DEFINE $\mathrm{q} : X \to Y$

We take $\mathrm{q} : X \to Y$ to be the quotient map: $\mathrm{q}(x) := [x]_{\sim}$. Of course q is surjective (all quotient maps are).

STEP 10: SHOW THAT q IS UNIFORMLY-CONTINUOUS
We apply Proposition 4.2.1.26 which characterizes uniform-continuity for functions whose codomain is a metric space. So, let $\varepsilon > 0$. We must find a uniform cover $\mathscr{U}$ of X such that for every $U \in \mathscr{U}$, whenever $x_1, x_2 \in U$, it follows that $|\mathrm{q}(x_1), \mathrm{q}(x_2)| < \varepsilon$. It suffices to show this for $\varepsilon := 2^{1-m}$. We show that $\mathscr{U}_m$ is a uniform cover that 'works'. So, let $U \in \mathscr{U}_m$ and let $x_1, x_2 \in U$. Then, in particular, $x_2 \in \mathrm{Star}_{\mathscr{U}_m}(x_1)$, and so

$$|\mathrm{q}(x_1), \mathrm{q}(x_2)| := |x_1, x_2| \leq 2^{1-m} =: \varepsilon. \tag{4.3.2.8}$$

STEP 11: FINISH THE PROOF BY PROVING THE DESIRED PROPERTY OF q
Let $S \subseteq Y$ and suppose that $\mathrm{diam}(S) < 1$. We wish to show that there is some $U \in \mathscr{U}_0$ such that $S \subseteq U$. It suffices to show that for $x_1, x_2 \in S$, there is some $U_{x_1,x_2} \in \mathscr{U}_1$ such that $x_1, x_2 \in U_{x_1,x_2}$. This is because, if this is true, then $S \subseteq \mathrm{Star}_{\mathscr{U}_1}(x_1)$, which in turn is contained in some element of $\mathscr{U}_0$ because $\mathscr{U}_1$ star-refines $\mathscr{U}_0$.

To show this, it suffices to show that if $|x_1, x_2| \leq 2^{1-m}$ for $m \in \mathbb{Z}^+$, then there is some $U \in \mathscr{U}_m$ such that $x_1, x_2 \in U$. So, let $x_1, x_2 \in X$ be such that $|x_1, x_2| \leq 2^{1-m}$. Then, without loss of generality, there is some path $\langle x^\infty, \ldots, x^n \rangle$ from x_1 to x_2 with

$$\ell(x^\infty, x^1) + \cdots + \ell(x^{n-1}, x^n) \leq 2^{1-m}. \tag{4.3.2.9}$$

It thus suffices to show that, whenever (4.3.2.9) holds, there is some $U \in \mathscr{U}_m$ with $x^\infty, x^n \in U$. We prove this by induction on n. For $n = 1$, (4.3.2.9) implies that $\ell(x_1, x_2), \ell(x_2, x_1) \leq 2^{1-m}$. As was mentioned above in Step 2, because X is T_0, at least one of these is strictly positive—without loss of generality suppose that $\ell(x_1, x_2) > 0$. Then, the fact that $\ell(x_1, x_2) \leq 2^{1-m}$ implies that

$$2^{1-\max\{o\in\mathbb{N}:x_2\in\mathrm{Star}_{\mathscr{U}_o}(x_1)\}} \leq 2^{1-m}, \tag{4.3.2.10}$$

so that

$$m \leq \max\{o \in \mathbb{N} : x_2 \in \mathrm{Star}_{\mathscr{U}_o}(x_1)\} \tag{4.3.2.11}$$

which implies that $x_2 \in \mathrm{Star}_{\mathscr{U}_o}(x_1)$ for some $o \geq m$, which implies that there is some $U \in \mathscr{U}_o$ such that $x_1, x_2 \in U$. As $\mathscr{U}_o$ star-refines $\mathscr{U}_m$, there is in particular some $U \in \mathscr{U}_m$ such that $x_1, x_2 \in U$. Thus, this does the case for $n = 1$. (Note that in fact we can take $U \in \mathscr{U}_o$—this will be important later.)

Now assume the result is true for all $k \leq n$. We wish to prove the result for $n + 1$.

We must have that $\ell(x^\infty, x^1) < 2^{1-m}$, because otherwise we would have to have that $\ell(x^k, x^{k+1}) = 0$ for $k \geq 1$, in which case x^k and x^{k+1} lie in some $U \in \mathscr{U}_o$ for o arbitrarily large. We can then guarantee that x^k for $k \geq 1$ are obtained in some element of $\mathscr{U}_o$, and hence, as x^∞ and x^1 are obtained in some element of $\mathscr{U}_o$, everything is contained in some element of $\mathscr{U}_m$.

Thus, without loss of generality assume that $\ell(x^\infty, x^1) < 2^{1-m}$, so that in fact $\ell(x^\infty, x^1) \leq 2^{-m}$ (because of the definition of ℓ (4.3.2.4)). Then, there is some k_0 such that

$$\ell(x^\infty, x^1) + \cdots + \ell(x^{k_0-1}, x^{k_0}) \leq 2^{-m} \tag{4.3.2.12}$$

(This is the same inequality with m one larger.) Take k_0 to be the largest such positive integer. Similarly, there is some (largest) l_0 such that

$$\ell(x^{k_0}, x^{k_0+1}) + \cdots + \ell(x^{l_0-1}, x^{l_0}) \leq 2^{-m} \tag{4.3.2.13}$$

By choice of k_0 and l_0, we have

$$\ell(x^\infty, x^1) + \cdots + \ell(x^{k_0-1}, x^{k_0}) + \ell(x^{k_0}, x^{k_0+1}) > 2^{-m}$$

and

$$\ell(x^{k_0}, x^{k_0+1}) + \cdots + \ell(x^{l_0-1}, x^{l_0}) + \ell(x^{l_0}, x^{l_0+1}) > 2^{-m}.$$

In order that (4.3.2.9) still be satisfied, it thus must be the case that

$$\ell(x^{l_0}, x^{l_0+1}) + \cdots + \ell(x^{n-1}, x^n) \leq 2^{-m}. \tag{4.3.2.14}$$

By the induction hypotheses, we then must have in particular that there are $U_1, U_2, U_3 \in \mathscr{U}_m$ such that $x^\infty, x^{k_0} \in U_1$, $x^{l_0}, x^n \in U_2$, and $x^{k_0}, x^{l_0} \in U_3$. There is some $U \in \mathscr{U}_m$ such that $\mathrm{Star}_{\mathscr{U}_{m+1}}(U_3) \subseteq U$. However, as $U_1, U_2 \subseteq \mathrm{Star}_{\mathscr{U}_{m+1}}(U_3)$, we have that $x^\infty, x^n \in U$, and this completes the proof. ■

[a]Proof adapted from [Isb64, pg. 8].

[b]The superscripts (as opposed to subscripts) are for the purpose of not conflicting with the subscripts on x_1 and x_2.

From this, that every T_0 uniform space is uniformly-completely-T_3 follows relatively easily.

Corollary 4.3.2.15 Let X be a T_0 uniform space. Then, X is uniformly-completely-T_3.

Proof. [a] Let X be a uniform space. We show that uniformly-continuous functions on X can separate closed sets from points. So, let $C \subseteq X$ be closed, and let $x_0 \in C^{\mathsf{C}}$. As C is closed, there must be some neighborhood of x_0 that does not intersect C (otherwise, x_0 would be an accumulation point of C). Then, because stars form a neighborhood base for the topology (Proposition 4.1.2.2), there is a uniform cover $\mathscr{U}$ such that

$$\mathrm{Star}_{\mathscr{U}}(x_0) \subseteq C^{\mathsf{C}}. \tag{4.3.2.16}$$

Now apply the previous theorem for the uniform cover $\mathscr{U}$, so that there is a metric space $\langle Y, |\cdot,\cdot|\rangle$ and a uniformly-continuous map $\mathrm{q}: X \to Y$ such that, if $\mathrm{diam}(S) < 1$ for

$S \subseteq Y$, it follows that $\mathrm{q}^{-1}(S)$ is contained in some element of $\mathscr{U}$. From (4.3.2.16), it follows that $C \cup \{x_0\}$ is not contained in any element of $\mathscr{U}$, and so

$$\operatorname{diam}(\mathrm{q}(C) \cup \{\mathrm{q}(x_0)\}) \geq 1. \tag{4.3.2.17}$$

Define $f\colon X \to [0,1]$ by

$$f(x) := {}^{b}\max\{\operatorname{dist}_C(x), 1\}. \tag{4.3.2.18}$$

This is uniformly-continuous because dist_C is. (4.3.2.17) implies that $\operatorname{dist}_C(x_0) \geq 1$, and so $f(x_0) = 1$. We showed in Proposition 4.2.1.15 that $\operatorname{dist}_C(C) = 0$.

Finally, we check that X is T_1. We know that X is T_0 by hypothesis, and so by Proposition 3.6.2.51 (regular T_0 spaces are T_2), X is T_2, hence T_1 (that uniformly-continuous functions separate closed sets from points in particular implies regularity). ■

[a] Proof adapted from [Isb64, pg. 8].
[b] See (4.2.1.16) for the definition of dist_C.

4.3.3 The counter-examples

We know from the diagram (3.6.3.1), that if we are to 'do any better' in terms of separation axioms, we would be able to prove that every uniform space is either perfectly-T_3 or completely-T_4 (which is equivalent to T_4 by Urysohn's Lemma (Theorem 3.6.2.106)). Unfortunately, however, there exist counter-examples to both these separation axioms.

■ **Example 4.3.3.1 — A uniform space that is not perfectly-T_3** The Uncountable Fort Space X of Example 3.6.1.24 will do just fine yet again. We already know that this space is not perfectly-T_3 from Example 3.6.2.77. Thus, all that remains to be done is to equip X with a uniformity that generates the Uncountable Fort Space Topology.

Exercise 4.3.3.2 Show that the uniform topology of the initial uniformity with respect to $\{f : X \to \mathbb{R} : f \in \mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})\}$ is the Uncountable Fort Space Topology.

Example 4.3.3.3 — A uniform space that is not T_4 Equip $\mathrm{Mor}_{\mathbf{Set}}(\mathbb{R}, \mathbb{R})$ with the topology of pointwise convergence—see Exercise 4.2.3.21.

Exercise 4.3.3.4 Show that this definition satisfies the axioms of Kelley's Convergence Theorem, and so defines a topology on $\mathrm{Mor}_{\mathbf{Set}}(\mathbb{R}, \mathbb{R})$.

Exercise 4.3.3.5 Show that $\langle \mathrm{Mor}_{\mathbf{Set}}(\mathbb{R}, \mathbb{R}), +\rangle$ is a topological group, where $+$ is defined pointwise:

$$[f_1 + f_2](x) := f_1(x) + f_2(x). \tag{4.3.3.6}$$

Thus, $\mathrm{Mor}_{\mathbf{Set}}(\mathbb{R}, \mathbb{R})$ is canonically a uniform space. Furthermore, as it is T_0,[a] it is uniformly-completely-T_3.

We now shows that $\mathrm{Mor}_{\mathbf{Set}}(\mathbb{R}, \mathbb{R})$ is not completely-T_4 with respect to this topology. To do this, we first show that $\mathrm{Mor}_{\mathbf{Set}}(\mathbb{R}, \mathbb{Z}) \subseteq \mathrm{Mor}_{\mathbf{Set}}(\mathbb{R}, \mathbb{R})$ is not completely-T_4.[b] Note that it is *not* immediate just from this that $\mathrm{Mor}_{\mathbf{Set}}(\mathbb{R}, \mathbb{R})$ is not T_4, as subspaces of T_4 spaces need not be T_4 in general (Example 3.6.2.99).

For $m \in \mathbb{Z}$, define

$$P_m := \left\{ f \in \mathrm{Mor}_{\mathbf{Set}}(\mathbb{R}, \mathbb{Z}) : f|_{[f^{-1}(m)]^{\mathsf{c}}} \text{ is injective.} \right\},$$

that is, the set of all functions that are injective 'modulo sending more than one point to m'. We show that P_0 and

P_1 are closed and disjoint, but cannot be separated by open neighborhoods.

We first check that P_0 is closed (the proof that P_1 is closed is nearly identical). So, let $\lambda \mapsto f_\lambda \in P_0$ converge to f_∞. Suppose that $f_\infty(x_1) = f_\infty(x_2)$ is distinct from 0. Then, by our definition of convergence and the fact that our functions are taking values in the integers, it must be the case that $\lambda \mapsto f_\lambda(x_i)$ is eventually equal to $f_\infty(x_i)$ for $i = 1, 2$. Then, in particular, we will have that $f_{\lambda_0}(x_1) = f_\infty(x_1) = f_\infty(x_2) = f_{\lambda_0}(x_2)$ for λ_0 sufficiently large, and hence $x_1 = x_2$. Thus, $f_\infty \in P_0$.

We now check that P_0 and P_1 are disjoint. If f is injective on the complement of $f^{-1}(0)$ (i.e. if $f \in P_0$), then this complement must be countable (because f will restrict to an injection from this complement into $\mathbb{Z}$, which is countable). In particular, there must be at least two elements in $f^{-1}(0)$, and so $f(x_1) = 0 = f(x_2)$ for $x_1 \neq x_2$. But then f cannot be injective on the complement of $f^{-1}(1)$, and so $f \notin P_1$. Thus, P_0 is disjoint from P_1.

Let $A := \{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ be a countably-infinite subset of $\mathbb{R}$, and for $S \subseteq A$ a finite and $f \in \mathrm{Mor}_{\mathbf{Set}}(\mathbb{R}, \mathbb{Z})$, let us define

$$U_{S,f} := \{g \in \mathrm{Mor}_{\mathbf{Set}}(\mathbb{R}, \mathbb{Z}) : g|_S = f|_S\}, \tag{4.3.3.7}$$

that is, the set of functions which agree with f on S. The complement of this is the collection of all functions which differ from f at at least one point of S. Because the functions take their value in $\mathbb{Z}$, however, if you take a net of such functions, the limit (if it has one) must still disagree with f at least one point. Therefore, $U_{S,f}^{\complement}$ is closed, and hence $U_{S,f}$ is open. Moreover, as

$$U_{S,f} \cap U_{T,f} = U_{S \cup T,f}, \tag{4.3.3.8}$$

it follows from Proposition 3.1.1.9 that this is a neighborhood base for f. If fact, if we restrict ourselves to only taking S from a given infinite subset of A, we still get a neighborhood base.

Now, let U and V be open neighborhoods of P_0 and P_1 respectively. We seek to show that U and V must intersect. To do so, we construct a sequence of functions $f_k \in U$ and a countably-infinite collection of finite subsets $B_k = \{\alpha_0, \ldots, \alpha_{m_k}\}$ of A such that $U_{B_k, f_k} \subseteq U$ and

$$f_k(x) := \begin{cases} k & \text{if } x = \alpha_k \in B_{k-1} \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.3.9}$$

We do so inductively. (We take $B_0 := \emptyset$.)

Take $f_1 := 0$, so that of course $f_1 \in P_0$. Thus, because $\left\{U_{S,f_1} : S \subseteq A \text{ finite.}\right\}$ is a neighborhood base at f_1, there must be some finite subset $B_1 \subseteq A$ such that $U_{B_1,f_1} \subseteq U$. In fact, we can enlarge B_1 so that it is of the form $B_1 = \{\alpha_0, \ldots, \alpha_{m_1}\}$ (as making B_1 larger makes the neighborhood smaller). Now define f_2 according to (4.3.3.9), that is

$$f_2(x) := \begin{cases} k & \text{if } x = \alpha_k \in B_1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.3.10}$$

Then, $f_2 \in P_0$, so that $f_2 \in U$, and so there must be some finite set $B_2 \subseteq A$ such that $U_{B_2,f_2} \subseteq U$. Again, by enlarging B_2 if necessary, we can guarantee it is of the form $B_2 = \{\alpha_0, \ldots, \alpha_{m_2}\}$ with $m_2 > m_1$. Then, we may define

$$f_3(x) := \begin{cases} k & \text{if } x = \alpha_k \in B_2 \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.3.11}$$

Then, for the same reason as before, $f_3 \in U$, and so there is some finite set $B_3 \subseteq A$ (that is without loss of generality of the form $B_3 = \{\alpha_0, \ldots, \alpha_{m_3}\}$ with $m_3 > m_2$) and $U_{B_3,f_3} \subseteq U$. Continue this process inductively.

Now define $g \in \mathrm{Mor}_{\mathbf{Set}}(\mathbb{R}, \mathbb{Z})$ by

$$g(x) := \begin{cases} k & \text{if } x = \alpha_k \in A \\ 1 & \text{otherwise.} \end{cases} \tag{4.3.3.12}$$

Then, $g \in P_1$, and so there is some finite set $B \subseteq A$ such that $U_{B,g} \subseteq V$. Let m be sufficiently large so that $B \subseteq B_m$. Then, $f_m \in U_{B,g} \subseteq V$ and $f_m \in U_{B_m,f_m} \subseteq U$, and so, in particular, lies in $U \cap V$. Thus, P_0 and P_1 are disjoint closed sets which cannot be separated by neighborhoods.

This shows that $\mathrm{Mor}_{\mathbf{Set}}(\mathbb{R}, \mathbb{Z})$ is not T_4, but we still must show that $\mathrm{Mor}_{\mathbf{Set}}(\mathbb{R},$
$R)$ itself is not T_4. This will follow from the following lemma.

Lemma 4.3.3.13 Let X be T_4 and let $C \subseteq X$ be closed. Then, C is T_4.

R Note that this doesn't hold in general. For example, this example shows that $\prod_{\mathbb{R}} \mathbb{R} \subseteq \prod_{\mathbb{R}}[-\infty, \infty]$ is not normal even though $\prod_{\mathbb{R}}[-\infty, \infty]$ is (it is compact by Exercise 3.6.2.28 and Tychonoff's Theorem (Theorem 3.5.3.14), and hence T_4 by Proposition 3.6.2.92).

Proof. Let $C_1, C_2 \subseteq C$ be disjoint and closed. Then, by definition of the subspace topology (Proposition 3.5.1.1), $C_1 = C_1' \cap C$ and $C_2 = C_2' \cap C$ for $C_1', C_2' \subseteq X$ closed, and so C_1 and C_2 are themselves closed in X because C is closed. Thus, because X is

T_4, C_1 and C_2 can be separated by neighborhoods in X, and hence can be separated by neighborhoods in C. ∎

[a] Recall that (Proposition 3.6.2.20) a space is T_2 iff limits are unique. As our definition of convergence obviously has this property, our space is automatically T_2, hence T_0.

[b] The proof of this is adapted from [Mun00, pg. 206].

Finally, we are ready to begin discussing Cauchyness and completeness in the general context of uniform spaces.

4.4 Cauchyness and completeness

Definition 4.4.1 — Cauchyness Let $\langle X, \widetilde{\mathscr{U}}\rangle$ be a uniform space and let $\lambda \mapsto x_\lambda$ be a net. Then, we say that $\lambda \mapsto x_\lambda$ is ***Cauchy*** iff for every $\mathscr{U} \in \widetilde{\mathscr{U}}$, there is some $U \in \mathscr{U}$ such that $\lambda \mapsto x_\lambda$ is eventually contained in U.

R You should compare this with our definition of Cauchyness in $\mathbb{R}$, Definition 2.4.3.1. As the collection of all ε-balls for all $\varepsilon > 0$ forms a uniform base, then the definition we gave before in Definition 2.4.3.1 will be a word-for-word special case of this definition (once we show that we can replace $\widetilde{\mathscr{U}}$ with any uniform base in the above definition—see Proposition 4.4.2). Indeed, part of the motivation for phrasing the definition in Definition 2.4.3.1 the way we did was to make the transition to this higher level of generality as transparent as possible—see the paragraphs that follow Definition 2.4.3.1 for a few more comments regarding this.

To check whether a net is Cauchy, it suffices to check on just a uniform base for the uniformity.

Proposition 4.4.2 Let X be a set, let $\widetilde{\mathscr{B}}$ be a uniform base on X, and let $\lambda \mapsto x_\lambda$ be a net. Then, $\lambda \mapsto x_\lambda$ is Cauchy iff for every $\mathscr{B} \in \widetilde{\mathscr{B}}$ there is some $B \in \mathscr{B}$ such that $\lambda \mapsto x_\lambda$ is eventually contained in B.

R With this equivalence, the definition we gave before for Cauchyness in $\mathbb{R}$ in Definition 2.4.3.1 is literally verbatim equivalent to this definition upon replacement of $\widetilde{\mathscr{B}}$ with $\{\mathscr{B}_\varepsilon : \varepsilon > 0\}$ and of $\mathscr{B}$ with $\mathscr{B}_\varepsilon := \{B_\varepsilon(x) : x \in \mathbb{R}\}$.

Proof. ($\Rightarrow$) There is nothing to check.

($\Leftarrow$) Suppose that for every $\mathscr{B} \in \widetilde{\mathscr{B}}$ there is some $B \in \mathscr{B}$ such that $\lambda \mapsto x_\lambda$ is eventually contained in B. Denote the uniformity on X by $\widetilde{\mathscr{U}}$. Let $\mathscr{U} \in \widetilde{\mathscr{U}}$. Then, there is some $\mathscr{B} \in \widetilde{\mathscr{B}}$ such that $\mathscr{B} \lll \mathscr{U}$. Thus, there is some $B \in \mathscr{B}$ such that $\lambda \mapsto x_\lambda$ is eventually contained in B. As $\mathscr{B} \lll \mathscr{U}$, there is some $U \in \mathscr{U}$ such that $\mathrm{Star}_{\mathscr{B}}(B) \subseteq U$. In particular, $B \subseteq U$, and so if $\lambda \mapsto x_\lambda$ is eventually contained in B, it is certainly eventually contained in U. ■

From this, we obtain relatively nice description of what it means to be Cauchy in our two large families of examples, namely semimetric spaces and topological groups.

Exercise 4.4.3 Let $\langle X, \mathscr{D}\rangle$ be a semimetric space and let $\lambda \mapsto x_\lambda \in X$ be a net. Show that $\lambda \mapsto x_\lambda$ is Cauchy iff for every $|\cdot,\cdot| \in \mathscr{D}$ and for every $\varepsilon > 0$, $\lambda \mapsto x_\lambda$ is eventually contained in $B_\varepsilon^{|\cdot,\cdot|}(x)$ for *some* $x \in X$.

R In other words, a net in a semimetric space is Cauchy iff it is Cauchy with respect to each semimetric.

Exercise 4.4.4 Let G be a topological group and let $\lambda \mapsto g_\lambda \in G$ be a net. Show that the following are equivalent.

(i). $\lambda \mapsto g_\lambda$ is Cauchy with respect to the left uniformity.
(ii). For every open neighborhood U of the identity there is *some* $g \in G$ such that $\lambda \mapsto g_\lambda$ is eventually contained in gU.

Similarly, show that the following are equivalent.

(i). $\lambda \mapsto g_\lambda$ is Cauchy with respect to the left uniformity.
(ii). For every open neighborhood U of the identity there is *some* $g \in G$ such that $\lambda \mapsto g_\lambda$ is eventually contained in Ug.

On the other hand, find an example of a topological group G and a net $\lambda \mapsto g_\lambda \in G$ that is Cauchy with respect to the left uniformity but not the right.

Just as continuous functions preserve convergence, so to do uniformly-continuous functions preserve Cauchyness.

Exercise 4.4.5 Let $f: X \to Y$ be uniformly-continuous and let $\lambda \mapsto x_\lambda \in X$ be Cauchy. Show that $\lambda \mapsto f(x_\lambda)$ is Cauchy.

W Warning: Continuous functions do *not* necessarily preserve Cauchyness.

■ **Example 4.4.6 — A continuous image of a Cauchy net need not be Cauchy** The net $\mathbb{Z}^+ \ni m \mapsto \frac{1}{m} \in \mathbb{R}^+$ is Cauchy. On the other hand, its image under the continuous map $\mathbb{R}^+ \ni x \mapsto \frac{1}{x} \in \mathbb{R}^+$ is $\mathbb{Z}^+ \ni m \mapsto m \in \mathbb{R}^+$, which is not even eventually bounded, much less Cauchy.

Of course, if we know what it means for nets to be Cauchy, then we likewise have a notion of what it means for uniform spaces to be *complete*.

Definition 4.4.7 — Completeness A uniform space is ***complete*** iff every Cauchy net converges.

R In case there might be some confusion (e.g. if the topology of the underlying uniform space comes from a totally-ordered set), then we shall say ***Cauchy-complete*** in contrast to *Dedekind complete*. We made a big deal about not merely saying "complete" when we meant Dedekind-complete. It's more important in that case as most of the Dedekind-complete things we ran into had canonical uniformities (though we didn't know it at the time), and so there is an ambiguity of Dedekind-completeness vs. Cauchy-completeness. On the other hand, most uniform spaces don't have an order structure, in which case there is no potential for ambiguity. In any case, besides being slightly more verbose, it doesn't hurt to clarify.

By now, you've probably got the impression that quasicompactness is kind-of a strong condition. Indeed, as we're about to see, quasicompactness implies completeness for uniform spaces. However, we can do one better, and get an "iff" statement by introducing another relatively natural condition, that of *total-boundedness*.

Definition 4.4.8 — Totally-bounded Let X be a uniform space. Then, X is ***totally-bounded*** iff every uniform cover of X has a finite subcover.

R This is just like the definition of quasicompactness, except that we only require that *uniform* covers have finite subcovers, instead of *all* covers.

The first thing we will want to know is that, as usual, it suffices to check this property on a uniform base.

Proposition 4.4.9 Let X be a uniform space and let $\widetilde{\mathscr{B}}$ be a uniform base for X. Then, X is totally-bounded iff every $\mathscr{B} \in \widetilde{\mathscr{B}}$ has a finite subcover.

Proof. ($\Rightarrow$) There is nothing to check.

($\Leftarrow$) Suppose that every $\mathscr{B} \in \widetilde{\mathscr{B}}$ has a finite subcover. Let $\mathscr{U}$ be a uniform cover of X. Then, there is a cover $\mathscr{B} \in \widetilde{\mathscr{B}}$ that star-refines $\mathscr{U}$. By hypothesis, there are $B_1, \ldots, B_m \in \mathscr{B}$ with $X = B_1 \cup \cdots \cup B_m$. From the definition of star-refinement, there is a $U_k \in \mathscr{U}$ such that $\operatorname{Star}_{\mathscr{B}}(B_k) \subseteq U_k$ for $1 \leq k \leq m$. In particular, $B_k \subseteq U_k$, and so $X = U_1 \cup \cdots \cup U_m$. Thus, $\{U_1, \ldots, U_m\}$ is a finite subcover of $\mathscr{U}$, and so every uniform cover of X has a finite subcover, as desired. ■

And now we present the result that was the motivation for the introduction of the condition "totally-bounded" in the first place: an equivalence between quasicompactness, and the two conditions of completeness and totally-bounded together.

Theorem 4.4.10. Let X be a uniform space. Then, X is quasicompact iff X is complete and totally-bounded.

> R This can loosely be thought of as a generalization of the Heine-Borel Theorem (Theorem 2.5.3.3), with "closed" being replaced by "complete" and "bounded" being replaced by "totally-bounded".

Proof. ($\Rightarrow$) Suppose that X is quasicompact. X is totally-bounded by definition, and so it suffices to show that X is complete. So, let $\lambda \mapsto x_\lambda \in X$ be Cauchy. As X is quasicompact, there is a subnet $\mu \mapsto x_{\lambda_\mu} \in X$ of $\lambda \mapsto x_\lambda$ converging to $x_\infty \in X$. We wish to show that $\lambda \mapsto x_\lambda$ converges to x_∞.

To do this, we apply Exercise 3.2.1.3 (characterization of convergence in topologies defined by neighborhood bases). So, let $\mathscr{U}$ be a uniform cover. We wish to show that $\lambda \mapsto x_\lambda$ is eventually contained in $\mathrm{Star}_{\mathscr{U}}(x_\infty)$.

Let $\mathscr{V}$ be a star-refinement of $\mathscr{U}$. As $\mu \mapsto x_{\lambda_\mu}$ converges to x_∞, $\mu \mapsto x_{\lambda_\mu}$ is eventually contained in $\mathrm{Star}_{\mathscr{V}}(x_\infty)$.

Furthermore, as $\lambda \mapsto x_\lambda$ is Cauchy, there is some $V \in \mathscr{V}$ such that $\lambda \mapsto x_\lambda$ is eventually contained in V. As V eventually contains $\lambda \mapsto x_\lambda$, by the definition of a subnet (Definition 3.2.9), it eventually contains $\mu \mapsto x_{\lambda_\mu}$. In particular, there is some λ_{μ_0} such that $x_{\lambda_{\mu_0}} \in V \cap V'$ for some $V' \in \mathscr{V}$ with $x_\infty \in V'$ (because $\mu \mapsto x_{\lambda_\mu}$ is also eventually contained in $\mathrm{Star}_{\mathscr{V}}(x_\infty)$).

Let $U \in \mathscr{U}$ be such that $\mathrm{Star}_{\mathscr{V}}(V) \subseteq U$. As $V' \in \mathscr{V}$ intersects V, we have that $V' \subseteq \mathrm{Star}_{\mathscr{V}}(V) \subseteq U$. In particular, $x_\infty \in U$, so that $U \subseteq \mathrm{Star}_{\mathscr{U}}(x_\infty)$. On the other hand, $\lambda \mapsto x_\lambda$ is eventually contained in $V \subseteq \mathrm{Star}_{\mathscr{V}}(V) \subseteq U \subseteq \mathrm{Star}_{\mathscr{U}}(x_\infty)$, and so $\lambda \mapsto x_\lambda$ is eventually contained in $\mathrm{Star}_{\mathscr{U}}(x_\infty)$, as desired.

($\Leftarrow$) Suppose that X is complete and totally-bounded. Let $\lambda \mapsto x_\lambda \in X$ be an ultra-net (Definition 3.8.2.9). To show that

X is quasicompact, it suffices to show that $\lambda \mapsto x_\lambda$ converges (Proposition 3.8.2.15). To do that, by completeness, it suffices to show that $\lambda \mapsto x_\lambda$ is Cauchy. So, let $\mathscr{U}$ be a uniform cover. To show that $\lambda \mapsto x_\lambda$ is Cauchy, it suffices to show that $\lambda \mapsto x_\lambda$ is eventually contained in some element of $\mathscr{U}$. Now, as X is totally bounded, there are finitely many $U_1, \ldots, U_m \in \mathscr{U}$ such that $X = U_1 \cup \cdots \cup U_m$. As $\lambda \mapsto x_\lambda$ is an ultra-net, it is eventually contained in U_1 or U_1^{C}. If it is eventually contained in U_1, we're done, so suppose that it is eventually contained in $U_1^{\mathsf{C}} = U_1^{\mathsf{C}} \cap (U_2 \cup \cdots \cup U_m)$. Once again, if it is eventually contained in U_2, we're done, so suppose that is not the case. Proceed inductively, we eventually find that $\lambda \mapsto x_\lambda$ is eventually contained in $U_1^{\mathsf{C}} \cap \cdots \cap U_{m-1}^{\mathsf{C}} \cap U_m$, in which case it is eventually contained in U_m, as desired. ■

4.4.1 Completeness of $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$

We mentioned back above in Example 4.2.3.22 that $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is complete. We now prove this.

Theorem 4.4.1.1. Let X be a topological space that has the property that a subset is open iff its intersection with each quasicompact subset K is open in K. Then, $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is complete.

R In particular, the limit of a uniformly convergent[a] net of continuous functions is continuous. This need not be the case if the convergence is just pointwise—see Example 4.2.3.19.

R This condition on X is called ***quasicompactly-generated***. For example, the cocountable topology on $\mathbb{R}$ is *not* quasicompactly-generated—see Example 4.4.1.7.

[a] Recall (Example 4.2.3.22) that "uniform convergence" is just the notion of convergence in the topological space $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ (or I suppose "uniform

convergence on quasicompact subsets" if you want to be verbose about it). As this is a semimetric space, concretely what this is means, $\lambda \mapsto f_\lambda \in \mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ converges to f_∞ iff for every quasicompact subset $K \subseteq X$, $\lambda \mapsto \sup_{x \in K} \{\|f_\lambda(x) - f_\infty(x)\|\}$ converges to 0.

Proof. STEP 1: PROVE THE RESULT FOR X QUASICOMPACT

We first take X to be quasicompact. In this case, the uniform structure on $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is the same as that generated by the single norm $\|\cdot\|_X$ (Exercise 4.2.3.25). Therefore, by Exercise 4.4.3 (Cauchyness in semimetric spaces), to show that $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is complete, it suffices to show that every net that is Cauchy with respect to $\|\cdot\|_X$ converges. So, suppose that $\lambda \mapsto f_\lambda \in \mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is Cauchy. As, for every $x \in X$,

$$\left|f_{\lambda_1}(x) - f_{\lambda_2}(x)\right| \leq \left\|f_{\lambda_1} - f_{\lambda_2}\right\|, \tag{4.4.1.2}$$

it follows that, for each $x \in X$, the net $\lambda \mapsto f_\lambda(x) \in \mathbb{R}$ is Cauchy. As $\mathbb{R}$ is complete, each of these nets has a limit. Call this limit $f_\infty(x)$. We need to check two things: (i) that $x \mapsto f_\infty(x)$ is continuous (so that indeed $f_\infty \in \mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$), and (ii) that $\lambda \mapsto f_\lambda$ converges to f_∞ in $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$.

We first show that f_∞ is continuous. Let $\varepsilon > 0$. Let λ_0 be such that, whenever $\lambda_1, \lambda_2 \geq \lambda_0$, it follows that $\left\|f_{\lambda_1} - f_{\lambda_2}\right\| < \varepsilon$. Let U be an open neighborhood of x_∞ such that $f_{\lambda_0}(U) \subseteq B_\varepsilon(f_{\lambda_0}(x_\infty))$. Let $x \in U$. Let $\lambda_1, \lambda_2 \geq \lambda_0$ be such that $\left|f_{\lambda_1}(x) - f_\infty(x)\right|, \left|f_{\lambda_2}(x_\infty) - f_\infty(x_\infty)\right| < \varepsilon$. Then,

$$\begin{aligned}
&|f_\infty(x) - f_\infty(x_\infty)| \\
&\qquad \leq \left|f_\infty(x) - f_{\lambda_1}(x)\right| + \left|f_{\lambda_1}(x) - f_{\lambda_0}(x)\right| \\
&\qquad\quad + \left|f_{\lambda_0}(x) - f_{\lambda_0}(x_\infty)\right| \\
&\qquad\quad + \left|f_{\lambda_0}(x_\infty) - f_{\lambda_2}(x_\infty)\right| \\
&\qquad\quad + \left|f_{\lambda_2}(x_\infty) - f_\infty(x_\infty)\right| \\
&\qquad < 5\varepsilon.
\end{aligned} \tag{4.4.1.3}$$

Thus, f_∞ is continuous.

We now check that $\lambda \mapsto f_\lambda$ converges to f_∞ in $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$. Let $\varepsilon > 0$. Now that we know that f_∞ is continuous, for each $x \in X$, there is some open neighborhood U_x of x such that $f_\infty(U_x) \subseteq B_\varepsilon(f_\infty(x))$. Then,

$$\{U_x : x \in X\} \tag{4.4.1.4}$$

is an open cover of X. Therefore, there is a finite subcover, $\{U_{x_1}, \ldots, U_{x_m}\}$. Thus, we may choose λ_0 such that, whenever $\lambda \geq \lambda_0$, it follows that $|f_\lambda(x_k) - f_\infty(x_k)| < \varepsilon$ for all $1 \leq k \leq m$. Now let $x \in X$ be arbitrary. Without loss of generality, assume that $x \in U_1$. Then, whenever $\lambda \geq \lambda_0$, it follows that

$$\begin{aligned} &|f_\lambda(x) - f_\infty(x)| \\ &\qquad \leq |f_\lambda(x) - f_\lambda(x_1)| + |f_\lambda(x_1) + f_\infty(x_1)| < 2\varepsilon. \end{aligned} \tag{4.4.1.5}$$

Taking the supremum over x, we find that

$$\| f_\lambda - f_\infty \| < 2\varepsilon, \tag{4.4.1.6}$$

so that indeed $\lambda \mapsto f_\lambda$ converges to f_∞.

Step 2: Prove the result in general

We now do the general case, in which case X is not necessarily quasicompact. So, let $\lambda \mapsto f_\lambda \in \mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ be Cauchy. Then, by Exercise 4.4.3 again, we must have that $\lambda \mapsto f_\lambda|_K\, \mathrm{Mor}_{\mathbf{Top}}(K, \mathbb{R})$ is Cauchy for each quasicompact subset $K \subseteq X$. Therefore, by the quasicompact case, $\lambda \mapsto f_\lambda$ converges to its pointwise limit f_∞ on each quasicompact subset. As f_∞ is continuous on each quasicompact subset, the intersection of the preimage of every open set with every quasicompact subset of X is open, and hence, by hypothesis, is open. Therefore, f_∞ is continuous. Furthermore, $\lambda \mapsto f_\lambda$ converges to f_∞ in $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ because it converges to f_∞ on each quasicompact subset. ∎

Now that we've just proven what goes right, we turn to the more interesting side of things—what can go wrong.

▪ **Example 4.4.1.7 — A discontinuous function that is continuous on every quasicompact subset** Take $X := \mathbb{R}$ and equip it with our good old-buddy, the cocountable topology. Recall that (Example 3.6.2.23) a subset of X is quasicompact iff it is finite.

Now let $f: X \to \mathbb{R}$ be any discontinuous function, for example, the Dirichlet Function (Example 2.5.1.13) is discontinuous with respect to the cocountable topology (because $\mathbb{Q}^{\mathsf{C}}$ is not closed). On the other hand, quasicompact subsets of X, that is, finite subsets of X are discrete, and so f restricted to finite subsets must be continuous.

We can use this trick to find an example of a space for which $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is not complete.

▪ **Example 4.4.1.8 — A topological space for which $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is not complete** Take $X := \mathbb{R}$ equipped with the cocountable extension topology. Recall that this means that the only closed sets are (i) X itself, (ii) countable subsets, and (iii) subsets which are closed in the usual topology of $\mathbb{R}$.

The same proof in the previous example shows that the only quasicompact subsets of X are the finite sets.[a] Let $f: X \to \mathbb{R}$ be the Dirichlet function. As $f^{-1}(0) = \mathbb{Q}^{\mathsf{C}}$ is not closed, f is not continuous. We construct a net of functions in $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ converging to f uniformly on each quasicompact (i.e. each finite) subset of X.

Our directed set Λ is the collection of all finite subsets of X ordered by inclusion. For $S \in \Lambda$, let us write $S = \{x_1, \ldots, x_m\}$

with $x_k < x_{k+1}$ and define

$$f_S(x) := \begin{cases} f(x) & \text{if } x \in S \\ f(x_1) & \text{if } x < x_1 \\ \frac{f(x_{k+1}-f(x_k)}{x_{k+1}-x_k}(x-x_k) & \text{for } x_k < x < x_{k+1} \\ f(x_m) & \text{if } x_m < x, \end{cases}$$

That is, it is a constant $f(x_1)$ for all $x \leq x_1$, and similarly for $x \geq x_m$. For x between x_k and x_{k+1}, is it a just the line segment going from $f(x_k)$ at $x = x_k$ to $f(x_{k+1})$ at $x = x_{k+1}$. By construction, this is continuous with respect to the usual topology, and hence continuous with respect to the cocountable extension topology.

Now, for $K \subseteq X$ quasicompact, K is finite, and hence $K \in \Lambda$. Thus, whenever $S \geq K$, that is, whenever $S \supseteq K$, we have that $f_S(x) = f(x)$ for all $x \in S$, so that $\| f_S - f \| = 0$. In particular, (i) $S \mapsto f_S$ is Cauchy; and (ii) if it converges in $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$, it must converge to $f \in \mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$. However, as f is not continuous, i.e. not an element of $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$, evidently $S \mapsto f_S$ does not converge, and so $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is not complete.

[a] In fact, the more open sets you have, the fewer quasicompact sets you have. Thus, as the cocountable extension topology is finer than the cocountable topology, there are fewer quasicompact sets in the cocountable extension topology in the sense that, if K is quasicompact for the cocountable extension topology, K is quasicompact for the cocountable topology, hence finite.

The key result of this subsection was that $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is complete for X quasicompactly-generated. It turns out that the converse of this is *false*.

▪ **Example 4.4.1.9 — A space for which $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is complete yet is not quasicompactly-generated** Take $X := \mathbb{R}$ equipped with the cocountable topology. We showed in Example 4.4.1.7 that X is not quasicompact-generated. It remains to show that $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is complete. To do this, we show that every element of $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is constant.

Let $f : X \to \mathbb{R}$ be continuous. If f is not constant, then $f^{-1}(a)$ is closed and proper for every $a \in \mathbb{R}$, and hence countable. The image cannot be countable then, because if it were, $\mathbb{R}$ would be a countable union of countable sets,[a] and hence countable. As $\mathbb{R} = \bigcup_{n\in\mathbb{Z}}[n, n+1]$ and the image is uncountable, some interval $[n, n+1]$ must intersect the image at uncountable many points. But then, either $[m, m+1]$ contains the image, or its preimage is a countable set, in which case we would have

$$\text{countable set} = f^{-1}([n, n+1]) = \bigcup_{a\in f(X)\cap[n,n+1]} f^{-1}(a)$$

is an uncountable union of nonempty disjoint sets, a contradiction. Therefore, it must be that $f(X) \subseteq [m, m+1]$, so that $f^{-1}([n, n+1]) = X$. Now, writing $[n, n+1] = [n, n+\frac{1}{2}] \cup [n+\frac{1}{2}, n+1]$ and applying the same logic again, we find, without loss of generality, that the image of f is contained in $[n, n+\frac{1}{2}]$. Applying this logic inductively, we deduce that the image of f is contained in an interval of length $\frac{1}{2^m}$ for all $m \in \mathbb{N}$. In particular, the image of f cannot contain more than one element, because two distinct points cannot fit inside an interval sufficiently small, a contradiction of the fact that f is not constant. Therefore, every $f : X \to \mathbb{R}$ continuous must be a constant function.

A Cauchy net of constant functions amounts to a Cauchy net of real numbers, which converges, and so the original Cauchy net of constant functions converges to this constant function. Therefore, $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is complete.

[a] Because $\mathbb{R} =: X = \bigcup_{a\in f(X)} f^{-1}(a)$.

In conclusion:

If a space is quasicompactly-generated, then $\mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})$ is complete. Furthermore, if $\mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})$ is complete, then X is a topological space.[a] Both of these implications are strict: the reals with the cocountable topology show that the first implication is strict, and the reals with the cocountable extension topology show that the second implication is strict.

[a] Uhm, duh. The content here is not in the implication, but rather in the counter-example.

There are a couple other important results about the space $\mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})$ that we present before moving to the next subsection.

Proposition 4.4.1.10 — Dini's Theorem Let X be a topological space, let $\lambda \mapsto f_\lambda \in \mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})$ be a net converging pointwise to $f_\infty \in \mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})$. Then, if $\lambda \mapsto f_\lambda(x)$ is uniformly eventually monotone,[a] then $\lambda \mapsto f_\lambda$ converges to f_∞.

R Of course, when we say "converges", we mean in $\mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})$, that is to say, "uniformly (on quasicompact subsets)".

R In particular, the conclusion is true if $\lambda \mapsto f_\lambda$ is eventually monotone in $\mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})$, where the order on $\mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})$ is defined by $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$.

R Note that it *is* allowed for $\lambda \mapsto f_\lambda(x)$ to be nondecreasing for some x and nonincreasing for others.

R Note that you do need to assume a priori that f_∞ is continuous. For example, the sequence $m \mapsto x^m$ convergence pointwise and is monotonic at every point, but its pointwise limit is not continuous (and so the convergence cannot be uniform by Theorem 4.4.1.1).

R Also note that you do not need X to be quasicompactly-generated for this result. X being quasicompactly-generated guarantees that $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is complete, but as we have to assume that f_∞ is continuous a priori anyways, we don't actually need to make this assumption.

R This is sort of like the Monotone Convergence Theorem (Proposition 2.4.3.21) for functions.

[a] Here, "uniformly eventually monotone" means that there is some λ_0 such that, whenever $\lambda \geq \lambda_0$, it follows that $\lambda \mapsto f_\lambda(x)$ is monotone for all $x \in X$. This is in contrast to requiring merely that $\lambda \mapsto f_\lambda(x)$ is monotone for every x—see Example 4.4.1.11.

Proof. Step 1: Prove the result for X quasicompact

Suppose that $\lambda \mapsto f_\lambda(x)$ is uniformly eventually monotone. By definition, this means that there is some λ_0 such that, whenever $\lambda \geq \lambda_0$, $\lambda \mapsto f_\lambda(x)$ is monotone for all $x \in X$. Define $g_\lambda := |f_\lambda - f_\infty|$. It follows that, whenever $\lambda \geq \lambda_0$, $\lambda \mapsto g_\lambda$ is nonincreasing. Without loss of generality, suppose that $\lambda \mapsto g_\lambda$ is *actually* nonincreasing. Not that $\lambda \mapsto g_\lambda$ converges pointwise to 0.

Denote the index set by Λ. Let $\varepsilon > 0$. Then, $X = \bigcup_{\lambda \in \Lambda} g_\lambda^{-1}([0, \varepsilon))$ because $\lambda \mapsto g_\lambda$ converges pointwise to 0. As X is quasicompact, there are finitely many $\lambda_1, \ldots, \lambda_m \in \Lambda$ such that $X = g_{\lambda_1}^{-1}([0, \varepsilon]) \cup \cdots \cup g_{\lambda_m}^{-1}$. Let $\lambda_0 \in \Lambda$ be at least as large as each λ_k. As $\lambda \mapsto g_\lambda$ is nonincreasing, it follows that $g_{\lambda_0}^{-1}([0, \varepsilon)) \supseteq g_{\lambda_k}^{-1}([0, \varepsilon))$ for each $1 \leq k \leq m$, and hence $X = g_{\lambda_0}^{-1}([0, \varepsilon))$.

Now suppose that $\lambda \geq \lambda_0$. Then, as $\lambda \mapsto g_\lambda$ is nonincreasing, we have that $g_\lambda \leq g_{\lambda_0}$, so that $g_\lambda(x) \leq g_{\lambda_0}(x) < \varepsilon$ for all $x \in X$, that is, $|f_\lambda(x) - f_\infty(x)| =: g_\lambda(x) < \varepsilon$ for all $x \in X$, and so $\|f_\lambda - f_\infty\| := \sup_{x\in X}|f_\lambda(x) - f_\infty(x)| < \varepsilon$. Thus, $\lambda \mapsto f_\lambda$ converges to f_∞ uniformly.

STEP 2: PROVE THE RESULT IN GENERAL
We now do the general case, in which case X is not necessarily quasicompact. So, suppose that $\lambda \mapsto f_\lambda$ is uniformly eventually monotone. Then certainly $\lambda \mapsto f_\lambda|_K$ is uniformly eventually monotone for all quasicompact $K \subseteq X$, and so by the previous step, $\lambda \mapsto f_\lambda|_K$ converges to $f_\infty|_K$ in $\mathrm{Mor}_{\mathbf{Top}}(K, \mathbb{R})$, and hence $\lambda \mapsto f_\lambda$ converges to f_∞ in $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$. ■

■ **Example 4.4.1.11 — A continuous pointwise limit of continuous functions for which the convergence is pointwise eventually monotone, but the convergence is not uniform** For $\lambda \in (0, 1)$, define $f_\lambda \colon [0, 1]$ by

$$f_\lambda(x) := \begin{cases} \frac{\lambda}{\lambda^{\frac{1}{\lambda}}} x^{\frac{1}{\lambda}} & \text{if } x \leq \lambda \\ \lambda & \text{if } \lambda \leq x \leq \frac{1}{2}(\lambda + \sqrt{\lambda}) \\ -\frac{\lambda}{\frac{1}{2}(\lambda+\sqrt{\lambda})-1}(x-1). & \end{cases}$$

This converges pointwise (as $\lambda \to 1$) to the constant 0 on $[0, 1]$, which of course is continuous, and furthermore, the convergence is eventually nonincreasing at each point; however, the convergence is *not* uniform (as we check next) because it is not *uniformly* eventually monotone.

If the convergence were uniform, then, in particular, for $\varepsilon := \frac{1}{2}$, there would be some λ_0 such that, whenever $\lambda \geq \lambda_0$, it follows that $f_\lambda(x) < \frac{1}{2}$ for all $x \in [0, 1]$. In particular, plugging in $x = \lambda$, we would have $\lambda_0 \leq \lambda < \frac{1}{2}$, so that $\lambda_0 < \frac{1}{2}$. Then, as $\frac{1}{2} \geq \lambda_0$, it would follows that $f_{\frac{1}{2}}(x) < \frac{1}{2}$ for all $x \in [0, 1]$, but this is not the case (e.g. for $x = \frac{1}{2}$).

Theorem 4.4.1.12 — Stone-Weierstrass Theorem. Let X be a topological space and let $\mathcal{A} \subseteq \mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ be a subalgebra.[a] Then, if for every $x_1, x_2 \in X$ with $x_1 \neq x_2$ there is some $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$, then either $\mathcal{A}$ is dense in $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ or there is some $x_0 \in X$ such that $\mathcal{A} = \left\{ f \in \mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R}) : f(x_0) = 0 \right\}$.

R If $\mathcal{A}$ satisfies the hypotheses of this theorem, then $\mathcal{A}$ is said to ***separate points***. Intuitively, the algebra $\mathcal{A}$ alone can 'tell the difference' between distinct points of X.

R Recall that (Definition 3.2.1.1) for $\mathcal{A}$ to be dense in $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ means that $\mathrm{Cls}(\mathcal{A}) = \mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$. Explicitly, for any continuous function $f \colon X \to \mathbb{R}$, there is a net $\lambda \mapsto f_\lambda \in \mathcal{A}$ converging (uniformly) to f. That is to say, you can uniformly approximate *any* real-valued continuous function on X with elements of $\mathcal{A}$.

R Note that we do *not* have to assume that X is quasicompactly-generated. This is for essentially the same reason as we did not have to assume this for Dini's Theorem.

R This generalizes to $\mathbb{C}$ nearly verbatim—the only difference in this case is you would have to additionally require that if $f \in \mathcal{A}$, then the complex conjugate (whatever the hell that is) of f is also in $\mathcal{A}$.

[a] For $\mathcal{A}$ to be a subalgebra of $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ is equivalent to saying that it is a subspace that is closed under multiplication with $1 \in \mathcal{A}$.

Proof. STEP 1: MAKE HYPOTHESES

Suppose that for every $x_1, x_2 \in X$ with $x_1 \neq x_2$ there is some $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$.

STEP 2: REDUCE TO THE QUASICOMPACT CASE

Suppose that we have proven the result for X quasicompact. Let $f \in \mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$. Let $K \subseteq X$ be quasicompact. By the quasicompact case, there is a net $\Lambda_K \ni \lambda \mapsto f_{K,\lambda} \in \mathcal{A}$ converging to f in $\mathrm{Mor}_{\mathbf{Top}}(K, \mathbb{R})$. Let $\mathcal{K}$ denote the collection of quasicompact subsets of X and regard it as a directed set by inclusion. We claim that

$$\mathcal{K} \times \prod_{K \in \mathcal{K}} \Lambda_K \ni \langle K, \lambda \rangle \mapsto f_{K,\lambda_K} \tag{4.4.1.13}$$

converges to $f \in \mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$. This will show that f is a limit point of $\mathcal{A}$, so that $f \in \mathrm{Cls}(\mathcal{A})$, as desired.

By Exercise 4.2.1.36 (characterization of convergence in semimetric spaces), to show that this converges to f, we need to show that this converges to f with respect to $\|\cdot\|_K$ for all $K \in \mathcal{K}$. So, let $K_0 \in \mathcal{K}$. Let $\varepsilon > 0$. As $\lambda \mapsto f_{K_0,\lambda}$ converges to f in $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$, there is some $[\lambda_0]_{K_0}$ such that, whenever $\lambda \geq [\lambda_0]_{K_0}$, it follows that $\left\| f_{K_0,\lambda} - f \right\|_{K_0} < \varepsilon$.

Now, suppose that $\langle K, \lambda \rangle \geq \langle K_0, \lambda_0 \rangle$. This means that $K \supseteq K_0$ and $\lambda_L \geq [\lambda_0]_L$ for every $L \in \mathcal{K}$. Hence, in this case,

$$\left\| f_{K,\lambda_K} - f \right\|_{K_0} \leq \left\| f_{K,\lambda_K} - f \right\|_K < \varepsilon. \tag{4.4.1.14}$$

Thus, this net does indeed converge to f with respect to each $\|\cdot\|_{K_0}$, and hence this net converges to f, as desired.

Thus, let us assume hereafter that X is quasicompact.

Step 3: Reduce to the case where $\mathcal{A}$ vanishes at no point

If there is some $x_0 \in X$ such that $\mathcal{A} = \left\{ f \in \mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R}) : f(x_0) = 0 \right\}$, we are done, so suppose that for every $x \in X$, there is some $f \in \mathcal{A}$ such that $f(x) \neq 0$.[a]

Step 4: Show that for every $x_1, x_2 \in X$ with $x_1 \neq x_2$ and $c_1, c_2 \in \mathbb{R}$ there is some $f \in \mathcal{A}$ such that $f(x_i) = c_i$

[b] By hypothesis, there is some $g \in \mathcal{A}$ such that $g(x_1) \neq g(x_2)$. From the previous step, there are $h, k \in \mathcal{A}$ such that $h(x_1) \neq 0$ and $k(x_2) \neq 0$. Define

$$\begin{aligned} f &:= \frac{c_1}{g(x_1)h(x_1) - g(x_2)h(x_1)}(gk - g(x_1)k) \\ &\quad + \frac{c_2}{g(x_2)k(x_2) - g(x_1)k(x_2)}(gk - g(x_1)k)\,. \end{aligned} \tag{4.4.1.15}$$

As $h(x_1) \neq 0$, $k(x_2) \neq 0$, and $g(x_1) \neq g(x_2)$, the denominators are not 0, and so this does indeed give an element of $\mathcal{A}$ (as it is an algebra). Furthermore, plugging in, we see that $f(x_1) = c_1$ and $f(x_2) = c_2$.

STEP 5: SHOW THAT $\mathrm{Cls}(\mathcal{A})$ IS ALSO A SUBALGEBRA

We leave this step as an exercise.

Exercise 4.4.1.16 Show that $\mathrm{Cls}(\mathcal{A})$ is a subalgebra of $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$.

STEP 6: SHOW THAT IF $f \in \mathcal{A}$, THEN $|f| \in \mathrm{Cls}(\mathcal{A})$

[c] Suppose that $f \in \mathcal{A}$. As X is quasicompact, by the Extreme Value Theorem (Theorem 3.8.2.2), $f(X)$ is quasicompact, hence bounded by the Heine-Borel Theorem (Theorem 2.5.3.3), so let $M > 0$ be such that $|f(x)| \leq M$ for all $x \in X$. Define $g := \frac{1}{M} f \in \mathcal{A}$. Note that $|g| \leq 1$. It suffices to show that $|g| \in \mathrm{Cls}(\mathcal{A})$. To do this, we construct a sequence $m \mapsto g_m \in \mathcal{A}$ converging to $|g|$ in $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$.

Define $g_0 := 0$ and for $m \in \mathbb{Z}^+$ define

$$h_m := h_{m-1} + \tfrac{1}{2}(g^2 - h_{m-1}^2). \tag{4.4.1.17}$$

As $\mathcal{A}$ is an algebra, it follows that $h_m \in \mathcal{A}$ for all $m \in \mathbb{N}$, and so it suffices to show that $\lim_m h_m = |g|$.

Note that, if we can prove that this sequence is Cauchy, so that, by completeness of $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$, it converges, its limit h_∞ must satisfy

$$h_\infty = h_\infty + \tfrac{1}{2}(g^2 - h_\infty^2), \tag{4.4.1.18}$$

and hence that $g^2 = h_\infty^2$. Then, if $h_\infty \geq 0$, it will follow that $h_\infty = |g|$. Thus, it suffices to show (i) that $m \mapsto h_m$ is Cauchy and (ii) that $m \mapsto h_m$ is eventually nonnegative.

We prove that $0 \leq h_m \leq |g|$ and $h_m \leq h_{m+1}$ for all $m \in \mathbb{N}$. We do this by induction. For $m = 0$, we have by definition $0 \leq 0 \leq |g|$ and

$$h_1 \coloneqq 0 + \tfrac{1}{2}(g^2 - 0) = g^2 \geq 0 =: h_0. \tag{4.4.1.19}$$

Thus, the result is true for $m = 0$. Now suppose the result is true for all $0 \leq k \leq m$. We first note that $h_{m+1} \geq h_m \geq 0$. Furthermore, note that

$$h_{m+2} - h_{m+1} \coloneqq \tfrac{1}{2}(g^2 - h_{m+1}^2) \geq \tfrac{1}{2}(g^2 - |g|^2) = 0. \tag{4.4.1.20}$$

Finally, we have that

$$\begin{aligned} h_{m+1} &\coloneqq h_m + \tfrac{1}{2}(g^2 - h_m^2) \\ &= h_m + \tfrac{1}{2}(|g| + h_m)(|g| - h_m) \\ &\leq h_m + \tfrac{1}{2}(|g| + |g|)(|g| - h_m) \\ &= h_m + |g|(|g| - h_m) \\ &\leq^d h_m + |g| - h_m \\ &= |g|. \end{aligned} \tag{4.4.1.21}$$

Thus, $m \mapsto h_m$ is a nondecreasing sequence in $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$ with $0 \leq h_m \leq |g|$. By Dini's Theorem (Proposition 4.4.1.10), the sequence $m \mapsto h_m$ converges. If the limit is denoted h_∞, then as each $h_m \geq 0$, $h_\infty \geq 0$ as well, and so as we have $g^2 = h_\infty^2$, we indeed have that $h_\infty = |g|$, as desired.

Step 7: Show that if $f \in \mathrm{Cls}(\mathcal{A})$, then $|f| \in \mathrm{Cls}(\mathcal{A})$

Suppose that $f \in \mathrm{Cls}(\mathcal{A})$. Then, there is a net $\Lambda \ni \lambda \mapsto f_\lambda \in \mathcal{A}$ converging to f. By the previous step, $|f_\lambda| \in \mathrm{Cls}(\mathcal{A})$. Furthermore, as

$$||f|(x) - |f_\lambda(x)|| \leq |f(x) - f_\lambda(x)|, \tag{4.4.1.22}$$

it follows that $\lambda \mapsto |f_\lambda| \in \mathrm{Cls}(\mathcal{A})$ converges to $|f|$, and so $|f| \in \mathrm{Cls}(\mathcal{A})$.

STEP 8: SHOW THAT IF $f, g \in \mathrm{Cls}(\mathcal{A})$, THEN $\max(f, g), \min(f, g) \in \mathrm{Cls}(\mathcal{A})$

Suppose that $f, g \in \mathrm{Cls}(\mathcal{A})$. Note that

$$\max(f, g) = \tfrac{1}{2}(f + g) + \tfrac{1}{2}|f - g|. \tag{4.4.1.23}$$

By the previous step and the fact that $\mathrm{Cls}(\mathcal{A})$ is itself an algebra (Step 5), it follows that $\max(f, g) \in \mathrm{Cls}(\mathcal{A})$. Similarly, $\min(f, g) \in \mathrm{Cls}(\mathcal{A})$.

STEP 9: SHOW THAT FOR $f \in \mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$, $x_0 \in X$, AND $\varepsilon > 0$, THERE IS SOME $f_{x_0} \in \mathrm{Cls}(\mathcal{A})$ SUCH THAT $f_{x_0}(x_0) = f(x_0)$ AND $f_{x_0} > f - \varepsilon$

^{e} By Step 4, for every $x \in X$ there is some $g_x \in \mathcal{A}$ such that $g_x(x_0) = f(x_0)$ and $g_x(x) = f(x)$. As g_x is continuous, there is an open neighborhood U_x of x such that $g_x(u) > f(u) - \varepsilon$ for all $u \in U_x$. As X is quasicompact, there are finitely many $U_{x_1}, \ldots, U_{x_m}$ that cover X. Define $f_{x_0} := \max(g_{x_1}, \ldots, g_{x_m})$. By the previous step, we have that $f_{x_0} \in \mathrm{Cls}(\mathcal{A})$. Furthermore, as $g_x(x_0) = f(x_0)$ for all $x \in X$, we certainly have that $f_{x_0}(x_0) = f(x_0)$. Finally, from the inequalities $g_{x_k}(u) > f(U) - \varepsilon$ for all $u \in U_{x_k}$ and the fact that $X = U_{x_1} \cup \cdots \cup U_{x_m}$, it follows that $f_{x_0} > f - \varepsilon$.

STEP 10: SHOW THAT $\mathrm{Cls}(\mathcal{A}) = \mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$

Let $f \in \mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$. Let $\varepsilon > 0$. We wish to find $g \in \mathrm{Cls}(\mathcal{A})$ such that $\|f - g\| < \varepsilon$. This will show that f is an accumulation point of $\mathrm{Cls}(\mathcal{A})$, and hence $f \in \mathrm{Cls}(\mathcal{A})$ as $\mathrm{Cls}(\mathcal{A})$ is closed.

By the previous step, for every $x \in X$, there is some $g_x \in \mathrm{Cls}(\mathcal{A})$ such that $g_x(x) = f(x)$ and $g_x > f - \varepsilon$. As g_x is continuous, there is an open neighborhood U_x of x such that $g_x(u) < f(u) + \varepsilon$ for all $u \in U_x$. As X is quasicompact, there are finitely many $U_{x_1}, \ldots, U_{x_m}$ that cover X. Define $g := \min(g_{x_1}, \ldots, g_{x_m})$. By Step 8, $g \in \mathrm{Cls}(\mathcal{A})$.

As each $g_x > f - \varepsilon$, it follows that $g > f - \varepsilon$. Furthermore, as $g_{x_k}(u) < f(u) + \varepsilon$ for all $u \in U_{x_k}$ and the fact that $X = U_{x_1} \cup \cdots \cup U_{x_m}$, it follows that $g < f + \varepsilon$. These inequalities together give $\| f - g \| < \varepsilon$, as desired. ■

[a] This is what we mean when we say "$\mathcal{A}$ vanishes at no point".
[b] Proof adapted from [Rud76, Theorem 7.31].
[c] Proof adapted from [Jr13, Theorem 3.3.8].
[d] Because $|g| \leq 1$.
[e] Remainder of proof adapted from [Rud76, Theorem 7.32].

We obtain as a near immediate corollary of this a result that is quite significant in its own right.

Corollary 4.4.1.24 — Weierstrass Approximation Theorem Let $D \subseteq \mathbb{R}^d$. Then, $\mathbb{R}[x_1, \ldots, x_d]$ is dense in $\mathrm{Mor}_{\mathbf{Top}}(D, \mathbb{R})$.

R Recall that (Definition 2.3.11) $\mathbb{R}[x]$ is the cring of all polynomials with real coefficients. It of course has additionally the canonical structure of an algebra (that is, you can scale polynomials). Similarly, $\mathbb{R}[x_1, \ldots, x_d]$ is the algebra of polynomials in the variables $x_1, \ldots, x_d$. For example, for $d = 3$, a typical element of $\mathbb{R}[x_1, x_2, x_3]$ might look like $\sqrt{2}x_1^2x_3 - \frac{2}{3}x_2^4x_3^5 + x_1x_2^7x_3^2$.

R Explicitly, given any continuous function $f\colon D \to \mathbb{R}$, then there is a net of polynomials $\lambda \mapsto p_\lambda$ converging (uniformly (on quasicompact subsets)) to f. That is, you can "approximate" continuous functions uniformly by polynomials.

R We're actually being a bit sloppy here: strictly speaking, we should be making a distinction between polynomials and polynomial *functions*. Elements of $\mathbb{R}[x]$ are technically just formal symbols that can be added and multiplied in the way you expect, but they themselves are *not* functions. Rather, a polynomial $p \in \mathbb{R}[x]$ *defines* a function, namely the function $\mathbb{R} \ni x \mapsto p(x) \in \mathbb{R}$. To see the difference, instead consider $\mathbb{Z}/2\mathbb{Z}[x]$.[a] $x(x+1) \in \mathbb{Z}/2\mathbb{Z}[x]$ is a polynomial with coefficients in $\mathbb{Z}/2\mathbb{Z}$ and it is *not* the zero polynomial. On the other hand, if you plug in both elements of $\mathbb{Z}/2\mathbb{Z}$ (namely 0 and 1), you get 0 both times. Thus, while the polynomial $x(x+1)$ is *not* zero, it *defines* the function that is identically 0 $\mathbb{Z}/2\mathbb{Z} \ni x \mapsto x(x+1) \in \mathbb{Z}/2\mathbb{Z}$.

In short, strictly speaking $\mathbb{R}[x]$ is *not* a subset of $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$, but rather *embeds* in $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$.

[a] See Example A.4.1.18 for the definition of $\mathbb{Z}/m\mathbb{Z}$.

Proof. By the Stone-Weierstrass Theorem, it suffices to show that $\mathbb{R}[x_1, \ldots, x_d]$ is an algebra that separates points.[a] That it is an algebra is immediate (sums of polynomials are polynomials, products of polynomials are polynomials, scalings of polynomials are polynomials, and both 0 and 1 are polynomials).

Exercise 4.4.1.25 Show that $\mathbb{R}[x_1, \ldots, x_d]$ separates points in $\mathrm{Mor}_{\mathbf{Top}}(D, \mathbb{R})$.

■

[a] Recall that (Theorem 4.4.1.12) this means that for every $x_1, x_2 \in X$ distinct, there is some $p \in \mathbb{R}[x]$ such that $p(x_1) \neq p(x_2)$.

Before moving on to the completion, we end with an exercise that will be needed later in Proposition 3.6.2.45.

Exercise 4.4.1.26 Let X be a topological space, let $\mathcal{B}(X,\mathbb{R})$ denote the collection of all bounded continuous real-valued functions on X, and for $f \in \mathcal{B}(X,\mathbb{R})$ define $\|f\| := \sup_{x\in X}|f(x)|$. Show that $\langle \mathcal{B}(X,\mathbb{R}), \|\cdot\|\rangle$ is a complete normed vector space.

> R Note that, unlike in Theorem 4.4.1.1, we do *not* need X to be quasicompactly-generated.

4.4.2 The completion

And now we show that every uniform space can be *completed*.[7]

Theorem 4.4.2.1 — Completion. Let X be a T_0 uniform space. Then, there exists a unique complete uniform space $\mathrm{Cmp}(X)$, the ***completion*** of X, such that

(i). $\mathrm{Cmp}(X)$ contains X; and
(ii). if Y is any other complete uniform space which contains X, then Y contains $\mathrm{Cmp}(X)$.

Furthermore, X is dense in $\mathrm{Cmp}(X)$.

> R You will see in the proof that this is the "Cauchy sequence construction" we mentioned in a footnote right before our proof of existence of the real numbers (Theorem 1.4.2.9). It turns out that, in the case of $\mathbb{R}$, this will gives us the right answer, but that the passage from $\mathbb{Q}$ to $\mathbb{R}$ should really be thought of as the *Dedekind-completion*, not the *Cauchy-completion*.

> R If X is not T_0, then, if you like, you can define the completion of X as $\mathrm{Cmp}(\mathrm{T0}(X))$—see Proposition 3.6.2.3. Note that in this case X is *not* (in general) contained in its completion.

[7] Of course, we already know that $\mathbb{Q}$ is not complete (see Propositions 2.4.3.49 and 2.4.3.58), and so in general there will most certainly be some 'completing' to be done.

Proof. STEP 1: DEFINE AN EQUIVALENCE RELATION $\sim$ ON CAUCHY NETS
Define first as a set

$$X' := \{\lambda \mapsto x_\lambda \in X : \lambda \mapsto x_\lambda \text{ is Cauchy.}\} \tag{4.4.2.2}$$

For $\lambda \mapsto x_\lambda$ and $\mu \mapsto y_\mu$ Cauchy nets, define $(\lambda \mapsto x_\lambda) \sim (\mu \mapsto y_\mu)$ iff open subsets of X eventually contain $\lambda \mapsto x_\lambda$ iff they eventually contain $\mu \mapsto y_\mu$.

STEP 2: SHOW THAT $\sim$ IS AN EQUIVALENCE RELATION
That $(\lambda \mapsto x_\lambda) \sim (\lambda \mapsto x_\lambda)$ is tautological. The definition of $\sim$ is $\lambda \mapsto x_\lambda \leftrightarrow \mu \mapsto y_\mu$ symmetric, and so of course $(\lambda \mapsto x_\lambda) \sim (\mu \mapsto y_\mu)$ implies $(\mu \mapsto y_\mu) \sim (\lambda \mapsto x_\lambda)$.

As for transitivity, suppose that $(\lambda \mapsto x_\lambda) \sim (\mu \mapsto y_\mu)$ and $(\mu \mapsto y_\mu) \sim (\nu \mapsto z_\nu)$. We wish to show that $(\lambda \mapsto x_\lambda) \sim (\nu \mapsto z_\nu)$. Let $U \subseteq X$ be open. We must show that U eventually contains $\lambda \mapsto x_\lambda$ iff it eventually contains $\nu \mapsto z_\nu$. By $\lambda \mapsto x_\lambda \leftrightarrow \nu \mapsto z_\nu$, it suffices to show only one of these directions. So, suppose that U eventually contains $\lambda \mapsto x_\lambda$. Then, U eventually contains $\mu \mapsto y_\mu$ (because $(\lambda \mapsto x_\lambda) \sim (\mu \mapsto y_\mu)$), and so U eventually contains $\nu \mapsto z_\nu$ (because $(\mu \mapsto y_\mu) \sim (\nu \mapsto z_\nu)$).

STEP 3: DEFINE $\mathrm{Cmp}(X)$ AS A SET
We define

$$\mathrm{Cmp}(X) := X'/\sim . \tag{4.4.2.3}$$

STEP 4: DEFINE A UNIFORMITY ON X'.
Denote the uniformity on X by $\widetilde{\mathscr{U}}$. For every uniform cover $\mathscr{U} \in \widetilde{\mathscr{U}}$, we define a corresponding cover $\mathscr{U}'$ of X' as follows.

For $\mathscr{U} \in \widetilde{\mathscr{U}}$ and $U \in \mathscr{U}$, define

$$\begin{aligned} U' &:= \{(\lambda \mapsto x_\lambda \in X' : \\ &\qquad \lambda \mapsto x_\lambda \text{ is eventually contained in } U.\} \\ \mathscr{U}' &:= \{U' : U \in \mathscr{U}\} \\ \widetilde{\mathscr{U}}' &:= \{\mathscr{U}' : \mathscr{U} \in \widetilde{\mathscr{U}}\}. \end{aligned} \tag{4.4.2.4}$$

We wish to show that $\widetilde{\mathscr{U}}'$ is a uniform *base* on X'. To show this (Proposition 4.1.3.2), we must show (i) that each element of $\widetilde{\mathscr{U}}'$ is in fact a cover of X' and (ii) that $\widetilde{\mathscr{U}}'$ is downward-directed with respect to star-refinement.

We first check that each $\mathscr{U}'$ is in fact a cover of X'. So, let $\mathscr{U}' \in \widetilde{\mathscr{U}}'$ be arbitrary and let $(\lambda \mapsto x_\lambda) \in X'$. As $\lambda \mapsto x_\lambda$ is Cauchy, there is some $U \in \mathscr{U}$ such that $\lambda \mapsto x_\lambda$ is eventually contained in U, so that $(\lambda \mapsto x_\lambda) \in U'$, and hence $\mathscr{U}'$ covers X'.

We now check that $\widetilde{\mathscr{U}}'$ is downward-directed with respect to star-refinement. So, let $\mathscr{U}', \mathscr{V}' \in \widetilde{\mathscr{U}}'$. Let $\mathscr{W}$ be a common star-refinement of $\mathscr{U}$ and $\mathscr{V}$. We wish to show that $\mathscr{W}'$ is a common star-refinement of $\mathscr{U}'$ and V'. By $\mathscr{U} \leftrightarrow \mathscr{V}$ symmetry, it suffices to prove that $\mathscr{W}'$ is a star-refinement of $\mathscr{U}'$. So, let $W_0' \in \mathscr{W}'$. As $\mathscr{W}$ is a star-refinement of $\mathscr{U}$, it follows that there is some $U_0 \in \mathscr{U}$ such that $\mathrm{Star}_{\mathscr{W}}(W_0) \subseteq U_0$. We wish to show that $\mathrm{Star}_{\mathscr{W}'}(W_0') \subseteq U_0'$. So, let $W' \in \mathscr{W}'$ intersect W_0'. Then, there is a net that is eventually contained in both W and W_0, so that, in particular, W and W_0 intersect. It follows that $W \subseteq \mathrm{Star}_{\mathscr{W}}(W_0) \subseteq U_0$, and hence in turn, that any net eventually contained in W is eventually contained in U_0, so that $W' \subseteq U_0'$, and hence $\mathrm{Star}_{\mathscr{W}'}(W_0') \subseteq U_0'$ as desired.

This completes the proof that $\widetilde{\mathscr{U}}'$ is a uniform base on X'.

STEP 5: SHOW THAT THE QUOTIENT MAP $\mathrm{q} : X' \to \mathrm{Cmp}(X)$ SATISFIES $\mathrm{q}^{-1}(\mathrm{q}(U')) = U'$

As it is always the case that $U' \subseteq \mathrm{q}^{-1}(\mathrm{q}(U'))$, it suffices to show that $\mathrm{q}^{-1}(\mathrm{q}(U')) \subseteq U'$. So, let $(\lambda \mapsto x_\lambda) \in \mathrm{q}^{-1}(\mathrm{q}(U'))$. Then,

$\mathrm{q}(\lambda \mapsto x_\lambda) \in \mathbb{Q}(U')$, that is, $\lambda \mapsto x_\lambda$ is a Cauchy net and there is another Cauchy net $(\mu \mapsto y_\mu) \in U'$ such that $(\lambda \mapsto x_\lambda) \sim (\mu \mapsto y_\mu)$. That is, open sets eventually contain $\lambda \mapsto x_\lambda$ iff they eventually contain $\mu \mapsto y_\mu$. However, as $(\mu \mapsto y_\mu) \in U'$, by definition of U' ((4.4.2.4)), $\mu \mapsto y_\mu$ is eventually contained in U, and so $\lambda \mapsto x_\lambda$ is eventually contained in U, and so $(\lambda \mapsto x_\lambda) \in U'$. Hence, $\mathrm{q}^{-1}(\mathrm{q}(U')) \subseteq U'$, and we are done.

Step 6: Define a uniform base on $\mathrm{Cmp}(X)$

For every uniform cover $\mathscr{U} \in \widetilde{\mathscr{U}}$, we define a corresponding cover $\mathrm{Cmp}(\mathscr{U})$ of $\mathrm{Cmp}(X)$. For $\mathscr{U} \in \widetilde{\mathscr{U}}$ and $U \in \mathscr{U}$, define[a]

$$\begin{aligned} \mathrm{Cmp}(U) &:= \mathrm{q}(U') \\ \mathrm{Cmp}(\mathscr{U}) &:= \mathrm{q}(\mathscr{U}') := \{\mathrm{Cmp}(U) : U \in \mathscr{U}\} \\ \widetilde{\mathrm{Cmp}(\mathscr{U})} &:= \mathrm{q}(\widetilde{\mathscr{U}}') := \{\mathrm{Cmp}(\mathscr{U}) : \mathscr{U} \in \widetilde{\mathscr{U}}\}. \end{aligned} \tag{4.4.2.5}$$

We claim that $\widetilde{\mathrm{Cmp}(\mathscr{U})}$ is a uniform base on $\mathrm{Cmp}(X)$. Certainly each $\mathrm{Cmp}(\mathscr{U})$ is a cover of $\mathrm{Cmp}(X)$ because $\mathscr{U}'$ is a cover of X' and q is surjective.

It follows from Proposition 4.1.1.52 that q preserves star-refinement (the purpose of the previous step was to verify the requisite hypotheses of Proposition 4.1.1.52), and so that $\mathrm{Cmp}(\widetilde{\mathscr{U}})$ is a uniform base follows from the fact that that $\widetilde{\mathscr{U}}'$ was a uniform base on X'.

Step 7: Show that $\mathrm{Cmp}(X)$ is complete.

Let $\lambda \mapsto \mathrm{q}(x^\lambda)$ be Cauchy in $\mathrm{Cmp}(X)$. By definition of Cauchyness and our uniformity on $\mathrm{Cmp}(X)$, this means that, for every uniform cover $\mathrm{Cmp}(\mathscr{U}) \in \mathrm{Cmp}(\widetilde{\mathscr{U}})$, there is some $\mathrm{Cmp}(U) \in \mathrm{Cmp}(\mathscr{U})$ such that $\lambda \mapsto \mathrm{q}(x^\lambda)$ is eventually contained in $\mathrm{Cmp}(U) := \mathrm{q}(U')$. Thus, for each $x^\lambda \in X'$ for λ sufficiently large, there is some $y^\lambda \in U'$ with $x^\lambda \sim y^\lambda$. However, by definition of U', U eventually contains y^λ, and hence, because $x^\lambda \sim y^\lambda$, eventually contains x^λ, and so in fact $x^\lambda \in U'$. Thus, we have a net $\lambda \mapsto x^\lambda \in X'$ that has the property that, for every uniform cover $\mathscr{U} \in \widetilde{\mathscr{U}}$, there is some $U \in \mathscr{U}$ such that $\lambda \mapsto x^\lambda$ is eventually contained in U'.

Let us denote the domain of $\lambda \mapsto x^\lambda$ by Λ. Now, each x^λ is itself a net, and so let us denote its domain by M^λ. Define $x^\infty \in X'$ to be the net

$$\Lambda \times \prod_{\lambda \in \Lambda} M^\lambda \ni (\langle \lambda, \mu \rangle \mapsto [x^\lambda]_{\mu^\lambda} \in X. \qquad (4.4.2.6)$$

We first must check that this is Cauchy, so that indeed $x^\infty \in X'$.

So, let $\mathscr{U} \in \widetilde{\mathscr{U}}$ be a uniform cover. Then, there is some $U \in \mathscr{U}$ such that $\lambda \mapsto x^\lambda$ is eventually contained in U'. So, let λ_0 be such that, whenever $\lambda \geq \lambda_0$, it follows that $x^\lambda \in U'$. For all such λ, the net x^λ itself must be eventually contained in U, so let μ_0^λ be such that, whenever $\mu^\lambda \geq \mu_0^\lambda$, it follows that $[x^\lambda]_{\mu^\lambda} \in U$. (For λ not at least λ_0, μ_0^λ may be anything). Now, suppose that $\langle \lambda, \mu \rangle \geq \langle \lambda_0, \mu_0 \rangle$. Then,

$$(x^\lambda)_{\mu^\lambda} \in U \qquad (4.4.2.7)$$

Thus, $\langle \lambda, \mu \rangle \mapsto (x^\lambda)_{\mu^\lambda}$ is Cauchy.

We now show that $\lambda \mapsto \mathrm{q}(x^\lambda)$ converges to $\mathrm{q}(x^\infty)$. To show this, as stars form neighborhood bases, it suffices to show that $\lambda \mapsto \mathrm{q}(x^\lambda)$ is eventually contained in $\mathrm{Star}_{\mathrm{Cmp}(\mathscr{U})}(\mathrm{q}(x^\infty))$ for all $\mathscr{U} \in \widetilde{\mathscr{U}}$. As q is surjective, the preimage of a star is equal to the star of the preimage (Proposition 4.1.1.18), and so it suffices to show that $\lambda \mapsto x^\lambda$ is eventually contained in $\mathrm{Star}_{\mathscr{U}'}(x^\infty)$ for all $\mathscr{U} \in \widetilde{\mathscr{U}}$. So, let $\mathscr{U} \in \widetilde{\mathscr{U}}$ be a uniform cover. Then, there is some $U \in \mathscr{U}$ such that $\lambda \mapsto x^\lambda$ is eventually contained in U'. Thus, we will be done if we can show that $x^\infty \in U'$ (so that then $U' \subseteq \mathrm{Star}_{\mathscr{U}'}(x^\infty)$). To show this, we must show that x^∞ is eventually contained in U. However, the exact same argument (see (4.4.2.7)) that was used above to show that x^∞ was Cauchy shows precisely this. Therefore, $\lambda \mapsto x^\lambda$ converges to x^∞.

Step 8: Show that $\mathrm{Cmp}(X)$ contains X

Of course, when we say that $\mathrm{Cmp}(X)$ "contains" X, what we really means is that there is a subset of $\mathrm{Cmp}(X)$ which is uniformly-homeomorphic to X. So, we define $\iota\colon X \to \mathrm{Cmp}(X)$, and show that it is a uniform-homeomorphism onto its image.[b]

Define $\mathrm{c}\colon X \to X'$ by $\mathrm{c}(x) := (\lambda \mapsto x_\lambda := x)$, that is, c sends x to the constant net with value x. (Constant nets converge, and hence are in particular Cauchy.) Then we define $\iota := \mathrm{q} \circ \mathrm{c}$. We first show that this is injective. Suppose that $\iota(x_1) = \iota(x_2)$. Then, every neighborhood that eventually contains the constant net x_1 eventually contains the constant net x_2. In other words, open sets in X contain x_1 iff they contain x_2, which implies that $x_1 = x_2$ because X is T_0.

We now check that ι is uniformly-continuous. We claim that $\iota^{-1}(\mathrm{Cmp}(\mathscr{U})) = \mathscr{U}$. However, using result that $\mathrm{q}^{-1}(\mathrm{q}(U')) = U'$ from Step 5, we have that

$$\begin{aligned}\iota^{-1}(\mathrm{Cmp}(\mathscr{U})) &:= \mathrm{c}^{-1}\left(\mathrm{q}^{-1}\left(\mathrm{q}(\mathscr{U}')\right)\right) = \mathrm{c}^{-1}(\mathscr{U}') \\ &:= \{\mathrm{c}^{-1}(U') : U \in \mathscr{U}\}.\end{aligned} \tag{4.4.2.8}$$

However, the only constant nets which are eventually contained in U are the elements of U themselves, and so $\mathrm{c}^{-1}(U') = U$, and so indeed

$$\iota^{-1}(\mathrm{Cmp}(\mathscr{U})) = \mathscr{U}. \tag{4.4.2.9}$$

The subspace uniformity induced on $\iota(X)$ is generated by

$$\left\{\mathrm{Cmp}(\mathscr{U}) \wedge \{\iota(X)\} : \mathscr{U} \in \widetilde{\mathscr{U}}\right\}, \tag{4.4.2.10}$$

that is, a cover of $\iota(X)$ is uniform iff it is a cover that is obtained from a uniform cover of $\mathrm{Cmp}(X)$ by simply restricting that cover to $\iota(X)$. Because the preimage of a cover with respect to the inverse of a function is the same as the image of that uniform cover, to show that the inverse of $\iota : X \to \iota(X)$ is uniformly-continuous, it suffices to show that $\iota(\mathscr{U})$ is a uniform cover on $\iota(X)$. By (4.4.2.10), it thus suffices to show that

$$\iota(\mathscr{U}) = \mathrm{Cmp}(\mathscr{U}) \wedge \{\iota(X)\}. \tag{4.4.2.11}$$

To show this, we first check that $\mathrm{q}(U') \cap \mathrm{q}(\mathrm{c}(X)) = \mathrm{q}(U' \cap \mathrm{c}(X))$. We always have that $\supseteq$ inclusion (Exercise A.3.27), and so it suffices to prove the $\subseteq$ inclusion. So, let $\lambda \mapsto x_\lambda \in X$ be a Cauchy net that is equivalent to a Cauchy net $\mu \mapsto y_\mu$ that is eventually contained in U and the constant Cauchy net $\nu \mapsto z_\infty \in X$. This implies in particular that $\nu \mapsto z_\infty$ is eventually contained in U, so that $(\nu \mapsto z_\infty) \in U' \cap \mathrm{c}(X)$, so that $\mathrm{q}(\lambda \mapsto x_\lambda) \in \mathrm{q}(U' \cap \mathrm{c}(X))$. Thus,

$$\begin{aligned} &\mathrm{Cmp}(\mathscr{U}) \wedge \{\iota(X)\} \\ &\qquad := \{\mathrm{q}(U') \cap \mathrm{q}(\mathrm{c}(X)) : U \in \mathscr{U}\} \\ &\qquad = \{\mathrm{q}(U' \cap \mathrm{c}(X)) : U \in \mathscr{U}\} \\ &\qquad = \{\iota(U) : U \in \mathscr{U}\}, \end{aligned} \tag{4.4.2.12}$$

which demonstrates the truth of (4.4.2.11).

Step 9: Show that any other complete uniform space that contains X contains $\mathrm{Cmp}(X)$

Let Y be some other complete uniform space that contains X. Let $(\lambda \mapsto x_\lambda) \in X'$ be a Cauchy net in X and let x_∞ be its (unique) limit in Y. Define $\kappa : \mathrm{Cmp}(X) \to Y$ by $\kappa(\mathrm{q}(\lambda \mapsto x_\lambda)) := x_\infty$.

Exercise 4.4.2.13 Show that κ is well-defined and is a uniform-homeomorphism onto its image.

Step 10: Show that $\mathrm{Cmp}(X)$ is unique

Let Y be another complete uniform space which (i) contains X and (ii) is contained in every other complete uniform space that contains X. From this, we know that Y is contained in

$\mathrm{Cmp}(X)$. On the other hand, we already knew that $\mathrm{Cmp}(X)$ was contained in Y. Therefore, $\mathrm{Cmp}(X) = Y$.

STEP 11: SHOW THAT X IS DENSE IN $\mathrm{Cmp}(X)$
Let $\mathrm{q}(\lambda \mapsto x_\lambda) \in \mathrm{Cmp}(X)$.

Exercise 4.4.2.14 Show that $\lambda \mapsto \iota(x_\lambda)$ converges to $\mathrm{q}(\lambda \mapsto x_\lambda)$ in $\mathrm{Cmp}(X)$.

It follows from this exercise that $\mathrm{Cls}(\iota(X)) = \mathrm{Cmp}(X)$, and so that X is dense in $\mathrm{Cmp}(X)$. ■

[a] As you might have guessed, this is just the quotient uniformity induced from the one on X' written out explicitly.

[b] Its image will be equipped with the subspace uniformity, that is, the initial uniformity (see Proposition 4.1.3.15) with respect to the inclusion into $\mathrm{Cmp}(X)$.

One thing that we will frequently want to do is extend a given function to its completion. If the codomain is complete as well, we can do this, and in fact, we can do it whenever we have a continuous function defined on a dense subspace.

Proposition 4.4.2.15 Let $S \subseteq X$ be a dense subset of a uniform space, let Y be a complete T_0 uniform space, and let $f\colon S \to Y$ be uniformly-continuous. Then, there exists a unique uniformly-continuous map $g\colon X \to Y$ such that $g|_S = f$.

R Mere continuity does not suffice, even in the nicest of cases—see the counter-example below (Example 4.4.2.19).

Proof. Let $x \in X$. As $\mathrm{Cls}(S) = X$, there is a net $\lambda \mapsto x_\lambda \in X$ converging to x. Pick any such net. By Exercise 4.4.5, $\lambda \mapsto f(x_\lambda)$ is Cauchy in Y, so that we may simply take its limit (because Y is complete and is T_0, and hence T_2, so that limits are unique—see Proposition 3.6.2.20). So, let us define $g\colon X\to Y$ by

$$g(x) := \lim_{\lambda} f(x_\lambda). \tag{4.4.2.16}$$

Exercise 4.4.2.17 Show that if $\mu\mapsto y_\mu$ also converges to x, then $\lim_\lambda f(x_\lambda) = \lim_\mu f(x_\mu)$.

> R This shows that g is well-defined, that is, then the definition did not depend on our choice of net converging to x.

Thus, the choice of net does not matter, and so for $x\in S$, we may simply take the constant net $\lambda\mapsto x_\lambda := x$, so that g is indeed an extension of f.

Exercise 4.4.2.18 Show that g is uniformly-continuous.

■

Example 4.4.2.19 — A real-valued continuous function on a dense subspace that does *not* extend In the notation of the previous proposition, take $S := \mathbb{R}$, $X := [-\infty,+\infty]$, $Y := \mathbb{R}$, and $f := \mathrm{id}_{\mathbb{R}}$. If this had an extension to all of X, then in particular the sequence $m\mapsto x_m := m$ would have to converge in $\mathbb{R}$.

> R In fact, if perhaps you thought you could make use of the fact that continuous functions restricted to quasicompact subsets are uniformly-continuous (Proposition 4.1.3.25) to prove the result in special cases, this even provides a counter-example in which every point of X has a compact neighborhood.

Among other things, the significance of this result is that group operations of topological groups extend uniquely to their completions.

Having shown that uniform spaces always have completions, finally, we may return to an unresolved issue all the way back from Chapter 1.

▪ **Example 4.4.2.20 — A nonzero totally-ordered Cauchy-complete field distinct from $\mathbb{R}$** Recall the field of rational functions with coefficients in the reals, $\mathbb{R}(x)$, from Example 2.3.16. Being a totally-ordered field, it is in particular a topological group (Exercise 4.2.2.11) with respect to its underlying commutative group $\langle \mathbb{R}(x), +, 0, - \rangle$, and so has a canonical uniform structure. Thus, we may complete to form the complete topological field $\mathrm{Cmp}(\mathbb{R}(x))$.

Exercise 4.4.2.21 Extend the order on $\mathbb{R}(x)$ to $\mathrm{Cmp}(\mathbb{R}(x))$ so that $\mathrm{Cmp}(\mathbb{R}(x))$ is a totally-ordered field containing $\mathbb{R}(x)$.

By construction then, $\mathrm{Cmp}(\mathbb{R}(x))$ is a nonzero totally-ordered Cauchy-complete field. Not only is it distinct from $\mathbb{R}$, but it cannot even embed in $\mathbb{R}$ as, if it did, then so to $\mathbb{R}(x)$ would embed in $\mathbb{R}$ (as it embeds in $\mathrm{Cmp}(\mathbb{R}(x))$), and hence $\mathbb{R}(x)$ would be archimedean, a contradiction of Example 2.3.16.

In terms of intuition, the completion is not unlike the closure. This is made precise by the following result.

Proposition 4.4.2.22 Let X be a complete T_0 uniform space and let $S \subseteq X$. Then, $\mathrm{Cmp}(S) = \mathrm{Cls}(S)$.

Proof. We leave this as an exercise.

Exercise 4.4.2.23 Prove the result yourself.

■

This allows us to present another example of a completion.

■ **Example 4.4.2.24 — $\mathrm{Cmp}(\mathbb{R}[x]) = \mathrm{Mor}_{\mathbf{Top}}(\mathbb{R}, \mathbb{R})$** Note that we are making the same abuse of notation here as described in a remark of the Weierstrass Approximation Theorem (Corollary 4.4.1.24).

By the previous result, $\mathrm{Cmp}(\mathbb{R}[x]) = \mathrm{Cls}(\mathbb{R}[x])$. However, by the Weierstrass Approximation Theorem (Corollary 4.4.1.24), $\mathrm{Cls}(\mathbb{R}[x]) = \mathrm{Mor}_{\mathbf{Top}}(\mathbb{R}, \mathbb{R})$, and so we have $\mathrm{Cmp}(\mathbb{R}[x]) = \mathrm{Mor}_{\mathbf{Top}}(\mathbb{R}, \mathbb{R})$.

■ **Example 4.4.2.25 — $\mathrm{Cmp}(\mathbb{Q}) = \mathbb{R}$** We've known for awhile now (Theorem 2.4.3.40) that $\mathbb{R}$ is Cauchy-complete; however, we never stopped to check that it is the 'smallest' complete uniform space that contains $\mathbb{Q}$. This, however now follows from Proposition 4.4.2.22 as $\mathrm{Cls}(\mathbb{Q}) = \mathrm{Cls}(\mathbb{R})$ (which itself was (hopefully) proven in Exercise 3.2.1.2).

4.4.3 Complete metric spaces

We present here two important results that are specific to complete metric spaces.

The Baire Category Theorem

The Baire Category Theorem is an important result concerning complete *metric* spaces. It has many important applications, most of which we haven't the time to present. One important application for us, however, is that it will allow us to do one of the remaining separation axiom counter-examples—see Example 3.6.2.93.

Theorem 4.4.3.1 — Baire Category Theorem. Let X be a complete metric space. Then,

(i). the countable intersection of open dense subsets of X is dense; and
(ii). the countable union of closed sets with empty interior has empty interior.

R These conclusions are equivalent, the equivalence being obtained by taking the complement of the conclusion. The former is arguably a bit more intuitive to prove, while the latter the form that is probably more frequently used in concrete situations.

W Warning: This is *false* in general for complete uniform spaces—see Example 4.4.3.7.

R This is often applied as follows. Let $C_m \subseteq X$ be closed and suppose that $X = \bigcup_{m \in \mathbb{N}} C_m$. Then, as X certainly has nonempty interior, there must be some $m_0 \in \mathbb{N}$ such that $\operatorname{Int}(C_{m_0}) \neq \emptyset$.

R The word "category" in the name has nothing to do with categories—the terminology it refers to is archaic.

Proof. For $m \in \mathbb{N}$, let $U_m \subseteq X$ be an open dense subset. We wish to show that

$$U := \bigcap_{m \in \mathbb{N}} U_m \tag{4.4.3.2}$$

is dense. The definition of density is that the closure is equal to all of X, in other words, that every point of X is an accumulation point, or in other words, that every open subset intersects the dense set. So, let $V \subseteq X$ be open. We wish to show that V intersects U.

As U_0 is dense, V intersects U_0, say at $x_0 \in U_0 \cap V$. As $U_0 \cap V$ is open, we can fit an ε-ball around x_0 inside $U_0 \cap V$. In fact, as X is perfectly-T_4, and in particular T_3, we can find some $\varepsilon_0 > 0$ such that

$$\operatorname{Cls}\left(B_{\varepsilon_0}(x_0)\right) \subseteq U_0 \cap V.^a \tag{4.4.3.3}$$

In fact, by making ε_0 smaller if necessary, we may without loss of generality assume that $\varepsilon_0 < 2^{-0} = 1$.

Now, because U_1 is dense, there is some $x_1 \in U_1 \cap B_{\varepsilon_0}(x_0)$, and so, just the same as before, there is some $0 < \varepsilon_1 < 2^{-1}$ such that

$$\operatorname{Cls}\left(B_{\varepsilon_1}(x_1)\right) \subseteq U_1 \cap B_{\varepsilon_0}(x_0). \tag{4.4.3.4}$$

Proceeding inductively, we can find $x_m \in X$ and $0 < \varepsilon_m < 2^{-m}$ such that

$$\operatorname{Cls}\left(B_{\varepsilon_m}(x_m)\right) \subseteq U_m \cap B_{\varepsilon_{m-1}}(x_{m-1}). \tag{4.4.3.5}$$

We now check that $m \mapsto x_m$ is Cauchy. Let $\varepsilon > 0$. Choose $m \in \mathbb{N}$ such that $\varepsilon_m < \varepsilon$. Suppose that $n \geq m$. Then, $x_n \in B_{\varepsilon_n}(x_n) \subseteq B_{\varepsilon_m}(x_m) \subseteq B_\varepsilon(x_m)$. Thus, $m \mapsto x_m$ is eventually contained in some ε ball, and is hence Cauchy. As X is complete, it converges (to a unique limit) and so we may define

$$x_\infty := \lim_m x_m. \tag{4.4.3.6}$$

We claim that $x \in U \cap V$. As explained above, this will complete the proof. $m \mapsto x_m$ is eventually contained in $\operatorname{Cls}(B_{\varepsilon_0}(x_0)) \subseteq V$, and so $x_\infty \in \operatorname{Cls}(B_{\varepsilon_0}(x_0)) \subseteq V$. Similarly, for $n \in \mathbb{N}$, $m \mapsto x_m$ is eventually contained in $\operatorname{Cls}\left(B_{\varepsilon_n}(x_n)\right) \subseteq U_n$, and so, same as before, $x_\infty \in U_n$. Hence, $x_\infty \in U$, and we are done. ■

[a] We have applied Proposition 3.6.2.102, which is what required we be T_3.

Example 4.4.3.7 — A complete uniform space which is not a Baire space [a] Define

$$X := \{f\colon \mathbb{Z}^+ \to [0,1] : \\ f(m) = 0 \text{ for all but finitely many } m \in \mathbb{N}.\}.^b \tag{4.4.3.8}$$

We define a uniformity on X as follows. First of all, for $m \in \mathbb{Z}^+$, define

$$X_m := \underbrace{[0,1] \times \cdots \times [0,1]}_{m} \tag{4.4.3.9}$$

equipped with the product uniformity. Note that X_m embeds in X via $\iota_m : X_m \to X$ defined by

$$\iota_m(\langle x_1, \ldots, x_m \rangle) := \left(k \mapsto \begin{cases} x_k & \text{if } k \leq m \\ 0 & \text{otherwise,} \end{cases} \right) \tag{4.4.3.10}$$

that is, $\langle x_1, \ldots, x_m \rangle$ is sent to the function from $\mathbb{Z}^+$ into $[0,1]$ which sends k to x_k for $k \leq m$ and 0 otherwise. We then equip X with the final uniformity (Proposition 4.1.3.20) with respect to the collection $\{\iota_m : m \in \mathbb{Z}^+\}$.

Let $\Lambda \ni \lambda \mapsto x_\lambda \in X$ be Cauchy. We first wish to show that $\lambda \mapsto x_\lambda$ is eventually contained in X_{m_0} for some $m_0 \in \mathbb{Z}^+$. To show this, we proceed by contradiction: suppose that for every $m \in \mathbb{N}$ and every λ there is some $\lambda_{m,\lambda} \geq \lambda$ such that $x_{\lambda_{m,\lambda}} \notin X_m$. Define $\Lambda' := \mathbb{Z}^+ \times \Lambda$. Then, $\Lambda' \ni \langle m, \lambda \rangle \mapsto x_{\lambda_{m,\lambda}}$ is a subnet of $\lambda \mapsto x_\lambda$, and hence is in turn Cauchy. Thus, for every $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots > 0$, there is some $\langle m_0, \lambda_0 \rangle$ such that, whenever $\langle m, \lambda \rangle, \langle n, \mu \rangle \geq \langle m_0, \lambda_0 \rangle$, it follows that

$$\left| x_{\lambda_{m,\lambda}}(k) - x_{\lambda_{n,\mu}}(k) \right| < \varepsilon_k \tag{4.4.3.11}$$

for all $k \in \mathbb{Z}^+$. As $x_{\lambda_{m_0,\lambda_0}}(k) = 0$ for all k sufficiently large, we find that

$$x_{\lambda_{n,\mu}}(k) < \varepsilon_k \tag{4.4.3.12}$$

for all k sufficiently large, say for $k \geq k_0$, and all $\langle n, \mu \rangle \geq \langle m_0, \lambda_0 \rangle$. Take $n := \max\{k_0, m_0\}$. As $x_{\lambda_{n,\mu}} \notin X_n$, there is some $l > n \geq k_0$ such that $x_{\lambda_{n,\mu}}(l) \neq 0$. Then,

$$0 < x_{\lambda_{n,\mu}}(l) < \varepsilon_l. \tag{4.4.3.13}$$

As ε_l is arbitrary, this is a contradiction. Thus, there is some $m_0 \in \mathbb{Z}^+$ such that $\lambda \mapsto x_\lambda$ is eventually contained in X_{m_0}.

However, X_{m_0}, being a finite product of compact metric spaces, is a compact metric space, and hence complete, so that $\lambda \mapsto x_\lambda$ converges in X_{m_0}, and hence in X. Therefore, X is complete.

We now wish to check that X is not a Baire space. From the definition, we have that

$$X = \bigcup_{m \in \mathbb{Z}^+} X_m. \tag{4.4.3.14}$$

Exercise 4.4.3.15 Show that X_m is closed in X.

Thus, to show that X is not a Baire space, it suffices to show that each X_m has empty interior. So, let $x \in X_m$. We show that every open neighborhood around x contains an element of X_{m+1} that is not contained in X_m. Let us write

$$x = \langle x_1, x_2, \ldots, x_m, 0, 0, 0, \ldots \rangle \tag{4.4.3.16}$$

Then, for every neighborhood U of x, there will be some $\varepsilon_0 > 0$ sufficiently small so that

$$x = \langle x_1, x_2, \ldots, x_m, \varepsilon_0, 0, 0, \ldots \rangle \in U, \tag{4.4.3.17}$$

so that x is not in the interior of X_m, so that each X_m has empty interior.

[a] A ***Baire space*** is a topological space in which the conclusion of the Baire Category Theorem holds. This example was inspired by priel's answer on mathoverflow.net. Thanks to Nate Eldredge for nudging me towards the correct proof.

[b] I am thinking of this as the set of all sequences which eventually terminate in all 0s.

Banach Fixed-Point Theorem

We finish the chapter with an application of the Baire Category Theorem that will prove useful to us later when we study differentiation.

Theorem 4.4.3.18 — Banach Fixed-Point Theorem. Let $\langle X, |\cdot,\cdot|\rangle$ be a metric space and let $f\colon X \to X$ be such that

$$|f(x_1), f(x_2)| \leq M|x_1, x_2| \tag{4.4.3.19}$$

for some $0 \leq 1 < M$. Then, there is *at most one* point $x_0 \in X$ such that $f(x_0) = x_0$. If X is nonempty and complete, then there is *exactly one* $x_0 \in X$ such that $f(x_0) = x_0$.

R Such an x_0 is called a ***fixed-point***, hence the name of the theorem. Thus, the theorem tells us that (i) fixed-points, if they exist, have to be unique; and (ii) in the (nonempty) complete case, there has to be some fixed-point (and hence exactly one fixed-point).

R (4.4.3.19) is just the statement that f is Lipschitz-continuous *for a constant* $M < 1$. Such maps are called ***contraction-mappings***, hence the alternative name for this theorem, the ***Contraction-Mapping Theorem***.

Proof. Let $x_1, x_2 \in X$ be two fixed points of f. Then,

$$|x_1, x_2| = |f(x_1), f(x_2)| \leq M|x_1, x_2|, \tag{4.4.3.20}$$

and hence, if $|x_1, x_2| \neq 0$, we would have $1 \leq M$: a contradiction. Therefore, $|x_1, x_2| = 0$, and hence $x_1 = x_2$.

Now take X to be nonempty and complete. We construct an actual fixed point of f. Let $x_0 \in X$ (there is such a point because X is nonempty). For $m \in \mathbb{Z}^+$, define

$$x_m := f(x_{m-1}). \tag{4.4.3.21}$$

We wish to show that the sequence $m \mapsto x_m$ is Cauchy. If we can do so, then its limit x_∞ must exist, and so by taking the limit of the previous equation, we would find that $x_\infty = f(x_\infty)$ (f is Lipschitz-continuous, hence uniformly-continuous, hence continuous). Thus, it suffices to show that $m \mapsto x_m$ is Cauchy.

To see this, we first notice that[a]

$$\begin{aligned} |y_1, y_2| &\leq |y_1, f(y_1)| + |f(y_1), f(y_2)| + |f(y_2), y_2| \\ &\leq |y_1, f(y_1)| + M|y_1, y_2| + |f(y_2), y_2|, \end{aligned}$$

and so

$$|y_1, y_2| \leq \frac{1}{1-M}\left(|y_1, f(y_1)| + |f(y_2), y_2|\right) \tag{4.4.3.22}$$

Also note that

$$|f^m(y_1), f^m(y_2)| \leq M^m |y_1, y_2|, \tag{4.4.3.23}$$

which follows of course from just applying (4.4.3.19) inductively. Taking $y_1 := f^m(x_0)$ and $y_2 := f^n(x_0)$ in (4.4.3.22), we find

$$\begin{aligned} |x_m, x_n| &\leq \frac{1}{1-M}\left(|x_m, x_{m+1}| + |x_{n+1}, x_n|\right) \\ &\leq \frac{1}{1-M}\left(M^m|x_0, f(x_0)| + M^n|f(x_0), x_0|\right) \\ &= \frac{M^m + M^n}{1-M}|x_0, f(x_0)|. \end{aligned}$$

Because $M < 1$, we can make $\frac{M^m+M^n}{1-M}$ arbitrarily small by taking m and n sufficiently large.[b] Hence, this sequence is Cauchy, and we are done. ■

[a] We're using ys instead of xs because those symbols are already used-up.

[b] If you are not comfortable with this amount of detail, I suggest you fill in the gaps. You will want to get to the point where you feel comfortable just asserting Cauchyness after obtaining an inequality like this.

■ **Example 4.4.3.24 — A contraction mapping with no fixed-point** Take $X := \mathbb{R}^+$ and define $f(x) := \frac{1}{2}x$. This is certainly a contraction mapping, but if $x \in X$ were a fixed point, that is, if $\frac{1}{2}x =: f(x) = x$, then we would have $x = 0 \notin X$. Therefore, there is no fixed point (in X).

5. Integration

So, first things first—fuck the Riemann integral. Seriously. The only argument pro-Riemann integral is that it is easier. What a ridiculous argument. This is math, dude. If you choose to do things because they're easy, you're in the wrong subject. Moreover, I would argue that this is not even true—if you set things up right, you can literally *define* the (Lebesgue) integral to be the area (measure) under the curve. Or, if you prefer, you can take a limit over the size of a partition of the sum of the areas of the rectangles corresponding to the subsets of the partition (the Riemann integral). Are you really going to sit here and try to argue that this is easier to teach? I call bullshit. And besides, if you're going to become a mathematician, you have to learn the Lebesgue integral at some point anyways... why learn something only to have to relearn it later?

Okay, so now that my rant is out of the way, let's actually do some mathematics.

5.1 Measure theory

All of integration theory ultimately boils down to measure theory. The definition of the integral itself is relatively easy. In fact, the definition of abstract measure spaces is even easier. There's really no question that writing down the definition of the Lebesgue integral is *significantly* easier than that of the Riemann integral. What is a bit tricky, however, is constructing specific measures. In our case, we will primarily be concerned with constructing Lebesgue measure (on $\mathbb{R}^d$), and this is really the only part that is a bit tricky. Fortunately, there is a *huge* theorem that will just spit out Lebesgue measure for us, the *Haar-Howes Theorem*.

5.1.1 Measures

The intuition behind measure is actually quite easy—a measure is just an axiomatization of our intuition about notion of things like length, area, and volume. Before we define "measure", however, it will be convenient to introduce a couple of terms.

Definition 5.1.1.1 — Subadditivity and additivity Let X be a set, let $\mathrm{m}\colon 2^X \to [0,\infty]$, and let $\mathscr{M} \subseteq 2^X$.

(i). m is ***subadditive*** on $\mathscr{M}$ iff for $\{M_m : m \in \mathbb{N}\} \subseteq \mathscr{M}$ we have

$$\mathrm{m}\left(\bigcup_{m\in\mathbb{N}} M_m\right) \leq \sum_{m\in\mathbb{N}} \mathrm{m}(M_m); \tag{5.1.1.2}$$

(ii). m is ***additive*** on $\mathcal{M}$ iff for $\{M_m : m \in \mathbb{N}\} \subseteq \mathcal{M}$ a *disjoint* collection we have

$$\mathrm{m}\left(\bigcup_{m\in\mathbb{N}} M_m\right) = \sum_{m\in\mathbb{N}} \mathrm{m}(M_m). \tag{5.1.1.3}$$

m is simply just subadditive (resp. additive) if it is subadditive (resp. additive) on all of 2^X.

R You might think that we should always have additivity, or at the very least, we should have finite additivity: if S and T are disjoint, then $\mathrm{m}(S \cup T) = \mathrm{m}(S) + \mathrm{m}(T)$. Unfortunately, this is *false* for Lebesgue measure, our measure of primary interest—see Example 5.1.5.42. We will have additivity on a very large class of sets, however, the so-called *measurable sets*—see Definition 5.1.1.19.

Exercise 5.1.1.4 Let $\mathscr{M}$ be a collection of sets that is closed under union, intersection, and complementation. Show that if m is additive on $\mathscr{M}$ then it is subadditive on $\mathscr{M}$.

R There is something to show here. While (5.1.1.3) itself is obviously a stronger condition than (5.1.1.2), it is also only assumed for *disjoint* collections. The problem then is to show that, if (5.1.1.3) holds for disjoint collections, then (5.1.1.2) holds for *all* collections.

Definition 5.1.1.5 — Measure Let X be a set. A ***measure*** on X is a function $\mathrm{m} : 2^X \to [0, \infty]$ such that

(i). $\mathrm{m}(\emptyset) = 0$;
(ii). (Nondecreasing) $\mathrm{m} : \langle 2^X, \subseteq \rangle \to [0, \infty]$ is nondecreasing;[a] and
(iii). (Subadditivity) m is subadditive.

A set equipped with measure is a ***measure space***.

R Note that we allow the measure of sets to be infinite. This is incredibly important—for example, we will want $\mathrm{m}(\mathbb{R}) = \infty$ (for Lebesgue measure anyways).

R As a consequence of this, we needn't worry about convergence in the third axiom. As a matter of fact, we definitely want to allow this sum to diverge—think about what the measure of $\bigcup_{m \in \mathbb{Z}} (m, m+1)$ should be.

R Most other sources will likely refer to m as an ***outer-measure***. For them, a measure would be considered a *measure* iff it were additionally additive (and not just subadditive). However, as we work exclusively with what they would refer to as an "outer-measure", we simply just the the term "measure" as "outer-measure" is unnecessarily verbose.[b]

R Measure is sometimes also called ***exterior-measure***.

[a] Concretely, this means that $\mathrm{m}(S) \leq \mathrm{m}(T)$ if $S \subseteq T$.

[b] Incidentally, as we shall see shortly, m restricted to the collection of measurable sets (Definition 5.1.1.19) will be additive (Carathéodory's Theorem (Theorem 5.1.1.23)), that is, a "measure" in their sense of the term. That is, every measure in our sense of the word determines a "measure" in their sense of the term, and so the distinction is not really that big of a deal—I just find it quite a bit cleaner to be working with m defined on all of 2^X than merely just the measurable sets.

Example 5.1.1.6 — The Zero Measure Let X be a set and define $\mathrm{m} : 2^X \to [0, \infty]$ by $\mathrm{m}(S) := 0$. How terribly interesting.

Example 5.1.1.7 — The Infinite Measure Let X be a set and define $\mathrm{m} : 2^X \to [0, \infty]$ by

$$\mathrm{m}(S) := \begin{cases} 0 & \text{if } S = \emptyset \\ \infty & \text{otherwise.} \end{cases} \tag{5.1.1.8}$$

Dear god, this example is even more interesting than the last one.

Example 5.1.1.9 — The Unit Measure Let X be a set and define $\mathrm{m} : 2^X \to [0, \infty]$ by

$$\mathrm{m}(S) := \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{otherwise.} \end{cases} \tag{5.1.1.10}$$

R Note that this is *never* additive (unless of course X is either empty or a single point). This makes it useful for producing counter-examples (see Example 5.1.2.4), and not much else.

Example 5.1.1.11 — The counting measure Let X be a set and for $S \subseteq X$ define $\mathrm{m}(S) := |S|$, that is, the cardinality of S.

R This is the ***counting measure*** on X.

R This is actually incredibly important, as we shall see (Proposition 5.2.3.23) that sums are just integrals with respect to the counting measure.

In measure theory, things almost always matter only 'up to' sets of measure 0. This concept is so important that there is a term for it.

Meta-definition 5.1.1.12 — Almost-everywhere XYZ Let $f\colon \langle X, \mathrm{m}\rangle \to Y$ be a function on a measure space. Then, f is ***almost-everywhere XYZ*** iff

$$\mathrm{m}\left(\{x \in X : f(x) \text{ is not XYZ.}\}\right) = 0. \tag{5.1.1.13}$$

R If we need to clarify the measure we're working with, we will write "m-almost-everywhere XYZ".

R Of particular importance is the condition "$f = g$ almost-everywhere",[a] which we shall denote as $f \sim_{\mathbf{AIE}} g$, and explicitly means that

$$\mathrm{m}\left(\{x \in X : f(x) \neq g(x)\}\right) = 0. \tag{5.1.1.14}$$

[a] Though see the following exercises for other important "almost-everywhere" definitions.

Exercise 5.1.1.15 Let X be a measure space and let Y be a set. Show that $\sim_{\mathbf{AIE}}$ is an equivalence relation on $\mathrm{Mor}_{\mathbf{Set}}(X, Y)$.

Exercise 5.1.1.16 Let X be a measure space, let $\langle Y, \leq \rangle$ be a preordered set, and for $f, g \in \mathrm{Mor}_{\mathbf{Set}}(X, Y)/\sim_{\mathbf{AlE}}$ define $f \leq g$ iff $f(x) \leq g(x)$ almost-everywhere. Show that $\leq$ is well-defined.

> **R** Recall that $\mathrm{Mor}_{\mathbf{Set}}(X, Y)/\sim_{\mathbf{AlE}}$ is our notation for the quotient set, that is, the set of equivalence classes—see Definition A.3.2.16.

Exercise 5.1.1.17 Let X be a measure space, let Y be a topological space, let $\lambda \mapsto f_\lambda \in \mathrm{Mor}_{\mathbf{Set}}(X, Y)/\sim_{\mathbf{AlE}}$, let $f_\infty \in \mathrm{Mor}_{\mathbf{Set}}(X, Y)$, and define "$\lambda \mapsto f_\lambda$ converges to f_∞ in $\mathrm{Mor}_{\mathbf{Set}}(X, Y)$" iff $\lambda \mapsto f_\lambda(x)$ converges to $f_\infty(x)$ almost-everywhere. Show that this definition of convergence is well-defined and defines a topology on $\mathrm{Mor}_{\mathbf{Set}}(X, Y)/\mathbf{AlE}$ via Kelley's Convergence Theorem (Theorem 3.4.2.1)

> **R** Note that this topology isn't terribly useful. For example, the integral will not be continuous with respect to this topology—see Example 5.2.3.17.

Whenever X is equipped with a measure, *all relations on* $\mathrm{Mor}_{\mathbf{Set}}(X, Y)$ *are defined only up to measure zero*. For example, if we write $f = g$, what we really mean is that $\mathrm{m}(\{x \in X : f(x) \neq g(x)\}) = 0$. Similarly, if we write $f \leq g$, what we really mean is that $\mathrm{m}(\{x \in X : f(x) \nleq g(x)\}) = 0$. Etc.. Unfortunately, there is no reasonable way to make this into a category without imposing extra conditions on our functions.[a]

[a] And it is certainly possible to do so—see Definition 5.1.1.39—but this condition is overly restrictive to be of much practical use—see Example 5.1.5.54.

■ **Example 5.1.1.18 — Composition is not well-defined almost-everywhere** $\sim_{\text{AIE}}$ is an equivalence relation by Exercise 5.1.1.15. Our claim is that composition is *not* well-defined with respect to this equivalence relation. Precisely, we give an examples of functions $f_1 \sim f_2$ and $g_1 \sim g_2$, with $g_1 \circ f_1 \nsim g_2 \circ f_2$.

Define $X \coloneqq \{x_0\}$, $Y \coloneqq \{y_0\}$, and $Z \coloneqq \{z_1, z_2\}$, and equip X with the Infinite Measure, Y with the Zero Measure, and Z with the Infinite Measure.[a]

There is only one function from X to Y—let $f_1 = f_2$ be this unique function. On the other hand, let $g_i \colon Y \to Z$ be the unique function that sends y_0 to z_i. f_1 and f_2 are equal everywhere, and so certainly equal almost-everywhere. On the other hand, Y has the Zero Measure, and so any two functions with domain Y are going to be equal almost-everywhere. On the other hand, $g_i \circ f_i$ is the unique function from X to Z that sends x_0 to z_i, and so $g_1 \circ f_1$ and $g_2 \circ f_2$ disagree on all of X, which has infinite measure! In particular, $g_1 \circ f_1 \nsim g_2 \circ f_2$.

[a] Though the measure Z is equipped with doesn't matter so much.

While we will be assigning a measure to every set, not all of them will be considered to be *measurable*. In general, we will *not* have additivity; however, when we restrict our (outer) measures to the collection of what are called *measurable sets*, we *will* have additivity. This is more or less the point of talking about measurable sets—additivity is a nice thing to have.

Definition 5.1.1.19 — Measurable (set) Let $\mathrm{m} : 2^X \to [0, \infty]$ be a measure on a set X and let $M \subseteq X$. Then, M is ***measurable*** iff

$$\mathrm{m}(S) = \mathrm{m}(S \cap M) + \mathrm{m}(S \cap M^{\mathsf{C}}) \tag{5.1.1.20}$$

for all sets $S \subseteq X$.

R Think about what this means: M is chopping up S into two pieces, the set of points in M and the set of points not in M. M is measurable, then, if the measure of S is the sum of the measure of these two pieces *for all* S. In particular, S itself is definitely not required to be measurable.[a]

R By subadditivity, we *always* have that $\mathrm{m}(S) \leq \mathrm{m}(S \cap M) + \mathrm{m}(S \cap M^{\mathsf{C}})$. Therefore, in fact, M is measurable iff

$$\mathrm{m}(S) \geq \mathrm{m}(S \cap M) + \mathrm{m}(S \cap M^{\mathsf{C}}) \quad (5.1.1.21)$$

for all $S \subseteq X$.

R You might say that the motivation for the definition is that, for sets S, T that satisfy this property, we should have at least finite additivity (i.e. S, T disjoint implies $\mathrm{m}(S \cup T) = \mathrm{m}(S) + \mathrm{m}(T)$). It turns-out that this is true, but in fact, perhaps surprisingly, we have much more than this—we actually have *(countable) additivity*–see Theorem 5.1.1.23. By asking for finite additivity, we get countable additivity for free!

W Warning: There definitely exist sets that are not measurable in general! In fact, nonmeasurable sets are easy to find in 'artificial' spaces—see Example 5.1.2.4. But such pathologies exist even for the nicest of measures. Indeed, see Example 5.1.5.33 for a set that is not measurable with respect to Lebesgue measure.

R This is also what is sometimes referred to ***Carathéodory measurable***.

[a] For one thing, this would make the definition circular.

Exercise 5.1.1.22 Show that if $\mathrm{m}(M) = 0$, then all subsets of M are measurable.

R In particular, M itself is measurable.

Theorem 5.1.1.23 — Carathéodory's Theorem. Let $m : 2^X \to [0,\infty]$ be a measure. Then,

(i). m is additive on the collection of measurable sets;
(ii). the countable union of measurable sets is measurable;
(iii). the countable intersection of measurable sets is measurable;
(iv). the complement of a measurable set is measurable;
(v). $\emptyset$ and X are measurable.

> R Note that De Morgan's Laws (Exercise A.2.3.12) imply that (ii) are (iii) are equivalent if (iv) is true. A collection of sets which satisfies (ii)–(v) is called a ***σ-algebra***. We do not use this language, but it is important to know for consulting other references.

Proof. STEP 1: SHOW (IV)
The definition of measurability is $S \leftrightarrow S^{\mathsf{C}}$ symmetric, so (iv) is automatically true.

STEP 2: SHOW (V)
The empty-set is measurable by the previous exercise, and hence by (iv), X is measurable as well, which establishes (v).

STEP 3: REDUCE THE PROOF OF (III) TO THE PROOF OF (II)
As was explained in a remark, we need not show (iii) itself—it will now follow if we can show (ii).

STEP 4: PROVE (II) FOR FINITE UNIONS
We first show that the union of finitely many measurable sets is measurable. It suffices of course to then just show that the union of two measurable sets is measurable. So, let $M_1, M_2 \subseteq X$ be

measurable and let $S \subseteq X$. Then,

$$\begin{aligned}
\mathrm{m}(S) &=^{a} \mathrm{m}(S \cap M_2) + \mathrm{m}(S \cap M_2^{\complement}) \\
&=^{b} \mathrm{m}(S \cap M_2 \cap M_1) + \mathrm{m}(S \cap M_2 \cap M_1^{\complement}) \\
&\qquad + \mathrm{m}(S \cap M_2^{\complement} \cap M_1) + \mathrm{m}(S \cap M_2^{\complement} \cap M_1^{\complement}) \\
&\geq^{c} \mathrm{m}(S \cap (M_1 \cup M_2)) + \mathrm{m}(S \cap (M_1 \cup M_2)^{\complement})
\end{aligned}$$

Thus, indeed, $M_1 \cup M_2$ is measurable.

STEP 5: COMPLETE THE PROOF OF (II)

Let $\{M_m : m \in \mathbb{N}\}$ be a countable collection of measurable sets. We wish to show that

$$\bigcup_{m \in \mathbb{N}} M_m \tag{5.1.1.24}$$

is measurable. First of all, define

$$M_m' := M_m \setminus \bigcup_{k=0}^{m-1} M_k. \tag{5.1.1.25}$$

Note that each M_m' is measurable because we already know that finite unions, complements, and hence also finite intersections, of measurable sets are measurable.

Exercise 5.1.1.26 Show that (i) $M_m' \subseteq M_m$, (ii) the collection $\{M_m' : m \in \mathbb{N}\}$ is disjoint, and (iii) $\bigcup_{m \in \mathbb{N}} M_m = \bigcup_{m \in \mathbb{N}} M_m'$.

R This trick (the one in (5.1.1.25), that is) is important. Don't forget it.

Thus, as

$$\bigcup_{m \in \mathbb{N}} M_m = \bigcup_{m \in \mathbb{N}} M_m', \tag{5.1.1.27}$$

it suffices to prove this step in the case where $\{M_m : m \in \mathbb{N}\}$ is itself disjoint (just rename M_m to now be M_m'). Thus, we now without loss of generality assume that $\{M_m : m \in \mathbb{N}\}$ is disjoint.

Now define

$$N_m := \bigcup_{k=0}^{m} M_k \text{ and } N := \bigcup_{k\in M} M_m. \tag{5.1.1.28}$$

so that, by the previous step, we have that N_m is measurable.

Then, for $S \subseteq X$,

$$\begin{aligned}\mathrm{m}(S \cap N_m) &=^{d} \mathrm{m}(S \cap N_m \cap M_m) \\ &\qquad + \mathrm{m}(S \cap N_m \cap M_m^{\mathsf{C}}) \\ &=^{e} \mathrm{m}(S \cap M_m) + \mathrm{m}(S \cap N_{m-1}) \\ &=^{f} \sum_{k=0}^{m} \mathrm{m}(S \cap M_k).\end{aligned} \tag{5.1.1.29}$$

Thus,

$$\begin{aligned}\mathrm{m}(S) &= \mathrm{m}(S \cap N_m) + \mathrm{m}(S \cap N_m^{\mathsf{C}}) \\ &= \sum_{k=0}^{m} \mathrm{m}(S \cap M_k) + \mathrm{m}(S \cap N_m^{\mathsf{C}}) \\ &\geq^{g} \sum_{k=0}^{m} \mathrm{m}(S \cap M_k) + \mathrm{m}(S \cap N^{\mathsf{C}}).\end{aligned} \tag{5.1.1.30}$$

Hence, taking the limit of this inequality as $m \to \infty$, we have

$$\begin{aligned}\mathrm{m}(S) &\geq \sum_{m\in\mathbb{N}} \mathrm{m}(S \cap M_m) + \mathrm{m}(S \cap N^{\mathsf{C}}) \\ &\geq^{h} \mathrm{m}(S \cap N) + \mathrm{m}(S \cap N^{\mathsf{C}}),\end{aligned} \tag{5.1.1.31}$$

and so indeed N is measurable.

Step 6: Prove (i)

Let N and S be as in the previous step and take $S := N$. Then, by (5.1.1.31), we have that

$$\mathrm{m}\left(\bigcup_{m\in\mathbb{N}} M_m\right) \geq \sum_{m\in\mathbb{N}} \mathrm{m}(M_m). \tag{5.1.1.32}$$

The other inequality is automatic from subadditivity, and so indeed, we have equality. ■

[a] Because M_2 is measurable.

[b] Because M_1 is measurable (applied twice).

[c] By subadditivity and the fact that $M_1 \cup M_2 = (M_1 \cap M_2) \cup (M_1 \cap M_2^{\mathsf{C}}) \cup (M_1^{\mathsf{C}} \cap M_2)$.

[d] Because E_m is measurable.

[e] Because the collection $\{M_m : m \in \mathbb{N}\}$ is disjoint.

[f] Apply this trick inductively.

[g] Because $N^{\mathsf{C}} \subseteq N_m^{\mathsf{C}}$.

[h] By subadditivity.

Exercise 5.1.1.33 Let $\mathrm{m} : 2^X \to [0, \infty]$ be a measure and let $S \subseteq T \subseteq X$. Show that if S is measurable with finite measure, then

$$\mathrm{m}(T \setminus S) = \mathrm{m}(T) - \mathrm{m}(S). \tag{5.1.1.34}$$

R The only reason you need S to have finite measure otherwise the right-hand side of this equation will be undefined if T also has infinite measure. In particular, the rearranged equation $\mathrm{m}(S) + \mathrm{m}(T \setminus S) = \mathrm{m}(T)$ holds even in the case $\mathrm{m}(S) = \infty$.

R Note that you do *not* need T to be measurable for this to hold.

Exercise 5.1.1.35 — "Continuity from below" Let $M_0 \subseteq M_1 \subseteq \cdots$ be an nondecreasing countable collection of measurable sets. Show that

$$\mathrm{m}\left(\bigcup_{k=0}^{\infty} M_k\right) = \lim_m \mathrm{m}\left(\bigcup_{k=0}^{m} M_k\right) = \lim_m \mathrm{m}(M_k). \quad (5.1.1.36)$$

Exercise 5.1.1.37 — "Continuity from above" Let $M_0 \supseteq M_1 \supseteq \cdots$ be a nonincreasing countable collection of measurable sets. Show that, *if at least one M_k has finite measure*,

$$\mathrm{m}\left(\bigcap_{k=0}^{\infty} M_k\right) = \lim_m \mathrm{m}\left(\bigcap_{k=0}^{m} M_k\right) = \lim_m \mathrm{m}(M_k). \quad (5.1.1.38)$$

R Though we technically have not defined measure on $\mathbb{R}$ yet, it is easy to see intuitively why we would neither expect nor want this to hold if the measure of each M_k were infinite. For example, take $M_k := (-\infty, -k)$. Then, on one hand, $\mathrm{m}(M_k) = \infty$ for all k, but yet $\mathrm{m}(\bigcap_{k=0}^{\infty} M_k) = \mathrm{m}(\emptyset) = 0$.

Just as we have a notion of measurable set, so too do we have a notion of measurable *function*

Definition 5.1.1.39 — Measurable (function)
Let $f\colon \langle X_1, \mathrm{m}_1\rangle \to \langle X_2, \mathrm{m}_2\rangle$ be a function between measure spaces. Then, f is ***measurable*** iff

(i). the preimage of every measurable set is measurable; and

(ii). the preimage of a set of measure 0 has measure 0.

R This is a *very strong* condition. For example, there are *uniform-homeomorphisms* on $\mathbb{R}$ that are not measurable—see Example 5.1.5.54 (the Cantor Function).

R Neither of these conditions imply one another—see the following examples.

Example 5.1.1.40 — A function which preserves measurability but not measure 0 Precisely, we give a function that has the property that the preimage of every set is measurable but the preimage of a set of measure 0 does not have measure 0.

Let $X_1 \coloneqq \{x_1\}$ be a one point set and let m_1 be the Infinite Measure (Example 5.1.1.7) on X_1. Let $X_2 \coloneqq \{x_2\}$ be a one point set and let m_2 be the Zero Measure (Example 5.1.1.6) on X_2. Let $f\colon X_1 \to X_2$ be the only function that exists from X_1 to X_2.

Every subset of X_1 is measurable, and so trivially the preimage of every subset of X_2 is measurable. On the other hand, $\mathrm{m}_2(\{x_2\}) = 0$, but $\mathrm{m}_1\left(f^{-1}(\{x_2\})\right) = \infty \neq 0$.

Example 5.1.1.41 — A function which preserves measure 0 **but not measurability** Precisely, we give a function that has the property that the preimage of every set of measure 0 has measure 0 but the preimage of some measurable set is not measurable.

Let $X_1 \coloneqq \{x_1, x_2\} =: X_2$ be a two point set. Equip X_1 with the Unit Measure (Example 5.1.1.9) and equip X_2 with the Infinite Measure. Let $f \coloneqq \mathrm{id}_{\{x_1,x_2\}}$.

Exercise 5.1.1.42 Show that $\{x_1\}, \{x_2\} \subseteq X_1$ are *not* measurable with respect to the Unit Measure.

There is only one subset of X_2 with measure 0, namely $\emptyset$, and of course the preimage of the empty-set (the empty-set itself) also has measure 0. On the other hand, $\{x_1\}$ is measurable with respect to the Infinite Measure, but not the Unit Measure, and so its preimage (namely itself) is not measurable.

I should probably mention at this point that the way I am presenting measure theory goes against the orthodoxy. (You can skip this comment if you don't plan to look in other sources.) Every other author I am aware of *only works with measurable sets*. For them, a *measure space* is a set, together with a σ-algebra, the "measurable sets", along with a set function that is additive on the σ-algebra (and sends the empty-set to 0). For them, measures are merely tools for constructing measures (a là Mr. Carathéodory). Of course, their "measure"[a] is just the restriction of the measure to the collection of all measurable sets. You might say they simply 'forget' that they had ever assigned a measure to the nonmeasurable sets. I find this unnecessarily complicated and messy. For one thing, if you do things the way I have presented them, you don't have to worry about σ-algebras (at least not explicitly). The disadvantage is that now I have to add the hypothesis "These sets are measurable" to a lot of my theorems. Meh. It's a trade-off, but I quite like not having to ever worry about σ-algebras explicitly.

[a] The quotes are to indicate that this is what they call it—I myself have not defined this term.

At some point in the near future, we will be doing arithmetic with ∞—for example, what should the measure of $\mathbb{R} \times \{0\}$ in $\mathbb{R}^2$ be? Of course, from our definition of product measures, this will turn out to be $\infty \cdot 0$. We hence declare that

$$\infty \cdot 0 := 0 =: 0 \cdot \infty. \tag{5.1.1.43}$$

There are other arithmetic notions we have to technically define (e.g. $x + \infty = \infty$ for x finite), but this is the only nonobvious one. On thing we leave *un*defined, however, is the quantity $\infty + (-\infty)$ (and likewise for $-\infty + \infty$).

5.1.2 Measures on topological and uniform spaces

One can go ahead and develop the theory for measures on arbitrary sets, but in practice, we will only be working with measures defined on spaces with *a lot* of extra structure. This motivates us to investigate measures on topological and uniform spaces, in which case we are of course going to require our measures to be compatible with this extra structure.

Measures on topological spaces

There are definitely measures on topological spaces for which the open sets are not measurable—see Example 5.1.2.4—in fact, this can even be the case for regular measures (the example in Example 5.1.2.4 is regular—see Definition 5.1.2.2 for the definition of a regular measure), but most measures on topological spaces which are not cooked-up for the sole purpose of producing counter-examples have the property that open sets are measurable. We have a name for such measures: *Borel measures*.

Definition 5.1.2.1 — Borel measure Let X be a topological space and let $\mathrm{m} : 2^X \to [0, \infty]$ be a measure. Then, m is ***Borel*** iff every open set is measurable. A topological space equipped with a Borel measure is a ***Borel measure space***.

R By Carathéodory's Theorem, it follows that the "σ-algebra" generated by the open sets likewise consists of measurable sets. The term for the sets in this σ-algebra is ***Borel set***, hence the term, *Borel measure*—a Borel measure is a measure in which the Borel sets are measurable (which is of course equivalent to the open sets being measurable).

R In particular, as the complements of measurable sets are measurable, closed sets are likewise measurable for Borel measures. Then, as countable unions and countable intersections of measurable sets are measurable, it follows that in fact all G_δ and F_σ (Definition 3.1.3) sets are measurable.

Definition 5.1.2.2 — Regular measure Let X be a topological space and let $\mathrm{m} : 2^X \to [0, \infty]$ be a measure. Then, m is ***regular*** iff

(i). m is finite on quasicompact subsets;
(ii). (Outer-regular) for $S \subseteq X$,

$$\mathrm{m}(S) = \inf\{\mathrm{m}(U) : S \subseteq U,\ U \text{ open.}\}; \text{ and} \tag{5.1.2.3}$$

(iii). (Inner-regular on open sets) for $U \subseteq X$ open,

$$\mathrm{m}(U) = \sup\{\mathrm{m}(K) : K \subseteq U,\ K \text{ quasicompact.}\}.$$

A topological space equipped with regular measure is a ***regular measure space***.

W Warning: As mentioned before, regular measures need not be Borel—see the following counterexample (Example 5.1.2.4).

■ **Example 5.1.2.4 — A regular measure that is not Borel** Define $X := \{x_1, x_2\}$, and equip it with the discrete topology and the Unit Measure (Example 5.1.1.9). Of course the measure of every quasicompact subset is finite. It is automatically outer-regular and inner-regular on open sets because every subset is both open and quasicompact. It is hence regular. On the other hand,

$$\mathrm{m}(\{x_1, x_2\}) = 1 < 1 + 1 = \mathrm{m}(\{x_1\}) + \mathrm{m}(\{x_2\}). \quad (5.1.2.5)$$

Therefore, one of the open sets $\{x_1\}$ or $\{x_2\}$ must have been nonmeasurable.

Of course, there is a counter-example to the other potential implication as well.

Example 5.1.2.6 — A Borel measure that is not regular Consider the counting measure on $\mathbb{R}$. The set $[0, 1]$ is quasicompact, but has infinite measure. Therefore, the counting measure on $\mathbb{R}$ is not regular. On the other hand, the cardinality of the union of two disjoint sets is the sum of the cardinalities of those two sets,[a] and so we always have $\mathrm{m}(S) = \mathrm{m}(S \cap M) + \mathrm{m}(S \cap M^{\mathsf{C}})$, that is to say, every set is measurable, and so certainly the open sets are measurable.

[a] This is how we defined addition of cardinals!

In general topology, I really dislike imposing unnecessary countability assumptions. On the other hand, countability is something fundamental in measure theory, simply because of the conditions of additivity and subadditivity—see Definition 5.1.1.5. Furthermore, as explained there, we *don't want* any stronger additivity assumptions. Thus, in the context of *measures* on topological spaces, it makes sense to impose countability conditions on our spaces. This leads us to the following definition.

Definition 5.1.2.7 — σ-quasicompact A topological space is ***σ-quasicompact*** iff it is the countable union of quasicompact sets. A topological space is ***σ-compact*** iff it is the countable union of compact sets.

R Recall that compact is synonymous (by definition) with T_2 and quasicompact. In particular, compact sets are *closed* (and so measurable for Borel measures).

Example 5.1.2.8 — A subspace of a σ-quasicompact space that is not σ-quasicompact The example is the Sorgenfrey Line S from Example 3.6.2.125.[a]

Exercise 5.1.2.9 Show that S is not σ-quasicompact.

We showed in Example 3.6.2.125 that S is perfectly-T_4. Thus, by the Tychonoff Embedding Theorem (Theorem 3.6.2.113), it is (homeomorphic to) a subspace of a product of $[0,1]$, which in particular is a compact space, hence trivially σ-quasicompact. Thus, this product is σ-quasicompact, but this subspace homeomorphic to S is not σ-quasicompact.

[a] Refer to the example for more detail, but in brief, the Sorgenfrey Line is, as a set, the real numbers, whose topology has as a base the set of all closed-open intervals $[a,b)$.

Exercise 5.1.2.10 Show that the product of two σ-quasicompact spaces is σ-quasicompact. Show that the product of two σ-compact spaces is σ-compact. Find an example of an infinite product of σ-compact spaces that is not σ-quasicompact.

Definition 5.1.2.11 — Topological measure space A ***topological measure space*** is a σ-compact topological space equipped with a regular Borel measure. If $\langle X, \mathrm{m}\rangle$ is a topological measure space, then we say that m is ***topological***.

Exercise 5.1.2.12 Show that a topological measure space can be written as the countable *disjoint* union of measurable sets each of which is contained in a compact set.

R In particular a topological measure space can be written as the countable disjoint union of measurable sets of finite measure. A measure space that can be written as the countable union of measurable sets of finite measure is called ***σ-finite***.

R By regularity and σ-compactness, you by definition have that topological measure spaces can be written as the countable union of compact sets. It is your job to turn that union into a disjoint one.

R I told you once upon a time not to forget a trick. You haven't forgotten it, have you?

Proposition 5.1.2.13 Let X be σ-compact topological space, let m be a measure on X, and let $K \subseteq X$ be quasicompact. Then, if m is Borel, then K is measurable.

R In particular, quasicompact sets are measurable in topological measure spaces.

Proof. Suppose that m is Borel. As X is σ-compact, we may write $X = \bigcup_{m\in\mathbb{N}} K_m$ for $K_m \subseteq X$ compact. Each K_m is in particular closed, and so $K \cap K_m$ is quasicompact by Proposition 3.5.1.6 (intersection of a quasicompact set and a closed set is quasicompact), hence closed as it is a subspace of a T_2 space (namely K_m), hence measurable. ∎

The following is nice characterization of measurability in topological measure spaces. It is arguably the reason why we give the conditions "σ-compact, regular, Borel" a name in the first place.

Proposition 5.1.2.14 Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and $M \subseteq X$. Then, the following are equivalent.

(i). M is measurable.

(ii). For every $\varepsilon > 0$, there is an open set U_ε and a closed set C_ε such that

$$C_\varepsilon \subseteq M \subseteq U_\varepsilon \text{ and } \mathrm{m}(U_\varepsilon \setminus C_\varepsilon) < \varepsilon. \quad (5.1.2.15)$$

(iii). There is a G_δ set G and an F_σ set F such that

$$F \subseteq M \subseteq G \text{ and } \mathrm{m}(G \setminus F) = 0. \quad (5.1.2.16)$$

Proof. Write $X = \bigcup_{m\in\mathbb{N}} K_m$ for $K_m \subseteq X$ compact.

($(i) \Rightarrow (ii)$) Suppose that M is measurable. Define $M_m := M \cap K_m$. As $M_m \subseteq K_m$, M_m has finite measure because m is regular. Let $\varepsilon > 0$. By outer-regularity, there is some open U_m containing M_m such that

$$\mathrm{m}(M_m) \leq \mathrm{m}(U_m) < \mathrm{m}(M_m) + \tfrac{\varepsilon}{2^m}. \quad (5.1.2.17)$$

Because M is measurable, it in turn follows that[a]

$$\mathrm{m}(U_m - M_m) < \tfrac{\varepsilon}{2^m}. \quad (5.1.2.18)$$

Define $U := \bigcup_{m\in\mathbb{N}} U_m$. Then,

$$\mathrm{m}(U \setminus M) \leq \sum_{m\in\mathbb{N}} \mathrm{m}(U_m - M_m) < 2\varepsilon. \quad (5.1.2.19)$$

Applying this same logic to M^{C}, we can find an open set V containing M^{C} such that

$$\mathrm{m}(V \setminus M^{\mathsf{C}}) < 2\varepsilon. \quad (5.1.2.20)$$

Then, V^{C} is then a closed subset of M and

$$\begin{aligned} \mathrm{m}(U \setminus V^{\mathsf{C}}) &= \mathrm{m}(U \cap V) \\ &\leq \mathrm{m}(U \cap V \cap M) + \mathrm{m}(U \cap V \cap M^{\mathsf{C}}) \quad (5.1.2.21) \\ &\leq \mathrm{m}(V \setminus M^{\mathsf{C}}) + \mathrm{m}(U \setminus M) < 4\varepsilon. \end{aligned}$$

$((ii) \Rightarrow (iii))$ Suppose that for every $\varepsilon > 0$, there is an open set U_ε and a closed set C_ε such that

$$C_\varepsilon \subseteq M \subseteq U_\varepsilon \text{ and } \mathrm{m}(U_\varepsilon \setminus C_\varepsilon) < \varepsilon. \tag{5.1.2.22}$$

For $\varepsilon := \frac{1}{m}$, $m \in \mathbb{Z}^+$, let $U_m \supseteq M$ be open and $C_m \subseteq M$ be closed and such that $\mathrm{m}(U_m \setminus C_m) < \frac{1}{m}$. Now define

$$G := \bigcap_{m \in \mathbb{N}} U_m \text{ and } F := \bigcup_{m \in \mathbb{N}} C_m. \tag{5.1.2.23}$$

Then, G is G_δ, F is F_σ (Definition 3.1.3), $F \subseteq M \subseteq G$, and

$$\mathrm{m}(G \setminus F) \leq^b \mathrm{m}(U_m \setminus C_m) < \frac{1}{m}, \tag{5.1.2.24}$$

and hence $\mathrm{m}(G \setminus F) = 0$.

$((iii) \Rightarrow (i))$ Suppose that there is a G_δ set G and an F_σ set F such that

$$F \subseteq M \subseteq G \text{ and } \mathrm{m}(G \setminus F) = 0. \tag{5.1.2.25}$$

Note that closed sets are measurable because the measure is Borel, and hence F_σ sets are measurable, being the countable union of closed (that is, measurable) sets. Thus, $M = F \cup (M \setminus F)$ will be measurable if we can show that $M \setminus F$ is measurable. However, $\mathrm{m}(M \setminus F) \leq \mathrm{m}(G \setminus F) = 0$, and so $M \setminus F$ is measurable by Exercise 5.1.1.22 (sets of measure zero are measurable), as desired. ■

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[a]See Exercise 5.1.1.33. We are also using here the fact that M_m is measurable, because M and K_m are measurable, K_m being measurable because it is compact, hence closed, and the measure is Borel (by hypothesis).

[b]Simply because $G \subseteq U_m$ and $F \supseteq C_m$ for all $m \in \mathbb{Z}^+$.

By definition, regularity requires inner-regularity on open sets. In fact, however, for topological measure spaces, we also get inner-regularity on measurable sets of finite measure.

Proposition 5.1.2.26 Let $\langle X, \mathrm{m}\rangle$ be a topological measure space. Then, m is inner-regular on measurable sets measure.

Proof. Step 1: Prove the result for sets of finite measure

Let $S \subseteq X$ be measurable. Suppose that $\mathrm{m}(S) < \infty$. By Exercise 5.1.2.12, write $X = \bigcup_{m\in\mathbb{N}} F_m$ for F_m measurable and contained in a compact set with $\{F_m : m \in \mathbb{N}\}$ disjoint. Let $S \subseteq X$ be measurable and of finite measure. Define $S_m := S \cap F_m$, so that each S_m is measurable, contained in a compact set, and with $\{S_m : m \in \mathbb{N}\}$ disjoint.

Let $\varepsilon > 0$. Then, by Proposition 5.1.2.14, there are open sets $U_{m,\varepsilon}$ and closed sets $C_{m,\varepsilon}$ such that

$$C_{m,\varepsilon} \subseteq S_m \subseteq U_{m,\varepsilon} \text{ and } \mathrm{m}(U_{m,\varepsilon} \setminus C_{m,\varepsilon}) < \frac{\varepsilon}{2^m}. \tag{5.1.2.27}$$

Note that as $C_m \subseteq S_m$ and S_m is contained in a compact set, C_m itself is compact, being a closed subset of a compact set.

As m is inner-regular on open sets, there are quasicompact sets $K'_{m,\varepsilon} \subseteq U_{m,\varepsilon}$ such that

$$\mathrm{m}(U_{m,\varepsilon}) - \frac{\varepsilon}{2^m} < \mathrm{m}(K'_{m,\varepsilon}) \leq \mathrm{m}(U_{m,\varepsilon}). \tag{5.1.2.28}$$

Note that $K'_{m,\varepsilon}$ is measurable by Proposition 5.1.2.13. Define $K_{m,\varepsilon} := K'_{m,\varepsilon} \cap C_{m,\varepsilon}$. Note that $K_{m,\varepsilon}$ is quasicompact by Proposition 3.5.1.6 (intersection of quasicompact set with closed set is quasicompact), and hence actually compact because $C_{m,\varepsilon}$ is compact (because subspaces of T_2 spaces are T_2). We then have that

$$\begin{aligned}
\mathrm{m}(U_{m,\varepsilon} \setminus K_{m,\varepsilon}) &:= \mathrm{m}(U_{m,\varepsilon} \cap (K'_{m,\varepsilon} \cap C'_{m,\varepsilon})^{\mathsf{C}}) \\
&\leq \mathrm{m}(U_{m,\varepsilon} \cap K'^{\mathsf{C}}_{m,\varepsilon}) + \mathrm{m}(U_{m,\varepsilon} \cap C^{\mathsf{C}}_{m,\varepsilon}) \\
&= \mathrm{m}(U_{m,\varepsilon} \setminus K'_{m,\varepsilon}) + \mathrm{m}(U_{m,\varepsilon} \setminus C_{m,\varepsilon}) \\
&<^{a} \tfrac{\varepsilon}{2^m} + \tfrac{\varepsilon}{2^m} = \tfrac{\varepsilon}{2^{m-1}}.
\end{aligned}$$

Now, $L_{m,\varepsilon} := \bigcup_{k=0}^{m} K_{k,\varepsilon}$ is likewise quasicompact and

$$\begin{aligned} \mathrm{m}(S) - \mathrm{m}(L_{m,\varepsilon}) &= {}^{b} \sum_{k=0}^{\infty} \mathrm{m}(S_k) - \sum_{k=0}^{m} \mathrm{m}(K_{k,\varepsilon}) \\ &= \sum_{k=m+1}^{\infty} \mathrm{m}(S_k) + \sum_{k=0}^{m} \big(\mathrm{m}(S_k) - \mathrm{m}(K_{k,\varepsilon})\big) \\ &\leq \sum_{k=m+1}^{\infty} \mathrm{m}(S_k) + \sum_{k=0}^{m} \big(\mathrm{m}(U_{k,\varepsilon}) - \mathrm{m}(K_{k,\varepsilon})\big) \qquad (5.1.2.29) \\ &< \sum_{k=m+1}^{\infty} \mathrm{m}(S_k) + \sum_{k=0}^{m} \frac{\varepsilon}{2^{k-1}} \\ &\leq \sum_{k=m+1}^{\infty} \mathrm{m}(S_k) + 4\varepsilon \end{aligned}$$

As $\mathrm{m}(S) = \sum_{k=0}^{\infty} \mathrm{m}(S_k)$ is finite, we can make this arbitrarily small by taking m sufficiently large. As $L_{m,\varepsilon} \subseteq S$ is quasicompact, we thus have that

$$\mathrm{m}(S) = \sup\{\mathrm{m}(K) : K \subseteq S,\ K \text{ quasicompact}\}, \qquad (5.1.2.30)$$

and so m is inner-regular on S.

STEP 2: PROVE THE RESULT FOR SETS OF INFINITE MEASURE

Let $S \subseteq X$ be measurable. Suppose that $\mathrm{m}(S) = \infty$. This time, write $X = \bigcup_{m\in\mathbb{N}} K_m$ for $K_m \subseteq X$ compact and $K_m \subseteq K_{m+1}$.[c] Define $S_m := S \cap K_m$. Then, $K_m \setminus S_m$ has finite measure, and so by outer-regularity, there is some open $U_m \subseteq X$ such that $K_m \setminus S_m \subseteq U_m$ and $\mathrm{m}\left(U_m \setminus (K_m \setminus S_m)\right) < 1$.[d] Now define $L_m := K_m \cap U_m^{\mathsf{C}}$. L_m is compact and $L_m \subseteq S_m$. Furthermore,

$$\begin{aligned} \mathrm{m}(S_m \setminus L_m) &:= \mathrm{m}(S_m \cap (K_m^{\mathsf{C}} \cup U_m) \\ &= \mathrm{m}(S_m \cap U_m) \qquad (5.1.2.31) \\ &\leq \mathrm{m}(U_m \setminus (K_m \setminus S_m)) < 1. \end{aligned}$$

It follows that $\mathrm{m}(L_m) > \mathrm{m}(S_m) - 1$. As each S_m is measurable, $S_m \subseteq S_{m+1}$, and $S = \bigcup_{m\in\mathbb{N}} S_m$, we have

$$\lim_m \mathrm{m}(S_m) = \mathrm{m}\left(\bigcup_{m\in\mathbb{N}} S_m\right) = \mathrm{m}(S) = \infty, \qquad (5.1.2.32)$$

and so in turn we have $\lim_m \mathrm{m}(L_m) = \infty$, so that

$$\sup\{\mathrm{m}(K) : K \subseteq S,\ K \text{ quasicompact.}\} = \infty, \quad (5.1.2.33)$$

as desired. ■

[a] Here we are applying Exercise 5.1.1.33 together with (5.1.2.27) and (5.1.2.28). This is why we neeed $K'_{m,\varepsilon}$ to be measurable.

[b] Here we are using the fact that the F_ks are disjoint, so that the S_ks and $K_{k,\varepsilon}$s are in turn disjoint.

[c] By definition, σ-compact just means we can write $X = \bigcup_{m\in\mathbb{N}} K'_m$ for $K_m \subseteq X$ compact, not necessarily with $K'_m \subseteq K'_{m+1}$. Given this, we may define $K_m := \bigcup_{k=0}^m K'_m$ so that now $X = \bigcup_{m\in\mathbb{N}}$, $K_m \subseteq X$ is compact, and $K_m \subseteq K_{m+1}$.

[d] Here, we have taken $\varepsilon = 1$.

A related result is the following.

Proposition 5.1.2.34 Let X be a σ-quasicompact topological space, let m be a regular measure, and let $S \subseteq X$. Then, if $\mathrm{m}(S) = \infty$, then there is a subset $T \subseteq S$ with $0 < \mathrm{m}(T) < \infty$.

R This condition, that every set of infinite measure has a subset of finite positive measure is sometimes called ***semifinite***.

R In particular, topological measure spaces are semifinite.

Proof. Suppose that $\mathrm{m}(S) = \infty$. Write $X = \bigcup_{m\in\mathbb{N}} K_m$ for K_m quasicompact. Define $S_m := S \cap K_m$. As K_m is quasicompact, it has finite measure, and so S_m has finite measure. We also have that

$$\infty = \mathrm{m}(S) = \mathrm{m}\left(\bigcup_{m\in\mathbb{N}} S_m\right) \leq \sum_{m\in\mathbb{N}} \mathrm{m}(S_m), \qquad (5.1.2.35)$$

which means that the measure of at least some S_m is strictly positive. ■

Proposition 5.1.2.36 Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and let $S \subseteq X$. Then, if m is inner-regular for S, then

(i).

$$\mathrm{m}(S) = \sup\left\{\mathrm{m}(K) : K \subseteq S,\ K \text{ compact.}\right\}; \qquad (5.1.2.37)$$

and

(ii).

$$S = \bigcup_{m\in\mathbb{N}} K_m \cup Z, \qquad (5.1.2.38)$$

where $K_0 \subseteq K_1 \subseteq \cdots$ is a nondecreasing countable collection of compact sets and $\mathrm{m}(Z) = 0$.

R To clarify, when we say that "m is inner-regular for S", we mean that

$$\mathrm{m}(S) = \sup\left\{\mathrm{m}(K) : K \subseteq S,\ K \text{ quasicompact.}\right\}.$$

Thus, the problem is to show you can can approximate S not just with quasicompact sets, but in fact actual compact sets.

R The second part says that, 'modulo a set of measure 0', S is a (countable nondecreasing) union of compact sets.

R In particular, in topological measure spaces, this works for S open (Definition 5.1.2.2) and for S measurable (Proposition 5.1.2.26).

Proof. Suppose that m is inner-regular for S.

(i) Define

$$\mathcal{S} := \{\mathrm{m}(K) : K \subseteq S,\ K \text{ compact.}\}\,. \tag{5.1.2.39}$$

Certainly $\mathrm{m}(S)$ is an upper-bound for $\mathcal{S}$ because m is nondecreasing. To show that it is the *least* upper-bound, let $\varepsilon > 0$. By hypothesis, there is then a *quasi*compact subset $K \subseteq S$ with $\mathrm{m}(S) - \varepsilon < \mathrm{m}(K) \le \mathrm{m}(S)$. Write $X = \bigcup_{m\in\mathbb{N}} L_m$ as a nondecreasing union of compact subsets $L_m \subseteq X$ and define $K_m := K \cap L_m$, so that $K = \bigcup_{m\in\mathbb{N}} K_m$, and hence $\mathrm{m}(K) = \lim_m \mathrm{m}(K_m)$. Thus, there is some $m_0 \in \mathbb{N}$ such that, whenever $m \ge m_0$, it follows that $|\mathrm{m}(K_m) - \mathrm{m}(K)| < \varepsilon$. From this, we have

$$\mathrm{m}(S) - 2\varepsilon < \mathrm{m}(K) - \varepsilon < \mathrm{m}(K_m) \le \mathrm{m}(S), \tag{5.1.2.40}$$

and hence $\mathrm{m}(S) = \sup(\mathcal{S})$, as desired.

(ii) Using the result of the previous part, let $K_0 \subseteq S$ be compact and such that $\mathrm{m}(S) - 2^{-0} < \mathrm{m}(K_0)$. Similarly, let $K_1 \subseteq S \setminus K_0$ be compact and such that $\mathrm{m}(S \setminus K_0) - 2^{-1} < \mathrm{m}(K_1)$. Similarly, let $K_2 \subseteq S \setminus (K_0 \cup K_1)$ be compact and such that $\mathrm{m}(S \setminus (K_1 \cup K_2) - 2^{-2} < \mathrm{m}(K_2)$. Proceeding inductively, for every $m \in \mathbb{N}$, choose $K_m \subseteq S \setminus \bigcup_{k=0}^{m-1} K_k$ compact and such that

$$\mathrm{m}\left(S \setminus \bigcup_{k=0}^{m-1} K_k\right) - 2^{-m} < \mathrm{m}(K_m). \tag{5.1.2.41}$$

Define $L_m := \bigcup_{k=0}^{m} K_m$, $L := \bigcup_{m\in\mathbb{N}} L_m = \bigcup_{m\in\mathbb{N}} K_m$, and $Z := S \setminus L$. Then, $L_0 \subseteq L_1 \subseteq \cdots$ is a nondecreasing countable collection of compact sets and

$$S = \bigcup_{m\in\mathbb{N}} L_m \cup Z, \tag{5.1.2.42}$$

and so it only remains to show that $\mathrm{m}(Z) = 0$ However,

$$\mathrm{m}(Z) := \mathrm{m}\left(S \setminus \bigcup_{m\in\mathbb{N}}\right) \leq \mathrm{m}\left(S \setminus \bigcup_{k=0}^{m} K_k\right) < 2^{-m}. \quad (5.1.2.43)$$

As this holds for every $m \in \mathbb{N}$, we have $\mathrm{m}(Z) = 0$, as desired. ■

In a topological measure space, we can approximate *every* set from the outside by open sets, and we can approximate open sets from the inside by quasicompact sets. One would then hope that every set is measurable 'modulo a set of measure 0'. Unfortunately, this need not be the case.

Example 5.1.2.44 — A topological measure space in which not every subset is measurable 'modulo a set of measure 0' Precisely, what we mean by "measurable 'modulo a set of measure 0'" is as follows.

Let $S \subseteq X$. Then, there is a measurable set $M \subseteq X$ such that $\mathrm{m}(S \cap M^{\mathsf{C}}) + \mathrm{m}(S^{\mathsf{C}} \cap M) = 0$.[a]

[a] In the 'Venn diagram' of S and M, $S \cap M^{\mathsf{C}} \cup S^{\mathsf{C}} \cap M$ is everything outside of $S \cap M$. We are saying that, in particular, this set has measure 0. This is what we mean when we say that *every* set is measurable 'modulo' a set of measure 0.

We present an example of a topological measure space X in which this is *false*.

Define $X := \{x_1, x_2\}$, equip X with the indiscrete topology, and let m be the Unit Measure on X. X is a finite set, and so certainly σ-compact (it is the union of its points, and points are always compact spaces in the subspace topology). The only open sets are $\emptyset$ and X, which are always measurable, and so the measure is Borel. Every subset has finite measure, and so in particular quasicompact sets have finite measure.

Let $S \subseteq X$. If $S = \emptyset$, then as $\emptyset$ is itself open, we certainly have $\mathrm{m}(S) = \inf\{\mathrm{m}(U) : S \subseteq U,\ U \text{ open.}\}$. Otherwise, the only open set containing S is X itself, in which case we have $1 = \mathrm{m}(S) = \inf\{\mathrm{m}(U) : S \subseteq U,\ U \text{ open.}\}$ as well. Thus, m is outer-regular.

A similar check shows that it is inner-regular on open sets (there are only two cases to check of course, $U = \emptyset$ and $U = X$).

Thus, $\langle X, \mathrm{m}\rangle$ is a topological measure space.

On the other hand, the only measurable subsets are $\emptyset$ and X itself, and for either $M = \emptyset$ or $M = X$, we have

$$\mathrm{m}(\{x_1\} \cap M^{\mathsf{C}}) + \mathrm{m}(\{x_1\}^{\mathsf{C}} \cap M) = 1. \tag{5.1.2.45}$$

Thus, it is not the case that $\{x_1\}$ is measurable 'up to sets of measure 0'.

One likewise might expect that nonempty open sets have positive measure in topological measure spaces. This is not true.

Example 5.1.2.46 — A measure on $\mathbb{R}$ for which $\langle\mathbb{R}, \mathrm{m}\rangle$ is a topological measure space, but is not strictly-positive—the Dirac Measure [a] Define $\mathrm{m} : 2^{\mathbb{R}} \to [0,\infty]$ by

$$\mathrm{m}(S) := \begin{cases} 0 & \text{if } 0 \notin S \\ 1 & \text{if } 0 \in S. \end{cases} \tag{5.1.2.47}$$

$\mathrm{m}((0,1)) = 0$ for example, so this is definitely not strictly-positive. What we need to check is that it is a topological measure.

We first check that it is regular. m itself is finite, and so certainly finite on quasicompact sets. Let $U \subseteq \mathbb{R}$ be open. If $0 \in U$, then as $\{0\}$ is quasicompact, we have that

$$\begin{aligned} \mathrm{m}(U) &= 1 \\ &= \sup\{\mathrm{m}(K) : K \subseteq U,\ K \text{ quasicompact}\}. \end{aligned} \tag{5.1.2.48}$$

On the other hand, if $0 \notin U$, then no subset of U will contain 0, and so once again we have

$$\begin{aligned} \mathrm{m}(U) &= 0 \\ &= \sup\{\mathrm{m}(K) : K \subseteq U,\ K \text{ quasicompact}\}. \end{aligned} \tag{5.1.2.49}$$

Thus, m is inner-regular on opens. Now let $S \subseteq \mathbb{R}$ be arbitrary. If $0 \in S$, then every open set which contains S will also contain 0, and so we definitely have

$$\mathrm{m}(S) = 1 = \inf\{\mathrm{m}(U) : S \subseteq U\, U \text{ open}\}. \tag{5.1.2.50}$$

On the other hand, if S does not contain 0, then $\{0\}^{\mathsf{C}}$ is an open set containing S, and so

$$\mathrm{m}(S) = 0 = \inf\{\mathrm{m}(U) : S \subseteq U\, U \text{ open}\}. \tag{5.1.2.51}$$

Thus, m is outer-regular, and hence regular.

We now check that m is Borel. To show this, obviously it suffices to show that every subset of $\mathbb{R}$ is measurable with respect to m.

So, let $M, S \subseteq \mathbb{R}$. We would like to show that M is measurable, and so we need to show that

$$\mathrm{m}(S) = \mathrm{m}(S \cap M) + \mathrm{m}(S \cap M^{\mathsf{C}}). \tag{5.1.2.52}$$

There are two cases: either $0 \in M$ or $0 \notin M$. By $M \leftrightarrow M^{\mathsf{C}}$, we may as well assume that $0 \in M$. Thus, we need to show that

$$\mathrm{m}(S) = \mathrm{m}(S \cap M). \tag{5.1.2.53}$$

There are two cases: $0 \in S$ or $0 \notin S$. In the former case, this equation reads $1 = 1$, an in the later case it reads $0 = 0$. Either way, it is true, and so M is measurable.

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[a] ***Strictly-positive*** means just that—that every nonempty open set has positive measure. The term here presumably comes from the Dirac Delta Function.

Before moving onto discussion of measures on uniform spaces, we must first at least discuss the property of *topological-additivity*.

Definition 5.1.2.54 — Topological-additivity Let X be a topological space and let m be a measure on X. Then, m is ***topologically-additive*** iff whenever $S, T \subseteq X$ are separated by neighborhoods, it follows that $\mathrm{m}(S \cup T) = \mathrm{m}(S) + \mathrm{m}(T)$.

R Note that the Unit Measure on a two point space with the discrete topology (see Example 5.1.2.4) is an example of a regular measure that is not topologically-additive. The point is, we do not get topological-additivity for free. On the other hand, note that Borel measures are automatically topologically-additive—see Exercise 5.1.2.55. In particular, topological measure spaces are topologically-additive.

Exercise 5.1.2.55 Let m be a Borel measure on X. Show that if $S, T \subseteq X$ are topologically-distinguishable, then $\mathrm{m}(S \cup T) = \mathrm{m}(S) + \mathrm{m}(T)$.

R In particular, Borel measure are topologically-additive. In fact, the converse is true for regular measures on T_2 spaces—see the next result.

Proposition 5.1.2.56 Let m be a regular topologically-additive measure on a T_2 space. Then, m is Borel.

R Thus, by the previous exercise, for T_2 spaces, Borel is equivalent to topological additivity.

W Warning: This will fail if the space is not T_2—see Example 5.1.2.69.

Proof. [a] Let $U \subseteq X$ be open and let $A \subseteq X$ be arbitrary. We wish to show that

$$\mathrm{m}(A) \geq \mathrm{m}(A \cap U) + \mathrm{m}(A \cap U^{\mathsf{C}}). \tag{5.1.2.57}$$

If $\mathrm{m}(A) = \infty$, this is automatically satisfied, so we may as well assume that $\mathrm{m}(A) < \infty$.

Let $\varepsilon > 0$. Then, by outer-regularity, there is some open set U_ε that contains A and

$$\mathrm{m}(A) \leq \mathrm{m}(U_\varepsilon) < \mathrm{m}(A) + \varepsilon. \tag{5.1.2.58}$$

Then, by inner-regularity on opens, there is some quasicompact $K_\varepsilon \subseteq U_\varepsilon \cap U$ such that

$$\mathrm{m}(U \cap U_\varepsilon) - \varepsilon < \mathrm{m}(K_\varepsilon) \leq \mathrm{m}(U \cap U_\varepsilon). \tag{5.1.2.59}$$

As X is T_2, K_ε is closed, and so that $U_\varepsilon \cap K_\varepsilon^{\mathsf{C}}$ is open, and so there is some compact $L_\varepsilon \subseteq U_\varepsilon \cap K_\varepsilon^{\mathsf{C}}$ such that

$$\mathrm{m}(U_\varepsilon \cap K_\varepsilon^{\mathsf{C}}) - \varepsilon < \mathrm{m}(L_\varepsilon) \leq \mathrm{m}(U_\varepsilon \cap K_\varepsilon^{\mathsf{C}}). \tag{5.1.2.60}$$

Hence,

$$\begin{aligned} \mathrm{m}(A) &> \mathrm{m}(U_\varepsilon) - \varepsilon \geq^{b} \mathrm{m}(K_\varepsilon \cup L_\varepsilon) - \varepsilon \\ &=^{c} \mathrm{m}(K_\varepsilon) + \mathrm{m}(L_\varepsilon) - \varepsilon \\ &> \mathrm{m}(U \cap U_\varepsilon) + \mathrm{m}(U_\varepsilon \cap K_\varepsilon^{\mathsf{C}}) - 3\varepsilon \\ &\geq^{d} \mathrm{m}(U \cap U_\varepsilon) + \mathrm{m}(U_\varepsilon \cap U^{\mathsf{C}}) - 3\varepsilon \\ &\geq^{e} \mathrm{m}(A \cap U) + \mathrm{m}(A \cap U^{\mathsf{C}}) - 3\varepsilon. \end{aligned} \tag{5.1.2.61}$$

As ε is arbitrary, we have that $\mathrm{m}(A) \geq \mathrm{m}(A \cap U) + \mathrm{m}(A \cap U^{\mathsf{C}})$, and so U is measurable. ■

[a] Proof adapted from [Coh13, pg. 194].

[b] Because $K_\varepsilon \cup L_\varepsilon \subseteq U_\varepsilon$.

[c] You can separate by neighborhoods disjoint compact subsets of T_2 spaces (Exercise 3.6.2.21). Then we apply the fact that m is topologically-additive.

[d] Because $U_\varepsilon \cap U^{\mathsf{C}} \subseteq U_\varepsilon \cap K_\varepsilon^{\mathsf{C}}$.

[e] Because $A \subseteq U_\varepsilon$.

Measures on uniform spaces

Of course, there is an analogue of topological-additivity (Definition 5.1.2.54) for uniform spaces.

Definition 5.1.2.62 — Uniform-additivity Let X be a uniform space and let m be a measure on X. Then, m is ***uniformly-additive*** iff whenever $S, T \subseteq X$ are uniformly-separated by neighborhoods, it follows that $\mathrm{m}(S \cup T) = \mathrm{m}(S) + \mathrm{m}(T)$.

W Warning: The term "uniform" most of the time is something strictly stronger than something only topological. This is not the case here: topological-additivity is superficially stronger than uniform-additivity because it is easier to be separated by neighborhoods than it is to be uniformly-separated by neighborhoods. In particular, topologically-additive measures on uniform spaces are automatically uniformly-additive.

R The second condition is a generalization of the defining condition of what is called a ***metric measure***. In particular, if X is a metric space, then any uniform measure is (by definition) a metric measure.

Exercise 5.1.2.63 Can you find an example of a regular measure m on a uniform space X with $S, T \subseteq X$ uniformly-separated by neighborhoods, but $\mathrm{m}(S \cup T) \neq \mathrm{m}(S) + \mathrm{m}(T)$.

R The point is: do we really need to assume uniform-additivity, or can we get it for free?

R Hint: We have already encountered a counter-example that will do the trick.

Proposition 5.1.2.64 Let m be a regular measure on a T_0 uniform space. Then, the following are equivalent.

(i). m is topologically-additive.

(ii). m is uniformly-additive.
(iii). m is Borel.

Proof. Let X be the T_0 uniform space that m is a regular measure on.

((i) $\Leftrightarrow$ (iii)) T_0 uniform spaces are T_2 (Corollary 4.3.2.15). Therefore, by Exercise 5.1.2.55 and Proposition 5.1.2.56, topological-additivity is equivalent to being Borel.

((i) $\Rightarrow$ (ii)) Suppose that m is topologically-additive. Let $S, T \subseteq X$ be uniformly-separated by neighborhoods. Then, S and T are separated by neighborhoods, and so $\mathrm{m}(S \cup T) = \mathrm{m}(S) + \mathrm{m}(T)$. Thus, m is uniformly-additive.

((ii) $\Rightarrow$ (i)) Suppose that m is uniformly-additive. Let $S, T \subseteq X$ be separated by neighborhoods. By definition, we want to show that $\mathrm{m}(S \cup T) = \mathrm{m}(S) + \mathrm{m}(T)$. By subadditivity, it suffices to show that $\mathrm{m}(S \cup T) \geq \mathrm{m}(S) + \mathrm{m}(T)$. If either one of the sets has infinite measure, then this inequality reads $\infty \geq \infty$, and so is automatically satisfied. Thus, we may as well assume that $\mathrm{m}(S)$ and $\mathrm{m}(T)$ are finite. Then, by outer-regularity, for every $\varepsilon > 0$, there is an open set W_ε with $S \cup T \subseteq W_\varepsilon$ and

$$\mathrm{m}(S \cup T) \leq \mathrm{m}(W_\varepsilon) < \mathrm{m}(S \cup T) + \varepsilon. \tag{5.1.2.65}$$

On the other hand, because S and T are separated by neighborhoods, we know that there are disjoint open sets U and V with $S \subseteq U$ and $T \subseteq V$. Define $U_\varepsilon := U \cap W_\varepsilon$ and $V_\varepsilon := V \cap W_\varepsilon$, so that U_ε and V_ε are both disjoint and open. By inner-regularity on open sets, there are quasicompact $K_\varepsilon \subseteq U_\varepsilon$ and $L_\varepsilon \subseteq V_\varepsilon$ such that

$$\mathrm{m}(U_\varepsilon) - \varepsilon < \mathrm{m}(K_\varepsilon) \leq \mathrm{m}(U_\varepsilon) \tag{5.1.2.66}$$

and

$$\mathrm{m}(V_\varepsilon) - \varepsilon < \mathrm{m}(L_\varepsilon) \leq \mathrm{m}(V_\varepsilon). \tag{5.1.2.67}$$

As X is T_2, K_ε and L_ε are closed (Exercise 3.6.2.22), and so as X is uniformly-T_3 (Corollary 4.3.2.15), K_ε and L_ε are uniformly-separated by neighborhoods. Thus, by hypothesis, we have $\mathrm{m}(K_\varepsilon \cup L_\varepsilon) = \mathrm{m}(K_\varepsilon) + \mathrm{m}(L_\varepsilon)$.

Now,

$$\begin{aligned} \mathrm{m}(S) + \mathrm{m}(T) &\leq \mathrm{m}(U_\varepsilon) + \mathrm{m}(V_\varepsilon) \\ &< \mathrm{m}(K_\varepsilon) + \mathrm{m}(L_\varepsilon) + 2\varepsilon \\ &= \mathrm{m}(K_\varepsilon \cup L_\varepsilon) + 2\varepsilon \\ &\leq \mathrm{m}(U_\varepsilon \cup V_\varepsilon) + 2\varepsilon \\ &\leq \mathrm{m}(W_\varepsilon) + 2\varepsilon < \mathrm{m}(S \cup T) + 2\varepsilon. \end{aligned} \tag{5.1.2.68}$$

As ε is arbitrary, it follows that $\mathrm{m}(S) + \mathrm{m}(T) \leq \mathrm{m}(S \cup T)$, as desired. ■

▪ Example 5.1.2.69 — A topologically-additive regular measure on a uniform space that is not Borel [a]

R The point is, you cannot drop the hypothesis of T_0 in Proposition 5.1.2.64.

Define $X := \{x_1, x_2, x_3\}$, and equip it the uniformity defined by the uniform base with just one cover, $\mathcal{B} := \{\{x_1, x_2\}, \{x_3\}\}$. (This is a uniform base because this single cover star-refines itself.) The neighborhood bases defined by this uniform base are

$$\begin{aligned} \mathcal{B}_{x_1} &= \{\mathrm{Star}_{\mathcal{B}}(x_1)\} = \{\{x_1, x_2\}\} \\ \mathcal{B}_{x_2} &= \{\mathrm{Star}_{\mathcal{B}}(x_2)\} = \{\{x_1, x_2\}\} \\ \mathcal{B}_{x_3} &= \{\mathrm{Star}_{\mathcal{B}}(x_3)\} = \{\{x_3\}\}. \end{aligned} \tag{5.1.2.70}$$

In particular, the open sets are[b]

$$\emptyset, X, \{x_1, x_2\}, \{x_3\}. \tag{5.1.2.71}$$

We define a measure $\mathrm{m} : 2^X \to [0, \infty]$ by

$$\mathrm{m}(S) := \begin{cases} 0 & \text{if } S = \emptyset, \{x_3\} \\ 1 & \text{otherwise.} \end{cases} \tag{5.1.2.72}$$

Every set has finite measure, and so certainly every quasicompact set does. There are only four open sets to check, and they are all quasicompact (because they are finite), and so certainly m is inner-regular on each one of them. Measures always are outer-regular on open sets, and so we only need to check outer-regularity on the remaining $8 - 4 =$ sets that are not open. Once again, there are only four things to check, and so you can just do so by hand.

We now check that this measure is uniformly-additive. If S and T are uniformly-separated by neighborhoods, then, without loss of generality, we have that $S \subseteq \{x_1, x_2\}$ and $T \subseteq \{x_3\}$. We automatically have that $\mathrm{m}(S \cup T) = \mathrm{m}(S) + \mathrm{m}(T)$ if either S or T is empty, so we may without loss of generality assume that $T = \{x_3\}$ and $S \subseteq \{x_1, x_2\}$ is nonempty. Then,

$$\mathrm{m}(S \cup T) = 1 = 1 + 0 = \mathrm{m}(S) + \mathrm{m}(T), \qquad (5.1.2.73)$$

and so the measure is uniformly-additive.

On the other hand, it is not Borel, because $U := \{x_1, x_2\}$ is not measurable, as we now check. Take $S := \{x_1\}$. Then,

$$\mathrm{m}(S) = 1 \neq 1 + 1 = \mathrm{m}(S \cap U) + \mathrm{m}(S \cap U^{\mathsf{C}}). \quad (5.1.2.74)$$

[a]Topologically-additive measures on uniform spaces are automatically uniformly-additive—see the remark in Definition 5.1.2.62.

[b]Recall that (Proposition 3.1.1.8), for a topology defined by a neighborhood base, a set U is open iff every point $x \in U$ has an element $B \in \mathcal{B}_x$ such that $x \in B \subseteq U$. As our set only contains 3 points, you can simply verify by hand that these are the only open sets.

A summary

Before finally moving on to the Haar-Howes Theorem, let us briefly summarize.

(i). Borel means that open sets are measurable—see Definition 5.1.2.1.

(ii). Regular means quasicompact sets have finite measure, inner-regularity on opens, and outer-regularity on everything—see Definition 5.1.2.2.

(iii). A topological space is σ-(quasi)compact iff it is the countable union of (quasi)compact sets—see Definition 5.1.2.7.

(iv). A topological measure space is a σ-compact topological space equipped with a regular Borel measure—see Definition 5.1.2.11.

(v). A subset M of a topological measure space is measurable iff for every $\varepsilon > 0$ there is an open set U_ε containing M and a closed set C_ε contained in M such that $\mathrm{m}(U_\varepsilon \setminus C_\varepsilon) < \varepsilon$.

(vi). Topological-additivity means that the measure is additive on two sets which are separated by neighborhoods—see Definition 5.1.2.54.

(vii). Uniform-additivity means that the measure is additive on two sets which are uniformly-separated by neighborhoods—see Definition 5.1.2.62.

(viii). Topological-additivity and uniform-additivity are both equivalent to being Borel for T_2 spaces—see Proposition 5.1.2.64.

(ix). On the other hand, neither regular topologically-additive nor regular uniformly-additive measures need be Borel (Example 5.1.2.69).

5.1.3 The Haar-Howes Theorem

It turns out that there is a theorem, the Haar-Howes Theorem,[1] that is quite general and will just spit out regular uniformly-additive for us. This is how we will construct Lebesgue measure. That being said, it doesn't just work for any old uniform space. We're going to need extra structure:

Definition 5.1.3.1 — Isogeneous space An ***isogeneous space*** is a uniform space X equipped with a group of uniform-

[1]Warning: This is the name I have chosen to call it, because I am unaware of another name for this result. In particular, don't expect others to know what you're talking about if you reference this theorem by name. (It is a generalization of the existence and essential uniqueness of haar measure, in case you are more familiar with that.) While the proof of the result I cobbled together from other sources, the formulation of the result I found essentially in [How91], in which he claims "the author" established essential uniqueness, as well as existence, which he and another mathematician (Izkowitz) established independently.

homeomorphisms Φ such that

$$\widetilde{\mathscr{B}}_{\Phi} := \{\mathcal{B}_U\} \tag{5.1.3.2}$$

where

$$\mathcal{B}_U := \{\phi(U) : \phi \in \Phi\} \text{ and } U \subseteq X \text{ nonempty open.}$$

is a uniform base for X.

R That is to say, for every open subset $U \subseteq X$, you obtain a single uniform cover in $\widetilde{\mathscr{B}}_{\Phi}$, and that single cover is given by 'translating' U around using Φ.

R Φ is called the ***group of symmetries*** of X. It is a subgroup of $\mathrm{Aut}_{\mathbf{Uni}}(X)$. $\widetilde{\mathscr{B}}_{\Phi}$ is the ***isogeneous base*** and each $\mathcal{B}_U$ is an ***isogeneous cover***.

R The example you should have in mind here is that of a topological group G. In this case, take Φ to be the set of all uniform-homeomorphisms given by left multiplication $\Phi := \{L_g : g \in G\}$.[a] Then, the corresponding isogeneous base is the canonical one, $\widetilde{\mathscr{B}} := \{\mathcal{B}_U\}$ with $\mathcal{B}_U := \{gU : g \in U\}$ for U an open neighborhood of the identity.[b] Of course, you can also choose right-translations over left-translations if you so desire.

R The example of metric spaces with Φ the group of isometries should also guide your intuition—see Exercise 5.1.3.6.

R What do you think the morphisms of isogeneous spaces should be?

[a] $L_g : G \to G$ is defined by $L_g(x) := gx$.

[b] What happened to the uniform covers for $U \subseteq G$ open but not necessarily containing the identity?

Exercise 5.1.3.3 Let X be a uniform space with uniform topology $\mathcal{U}$, let $\mathcal{B}$ be a base for the topology, and let Φ be a subgroup of $\mathrm{Aut}_{\mathbf{Uni}}(X)$. Show that

$$\{\mathcal{B}_B : B \in \mathcal{B}\} \tag{5.1.3.4}$$

is a uniform base iff $\widetilde{\mathscr{B}}_\Phi$ is, and that these two uniform bases both define the same uniformity.

> R The point is that, in order to show that Φ makes X into an isogeneous space, we only need to check that the smaller collection of covers in (5.1.3.4), the covers coming from only elements of the base instead of *all* open sets, form a uniform base.

Exercise 5.1.3.5 Let G be a topological group and let $\Phi := \{\phi_g : G \in G\}$, where $\phi_g : G \to G$ is defined by $\phi_g(x) := gx$. Show that $\langle G, \Phi\rangle$ is an isogeneous space.

Exercise 5.1.3.6 Let $\langle X, |\cdot,\cdot|\rangle$ be a metric space and let $\Phi \subseteq \mathrm{Aut}_{\mathbf{Uni}}(X)$ be the group of isometries. Show that $\langle X, \Phi\rangle$ is an isogeneous space.

> R A function $f : \langle X, |\cdot,\cdot|\rangle \to \langle Y, |\cdot,\cdot|\rangle$ between metric spaces is an ***isometry*** iff it is an isomorphism in the category of metric spaces. Concretely, this means that it is bijective[a] and satisfies $|f(x_1), f(x_2)| = |x_1, x_2|$ for all $x_1, x_2 \in X$.

> R Hint: Apply Exercise 5.1.3.3 to the base consisting of all ε-balls.

[a] This condition implies that f must be injective, but it need not be surjective.

Definition 5.1.3.7 — Uniformly-measurable Let $\mathrm{m}\colon 2^X \to [0,\infty]$ be a measure on a set X and let $\mathcal{U}$ is a cover of X. Then, $\mathcal{U}$ is ***uniformly-measurable*** iff m is constant on $\mathcal{U}$. A uniform base consisting of uniformly-measurable covers is a ***uniformly-measurable base***.

R Think about what having a uniform base of uniformly-measurable covers means for a metric space—if we take as a uniform base the collection of all covers by ε-balls, then the statement that this uniform base is uniformly-measurable[a] is just the statement that every ε-ball has to have the same measure.

R Note that you definitely do not want to require *every* uniform cover be uniformly-measurable. For example, in a metric space, by upward-closedness the collection of all ε-balls together with a single 2ε-ball will also be a uniform-cover—we definitely do not want to require that a 2ε-ball has the same measure as an ε-ball.

[a] It doesn't have to be of course—it will depend on our choice of measure.

Definition 5.1.3.8 — Isogeneous measure Let $\langle X,\Phi\rangle$ be an isogeneous space. An ***isogeneous measure*** is a uniformly-additive regular measure for which $\widetilde{\mathscr{B}}_\Phi$ is a uniformly-measurable base.

R Explicitly, this means that

(i). m is uniformly-additive;
(ii). m is regular; and
(iii). $\mathrm{m}(\phi(U)) = \mathrm{m}(U)$ for $\phi\in\Phi$ and $U\subseteq X$ open.

Exercise 5.1.3.9 Let m be an isogeneous measure on an isogeneous space $\langle X,\Phi\rangle$. Show that $\mathrm{m}(\phi(S)) = \mathrm{m}(S)$ for all $\phi\in\Phi$ and $S\subseteq X$.

R In other words, this holds for *all* S, not just open S.

Exercise 5.1.3.10 Let m be an isogeneous measure on an isogeneous space $\langle X, \Phi \rangle$, let $M \subseteq X$ be measurable, and let $\phi \in \Phi$. Show that $\phi(M)$ is measurable.

R In other words, the symmetries of the isogeneous space preserve measurability (in particular, for Lebesgue measure, we will have that isometries of $\mathbb{R}^d$ preserve measurability—see Definition 5.1.3.50 (the definition of Lebesgue measure)).

And finally now the key result that will allow us to define basically every measure we work with in these notes (and much more).

Theorem 5.1.3.11 — Haar-Howes Theorem. Let $\langle X, \Phi \rangle$ be a locally compact[a] isogeneous space and let $K \subseteq X$ be quasicompact with nonempty interior. Then, there exists a unique isogeneous measure m on X such that $\mathrm{m}(K) = 1$.

R You should think of K has a set with which we can 'compare' all other sets to get a "measure" of 'size'. The condition that it be quasicompact you can think of the condition that the measure of K be finite, and the condition that it have nonempty interior you can think of the condition that the measure of K be positive. If $\mathrm{m}(K)$ is neither infinite nor zero, then we can 'normalize' to get $\mathrm{m}(K) = 1$.

R Classically, the term "Haar measure" is reserved for G a T_0 locally quasicompact group with Φ the set of all left-translations. In particular, *Lebesgue measure* will be the Haar measure for the topological group $\langle \mathbb{R},^d +, 0, - \rangle$.[b]

R Yes, there are σ-compact spaces which are not locally quasicompact and vice-versa—see Exercise 5.1.4.57.

[a] Equivalently, T_0 and locally quasicompact. Locally compact implies locally T_2 implies T_1 implies T_0. Conversely, T_0 implies uniformly-completely-T_3 implies T_2 and subspaces of T_2 spaces are T_2.

[b] If the group is commutative, the symmetries by left translation and right translations are the same, so left vs. right does not matter.

Proof. STEP 1: MAKE HYPOTHESES AND INTRODUCE NOTATION
Let $K_0 \subseteq X$ be a quasicompact subset with nonempty interior. Denote the uniform topology on X by $\mathscr{U}$ and denote the collection of all quasicompact subsets of X by $\mathscr{K}$

STEP 2: DEFINE $(K : U)$ FOR $K \in \mathscr{K}$ AND $U \in \mathscr{U}$
The cover $\mathscr{B}_U := \{\phi(U) : h \in \Phi\}$ is an open cover of K, and so there is a finite subcover. Let $(K : U)$ denote the cardinality of the smallest such subcover.

STEP 3: DEFINE $\mathrm{H}_U : \mathscr{K} \to \mathbb{R}_0^+$
For $U \in \mathscr{U}$, define $\mathrm{H}_U : \mathscr{K} \to \mathbb{R}_0^+$ by

$$\mathrm{H}_U(K) := \frac{(K : U)}{(K_0 : U)}.^{a} \tag{5.1.3.12}$$

STEP 4: SHOW THAT $\mathrm{H}_U(K) \leq (K : \mathrm{Int}(K_0))$
We now check that $\mathrm{H}_U(K) \leq (K : \mathrm{Int}(K_0))$, that is, $(K : U) \leq (K : \mathrm{Int}(K_0))(K_0 : U)$. Let us temporarily write $m := (K : \mathrm{Int}(K_0))$ and $n := (K_0 : U)$. There are thus $\phi_1, \ldots, \phi_m \in \Phi$ such that $\{\phi_1(\mathrm{Int}(K_0)), \ldots, \phi_m(\mathrm{Int}(K_0))\}$ covers K, and $\phi_1', \ldots, \phi_n' \in \Phi$ such that $\{\phi_1'(U), \ldots, \phi_n'(U)\}$ covers K_0. Therefore,

$$\begin{aligned} K &\subseteq \bigcup_{k=1}^{m} \phi_k(\mathrm{Int}(K_0)) \subseteq \bigcup_{k=1}^{m} \phi_k(K_0) \\ &\subseteq \bigcup_{k=1}^{m} \phi_k\left(\bigcup_{l=1}^{n} \phi_l'(U)\right) = \bigcup_{k=1}^{m}\bigcup_{l=1}^{n} [\phi_k \circ \phi_l'](U) \end{aligned} \tag{5.1.3.13}$$

Hence, K is covered by mn elements of $\mathcal{B}_U$, and hence $(K : U) \leq mn := (K : \mathrm{Int}(K_0))(K_0 : U)$.

STEP 5: DEFINE $\mathrm{H} : \mathcal{K} \to \mathbb{R}_0^+$

Define $\mathcal{H} := \prod_{K\in\mathcal{K}}[0,(K : \operatorname{Int}(K_0))]$. Each H_U may be thought of as a point in $\mathcal{H}$, whose component at $K \in \mathcal{K}$ is $\mathrm{H}_U(K) \in [0,(K : K_0)]$.[b] Thus, for $U \in \mathcal{U}$, let us define

$$C_U := \operatorname{Cls}(\{\mathrm{H}_V : \mathscr{U} \ni V \subseteq U\}) \tag{5.1.3.14}$$

and

$$\mathscr{C} := \{C_U : U \in \mathscr{U}\}. \tag{5.1.3.15}$$

We wish to show that the intersection of any finitely many elements of $\mathscr{C}$ is nonempty. Then, because $\mathcal{H}$ is quasicompact by Tychonoff's Theorem (Theorem 3.5.3.14), it will follow that the intersection over *all* elements in $\mathscr{C}$ will be nonempty (Proposition 3.2.57).

This is actually really easy, however, because for $U_1,\ldots,U_m \in \mathscr{U}$, we have that

$$\mathrm{H}_{U_1\cap\cdots\cap U_m} \in \bigcap_{k=1}^m C_{U_k}. \tag{5.1.3.16}$$

Therefore, by quasicompactness, there is some

$$\mathrm{H} \in \bigcap_{U\in\mathscr{U}} C_U. \tag{5.1.3.17}$$

Though H is an element of the product $\mathcal{H} := \prod_{K\in\mathscr{K}}[0,(K,\operatorname{Int}(K_0))]$, we can regard H as a function $\mathrm{H}\colon \mathscr{K} \to \mathbb{R}_0^+$ by defining $\mathrm{H}(K) := \mathrm{H}_K$, that is, the value of H at $K \in \mathscr{K}$ is the K-component of $\mathrm{H} \in \prod_{K\in\mathscr{K}}[0,(K,\operatorname{Int}(K_0))]$.

Step 6: Show that $\mathrm{H}(K_1) \leq \mathrm{H}(K_2)$ if $K_1 \subseteq K_2$

Let $K_1, K_2 \in \mathscr{K}$ be such that $K_1 \subseteq K_2$. We first show that, for each $U \in \mathscr{U}$, $\mathrm{H}_U(K_1) \leq \mathrm{H}_U(K_2)$. But this is trivial, because the covering of K_2 with $(K_2 : U)$ elements of $\mathscr{B}_U$ is also a covering of K_1 with $(K_2 : U)$ elements of $\mathcal{B}_U$, so that $(K_1 : U) \leq (K_2 : U)$, and hence $\mathrm{H}_U(K_1) \leq \mathrm{H}_U(K_2)$.

Thinking of elements f of $\mathcal{H}$ as functions from $\mathscr{K}$ to $\mathbb{R}$, consider the map[c] that sends $f \in \mathcal{H}$ to $f(K_2) - f(K_1)$. This is a composition of continuous functions, and hence continuous.[d] This map is also nonnegative on each C_U because $\mathrm{H}_U(K_1) \leq \mathrm{H}_U(K_2)$ for each $U \in \mathcal{U}$ (we need continuity so that we know it is nonnegative on the *closure* of $\{\mathrm{H}_V : V \subseteq U\}$). As H is an element of each C_U, it follows that this map is also nonnegative at H, so that $\mathrm{H}(K_1) \leq \mathrm{H}(K_2)$.

Step 7: Show that $\mathrm{H}(K_1 \cup K_2) \leq \mathrm{H}(K_1) + \mathrm{H}(K_2)$

Let $K_1, K_2 \in \mathscr{K}$. We first show that $\mathrm{H}_U(K_1 \cup K_2) \leq \mathrm{H}_U(K_1) + \mathrm{H}_U(K_2)$ for each $U \in \mathscr{U}$. This is trivial, because a covering of K_1 with $(K_1 : U)$ elements of $\mathscr{B}_U$ together with a covering of K_2 with $(K_2 : U)$ elements of $\mathscr{B}_U$ is a cover of $K_1 \cup K_2$ with $(K_1 : U) + (K_2 : U)$ elements of $\mathscr{B}_U$, so that $(K_1 \cup K_2 : U) \leq (K_1 : U) + (K_2 : U)$. It follows that $\mathrm{H}_U(K_1 \cup K_2) \leq \mathrm{H}_U(K_1) + \mathrm{H}_U(K_2)$.

Proceeding similarly as in Step 6, the map that sends $f \in \mathcal{H}$ to $f(K_1) + f(K_2) - f(K_1 \cup K_2)$ is continuous and nonnegative on each C_U, and hence is nonnegative for $\mathrm{H} \in \mathcal{H}$. Thus, $\mathrm{H}(K_1 \cup K_2) \leq \mathrm{H}(K_1) + \mathrm{H}(K_2)$.

Step 8: Show that $\mathrm{H}_U(K_1 \cup K_2) = \mathrm{H}_U(K_1) + \mathrm{H}_U(K_2)$ if K_1 and K_2 are uniformly-separated by neighborhoods

Let $K_1, K_2 \in \mathcal{K}$ be uniformly-separated by neighborhoods with respect to $\mathscr{B}_U$. We have already shown that $\mathrm{H}_U(K_1 \cup K_2) \leq \mathrm{H}_U(K_1) + \mathrm{H}_U(K_2)$, so it suffices to show that $\mathrm{H}_U(K_1) + \mathrm{H}_U(K_2) \leq \mathrm{H}_U(K_1 \cup K_2)$. In other words, it suffices to show that $(K_1 : U) + (K_2 : U) \leq (K_1 \cup K_2 : U) =: m$. Let $\phi_1(U), \ldots, \phi_m(U) \in \mathscr{B}_U$ be a cover of $K_1 \cup K_2$. By hypothesis,[e] every single one of these can only intersect U_1 or U_2, but not both. Thus, after relabeling if necessary, the first k of these guys will form a cover of K_1 and the latter $m - k$ will form a cover of K_2. Thus, $(K_1 : U) \leq k$ and $(K_2 : U) \leq m - k$, and so $(K_1 : U) + (K_2 : U) \leq k + (m - k) = m := (K_1 \cup K_2 : U)$, which completes this step.

STEP 9: SHOW THAT $\mathrm{H}(K_1 \cup K_2) = \mathrm{H}(K_1) + \mathrm{H}(K_1)$ IF K_1 AND K_2 ARE UNIFORMLY-SEPARATED

Proceeding similarly as in Step 7, the map that sends $f \in \mathcal{H}$ to $f(K_1) + f(K_2) - f(K_1 \cup K_2)$ is continuous and vanishes on each C_U, and hence vanishes for $\mathrm{H} \in \mathcal{H}$. Thus, $\mathrm{H}(K_1 \cup K_2) = \mathrm{H}(K_1) + \mathrm{H}(K_2)$.

STEP 10: DEFINE THE MEASURE m ON ALL OPEN SUBSETS OF X

For $U \subseteq X$ open, define

$$\mathrm{m}(U) := \sup\{\mathrm{H}(K) : K \subseteq U,\ K \in \mathcal{K}\}. \tag{5.1.3.18}$$

STEP 11: EXTEND m TO ALL SUBSETS OF X

Now, for an arbitrary subsets S of X, define

$$\mathrm{m}(S) := \inf\{\mathrm{m}(U) : S \subseteq U,\ U \in \mathcal{U}\}. \tag{5.1.3.19}$$

Exercise 5.1.3.20 Show that this agrees with (5.1.3.18) when S is open, so that this is indeed an extension.

STEP 12: SHOW THAT m IS A MEASURE

Exercise 5.1.3.21 Check that $\mathrm{m}(\emptyset) = 0$ and that m is nondecreasing.

We now check that it is subadditive. To prove this, we will first need a lemma.

Lemma 5.1.3.22 Let X be locally compact, let $K \subseteq X$ be quasicompact, and let $U_1, U_2 \subseteq X$ be open and such that $K \subseteq U_1 \cup U_2$. Then, there are quasicompact subsets $K_1, K_2 \subseteq X$ such that (i) $K_1 \subseteq U_1$, (ii) $K_2 \subseteq U_2$, and (iii) $K = K_1 \cup K_2$.

R Note that nothing here is necessarily disjoint.

Proof. For each $x \in U_1$ and $y \in U_2$, let U_x and V_x be open neighborhoods of x and y contained in U_1 and U_2 respectively. By Proposition 3.7.6, there are open sets $U'_x \subseteq X$ and $V'_x \subseteq X$ with compact closure such that $x \in U'_x \subseteq \mathrm{Cls}(U'_x) \subseteq U_1$ and similarly for V'_x. Thus, without loss of generality, suppose that each U_x and V_x has compact closure and $\mathrm{Cls}(U_x) \subseteq U_1$ and $\mathrm{Cls}(V_y) \subseteq U_2$.

The U_xs and V_ys together cover $U_1 \cup U_2$, and so cover K, and so there are finitely many $x_1, \ldots, x_m \in U_1$ and $y_1, \ldots, y_n \in U_2$ such that

$$U_{x_1} \cup \cdots \cup U_{x_m} \cup V_{y_1} \cup \cdots \cup V_{y_n}. \tag{5.1.3.23}$$

Define

$$K_1 := \left(\mathrm{Cls}(U_{x_1}) \cup \cdots \cup \mathrm{Cls}(U_{x_m})\right) \cap K \tag{5.1.3.24a}$$
$$K_2 := \left(\mathrm{Cls}(V_{x_1}) \cup \cdots \cup \mathrm{Cls}(V_{x_n})\right) \cap K. \tag{5.1.3.24b}$$

These are both compact, $K_i \subseteq U_i$, and $K = K_1 \cup K_2$, as desired. ■

As X is a T_0 uniform space, it is in particular T_2, and so it is not just locally quasicompact, but in fact locally compact, so that we may indeed apply the lemma.

We now show subadditivity for *open* sets. (We will then prove subadditivity in general.) So, let $\{U_m : m \in \mathbb{N}\}$ be a countable collection of open sets of X. Let $K \subseteq \bigcup_{m\in\mathbb{N}} U_m$.

Then, there is some $m_K \in \mathbb{N}$ such that $K \subseteq \bigcup_{k=1}^{m_K} U_k$. By applying this lemma inductively then, we may find quasicompact sets $K_1, \ldots, K_m$ such that (i) $K_k \subseteq U_k$ for $0 \leq k \leq m$ and $K = \bigcup_{k=1}^{m} K_k$. Using the fact that we have already proved finite 'subadditivity' (of H) for quasicompact sets (Step 7), we find that

$$\mathrm{H}(K) \leq \sum_{k=1}^{m} \mathrm{H}(K_k) \leq \sum_{k=1}^{m} \mathrm{m}(U_k) \leq \sum_{m\in\mathbb{N}} \mathrm{m}(U_m). \quad (5.1.3.25)$$

Taking the sup over $K \in \mathscr{K}$ such that $K \subseteq \bigcup_{m\in\mathbb{N}} U_m$, we find that

$$\begin{aligned}\mathrm{m}\left(\bigcup_{m\in\mathbb{N}} U_m\right) \\ &:= \sup\left\{\mathrm{H}(K) : K \subseteq \bigcup_{m\in\mathbb{N}} U_m,\ K \in \mathscr{K}\right\} \quad (5.1.3.26)\\ &\leq \sum_{m\in\mathbb{N}} \mathrm{m}(U_m).\end{aligned}$$

Having proved subadditivity for open sets, we now prove it for arbitrary sets. So, let $\{S_m : m \in \mathbb{N}\}$ be an arbitrary countable collection of subsets of X. If $\sum_{m\in\mathbb{N}} \mathrm{m}(S_m) = \infty$, then there is nothing to show, and so we may as well suppose that $\sum_{m\in\mathbb{N}} \mathrm{m}(S_m) < \infty$. Let $\varepsilon > 0$ and for each $m \in \mathbb{N}$ pick an open set U_m such that (i) $S_m \subseteq U_m$ and (ii) $\mathrm{m}(S_m) \leq \mathrm{m}(U_m) < \mathrm{m}(S_m) + \frac{\varepsilon}{2^m}$. Then, using subadditivity for open sets, we find

$$\begin{aligned}\mathrm{m}\left(\bigcup_{m\in\mathbb{N}} S_m\right) &\leq \mathrm{m}\left(\bigcup_{m\in\mathbb{N}} U_m\right) \leq \sum_{m\in\mathbb{N}} \mathrm{m}(U_m)\\ &< \sum_{m\in\mathbb{N}} \left[\mathrm{m}(S_m) + \tfrac{\varepsilon}{2^m}\right] \quad (5.1.3.27)\\ &= \sum_{m\in\mathbb{N}} \mathrm{m}(S_m) + 2\varepsilon.\end{aligned}$$

Hence, as $\varepsilon > 0$ was arbitrary, we have that

$$\mathrm{m}\left(\bigcup_{m\in\mathbb{N}} S_m\right) \leq \sum_{M\in\mathbb{N}} \mathrm{m}(S_m). \tag{5.1.3.28}$$

Thus, m is a measure on X.

STEP 13: SHOW THAT EACH $\mathscr{B}_U$ IS UNIFORMLY-MEASURABLE WITH RESPECT TO m

Let $\phi \in \Phi$. We want to show that $\mathrm{m}(\phi(U)) = \mathrm{m}(U)$. Then, for any other $\phi' \in \Phi$, we will have that $\mathrm{m}(\phi(U)) = \mathrm{m}(U) = \mathrm{m}(\phi'(U))$, so that indeed every element of $\mathscr{B}_U$ has the same measure.

However, K is a quasicompact set contained in U iff $\phi(K)$ is a quasicompact set contained in $\phi(U)$. Therefore, by the definition of $\mathrm{m}(U)$ (5.1.3.18) it suffices to show that $\mathrm{H}(K) = \mathrm{H}(\phi(K))$ for all $K \in \mathscr{K}$. To show this, we first show that $\mathrm{H}_U(K) = \mathrm{H}_U(h(K))$ for all $U \in \mathscr{U}$. That is, we would like to show that $(K : U) = (\phi(K) : U)$. However, every cover of K by elements of $\mathscr{B}_U$, $\phi_1(U), \ldots, \phi_m(U)$, gives a cover of $\phi(K)$ by elements of $\mathscr{B}_U$ of the same cardinality, $\phi(\phi_1(U)), \ldots, \phi(\phi_m(U))$. It thus follows that $\mathrm{H}_U(K) = \mathrm{H}_U(h(K))$.

For $\phi \in \Phi$ fixed, consider the map from $\mathcal{H}$ to $\mathbb{R}$ that sends f to $f(\phi(K)) - f(K)$. We just showed that this is 0 on each $\mathrm{H}_U \in T$, and so it is 0 on C_U, and so it is 0 on H, that is, $\mathrm{H}(K) = \mathrm{H}(\phi(K))$.

STEP 14: SHOW THAT IF S AND T ARE UNIFORMLY-SEPARATED, THEN $\mathrm{m}(S \cup T) = \mathrm{m}(S) + \mathrm{m}(T)$

Note that we always have that $\mathrm{m}(S \cup T) \leq \mathrm{m}(S) + \mathrm{m}(T)$, and so it suffices to show that $\mathrm{m}(S \cup T) \geq \mathrm{m}(S) + \mathrm{m}(T)$.

We first prove this for open sets. So, let $U, V \in \mathscr{U}$. If either U or V s has infinite measure, then this inequality just reads $\infty \geq \infty$, and is so automatically satisfied. Thus, without loss of generality, assume that $\mathrm{m}(U), \mathrm{m}(V) < \infty$. Let $\varepsilon > 0$. Then, there is are some $K, L \in \mathscr{K}$ such that $K \subseteq U$, $L \subseteq V$, and

$$\mathrm{m}(U) - \varepsilon < \mathrm{H}(K) \leq \mathrm{m}(U) \text{ and } \mathrm{m}(V) - \varepsilon < \mathrm{H}(L) \leq \mathrm{m}(V).$$

If U and V are uniformly-separated, then certainly K and L are uniformly-separated, and so by Step 9, we have that

$$\mathrm{H}(K \cup L) = \mathrm{H}(K) + \mathrm{H}(L), \tag{5.1.3.29}$$

and so

$$\begin{aligned} \mathrm{m}(U \cup V) &\geq \mathrm{H}(K \cup L) = \mathrm{H}(K) + \mathrm{H}(L) \\ &> \mathrm{m}(U) + \mathrm{m}(V) - 2\varepsilon. \end{aligned} \tag{5.1.3.30}$$

Hence, $\mathrm{m}(U \cup V) \geq \mathrm{m}(U) + \mathrm{m}(V)$.

We now do the general case. Once again, if either S or T has infinite measure, we are done, so we may as well suppose that $\mathrm{m}(S), \mathrm{m}(T) < \infty$.

Our first order of business it to show that there are *some* open sets containing S and T respectively which are uniformly-separated.

Look at any open cover $\mathscr{B}$ which uniformly-separates S and T, and take an open star-refinement $\mathscr{C}$ of this[f]. Define $U :=$ $\mathrm{Star}_{\mathscr{C}}(S)$ and $V := \mathrm{Star}_{\mathscr{C}}(T)$. We wish to show that U and V are uniformly-separated with respect to $\mathscr{C}$. Because $\mathcal{B}$ uniformly-separates S and T, by definition (see Definition 4.3.1.3), we have that $\mathrm{Star}_{\mathscr{B}}(S)$ and $\mathrm{Star}_{\mathscr{B}}(T)$ are disjoint. Therefore, it suffices to show that $\mathrm{Star}_{\mathscr{C}}(U) \subseteq \mathrm{Star}_{\mathscr{B}}(S)$ (and similarly for V). So, suppose that $C \in \mathscr{C}$ intersects U. Then, by definition of U, it must intersect some element $C' \in \mathscr{C}$ which intersects S. Let $B \in \mathscr{B}$ be such that $\mathrm{Star}_{\mathscr{C}}(C') \subseteq B$. We then have that

$$C \subseteq^{g} \mathrm{Star}_{\mathscr{C}}(C') \subseteq B \subseteq^{h} \mathrm{Star}_{\mathscr{B}}(S). \tag{5.1.3.31}$$

Thus, indeed, $\mathrm{Star}_{\mathscr{C}}(U) \subseteq \mathrm{Star}_{\mathscr{B}}(S)$.

So, let $U, V \in \mathscr{U}$ be open sets containing S and T respectively which are uniformly-separated. Let $\varepsilon > 0$, and choose $W \in \mathscr{U}$ that contains $S \cup T$ and satisfies

$$\mathrm{m}(S \cup T) \leq \mathrm{m}(W) < \mathrm{m}(S \cup T) + \varepsilon. \tag{5.1.3.32}$$

Let us replace U and V by $U \cap W$ and $V \cap W$—upon doing so, it will still be the case that $U, V \in \mathscr{U}$, it will still be the case that $S \subseteq U$ and $T \subseteq V$, and it will still be the case that U and V are uniformly-separated, but now we will also have that $\mathrm{m}(U \cup V) \leq \mathrm{m}(W)$. Then we have

$$\begin{aligned} \mathrm{m}(S) + \mathrm{m}(T) + \varepsilon &\leq \mathrm{m}(U) + \mathrm{m}(V) + \varepsilon \\ &= \mathrm{m}(U \cup V) + \varepsilon \\ &\leq \mathrm{m}(W) + \varepsilon < \mathrm{m}(S \cup T) + 2\varepsilon. \end{aligned} \tag{5.1.3.33}$$

As ε is arbitrary, we have that

$$\mathrm{m}(S) + \mathrm{m}(T) \leq \mathrm{m}(S \cup T), \tag{5.1.3.34}$$

as desired.

In particular, we have now shown that $\widetilde{\mathscr{B}}_\Phi$ is a uniformly-measurable base for m and that m is additive for uniformly-separated sets, so that indeed m is a uniformly-additive on X.

It remains to show that m is regular.

Step 15: Show that $\mathrm{H}(K) \leq \mathrm{m}(K)$

Let $U \in \mathscr{U}$ contain K. Then, by the definition of $\mathrm{m}(U)$, (5.1.3.18), we have that $\mathrm{H}(K) \leq \mathrm{m}(U)$. Taking the infimum over all such U, we obtain $\mathrm{H}(K) \leq \mathrm{m}(K)$.

Step 16: Show that m is regular

The first thing we check is that $\mathrm{m}(K) < \infty$ for K quasicompact. By Proposition 3.7.6, there is some open $U \subseteq X$ containing

K with compact closure. Hence, for $K \subseteq K' \subseteq U$, with K' quasicompact, we have

$$\mathrm{H}(K') \leq {}^{i}\mathrm{H}(\mathrm{Cls}(U)) < \infty.^{j} \tag{5.1.3.35}$$

Taking the supremum over such K', we have that $\mathrm{m}(U) \leq \mathrm{H}(\mathrm{Cls}(U))$, and so, as $\mathrm{m}(K) \leq \mathrm{m}(U)$ (because m is a measure), $\mathrm{m}(K)$ is finite.

m is outer-regular by definition (5.1.3.19).

We now turn to inner-regular on open subsets. This is *almost* true by the definition (5.1.3.18), but we don't have $\mathrm{H}(K) = \mathrm{m}(K)$. However, we actually don't need this—we only need $\mathrm{H}(K) \leq \mathrm{m}(K)$. To show this, let $U \in \mathscr{U}$ contain K. Then, by the definition of $\mathrm{m}(U)$, (5.1.3.18), we have that $\mathrm{H}(K) \leq \mathrm{m}(U)$. Taking the infimum over all such U, we obtain $\mathrm{H}(K) \leq \mathrm{m}(K)$.

Thus, m is regular.

STEP 17: CONCLUDE THAT m IS AN ISOGENEOUS MEASURE

We know that m is regular from the previous step and uniformly-additive from Step 14. By Step 13, $\widetilde{\mathscr{B}}_{\Phi}$ is a uniformly-measurable base, and hence m is an isogeneous measure.

STEP 18: SHOW THAT $\mathrm{m}(K_0) = 1$

First of all, from the definition, we have that $\mathrm{H}_U(K_0) = 1$ for all $U \in \mathscr{U}$. Perhaps we could try harder and show that we already do have that $\mathrm{m}(K_0) = 1$; however, this enough is to show simply that $\mathrm{m}(K_0) > 0$, and so by simply dividing by $\mathrm{m}(K_0)$ if necessary, we obtain a regular uniformly-additive measure with uniformly-measurable base $\widetilde{\mathscr{B}}_{\Phi}$ and $\mathrm{m}(K_0) = 1$.

STEP 19: DEFINE $S \Subset T$

For the rest of this proof, let us write $S \Subset T$ iff there is some uniform cover $\mathscr{B} \in \widetilde{\mathscr{B}}_\Phi$ for which $\mathrm{Star}_{\mathscr{B}}(S) \subseteq T$.

STEP 20: SHOW THAT IF $S \Subset T$, THEN $\mathrm{Cls}(S) \subseteq \mathrm{Int}(T)$
Suppose that $S \Subset T$ and let $x \in X$ be an accumulation point of S. By definition, there is some uniform cover $\mathscr{B} \in \widetilde{\mathscr{B}}$ such that $\mathrm{Star}_{\mathscr{B}}(S) \subseteq T$. Because x is an accumulation point of S and every element of $\mathscr{B}$ is open, every element of $\mathscr{B}$ that contains x (of which there must be at least one, say $B \in \mathscr{B}$), must intersect S, and so we have that $x \in B \subseteq \mathrm{Star}_{\mathscr{B}}(S) \subseteq T$, which implies that $x \in \mathrm{Int}(T)$.

STEP 21: SHOW THAT m IS UNIQUE
Let m′ be another regular uniformly-additive on measure X with uniformly-measurable base $\widetilde{\mathscr{B}}_\Phi$ and $\mathrm{m}(K_0) = 1$. As both m and m′ are outer-regular, it suffices to show that they agree on open sets. Then, because they are both inner-regular on open sets, it suffices to show that they both agree on quasicompact subsets. However, on account of Proposition 3.7.6 and outer-regularity, it in turn suffices to show that they agree on open sets with compact closure. So, let $\mathscr{G}$ denote the collection of all open sets with compact closures. Furthermore, let us define

$$\begin{aligned} &Z(\mathscr{G}) := \\ &\big\{S \in 2^X : \text{for every } \varepsilon > 0 \text{ there are } C_\varepsilon \text{ closed} \\ &\qquad\qquad \text{and } U_\varepsilon \in \mathcal{G} \text{ such that} \\ &\qquad\qquad\quad C_\varepsilon \subseteq S \subseteq U_\varepsilon \text{ and} \\ &\qquad\qquad\quad \mathrm{m}(U_\varepsilon \setminus C_\varepsilon), \mathrm{m}'(U_\varepsilon \setminus C_\varepsilon) < \varepsilon\big\}\,. \end{aligned} \tag{5.1.3.36}$$

To prove that they agree on $\mathscr{G}$, we will first show that they agree on $Z(\mathscr{G}) \cap \mathscr{G}$. To show that this is in fact sufficient, we prove that

$$\mathrm{m}(U) = \sup\{\mathrm{m}(V) : V \in \mathscr{G} \cap Z(\mathscr{G}),\ V \Subset U\} \tag{5.1.3.37}$$

for $U \in \mathscr{G}$ open (the exact same proof will work for m'). Of course, we only need to show that $\leq$ inequality (because $\mathrm{m}(U) \geq \mathrm{m}(V)$ for $V \Subset U$). As m is inner-regular on open sets (and so in particular on elements of $\mathscr{G}$), we have that

$$\mathrm{m}(U) = \sup\{\mathrm{m}(K) : K \subseteq U,\ K \in \mathscr{K}.\}, \qquad (5.1.3.38)$$

and so to show that this suprema is at most the suprema in (5.1.3.37), it suffices to show that, for every $K \subseteq U$ quasicompact, there is some $V_K \in \mathscr{G} \cap Z(\mathscr{G})$ with $K \subseteq V_K \Subset U$.

So, let $U \subseteq X$ be open and let $K \subseteq U$ be quasicompact. We want to find such a V_K. As X is T_0, it is uniformly-completely-T_3, and so we may uniformly-separate quasicompact sets from closed sets, and so there is some uniform cover $\mathscr{B} \in \widetilde{\mathscr{B}}_{\Phi}$ such that $\mathrm{Star}_{\mathscr{B}}(K)$ is disjoint from $\mathrm{Star}_{\mathscr{B}}(U^{\mathsf{C}})$. Because X is a locally compact T_2 isogeneous space, by taking a star-refinement if necessary, we can without loss of generality assume that each element of $\mathscr{B}$ has compact closure (i.e. is an element of $\mathscr{G}$).[k] Take a star-refinement $\mathscr{C} \in \widetilde{\mathscr{B}}_{\Phi}$. Once again, every element of $\mathscr{C}$ has compact closure. By quasicompactness of K, there are finitely many $C_1, \ldots, C_m \in \mathscr{C}$ that cover K. Define $C \coloneqq C_1 \cup \cdots \cup C_m \in \mathscr{G}$. Furthermore,

$$K \subseteq C \subseteq \mathrm{Star}_{\mathscr{C}}(C) \subseteq \mathrm{Star}_{\mathscr{B}}(K) \subseteq \mathrm{Star}_{\mathscr{B}}(U^{\mathsf{C}})^{\mathsf{C}} \subseteq U.$$

Thus, we have shown that for $U \in \mathscr{G}$ and $K \subseteq U$ quasicompact, there is some $V_K \in \mathscr{G}$ with $K \subseteq V_K \Subset U$. However, we still need to show that we can find such a $V_K \in \mathscr{G} \cap Z(\mathscr{G})$. From what we have just shown, it suffices to show that, for every $U, V \in \mathscr{G}$ with $U \Subset V$, there is some $W \in Z(\mathscr{G}) \cap \mathscr{G}$ with $U \Subset W \Subset V$.

So, let $U, V \in \mathcal{G}$ with $U \Subset V$. We showed before in Step 20 that this implies that $\mathrm{Cls}(U) \subseteq \mathrm{Int}(V)$. Then, because we may uniformly separate quasicompact sets from closed sets in uniformly-completely-T_3 spaces, there is some uniform cover $\mathscr{B} \in \widetilde{\mathscr{B}}_{\Phi}$ such that $\mathrm{Star}_{\mathscr{B}}(\mathrm{Cls}(U))$ is disjoint from $\mathrm{Star}_{\mathscr{B}}(V^{\mathsf{C}})$.

Define

$$W := \mathrm{Star}_{\mathscr{B}}(\mathrm{Cls}(U)). \tag{5.1.3.39}$$

W is open as every element of $\mathscr{B}$ is open. It also has compact closure as

$$W \subseteq \mathrm{Star}_{\mathscr{B}}(V^{\mathsf{C}})^{\mathsf{C}} \subseteq (V^{\mathsf{C}})^{\mathsf{C}} = V, \tag{5.1.3.40}$$

and so its closure is contained in $\mathrm{Cls}(V)$, which is compact. By definition, we have that $U \Subset W$. We check that also $W \Subset V$. To show this, we show that $\mathrm{Star}_{\mathscr{B}}(W) \subseteq V$. So, let $B \in \mathscr{B}$ intersect W. We proceed by contradiction: suppose that B intersects V^{C}. Then, it is contained in $\mathrm{Star}_{\mathscr{B}}(V^{\mathsf{C}})$, which is disjoint from W: a contradiction. Thus, $U \Subset W \Subset V$ and $W \in \mathcal{G}$. It remains to show that $W \in Z(\mathscr{G})$. This, however, follows from the fact that that W is measurable with respect to both m and m' (because it is open—see Proposition 5.1.2.14), the fact that it has finite measure for both m and m' (because its closure is quasicompact and the measure is regular), and Proposition 5.1.2.14 (we can force U_ε there to have compact closure by intersecting it with V). This finally establishes (5.1.3.37), and so finishes our proof that it suffices to show that m and m' agree on $\mathscr{G} \cap Z(\mathscr{G})$.

For the rest of the proof, take note that everything we know about m' is likewise true about m. Therefore, everything we prove to be true about m' will also be true about m. Thus, hereafter, if we prove facts about either m or m', we shall prove them about m'—they are then automatically true about m.

For $U \in \mathscr{G} \cap Z(\mathscr{G})$, every cover $\mathcal{B}_V$ has a finite subcover of U (because $\mathrm{Cls}(U)$ is compact), and so just as we did for K compact, we may define $(U : V)$ to be the the cardinality of the smallest such subcover. We shall use this notation in a moment.

Exercise 5.1.3.41 Let $U \in \mathscr{G} \cap Z(\mathscr{G})$. Show that, for every $\varepsilon > 0$, there is some open U_ε, such that

$$\mathrm{m}'\left(\mathrm{Star}_{\mathscr{B}_{U_\varepsilon}}(U) - U\right) < \varepsilon \tag{5.1.3.42}$$

Now, fix $U_0 \in Z(\mathscr{G}) \cap \mathscr{G}$ and let U open be arbitrary.

Exercise 5.1.3.43 Show that there is some $U' \Subset U$ such that, for all V sufficiently small (with respect to $\Subset$),

$$\frac{\mathrm{m}'(U_0)}{\mathrm{m}'(U)} \leq \frac{(U_0' : V)}{(U' : V)} \tag{5.1.3.44}$$

whenever $\mathrm{Star}_{\mathscr{B}_{U'}}(U_0) \subseteq U_0'$.

R Hint: See (10.2) in [How91].

Exercise 5.1.3.45 Show that there is some $U'' \Subset U'$ such that, for all V sufficiently small (with respect to $\Subset$),

$$\frac{\mathrm{m}'(U_0)}{\mathrm{m}'(U'')} \geq \frac{(U_0' : V)}{(U' : V)} \tag{5.1.3.46}$$

whenever $\mathrm{Star}_{\mathscr{B}_{U'}}(U_0') \subseteq U_0$.

R Hint: See (10.3) in [How91].

Exercise 5.1.3.47 Combine the last three exercises (and the fact that $\mathrm{m}(K_0) = 1 = \mathrm{m}'(K_0)$) to show that m' and m agree on $\mathscr{G} \cap Z(\mathscr{G})$.

R Hint: You should be able to combine these results to get an expression for $\mathrm{m}'(U_0)$ that, besides factors of $\mathrm{m}'(U'')$ and $\mathrm{m}'(U')$, will be completely independent of m'. As explained above, everything true of m' must also be true of m, and so, we will have the same expression for m, with exception of the fact that the factors $\mathrm{m}'(U')$ and $\mathrm{m}'(U'')$ will be different.

■

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[a] K_0 is nonempty, and so cannot be covered by anything empty. Therefore, $(K_0 : U) \geq 1$, and in particular, is not 0.

[b] That was sort of the point of the previous step.

[c] For each $K_1, K_2 \in \mathscr{K}$ with $K_1 \subseteq K_2$, we have such a map.

[d] The first map from $\mathcal{H}$ into $\mathbb{R} \times \mathbb{R}$ is the projection of $f \in \mathcal{H}$ onto the K_1^{th} coordinate in the first coordinate and the projection of $f \in \mathcal{H}$ onto K_2^{th} coordinate in the second coordinate. This map is continuous because it is continuous in each coordinate (each coordinate is continuous because projections are continuous). The first map is followed by the map from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$ given by subtraction, which is continuous because we know that $\langle R, +, 0, - \rangle$ is a topological group.

[e] This is the definition of uniformly-separated—see Definition 4.3.1.3.

[f] Note that every $\mathscr{B}_U$ is an open cover—both $\mathscr{B}$ and $\mathscr{C}$ are secretly of the form $\mathscr{B}_U$ and $\mathscr{B}_V$ for some $U, V \subseteq X$ open—so there is no need to say "open" here—it is just to clarify. (We don't write $\mathscr{B}_U$ or $\mathscr{B}_V$ to simplify the notation (and also because we will want to write U for something else).)

[g] Because C and C' intersect.

[h] Because B contains C', which intersects S.

[i] H is nondecreasing by Step 6.

[j] Recall that H is always finite, by definition.

[k] Take any open set U with compact closure (which exists by local compactness). Then, $\mathscr{B}_U$ will be a cover whose elements are in $\mathscr{G}$. Take a common star-refinement of this and $\mathscr{B}$. The closures of elements of this new cover will be contained in the elements of $\mathscr{B}_U$, which themselves will be compact as closed sets of compact sets are compact.

Dayyyuuummm. That was a hard theorem. Probably the hardest in these notes. But holy Jesus was it worth it. Check out this epic definition.

Exercise 5.1.3.48 Define $X := \mathbb{R}^d$[a] and

$$\Phi := \{\phi\colon X \to X : \phi \text{ an isometry.}\}\,. \tag{5.1.3.49}$$

Show that $\langle X, \Phi\rangle$ is an isogeneous space.

> **R** You have to check that (i) $\mathbb{R}^d$ is σ-compact, (ii) $\mathbb{R}^d$ is T_0, (iii) $\mathbb{R}^d$ is locally quasicompact, and (iv) the group of isometries actually generate a uniform base for $\mathbb{R}^d$ via (5.1.3.2). These are actually all quite trivial.[b] For example, (iv) is the most nontrivial, but this is true essentially just because the isometric image of an ε-ball is—gasp—another ε-ball!

[a] With the usual uniformity.
[b] Though definitely convince yourself that they are true!

Definition 5.1.3.50 — Lebesgue measure ***Lebesgue measure*** m on $\mathbb{R}^d$ is unique isogeneous measure with respect to the symmetry group of all isometries[a] such that $\mathrm{m}([0,1]\times\cdots\times[0,1]) = 1$.

> **R** This is actually much better than defining Lebesgue measure to be 'classical' Haar measure (i.e. Haar measure with respect to the topological group structure). With this definition we *automatically* get that Lebesgue measure is invariant under rotations for free, whereas with the "classical" definition, this requires some work.

> **R** Besides using Haar measure to define Lebesgue measure, it is also common to use something called *Carathéodory's Extension Theorem*, which, while not that bad, has the problem that it lacks a uniqueness result.[b] Moreover, Carathéodory doesn't even give us a regular measure—it actually just gives us

a measure. We would then have to go through by hand and check that the measure defined in this way is finite on quasicompact sets, inner-regular on open sets, outer-regular, has a uniformly-measurable base, is uniformly-additive, is invariant under translation, is invariant under rotation, is invariant under reflection, and that translations, rotations, and reflections actually give us all the isometries. Ew. The theory is just so much prettier when all of this hard work is done for us by *one* result, instead of by fifty-bajillion separate ones.

[a]In the case of $\mathbb{R}^d$, all rotations, reflections, and translations are isometries. (In fact, one can show that these generate all isometries.)

[b]At least in general. I think in certain nice cases it can be made to work.

We can even define counting measure using the Haar-Howes Theorem.[2]

Definition 5.1.3.51 — Counting measure Let X be a set equipped with the discrete uniformity and take $\Phi := \mathrm{Aut}_{\mathbf{Set}}(X)$ to be the group of all bijections from X to itself. Then,

$$\begin{aligned}\{\mathcal{B}_{\{x_0\}} : x_0 \in X\} &:= \{\{h(\{x_0\}) : h \in H\}\} \\ &= \{\{\{x\} : x \in X\}\}^{a}\end{aligned} \tag{5.1.3.52}$$

is a uniform base, and so, because $\mathcal{B} := \{\{x_0\} : x_0 \in X\}$ is a base for this topology, by Exercise 5.1.3.3, $\langle X, H\rangle$ is an isogeneous space. Therefore, by the Haar-Howes Theorem, there is a unique isogeneous measure m, the *counting measure*, on X such that $\mathrm{m}(\{x_0\}) = 1$.

[a]This is the uniform base which has a single cover, namely, the cover by singletons. I say this because what is going on here is actually very easy, even though the notation may be a bit hard to parse.

[2]Though this is a bit like using a sledgehammer (or maybe a nuke?) 'swat' a fly.

5.1.4 The product measure

As you know, the integral is 'supposed' to be the "area under the curve". The functions we will be integrating will take values in the reals, and so the integral will spit out a real number. This is probably nothing new to you. What is almost certainly new to you, however, is that now the *domains* of the functions we will be integrating will be topological measure spaces.[3] That is, we will be integrating functions $f\colon X \to \mathbb{R}$, for X a topological measure space. We will then simply define the integral to be the measure of the set[4]

$$\{\langle x, y\rangle \in X \times \mathbb{R} : 0 \leq y < f(x)\}\,. \tag{5.1.4.1}$$

To do this, of course, we must first define a measure on $X \times \mathbb{R}$. In fact, we will much more generally define a measure on $X_1 \times X_2$ for X_1 and X_2 any topological measure spaces: the *product measure*.

The product measure is—you guessed it—a measure on the product. As the integral is the 'area under the curve' and the 'area under the curve' is a subset of $X \times [0, \infty]$ (for $f\colon X \to [0, \infty]$), it is necessary for our development to put a measure on $X \times [0, \infty]$.[5]

Theorem 5.1.4.2 — Product measure. Let $\langle X_1, \mathrm{m}_1\rangle$ and $\langle X_2, \mathrm{m}_2\rangle$ be topological measure spaces. Then, there exists a unique topological measure $\mathrm{m}_1 \times \mathrm{m}_2$ on $X_1 \times X_2$, the ***product measure***, that satisfies $[\mathrm{m}_1 \times \mathrm{m}_2](K_1 \times K_2) = \mathrm{m}_1(K_1)\,\mathrm{m}_2(K_2)$ for $K_i \subseteq X_i$ compact.

Furthermore, it satisfies

(i). $[\mathrm{m}_1 \times \mathrm{m}_2](S_1 \times S_2) = \mathrm{m}_1(S_1)\,\mathrm{m}_2(S_2)$ for all $S_i \subseteq X_i$[a];

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[3] You can do things much more generally than this, but for us,there is no need. As a rule, it is usually best to do things in the nicest theory that encompasses every example you're interested in, and for us, we will not be interested in measures that are not topological (except perhaps when it comes to counter-examples).

[4] At least when f is nonnegative—we'll have to work just a teensy bit harder to take the 'signed area' if the function is negative somewhere.

[5] Of course, you don't *have* to do things this way, and indeed, this is not how it's usually done. Doing things the other way, however, is arguably one reason why people think the Lebesgue integral is "not geometric". On the other hand, I can't believe anybody would argue that the "area under the curve" is not geometric or intuitive—this is in fact how we explain things to high school students, after all.

(ii).

$$[\mathrm{m}_1 \times \mathrm{m}_2](U) = \sup \left\{ \sum_{k=0}^{m} \mathrm{m}_1(K_{1,k})\, \mathrm{m}_2(K_{2,k}) : m \in \mathbb{N} \right. \quad K_{i,k} \subseteq X_i \text{ compact}, \quad \{K_{1,k} \times K_{2,k} : 0 \le k \le m\} \text{ is disjoint}, \quad \left. \bigcup_{k=0}^{m} K_{1,k} \times K_{2,k} \subseteq U \right\} \tag{5.1.4.3}$$

for $U \subseteq X_1 \times X_2$ open;

and

(iii).

$$[\mathrm{m}_1 \times \mathrm{m}_2](S) = \inf \left\{ \sum_{m \in \mathbb{N}} \mathrm{m}_1(U_{1,m})\, \mathrm{m}_2(U_{2,m}) : \right. \quad \left. U_{i,m} \subseteq X_i \text{ open}, \; S \subseteq \bigcup_{m \in \mathbb{N}} U_{1,m} \times U_{2,m} \right\}. \tag{5.1.4.4}$$

R So those formulas look ridiculous, but I promise, it's not that complicated. The first simply says that we can approximate open sets from the inside with compact sets of a special form (namely, finite disjoint unions of compact rectangles)—essentially just a nice version of inner-regularity. The second simply says that we can approximate *any* set from the outside with open sets of a special form (namely, countable unions of open rectangles)—essentially just a nice version of outer-regularity. Perhaps one thing to note is that in we do not need to require disjointness explicitly in (5.1.4.4)—not having disjointness makes the sum larger, and so the infimum 'doesn't care' about these cases, so to speak.

R Also note that it says *compact* in (5.1.4.3) instead of just *quasicompact* (though certainly we still have equality if we had used the word "quasicompact")—this is for essentially the same fact as given in Proposition 5.1.2.36. This will be used repeatedly throughout the proof, probably nearly every time we make use of inner-regularity of X_1 or X_2, and so we mention this once only here, instead of referencing it every time we use it.

[a]That is, you assume that 'area is base times height' for *quasicompact* rectangles, and then you get that 'area is base times height' for *all* rectangles—you don't even need S_1 and S_2 to be measurable.

Proof. STEP 1: CHECK THAT $X_1 \times X_2$ IS σ-COMPACT
You (hopefully) already showed in Exercise 5.1.2.10 that the product of two σ-compact spaces is σ-compact, and so $X_1 \times X_2$ is σ-compact.

STEP 2: DEFINE $\mathrm{m}_1 \times \mathrm{m}_2$
We define

$$\begin{aligned} \mathrm{m}_{\mathrm{K}}(K_1 \times K_2) &:= \mathrm{m}_1(K_1)\,\mathrm{m}_2(K_2) \text{ for } K_1 \subseteq X_1,\ K_2 \subseteq X_2 \text{ compact} \\ \mathrm{m}_{\mathrm{U}}(U) &:= \sup\left\{ \sum_{k=0}^{m} \mathrm{m}_K(K_{1,k} \times K_{2,k}) : m \in \mathbb{N} \right. \\ &\qquad K_{1,k} \subseteq X_1,\ K_{2,k} \subseteq X_2 \text{ compact}, \\ &\qquad \{K_{1,k} \times K_{2,k} : 0 \leq k \leq m\} \text{ is disjoint}, \\ &\qquad \left. \bigcup_{k=0}^{m} K_{1,k} \times K_{2,k} \subseteq U \right\} \text{ for } U \subseteq X_1 \times X_2 \text{ open} \\ [\mathrm{m}_1 \times \mathrm{m}_2](S) &:= \inf\{\mathrm{m}_{\mathrm{U}}(U) : S \subseteq U,\ U \text{ open.}\}. \end{aligned} \tag{5.1.4.5}$$

We will eventually show that all these formulas agree in the case that more than one applies to a given set (e.g. the second and third both apply to any open set). After which time, we shall simply denote $\mathrm{m}_1 \times \mathrm{m}_2$ for everything.

STEP 3: SHOW THAT $\mathrm{m}_1 \times \mathrm{m}_2$ IS NONDECREASING

Exercise 5.1.4.6 Check that m_{K} is nondecreasing on sets of the form $K_1 \times K_2$ for $K_i \subseteq X_i$ compact.

Exercise 5.1.4.7 Check that m_{U} is nondecreasing on open sets.

Exercise 5.1.4.8 Check that $\mathrm{m}_1 \times \mathrm{m}_2$ itself is nondecreasing.

STEP 4: SHOW THAT $\mathrm{m}_{\mathrm{U}}(U_1 \times U_2) = \mathrm{m}_1(U_1)\,\mathrm{m}_2(U_2)$ FOR $U_i \subseteq X_i$ OPEN

Let $U_i \subseteq X_i$ be open. Let us first do the case where one has measure 0. Without loss of generality, take $\mathrm{m}_1(U_1) = 0$. Then, whenever we have

$$\bigcup_{k=0}^{m} K_{1,k} \times K_{2,k} \subseteq U_1 \times U_2, \tag{5.1.4.9}$$

we must have that $K_{1,k} \subseteq U_1$ for all $0 \leq k \leq m$,[a] which forces $\mathrm{m}_1(K_{1,k}) = 0$, and so in turn it forces

$$\sum_{k=0}^{m} \mathrm{m}_{\mathrm{K}}(K_{1,k} \times K_{2,k}) = 0, \tag{5.1.4.10}$$

and hence $\mathrm{m}_{\mathrm{U}}(U_1 \times U_2) = 0 = \mathrm{m}_1(U_1)\,\mathrm{m}_2(U_2)$.

Let us now suppose that one of U_1 and U_2 has infinite measure and the other is positive. Without loss of generality, suppose that $\mathrm{m}_1(U_1) = \infty$ and $\mathrm{m}_2(U_2) > 0$. Then, by inner-regularity on opens, for every $M > 0$, there is a compact subset $K_1 \subseteq U_1$ with $\mathrm{m}_1(K_1) > M$. Similarly, there is a compact subset $K_2 \subseteq U_2$ with $\mathrm{m}_2(K_2) > 0$. Then,

$$\begin{aligned} \mathrm{m}_{\mathrm{U}}(U_1 \times U_2) \geq \mathrm{m}_{\mathrm{K}}(K_1 \times K_2) &\coloneqq \mathrm{m}_1(K_1)\,\mathrm{m}_2(K_2) \\ &> M\,\mathrm{m}_2(K_2), \end{aligned} \tag{5.1.4.11}$$

and so, as $M > 0$ is arbitrary and $\mathrm{m}_2(K_2) > 0$, $\mathrm{m}_{\mathrm{U}}(U_1 \times U_2) = \infty$.

Finally, consider the case where both $0 < \mathrm{m}(U_1), \mathrm{m}(U_2) < \infty$. Let $\varepsilon > 0$ be such that

$$(\mathrm{m}_1(U_1)+\mathrm{m}_2(U_2))\cdot\min\{\mathrm{m}_1(U_1), \mathrm{m}_2(U_2)\} > \varepsilon > 0 \tag{5.1.4.12}$$

and choose $K_i \subseteq U_i$ compact so that

$$\mathrm{m}_i(U_i) - \frac{\epsilon}{\mathrm{m}_1(U_2) + \mathrm{m}_2(U_2)} < \mathrm{m}(K_i) \leq \mathrm{m}(U_i). \quad (5.1.4.13)$$

Then,

$$\begin{aligned}
\mathrm{m}_1(U_1)\,\mathrm{m}_2(U_2) &\geq \mathrm{m}_1(K_1)\,\mathrm{m}_2(K_2) \\
&> \left(\mathrm{m}_1(U_1) - \frac{\epsilon}{\mathrm{m}_1(U_2) + \mathrm{m}_2(U_2)}\right) \\
&\quad \cdot \left(\mathrm{m}_2(U_2) - \frac{\epsilon}{\mathrm{m}_1(U_1) + \mathrm{m}_2(U_2)}\right) \qquad (5.1.4.14)\\
&= \mathrm{m}_1(U_1)\,\mathrm{m}_2(U_2) - \epsilon \\
&\quad + \frac{\epsilon^2}{(\mathrm{m}_1(U_1) + \mathrm{m}_2(U_2))^2} \\
&> \mathrm{m}_1(U_1)\,\mathrm{m}_2(U_2) - \epsilon,
\end{aligned}$$

whence it follows[b] that $\mathrm{m}_{\mathrm{U}}(U_1 \times U_2) = \mathrm{m}_1(U_1)\,\mathrm{m}_2(U_2)$.

Step 5: Show that $[\mathrm{m}_1 \times \mathrm{m}_2](U) = \mathrm{m}_{\mathrm{U}}(U)$
Let $S \subseteq X_1 \times X_2$ be open. We must show that $[\mathrm{m}_1 \times \mathrm{m}_2](S) = \mathrm{m}_{\mathrm{U}}(S)$. Of course, $\mathrm{m}_{\mathrm{U}}(S) \in \{\mathrm{m}_{\mathrm{U}}(U) : S \subseteq U,\ U \text{ open}\}$, which gives us that $[\mathrm{m}_1 \times \mathrm{m}_2](S) \leq \mathrm{m}_{\mathrm{U}}(S)$.[c] On the other hand, because m_{U} is nondecreasing on open sets, we have that $\mathrm{m}_{\mathrm{U}}(S)$ is a lower-bound for $\{\mathrm{m}_{\mathrm{U}}(U) : S \subseteq U,\ U \text{ open}\}$, which gives us the other inequality.

Step 6: Show that $[\mathrm{m}_1 \times \mathrm{m}_2](S_1 \times S_2) = \mathrm{m}_1(S_1)\,\mathrm{m}_2(S)$ for $\mathrm{m}_i(S_i) < \infty$
Let $S_i \subseteq X_i$ with $\mathrm{m}_i(S_i) < \infty$. We want to show that $[\mathrm{m}_1 \times \mathrm{m}_2](S) = \mathrm{m}_1(S_1)\,\mathrm{m}_2(S_2)$. In other words, we want to show that

$$\mathrm{m}_1(S_1)\,\mathrm{m}_2(S_2) = \inf\{\mathrm{m}_{\mathrm{U}}(U) : S_1 \times S_2 \subseteq U,\ U \text{ open}\}.$$

Let $\varepsilon > 0$. Choose $\delta > 0$ such that $\delta(\mathrm{m}_1(S_1)+\mathrm{m}_2(S_2))+\delta^2 < \varepsilon$. Because m_i is regular, there is some open $U_i \subseteq X_i$ with $S_i \subseteq U_i$ and

$$\mathrm{m}_i(S_i) \leq \mathrm{m}_i(U_i) < \mathrm{m}_i(S_i) + \delta. \tag{5.1.4.15}$$

Then, $S_1 \times S_2 \subseteq U_1 \times U_2$, $U_1 \times U_2$ is open, and

$$\begin{aligned} \mathrm{m}_1(S_1)\,\mathrm{m}_2(S_2) &\leq \mathrm{m}_1(U_1)\,\mathrm{m}_2(U_2) = {}^{d}\, \mathrm{m}_{\mathrm{U}}(U_1 \times U_2) \\ &< \mathrm{m}_1(S_1)\,\mathrm{m}_2(S_2) + \delta(\mathrm{m}_1(S_1) + \mathrm{m}_2(S_2)) + \delta^2 \\ &< \mathrm{m}_1(S_1)\,\mathrm{m}_2(S_2) + \varepsilon, \end{aligned} \tag{5.1.4.16}$$

whence it follows[e] that $[\mathrm{m}_1 \times \mathrm{m}_2](S_1 \times S_2) = \mathrm{m}_1(S_1)\,\mathrm{m}_2(S_2)$.

Step 7: Show that $\mathrm{m}_{\mathrm{U}}(K_1 \times K_2) = \mathrm{m}_{\mathrm{K}}(K_1 \times K_2)$
Now suppose that $U := K_1 \times K_2$ is open for $K_i \subseteq X_i$ compact.[f] We must show that $\mathrm{m}_{\mathrm{K}}(U) = \mathrm{m}_{\mathrm{U}}(U)$. That is, we must show that $\mathrm{m}_{\mathrm{U}}(U) = \mathrm{m}_1(K_1)\,\mathrm{m}_2(K_2)$. However, from the definition of m_{U} as a supremum, we have that $\mathrm{m}_{\mathrm{U}}(U) \geq \mathrm{m}_1(K_1)\,\mathrm{m}_2(K_2)$. To show the other inequality, let

$$\bigcup_{k=0}^{m} K_{1,m} \times K_{2,m} \subseteq K_1 \times K_2 \tag{5.1.4.17}$$

be a disjoint union for $K_{i,k} \subseteq X_i$ compact.[g] We need to show that

$$\sum_{k=1}^{m} \mathrm{m}_1(K_{1,k})\,\mathrm{m}_2(K_{2,k}) \leq \mathrm{m}_1(K_1)\,\mathrm{m}_2(K_2), \tag{5.1.4.18}$$

as the supremum of the left-hand side over all possibilities for the $K_{i,k}$s is precisely $\mathrm{m}_{\mathrm{U}}(U)$—see (5.1.4.5). $K_{i,k} \subseteq K_i$ for $0 \leq k \leq m$, and so

$$\left(\bigcup_{k=0}^{m} K_{1,k}\right) \times \left(\bigcup_{k=0}^{m} K_{2,k}\right) \subseteq K_1 \times K_2.^{h} \tag{5.1.4.19}$$

Now, here is where the notation becomes atrocious, but the idea is only a little bit tricky. The intuition is that we break up the bounding rectangle into a grid of smaller rectangles so that each of the original rectangles is a disjoint union of the 'grid' rectangles. Let $S \subseteq \{1, \ldots, k-1, k+1, \ldots, m\} =: \mathcal{S}_k$ and define

$$K_{i,k,S} := K_{i,k} \cap \bigcap_{l \in S} K_{i,l} \cap \bigcap_{l \notin S} K_{i,l}^{\complement}, \tag{5.1.4.20}$$

so that

$$K_{i,k} = \bigsqcup_{S \subseteq \mathcal{S}_k} K_{i,k,S} \tag{5.1.4.21}$$

and

$$L_i := \bigsqcup_{k=0}^{m} K_{i,k} = \bigsqcup_{k=0}^{m} \bigsqcup_{S \subseteq \mathcal{S}_k} K_{i,k,S} \tag{5.1.4.22}$$

are *disjoint* unions. Then, we have that

$$\begin{aligned}
&\sum_{k=0}^{m} \mathrm{m}_1(K_{1,k})\, \mathrm{m}_2(K_{2,k}) \\
&\quad = \sum_{k=0}^{m} \Bigg(\sum_{S \subseteq \mathcal{S}_k} \mathrm{m}_1(K_{1,k,S}) \Bigg) \Big(\sum_{S \subseteq \mathcal{S}_k} \mathrm{m}_2(K_{2,k,S}) \Bigg) \\
&\quad \leq \Bigg(\sum_{k=0}^{m} \sum_{S \subseteq \mathcal{S}_k} \mathrm{m}_1(K_{1,k,S}) \Bigg) \Bigg(\sum_{k=0}^{m} \sum_{S \subseteq \mathcal{S}_k} \mathrm{m}_2(K_{2,k,S}) \Bigg) \\
&\quad = \mathrm{m}_1(L_1)\, \mathrm{m}_2(L_2) \leq^{i} \mathrm{m}_1(K_1)\, \mathrm{m}_2(K_2),
\end{aligned}$$

as was to be shown.

Step 8: Show that $[\mathrm{m}_1 \times \mathrm{m}_2](K_1 \times K_2) = \mathrm{m}_{\mathrm{K}}(K_1 \times K_2)$
In other words, we want to show that $[\mathrm{m}_1 \times \mathrm{m}_2](K_1 \times K_2) = \mathrm{m}_1(K_1)\, \mathrm{m}_2(K_2)$. However, this is true by Step 6 as $\mathrm{m}_i(K_i) < \infty$ by regularity.

At this point, we have shown that all of m_K, m_U, and $\mathrm{m}_1 \times \mathrm{m}_2$ agree on their common domains of definition, and so hereafter we shall only write $\mathrm{m}_1 \times \mathrm{m}_2$.

STEP 9: SHOW THAT $\mathrm{m}_1 \times \mathrm{m}_2$ IS A MEASURE

That $[\mathrm{m}_1 \times \mathrm{m}_2](\emptyset) = 0$ follows immediately from the definition. We already showed that it is nondecreasing.

Exercise 5.1.4.23 Check that $\mathrm{m}_1 \times \mathrm{m}_2$ is subadditive.

STEP 10: SHOW THAT $\mathrm{m}_1 \times \mathrm{m}_2$ IS REGULAR

That it is outer-regular follows immediately from the definition.[j]

Inner-regularity also follows quite easily from the definition.

$$\begin{aligned}
[\mathrm{m}_1 \times \mathrm{m}_2](U) \coloneqq \sup \Bigg\{ & \sum_{k=0}^{m} [\mathrm{m}_1 \times \mathrm{m}_2](K_{1,k} \times K_{2,k}) : m \in \mathbb{N} \\
& K_{1,k} \subseteq X_1, \\
& K_{2,k} \subseteq X_2 \text{ compact}, \\
& \left\{ K_{1,k} \times K_{2,k} : 0 \leq k \leq m \right\} \text{ is disjoint}, \\
& \bigcup_{k=0}^{m} K_{1,k} \times K_{2,k} \subseteq U \Bigg\} \\
\leq{}& \sup\{[\mathrm{m}_1 \times \mathrm{m}_2](K) : K \subseteq U,\ K \text{ compact}\} \\
\leq{}& {}^{k}[\mathrm{m}_1 \times \mathrm{m}_2](U),
\end{aligned}$$

and so all of these inequalities must be equalities.

We now check that it is finite on quasicompact sets. Let $K \subseteq X_1 \times X_2$ be quasicompact. Define $K_i \coloneqq \pi_i(K) \subseteq X_i$. This is quasicompact by the Extreme Value Theorem, and so $\mathrm{m}_i(K_i) < \infty$. However, $K \subseteq K_1 \times K_2$, and hence $[\mathrm{m}_1 \times \mathrm{m}_2](K) \leq \mathrm{m}_1(K_1)\,\mathrm{m}_2(K_2) < \infty$.

STEP 11: SHOW THAT $\mathrm{m}_1 \times \mathrm{m}_2$ IS ADDITIVE ON FINITE DISJOINT UNIONS OF COMPACT RECTANGLES

Let $K_{i,k} \subseteq X_i$ be compact for $0 \leq k \leq m$ be such that $\bigcup_{k=0}^{m} K_{1,k} \times K_{2,k}$ is a disjoint union. We wish to show that

$$\begin{aligned}[\mathrm{m}_1 \times \mathrm{m}_2]\left(\bigcup_{k=0}^{m} K_{1,k} \times K_{2,k}\right) \\ = \sum_{k=0}^{m} \mathrm{m}_1(K_{1,k})\, \mathrm{m}_2(K_{2,k}).\end{aligned} \tag{5.1.4.24}$$

The $\leq$ equality follows from subadditivity. To show the other inequality, let $\varepsilon > 0$, and let $U \subseteq X_1 \times X_2$ be open and such that

$$\bigcup_{k=0}^{m} K_{1,k} \times K_{2,k} \subseteq U \text{ and } [\mathrm{m}_1 \times \mathrm{m}_2](U) \\ < [\mathrm{m}_1 \times \mathrm{m}_2]\left(\bigcup_{k=0}^{m} K_{1,k} \times K_{2,k}\right) + \varepsilon.$$

Then,

$$\begin{aligned}[\mathrm{m}_1 \times \mathrm{m}_2]\left(\bigcup_{k=0}^{m} K_{1,k} \times K_{2,k}\right) > [\mathrm{m}_1 \times \mathrm{m}_2](U) - \varepsilon \\ \geq {}^{l}\sum_{k=0}^{m} \mathrm{m}_1(K_{1,k})\, \mathrm{m}_2(K_{2,k}) - \varepsilon,\end{aligned} \tag{5.1.4.25}$$

and so as $\varepsilon > 0$ is arbitrary, we have the other inequality.

Step 12: Show that $\mathrm{m}_1 \times \mathrm{m}_2$ is Borel
Now, let $U \subseteq X_1 \times X_2$ be open. We wish of course to show that U is open. Write $X_i = \bigcup_{m \in \mathbb{N}} J_{i,m}$ for $J_{i,m}$ compact and define $U_{k,l} := U \cap (J_{1,k} \times J_{2,l})$. Then, as $U = \bigcup_{k,l \in \mathbb{N}} U_{k,l}$, it suffices to show that $U_{k,l}$ is measurable for $k, l \in \mathbb{N}$. Thus, for this step, without loss of generality, suppose that X_1 and X_2 themselves are compact.

We first show that $K_1 \times K_2$ is measurable for $K_i \subseteq X_i$ compact. So, let $S \subseteq X_1 \times X_2$ be arbitrary. We wish to show that

$$\begin{aligned}[\mathrm{m}_1 \times \mathrm{m}_2](S) \geq [\mathrm{m}_1 \times \mathrm{m}_2]&(S \cap (K_1 \times K_2)) \\ &+ [\mathrm{m}_1 \times \mathrm{m}_2](S \cap (K_1 \times K_2)^{\mathsf{C}}).\end{aligned} \tag{5.1.4.26}$$

If $[\mathrm{m}_1 \times \mathrm{m}_2](S) = \infty$, we are done, so we may as well assume that $[\mathrm{m}_1 \times \mathrm{m}_2](S) < \infty$. Let $\varepsilon > 0$, and let $U \subseteq X_1 \times X_2$ be open and such that

$$S \subseteq U \text{ and } [\mathrm{m}_1 \times \mathrm{m}_2](S) \leq [\mathrm{m}_1 \times \mathrm{m}_2](U) < [\mathrm{m}_1 \times \mathrm{m}_2](S) + \varepsilon .$$

$K_1 \times K_2$ is closed, and so $U \cap (K_1 \times K_2)^{\mathsf{C}}$ is open. Therefore, there is a disjoint union

$$\bigcup_{k=0}^{m} L_{1,k} \times L_{2,k} \subseteq U \cap (K_1 \times K_2)^{\mathsf{C}} \tag{5.1.4.27}$$

for $L_{i,k} \subseteq X_i$ compact such that

$$[\mathrm{m}_1 \times \mathrm{m}_2](U \cap (K_1 \times K_2)^{\mathsf{C}}) - \varepsilon < \sum_{k=0}^{m} [\mathrm{m}_1 \times \mathrm{m}_2](L_{1,k} \times L_{2,k}).$$

Similarly, there is a disjoint union

$$\bigcup_{k=0}^{n} M_{1,k} \times M_{2,k} \subseteq U \cap \left(\bigcup_{k=0}^{m} L_{1,k} \times L_{2,k} \right)^{\mathsf{C}} \tag{5.1.4.28}$$

for $M_{i,k} \subseteq X_i$ compact such that

$$\begin{aligned}[\mathrm{m}_1 \times \mathrm{m}_2] \left(U \cap \left(\bigcup_{k=0}^{m} L_{1,k} \times L_{2,k} \right)^{\mathsf{C}} \right) - \varepsilon \\ < \sum_{k=0}^{n} [\mathrm{m}_1 \times \mathrm{m}_2](M_{1,k} \times M_{2,k}).\end{aligned} \tag{5.1.4.29}$$

Hence,

$$\begin{aligned}
[\mathrm{m}_1 \times \mathrm{m}_2](S) &> [\mathrm{m}_1 \times \mathrm{m}_2](U) - \varepsilon \\
&\geq [\mathrm{m}_1 \times \mathrm{m}_2]\left(\bigcup_{k=0}^{m} L_{1,k} \times L_{2,k} \cup \bigcup_{k=0}^{n} M_{1,k} \times M_{2,k}\right) - \varepsilon \\
&= {}^{m} \sum_{k=0}^{m} [\mathrm{m}_1 \times \mathrm{m}_2](L_{1,k} \times L_{2,k}) \\
&\qquad + \sum_{k=0}^{n} [\mathrm{m}_1 \times \mathrm{m}_2](M_{1,k} \times M_{2,k}) - \varepsilon \\
&> [\mathrm{m}_1 \times \mathrm{m}_2] \left(U \cap (K_1 \times K_2)^{\mathsf{C}}\right) \\
&\qquad + [\mathrm{m}_1 \times \mathrm{m}_2] \left(U \cap \left(\bigcup_{k=0}^{m} L_{1,k} \times L_{2,k}\right)^{\mathsf{C}}\right) \\
&\qquad - 3\varepsilon \\
&\geq [\mathrm{m}_1 \times \mathrm{m}_2](U \cap (K_1 \times K_2)^{\mathsf{C}}) \\
&\qquad + [\mathrm{m}_1 \times \mathrm{m}_2](U \cap (K_1 \times K_2)) - 3\varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we obtain the desired inequality.

As X_1 and X_2 are compact, $[\mathrm{m}_1 \times \mathrm{m}_2](X_1 \times X_2) = \mathrm{m}_1(X_1)\,\mathrm{m}_2(X_2) < \infty$, and so $[\mathrm{m}_1 \times \mathrm{m}_2](U) < \infty$. So, let $\varepsilon > 0$, and let $K_{i,k} \subseteq X_i$ be compact such that $\{K_{1,k} \times K_{2,k} : 0 \leq k \leq m\}$ is disjoint and

$$\begin{aligned}
&[\mathrm{m}_1 \times \mathrm{m}_2](U) - \varepsilon \\
&\quad < \sum_{k=0}^{m} [\mathrm{m}_1 \times \mathrm{m}_2](K_{1,k} \times K_{2,k}) \leq [\mathrm{m}_1 \times \mathrm{m}_2](U).
\end{aligned} \tag{5.1.4.30}$$

Applying Exercise 5.1.1.33 and using the fact that $\cup_{k=0}^{m} K_{1,k} \times K_{2,k}$ is measurable, we have that

$$[\mathrm{m}_1 \times \mathrm{m}_2]\left(U \setminus \left(\bigcup_{k=0}^{m} K_{1,k} \times K_{2,k}\right)\right) < \varepsilon. \tag{5.1.4.31}$$

Picking such a disjoint union of rectangles for $\varepsilon = \frac{1}{m}$ and taking the union of these over $m \in \mathbb{Z}^+$ gives a measurable set

F with $[\mathrm{m}_1 \times \mathrm{m}_2](U \setminus F) = 0$, and so $U \setminus F$ is measurable, and so $U = F \cup (U \setminus F)$ is measurable.

STEP 13: SHOW UNIQUENESS

To show that two regular measures agree, by outer-regularity, it suffices to show that they agree on on open sets. By inner-regularity, to show this, it suffices to show that

$$\begin{aligned}
&\sup\{\mathrm{m}(K) : K \subseteq U,\ K \text{ quasicompact}\} \\
&\qquad = \sup\left\{\sum_{k=0}^{m}[\mathrm{m}_1 \times \mathrm{m}_2](K_{1,k} \times K_{2,k}) : m \in \mathbb{N}\right. \\
&\qquad\qquad K_{1,k} \subseteq X_1,\ K_{2,k} \subseteq X_2 \text{ quasicompact}, \\
&\qquad\qquad \left\{K_{1,k} \times K_{2,k} : 0 \leq k \leq m\right\} \text{ is disjoint}, \\
&\qquad\qquad \left.\bigcup_{k=0}^{m} K_{1,k} \times K_{2,k} \subseteq U\right\}.^n
\end{aligned} \tag{5.1.4.32}$$

The left-hand side is automatically at least as large as the right-hand side because the set inside the sups on the left-hand side is a superset of the set inside the sup right-hand side. On the other hand, for $K \subseteq X_1 \times X_2$ quasicompact, $K_i := \pi_i(K)$ is quasicompact by the Extreme Value Theorem, and as $K \subseteq K_1 \times K_2$, we have that $\mathrm{m}(K) \leq \mathrm{m}_1(K_1)\,\mathrm{m}_2(K_2)$, which shows that the left-hand side is no larger than the right-hand side, which gives equality.

STEP 14: SHOW 'RECTANGLE OUTER-REGULARITY'

We now seek to show (5.1.4.4), which we reproduce here for convenience.

$$\begin{aligned}
[\mathrm{m}_1 \times \mathrm{m}_2](S) = \inf&\left\{\sum_{m\in\mathbb{N}} \mathrm{m}_1(U_{1,m})\,\mathrm{m}_2(U_{2,m}) :\right. \\
&\left. U_{i,m} \subseteq X_i \text{ open}, S \subseteq \bigcup_{m\in\mathbb{N}} U_{1,m} \times U_{2,m}\right\}.
\end{aligned} \tag{5.1.4.33}$$

We first reduce it to the case where S is open. So, suppose we have proven the result for open sets, and let $S \subseteq X_1 \times X_2$ be arbitrary. Note that the right-hand side of (5.1.4.33) is at least as large as $\inf \{[\mathrm{m}_1 \times \mathrm{m}_2](U) : S \subseteq U,\ U \text{ open.}\}$, and so in particular, if $[\mathrm{m}_1 \times \mathrm{m}_2](S) = \infty$, there is nothing to prove, so suppose that $[\mathrm{m}_1 \times \mathrm{m}_2](S) < \infty$.

Let $\varepsilon > 0$ and pick $U \supseteq S$ open and such that

$$[\mathrm{m}_1 \times \mathrm{m}_2](S) \leq [\mathrm{m}_1 \times \mathrm{m}_2](U) < [\mathrm{m}_1 \times \mathrm{m}_2](S)+\varepsilon. \tag{5.1.4.34}$$

If we have proven the result for U open, then there are $U_{i,m} \subseteq X_i$ open such that $U \subseteq \bigcup_{m\in\mathbb{N}} U_{1,m} \times U_{2,m}$ and

$$\begin{aligned}[\mathrm{m}_1 \times \mathrm{m}_2](U) &\leq \sum_{m\in\mathbb{N}} \mathrm{m}_1(U_{1,m})\, \mathrm{m}_2(U_{2,m}) \\ &< [\mathrm{m}_1 \times \mathrm{m}_2](U) + \varepsilon.\end{aligned} \tag{5.1.4.35}$$

Hence,

$$\begin{aligned}[\mathrm{m}_1 \times \mathrm{m}_2](S) &\leq [\mathrm{m}_1 \times \mathrm{m}_2](U) \\ &\leq \sum_{m\in\mathbb{N}} \mathrm{m}_1(U_{1,m})\, \mathrm{m}_2(U_{2,m}) \\ &< [\mathrm{m}_1 \times \mathrm{m}_2](U) + \varepsilon \\ &< [\mathrm{m}_1 \times \mathrm{m}_2](S) + 2\varepsilon,\end{aligned} \tag{5.1.4.36}$$

which gives (5.1.4.33), as desired.

It thus suffices to prove (5.1.4.33) for $S = U \subseteq X$ open. So, let $U \subseteq X_1 \times X_2$ be open. First suppose that $[\mathrm{m}_1 \times \mathrm{m}_2](U) = \infty$. We wish to show that $\sum_{m\in\mathbb{N}} \mathrm{m}_1(U_{1,m})\, \mathrm{m}_2(U_{2,m}) = \infty$ whenever $\bigcup_{m\in\mathbb{N}} U_{1,m} \times U_{2,m} \supseteq U$ and $U_{i,m} \subseteq X_i$ is open. As this sum is at least as large as $[\mathrm{m}_1 \times \mathrm{m}_2]\left(\bigcup_{m\in\mathbb{N}} U_{1,m} \times U_{2,m}\right)$, it suffices to show that this measure is infinite. However, this is infinite because it contains U, which is of infinite measure. Thus, it suffices to prove the result in the case U has finite measure.

So now suppose that $[\mathrm{m}_1 \times \mathrm{m}_2](U) < \infty$. Let $K_0 \subseteq U$ be a finite disjoint union of compact rectangles such that

$[\mathrm{m}_1 \times \mathrm{m}_2](U \setminus K_0) < 2^{-0}$. $U \setminus K_0$ is open, and so again pick a finite disjoint union of compact rectangles $K_1 \subseteq U \setminus K_0$ such that $[\mathrm{m}_1 \times \mathrm{m}_2]((U \setminus K_0) \setminus K_1) < 2^{-1}$. Proceeding inductively, let $K_{m+1} \subseteq U \setminus (K_0 \cup K_1 \cup \cdots \cup K_m)$ be a finite disjoint union of compact rectangles such that $[\mathrm{m}_1 \times \mathrm{m}_2](U \setminus (K_0 \cup \cdots \cup K_m) \setminus K_{m+1}) < 2^{-(m+1)}$. It follows that $[\mathrm{m}_1 \times \mathrm{m}_2]\left(U \setminus \bigcup_{m \in \mathbb{N}} K_m\right) = 0$, and so, as we can approximate the countably many disjoint compact rectangles appearing in $\bigcup_{m \in \mathbb{N}} K_m$ with open rectangles as good as we like, if we can prove the results for G_δ sets of measure 0 (with the open sets appearing in the intersection of finite measure), we will be done.

So, let $G \subseteq X_1 \times X_2$ be G_δ of measure 0, and write $G = \bigcap_{m \in \mathbb{N}} U_m$ for $U_m \subseteq X_1 \times X_2$ open of finite measure. By replacing U_m with $\bigcap_{k=0}^{m} U_k$, we can without loss of generality assume that this is a nonincreasing sequence $U_m \supseteq U_{m+1}$. Then, as these sets have finite measure, we have that

$$0 = [\mathrm{m}_1 \times \mathrm{m}_2](G) = \lim_m [\mathrm{m}_1 \times \mathrm{m}_2](U_m). \qquad (5.1.4.37)$$

Thus, for every $\varepsilon > 0$, we can pick U_{m_0} with measure at most ε. Now, using the fact that we have proven the result for open sets of finite measure, cover U_{m_0} with open rectangles the sum of whose measures is within ε of $[\mathrm{m}_1 \times \mathrm{m}_2](U_{m_0})$. Then, the sum of these measures will be within 2ε of 0, proving the result of G, thereby completing the proof of this step.

Step 15: Show that $[\mathrm{m}_1 \times \mathrm{m}_2](S_1 \times S_2) = \mathrm{m}_1(S_1)\,\mathrm{m}_2(S_2)$ for $S_i \subseteq X_i$

We have actually already done the case where $\mathrm{m}_i(S_i) < \infty$—see Step 6. So, suppose that at least one of S_1 and S_2 has infinite measure. Without loss of generality, suppose that $\mathrm{m}_1(S_1) = \infty$.

Let us first suppose that $\mathrm{m}_2(S_2) > 0$. In this case, we wish to show that $[\mathrm{m}_1 \times \mathrm{m}_2](S_1 \times S_2) = \infty$. Let $U \supseteq S_1 \times S_2$ be open. By outer-regularity, it suffices to show that $[\mathrm{m}_1 \times \mathrm{m}_2](U) = \infty$.

First of all, for $\langle x, y\rangle \in U$, let $U_{x,y} \times V_{x,y} \subseteq U$ be an open neighborhood of $\langle x, y\rangle$. Define

$$P_x := \{y \in X_2 : \langle x, y\rangle \in U\} . \tag{5.1.4.38}$$

for $x \in S_1$. As $U \supseteq S_1 \times S_2$, we have that

$$\begin{aligned} P_x &\supseteq \{y \in X_2 : \langle x, y\rangle \in S_1 \times S_2\} \\ &= \begin{cases} S_2 & \text{if } x \in S_1 \\ \emptyset & \text{if } x \notin S_1 . \end{cases} \end{aligned} \tag{5.1.4.39}$$

Note that P_x is open (and has positive measure for $x \in S_1$ because S_2 has positive measure). So, for $x \in S_1$, let $M_x < \mathrm{m}_2(P_x)$ be arbitrary and choose some $L_x \subseteq P_x$ compact such that $\mathrm{m}(L_x) > M_x$. $\{V_{x,y} : y \in P_x\}$ is an open cover of L_x, and so there are finitely many $y_1, \ldots , y_{n_x} \in P_x$ such that

$$L_x \subseteq V_{x,y_1} \cup \cdots \cup V_{x,y_{n_x}} . \tag{5.1.4.40}$$

Define

$$U_x := U_{x,y_1} \cap \cdots \cap U_{x,y_{n_x}} \text{ and } V_x := V_{x,y_1} \cup \cdots \cup V_{x,y_{n_x}},$$

and

$$U_1 := \bigcup_{x \in S_1} U_x . \tag{5.1.4.41}$$

Note that for every $x \in S_1$

$$\begin{aligned} U_x \times V_x &= \bigcup_{k=1}^{n_x} \left(U_{x,y_1} \cap \cdots \cap U_{x,y_{n_x}}\right) \times V_{x,y_k} \\ &\subseteq \bigcup_{k=1}^{n_x} U_{x,y_k} \times V_{x,y_k} \subseteq U. \end{aligned} \tag{5.1.4.42}$$

As $x \in U_x$, $U_1 \supseteq S_1$, and so $\mathrm{m}_1(U_1) = \infty$ because $\mathrm{m}_1(S_1)$. So, for every $M > 0$, let $K_M \subseteq U_1$ be compact and such that

$\mathrm{m}_1(K_M) > M$. $\{U_x : x \in S_1\}$ is an open cover of K_M, and so there are finitely may $x_1, \dots, x_m \in S_1$ such that

$$K_M \subseteq U_{x_1} \cup \cdots \cup U_{x_m}. \tag{5.1.4.43}$$

Make this union disjoint by defining $U'_k := U_{x_k} \setminus \bigcup_{l=0}^{k-1} U_{x_l}$, so that we have

$$\begin{aligned}\sum_{k=1}^m \mathrm{m}_1(U'_k) = \mathrm{m}_1\left(\bigcup_{k=0}^m U'_k\right) &= \mathrm{m}_1\left(\bigcup_{k=0}^m U_{x_k}\right) \\ &\geq \mathrm{m}_1(K_M) > M.\end{aligned} \tag{5.1.4.44}$$

Note that

$$\mathrm{m}_2(V_{x_k}) \geq \mathrm{m}_2(L_{x_k}) > \min\{M_{x_1}, \dots, M_{x_m}\}. \tag{5.1.4.45}$$

As $M_{x_k} < \mathrm{m}_2(P_{x_k})$ was arbitrary, in fact we have

$$\mathrm{m}_2(V_{x_k}) \geq \min\{\mathrm{m}_2(P_{x_1}), \dots, \mathrm{m}_2(P_{x_m})\} \geq \mathrm{m}_2(S_2).$$

By (5.1.4.42), we have that $\bigcup_{k=1}^m U'_k \times V_{x_k} \subseteq U$ (because $U'_k \subseteq U_{x_k}$), and so, as this union is disjoint (because the U'_ks are disjoint), we have that

$$\begin{aligned}[\mathrm{m}_1 \times \mathrm{m}_2](U) \geq [\mathrm{m}_1 \times \mathrm{m}_2]&\left(\bigcup_{k=1}^m U'_k \times V_{x_k}\right) \\ &= \sum_{k=1}^m [\mathrm{m}_1 \times \mathrm{m}_2](U'_k \times V_{x_k}) \\ &= \sum_{k=1}^m \mathrm{m}_1(U'_k)\,\mathrm{m}_2(V_{x_k}) \\ &\geq \mathrm{m}_2(S_2) \sum_{k=1}^m \mathrm{m}_1(U'_k) \\ &> \mathrm{m}_2(S_2) M.\end{aligned} \tag{5.1.4.46}$$

As $\mathrm{m}_2(S_2) > 0$ and $M > 0$ is arbitrary, it follows that $[\mathrm{m}_1 \times \mathrm{m}_2](U) = \infty$, as desired.

Now assume that $\mathrm{m}_2(S_2) = 0$. As $[\mathrm{m}_1 \times \mathrm{m}_2](S_1 \times S_2) \leq [\mathrm{m}_1 \times \mathrm{m}_2](X_1 \times S_2)$, it suffices to show that $[\mathrm{m}_1 \times \mathrm{m}_2](X_1 \times S_2) = 0$.

Write $X_1 = \bigcup_{m \in \mathbb{N}} K_m$ for $K_m \subseteq X_1$ compact. Then,

$$\begin{aligned}[\mathrm{m}_1 \times \mathrm{m}_2](X_1 \times S_2) &\leq \sum_{m \in \mathbb{N}} [\mathrm{m}_1 \times \mathrm{m}_2](K_m \times S_2) \\ &=^{o} \sum_{m \in \mathbb{N}} \mathrm{m}_1(K_m)\, \mathrm{m}_2(S_2) = 0,\end{aligned} \tag{5.1.4.47}$$

and so $[\mathrm{m}_1 \times \mathrm{m}_2](X_1 \times S_2) = 0$, as desired. ■

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[a]Okay, you caught me. This is not necessarily true if $K_{2,k} = \emptyset$. Congratulations. Would you like a cookie?

[b]Note that, after throwing away the intermediate steps, this inequality reads $\mathrm{m}_1(U_1)\, \mathrm{m}_2(U_2) - \varepsilon < \mathrm{m}_1(K_1)\, \mathrm{m}_2(K_2) \leq \mathrm{m}_1(U_1)\, \mathrm{m}_2(U_2)$, which, by our good old buddy from Chapter 1, Proposition 1.4.1.13, is precisely the statement that $\mathrm{m}_1(U_1)\, \mathrm{m}_2(U_2)$ is the supremum of (5.1.4.5).

[c]Because $[\mathrm{m}_1 \times \mathrm{m}_2](S)$ is defined to be the infimum of this set—see (5.1.4.5).

[d]By Step 4.

[e]Similarly as in (5.1.4.14), note that, after throwing away the intermediate steps, this inequality reads $\mathrm{m}_1(S_1)\, \mathrm{m}_2(S_2) \leq \mathrm{m}_{\mathrm{U}} < \mathrm{m}_1(S_1)\, \mathrm{m}_2(S_2) + \varepsilon$, which, by our good old buddy from Chapter 1, Proposition 1.4.1.13, is precisely the statement that $\mathrm{m}_1(S_1)\, \mathrm{m}_2(S_2)$ is the infimum of (5.1.4.5).

[f]This probably seems a bit awkward, but keep in mind that m_{K} was only defined for compact rectangles and m_{U} was only defined for open sets, so in order to even ask the question "Are these equal?", we must be dealing with a set that is both a compact rectangle *and* open.

[g]Imagine a finite disjoint union of rectangles contained inside another big rectangle.

[h]This is the statement that the 'bounding rectangle' of the union of rectangles will still be contained inside the big rectangle.

[i]By (5.1.4.19).

[j]Indeed, the definition was designed for the purpose of being regular.

[k]Because $\mathrm{m}_1 \times \mathrm{m}_2$ is nondecreasing.

[l]Because of the definition of $\mathrm{m}_{\mathrm{U}}(U) = [\mathrm{m}_1 \times \mathrm{m}_2](U)$ as the suprema of this sum over all finite disjoint unions of compact rectangles contained in U.

[m]By the previous step, because these two collections of rectangles are disjoint.

[n]Here, m denotes *any* regular borel measure on $X_1 \times X_2$ such that $\mathrm{m}(K_1 \times K_2) = \mathrm{m}_1(K_1)\, \mathrm{m}_2(K_2)$ for $K_i \subseteq X_i$ compact.

[o]By Step 6—the case when both sets have finite measure.

Be careful: If you've studied product measures before with the use of σ-algebras, there is going to be what will seem like a little bit weird behavior.

Example 5.1.4.48 — A measurable set $M \subseteq X_1 \times X_2$ for which $M_{x_1} \subseteq X_2$ is not measurable for any x_1 First of all, we have used the notation here

$$M_{x_1} := \{x_2 \in X_2 : \langle x_1, x_2\rangle \in M\} . \tag{5.1.4.49}$$

Let m_1 be the Zero Measure on X_1 (which, for the sake of concreteness, you can take to be $\mathbb{R}$ if you like), and let m_2 be Lebesgue measure on $\mathbb{R}$. Let $N \subseteq X_2$ be a nonmeasurable set,[a] and define $M := X_1 \times N$.

The product measure $\mathrm{m}_1 \times \mathrm{m}_2$ is the Zero Measure again, and so every subset of $X_1 \times X_2$ is measurable. In particular, M is measurable. On the other hand, $M_{x_1} = N \subseteq X_2$ is not measurable for all $x_1 \in X_1$.

R On the other hand, this is true for *almost-every* x_1—Exercise 5.2.3.51.

[a]Such a set exists by Example 5.1.5.33.

The product of two isogeneous spaces is canonically an isogeneous spaces, and if they are both T_0 locally quasicompact, so too will the product be. Hence, the Haar-Howes Theorem tells us that the product will obtain a measure. The question, then, is whether this measure agrees with the product measure as defined above. Fortunately, the answer is in the affirmative.

Definition 5.1.4.50 — Product isogeneous structure Let $\langle X_1, \Phi_1\rangle$ and $\langle X_2, \Phi_2\rangle$ be isogeneous spaces. Then, the ***product isogeneous structure*** on $X_1 \times X_2$ is given by

$$\Phi_1 \times \Phi_2 := \{\phi_1 \times \phi_2 : \phi_1 \in \Phi_1,\ \phi_2 \in \Phi_2\}, \tag{5.1.4.51}$$

where $\phi_1 \times \phi_2 : X_1 \times X_2 \to X_1 \times X_2$ is defined by

$$[\phi_1 \times \phi_2](\langle x_1, x_2 \rangle) := \langle \phi_1(x_1), \phi_2(x_2) \rangle. \tag{5.1.4.52}$$

Exercise 5.1.4.53 Show that this in fact gives an isogeneous structure.

R In other words, you need to show that

$$\begin{aligned} \{&\mathcal{B}_{U_1 \times U_2} : \\ &U_i \subseteq X_i \text{ nonempty open.}\} \end{aligned} \tag{5.1.4.54}$$

is a uniform base, where

$$\begin{aligned} \mathcal{B}_{U_1 \times U_2} := \{&\phi_1(U_1) \times \phi_2(U_2) : \\ &\phi_1 \in \Phi_1,\ \phi_2 \in \Phi_2\}. \end{aligned}$$

Proposition 5.1.4.55 Let $\langle X_1, \Phi_1 \rangle$ and $\langle X_2, \Phi_2 \rangle$ be T_0 locally quasicompact isogeneous spaces, let $K_i \subseteq X_i$ be quasicompact with nonempty interior, and let m_i be the unique isogeneous measure on X_i such that $\mathrm{m}_i(K_i) = 1$. Then, $\mathrm{m}_1 \times \mathrm{m}_2$ is the unique isogeneous measure on $\langle X_1 \times X_2, \Phi_1 \times \Phi_2 \rangle$ such that $[\mathrm{m}_1 \times \mathrm{m}_2](K_1 \times K_2) = 1$.

Proof. We leave this as an exercise.

Exercise 5.1.4.56 Prove this yourself.

■

Finally, we mentioned tangentially in a remark of the Haar-Howes Theorem that neither one of locally (quasi)compact and σ-(quasi)compact imply the other. We end this section with the relevant counter-examples.

Exercise 5.1.4.57

(i). Find an example of a topological space that is locally compact but not σ-quasicompact.
(ii). Find an example of a topological space that is σ-compact but not locally quasicompact.

R Hint: See [SJ70, p. 58] for (ii). (i) should be quite a bit easier.

5.1.5 Lebesgue measure

Before continuing on with integration, we prove some properties about Lebesgue measure. Knowing that Lebesgue measure on $\mathbb{R}^d$ is just the product of the Lebesgue measure on $\mathbb{R}$ is actually incredibly useful,[6] and so now that we know this, we take advantage of this fact.

Proposition 5.1.5.1 The Lebesgue measure of a point is 0.

W Warning: This is most definitely not true for general measures. Counter-example?

Proof. The argument is the exact same in $\mathbb{R}^d$ as it is in $\mathbb{R}$, so we write down the argument in $\mathbb{R}$ and save us from some $\cdots$.

The first thing to notice is that, by translation invariance, the measure of every point is the same. Let us denote this measure by M.

[6]Lebesgue measure on $\mathbb{R}^d$ is defined to be the unique isogeneous measure with respect to the group of isometries on $\mathbb{R}^d$ (that assign measure 1 to the unit 'cube'). On the other hand, by Proposition 5.1.4.55, this is just the product of Lebesgue measure on $\mathbb{R}$.

Points are closed, and hence measurable, and so m is additive on points. Therefore, we have that

$$\begin{aligned} 1 = \mathrm{m}([0,1]) &\geq \mathrm{m}\left(\bigcup_{x\in[0,1]\cap\mathbb{Q}} \{x\} \right) \\ &= \sum_{x\in[0,1]\cap\mathbb{Q}} \mathrm{m}(\{x\}) = \sum_{m\in\mathbb{N}} M. \end{aligned} \tag{5.1.5.2}$$

This equation forces $M = 0$. ■

Corollary 5.1.5.3 $\mathrm{m}([0,1)\times\cdots\times[0,1)) = 1$.

R Sets of the form

$$[a_1, b_1)\times\cdots\times[a_d, b_d) \tag{5.1.5.4}$$

are important in measure theory because, for example,

$$[0,2) = [0,1)\cup[1,2) \tag{5.1.5.5}$$

is a *disjoint* union. If we tried replacing everything here with all open intervals we we would have that $(0,2) = (0,1)\cup(1,2)$, which is just plain false, and if we tried replacing everything here with all closed intervals the union would not be disjoint ($[0,1]$ and $[1,2]$ intersect at 1). The disjointness is important in measure theory of course because of additivity (on measurable sets). Sets of the form (5.1.5.4) are called ***half-open rectangles*** or ***closed-open rectangles***.

Proof. Now that we know that

$$\mathrm{m}([0,1)\times\cdots\times[0,1)) = \mathrm{m}([0,1))\cdots\mathrm{m}([0,1)), \tag{5.1.5.6}$$

it suffices to prove this result in $\mathbb{R}$.

That it is true in $\mathbb{R}$ follows from the fact that

$$1 = \mathrm{m}([0,1]) = \mathrm{m}([0,1)) + \mathrm{m}(\{1\}) = \mathrm{m}([0,1)). \tag{5.1.5.7}$$

■

Proposition 5.1.5.8 $\mathrm{m}([a_1, b_1] \times \cdots \times [a_d, b_d]) = (b_1 - a_1) \cdots (b_d - a_d)$.

Proof. Now that we know that it is a product measure, it suffices to show the one-dimensional case, and so we prove that $\mathrm{m}([a, b]) = b - a$.

Exercise 5.1.5.9 Prove the cases where at least one of a of b is infinite.

By translation invariance, it suffices to show that $\mathrm{m}([0, b]) = b$.

Because the measure of points is 0, it suffices to show that $\mathrm{m}([0, b)) = b$.

Exercise 5.1.5.10 Show that $\mathrm{m}([0, m)) = m$ for $m \in \mathbb{N}$.

Exercise 5.1.5.11 Show that $\mathrm{m}([0, \frac{p}{q})) = \frac{p}{q}$ for $p, q \in \mathbb{Z}^+$.

Then, we have that

$$q = \mathrm{m}([0, q)) \leq \mathrm{m}([0, b)) \leq \mathrm{m}([0, r)) = r \qquad (5.1.5.12)$$

for all $p, q \in \mathbb{Q}^+$ with $q \leq b \leq r$. It follows that $\mathrm{m}([0, b)) = b$. ■

Exercise 5.1.5.13 Show that any open set in $\mathbb{R}^d$ can be written at the countable disjoint union of half-open rectangles.

Exercise 5.1.5.14 Let X be a regular measure space and for $S \subseteq X \times \mathbb{R}^d$ and $a \in \mathbb{R}$, define

$$aS := \{\langle x, ay \rangle : \langle x, y \rangle \in S\}, \qquad (5.1.5.15)$$

where $x \in X$ and $y \in \mathbb{R}^d$. Show that

(i).

$$m(aS) = |a|^d \, m(S); \tag{5.1.5.16}$$

and

(ii). if S is measurable, then aS is measurable.

R Hint: Prove it for $\mathbb{R}^d$ alone first (i.e. if X is just a point, so that $X \times \mathbb{R}^d = \mathbb{R}^d$). For this case, prove it for S open using the previous exercise and the fact that we already know it is true for half-open rectangles. Then prove it in general for $\mathbb{R}^d$ by outer-regularity. Then generalize this to the case $X \times \mathbb{R}^d$ for X not-necessarily a point using the definition of product measures.

Exercise 5.1.5.17 Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a linear transformation and let $S \subseteq \mathbb{R}^d$. Show that

$$m(T(S)) = |\det(T)| \, m(S). \tag{5.1.5.18}$$

R Hint: Recall your matrix decompositions from linear algebra and use the fact that we already know (by Haar-Howes Theorem (Theorem 5.1.3.11)) that Lebesgue measure is invariant under isometries together with the result of the previous exercise.

Exercise 5.1.5.19 Show that countable sets have Lebesgue measure 0.

R In particular, $\mathbb{Q}$ has 0 measure!

R Note that the converse is false—see Example 5.1.5.28.

Generalized Cantor sets

We next turn to a couple relatively famous subsets of the real line with particularly interesting properties. One such such, the *Cantor Set*, will serve as our first example of an uncountable set of measure zero.[7] Another such set, the *Cantor-Smith-Volterra Set*, will serve as our first example of a set with empty interior but positive measure.[8] Both of these sets are in fact just particular cases of a family of sets known as *Generalized Cantor sets*, and we begin with these.

Example 5.1.5.20 — Generalized Cantor sets For $m \in \mathbb{N}$, let $\alpha_m \in (0,1)$. For every distinct choice of the α_ms, we obtain a set C defined as below. Sets of this form are known as ***generalized Cantor sets***.

We shall define a decreasing sequence of sets $[0,1] =: C_0 \supset C_1 \supset C_2 \supset \cdots$ recursively in stages, and then finally we will define $C := \bigcap_{m\in\mathbb{N}} C_m$.

We will see that C_k is a disjoint union of 2^k closed intervals, all of the same length. C_{k+1} will then be obtained from C_k by removing the middle α_k^{th} of each of these 2^k intervals. That is, α_k is the fraction of the length that you are removing, so, for example, if the length of each interval in C_k is $\frac{1}{6}$ and $\alpha_k = \frac{1}{5}$, then you will be removing the middle open intervals of length $\frac{1}{5} \cdot \frac{1}{6} = \frac{1}{30}$. This is the basic idea anyways. We now turn to the precise construction.

First of all, note that the set obtained from the interval $[a,b]$ of length $L := b-a$ by removing the middle α^{th} of the interval is

$$\begin{aligned} &[a,b] \setminus \left(\tfrac{a+b}{2} - \alpha\tfrac{L}{2}, \tfrac{a+b}{2} + \alpha\tfrac{L}{2}\right) \\ &\quad = \left[a, \tfrac{a+b}{2} - \alpha\tfrac{L}{2}\right] \cup \left[\tfrac{a+b}{2} + \alpha\tfrac{L}{2}, b\right]. \end{aligned} \tag{5.1.5.21}$$

[7] We already know that countable sets must have Lebesgue measure zero. The existence of the Cantor Set thus shows that the converse is false.

[8] We already know (Why?) that sets of Lebesgue measure zero must have empty interior. The existence of the Cantor-Smith-Volterra Set thus shows that the converse is false.

Note that this is the disjoint union of two closed intervals each with length $\frac{1-\alpha}{2}L$. Using this, we now turn to the actual definitions.

Certainly, $C_0 := [0,1]$ is the disjoint union of $2^0 = 1$ closed intervals all of the same length. Inductively, assuming that C_k is the disjoint union of 2^k closed intervals each of length L, we write

$$C_k = \bigcup_{i=1}^{2^k} [a_i, b_i] \tag{5.1.5.22}$$

and define

$$C_{k+1} := \bigcup_{i=1}^{2^k} \left(\left[a_i, \tfrac{a_i+b_i}{2} - \alpha_k L \right] \cup \left[\tfrac{a_i+b_i}{2} + \alpha_k L, b_i \right] \right). \tag{5.1.5.23}$$

Note that C_{k+1} is again the disjoint union of 2^{k+1} intervals all of the same length $\frac{1-\alpha_k}{2}L$, and so this recursive definition is makes sense. As mentioned before, we now define

$$C := \bigcap_{m\in\mathbb{N}} C_m. \tag{5.1.5.24}$$

Having now defined C, we investigate some of its properties, the first of which being its Lebesgue measure.

From the above, if C_k is the disjoint union of 2^k intervals each of length L, then C_{k+1} will be the disjoint union of 2^{k+1} intervals each of length $\frac{1-\alpha_k}{2}L$. As $\mathrm{m}(C_0) = 1$, we thus inductively find that

$$\mathrm{m}(C_m) = 2^m \cdot \tfrac{1}{2^m} \prod_{k=0}^{m-1} (1-\alpha_k) = \prod_{k=0}^{m-1} (1-\alpha_k). \tag{5.1.5.25}$$

By Exercise 5.1.1.37 ("continuity from above") then, it follows that

$$\begin{aligned} \mathrm{m}(C) &:= \mathrm{m}\left(\bigcap_{m\in\mathbb{N}} C_m\right) = \lim_m (C_m) \\ &= \lim_m \prod_{k=0}^{m-1}(1-\alpha_k) =: \prod_{m\in\mathbb{N}}(1-\alpha_m). \end{aligned} \tag{5.1.5.26}$$

We next show that C is uncountable. We do this in two steps.

Exercise 5.1.5.27 Show that C is perfect.

R Recall that this means that C is equal to its set of accumulation points—see Definition 2.5.2.18.

Thus, in summary,

> C is a nonempty perfect subset of $[0, 1]$ with measure $\prod_{m\in\mathbb{N}}(1-\alpha_m)$.[a]
>
> [a] In particular, it is closed, hence compact, and uncountable (Proposition 2.5.2.20).

Having introduced the family of generalized Cantor sets, we turn to two important special cases.

■ **Example 5.1.5.28 — An uncountable set of zero measure—the Cantor Set** The ***Cantor Set*** is a generalized cantor set with $\alpha_m = \frac{1}{3}$ for all $m \in \mathbb{N}$. In fact, there is nothing particularly special about $\frac{1}{3}$—what is important is that all the α_ms are equal. We do this more general case instead. So, let $L \in (0, 1)$, define $\alpha_m := L$ for all $m \in \mathbb{N}$, and denote by C the resulting generalized Cantor set.

We know from Example 5.1.5.20 that C is nonempty and perfect, hence uncountable by Proposition 2.5.2.20. Furthermore, as $0 < 1 - L < 1$,

$$\mathrm{m}(C) = \lim_m \prod_{k=0}^{m-1} (1 - L) = \lim_m (1 - L)^m = 0. \quad (5.1.5.29)$$

Thus, C is indeed an uncountable set of measure 0.

Example 5.1.5.30 — A set with empty interior of positive measure—the Cantor-Smith-Volterra Set The Cantor-Smith-Volterra Set is in fact a generalized Cantor set, though that is not how we shall define it.[a]

Exercise 5.1.5.31 Modify construction of a generalized Cantor set starting with $C_0 := [0, 1]$ again, but upon constructing C_{k+1} from C_k, remove the middle open interval of length $\frac{1}{4^{k+1}}$ from each closed interval of C_k. The resulting set is the ***Cantor-Smith-Volterra Set***. Show that it has measure $\frac{1}{2}$, but has empty interior.

R Note that generalized Cantor sets were constructed by removing a given *fraction* of each interval at each step, whereas in this case, what you are removing is not 'relative', but rather, 'absolute': at the k^{th} step, you remove intervals of length $\frac{1}{4^k}$ *period*.

R The Cantor-Smith-Volterra Set is still uncountable of course, just as the Cantor Set likewise had empty-interior. It's just that it's not surprising for a set with empty interior to have measure 0—what is surprising however is a set with empty interior of *positive measure*. Similarly, it's not surprising for a uncountable set to have positive measure—what is surprising is an uncountable set of *measure zero*.

Exercise 5.1.5.32 Show that the Cantor-Smith-Volterra Set is in fact a generalized Cantor set.

> R That is, find $\alpha_m \in (0,1)$ for $m \in \mathbb{N}$ so that the resulting generalized Cantor set is the Cantor-Smith-Volterra set.

[a] You'll see why in a moment—the definition is straightforward, and finding the appropriate α_ms is unnecessary tedium.

■ **Example 5.1.5.33 — A set that is not Lebesgue-measurable** [a] Define $\mathrm{q}: \mathbb{R} \to [0,1)$ by $\mathrm{q}(x) := x - \lfloor x \rfloor$. In other words, this is just the "fractional part" of the real number (intuitively, you drop of the integer in front of its decimal expansion). Note that 1 gets sent to 0, and so, if you like, you can think of the image as a circle, with 1 'glued to' 0, if this helps your intuition.

Now fix $\theta \in \mathbb{Q}^{\mathsf{C}}$ and define $R: \mathbb{R} \to \mathbb{R}$ by $R(x) := x + \theta$.[b] For $x_1, x_2 \in \mathbb{R}$, define $x_1 \sim x_2$ iff there is some $m \in \mathbb{Z}$ such that $x_1 = R^m(x_2) := x_2 + m\theta$.

Exercise 5.1.5.34 Show that $\sim$ is an equivalence relation on $\mathbb{R}$.

Denote by O_x the equivalence class of $x \in \mathbb{R}$ with respect to $\sim$.[c]

Exercise 5.1.5.35 Show that $\mathrm{q}(O_x)$ is dense in $[0,1)$.

> R Hint: This is why we needed θ to be *irrational*.

Now, let $N \subseteq \mathbb{R}$ be any set which contains exactly one point from each equivalence class with respect to $\sim$. We claim that

$q(N) \subseteq [0,1)$ is not measurable. We proceed by contradiction: suppose that $q(N)$ is measurable.

Exercise 5.1.5.36 Use the fact that $\sim$ is an equivalence relation to show that $q(R^m(N))$ and $q(R^n(N))$ are disjoint iff $m \neq n$.

R Hint: See Proposition A.3.2.10—equivalence classes form a partition of the set.

Exercise 5.1.5.37 Show that

$$[0,1) = \bigcup_{m\in\mathbb{Z}} q(R^m(N)) \tag{5.1.5.38}$$

R Hint: Once again, uses the fact that equivalence classes form a partition.

Exercise 5.1.5.39 Let $S \subseteq \mathbb{R}$. Show that $q(S)$ is measurable iff $q(R(S))$ is measurable.

Exercise 5.1.5.40 Let $S \subseteq \mathbb{R}$. Show that $m(q(S)) = m(q(R(S)))$.

Thus, these exercises give us that

$$[0,1) = \bigcup_{m\in\mathbb{Z}} q(R^m(N)) \tag{5.1.5.41}$$

is a disjoint union of measurable sets, all of which have the same measure $M := q(N)$. If $M = 0$, then, by additivity, we have $m([0,1)) = 0$: a contradiction. On the other hand, if $M > 0$, by additivity again, we have $m([0,1)) = \infty$: a contradiction. Therefore, it cannot be the case that N is measurable.

[a]Construction adapted from [Pug02, pg. 407].

[b]The "R" is for "rotation" because in the "circle" picture, this will correspond to a rotation by angle θ.

[c]The "O" is for "orbit"—you can imagine the point x "orbiting" around the circle as you apply R to it over-and-over.

Example 5.1.5.42 — Two disjoint sets S, T with $\mathrm{m}(S \cup T) \neq \mathrm{m}(S) + \mathrm{m}(T)$ Let N be as in the previous example. We showed there that it was not measurable. Therefore, there must exist some $S \subseteq \mathbb{R}$ such that

$$\begin{aligned}\mathrm{m}&\left((S \cap N) \cup (S \cap N^{\mathsf{C}})\right) \\ &= \mathrm{m}(S) \\ &\neq \mathrm{m}(S \cap N) + \mathrm{m}(S \cap N^{\mathsf{C}}).\end{aligned} \tag{5.1.5.43}$$

As a matter of fact, a similar sort of trick can be adapted to prove that *every* set of positive measure has a nonmeasurable subset.

Proposition 5.1.5.44 Let $S \subseteq \mathbb{R}^d$. Then, if $\mathrm{m}(S) > 0$, then S has a nonmeasurable subset.

W Warning: The analogous result for "measurable" in place of "not measurable" is false. Obviously, every set of positive measure (in fact, every set period) has a measurable subset: the empty-set. Instead, what you might guess is that every set of positive measure has a *measurable* subset of *positive measure*. This, however, is false—see Example 5.1.5.50.

Proof. [a] Suppose that $\mathrm{m}(S) > 0$. If $\mathrm{m}(S) = \infty$, then because Lebesgue measure is semifinite (Proposition 5.1.2.34), there is some subset $T \subseteq S$ with $0 < \mathrm{m}(T) < \infty$. If T contains a set that is not measurable, then of course so does S, so it suffices to prove the case where S has *finite* measure. Thus, without loss of generality, suppose that $\mathrm{m}(S) < \infty$.

Consider the collection of cosets[b] $\{x + \mathbb{Q}^d : x \in \mathbb{R}^d\}$. Let $N \subseteq \mathbb{R}^d$ be a set that contains precisely one element of each coset. It follows that[c]

$$\mathbb{R}^d = \bigcup_{r \in \mathbb{Q}^d} (r + N) \tag{5.1.5.45}$$

is a disjoint union, and so it in turn follows that

$$S = \bigcup_{r \in \mathbb{Q}^d} [S \cap (r + N)] \tag{5.1.5.46}$$

is a disjoint union.

We proceed by contradiction: suppose that every subset of S is measurable. Then, the previous equality implies that

$$\mathrm{m}(S) = \sum_{r \in \mathbb{Q}^d} \mathrm{m}(S \cap (r + N)) \tag{5.1.5.47}$$

We show that $\mathrm{m}(S \cap (r + N)) = 0$ for all $r \in \mathbb{Q}$, which will gives us that $\mathrm{m}(S) = 0$: a contradiction.

As $\mathrm{m}(S \cap (r + N)) < \infty$ and $\mathbb{R}$ is inner-regular on measurable sets (Proposition 5.1.2.26), it suffices to show that every quasicompact subset of $S \cap (r + N)$ has measure 0. So, let $K \subseteq S \cap (r + N)$ be quasicompact.

Define

$$H \coloneqq \bigcup_{s \in \mathbb{Q}^d \cap [0,1]} (s + K). \tag{5.1.5.48}$$

As $K \subseteq r + N$, this is a disjoint union (for the same reason that (5.1.5.45) is a disjoint union), and so we have

$$\mathrm{m}(H) = \sum_{s \in \mathbb{Q}^d \cap [0,1]} \mathrm{m}(s + K) =^d \sum_{s \in \mathbb{Q}^d \cap [0,1]} \mathrm{m}(K). \tag{5.1.5.49}$$

H is bounded, and so has finite measure. Thus, the above equality forces $\mathrm{m}(K) = 0$, and we are done. ■

[a] Proof adapted from [Rud87, pg. 53].

[b] See Proposition A.4.1.1 for the definition of cosets, though this definition will not actually be important for us here.

[c] This follows from the fact that the cosets form a partition of $\mathbb{R}$, which in turn follows from the fact that equivalence classes form partitions—see Corollary A.3.2.11.

[d] By translation invariance.

Example 5.1.5.50 — A set with no measurable subsets of positive measure Let $\mathrm{q}\colon \mathbb{R} \to [0,1)$ and N be as in Example 5.1.5.33. Recall that $\mathrm{q}(x) := [x] - \lfloor x \rfloor$ and that N is a set that contains exactly one element of each equivalence class with respect to the equivalence relation $x \sim y$ iff there is some $m \in \mathbb{Z}$ such that $x_1 - x_2 = m\theta$, where $\theta \in \mathbb{Q}^{\mathsf{C}}$ is some fixed irrational number. We claim that $\mathrm{q}(N)$ has no measurable subset of positive measure.

So, let $M \subseteq \mathrm{q}(N)$ be measurable. We wish to show that $\mathrm{m}(M) = 0$. Define $M' := \mathrm{q}^{-1}(M)$, so that $\mathrm{q}(M') = M$. In Example 5.1.5.33, we showed that

$$[0,1) = \bigsqcup_{m\in\mathbb{Z}} \mathrm{q}(R^m(N)) \tag{5.1.5.51}$$

is a disjoint union, where $R\colon \mathbb{R} \to \mathbb{R}$ is defined by $R(x) := x + \theta$. It follows that

$$\bigsqcup_{m\in\mathbb{Z}} \mathrm{q}(R^m(M')) \tag{5.1.5.52}$$

is a disjoint union contained in $[0,1)$, and furthermore, a disjoint union of measurable sets as M is measurable. Therefore,

$$\begin{aligned} 1 &\geq \sum_{m\in\mathbb{Z}} \mathrm{m}(\mathrm{q}(R^m(M')) = \sum_{m\in\mathbb{Z}} \mathrm{m}(\mathrm{q}(R^0(M')) \\ &= \sum_{m\in\mathbb{Z}} \mathrm{m}(M), \end{aligned} \tag{5.1.5.53}$$

which implies that $\mathrm{m}(M) = 0$, as desired.

We mentioned awhile ago in the definition of measurable functions (Definition 5.1.1.39) the existence of a uniform-homeomorphism of $\mathbb{R}$ that preserves neither measurability not measure 0. It is time we return to this.

Example 5.1.5.54 — A uniform-homeomorphism of $\mathbb{R}$ that preserves neither measurability nor measure 0—the Cantor Function We first define a uniformly-continuous function $f : [0,1] \to [0,1]$. This function will be nondecreasing, and so $g := f + \mathrm{id}_{[0,1]} : [0,1] \to [0,2]$ will be increasing, and hence injective. It will turn out that $f(0) = 0$ and $f(1) = 1$, so that $g(0) = 0$ and $g(1) = 2$, and so we will then extend g to all of $\mathbb{R}$ 'periodically', defining $g(x)$ to be $g(x-1)+2$ for $x \in [1,2]$, etc..[a]

R f is the ***Devil's Staircase***, and g is the ***Cantor Function***.[b]

We define $f : [0,1] \to [0,1]$ as the uniform limit of a sequence of continuous functions. It will then be continuous, because $\mathrm{Mor}_{\mathbf{Top}}([0,1],\mathbb{R})$ is complete, and hence uniformly-continuous because $[0,1]$ is quasicompact (by the Cantor-Heine Theorem (Proposition 4.1.3.25)). We then must check that f is nondecreasing. Once we do so, we will have that g is increasing, and hence injective. Then, because the domain $[0,1]$ is quasicompact and the codomain $[0,2]$ is T_2, we will have that its inverse is continuous (Exercise 3.6.2.17 does this for us), and hence uniformly-continuous, and hence a uniform-homeomorphism. Extending g 'periodically' has no effect on this, that is to say, the period extension to all of $\mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, will still be a uniform-homeomorphism.

So, let us get started.[c] Define $f_0 : [0,1] \to [0,1]$ to be the identity function. We define $f_m : [0,1] \to \mathbb{R}$ for $m > 0$

recursively:

$$f_m(x) := \begin{cases} \frac{1}{2} f_{m-1}(3x) & \text{if } x \in [0, \frac{1}{3}] \\ \frac{1}{2} & \text{if } x \in [\frac{1}{3}, \frac{2}{3}] \\ \frac{1}{2} + \frac{1}{2} f_{m-1}(3x-2) & \text{if } x \in [\frac{2}{3}, 1]. \end{cases} \qquad (5.1.5.55)$$

Exercise 5.1.5.56 Show that f_m is well-defined for all $m \in \mathbb{N}$.

R You must check that the two expressions agree for $x = \frac{1}{3}$ and $x = \frac{2}{3}$.

Exercise 5.1.5.57 Show that $f_m(0) = 0$ and $f_m(1) = 1$ for all $m \in \mathbb{N}$.

Exercise 5.1.5.58 Show that $f_m(x) \in [0, 1]$, so that indeed f_m defines a function into $[0, 1]$.

Exercise 5.1.5.59 Show that f_m is continuous.

R Hint: Use induction and apply the Pasting Lemma (Proposition 3.1.3.14).

Exercise 5.1.5.60 Show that f_m is nondecreasing.

We now check that $m \mapsto f_m \in \mathrm{End}_{\mathbf{Top}}([0, 1])$ is Cauchy.[d] To do this, we will show by induction that

$$\|f_{m+1} - f_m\| \leq \left(\frac{1}{2}\right)^{m+1}. \qquad (5.1.5.61)$$

It will then follow from the Triangle Inequality that

$$\begin{aligned}\|f_n - f_m\| &\le \sum_{k=m+1}^{n}\left(\frac{1}{2}\right)^k \le \sum_{k=m+1}^{\infty}\left(\frac{1}{2}\right)^k \\ &= \frac{1}{1-\frac{1}{2}} - \frac{1-(\frac{1}{2})^{m+1}}{1-\frac{1}{2}} = \left(\frac{1}{2}\right)^m,\end{aligned} \tag{5.1.5.62}$$

from which Cauchyness follows immediately.

Using the definition (5.1.5.55), we have that

$$f_1(x) := \begin{cases} \frac{1}{2}x & \text{if } x \in [0, \frac{1}{3}] \\ \frac{1}{2} & \text{if } x \in [\frac{1}{3}, \frac{2}{3}] \\ \frac{1}{2}(3x-1) & \text{if } x \in [\frac{2}{3}, 1], \end{cases} \tag{5.1.5.63}$$

and so

$$|f_1(x) - f_0(x)| = \begin{cases} \frac{1}{2}x & \text{if } x \in [0, \frac{1}{3}] \\ \left|x - \frac{1}{2}\right| & \text{if } x \in [\frac{1}{3}, \frac{2}{3}] \\ \frac{1}{2}(1-x) & \text{if } x \in [\frac{2}{3}, 1]. \end{cases} \tag{5.1.5.64}$$

It follows that

$$\|f_1 - f_0\| = \tfrac{1}{6} \le \left(\tfrac{1}{2}\right)^{0+1}. \tag{5.1.5.65}$$

This completes the base case. As for the inductive case, fix $m \ge 1$ and assume that (5.1.5.61) holds for $0 \le n < m$. We prove that it holds for m as well. From the definition (5.1.5.55) again, we have that

$$\begin{aligned}&|f_{m+1}(x) - f_m(x)| \\ &= \tfrac{1}{2}\cdot\begin{cases} |f_m(3x) - f_{m-1}(3x)| & \text{if } x \in [0, \frac{1}{3}] \\ 0 & \text{if } x \in [\frac{1}{3}, \frac{2}{3}] \\ |f_m(3x-2) - f_{m-1}(3x-2)| & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}\end{aligned}$$

Thus, by the induction hypothesis,

$$\|f_{m+1} - f_m\| \le \tfrac{1}{2}\|f_m - f_{m-1}\| = \left(\frac{1}{2}\right)^{m+1}. \tag{5.1.5.66}$$

We may now define $f := \lim_m f \in \mathrm{End}_{\mathbf{Top}}([0,1])$. You showed that each f_m is nondecreasing, and so, as limits preserve inequalities, f itself is nondecreasing. Now define $g(x) := f(x)+x$. As described at the beginning of the example, $g\colon [0,1] \to [0,2]$ is a uniform-homeomorphism.

We now show that g preserves neither measurability nor measure 0. Let $C \subset [0,1]$ denote the ($L = \frac{1}{3}$) Cantor Set and let C_m denote the set defined in the m^{th} step of the construction of the Cantor Set—see Example 5.1.5.28 if you don't know what we're referring to. For convenience of notation, define $D_m := [0,1] \setminus C_m$ and $D := [0,1] \setminus C$.

Exercise 5.1.5.67 Show that $f_k|_{D_m} = f_m|_{D_m}$ is constant[a] on each component of D_m for all $k \geq m$. In particular, the image of f_k on D_m is a finite set of points for all $k \geq m$.

R Though it's perhaps not clear from the formulas, the f_ms were defined so that precisely this is true: even though $f(0) = 0$ and $f(1) = 1$, f does all of increasing on a set of measure 0, namely the Cantor Set.

[a]Note that this constant will depend on the component.

From this, it follows that f itself is constant on each component of D.

Recall that each D_m is a disjoint union of open intervals. As f is constant on each one of these intervals, the measure of the image of each one of these intervals under g is just the length of that interval (the image is the interval itself plus whatever constant f happened to be on that interval). Furthermore, by injectivity, the images of each of these intervals must be disjoint, and hence, $\mathrm{m}(g(D))$ is the sum of the measures of all these intervals, namely, $\mathrm{m}(D) = 1$. As $g([0,1]) = [0,2]$, we thus have that

$$\begin{aligned} \mathrm{m}(g(C)) &\geq \mathrm{m}(g([0,1])) - \mathrm{m}(g([0,1] \setminus C)) \\ &= \mathrm{m}([0,2]) - \mathrm{m}(g(D)) = 2 - 1 = 1. \end{aligned} \tag{5.1.5.68}$$

Thus, $\mathrm{m}(g(C)) \geq 1$, despite the fact that $\mathrm{m}(C) = 0$.[e]

Every subset of $\mathbb{R}$ of positive measure contains a nonmeasurable set (Proposition 5.1.5.44), so let $N \subseteq g(C)$ be some such set. Then, $g^{-1}(N) \subseteq C$ is measurable because it has measure 0. Thus, $M := g^{-1}(N)$ is measurable, but yet $g(M) = N$ is not.[f]

---

[a]Of course, we have to shift the graph up to maintain continuity.

[b]Some people use both these terms to refer to f. I prefer this convention so that each has its own name. Furthermore, f is also known by the names of the ***Cantor-Lebesgue Function***, the ***Lebesgue Function***, among other things—see the Wikipedia page for a more complete list of aliases.

[c]It will probably help to find a picture on this internet of what this is supposed to look like. (Sorry. Making diagrams takes awhile and I have not had the time.) Let C_m denote the 'm^{th} step' in the construction of the Cantor Set (see Example 5.1.5.28). In words, f_m is supposed to be the function that is constant $[0,1] \setminus C_m$ and increases linearly on the intervals that remain in C_m.

[d]Recall that $\mathrm{End}_{\mathbf{Top}}([0,1]) := \mathrm{Mor}_{\mathbf{Top}}([0,1],[0,1])$—see Definition B.2.2. To simplify notation, we shall denote the norm on $\mathrm{End}_{\mathbf{Top}}([0,1])$ simply as $\lVert\cdot\rVert$.

[e]This shows that the preimage of a set of measure 0 under $h := g^{-1}$ need not have measure 0, even though h is a uniform-homeomorphism.

[f]This shows that the preimage of a measurable set under $h := g^{-1}$ need not be measurable, even though h is a uniform-homeomorphism.

Before we finally move on to the integral, we tie up a loose end: we defined $0^0 := 1$ way back in Chapter 2 (see Theorem 2.5.1.20), but we never justified it—it is about time we do so.

Exercise 5.1.5.69 — $0^0 := 1$ Show that, for every $\varepsilon > 0$, there is some open neighborhood U of $\langle 0,0\rangle \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ such that

$$\frac{\mathrm{m}\left(\{\langle x,y\rangle \in U : |x^y - 1| \geq \varepsilon\}\right)}{\mathrm{m}(U)} < \varepsilon. \tag{5.1.5.70}$$

R In words, for every $\varepsilon > 0$, there is an open neighborhood of $\langle 0, 0\rangle$ on which the fraction of the measure of the set on which x^y is more than ε away from 1 is less than ε.

R I totally understand if people wish to leave the symbol 0^0 undefined, but I think this result makes it clear that, *if* 0^0 is to represent any real number at all, then that real number should be 1.

R Perhaps it's still worth keeping in mind that, for any real number $a \in [0, 1]$, there is a net $\lambda \mapsto \langle x_\lambda, y_\lambda\rangle \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ that converges the origin, but for which $\lambda \mapsto (x_\lambda)^{y_\lambda}$ converges to a.

5.2 The integral

5.2.1 Characteristic, simple, Borel, and integrable functions

An important type of function in measure theory are the *simple functions*, of which the *characteristic functions* are an important special case.

Definition 5.2.1.1 — Characteristic function Let X be a set and let $S \subseteq X$. Then, the ***characteristic function*** of S, $\chi_S : X \to \{0, 1\}$, is defined by

$$\chi_S(x) := \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases} \tag{5.2.1.2}$$

R One reason that characteristic functions are important in measure theory is because the integral of χ_S will turn out to simply be $\mathrm{m}(S)$. In particular, *if you know the integral, you know the measure*.

R We have seen a characteristic function before: the Dirichlet Function is the characteristic function of $\mathbb{Q} \subseteq \mathbb{R}$.

R You might say that the Lebesgue integral is to characteristic functions as the Riemann integral is to rectangles—one can define the Lebesgue integral (though we will not) by approximating functions with characteristic functions,[a] just as one defines the Riemann integral by approximating functions with rectangles (aka (scalar multiples of) characteristic functions of intervals).[b]

[a]Well, actually finite linear combinations of characteristic functions, the so-called *simple functions*.

[b]I actually find this approach a bit messy, but it's just yet more evidence that the Lebesgue integral is hardly more difficult than the Riemann integral—it's the same thing with arbitrary (measurable) sets instead of intervals.

Definition 5.2.1.3 — Simple function Let $f\colon X \to Y$ be a function. Then, f is ***simple*** iff $\operatorname{Im}(f)$ is finite.

R That is, iff $\operatorname{Im}(f)$ has a finite number of elements.

R We will only be concerned with the case $Y = [-\infty, \infty]$, in which case this is equivalent to the statement that f a finite nonnegative linear combination of characteristic functions of (disjoint) sets.

In principle, we can integrate *any* function.[9] The 'problem' with this, of course, is that the integral will not satisfy certain properties we would like it to (e.g. $\int \mathrm{d}x\, [f(x)+g(x)] = \int \mathrm{d}x\, f(x) + \int \mathrm{d}x\, g(x)$) if we don't restrict the functions we integrate. This is exactly analogous to how we do *not* have $\mathrm{m}(S \cup T) = \mathrm{m}(S) + \mathrm{m}(T)$ for S and T disjoint—in general we have to assume that S and T are measurable. We thus seek a condition for functions that is analogous to the condition of measurability for sets. The condition we are looking for is what is called *Borel*.

[9]Literally. You don't need the function to be measurable or Borel or whatever—you can always talk about "measure of the 'area' under the curve", it's just that, unless you make some assumptions, this thing will be poorly behaved.

Definition 5.2.1.4 — Borel functions Let X be a measure space and let $f\colon X \to [-\infty, \infty]$ be a function. Then, f is ***Borel*** iff

$$\{\langle x, y\rangle \in X \times [-\infty, \infty] : 0 \leq y < f(x)\} \tag{5.2.1.5}$$

and

$$\{\langle x, y\rangle \in X \times [-\infty, \infty] : f(x) \leq y < 0\} \tag{5.2.1.6}$$

are measurable.

R The collection of all Borel functions $X \to [-\infty, \infty]$ is denoted $\mathrm{Bor}(X)$. The collection of all nonnegative Borel functions on X is denoted $\mathrm{Bor}_0^+(X)$.

R The two sets in (5.2.1.5) are the "area 'under' the curve" and the "area 'below' the curve" respectively. The integral is going to be defined to be (Definition 5.2.2.1) the measure of the first set minus the measure of the second set. Thus, the condition of "Borel" is going to be a condition commonly imposed on are functions so that the integral satisfies the properties you would expect it to.[a]

R The reason for the difference of the inequalities (i.e. "$<$" vs. "$\leq$") is essentially the same as the reason working with closed-open intervals is convenient: the union of the two of these sets is a disjoint union.

R We explain the reason this is called "Borel" in Proposition 5.2.1.13 below.

---

[a] For example, just as we need not have $\mathrm{m}(S \cup T) = \mathrm{m}(S) + \mathrm{m}(T)$ for S and T disjoint in general unless S and T are measurable, so to we need not have that $\int_X \mathrm{d}\,\mathrm{m}(x)\,[f(x) + g(x)] = \int_X \mathrm{d}\,\mathrm{m}(x)\, f(x) + \int_X \mathrm{d}\,\mathrm{m}(x)\, g(x)$ in general unless f and g are Borel.

Exercise 5.2.1.7 Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and let $f, g\colon X \to [-\infty, \infty]$. Then, if $f(x) = g(x)$ almost-everywhere, then f is Borel iff g is Borel.

It is most convenient to work with

$$\{\langle x, y\rangle \in X \times [-\infty, \infty] : 0 \leq y < f(x)\}, \tag{5.2.1.8}$$

for similar reasons as it is more convenient to work with $[a, b)$ than it is $[a, b]$, but it actually makes no difference, as we now check.

Proposition 5.2.1.9 Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and let $f\colon X \to [0, \infty]$ be a function. Then,

$$\{\langle x, y\rangle \in X \times [0, \infty] : y < f(x)\} \tag{5.2.1.10}$$

is measurable iff

$$\{\langle x, y\rangle \in X \times [0, \infty] : y \leq f(x)\} \tag{5.2.1.11}$$

is. Furthermore, they both have the same measure.

Proof. We leave this as an exercise.

Exercise 5.2.1.12 Prove the result yourself.

R Hint: For the $\Rightarrow$ direction, consider the sequence $m \mapsto \left(1 + \frac{1}{m}\right) f$. For the $\Leftarrow$ direction, consider the sequence $m \mapsto \left(1 - \frac{1}{m}\right) f$.

■

There is a very important characterization of Borel functions. In fact, this characterization is the reason Borel functions are called "*Borel* functions".

Proposition 5.2.1.13 Let $\langle X, \mathrm{m}\rangle$ be a topological measure space. Then, the following are equivalent.

(i). $f \in \operatorname{Bor}(X)$.
(ii). $f^{-1}(U)$ is measurable for every $U \subseteq [-\infty, \infty]$ open.
(iii). $f^{-1}(C)$ is measurable for every $C \subseteq [-\infty, \infty]$ closed.

R As preimages preserves unions, intersections, and complements, it follows that the preimage of any Borel set (Definition 5.1.2.1) will likewise be measurable. This is why we call Borel functions "*Borel* functions".

R A lot of authors call these *measurable functions*. This conflicts with other standard terminology (which we do make use of—see Definition 5.1.1.39), and so I recommend you not use it. There is a way to make this not inconsistent, but it's awkward: when you have a function $f\colon \mathbb{R} \to \mathbb{R}$, for example, you simply declare that the "measurable" sets in the codomain are the Borel sets but the "measurable" sets in the domain are all (Lebesgue) measurable sets. Ew.

Proof. [a] ($(i) \Rightarrow (ii)$) Suppose that $f \in \operatorname{Bor}(X)$. Let us write

$$\Gamma_f := \{\langle x, y\rangle \in X \times [0, \infty] : y < f(x)\}. \tag{5.2.1.14}$$

We first do the case where f is bounded, say by $C \in [0, \infty)$, and is 0 outside a set of finite measure $M \subseteq X$. In this case, $[\mathrm{m} \times \mathrm{m}_{\mathrm{L}}](\Gamma_f)$ is finite,[b] and so by inner-regularity on measurable subsets of finite measure, there is an increasing sequence of compact sets $K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq \Gamma_f$ such that $[\mathrm{m} \times \mathrm{m}_{\mathrm{L}}](\Gamma_f \setminus C) = 0$, where $C := \bigcup_{m \in \mathbb{N}} K_m$ (Proposition 5.1.2.26). Now, define $g_m, h_m : X \to [0, \infty]$ by

$$g_m(x) := \begin{cases} \sup_{\langle x,y\rangle \in K_m} \{y\} & \text{if } K_m \cap (\{x\} \times [0, \infty]) \neq \emptyset \\ 0 & \text{if } K_m \cap (\{x\} \times [0, \infty]) = \emptyset \end{cases} \tag{5.2.1.15}$$

and

$$h_m(x) := \begin{cases} \inf_{\langle x,y\rangle \in (X \times \mathbb{R}) \setminus K_m} \{y\} & \text{if } K_m \cap (\{x\} \times [0, \infty]) \neq \emptyset \\ 0 & \text{if } K_m \cap (\{x\} \times [0, \infty]) = \emptyset. \end{cases} \tag{5.2.1.16}$$

Note that, as $K_m \subseteq \Gamma_f$, $g_m(x) \leq f(x)$ and $f(x) \leq h_m(x)$. Furthermore, for each fixed $x \in X$, the sequence $m \mapsto g_m(x)$ is nondecreasing, and so by the Monotone Convergence Theorem (Proposition 2.4.3.21), we may define $g(x) := \lim_m g_m(x)$. Similarly, we may define $h(x) := \lim_m h_m(x)$.

Exercise 5.2.1.17 Show that the preimage under g_m of every open set is measurable by showing that the preimage of $(-\infty, b)$ is open for all $b \in \mathbb{R}$. Show that the preimage under h_m of every open set is measurable by showing that the preimage of (a, ∞) is open for all $a \in \mathbb{R}$.

Exercise 5.2.1.18 Use this to deduce that the preimage of every open set under both g and h measurable.

Exercise 5.2.1.19 Show that $[\mathrm{m} \times \mathrm{m_L}](\Gamma_{h-g}) = 0$.

Exercise 5.2.1.20 Use this, together with the fact that the preimage of every open set under both g and h is measurable, to show that $g(x) = h(x)$ for almost-every x.

As $g \leq f \leq h$, it follows that f itself is equal almost-everywhere to both g and h.

Exercise 5.2.1.21 Use the fact that $f(x) = g(x)$ almost-everywhere and the fact that the preimage of every open set under g is measurable to show that the preimage of every open set under f is measurable.

This completes the proof in the case that f is bounded and vanishes outside a set of finite measure.

We now do the proof for $f \in \mathrm{Bor}_0^+(X)$ be arbitrary. Write $X = \bigcup_{m \in \mathbb{N}} M_m$ as the increasing union of measurable sets of finite measure, so that

$$X \times [0,\infty] = \bigcup_{m\in\mathbb{N}} M_m \times [0,m) \cup \bigcup_{m\in\mathbb{N}} M_m \times \{\infty\}. \quad (5.2.1.22)$$

Thus,

$$\begin{aligned}\Gamma_f = & \bigcup_{m\in\mathbb{N}} \Gamma_f \cap (M_m \times [0,m)) \\ & \cup \bigcup_{m\in\mathbb{N}} \Gamma_f \cap (M_m \times \{\infty\}).\end{aligned} \quad (5.2.1.23)$$

Note that

$$\begin{aligned}&\Gamma_f \cap (M_m \times [n,n+1)) \\ &\quad = \{\langle x,y\rangle \in X \times [-\infty,\infty] : \\ &\qquad y < f(x),\ x \in M_m,\ n \leq y < n+1\} \\ &\quad = \Gamma_{f_m},\end{aligned} \quad (5.2.1.24)$$

where $f_m : X \to \mathbb{R}$ is defined by

$$f_m(x) := \begin{cases} f(x) & \text{if } x \in M_m \text{ and } f(x) < m \\ m & \text{if } x \in M_m \text{ and } f(x) \geq m \\ 0 & \text{otherwise.} \end{cases} \quad (5.2.1.25)$$

Γ_{f_m} is measurable because Γ_f is. Furthermore, f_m is bounded and vanishes outside a set of finite measure, and so the previous case applies, and we have that the preimage of open sets under f_m are measurable.

Exercise 5.2.1.26 Use this to show that the preimage of open sets under f are measurable.

Finally, for $f \in \mathrm{Bor}(X)$ arbitrary (not necessarily nonnegative), each of the nonnegative and nonpositive parts of f are Borel (by definition), and so the previous case shows that the preimage under either of these functions of open sets is measurable. It then follows that the preimage under f of open sets are measurable.

($(ii) \Rightarrow (iii)$) Suppose that $f^{-1}(U)$ is measurable for every $U \subseteq [-\infty, \infty]$ open. Let $C \subseteq [-\infty, \infty]$ be closed. Then, C^{C} is open, and so $f^{-1}(C^{\mathsf{C}}) = f^{-1}(C)^{\mathsf{C}}$ is measurable, and so the complement of this, $f^{-1}(C)$, is measurable.

($(iii) \Rightarrow (i)$) Suppose that $f^{-1}(C)$ is measurable for every $C \subseteq [-\infty, \infty]$ closed. It follows that the preimage of half-open rectangles are measurable,[c] and so

$$\begin{aligned}\{\langle x, y\rangle \in X \times [0, \infty] : 0 \leq y < f(x)\} \\ = \bigcup_{r \in \mathbb{Q}} f^{-1}([r, \infty)) \times [0, r)\end{aligned} \tag{5.2.1.27}$$

is measurable. Similarly the 'negative' version of this set is measurable too, and so f is Borel. ■

[a] Proof adapted from [Pug02, pg. 384].

[b] m_{L} denotes Lebesgue measure on $[0, \infty]$, to distinguish it from the measure on X.

[c] Why?

Proposition 5.2.1.28 Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and let $\{f_m : m \in \mathbb{N}\} \subseteq \mathrm{Bor}(X)$ be a countable collection of measurable functions. Then, $x \mapsto \sup_{m \in \mathbb{N}} f_m(x)$ and $x \mapsto \inf_{m \in \mathbb{N}} f_m(x)$ are Borel.

Proof. We leave this as an exercise.

> **Exercise 5.2.1.29** Prove the result yourself.

■

Proposition 5.2.1.30 Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and let $m \mapsto f_m \in \mathrm{Bor}(X)$ be a sequence converging to $f_\infty \colon X \to [-\infty, \infty]$ in $\mathrm{Mor}_{\mathbf{Set}}(X, [-\infty, \infty])/{\sim_{\mathbf{AlE}}}$. Then, $f_\infty \in \mathrm{Bor}(X)$.

R Saying that it converges

$$\text{“in } \mathrm{Mor}_{\mathbf{Set}}(X, [-\infty, \infty])/{\sim_{\mathbf{AlE}}}\text{ ”} \qquad (5.2.1.31)$$

is a fancy way of saying that it converges to $f_\infty(x)$ almost-everywhere.

W Warning: This fails for nets in general—see the following counter-example (Example 5.2.1.33).

Proof. We leave this as an exercise.

Exercise 5.2.1.32 Prove the result yourself.

R Hint: Use the previous result.

■

▪ Example 5.2.1.33 — A limit of a Borel function which is not Borel Let $N \subseteq \mathbb{R}$ be a set that is not measurable and define

$$\mathscr{S} := \{S \subseteq N : S \text{ is measurable.}\}. \qquad (5.2.1.34)$$

This is a directed set with respect to inclusion. Certainly each χ_S for $S \in \mathscr{S}$ is measurable. I claim that $\lim_{S\in\mathscr{S}} \chi_S = \chi_N$ point-wise, which will show that $S \mapsto \chi_S$ is a net of Borel functions converging to a function which is not Borel.[a]

So, let $x \in \mathbb{R}$. If $x \notin N$, then $\chi_S(x) = 0$ for all $S \in \mathscr{S}$, and so certainly $S \mapsto \chi_S(x)$ converges to $\chi_N(x) = 0$. On the other hand, if $x \in N$, then, whenever $S \supseteq \{x\}$,[b] it follows that $x \in S$ and hence $\chi_S(x) = 1$. Thus, in this case we will have $S \mapsto \chi_S(x)$ converges to $\chi_N(x) = 1$ as well.

[a]In fact, it is a nondecreasing sequence and converges *everywhere* (not just almost-everywhere).

[b]Note that $\{x\} \in \mathscr{S}$.

In the course of showing uniqueness of the integral, it will be important to know that we can approximate Borel functions by finite linear combinations of characteristic functions.

Proposition 5.2.1.35 Let X be a topological measure space and let $f \colon X \to [-\infty, \infty]$. Then, the following are equivalent.

(i). f is nonnegative (Borel).

(ii). There exists a nondecreasing sequence $m \mapsto s_m \in \mathrm{Mor}_{\mathbf{Set}}(x, [0, \infty])$ of (Borel) simple functions such that $\lim_m s_m = f$ in $\mathrm{Mor}_{\mathbf{Set}}(X, [0, \infty])/\sim_{\mathbf{AlE}}$.

(iii). There are (measurable) sets $M_m \subseteq X$ and positive real numbers $c_m > 0$ such that $f = \sum_{m \in \mathbb{N}} c_m \chi_{M_m}$.

R To clarify, there are really *two* sets of equivalent conditions here, one including the words in parentheses and one without.

Proof. [a] ((*i*) $\Rightarrow$ (*ii*)) Suppose that f is nonnegative (Borel). Write $X = \bigcup_{m \in \mathbb{N}} K_m$ for $K_m \subseteq X$ compact. Define $f_m : X \to [0, \infty]$ by

$$f_m(x) := \begin{cases} f(x) & \text{if } x \in K_m \text{ and } f(x) \leq m \\ m & \text{if } x \in K_m \text{ and } f(x) > m \\ 0 & \text{otherwise.} \end{cases} \tag{5.2.1.36}$$

For $n \in \mathbb{N}$ and $0 \leq o < mn$, define

$$S_{m,n,o} := \left\{ x \in K_m : \tfrac{o}{m} < f_n(x) \leq \tfrac{o+1}{m} \right\}. \tag{5.2.1.37}$$

Finally, define $s_m : X \to [0, \infty]$ by

$$s_m := \sum_{o=0}^{m^2} \tfrac{o}{m} \chi_{S_{m,m,o}}. \tag{5.2.1.38}$$

Exercise 5.2.1.39 Show that $m \mapsto s_m$ is a nondecreasing sequence of (Borel) simple functions converging pointwise to f

($(ii) \Rightarrow (iii)$) Suppose that there exists a nondecreasing sequence $m \mapsto s_m \in \mathrm{Mor}_{\mathbf{Set}}(X, [0, \infty])$ of (Borel) simple functions such that $\lim_m s_m = f$ in $\mathrm{Mor}_{\mathbf{Set}}(X, [0, \infty])/\mathbf{AlE}$. Let $c\chi_M$ be a term appearing in s_m, with $M \subseteq X$ (measurable) and $c > 0$. As $s_{m+1} \geq s_m$, there must be some term $c'\chi_{M'}$ appearing in s_{m+1} with $M' \subseteq X$ (measurable), $c' > 0$, $M \subseteq M'$, and $c \leq c'$. Then note that $c'\chi_{M'} - c\chi_M = (c' - c)\chi_M + c'\chi_{M' \setminus M}$. It follows that $s_{m+1} - s_m$ is again a simple (Borel) function.[b] It follows in turn that

$$\begin{aligned} s_m &= (s_m - s_{m-1}) + (s_{m-1} - s_{m-2}) + \cdots + (s_1 - s_0) + s_0 \\ &= \sum_{k=0}^{n_m} c_k \chi_{M_k} \end{aligned}$$

for $M_k \subseteq X$ (measurable), $c_k > 0$, and some $n_m \in \mathbb{N}$. We thus have that $f = \sum_{m \in \mathbb{N}} c_m \chi_{M_m}$, as desired.

($(iii) \Rightarrow (i)$) Suppose that there are (measurable) sets $M_m \subseteq X$ and positive real numbers $c_m > 0$ such that $f = \sum_{m \in \mathbb{N}} c_m \chi_{M_m}$. Then, f is nonnegative (Borel) because sums of nonnegative (Borel) functions are nonnegative (Borel (Exercise 5.2.1.43)) and limits of sequences of nonnegative (Borel) functions are nonnegative (Borel (Proposition 5.2.1.30)). ■

[a] Proof adapted from [SS05, pg. 31].

[b] Recall that (Definition 5.2.1.3) a simple function is a *nonnegative* linear combination of characteristic functions, so we needed to check that we could rewrite $s_{m+1} - s_m$ in such a way so that no negative coefficients appeared.

One technique that will prove invaluable to us is the ability to decompose a function into its nonnegative and nonpositive parts. A common strategy will be to prove results for nonnegative functions, and then use the below decomposition to deduce the result in the general case.

Definition 5.2.1.40 Let X be a set and let $f\colon X \to [-\infty, \infty]$ be a function. Then, we write

$$f_+ := \max\{f, 0\} \text{ and } f_- := -\min\{f, 0\}. \tag{5.2.1.41}$$

R That is, f_+ is equal to f if f is positive and 0 otherwise; likewise, f_- is equal to f if $-f$ is negative and 0 otherwise.

R Note that *always* $f_+, f_- \geq 0$.

R Also note that $f = f_+ - f_-$ and $|f| = f_+ + f_-$.

Exercise 5.2.1.42 Let X be a measure space and let $f\colon X \to [-\infty, \infty]$ be a function. Show that f is Borel iff f_+ and f_- are Borel.

Of course, we certainly want it to be the case case that sums and products of Borel functions are Borel.

Exercise 5.2.1.43 Let X be a topological measure space and let $f, g \in \mathrm{Bor}(X)$.

(i). Show that $f + g \in \mathrm{Bor}(X)$.
(ii). Show that $fg \in \mathrm{Bor}(X)$.

R Hint: First prove it for nonnegative Borel functions, and then use the decomposition $f = f_+ - f_-$ to prove the result for arbitrary Borel functions.

5.2.2 The integral itself

And finally, I present unto thee, the *integral*.

Definition 5.2.2.1 — Integral Let $\langle X, \mathrm{m}\rangle$ be a topological measure space, let $f\colon X \to [-\infty, \infty]$, and write

$$\mathrm{I}_+(f) := [\mathrm{m}\times\mathrm{m}_{\mathrm{L}}](\{\langle x, y\rangle \in X\times[-\infty,\infty] : 0 \leq y < f(x)\}) \tag{5.2.2.2}$$

and

$$\mathrm{I}_-(f) := [\mathrm{m}\times\mathrm{m}_{\mathrm{L}}](\{\langle x, y\rangle \in X\times[-\infty,\infty] : f(x) \leq y < 0\}), \tag{5.2.2.3}$$

where m_{L} is Lebesgue measure.

Then, f is ***∞-integrable*** iff at least one of I_+ and I_- is finite, in which case the ***integral*** of f, $\int_X \mathrm{d}\,\mathrm{m}(x)\, f(x)$, is defined by

$$\int_X \mathrm{d}\,\mathrm{m}(x)\, f(x) := \mathrm{I}_+(f) - \mathrm{I}_-(f). \tag{5.2.2.4}$$

Furthermore, if both I_+ and I_- are finite, then f is ***integrable***.

R We say that $\int_X \mathrm{d}\,\mathrm{m}(x)\, f(x)$ ***converges*** iff f is integrable.

R For $S \subseteq X$, we define

$$\int_S \mathrm{d}\,\mathrm{m}(x)\, f(x) := \int_X \mathrm{d}\,\mathrm{m}(x)\, \chi_S(x) f(x). \tag{5.2.2.5}$$

If $X = \mathbb{R}$ and $S = [a, b]$, we define

$$\int_a^b \mathrm{d}x\, f(x) := \int_{[a,b]} \mathrm{d}\,\mathrm{m}_{\mathrm{L}}(x)\, f(x) \tag{5.2.2.6}$$

Warning: Many results require the integrand to be Borel, which means that, if you want to apply the result to $\int_S \mathrm{d}\,\mathrm{m}(x)\, f(x)$, you need f to be Borel *and* S to be measurable. Don't forget about S!

R The two sets in (5.2.2.2) and (5.2.2.3) are the "area 'under' the curve" and the "area 'below' the curve" respectively. Thus, the condition "∞-integrable" is exactly the condition needed for this difference, that is, $\int_X \mathrm{d}\,\mathrm{m}(x)\, f(x)$, to *make sense*, and the condition "integrable" is exactly the condition needed for this difference to make sense and be *finite* (that is, if you have the prefix "∞", you allow for the integral to be infinite).

R Note that we do *not* require (∞-)integrable functions be Borel. Indeed, the assumption that f be ∞-integrable is essentially the bare-minimum assumption we need to make in order for this to make sense, and the measure of these sets makes sense irrespective of whether or not they are measurable (i.e. whether or not f is Borel). If you're wondering "But are there any functions that are not Borel that I actually want to be able to integrate?", the answer is "Almost certainly not.". Despite this, however, it is convenient to have the integral make sense for not-necessarily-Borel functions because, while in practice the functions you're working with are Borel anyways, it is nice that you don't have to check this before writing down the integral, for example, see the remark in Fubini's Theorem (Theorem 5.2.3.27).

R We should probably mention that this convention is nonstandard—for most authors, the term "integrable" would be equivalent to "Borel and integrable" in our terminology. Furthermore, most authors just don't use a term analogous to ∞-integrable.[a]

R If the measure is clear from context, we may just simply write $\mathrm{d}x$ instead of $\mathrm{d}\,\mathrm{m}(x)$. We may even write $\int_X \mathrm{d}\,\mathrm{m}\; f$ or just $\int_X f$ if 'the variable of integration' is irrelevant.

R I happen to prefer to put the $\mathrm{d}x$ in front of the integral,[b] but don't let that confuse you—the meaning is just the same as $\int f(x)\,\mathrm{d}x$.

In earlier versions of the notes, I wanted to extend the definition of the integral to other functions whose integral 'should' exist, but does not exist in this sense. One classic example of this is the function $\mathbb{R} \ni x \mapsto \frac{\sin(x)}{x}$. On one hand, this is not even∞-integrable; however, $\lim_{a\to\infty} \int_{-a}^{a} \mathrm{d}x\, \frac{\sin(x)}{x}$ *does* converge. The idea was then to define the integral of such functions to be $\lim_{K\in\mathscr{K}} \int_K \mathrm{d}\,\mathrm{m}(x)\, f(x)$, where $\mathscr{K}$ is the collection of quasicompact subsets of X. Unfortunately, however, this just doesn't work: if f is ∞-integrable (and Borel), then it converges to the usual integral, but otherwise the limit does not exist. However we will have to wait awhile to see this—see Theorem 5.2.2.36.

Needless to say, the Riemann integral has this 'defect' as well. In fact, it's *much* worse: the Riemann integral is not defined (at least not without taking limits) over *any* unbounded set for *any* function. On the other hand, there are definitions of the integral that agree with the Lebesgue integral for all Lebesgue integrable functions, and furthermore, will assign the 'correct' value to the integral of functions like $x \mapsto \frac{\sin(x)}{x}$.[c] Unfortunately, however, all such definitions I am aware of are $\mathbb{R}^d$-specific, and this is nowhere even close to being powerful enough to cover all cases of interest. It is a very tiny inconvenience to have to use $\lim_{a\to\infty} \int_{-a}^{a} \mathrm{d}x\, f(x)$ instead of just $\int_{\mathbb{R}} \mathrm{d}x\, f(x)$; on the other hand, it is a *huge* inconvenience to not be able to integrate over anything besides Euclidean space.

Note that

$$\int_X \mathrm{d}\,\mathrm{m}(x)\, f(x) = \int_X \mathrm{d}\,\mathrm{m}(x)\, f_+(x) - \int_X \mathrm{d}\,\mathrm{m}(x)\, f_-(x)$$

even if f is not Borel.[d] This is important because it means a lot of properties you can prove about the integral by first proving the result for nonnegative functions, and then using the decomposition $f = f_+ - f_-$ to extend the result for *all* functions.

R If one needs to specify, this is the ***Lebesgue integral***. You may have heard of the ***Lebesgue-Stieltjes integral***. This is in fact no more general than the Lebesgue integral as the Lebesgue-Stieltjes integral with respect to a function h (that has to satisfy some hypotheses in order for things to make sense) of f is just the (Lebesgue) integral of f with respect to a measure that depends on h. We could study this measure if we wanted to, but we have no need. The point is to reassure you that you're not 'missing out' on something more general.[e]

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[a]By this point, it probably goes without saying that I prefer the terminology I do because it allows me to be more precise. For example, I now have terms for four classes of functions: integrable, ∞-integrable, Borel integrable, and Borel ∞-integrable; whereas with the other convention we would only have the terminology to speak of just one class of functions.

[b]For two reasons: (i) it tells you immediately what the variable you are integrating with respect to is (just a slight convenience, especially when the integrand is complicated) and (ii) it is more analogous to how we write the derivative—no one writes $\mathrm{d}f(x)/\mathrm{d}x$—people write $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$.

[c]The so-called *Henstock integral* is such an example.

[d]We mention this because in general you *do* need f and g to be Borel to have $\int_X \mathrm{d}\,\mathrm{m}(x)\,[f(x)+g(x)] = \int_X \mathrm{d}\,\mathrm{m}(x)\,f(x) + \int_X \mathrm{d}\,\mathrm{m}(x)\,g(x)$—see Theorem 5.2.2.7.

[e]In fact, I know of no definition of the integral which strictly generalizes the Lebesgue integral (in a way that makes no direct reference to the Lebesgue integral). The parenthetical comment refers to the fact that you can define integrals with values in more general spaces (e.g. the ***Gelfand-Pettis integral*** for topological vector spaces), but in order to define such integrals you ultimately reduce their definitions to the Lebesgue integral.

I am not sure how to uniquely characterize the definition of the integral on *all* (∞-integrable) functions, *but*, for Borel functions, there are a couple of relatively easy properties that uniquely characterize the integral.

Theorem 5.2.2.7 — Fundamental Theorem of the Integral. Let $\langle X, \mathrm{m}\rangle$ be a topological measure space. Then, $f \mapsto \int_X \mathrm{d}\,\mathrm{m}(x)\,f(x)$ is the unique function $\mathrm{I}\colon \mathrm{Bor}_0^+(X) \to [0,\infty]$ such that (i). (Normalization) $\mathrm{I}(\chi_S) = \mathrm{m}(S)$ for $S \subseteq X$ measurable;[a]

(ii). (Additivity) $\mathrm{I}(f+g) = \mathrm{I}(f) + \mathrm{I}(g)$ for $f, g \in \mathrm{Bor}_0^+(X)$;

(iii). (Nonnegative-homogeneity) $\mathrm{I}(af) = a\mathrm{I}(f)$ for $f \in \mathrm{Bor}_0^+(X)$ and $a \geq 0$;[b] and

(iv). (Lebesgue's Monotone Convergence Theorem) whenever $m \mapsto f_m$ is a nondecreasing sequence of Borel functions then

$$\lim_m \mathrm{I}(f_\lambda) = \mathrm{I}(\lim_m f_m).^c \tag{5.2.2.8}$$

R Furthermore, we always have $\mathrm{I}(f+g) \leq \mathrm{I}(f) + \mathrm{I}(g)$ and $\lim_\lambda \mathrm{I}(f_\lambda) \leq \mathrm{I}(\lim_\lambda f_\lambda)$, even if the functions involved are not necessarily Borel and even if the net is not a sequence, though, oddly, we have to wait to prove this—see Theorem 5.2.3.68.

W Warning: $\lim_m \int_X f_m = \int_X \lim_m f_m$ can fail if the convergence is not monotone, or even if it is monotone *decreasing*—see the Example 5.2.3.17.

R First of all, we are characterizing the integral uniquely via certain properties it satisfies, and then furthermore, it turns out that this unique such function is simply given by *the 'area' under the curve*. What do you think of that, Riemann?[d]

[a] χ_S won't be Borel unless S is measurable, though certainly this result is true just the same.

[b] Of course, this is actually true for *all* $a \in \mathbb{R}$, but for the time being, as we are currently regarding I as a function of *nonnegative* Borel functions, we need $a \geq 0$ in order that af itself is nonnegative.

[c] Note that $\lim_m f(x)$ *always* exists, by the usual Monotone Convergence Theorem—either it is bounded, in which case it converges in $\mathbb{R}$, or it is unbounded, in which case it converges to ∞. (I lied a bit. . . as it is only nondecreasing almost-everywhere, then the limit exists only almost-everywhere.) Also note that $f_\infty \in \mathrm{Bor}_0^+(X)$ by Proposition 5.2.1.30.

[d] Seriously, people actually claim that the Lebesgue integral is not "geometric". Da fuq? Can someone please explain to me how limits of partitions of sums of areas of rectangles is more geometric than the area under the curve?

Proof. [a] STEP 1: INTRODUCE NOTATION
As we will be making use of the set quite a bit, it will be useful to introduce the notation

$$\Gamma_f := \{\langle x, y\rangle \in X \times [0, \infty] : y < f(x)\} . \qquad (5.2.2.9)$$

STEP 2: DEFINE I
Let $f \colon X \to \mathbb{R}_0^+$ and define

$$\mathrm{I}(f) := \int_X \mathrm{d}\,\mathrm{m}(x)\, f(x) = [\mathrm{m} \times \mathrm{m_L}](\Gamma_f). \qquad (5.2.2.10)$$

STEP 3: SHOW THAT $\mathrm{I}(\chi_S) = \mathrm{m}(S)$

$$\Gamma_{\chi_S} := \{\langle x, y\rangle \in X \times \mathbb{R} : y < \chi_S(x)\} = S \times [0, 1) \quad (5.2.2.11)$$

and so

$$\mathrm{I}(\chi_S) := [\mathrm{m} \times \mathrm{m_L}](\Gamma_{\chi_S}) = \mathrm{m}(S) \cdot 1 = \mathrm{m}(S). \quad (5.2.2.12)$$

STEP 4: SHOW THAT I IS ADDITIVE
We wish to show that $\mathrm{I}(f + g) = \mathrm{I}(f) + \mathrm{I}(g)$. If either $\mathrm{I}(f)$ or $\mathrm{I}(g)$ is infinite, then this is trivially satisfied as this equation then reads $\infty = \infty$. Thus, without loss of generality, we may assume that $\mathrm{I}(f), \mathrm{I}(g) < \infty$.

For $f \colon X \to [0, \infty]$, define $\tau_f : X \times [0, \infty] \to X \times [0, \infty]$ by

$$\tau_f(\langle x, y\rangle) := \langle x, f(x) + y\rangle. \qquad (5.2.2.13)$$

This definition was made so that we have

$$\Gamma_{f+g} = \Gamma_f \cup \tau_f(\Gamma_g) \qquad (5.2.2.14)$$

is a disjoint union.[b] Thus, it suffices to show that (i) $\tau_f(M)$ is measurable if M is and (ii) that $[\mathrm{m}\times\mathrm{m}_{\mathrm{L}}](\tau_f(M)) = [\mathrm{m}\times\mathrm{m}_{\mathrm{L}}](M)$ for $M\subseteq X\times[0,\infty]$ measurable.

First of all note that, while we don't immediately know that $\tau_f(M)$ is measurable for *all* measurable M, we do know that

$$\tau_f(\Gamma_g) = \Gamma_{f+g}\setminus\Gamma_f, \tag{5.2.2.15}$$

is measurable, because sums of Borel functions are Borel (Exercise 5.2.1.43).

We now show (ii), that is, that $[\mathrm{m}\times\mathrm{m}_{\mathrm{L}}](\tau_f(M)) = [\mathrm{m}\times\mathrm{m}_{\mathrm{L}}](M)$ for $M\subseteq X\times[0,\infty]$ measurable. To show this, we apply Proposition 5.1.2.14 (sets in topological measure spaces are measurable iff you can approximate them with open and closed sets—it is thus very important that the product measure is regular and Borel).

We prove this in stages. First of all, take $M = M_X\times[0,b)$ for $M_X\subseteq X$ measurable. Then,

$$M = \Gamma_g \text{ for } g := b\chi_{M_X}, \tag{5.2.2.16}$$

and so we have that (by (5.2.2.14))

$$\Gamma_f\cup\tau_f(M) = \Gamma_{f+g} = \Gamma_{g+f} = M\cup\tau_g(\Gamma_f). \tag{5.2.2.17}$$

Furthermore,

$$\begin{aligned}\tau_g(\Gamma_f) &= \tau_g\big(\{\langle x,y\rangle\in\Gamma_f : x\in M_X\} \\ &\qquad\cup\{\langle x,y\rangle\in\Gamma_f : x\notin M_X\}\big) \\ &= \{\langle x,y+b\rangle\in\Gamma_f : x\in M_X\} \\ &\qquad\cup\{\langle x,y\rangle\in\Gamma_f : x\notin M_X\},\end{aligned} \tag{5.2.2.18}$$

and so by translation invariance (and the measurability of M_X), we have

$$\begin{aligned}
[\mathrm{m} \times \mathrm{m_L}](\tau_g(\Gamma_f)) \\
&= [\mathrm{m} \times \mathrm{m_L}]\left(\{\langle x, y+b\rangle \in \Gamma_f : x \in M_X\}\right. \\
&\qquad \left.\cup \{\langle x, y\rangle \in \Gamma_f : x \notin M_X\}\right) \\
&= [\mathrm{m} \times \mathrm{m_L}]\left(\{\langle x, y\rangle \in \Gamma_f : x \in M_X\}\right. \\
&\qquad \left.\cup \{\langle x, y\rangle \in \Gamma_f : x \notin M_X\}\right) \\
&= [\mathrm{m} \times \mathrm{m_L}](\Gamma_f),
\end{aligned} \tag{5.2.2.19}$$

and hence (by (5.2.2.17) and the fact that these sets (in particular, $\tau_f(M)$) are measurable)

$$\begin{aligned}
[\mathrm{m} \times \mathrm{m_L}](\Gamma_f) + [\mathrm{m} \times \mathrm{m_L}](\tau_f(M)) \\
&= [\mathrm{m} \times \mathrm{m_L}](M) + [\mathrm{m} \times \mathrm{m_L}](\tau_g(\Gamma_f)) \\
&= [\mathrm{m} \times \mathrm{m_L}](M) + [\mathrm{m} \times \mathrm{m_L}](\Gamma_f),
\end{aligned} \tag{5.2.2.20}$$

which yields (because $[\mathrm{m} \times \mathrm{m_L}](\Gamma_f) < \infty$ by assumption)

$$[\mathrm{m} \times \mathrm{m_L}](\tau_f(M)) = [\mathrm{m} \times \mathrm{m_L}](M). \tag{5.2.2.21}$$

Now take $M = M_X \times [a, b)$. Then,

$$M = \Gamma_g \setminus \Gamma_h \text{ for } g := b\chi_{M_X} \text{ and } h := a\chi_{M_X}. \tag{5.2.2.22}$$

Now the fact that $[\mathrm{m} \times \mathrm{m_L}](\tau_f(M)) = [\mathrm{m} \times \mathrm{m_L}](M)$ follows from the previous case:

$$\begin{aligned}
[\mathrm{m} \times \mathrm{m_L}](\tau_f(M)) \\
&= [\mathrm{m} \times \mathrm{m_L}](\tau_f(\Gamma_g)) - [\mathrm{m} \times \mathrm{m_L}](\tau_f(\Gamma_h)) \\
&= [\mathrm{m} \times \mathrm{m_L}](\Gamma_g) - [\mathrm{m} \times \mathrm{m_L}](\Gamma_h) \\
&= [\mathrm{m} \times \mathrm{m_L}](\Gamma_g \setminus \Gamma_h) = [\mathrm{m} \times \mathrm{m_L}](M).
\end{aligned} \tag{5.2.2.23}$$

For the general case, let $M \subseteq X \times [0, \infty]$ be arbitrary measurable. Write $X = \bigcup_{m \in \mathbb{N}} R_m$ as the disjoint union of measurable sets of finite measure (this is an application of

Exercise 5.1.2.12), and define $M_{m,n} \coloneqq M \cap (R_m \times [n, n+1))$ (as well as $M_{m,\infty} \coloneqq M \cap (R_m \times \{\infty\})$ if you really like, though this will not matter as everything here has measure 0). Then, M is the disjoint union of the $M_{m,n}$s, and so it suffices to prove the result for subsets of $R_m \times [n, n+1)$.

So, now, let us change notation, let $M_X \subseteq X$ be measurable of finite measure and take M to be a measurable subset of $M_X \times [n, n+1)$. Let $\varepsilon > 0$. By 'rectangle outer-regularity' (see (5.1.4.4)), there is a countable cover of M by open rectangles $\bigcup_{m\in\mathbb{N}} U_{1,m} \times U_{2,m}$ such that

$$\sum_{m\in\mathbb{N}} \mathrm{m}(U_{1,m})\,\mathrm{m}_{\mathrm{L}}(U_{2,m}) - \varepsilon < [\mathrm{m}\times\mathrm{m}_{\mathrm{L}}](M). \quad (5.2.2.24)$$

Without loss of generality,[c] we may assume that each $U_{2,m} = [a_m, b_m)$, in which case the previous case applies, so that

$$\begin{aligned}
[\mathrm{m}\times\mathrm{m}_{\mathrm{L}}](\tau_f(M)) &\leq \sum_{m\in\mathbb{N}} [\mathrm{m}\times\mathrm{m}_{\mathrm{L}}](\tau_f(U_{1,m}\times U_{2,m})) \\
&= \sum_{m\in\mathbb{N}} [\mathrm{m}\times\mathrm{m}_{\mathrm{L}}](U_{1,m}\times U_{2,m}) \\
&< [\mathrm{m}\times\mathrm{m}_{\mathrm{L}}](M) + \varepsilon,
\end{aligned}$$

and so as $\varepsilon > 0$ is arbitrary

$$[\mathrm{m}\times\mathrm{m}_{\mathrm{L}}](\tau_f(M)) \leq [\mathrm{m}\times\mathrm{m}_{\mathrm{L}}](M). \quad (5.2.2.25)$$

Similarly,

$$\begin{aligned}
&[\mathrm{m}\times\mathrm{m}_{\mathrm{L}}]\left(\tau_f\left((M_X\times[n,n+1))\setminus M\right)\right) \\
&\qquad \leq [\mathrm{m}\times\mathrm{m}_{\mathrm{L}}]((M_X\times[n,n+1))\setminus M).
\end{aligned} \quad (5.2.2.26)$$

Hence,

$$\begin{aligned}
&[\mathrm{m}\times\mathrm{m_L}](\tau_f(M)) + [\mathrm{m}\times\mathrm{m_L}]\left(\tau_f\left((M_X\times[n,n+1))\setminus M\right)\right) \\
&\qquad \leq [\mathrm{m}\times\mathrm{m_L}](M) + [\mathrm{m}\times\mathrm{m_L}]((M_X\times[n,n+1))\setminus M) \\
&\qquad = [\mathrm{m}\times\mathrm{m_L}](M_X\times[n,n+1)) \\
&\qquad = [\mathrm{m}\times\mathrm{m_L}](\tau_f(M_X\times[n,n+1))) \\
&\qquad \leq [\mathrm{m}\times\mathrm{m_L}](\tau_f(M)) \\
&\qquad\qquad + [\mathrm{m}\times\mathrm{m_L}]\left(\tau_f(M_X\times[n,n+1))\setminus\tau_f(M)\right) \\
&\qquad = [\mathrm{m}\times\mathrm{m_L}](\tau_f(M)) \\
&\qquad\qquad + [\mathrm{m}\times\mathrm{m_L}]\left(\tau_f\left((M_X\times[n,n+1))\setminus M\right)\right).
\end{aligned}$$

Thus, all of the inequalities must be inequalities, which in particular implies that

$$\begin{aligned}
&[\mathrm{m}\times\mathrm{m_L}](\tau_f(M_X\times[n,n+1))) \\
&\qquad = [\mathrm{m}\times\mathrm{m_L}](\tau_f(M)) \\
&\quad + [\mathrm{m}\times\mathrm{m_L}]\left(\tau_f(M_X\times[n,n+1))\setminus\tau_f(M)\right).
\end{aligned} \tag{5.2.2.27}$$

Exercise 5.2.2.28 Use this to conclude that $\tau_f(M)$ is measurable.

R Hint: See [Pug02, Theorem 6.4.13].

Hence,

$$\begin{aligned}
0 &= [\mathrm{m}\times\mathrm{m_L}](M_X\times[n,n+1)) - [\mathrm{m}\times\mathrm{m_L}](\tau_f(M_X\times[n,n+1))) \\
&= \left([\mathrm{m}\times\mathrm{m_L}](M) + [\mathrm{m}\times\mathrm{m_L}]((M_X\times[n,n+1))\setminus M)\right) \\
&\qquad - \left([\mathrm{m}\times\mathrm{m_L}](\tau_f(M)) + [\mathrm{m}\times\mathrm{m_L}](\tau_f(M_X\times[n,n+1))\setminus\tau_f(M))\right) \\
&= \left([\mathrm{m}\times\mathrm{m_L}](M) - [\mathrm{m}\times\mathrm{m_L}](\tau_f(M))\right) \\
&\qquad + \left([\mathrm{m}\times\mathrm{m_L}]((M_X\times[n,n+1))\setminus M)\right. \\
&\qquad \left. - [\mathrm{m}\times\mathrm{m_L}]\left(\tau_f\left((M_X\times[n,n+1))\setminus M\right)\right)\right).
\end{aligned}$$

We already showed in (5.2.2.25) that $[\mathrm{m}\times\mathrm{m_L}](M) \leq [\mathrm{m}\times\mathrm{m_L}](\tau_f(M))$ (and similarly for the complement), so that both of these terms are nonnegative, and hence 0. This gives us $[\mathrm{m}\times\mathrm{m_L}](\tau_f(M)) = [\mathrm{m}\times\mathrm{m_L}](M)$, as desired.

STEP 5: SHOW THAT I IS NONNEGATIVE HOMOGENEOUS

Note that

$$\begin{aligned}\Gamma_{af} &\coloneqq \{\langle x,y\rangle \in X\times\mathbb{R} : y < af(x)\} \\ &= \{\langle x,ay\rangle \in X\times\mathbb{R} : y < f(x)\} = a\Gamma_f.\end{aligned} \tag{5.2.2.29}$$

Then, it follows from Exercise 5.1.5.14 that

$$\mathrm{I}(af) \coloneqq \mathrm{m}(\Gamma_{af}) = \mathrm{m}(a\Gamma_f) = a\,\mathrm{m}(\Gamma_f) \eqqcolon a\mathrm{I}(f). \tag{5.2.2.30}$$

Step 6: Prove Lebesgue's Monotone Convergence Theorem

Let $m\mapsto f_m$ be a nondecreasing sequence of functions (it is implicit that the it is nondecreasing *almost-everywhere*). Let us define $f_\infty : X\to[0,\infty]$ by $f_\infty(x)\coloneqq \lim_m f(x)$. For almost every $x\in X$, the sequence $m\mapsto f_m(x)$ is nondecreasing, and so by the usual Monotone Convergence Theorem, this limit $f_\infty(x)$ exists in $[0,\infty]^d$ for almost-every x. For all the points where we do not have monotonicity, we may without loss of generality take f_∞ to be infinite there.

We wish to show that

$$\lim_m \mathrm{I}(f_m) = \mathrm{I}(f_\infty). \tag{5.2.2.31}$$

However, note that

$$\Gamma_{f_m}\subseteq\Gamma_{f_{m+1}} \text{ and } \bigcup_{m\in\mathbb{Z}^+}\Gamma_{f_m} = \Gamma_{f_\infty} \tag{5.2.2.32}$$

for all m, and hence by Exercise 5.1.1.35 ("continuity from below")

$$\begin{aligned}\lim_m \mathrm{I}(f_m) &\eqqcolon \lim_m[\mathrm{m}\times\mathrm{m_L}](\Gamma_{f_m}) \\ &= \lim_m[\mathrm{m}\times\mathrm{m_L}]\left(\bigcup_{k=1}^m\Gamma_{f_k}\right) \\ &= [\mathrm{m}\times\mathrm{m_L}]\left(\bigcup_{m\in\mathbb{Z}^+}\Gamma_{f_m}\right) \\ &= [\mathrm{m}\times\mathrm{m_L}](\Gamma_{f_\infty}) \eqqcolon \mathrm{I}(f_\infty).^e\end{aligned} \tag{5.2.2.33}$$

STEP 7: SHOW UNIQUENESS

Now, let $\mathrm{I} : \mathrm{Bor}_0^+(X) \rightarrow [0, \infty]$ be an additive nonnegative-homogeneous map that satisfies Lebesgue's Monotone Convergence Theorem and $\mathrm{I}(\chi_S) = \mathrm{m}(S)$. It follows immediately that they agree on Borel simple functions. However, we already know (Proposition 5.2.1.35) that *any* Borel function can be written as the limit monotone limit of a sequence of simple Borel functions, so that, by Lebesgue's Monotone Convergence Theorem, we actually have equality of I with the integral on *all* Borel functions. ■

[a]Proof adapted from [Pug02, pg. 377].

[b]This requires the use of the *strict* inequality in the definition of Γ_f and Γ_g.

[c]Because every open set can be written as the countable disjoint union of closed-open intervals—see Exercise 5.1.5.13.

[d]If the sequence is bounded, the Monotone Convergence Theorem tells us it converges in $\mathbb{R}$. Otherwise, it converges to ∞.

[e]I find it fascinating how relatively trivial this is compared to additivity, despite this having a name attached to it and additivity practically taken for granted. On the contrary: Lebesgue's Monotone Convergence Theorem requires just basic theory of measures—additivity on the other hand requires a reasonably well-developed theory of topological measure spaces!

Exercise 5.2.2.34 Let $\langle X, \mathrm{m}\rangle$ be a topological measure space, let $f : X \rightarrow [-\infty, \infty]$ be integrable and Borel, and let $a \in [-\infty, \infty]$. Show that

$$\int_X \mathrm{d}\,\mathrm{m}(x)\, a f(x) = a \int_X \mathrm{d}\,\mathrm{m}(x)\, f(x). \tag{5.2.2.35}$$

R Of course, this is just the same as the "Nonnegative-homogeneity" listed above, except without the requirement that a be nonnegative.

We now turn to the issue mentioned in a remark of the definition of the integral (Definition 5.2.2.1) regarding the extension of the definition of the integral to functions whose integral 'should' exist, but doesn't.

Theorem 5.2.2.36. Let $\langle X, \mathrm{m}\rangle$ be a topological measure space, let $f\colon X \to [-\infty, \infty]$ be Borel, and let $\mathscr{K}$ denote the collection of quasicompact subsets of X. Then, if $f|_K$ is ∞-integrable for all $K \in \mathscr{K}$, then f is ∞-integrable iff $\lim_{K\in\mathscr{K}} \int_K \mathrm{d}\,\mathrm{m}(x)\, f(x)$ exists, and furthermore, in this case, we have

$$\int_X \mathrm{d}\,\mathrm{m}(x)\, f(x) = \lim_{K\in\mathscr{K}} \int_K \mathrm{d}\,\mathrm{m}(x)\, f(x). \tag{5.2.2.37}$$

R The condition that $f|_K$ be ∞-integrable for all quasicompact $K \subseteq X$ is imposed just so that the terms appearing in the limit $\int_K \mathrm{d}\,\mathrm{m}(x)\, f(x)$ make sense. It is the other hypothesis, that $\int_X \mathrm{d}\,\mathrm{m}(x)\, f_+(x) = \infty = \int_X \mathrm{d}\,\mathrm{m}(x)\, f_-(x)$, that is the significant one.

R Incidentally, in earlier versions of these notes, I called this condition[a] "quasicompactly ∞-integrable", though as I don't really use this condition anywhere else in the current version of the notes, it didn't make sense to give it a name.

In the non-∞ case this is just the condition that the integral of f over every quasicompact set be finite. One significance of this condition is that, while even something like the identity function on $\mathbb{R}$ need not be integrable, *every* continuous function is necessarily quasicompactly integrable (by the Extreme Value Theorem (Theorem 3.8.2.2) and the fact that quasicompact sets have finite measure). This is more often called ***locally integrable***, but I find this to be misleading terminology—to mean, "locally integrable" *should* mean "Every point has a neighborhood base consisting of measurable sets on which the function is integrable.".

R If at least one of $\int_X \mathrm{d}\,\mathrm{m}(x)\, f_\pm(x)$ is finite, then the integral of f is just $\int_X \mathrm{d}\,\mathrm{m}(x)\, f_+(x) - \int_X \mathrm{d}\,\mathrm{m}(x)\, f_-(x)$, which *does* make sense, that is, there is no reason to extend the definition of the integral to take into account such functions. The only functions which require any extra work at all are those which have $\int_X \mathrm{d}\,\mathrm{m}(x)\, f_\pm(x) = \infty$. However, this result shows that, in this case, $\lim_{K\in\mathscr{K}} \int_K \mathrm{d}\,\mathrm{m}(x)\, f(x)$ just doesn't make sense, and so you cannot use this to extend the definition of the integral for functions like $x \mapsto \frac{\sin(x)}{x}$.

[a] That is, the condition that $f|_K$ be ∞-integrable for all quasicompact $K \subseteq X$.

Proof. Suppose that $f|_K$ is ∞-integrable for all $K \in \mathscr{K}$.

($\Rightarrow$) Suppose that f is ∞-integrable.

Let us reduce it to the case where f is nonnegative. So, suppose we have proven the result for f nonnegative, Borel, and ∞-integrable. Then, for f arbitrary Borel and ∞-integrable, f_+ and f_- are nonnegative Borel, and so we have

$$\begin{aligned}\int_X \mathrm{d}\,\mathrm{m}(x)\, f(x) &= \int_X \mathrm{d}\,\mathrm{m}(x)\, f_+(x) - \int_X \mathrm{d}\,\mathrm{m}(x)\, f_-(x) \\ &= \lim_{K\in\mathscr{K}} \int_K \mathrm{d}\,\mathrm{m}(x)\, f_+(x) - \lim_{K\in\mathscr{K}} \int_K \mathrm{d}\,\mathrm{m}(x)\, f_-(x) \\ &= \lim_{K\in\mathscr{K}} \int_K \mathrm{d}\,\mathrm{m}(x)\, f(x),\end{aligned}$$

as desired.

It thus remains to prove the result in the nonnegative case. So, let $f\colon X \to [-\infty,\infty]$ be nonnegative and Borel. We wish to show that

$$\int_X \mathrm{d}\,\mathrm{m}(x)\, f(x) = \lim_{K\in\mathscr{K}} \int_K \mathrm{d}\,\mathrm{m}(x)\, f(x). \tag{5.2.2.38}$$

To show this, we show that every cofinal subnet $\lambda \mapsto \int_{K_\lambda} \mathrm{d}\,\mathrm{m}(x)\, f(x)$ has in turn a subnet that converges to $\int_X \mathrm{d}\,\mathrm{m}(x)\, f(x)$. So, let $\lambda \mapsto \int_{K_\lambda} \mathrm{d}\,\mathrm{m}(x)\, f(x)$ be a cofinal subnet of $K \mapsto \int_X \mathrm{d}\,\mathrm{m}(x)\, f(x)$. Write $X = \bigcup_{m\in\mathbb{N}} K_m$ with $L_m \subseteq X$ compact and $L_m \subseteq L_{m+1}$.[a] As the subnet is strict, the 'indices' $\{K_\lambda : \lambda\}$ are cofinal in $\mathscr{K}$, which in particular implies that, for every $m \in \mathbb{N}$, there is some $K_{\lambda_m} \supseteq K_m$. Then, the sequence $m \mapsto \chi_{K_{\lambda_m}} f$ is nondecreasing and converges to f, and so by Lebesgue's Monotone Convergence Theorem, we have that

$$\lim_m \int_{K_{\lambda_m}} \mathrm{d}\,\mathrm{m}(x)\, f(x) = \int_X, \tag{5.2.2.39}$$

as desired.

($\Leftarrow$) Suppose that $\lim_{K\in\mathscr{K}} \int_K \mathrm{d}\,\mathrm{m}(x)\, f(x)$ exists. We proceed by contradiction: suppose that f is not ∞-integrable. By definition, this means that $\int_X \mathrm{d}\,\mathrm{m}(x)\, f_+(x) = \infty = \int_X \mathrm{d}\,\mathrm{m}(x)\, f_-(x)$.

Write $X = \bigcup_{m\in M} F_m$ as a countable union of disjoint measurable sets of finite measure. Define $P := f^{-1}((0,\infty])$ and $N := f^{-1}([-\infty,0))$, as well as $P_m := P\cap F_m$ and $N_m := N\cap F_m$. Each P_m and N_m, as well as P and N themselves, are measurable because f is Borel.

Also note that

$$\sum_{m\in\mathbb{N}} \int_{P_m} \mathrm{d}\,\mathrm{m}(x)\, f_+(x) =^b \int_P \mathrm{d}\,\mathrm{m}(x)\, f_+(x) = \infty. \tag{5.2.2.40}$$

Similarly,

$$\sum_{m\in\mathbb{N}} \int_{N_m} \mathrm{d}\,\mathrm{m}(x)\, f_-(x) = \infty. \tag{5.2.2.41}$$

Define

$$a_m := \int_{P_m} \mathrm{d}\,\mathrm{m}(x)\, f_+(x) - \int_{N_m} \mathrm{d}\,\mathrm{m}(x)\, f_-(x). \tag{5.2.2.42}$$

It follows from Theorem 2.4.4.60 that there is some rearrangement which converges to any real number we like (as well as $\pm\infty$). Given any such rearrangement, we will find nondecreasing collection of compact sets $K_0 \subseteq K_1 \subseteq \cdots$ with $X = \bigcup_{m\in\mathbb{N}} K_m$ that has the property that $m \mapsto \int_{K_m} \mathrm{d}\,\mathrm{m}(x)\, f(x)$ converges to the value of this rearrangement. This will in particular imply that $\mathscr{K} \ni K \mapsto \int_K \mathrm{d}\,\mathrm{m}(x)\, f(x)$ has subnets converging to distinct values, whence it follows that the net $\mathscr{K} \ni K \mapsto \int_K \mathrm{d}\,\mathrm{m}(x)\, f(x)$ itself cannot converge. This will give us our contradiction, thereby completing the proof.

So, let $\alpha \in \mathbb{R}$, and pick some rearrangement which converges to α. Thus, after reindexing if necessary, we have that $\sum_{m\in\mathbb{N}} a_m = \alpha$. As just explained, we seek to find a nondecreasing collection of compact sets $K_0 \subseteq K_1 \subseteq \cdots$ with $X = \bigcup_{m\in\mathbb{N}} K_m$ and $\lim_m \int_{K_m} \mathrm{d}\,\mathrm{m}(x)\, f(x) = \alpha$.

By Proposition 5.1.2.36, we can write

$$P_m = \bigcup_{n\in\mathbb{N}} K_{m,n} \cup Y_m \text{ and } N_m = \bigcup_{n\in\mathbb{N}} L_{m,n} \cup Z_m, \quad (5.2.2.43)$$

where each $K_{m,0} \subseteq K_{m,1} \subseteq \cdots$ and $L_{m,0} \subseteq L_{m,1} \subseteq \cdots$ is a nondecreasing countable collection of compact sets, and each Y_m and Z_m has measure 0. By Lebesgue's Monotone Convergence Theorem (and the fact that each Y_m and Z_m has measure 0), we have

$$\int_{P_m} \mathrm{d}\,\mathrm{m}(x)\, f_+(x) = \lim_n \int_{K_{m,n}} \mathrm{d}\,\mathrm{m}(x)\, f_+(x) \quad (5.2.2.44)$$

and

$$\int_{N_m} \mathrm{d}\,\mathrm{m}(x)\, f_-(x) = \lim_n \int_{L_{m,n}} \mathrm{d}\,\mathrm{m}(x)\, f_-(x). \quad (5.2.2.45)$$

Let $\varepsilon > 0$. For $m \in \mathbb{N}$, let $n_m \in \mathbb{N}$ be sufficiently large so that

$$\left| \int_{P_k} \mathrm{d}\,\mathrm{m}(x)\, f_+(x) - \int_{K_{k,n_m}} \mathrm{d}\,\mathrm{m}(x)\, f_+(x) \right| < \tfrac{\varepsilon}{2^m} \quad (5.2.2.46)$$

and

$$\left|\int_{N_k} \mathrm{d}\,\mathrm{m}(x)\, f_-(x) - \int_{L_{k,n_m}} \mathrm{d}\,\mathrm{m}(x)\, f_-(x)\right| < \tfrac{\varepsilon}{2^m} \quad (5.2.2.47)$$

for all $0 \leq k \leq m$, and define

$$M_m := \bigcup_{k=0}^{m} K_{k,n_m} \cup \bigcup_{k=0}^{m} L_{k,n_m}. \quad (5.2.2.48)$$

Certainly $M_0 \subseteq M_1 \subseteq \cdots$ is a nondecreasing countable collection of compact sets with $X = \bigcup_{m\in\mathbb{N}} M_m$. It thus only remains to show that $\lim_m \int_{M_m} \mathrm{d}\,\mathrm{m}(x)\, f(x) = \alpha$.

So, let $\varepsilon > 0$, and choose $m_0 \in \mathbb{N}$ such that, whenever $m \geq m_0$, it follows that $\left|\sum_{k=0}^{m} a_k - \alpha\right| < \varepsilon$. Then, whenever $m \geq m_0$, it follows that

$$\begin{aligned}
\left|\int_{M_m} \mathrm{d}\,\mathrm{m}(x)\, f(x) - \alpha\right| &=^{c} \left|\sum_{k=0}^{m}\left[\int_{K_{k,n_m}} \mathrm{d}\,\mathrm{m}(x)\, f_+(x)\right.\right. \\
&\qquad \left.\left. - \int_{L_{k,n_m}} \mathrm{d}\,\mathrm{m}(x)\, f_-(x)\right] - \alpha\right| \\
&\leq \sum_{k=0}^{m}\left|\int_{K_{k,n_m}} \mathrm{d}\,\mathrm{m}(x)\, f_+(x) - \int_{P_k} \mathrm{d}\,\mathrm{m}(x)\, f_+(x)\right| \\
&\quad + \sum_{k=0}^{m}\left|\int_{N_k} \mathrm{d}\,\mathrm{m}(x)\, f_-(x) - \int_{L_{k,n_m}} \mathrm{d}\,\mathrm{m}(x)\, f_-(x)\right| \\
&\quad + \left|\sum_{k=0}^{m}\left[\int_{P_k} \mathrm{d}\,\mathrm{m}(x)\, f_+(x) - \int_{N_k} \mathrm{d}\,\mathrm{m}(x)\, f_-(x)\right] - \alpha\right| \\
&< \sum_{k=0}^{m} \tfrac{\varepsilon}{2^m} + \sum_{k=0}^{m} \tfrac{\varepsilon}{2^m} + \left|\sum_{k=0}^{m} a_k - \alpha\right| \leq 2\varepsilon + 2\varepsilon + \varepsilon = 5\varepsilon.
\end{aligned}$$

Thus, $m \mapsto \int_{M_m} \mathrm{d}\,\mathrm{m}(x)\, f(x)$ converges to α, as desired. ■

[a]We can do this because, writing $X = \bigcup_{m\in\mathbb{N}} L'_m$, if we replace L'_m with $K_m := \bigcup_{k=0}^{m} L'_k$, we can ensure that this is a nondecreasing union.

[b]This is an implicit application of Lebesgue's Monotone Convergence Theorem.

[c]Note that $\{K_{k,n_m} : 0 \leq k \leq m\}$ is a disjoint collection because $\{P_k : 0 \leq k \leq m\}$ is. Similarly for $\{L_{k,n_m} : 0 \leq k \leq m\}$. Also note that f_- vanishes on each K_{k,n_m}, and likewise f_+ vanishes on each L_{k,n_m}.

One crucial fact is that the integral 'doesn't care' about sets of measure 0.

Proposition 5.2.2.49 Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and let $f, g \colon X \to [-\infty, \infty]$ be ∞-integrable and Borel. Then, if $f(x) = g(x)$ almost-everywhere, then $\int_X \mathrm{d}\,\mathrm{m}(x)\, f(x) = \int_X \mathrm{d}\,\mathrm{m}(x)\, g(x)$.

R In particular, if $f(x) = 0$ almost-everywhere, then $\int_X \mathrm{d}\,\mathrm{m}(x)\, f(x) = 0$.

R For example, the *integral of the Dirichlet Function is* 0![a]

R Note that, by Exercise 5.2.1.7, if f is Borel then so is g (and vice versa), so in fact you only need to assume a priori that one of f and g is Borel.

[a]Another way to see this is from the fact that $\int_X \mathrm{d}\,\mathrm{m}(x)\, \chi_S(x) = \mathrm{m}(S)$ for S measurable and the fact that $\mathrm{m}_{\mathrm{L}}(\mathbb{Q}) = 0$ because $\mathbb{Q}$ is countable—see Exercise 5.1.5.19.

Proof. STEP 1: REDUCE TO THE CASE IN WHICH f AND g ARE NONNEGATIVE

Suppose we have proven the result when f and g are nonnegative. If $f(x) = g(x)$ almost-everywhere, then $f_\pm(x) = g_\pm(x)$

almost-everywhere, and so

$$\begin{aligned}\int_X \mathrm{d}\,\mathrm{m}(x)\, f(x) & \\ &= \int_X \mathrm{d}\,\mathrm{m}(x)\, f_+(x) - \int_X \mathrm{d}\,\mathrm{m}(x)\, f_-(x) \\ &= \int_X \mathrm{d}\,\mathrm{m}(x)\, g_+(x) - \int_X \mathrm{d}\,\mathrm{m}(x)\, g_-(x) \\ &= \int_X \mathrm{d}\,\mathrm{m}(x)\, g(x).\end{aligned} \tag{5.2.2.50}$$

STEP 2: PROVE THE CASE WHERE f AND g ARE NONNEGATIVE

Suppose that f and g are nonnegative, ∞-integrable, and Borel. Then, by definition

$$\int_X \mathrm{d}\,\mathrm{m}(x)\, f(x) = [\mathrm{m}\times\mathrm{m}_{\mathrm{L}}](\Gamma_f), \tag{5.2.2.51}$$

where

$$\Gamma_f := \{\langle x, y\rangle \in X\times[0,\infty] : 0\le y < f(x)\}, \tag{5.2.2.52}$$

and similarly for g.

Define

$$S := \{x\in X : f(x)\neq g(x)\}. \tag{5.2.2.53}$$

By hypothesis, we have that $\mathrm{m}(S) = 0$. We also have that

$$\begin{aligned}\Gamma_f \cap \Gamma_g^{\mathsf{C}} &= \{\langle x, y\rangle \in X\times[0,\infty] : y < f(x) \text{ and } y \ge g(x)\} \\ &\subseteq S\times[0,\infty],\end{aligned}$$

and so

$$\begin{aligned}[\mathrm{m}\times\mathrm{m}_{\mathrm{L}}](\Gamma_f\cap\Gamma_g^{\mathsf{C}}) &\le [\mathrm{m}\times\mathrm{m}_{\mathrm{L}}](S\times[0,\infty]) \\ &= 0\cdot\infty := 0.\end{aligned} \tag{5.2.2.54}$$

Hence,

$$\begin{aligned}[\mathrm{m}\times\mathrm{m_L}](\Gamma_f) &\overset{a}{=} [\mathrm{m}\times\mathrm{m_L}](\Gamma_f\cap\Gamma_g)+[\mathrm{m}\times\mathrm{m_L}](\Gamma_f\cap\Gamma_g^{\complement}) \\ &= [\mathrm{m}\times\mathrm{m_L}](\Gamma_f\cap\Gamma_g).\end{aligned}$$

By $f\leftrightarrow g$ symmetry, we likewise have that $[\mathrm{m}\times\mathrm{m_L}](\Gamma_g)=[\mathrm{m}\times\mathrm{m_L}](\Gamma_f\cap\Gamma_g)$. Combining this with the previous equality gives us that $\int_X \mathrm{d}\,\mathrm{m}(x)\,f(x)=\int_X \mathrm{d}\,\mathrm{m}(x)\,g(x)$, as desired. ■

[a] Because Γ_g is measurable, which is precisely the definition of a borel function.

Exercise 5.2.2.55 What happens if we don't assume f and g to be Borel?

5.2.3 Properties of the integral

So obviously we have already discussed some properties of the integral, and admittedly there isn't really any hard and fast rules used to distinguish properties that came in the previous section The integral itself and this one, but loosely speaking, results in the previous section are fundamental enough that they conceivably could have been integrated (no pun intended) into the definition itself (for example, you could have defined the integral as a function on $\sim_{\mathrm{AIE}}$-equivalence classes of functions, which would have been well-defined by Proposition 5.2.2.49). In any case, this distinction doesn't really matter: they're all true, and you should know them.

Proposition 5.2.3.1 — Triangle Inequality Let $\langle X,\mathrm{m}\rangle$ be a topological measure space and let $f\colon X\to[-\infty,\infty]$ be ∞-integrable. Then,

$$\left|\int_X \mathrm{d}\,\mathrm{m}(x)\,f(x)\right|\leq\int_X \mathrm{d}\,\mathrm{m}(x)\,\left|f(x)\right|. \tag{5.2.3.2}$$

Proof. Using the fact that $f = f_+ - f_-$ and $|f| = f_+ + f_-$, we have

$$\begin{aligned}\left|\int_X \mathrm{d}\,\mathrm{m}(x)\, f(x)\right| &=^{a} \left|\int_X \mathrm{d}\,\mathrm{m}(x)\, f_+(x) - \int_X \mathrm{d}\,\mathrm{m}(x)\, f_-(x)\right| \\ &\leq \int_X \mathrm{d}\,\mathrm{m}(x)\, f_+(x) + \int_X \mathrm{d}\,\mathrm{m}(x)\, f_-(x) \\ &= \int_X \mathrm{d}\,\mathrm{m}(x)\, |f(x)|\,.\end{aligned}$$

■

[a] This requires that f be borel.

Exercise 5.2.3.3 — Fatou's Lemma Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and let $m \mapsto f_m \in \mathrm{Bor}_0^+(X)$ be a sequence. Show that

$$\begin{aligned}&\int_X \mathrm{d}\,\mathrm{m}(x)\, \liminf_m f_m(x) \leq \liminf_m \int_X \mathrm{d}\,\mathrm{m}(x)\, f_m(x) \\ &\quad \leq \limsup_m \int_X \mathrm{d}\,\mathrm{m}(x)\, f_m(x) \leq \int_X \mathrm{d}\,\mathrm{m}(x)\, \limsup_m f_m(x).\end{aligned}$$

Exercise 5.2.3.4 Find an example of a topological measure space $\langle X, \mathrm{m}\rangle$ and a sequence of nonnegative Borel functions $m \mapsto f_m \in \mathrm{Bor}_0^+(X)$ for which the inequalities in Fatou's Lemma are strict.

One of the most important results regarding the integral is the ***Dominated Convergence Theorem***.

Theorem 5.2.3.5 — Dominated Convergence Theorem. Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and let $m \mapsto f_m \in \mathrm{Bor}(X)$ be a sequence converging to $f_\infty \in \mathrm{Mor}_{\mathbf{Set}}(X, [-\infty, \infty])/\sim_{\mathbf{AlE}}$.[a] Then, if there is some integrable $g \in \mathrm{Mor}_{\mathbf{Set}}(X, [0, \infty])$ such that eventually $|f_m| \leq g$, then f_∞

is integrable and

$$\lim_\lambda \int_X \mathrm{d}\,\mathrm{m}(x)\,|f_\lambda(x) - f_\infty(x)| = 0. \tag{5.2.3.6}$$

R In particular, by the Triangle Inequality, it follows that

$$\lim_\lambda \int_X \mathrm{d}\,\mathrm{m}(x)\, f_\lambda(x) = \int_X \mathrm{d}\,\mathrm{m}(x)\, \lim_\lambda f_\lambda(x). \tag{5.2.3.7}$$

W Warning: It is not enough that f_∞ itself be integrable[b]—see Example 5.2.3.17. But it's worth than this: *even if your space has finite measure, and every f_m is integrable, and f_∞ is integrable, this can still fail*—see Example 5.2.3.20.

R It's worth noting that if you want to pull this shit (i.e. commute limits with integrals) with the Riemann integral you need *uniform convergence*. Holy Jesus is this an inconveniently strong condition.

[a]Of course, f_∞ is likewise Borel by Proposition 5.2.1.30.
[b]This is part of the conclusion, not part of the hypothesis

Proof. We leave the proof as a series of exercises.

Exercise 5.2.3.8 First of all check that, *if* the result is true, then f_∞ is indeed integrable.

Exercise 5.2.3.9 Reduce the proof to the case where each $f_m \geq 0$.

Exercise 5.2.3.10 Define $g_m := \inf_{k\geq m} f_k$ and $h_m := \sup_{k\geq m} f_k$.[a] Show that

$$\begin{aligned}\lim_m \int_X \mathrm{d}\,\mathrm{m}(x)\,\lvert g_m(x) - f_\infty(x)\rvert &= 0 \\ = \lim_m \int_X \mathrm{d}\,\mathrm{m}(x)\,\lvert h_m(x) - f_\infty(x)\rvert\,. \end{aligned} \tag{5.2.3.11}$$

[a] These are all Borel functions by Proposition 5.2.1.28.

Exercise 5.2.3.12 Use this to finish the proof.

■

Exercise 5.2.3.13 Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and let $f \in \mathrm{Bor}_0^+(X)$. Show that if $\int_X \mathrm{d}\,\mathrm{m}(x)\, f(x) = 0$, then $f(x) = 0$ almost-everywhere.

Exercise 5.2.3.14 Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and let $f, g \in \mathrm{Bor}(X)$ be ∞-integrable. Show that if $f \leq g$, then $\int_X \mathrm{d}\,\mathrm{m}(x)\, f(x) \leq \int_X \mathrm{d}\,\mathrm{m}(x)\, g(x)$.

Proposition 5.2.3.15 — Integral Test Let $m_0 \in \mathbb{N}$ and let $f\colon [m_0, \infty) \to [0, \infty)$ be nonincreasing. Then, $\sum_{m=m_0}^{\infty} f(m)$ converges iff $\int_{m_0}^{\infty} \mathrm{d}x\, f(x)$ converges.

Proof. Define $g\colon [m_0, \infty) \to [0, \infty)$ by $g(x) := f(\lfloor x\rfloor)$. Similarly define $h\colon [m_0, \infty) \to [0, \infty)$ by $h(x) := f(\lceil x\rceil)$. As f is nonincreasing and $\lfloor x\rfloor \leq x$, we have that $g(x) := f(\lfloor x\rfloor) \geq f(x)$. Similarly, we have $h(x) \leq f(x)$. By the previous exercise, this gives

$$\begin{aligned} f(m_0)+\sum_{m=m_0+1}^{\infty} f(m) &= \int_{m_0}^{\infty} \mathrm{d}x\, g(x) \\ &\geq \int_{m_0}^{\infty} \mathrm{d}x\, f(x) \geq \int_{m_0}^{\infty} \mathrm{d}x\, h(x) \\ &= \sum_{m=m_0}^{\infty} f(m). \end{aligned} \tag{5.2.3.16}$$

This implies the desired result. ■

▪ **Example 5.2.3.17 — A monotone sequence of functions which converges to 0 whose integrals converge to ∞** This is actually quite easy. Not even close to being exotic compared to some of the counter-examples we've see. For example, define $f_m : \mathbb{R} \to [0,\infty]$ by $f_m := \chi_{[m,\infty)}$. Then, $\lim_m f_m(x) = 0$ for all $x \in \mathbb{R}$, and so

$$\int_{\mathbb{R}} \mathrm{d}x\, \lim_m f_m(x) = 0. \tag{5.2.3.18}$$

On the other hand, $\int_{\mathbb{R}} \mathrm{d}x\, f_m(x) = \infty$ for all $m \in \mathbb{N}$, and so

$$\lim_m \int_{\mathbb{R}} \mathrm{d}x\, \lim_m f_m(x) = \infty. \tag{5.2.3.19}$$

Fortunately, however, we do have 'commutation' of limits with integrals in cases more than just a nondecreasing sequence of functions, because of, for example, the Dominated Convergence Theorem (Theorem 5.2.3.5). As mentioned in a remark of this theorem, integrability of the limit and each function in the sequence is *not* enough—you really need them *all* to be bounded by a *single* integrable function.

▪ **Example 5.2.3.20 — A sequence of integrable functions converging to an integrable function on a space**

of finite measure, but for which the limit of the integral is not the integral of the limit For $m \in \mathbb{N}$, define $f_m : [0,1] \to \mathbb{R}$ by $f_m(x) := (m+1)x^m$. Then, $\lim_m f_m(x) = 0$ almost-everywhere, and so we have

$$\int_0^1 \mathrm{d}x \lim_m f_m(x) = 0. \tag{5.2.3.21}$$

On the other hand, we have that $\int_0^1 \mathrm{d}x\, f_m(x) = 1$ for all $m \in \mathbb{N}$, and so

$$\lim_m \int_0^1 \mathrm{d}x\, f_m(x) = 1. \tag{5.2.3.22}$$

We mentioned awhile back that sums are really just integrals over the counting measure.

Proposition 5.2.3.23 Let m denote the counting measure on $\mathbb{N}$ and let $f \in \mathrm{Bor}_0^+(\langle \mathbb{N}, \mathrm{m}\rangle)$. Then,

$$\int_{\mathbb{N}} \mathrm{d}\,\mathrm{m}(m)\, f(m) = \sum_{m\in\mathbb{N}} f(m). \tag{5.2.3.24}$$

R If you're ever teaching a child how to add two numbers, say 3 and 5, don't tell them to integrate the function $f\colon \{x_1, x_2\} \to \mathbb{N}$, $f(x_1) = 3$ and $f(x_2) = 5$, over the two point space equipped with the counting measure. As elucidating as that may seem, it's circular—we needed to know how to add natural numbers to get to this point. It's probably best to teach them that it is the cardinality of the disjoint union of the set $\{0, 1, 2\}$ and $\{0, 1, 2, 3, 4\}$.[a]

[a] Don't forget to explain that a finite cardinal is an equivalence class of sets, all of which have the property that there is no bijection onto a proper subset, with respect to the equivalence relation that is isomorphism in the category sets. If you omit this, I fear you might risk confusing them.

Proof. For $m \in \mathbb{N}$, define

$$f_m := \sum_{k=0}^{m} \chi_{\{k\}} f(k). \tag{5.2.3.25}$$

Note that $m \mapsto f_m$ is nondecreasing and converges to f. Hence, by Lebesgue's Monotone Convergence Theorem,

$$\begin{aligned}
&\int_{\mathbb{N}} \mathrm{d}\,\mathrm{m}(m)\, f(m) \\
&\quad = \int_{\mathbb{N}} \mathrm{d}\,\mathrm{m}(m)\, \lim_n \sum_{k=0}^{n} \chi_{\{k\}}(m) f(k) \\
&\quad = \lim_n \int_{\mathbb{N}} \mathrm{d}\,\mathrm{m}(m)\, \sum_{k=0}^{n} \chi_{\{k\}}(m) f(k) \\
&\quad =^{a} \lim_n \sum_{k=0}^{n} f(k) \int_{\mathbb{N}} \chi_{\{k\}}(n) \\
&\quad = \lim_n \sum_{k=0}^{n} f(k)\, \mathrm{m}(\{k\}) = \lim_n \sum_{k=0}^{n} f(k) \\
&\quad = \sum_{m \in \mathbb{N}} f(m).
\end{aligned} \tag{5.2.3.26}$$

■

[a] By additivity and nonnegative-homogeneity.

Fubini's Theorem and its consequences

We next present Fubini's Theorem, which is absolutely crucial in the calculation of integrals in multivariable calculus. It essentially reduces integral multivariable calculus to single-variable calculus.

Theorem 5.2.3.27 — Fubini's Theorem. Let $\langle X_1, \mathrm{m}_1 \rangle$ and $\langle X_2, \mathrm{m}_2 \rangle$ be topological measure spaces, let $M_1 \subseteq X_1$ and let $M_2 \subseteq X_2$ be measurable, and let $f \colon X_1 \times X_2 \to [-\infty, \infty]$ be

∞-integrable. Then, if f is Borel or characteristic, then

$$\int_{M_1\times M_2} \mathrm{d}[\mathrm{m}_1\times\mathrm{m}_2](x)\, f(x) = \int_{M_1} \mathrm{d}\,\mathrm{m}_1(x_1) \int_{M_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, f(x_1, x_2) \tag{5.2.3.28}$$

and

$$\int_{M_1\times M_2} \mathrm{d}[\mathrm{m}_1\times\mathrm{m}_2](x)\, f(x) = \int_{M_2} \mathrm{d}\,\mathrm{m}_2(x_2) \int_{M_1} \mathrm{d}\,\mathrm{m}_1(x_1)\, f(x_1, x_2) \tag{5.2.3.29}$$

R That is to say, the result holds if either f is Borel or if $f = \chi_S$, for S *not necessarily measurable*.

R In particular, it is implicit that for every $x_1 \in X_1$, $x_2 \mapsto f(x_1, x_2)$ is ∞-integrable, and that $x_1 \mapsto \int_{M_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, f(x_1, x_2)$ is ∞-integrable, and similarly for the $1 \leftrightarrow 2$ case. (That is to say, the minimum requirements needed in order to ensure that the right-hand sides of (5.2.3.28) and (5.2.3.29) are defined are in fact true.)

R In other words, the "double integral" is equal to both "iterated integrals".

R The case where f is nonnegative is sometimes called ***Tonelli's Theorem***, in which case "Fubini's Theorem" is probably being used to refer to the case where f is integrable (in the nonnegative case, f would be allowed to be ∞-integrable in our terminology). By stating and proving the result for f ∞-integrable, we subsume both special cases (and more) into a single statement.

R This theorem is one motivation for us defining the integral for *all* (∞-integrable) functions, instead of just the Borel (∞-integrable) functions. While it *is* true that $x_1 \mapsto f(x_1, x_2)$ and $x_2 \mapsto f(x_1, x_2)$ are Borel if f is Borel (Proposition 5.2.3.55), we don't know this a priori, and so it is convenient that we can still write down these integrals and have them make sense without first having to prove that the integrands are Borel.

Proof. Step 1: Make hypotheses
Suppose that f is Borel or characteristic. First, consider the case where f is Borel. Ultimately, (Step 11) we reduce this case to proving the case for f characteristic as well, and so in that step we will finish the case f characteristic as well.

Step 2: Reduce to the case of (5.2.3.28)
By $1 \leftrightarrow 2$ symmetry, it suffices to prove (5.2.3.28).

Step 3: Reduce to the case in which $M_1 = X_1$ and $M_2 = X_2$
Suppose that we have proven the result in the case $M_1 = X_1$ and $M_2 = X_2$, and let $S_1 \subseteq X_1$ and $M_2 \subseteq X_2$ be measurable. Then, $\chi_{M_1 \times M_2}(x_1, x_2) = \chi_{M_1}(x_1)\chi_{M_2}(x_2)$ is Borel, and so we have that.

$$\begin{aligned}
&\int_{M_1 \times M_2} \mathrm{d}[\mathrm{m}_1 \times \mathrm{m}_2](x)\, f(x) \\
&\qquad := \int_{X_1 \times X_2} \mathrm{d}[\mathrm{m}_1 \times \mathrm{m}_2](x)\, \chi_{M_1 \times M_2}(x) f(x) \\
&\qquad = \int_{X_1} \mathrm{d}\,\mathrm{m}_1(x_1) \int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, \chi_{M_1 \times M_2}(x_1, x_2) f(x_1, x_2) \\
&\qquad = \int_{X_1} \mathrm{d}\,\mathrm{m}_1(x_1)\, \chi_{M_1}(x_1) \int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, \chi_{M_2}(x_2) f(x_1, x_2) \\
&\qquad =: \int_{M_1} \mathrm{d}\,\mathrm{m}_1(x_1) \int_{M_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, f(x_1, x_2).
\end{aligned}$$

Step 4: Reduce to the case in which f is nonnegative

Suppose we have proven the result in which f is nonnegative. If f is ∞-integrable, we have

$$\begin{aligned}
&\int_{X_1\times X_2} \mathrm{d}[\mathrm{m}_1\times\mathrm{m}_2](x)\, f(x) \\
&\qquad := \int_{X_1\times X_2} \mathrm{d}[\mathrm{m}_1\times\mathrm{m}_2](x)\, f_+(x) \\
&\qquad\quad - \int_{X_1\times X_2} \mathrm{d}[\mathrm{m}_1\times\mathrm{m}_2](x\, f_-(x) \\
&\qquad = \int_{X_1} \mathrm{d}\,\mathrm{m}_1(x_1) \int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, f_+(x_1,x_2) \\
&\qquad\quad - \int_{X_1} \mathrm{d}\,\mathrm{m}_1(x_1) \int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, f_-(x_1,x_2) \\
&\qquad =: \int_{X_1} \mathrm{d}\,\mathrm{m}_1(x_1) \int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, f(x_1,x_2),
\end{aligned} \tag{5.2.3.30}$$

as desired.

Step 5: Reduce to the case in which f is a characteristic function

Suppose that we have proven the result for characteristic functions. Then, by Proposition 5.2.1.35 (every nonnegative Borel function is a monotonic limit of simple functions), Lebesgue's Monotone Convergence Theorem, and linearity of the integral, the result follows for arbitrary $f \in \mathrm{Bor}_0^+(X_1\times X_2)$.

Step 6: Define a measure m on $X_1 \times X_2$

Define $\mathrm{m}\colon 2^{X_1\times X_2} \to [0,\infty]$ by

$$\mathrm{m}(S) := \int_{X_1} \mathrm{d}\,\mathrm{m}_1(x_1) \int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, \chi_{S_{x_1}}(x_2), \tag{5.2.3.31}$$

where

$$S_{x_1} := \{x_2 \in X_2 : \langle x_1, x_2\rangle \in S\}\,. \tag{5.2.3.32}$$

We wish to show that this is a regular Borel measure on $X_1 \times X_2$ such that $\mathrm{m}(K_1 \times K_2) = \mathrm{m}_1(K_1)\,\mathrm{m}_2(K_2)$ for $K_i \subseteq X_i$ quasicompact.

STEP 7: SHOW THAT $\mathrm{m}[K_1 \times K_2] = \mathrm{m}_1(K_1)\,\mathrm{m}_2(K_2)$
As

$$[K_1 \times K_2]_{x_1} = \begin{cases} K_2 & \text{if } x_1 \in K_2 \\ \emptyset & \text{if } x \in K_2^{\mathsf{C}}, \end{cases} \tag{5.2.3.33}$$

we have $\chi_{[K_1 \times K_2]_{x_1}}(x_2) = \ chi_{K_1}(x_1)\chi_{K_2}(x_2)$, and hence indeed $\mathrm{m}(K_1 \times K_2) := \int_{X_1}$. Thus, it suffices to show that this defines a regular Borel measure.

STEP 8: SHOW THAT m IS A MEASURE
It is immediate that $\mathrm{m}(\emptyset) = 0$.

m is nondecreasing by Exercise 5.2.3.14.

Let $M_m \subseteq X_1 \times X_2$ for $m \in \mathbb{N}$. Define $S_m := \bigcup_{k=0}^{m} M_k$ for $m \in \mathbb{N}$. Then

$$\chi_{\bigcup_{m \in \mathbb{N}} M_m}(x) = \lim_n \chi_{S_n}(x) \tag{5.2.3.34}$$

for all $x \in X_1 \times X_2$, and so, by Lebesgue's Monotone Convergence Theorem, we have that m is subadditive.

Thus, m is a measure.

STEP 9: SHOW THAT m IS REGULAR
We first check outer-regularity. Let $S \subseteq X_1 \times X_2$. If $\mathrm{m}(S) = \infty$, then

$$\inf \{\mathrm{m}(U) : S \subseteq U,\ U \text{ open.}\} = \infty \tag{5.2.3.35}$$

because m is nondecreasing, so we may without loss of generality assume that $\mathrm{m}(S) < \infty$. Let $\varepsilon > 0$. By the defining theorem of the product measure (Theorem 5.1.4.2),

$$[\mathrm{m}_1 \times \mathrm{m}_2](S) = \inf \left\{ \sum_{m\in\mathbb{N}} \mathrm{m}_1(U_{1,m})\, \mathrm{m}_2(U_{2,m}) : \right. \\ \left. U_{i,m} \subseteq X_i \text{ open}, S \subseteq \bigcup_{m\in\mathbb{N}} U_{1,m} \times U_{2,m} \right\},$$

and so there are open $U_{i,m} \subseteq X_i$ with $S \subseteq \bigcup_{m\in\mathbb{N}} U_{1,m} \times U_{2,m}$ such that

$$\begin{aligned} [\mathrm{m}_1 \times \mathrm{m}_2](S) - \varepsilon < \sum_{m\in\mathbb{N}} \mathrm{m}_1(U_{1,m})\, \mathrm{m}_2(U_{2,m}) \\ \leq [\mathrm{m}_1 \times \mathrm{m}_2](S). \end{aligned} \tag{5.2.3.36}$$

Without loss of generality,[a] assume that the $U_{1,m}$s and the $U_{2,m}$ are disjoint. Then,

$$\begin{aligned} &\mathrm{m}\left(\bigcup_{m\in\mathbb{N}} U_{1,m} \times U_{2,m} \right) \\ &:= \int_{X_1} \mathrm{d}\,\mathrm{m}_1(x_1) \int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, \chi_{\left[\bigcup_{m\in\mathbb{N}} U_{1,m}\times U_{2,m}\right]_{x_1}}(x_2) \\ &=^{b} \int_{X_1} \mathrm{d}\,\mathrm{m}_1(x_1) \int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2) \sum_{m\in\mathbb{N}} \chi_{U_{1,m}}(x_1)\chi_{U_{2,m}}(x_2) \\ &=^{c} \sum_{m\in\mathbb{N}} \int_{X_1} \mathrm{d}\,\mathrm{m}_1(x_1)\, \chi_{U_{1,m}}(x_1) \int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, \chi_{U_{2,m}}(x_2) \\ &= \sum_{m\in\mathbb{N}} \mathrm{m}_1(U_{1,m})\, \mathrm{m}_2(U_{2,m}). \end{aligned} \tag{5.2.3.37}$$

This, together with Equation (5.2.3.36) shows that m is outer-regular.

A very similar argument using

$$\begin{aligned}
&[\mathrm{m}_1 \times \mathrm{m}_2](U) \\
&\qquad := \sup \Bigg\{ \sum_{k=0}^{m} \mathrm{m}_1(K_{1,k})\, \mathrm{m}_2(K_{2,k}) : \\
&\qquad\quad m \in \mathbb{N},\ K_{i,k} \subseteq X_i \text{ quasicompact}, \\
&\qquad\quad \left\{K_{1,k} \times K_{2,k} : 0 \leq k \leq m\right\} \text{ is disjoint}, \\
&\qquad\quad \bigcup_{k=0}^{m} K_{1,k} \times K_{2,k} \subseteq U \Bigg\}
\end{aligned} \tag{5.2.3.38}$$

for $U \subseteq X_1 \times X_2$ open and

$$\begin{aligned}
&[\mathrm{m}_1 \times \mathrm{m}_2](S) \\
&\quad := \inf \left\{ [\mathrm{m}_1 \times \mathrm{m}_2](U) : S \subseteq U,\ U \text{ open} \right\};
\end{aligned} \tag{5.2.3.39}$$

for $S \subseteq X_1 \times X_2$ arbitrary, shows that m is inner-regular on open sets.

By Step 7, m is finite on quasicompact subsets.

Thus, m is regular.

Step 10: Show that m is Borel

Let $U \subseteq X_1 \times X_2$ be open. The argument for inner-regularity on open sets using (5.2.3.38) and (5.2.3.39) will show that the same equation holds with $\mathrm{m}_1 \times \mathrm{m}_2$ on the left-hand side replaced with m. Thus, modulo a set of measurable 0 (which is measurable), U is a countable disjoint union of quasicompact rectangles. Thus, it suffices to show that $\mathrm{m}(K_1 \times K_2)$ is measurable for $K_i \subseteq X_i$ quasicompact.

So, let $S \subseteq X_1 \times X_2$ be arbitrary. We wish to show that

$$\mathrm{m}(S) \geq \mathrm{m}(S \cap (K_1 \times K_2)) + \mathrm{m}(S \cap (K_1 \times K_2)^{\mathrm{C}}). \tag{5.2.3.40}$$

If $\mathrm{m}(S) = \infty$, we are done, so we may as well assume that $\mathrm{m}(S) < \infty$. Let $\varepsilon > 0$. Let $U \subseteq X_1 \times X_2$ be open and such that (i) $S \subseteq U$ and (ii) $\mathrm{m}(S) > \mathrm{m}(U) - \varepsilon$. Because the space is T_2,

$K_1 \times K_2$ is closed, and so $U \cap (K_1 \times K_2)^{\mathsf{C}}$ is open. Therefore, there is a disjoint union

$$\bigcup_{k=0}^{m} L_{1,k} \times L_{2,k} \subseteq U \cap (K_1 \times K_2)^{\mathsf{C}} \tag{5.2.3.41}$$

for $L_{i,k} \subseteq X_i$ quasicompact such that

$$\mathrm{m}(U \cap (K_1 \times K_2)^{\mathsf{C}}) - \varepsilon < \sum_{k=0}^{m} \mathrm{m}(L_{1,k} \times L_{2,k}). \tag{5.2.3.42}$$

Similarly, there is a disjoint union

$$\bigcup_{k=0}^{n} M_{1,k} \times M_{2,k} \subseteq U \cap \left(\bigcup_{k=0}^{m} L_{1,k} \times L_{2,k}\right)^{\mathsf{C}} \tag{5.2.3.43}$$

for $M_{i,k} \subseteq X_i$ quasicompact such that

$$\begin{aligned} &\mathrm{m}\left(U \cap \left(\bigcup_{k=0}^{m} L_{1,k} \times L_{2,k}\right)^{\mathsf{C}}\right) - \varepsilon \\ &\qquad < \sum_{k=0}^{n} [\mathrm{m}_1 \times \mathrm{m}_2](M_{1,k} \times M_{2,k}). \end{aligned} \tag{5.2.3.44}$$

Hence,

$$
\begin{aligned}
\mathrm{m}(S) &> \mathrm{m}(U) - \varepsilon \\
&\geq \mathrm{m}\left(\bigcup_{k=0}^{m} L_{1,k} \times L_{2,k} \cup \bigcup_{k=0}^{n} M_{1,k} \times M_{2,k}\right) - \varepsilon \\
&= {}^{d} \sum_{k=0}^{m} \mathrm{m}(L_{1,k} \times L_{2,k}) \\
&\qquad + \sum_{k=0}^{n} \mathrm{m}(M_{1,k} \times M_{2,k}) - \varepsilon \\
&> \mathrm{m}\left(U \cap (K_1 \times K_2)^{\mathsf{C}}\right) \\
&\qquad + \mathrm{m}\left(U \cap \left(\bigcup_{k=0}^{m} L_{1,k} \times L_{2,k}\right)^{\mathsf{C}}\right) - 3\varepsilon \\
&\geq \mathrm{m}(U \cap (K_1 \times K_2)^{\mathsf{C}}) \\
&\qquad + \mathrm{m}(U \cap (K_1 \times K_2)) - 3\varepsilon.
\end{aligned}
$$

As $\varepsilon > 0$ was arbitrary, we obtain the desired result.

STEP 11: DEDUCE THE RESULT FOR f A CHARACTERISTIC FUNCTION

Then, by uniqueness of the product measure, we will have

$$
\begin{aligned}
&\int_{X_1} \mathrm{d}\,\mathrm{m}_1(x_1) \int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, \chi_{S_{x_1}}(x_2) \\
&\quad =: \mathrm{m}(S) = [\mathrm{m}_1 \times \mathrm{m}_2](S) \qquad (5.2.3.45) \\
&\quad = \int_{X_1 \times X_2} \mathrm{d}[\mathrm{m}_1 \times \mathrm{m}_2](x)\, \chi_S(x),
\end{aligned}
$$

as desired. ■

R While the proof here is quite long and the details are not a particularly high priority the first time through the subject, the technique used here is quite important, and is worth taking note of. The technique I have in mind is proving the result in steps: first for characteristic functions of measurable sets, then for nonnegative Borel functions, and finally for Borel ∞-integrable functions.

---

[a]Why can we do this?

[b]Because $\left[\bigcup_{m\in\mathbb{N}} U_{1,m}\times U_{2,m}\right]_{x_1}$ is $\bigcup_{\substack{k\in\mathbb{N}\\ x_1\in U_{1,k}}} U_{2,k}$, the characteristic function of which is, by disjointness, $\sum_{\substack{k\in\mathbb{N}\\ x_1\in U_{1,k}}} \chi_{2,k}$, which itself is equal to $\sum_{m\in\mathbb{N}} \chi_{U_{1,m}}(x_1)\chi_{U_{2,m}}$.

[c]By Lebesgue's Monotone Convergence Theorem.

[d]Because m agrees with $\mathrm{m}_1\times\mathrm{m}_2$ on disjoint unions of quasicompact rectangles, and $\mathrm{m}_1\times\mathrm{m}_2$ is additive on quasicompact rectangles.

While certainly very useful for computation, Fubini's Theorem can usually not be used alone. To really make use of this result, we must first meet the Fundamental Theorem of Calculus (Theorems 6.4.3.3 and 6.4.3.6)—see Example 6.4.3.16. That said, we can use it to prove one result that you are almost certainly familiar with, though perhaps not by name: ***Cavalieri's Principle***.

Corollary 5.2.3.46 — Cavalieri's Principle Let $\langle X_1,\mathrm{m}_1\rangle$ and $\langle X_2,\mathrm{m}_2\rangle$ be topological measure spaces and let $S\subseteq X_1\times X_2$. Then,

$$\int \mathrm{d}\,\mathrm{m}_1(x_1)\;\mathrm{m}_2(S_{x_1})=[\mathrm{m}_1\times\mathrm{m}_2](S)=\int \mathrm{d}\,\mathrm{m}_2(x_2)\;\mathrm{m}_1(S_{x_2}),\tag{5.2.3.47}$$

where

$$S_{x_1}\coloneqq\{x_2\in X_2:\langle x_1,x_2\rangle\in S\}\tag{5.2.3.48}$$

and

$$S_{x_2}\coloneqq\{x_1\in X_1:\langle x_1,x_2\rangle\in S\}\tag{5.2.3.49}$$

for $x_i\in X_i$.

R Cavalieri's Principle is the statement the 'volume' of an object can be computed by integrating the 'area' of the cross-section as a function of the 'height'. For example, this is most likely what you would use to show that volume of a cone with radius r and height h is $\frac{1}{3}\pi r^2 h$.[a]

R Cavalieri's Principle is one good reason why we wanted to state Fubini's Theorem for f the characteristic function of a not-necessarily-measurable. The proof of this result essentially amounts to plugging in $f = \chi_S$ into Fubini's Theorem, and if we needed f to be Borel, we would need S to be measurable.

[a]Though god (and perhaps readers who have consulted Proposition 6.4.5.48) only knows what this wacky symbol "π" is.

Proof. By Fubini's Theorem, we have

$$\begin{aligned}
&[\mathrm{m}_1 \times \mathrm{m}_2](S) \\
&\quad = \int_{X_1 \times X_2} \mathrm{d}[\mathrm{m}_1 \times \mathrm{m}_2](x)\, \chi_S(x) \\
&\quad = \int_{X_1} \mathrm{d}\,\mathrm{m}_1(x_1) \int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, \chi_S(x_1, x_2) \\
&\quad = \int_{X_1} \mathrm{d}\,\mathrm{m}_1(x_1) \int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, \chi_{S_{x_1}}(x_2) \\
&\quad = \int_{X_1} \mathrm{d}\,\mathrm{m}_1(x_1)\; \mathrm{m}_2(S_{x_1}).
\end{aligned} \tag{5.2.3.50}$$

∎

Exercise 5.2.3.51 Let $\langle X_1, \mathrm{m}_1\rangle$ and $\langle X_2, \mathrm{m}_2\rangle$ be topological measure spaces and let $M \subseteq X_1 \times X_2$. Show that if M is measurable that

$$M_{x_1} \text{ and } M_{x_2} \tag{5.2.3.52}$$

are measurable for almost-every $x_i \in X_i$, where

$$M_{x_1} := \{x_2 \in X_2 : \langle x_1, x_2\rangle \in M\} \tag{5.2.3.53}$$

$$M_{x_2} := \{x_1 \in X_1 : \langle x_1, x_2\rangle \in M\}\,. \tag{5.2.3.54}$$

Warning: This is *not* true for *all* x_1 and x_2, only *almost*-all x_1 and x_2—see Exercise 5.2.3.51.

This allows us to prove the following well-known result about product measures.

Proposition 5.2.3.55 Let $\langle X_1, \mathrm{m}_1\rangle$ and $\langle X_1, \mathrm{m}_2\rangle$ be topological measure spaces, and let $f\colon X_1 \times X_2 \to [-\infty, \infty]$. Then, if f is Borel and ∞-integrable, then

$$X_1 \ni x_1 \mapsto \int_{X_2} \mathrm{d}\,\mathrm{m}(x_2)\, f(x_1, x_2) \in [-\infty, \infty] \quad (5.2.3.56)$$

and

$$X_2 \ni x_2 \mapsto \int_{X_1} \mathrm{d}\,\mathrm{m}_1(x_1)\, f(x_1, x_2) \in [-\infty, \infty] \quad (5.2.3.57)$$

are Borel.

Proof. STEP 1: MAKE HYPOTHESES
Suppose that f is Borel and ∞-integrable.

STEP 2: REDUCE TO THE CASE OF $x_1 \mapsto f(x_1, x_2)$
By $1 \leftrightarrow 2$ symmetry, it suffices to prove that $X \ni x_1 \mapsto \int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, f(x_1, x_2) \in [-\infty, \infty]$ is Borel.

STEP 3: REDUCE TO THE CASE WHERE f IS NONNEGATIVE
Suppose we have proven the result for nonnegative functions and write $f = f_+ - f_-$. Then, $x_1 \mapsto \int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, f_\pm(x_1, x_2)$ is Borel, and so the difference of these two functions, namely $x_1 \mapsto \int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, f(x_1, x_2)$ is Borel.

STEP 4: REDUCE TO THE CASE WHERE $f = \chi_M$ FOR $M \subseteq X_1 \times X_2$ MEASURABLE

Suppose that we have proven the result for characteristic functions of measurable sets, and let $f: X_1 \times X_2 \to [0, \infty]$ be Borel. By Proposition 5.2.1.35, f is the almost-everywhere pointwise limit of a nondecreasing sequence of simple Borel functions. Hence, as sums and limits of Borel functions are Borel, it follows that the $x_1 \mapsto f(x_1, x_2)$ is Borel because this is true for all characteristic functions.

STEP 5: REDUCE TO THE CASE WHERE $f = \chi_M$ FOR $M \subseteq X_1 \times X_2$ MEASURABLE OF FINITE MEASURE

Suppose we have proven the result characteristic functions of measurable sets with finite measure, and let $M \subseteq X_1 \times X_2$ be measurable (not necessarily of finite measure). Write $X_i = \bigcup_{m \in \mathbb{N}} K_{i,m}$ for $K_{i,m} \subseteq X_i$ compact and define

$$L_m := \bigcup_{k=0=l}^{m} K_{1,k} \times K_{2,l}, \tag{5.2.3.58}$$

so that each L_m is compact, $L_m \subseteq L_{m+1}$, and $X_1 \times X_2 = \bigcup_{m \in \mathbb{N}} L_m$. Now define $M_m := M \cap L_m$. It follows that $M = \bigcup_{m \in \mathbb{N}} M_m$, and hence $M_{x_1} = \bigcup_{m \in \mathbb{N}} [M_m]_{x_1}$, and hence $\chi_{M_{x_1}} = \lim_m \chi_{[M_m]_{x_1}}$, and hence $\mathrm{m}_2(M_{x_1}) = \lim_m \mathrm{m}_2([M_m]_{x_1})$. As M_m is measurable with finite measure, by assumption, $x_1 \mapsto \int_{X_2} \mathrm{d}\,\mathrm{m}(x_2)\, \chi_{[M_m]_{x_1}}(x_2) =$ [a] $\mathrm{m}_2([M_m]_{x_1})$ is Borel, and so $x_1 \mapsto \mathrm{m}_2(M_{x_1})$ is Borel by Proposition 5.2.1.30 (limits of Borel functions are Borel).

STEP 6: PROVE THE RESULT FOR $f = \chi_M$ FOR $M \subseteq X_1 \times X_2$ MEASURABLE OF FINITE MEASURE

Let $M \subseteq X_1 \times X_2$ be measurable of finite measure. Let $\varepsilon > 0$, and , using "rectangle outer regularity" ((5.1.4.4)), choose $U^{\varepsilon}_{i,m} \subseteq X_i$ open such that $M \subseteq \bigcup_{m \in \mathbb{N}} U^{\varepsilon}_{1,m} \times U^{\varepsilon}_{2,m} =: U^{\varepsilon}$ and

$$\sum_{m \in \mathbb{N}} \mathrm{m}_1(U^{\varepsilon}_{1,m})\, \mathrm{m}_2(U^{\varepsilon}_{2,m}) - \varepsilon < [\mathrm{m}_1 \times \mathrm{m}_2](M). \tag{5.2.3.59}$$

Using the usual trick,[b] we may without loss of generality assume that $\{U^{\varepsilon}_{i,m} : m \in \mathbb{N}\}$ are disjoint for $i = 1, 2$ (though they will likely not be open anymore). Thus,

$$U^{\varepsilon}_{x_1} = \begin{cases} U^{\varepsilon}_{2,m} & \text{if } x_1 \in U^{\varepsilon}_{1,m} \\ \emptyset & \text{otherwise,} \end{cases} \tag{5.2.3.60}$$

and so

$$\mathrm{m}_2(U^{\varepsilon}_{x_1}) = \sum_{m\in\mathbb{N}} \mathrm{m}_2(U^{\varepsilon}_{2,m})\chi^{\varepsilon}_{U_{1,m}}(x_1), \tag{5.2.3.61}$$

and so, being a limit of Borel functions, $x_1 \mapsto \mathrm{m}_2(U^{\varepsilon}_{x_1})$ is Borel.

Now, by Cavalieri's Principle, we have

$$\begin{aligned} \varepsilon &> [\mathrm{m}_1 \times \mathrm{m}_2](M \setminus U^{\varepsilon}) \\ &= \int_{X_1} \mathrm{d}\, \mathrm{m}_1(x_1) \left|\mathrm{m}_2(M_{x_1}) - \mathrm{m}_2(U^{\varepsilon}_{x_1})\right| \end{aligned} \tag{5.2.3.62}$$

Thus, this integral must vanish as $\varepsilon \to 0^+$, which implies that $\lim_{\varepsilon\to 0^+} \mathrm{m}_2(U^{\varepsilon}_{x_1}) = \mathrm{m}_2(M_{x_1})$ almost-everywhere (Exercise 5.2.3.13), and hence $x_1 \mapsto \mathrm{m}_2(M_{x_1})$ is Borel, once again, because pointwise almost-everywhere limits of Borel functions are Borel. ■

[a] Note that $[M_m]_{x_1}$ need not be measurable (Example 5.1.4.48), and so this is not true for *all* x_1. Instead, it is only true for *almost*-all x_1 by the previous exercise, which, fortunately, is all we need by Exercise 5.2.1.7 (functions equal to a borel function almost-everywhere are borel).

[b] The one you weren't supposed to get—see (5.1.1.25).

Once again, the special case of $f = \chi_M$ is worth stating separately.

Corollary 5.2.3.63 Let $\langle X_1, \mathrm{m}_1\rangle$ and $\langle X_2, \mathrm{m}_2\rangle$ be topological measure spaces and let $M \subseteq X_1 \times X_2$. Then, if M is measurable, then

$$X_1 \ni x_1 \mapsto \mathrm{m}_2(M_{x_1}) \in [0, \infty] \tag{5.2.3.64}$$

and

$$X_2 \ni x_2 \mapsto \mathrm{m}_1(M_{x_2}) \in [0, \infty] \tag{5.2.3.65}$$

are Borel, where

$$M_{x_1} \coloneqq \{x_2 \in X_2 : \langle x_1, x_2\rangle \in M\} \tag{5.2.3.66}$$

$$M_{x_2} \coloneqq \{x_1 \in X_1 : \langle x_1, x_2\rangle \in M\} . \tag{5.2.3.67}$$

Proof. Suppose that M is measurable. By $1 \leftrightarrow 2$ symmetry, it suffices to prove that $x_1 \mapsto \mathrm{m}_2(M_{x_1})$ is Borel. As M is measurable, χ_M is Borel and ∞-integrable, and so by the previous result,

$$\begin{aligned} x_1 \mapsto \int_{X_2} \mathrm{d}\,\mathrm{m}(x_2)\,\chi_M(x_1, x_2) &= \int_{X_2} \mathrm{d}\,\mathrm{m}(x_2)\,\chi_{M_{x_1}}(x_2) \\ &= \mathrm{m}_2(M_{x_1}) \end{aligned}$$

is Borel. ■

In Theorem 5.2.2.7, we gave a list of properties that uniquely characterized the integral on nonnegative Borel functions. We mentioned in a remark there that in fact weaker versions of the statements held even when the functions in question are not necessarily Borel. Cavalieri's Principle finally allows to to prove all of these.

Theorem 5.2.3.68. Let $\langle X, \mathrm{m}\rangle$ be a topological measure space. Then,

(i).

$$\int_X \mathrm{d}\,\mathrm{m}(x)\,[f(x)+g(x)] \leq \int_X \mathrm{d}\,\mathrm{m}(x)\,f(x)+\int_X \mathrm{d}\,\mathrm{m}(x)\,g(x)$$

for $f, g\colon X \to [0,\infty]$ ∞-integrable; and

(ii). whenever $\lambda \mapsto f_\lambda$ is a nondecreasing net of nonnegative ∞-integrable functions, then

$$\lim_\lambda \int_X \mathrm{d}\,\mathrm{m}(x)\,f_\lambda(x) \leq \int_X \mathrm{d}\,\mathrm{m}(x)\,\lim_\lambda f_\lambda(x). \quad (5.2.3.69)$$

R The point is that we do not need to require the functions be Borel or the net to be a sequence for these statements to hold. Of course, the inequalities become equalities if all functions involved are Borel—see Theorem 5.2.2.7.

Proof. (i) By definition, we have

$$\begin{aligned}&\int_X \mathrm{d}\,\mathrm{m}(x)\,f(x)\\ &\quad := [\mathrm{m}\times\mathrm{m_L}]\left(\{\langle x,y\rangle \in X\times[-\infty,\infty] : \right.\\ &\qquad\qquad \left. 0\leq y<f(x)\}\right),\end{aligned} \quad (5.2.3.70)$$

where $\mathrm{m_L}$ is Lebesgue measure. Writing

$$\Gamma_f := \{\langle x,y\rangle \in X\times[0,\infty] : 0\leq y<f(x)\}, \quad (5.2.3.71)$$

it thus suffices to show that

$$[\mathrm{m}\times\mathrm{m_L}](\Gamma_{f+g}) \leq [\mathrm{m}\times\mathrm{m_L}](\Gamma_f)+[\mathrm{m}\times\mathrm{m_L}](\Gamma_g). \quad (5.2.3.72)$$

Define $\tau_f : X\times[0,\infty]\to X\times[0,\infty]$ by

$$\tau_f(\langle x,y\rangle) := \langle x, y+f(x)\rangle. \quad (5.2.3.73)$$

This definition was made so that we have

$$\Gamma_{f+g} = \Gamma_f \cup \tau_f(\Gamma_g) \quad (5.2.3.74)$$

is a disjoint[a] union. From this, it follows that

$$\begin{aligned}[\mathrm{m}\times\mathrm{m_L}](\Gamma_{f+g}) \\ \leq [\mathrm{m}\times\mathrm{m_L}](\Gamma_f)+[\mathrm{m}\times\mathrm{m_L}](\tau_f(\Gamma_g)),\end{aligned} \tag{5.2.3.75}$$

and so it suffices to show that

$$[\mathrm{m}\times\mathrm{m_L}](\tau_f(\Gamma_g))=[\mathrm{m}\times\mathrm{m_L}](\Gamma_g). \tag{5.2.3.76}$$

However, by Cavalieri's Principle, we have

$$\begin{aligned}&[\mathrm{m}\times\mathrm{m_L}](\tau_f(\Gamma_g)) \\ &\quad=\int_X \mathrm{d}\,\mathrm{m}(x) \\ &\qquad[\mathrm{m_L}\left(\{y\in[0,\infty]:f(x)\leq y<g(x)+f(x)\}\right)] \\ &\quad=^b\int_X \mathrm{d}\,\mathrm{m}(x) \\ &\qquad[\mathrm{m_L}\left(\{y\in[0,\infty]:0\leq y<g(x)\}\right)] \\ &\quad=[\mathrm{m}\times\mathrm{m_L}](\Gamma_g),\end{aligned}$$

as desired.

(ii) For each λ, we have that

$$\begin{aligned}&\int_X \mathrm{d}\,\mathrm{m}(x)\,f_\lambda(x) \\ &\quad:=[\mathrm{m}\times\mathrm{m_L}]\left(\{\langle x,y\rangle\in X\times[0,\infty]:0\leq y<f_\lambda(x)\}\right) \\ &\quad\leq[\mathrm{m}\times\mathrm{m_L}]\left(\{\langle x,y\rangle\in X\times[0,\infty]:0\leq y<f_\infty(x)\}\right) \\ &\quad=:\int_X \mathrm{d}\,\mathrm{m}(x)\,f_\infty(x),\end{aligned}$$

where we have written $f_\infty(x):=\lim_\lambda f_\lambda(x)$. Of course, as the net is nondecreasing, we have $f_\lambda\leq f_\infty$ for all λ. Taking the limit of this inequality gives the desired result

$$\lim_\lambda\int_X \mathrm{d}\,\mathrm{m}(x)\,f_\lambda(x)\leq\int_X \mathrm{d}\,\mathrm{m}(x)\,\lim_\lambda f_\lambda(x). \tag{5.2.3.77}$$

■

[a] This is true, but actually we don't need disjointness for this proof.

[b] By translation invariance.

This in turn allows us to prove the following important 'continuity' result.

Proposition 5.2.3.78 Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and let $f\colon X \to [-\infty, \infty]$ be integrable and Borel. Then, for every $\varepsilon > 0$, there is some $\delta > 0$ such that, whenever $\mathrm{m}(S) < \delta$, it follows that $\int_S \mathrm{d}\,\mathrm{m}(x)\ |f(x)| < \varepsilon$.

Proof. For convenience, write $g := |f|$. Let $\varepsilon > 0$. For $m \in \mathbb{N}$, define

$$M_m := g^{-1}([0, m]) \tag{5.2.3.79}$$

Note that this is measurable because g is Borel (see Proposition 5.2.1.13). It follows that $g_m := g\chi_{M_m}$ is Borel. As $m \mapsto g_m$ is nondecreasing and converges to g pointwise, it follows from Lebesgue's Monotone Convergence Theorem that

$$\lim_m \int_X \mathrm{d}\,\mathrm{m}(x)\, g_m(x) = \int_X \mathrm{d}\,\mathrm{m}(x)\, g(x). \tag{5.2.3.80}$$

Thus, there is some $m_0 \in \mathbb{Z}^+$ such that, whenever $m \geq m_0$, it follows that[a]

$$\int_X \mathrm{d}\,\mathrm{m}(x)\ [g(x) - g_m(x)] < \varepsilon. \tag{5.2.3.81}$$

Define $\delta := \frac{\varepsilon}{m_0}$. Now, let $S \subseteq X$ be such that $\mathrm{m}(S) < \delta$. Then,

$$\begin{aligned}
&\int_S \mathrm{d}\,\mathrm{m}(x)\, |f(x)| \\
&\qquad =: \int_S \mathrm{d}\,\mathrm{m}(x)\, g(x) \\
&\qquad \leq^b \int_S \mathrm{d}\,\mathrm{m}(x)\ \left[g(x) - g_{m_0}(x)\right] \\
&\qquad\quad + \int_S \mathrm{d}\,\mathrm{m}(x)\, g_{m_0}(x) \\
&\qquad \leq^c \int_X \mathrm{d}\,\mathrm{m}(x)\ \left[g(x) - g_{m_0}(x)\right] + m_0\, \mathrm{m}(S) \\
&\qquad < \varepsilon + m_0\delta = 2\varepsilon.
\end{aligned} \tag{5.2.3.82}$$

■

---

[a] This is where we use the fact that f is integrable, so that $\int_X \mathrm{d}\,\mathrm{m}(x)\, g(x)$ is finite.

[b] Note that we may not have equality here if S is not measurable; however, the inequality still holds by virtue of Theorem 5.2.3.68.

[c] Because $g(x) - g_{m_0}(x) \geq 0$ and g_{m_0} is bounded above by m_0.

Exercise 5.2.3.83 Let $I \subseteq \mathbb{R}$ be an interval, let $f \in \mathrm{Bor}(I)$ be integrable, and let $a \in I$. Show that the map $x \mapsto \int_a^x \mathrm{d}\,t\, f(t)$ is uniformly-continuous.

In a similar spirit, we have the following result.

Proposition 5.2.3.84 Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and let $f\colon X \to [-\infty, \infty]$ be integrable and Borel. Then, for every $\varepsilon > 0$, there is some compact $K \subseteq X$ such that $\left|\int_{K^{\mathrm{c}}} \mathrm{d}\,\mathrm{m}(x)\, f(x)\right| < \varepsilon$.

Proof. Let $\varepsilon > 0$. As f is integrable, we have

$$\int_X \mathrm{d}\,\mathrm{m}(x)\, f(x) = \lim_{K \in \mathcal{K}} \int_K \mathrm{d}\,\mathrm{m}(x)\, f(x), \tag{5.2.3.85}$$

where $\mathcal{K}$ is the collection of quasicompact subsets of X. Write $X = \bigcup_{m \in \mathbb{N}} K_m$ as a nondecreasing collection of compact sets. Then, as subnets of convergent nets converge to the same thing, we have

$$\lim_X \mathrm{d}\,\mathrm{m}(x)\, f(x) = \lim_m \int_{K_m} \mathrm{d}\,\mathrm{m}(x)\, f(x). \tag{5.2.3.86}$$

So, let $m_0 \in \mathbb{N}$ be such that, whenever $m \geq m_0$, it follows that

$$\begin{aligned} &\left|\int_{K_m^{\mathrm{c}}} \mathrm{d}\,\mathrm{m}(x)\, f(x)\right| \\ &\quad =^{a} \left|\int_{K_m} \mathrm{d}\,\mathrm{m}(x)\, f(x) - \int_X \mathrm{d}\,\mathrm{m}(x)\, f(x)\right| \\ &\quad < \varepsilon. \end{aligned} \tag{5.2.3.87}$$

■

[a]This requires that f is borel.

Exercise 5.2.3.88 Show that we cannot replace

$$\text{“}\left|\int_{K^{\mathsf{C}}}\mathrm{d}\,\mathrm{m}(x)\,f(x)\right|<\varepsilon\text{”} \tag{5.2.3.89}$$

in the previous result with

$$\text{“}\int_{K^{\mathsf{C}}}\mathrm{d}\,\mathrm{m}(x)\,|f(x)|\text{”}. \tag{5.2.3.90}$$

More precisely, find an example of a topological measure space $\langle X,\mathrm{m}\rangle$ and an integrable Borel function $f\colon X\to[-\varepsilon,\varepsilon]$ for which it is *not* the case that for every $\varepsilon>0$ there is some compact $K\subseteq X$ such that $\int_{K^{\mathsf{C}}}|f(x)|<\varepsilon$.

We next present a result that is quite important in its own right, but can additionally be viewed as justification for the ‘naturality’ of the condition *regular*.

Theorem 5.2.3.91 — Riesz-Markov Theorem. Let X be a σ-compact topological space and let $\mathrm{I}\colon \mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})\to[-\infty,\infty]$ be linear.[a] Then, if X is locally quasicompact and I is nondecreasing, then there exists a unique topological measure m on X such that

$$\mathrm{I}(f)=\int_X \mathrm{d}\,\mathrm{m}(x)\,f(x) \tag{5.2.3.92}$$

for all $f\in\mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})$.

R “Nondecreasing” of course means that $f\leq g$ implies that $\mathrm{I}(f)\leq\mathrm{I}(g)$, where $f\leq g$ means that $f(x)\leq g(x)$ for all $x\in X$. In particular, $\mathrm{I}(f)\geq 0$ if $f\geq 0$.

R The point is that, if you start with something that a priori has nothing to do with measure theory at all, you can in fact obtain a measure, and not just any measure, but a *regular Borel* measure. Thus, you might view the condition of "regular" (and "Borel") as being a natural condition to impose on measures in the sense that these conditions arise from the theory naturally à la Riesz-Markov—you don't have to put them in "by hand".

[a] That is, we require that $\mathrm{I}(f+g)=\mathrm{I}(f)+\mathrm{I}(g)$ for all $f,g\in\mathrm{Mor}_{\mathbf{Top}}(X,\mathbb{R})$ except when one of these is $+\infty$ and the other $-\infty$.

Proof. We leave this as an exercise.

Exercise 5.2.3.93 Prove this yourself.

R Hint: This is one of those exercises that really shouldn't be an exercise, but it is anyways because I ran out of time. In particular, as the phrasing of the theorem is slightly non-standard, there's a possibility it's not correct verbatim. Thus, you should view it as part of the exercise to 'tweak' the statement to make it correct if it turns out that this is not completely correct as stated. See [Rud87, Theorem 2.14] to guide you.

■

The following is an important result that we will need to prove the Fundamental Theorem of Calculus in the next chapter. To state it, we will need to introduce a couple of definitions.

Definition 5.2.3.94 — Upper-semicontinuity and lower-semicontinuity Let $f\colon X\to[-\infty,\infty]$ be a function on a topological space X and let $x_0\in X$. Then, f is ***upper-semicontinuous*** at x_0 iff for every $\varepsilon>0$, there is an open neighborhood U of x_0 such that, whenever $x\in U$, it follows that $f(x)-f(x_0)<\varepsilon$. f is ***lower-semicontinuous*** at x_0 iff for every $\varepsilon > 0$, there is an open neighborhood U of x_0 such that, whenever $x \in U$, it follows that $f(x_0) - f(x) < \varepsilon$.

R Upper-semicontinuity means that you can force $f(x)$ to be as close as you like to $f(x_0)$ *from above* (by moving sufficiently close to x_0), but you have no control over what happens below—$f(x)$ can be arbitrarily negative in a neighborhood of x_0 and it can still be upper-semicontinuous. Similarly, for lower-semicontinuous.

Theorem 5.2.3.95 — Carathéodory-Vitali Theorem. Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and let $f\colon X \to [-\infty, \infty]$ be integrable and Borel. Then, for every $\varepsilon > 0$, there are $u, v \in \mathrm{Bor}(X)$ such that

(i). $u \leq f \leq v$;
(ii). u is upper-semicontinuous and bounded above;
(iii). v is lower-semicontinuous and bounded below; and
(iv).

$$\int_X \mathrm{d}\,\mathrm{m}(x)\,[v(x) - u(x)] < \varepsilon. \tag{5.2.3.96}$$

Proof. [a] We first prove the case for f nonnegative. In this case, by Proposition 5.2.1.35, we can write f as

$$f = \sum_{m\in\mathbb{N}} c_m \chi_{M_m}, \tag{5.2.3.97}$$

for $c_m > 0$ and $M_m \subseteq X$ measurable. Let $\varepsilon > 0$, and by inner and outer-regularity let $K_m \subseteq M_m \subseteq U_m$ be such that K_m is compact, U_m is open, and

$$c_m\,\mathrm{m}(U_m \setminus K_m) < \tfrac{\varepsilon}{2^m}. \tag{5.2.3.98}$$

As f was assumed to be integrable, it must be the case that

$$\sum_{m\in\mathbb{N}} c_m\,\mathrm{m}(M_m) = \int_X \mathrm{d}\,\mathrm{m}(x)\, f(x) \tag{5.2.3.99}$$

converges, and so there is some $m_0 \in \mathbb{N}$ such that

$$\sum_{m=m_0+1}^{\infty} c_m \, \mathrm{m}(M_m) < \varepsilon. \tag{5.2.3.100}$$

Finally, define

$$v := \sum_{m=0}^{\infty} c_m \chi_{U_m} \text{ and } u := \sum_{m=0}^{m_0} c_m \chi_{K_m}. \tag{5.2.3.101}$$

Exercise 5.2.3.102 Show that u and v satisfy the desired properties.

Exercise 5.2.3.103 Finish the proof by doing the general case by writing $f = f_+ - f_-$.

■

[a]Proof adapted from [Rud87, pg. 56].

5.2.4 Riemann integrability

Finally, as I am required to teach you the Riemann integral, I begrudgingly include a cop-out version just to say I taught it.

Definition 5.2.4.1 — Riemann integrable Let $f\colon [a,b] \to \mathbb{R}$. Then, f is ***Riemann integrable*** iff f is bounded and continuous almost-everywhere. In this case, its ***Riemann integral*** is defined to be $\int_a^b \mathrm{d}x\, f(x)$.

R This is usually a theorem, but because I think it's a waste of time to develop the Riemann integral when the Lebesgue integral does everything the Riemann integral does (and much more), I include it as a definition just to say I did it. Congratulations! You now know the Riemann integral!

5.3 L^p spaces

For X a topological measure space, $L^p(X)$ will wind up being a certain collection of Borel functions defined on X. While there are many reasons one might be interested in such objects, one possible motivation is that, if you care about topological measure spaces, then you can obtain information about X by a study of $L^p(X)$.[10] In this case, the space is a topological measure space, and so the relevant functions will be the Borel functions. In a perfect world, I suppose we could use the integral to put (semi)norms on $\mathrm{Bor}(X)$ itself, but unlike in the continuous case where we had theorems (namely the Extreme Value Theorem) to guarantee the finiteness of the supremum seminorms on quasicompact sets, there are no such theorems in this case—there's no getting around the fact that some Borel function will have infinite integral on any set of positive measure. And so, instead of studying all of the Borel functions at once, we study certain subsets of them. The L^p spaces are certain nice spaces of Borel functions, namely the ones whose p^{th} power is integrable.

Before we do anything else, we define the so-called L^p *norms*.

Definition 5.3.1 — L^p-norm Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and let $p \in [1, \infty]$. Then, the L^p ***norm***, $\|\cdot\|_p : \mathrm{Bor}(X) \to [0, \infty]$, is defined by

$$\|f\|_p := \begin{cases} \left(\int \mathrm{d}_X\, \mathrm{m}(x)\, |f(x)|^p\right)^{\frac{1}{p}} & p \neq \infty \\ \sup\left\{y \in \mathbb{R} : \mathrm{m}\left(f^{-1}([y, \infty])\right) > 0\right\} & p = \infty. \end{cases}$$

R If you take $X := \{1, \ldots, d\}$ to be a d point set with the counting measure, this reads

$$\|v\|_p = (|v_1|^p + \cdots + |v_d|^p)^{\frac{1}{p}}, \qquad (5.3.2)$$

where we have suggestively written $v_k := v(k)$. In particular, $L^2(\{1, \ldots, d\}) \cong_{\mathbf{Hil}_{\mathbb{R}}} \mathbb{R}^d$.[a]

[10] In fact, the idea of studying spaces by instead studying functions defined on those spaces is quite a pervasive theme in all of mathematics.

R $\|f\|_\infty$ is the ***essential supremum***. Intuitively, you're taking the supremum over all y-values that have the property that the measure of the set on which f is at least y is *strictly* positive. Intuitively speaking, it is the supremum norm 'modulo sets of measure 0'. For example, the function that is 0 on $\mathbb{Q}^{\mathrm{C}}$ and ∞ on $\mathbb{Q}$ has essential supremum 0—that it is infinite on an infinite set does not matter, as this infinite set has measure 0. Note that in the case that X is quasicompact, the supremum norm $\|f\|_\infty$ as defined in Example 4.2.3.22 agrees with the essential supremum norm $\|f\|_\infty$, so there is no ambiguity. The reason this is called the $p = \infty$ norm is because $\lim_{p\to\infty}\|f\|_p = \|f\|_\infty$ (at least when $\|f\|_{p_0} < \infty$ for some $p_0 < \infty$)—see Exercise 5.3.3.

R The "L" is for "Lebesgue". I have no idea what the p is for. In fact, I think it's a bad choice of letter, but we're pretty much stuck with it at this point.

---

[a] $\mathbf{Hil}_{\mathbb{R}}$ is the category of *(real) Hilbert spaces*. If you know what this is, great—here's another cookie **gives yet another cookie to the reader**. If not, don't worry about it for now—as long as you intuitively see why this is the same as the Euclidean norm, you're fine.

Exercise 5.3.3 Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and let $f \in \mathrm{Bor}(X)$. Show that if $\|f\|_{p_0} < \infty$ is finite for some $1 \leq p_0 < \infty$, then

$$\lim_{p\to\infty}\|f\|_p = \|f\|_\infty. \tag{5.3.4}$$

Definition 5.3.5 — Hölder conjugate Let $1 < p, q < \infty$. Then, p and q are ***Hölder conjugate*** iff

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{5.3.6}$$

1 and ∞ are Hölder conjugates.

R Note that 2 is Hölder conjugate to itself.

R One way to see that we don't want to[a] investigate the case $0 < p < 1$ is because then it does not have a Hölder conjugate.[b] This really breaks things because then, for example, Hölder's Inequality will not hold.

[a]Well, usually not.
[b]Or at least, its Hölder conjugate would be negative.

Theorem 5.3.7 — Hölder's Inequality. Let X be a topological measure space, let $1 \leq p, q \leq \infty$ be Hölder conjugates, and let $f, g \in \mathrm{Bor}(X)$. Then,

$$\|fg\|_1 \leq \|f\|_p \|g\|_q. \tag{5.3.8}$$

R The case $p = 2 = q$ is called the ***Cauchy-Schwarz Inequality***. In the case that X is a d point space with the counting measure, it is literally the statement that the 'dot product' of two vectors is at most the product of their Euclidean norms.

Proof. If both $\|f\|_p = \infty = \|g\|_q$, then the inequality is trivially satisfied. If one of $\|f\|_p$ and $\|g\|_q$ is 0, without loss of generality, say $\|f\|_p$, then we get that $|f(x)| = 0$ almost-everywhere by Exercise 5.2.3.13, and so $|f(x)g(x)| = 0$ almost-everywhere, and so $\|fg\|_1 = 0$,[a] and so in this case the inequality is satisfied as well. Thus, we may as well assume that $0 < \|f\|_p, \|g\|_q < \infty$. Then, replacing f by $\frac{f}{\|f\|_p}$ and $\frac{g}{\|g\|_q}$, we see that it suffices to show that $\|fg\|_1 \leq 1$ if $\|f\|_p = 1 = \|g\|_q$.

First take $1 < p, q < \infty$.

Exercise 5.3.9 Show that there are $s, t \in \mathrm{Bor}(X)$ such that $2^s = |f|^p$ and $2^t = |g|^q$.

Exercise 5.3.10 Show that that if $a + b = 1$ with $a, b \geq 0$, then $2^{ax+by} \leq a2^x + b2^y$.

R Note that this does not necessarily work if a or b is negative. Thus, is the point where the proof breaks down if $p < 1$ (because then either p is negative or q is negative).

R This is called ***convexity***.

Using this, we obtain

$$\begin{aligned} |f||g| &= 2^{s/p}2^{t/q} = 2^{s/p+t/q} \leq \tfrac{2^s}{p} + \tfrac{2^t}{q} \\ &= \tfrac{|f|^p}{p} + \tfrac{|g|^q}{q}. \end{aligned} \tag{5.3.11}$$

Integrating this inequality gives

$$|fg|_1 \leq \tfrac{1}{p}|f|_p^p + \tfrac{1}{q}|f|_q^q =^b = \tfrac{1}{p} + \tfrac{1}{q} = 1, \tag{5.3.12}$$

as desired.

Exercise 5.3.13 Prove the result for $p = 1$ and $q = \infty$.

■

[a] I feel as if this notation, while consistent, is a bit obtuse—$|f|$ is the function defined by $x \mapsto |f(x)|$, and $\|f\|_1$ is the number defined by $\int_X \mathrm{d}\,\mathrm{m}(x)\,|f(x)|$.

[b] Because we have assumed that $|f|_p = 1 = |f|_q$.

Theorem 5.3.14 — Minkowski's Inequality. Let $\langle X_1, \mathrm{m}_1\rangle$ and $\langle X_2, \mathrm{m}_2\rangle$ be topological measure spaces, let $f \in \mathrm{Bor}(X_1 \times X_2)$, and let $1 \leq p \leq \infty$. Then,

$$\left\|\int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, f(\cdot, x_2)\right\|_p \leq \int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, \|f(\cdot, x_2)\|_p. \tag{5.3.15}$$

R Explicitly, this reads

$$\left[\int_{X_1} \mathrm{d}\,\mathrm{m}_1(x_1) \left| \int_{X_2} \mathrm{d}\,\mathrm{m}_2(x_2)\, f(x_1, x_2) \right|^p \right]^{1/p} \leq \int \mathrm{d}_{X_2}\, \mathrm{d}\,\mathrm{m}_2(x_2) \left[\int_{X_1} \mathrm{d}\,\mathrm{m}_1(x_1)\, |f(x_1, x_2)|^p \right]^{1/p}.$$

R I remember this roughly as "You can bring the exponents 'in a level' if you switch the order of integration."

Proof. We leave this as an exercise.

Exercise 5.3.16 Prove the result.

R Hint: See [Fol99, p. 6.19].

■

Proposition 5.3.17 — Triangle Inequality Let $\langle X, \mathrm{m}\rangle$ be a topological measure space, let $f, g \in \mathrm{Bor}(X)$, and let $1 \leq p \leq \infty$. Then,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \tag{5.3.18}$$

R Sometimes this itself is referred to as "Minkowski's Inequality". I think that's a bit silly because (i) the more general inequality of the previous result is also called "Minkowski's Inequality" and (ii) this inequality already has a name—the Triangle Inequality!

Proof. Note that this equation is trivially satisfied if either one of $\|f\|_p$ and $\|g\|_p$ is infinite or if both of $\|f\|_p$ and $\|g\|_p$ are zero, so suppose that neither of these is the case.

First take $1 < p < \infty$. Then,

$$\begin{aligned}\|(f+g)^p\|_1 &= \left\|f(f+g)^{p-1} + g(f+g)^{p-1}\right\|_1 \\ &\leq^a \left\|f(f+g)^{p-1}\right\|_1 + \left\|g(f+g)^{p-1}\right\| \\ &\leq^b \|f\|_p \left\|(f+g)^{p-1}\right\|_q + \|g\|_p \left\|(f+g)^{p-1}\right\|_q \\ &= \left\|(f+g)^{p-1}\right\|_q \left(\|f\|_p + \|g\|_p\right),\end{aligned}$$

where q is the Hölder conjugate of p. This implies that $(p-1)q = p$, and so

$$\begin{aligned}\left\|(f+g)^{p-1}\right\|_q &:= \left(\int_X \mathrm{d}\,\mathrm{m}(x)\ \left|(f(x)+g(x))^{p-1}\right|^q\right)^{\frac{1}{q}} \\ &= \left(\int_X \mathrm{d}\,\mathrm{m}(x)\ |f(x)+g(x)|^p\right)^{\frac{1}{q}} \\ &=: \|(f+g)^p\|_1^{\frac{1}{q}}.\end{aligned}$$

So in fact, the previous inequality reads

$$\|(f+g)^p\|_1 \leq \|(f+g)^p\|_1^{\frac{1}{q}} \left(\|f\|_p + \|g\|_p\right), \tag{5.3.19}$$

so that

$$\begin{aligned}\|f+g\|_p := \|(f+g)^p\|_1^{\frac{1}{p}} &= \|(f+g)^p\|_1^{1-\frac{1}{q}} \\ &\leq \|f\|_p + \|g\|_p,\end{aligned} \tag{5.3.20}$$

as desired.

Exercise 5.3.21 Do the case $p = 1$ and $p = \infty$.

■

[a] By the usual Triangle Inequality.

[b] By Hölder's Inequality.

This is just the statement that the 'norm' $\|\cdot\|_p$ is in fact a norm, the Triangle Inequality being the only nonobvious axiom that needs to be satisfied. Now that we know this, we are free to defined the L^p spaces.

Definition 5.3.22 — L^p spaces Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and let $1 \leq p \leq \infty$. Then, $L^p(X)$ is the normed vector space defined by

$$L^p(X) := \{f \in \mathrm{Bor}(X)/\sim_{\mathbf{AIE}} : \|f\|_p < \infty\}, \qquad (5.3.23)$$

with the addition and scalar multiplication the same as that in $\mathrm{Bor}(X)$ equipped with the L^p norm, $\|\cdot\|_p$.

R I don't like this definition. I think it is artificially cooked up because people are too afraid to work with things that aren't metric spaces. A definition like this is analogous to studying only the *bounded* continuous functions on a topological space, instead of *all* continuous functions $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$. That is, you restrict your class of functions for no other reason that you want *one* norm, instead of a *family* of seminorms.[a] I think it is much more natural to look at *quasicompactly* integrable[b] functions. Then, just as in $\mathrm{Mor}_{\mathbf{Top}}(X, \mathbb{R})$, you would obtain a seminorm for each quasicompact subset. For one thing, continuous functions are all compactly integrable. Obviously this fails for the 'global' L^p spaces. For example, $x \mapsto x$ is not an element of $L^1(\mathbb{R})$. In fact, I prefer the local approach so much more that I think I may add it at a later date when I have more time. For the moment, however, the 'usual' L^p spaces it is.

[a] The supremum norm of a bounded function is always finite, even if your topological space is not necessarily quasicompact.

[b] See the remark in Theorem 5.2.2.36. In brief, this just means integrable over every quasicompact set.

One hugely important property of the L^p spaces is that they are *complete*.

Theorem 5.3.24 — Riesz-Fischer Theorem. Let $\langle X, \mathrm{m}\rangle$ be a topological measure space and let $1 \leq p \leq \infty$. Then, $L^p(X)$ is complete.

Proof. We leave this as an exercise.

Exercise 5.3.25 Prove this result yourself.

R Hint: Yet another one that should not be an exercise. This time, check-out [SS05, pg. 70].

■

Remember way back when we did Heine-Borel and Bolzano-Weierstrass? We mentioned there briefly that these results fail to hold in general, that is, there are spaces in which closed bounded sets are not necessarily quasicompact, and there are spaces with bounded infinite sets that do not have accumulation points. The L^p spaces provide examples for which both of these phenomena can happen.

■ **Example 5.3.26 — A closed bounded set in a normed vector space that is not quasicompact** We take our topological measure space to be $X := \mathbb{N}$ equipped with the counting measure. Note then that elements of $L^1(X)$ are just sequences $m \mapsto a_m$ for which

$$\sum_{m\in\mathbb{N}} |a_m| < \infty. \tag{5.3.27}$$

Consider the closed unit ball $D_1(0) := \{f \in L^1(X) : \|f\|_1 \leq 1\}$.

Exercise 5.3.28 Check that $D_1(0)$ is closed.

It is clearly bounded, by definition. We show that $D_1(0)$ is not quasicompact. To do so, we construct a sequence in $D_1(0)$ which has no convergent subnet. For $m \in \mathbb{N}$, let a^m be the sequence which sends m to 1 and everything else to 0. So, for example,

$$\begin{aligned} a^0 &:= \langle 1, 0, 0, 0, \ldots \rangle \\ a^1 &:= \langle 0, 1, 0, 0, \ldots \rangle \\ a^2 &:= \langle 0, 0, 1, 0, \ldots \rangle \end{aligned} \tag{5.3.29}$$

etc.

We then have that

$$\|a^m - a^n\|_1 = \begin{cases} 2 & \text{if } m \neq n \\ 0 & \text{if } m = n. \end{cases} \tag{5.3.30}$$

This shows that there is no Cauchy subnet, and hence certainly no convergent subnet.

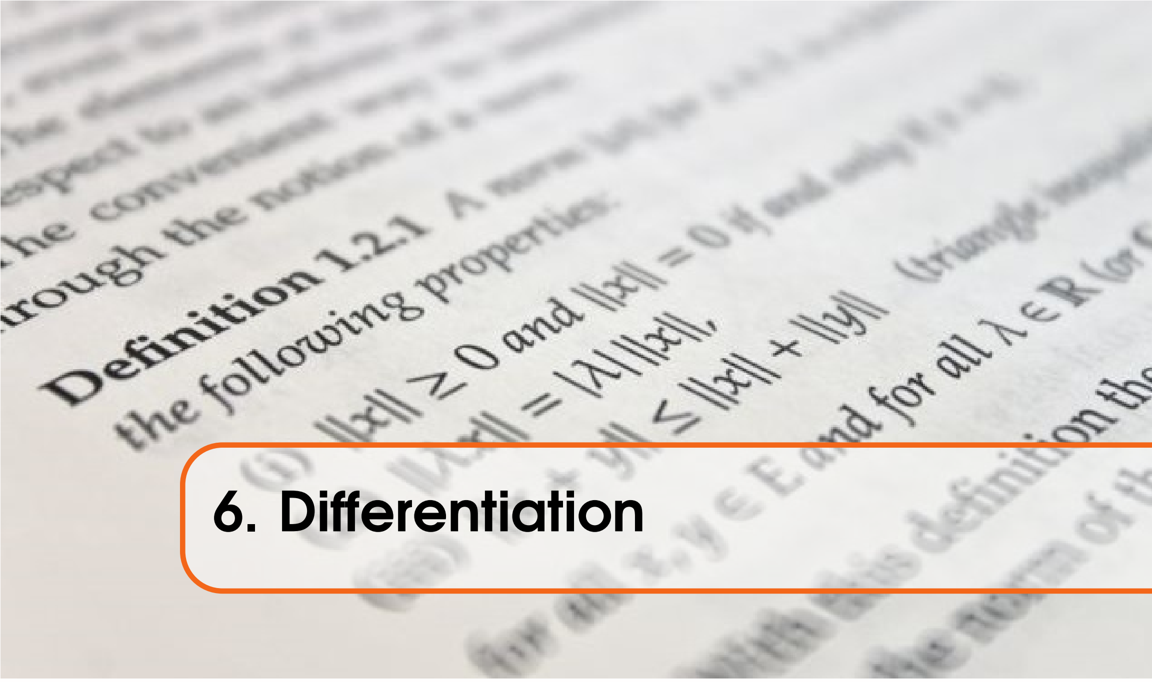
■ **Example 5.3.31 — A bounded infinite set without an accumulation point** Let $m \mapsto a_m \in L^1(X)$ be as in the previous example.[a] Then set $\{a_m : m \in \mathbb{N}\}$ is certainly bounded and infinite, so it suffices to show that it doesn't have an accumulation point. As all the points in $\{a_m : m \in \mathbb{N}\}$ are at least a distance of 2 from each other, no net whose terms come from $\{a_m : m \in \mathbb{N}\}$ can be Cauchy unless it is eventually constant. It follows that this set has no limit points, and so hence of course no accumulation points.

[a] Briefly, the 'standard unit vectors' in $L^1(\mathbb{N})$.

Exercise 5.3.32 Let $\langle X, \mathrm{m} \rangle$ be a topological measure space and let $1 \leq p \leq q \leq \infty$.

(i). Show that if $\mathrm{m}(X) < \infty$, then $L^q(X) \subseteq L^p(X)$.

(ii). Find a counter-example to show that this is false if $\mathrm{m}(X) = \infty$.



6. Differentiation

Finally we are ready to begin our study of differentiation. We could have started this awhile ago, but we really needed certain facts about continuity, uniform convergence, and even integration, before we could address all the facts we care about related to differentiation. This thus led us to a relatively broad study of the most general spaces[1] in which these notions makes sense. For the time being, however, we return to $\mathbb{R}^d$.

6.1 Tensors and abstract index notation

6.1.1 Motivation and introduction

Let $f\colon \mathbb{R}^d \to \mathbb{R}$ and let $x \in \mathbb{R}^d$. We will then think of the derivative of f at x as a thing that takes in "tangent vector" and spits out numbers, that number being thought of as the directional derivative of f in that direction. From this perspective, the derivative of f at x is not itself a vector but rather a *covector*, that is, a thing which takes in vectors and spits out numbers.

The second derivative will then be a thing which takes in *two* vectors and spits out a number, that number being interpreted as

[1] Meh, basically anyways

the directional derivative in the direction of the first vector of the directional derivative in the direction of the second vector of f. Such an object will be called a *covariant tensor of rank* 2. Similarly, the third derivative will be a covariant tensor of rank 3, and so on.

What happens if f isn't just scalar-valued, but vector valued, for example, $f\colon \mathbb{R}^d \to \mathbb{R}^e$. Then, the derivative of f will not be an object that takes in a vector and spits out a *vector*—the interpretation is the same, that is, this is still thought of as the directional derivative of f in the direction of the given vector, but because f is vector-valued, this "rate-of-change" is itself a vector. This is what will be called a *tensor of rank* $\langle 1, 1\rangle$. Likewise, the second derivative will now be a thing which takes in two vectors and spits out a vector—a tensor of rank $\langle 1, 2\rangle$.

Thus, the reason we introduce tensors is because derivatives are naturally thought of as tensors. You can 'cheat' and ignore the issue if you only care about the first derivative, but if you care about higher derivatives, there really isn't any way to get around tensors.[2]

We do define tensors in the appendix—see Definition D.3.2.41—however, we reproduce here the necessary facts we need to know about tensors. The definition given in the appendix is the 'right' one, and we develop things (technically) from scratch, but unless you've seen it before, it's probably a bit unapproachable. This is really something that should be studied in its own right in a linear algebra course. Fortunately, tensors are a bit like the integral in the sense that you can use them without being able to define them. In a similar way that people "black-box" the integral when first teaching calculus (that is, they tell you the rules it satisfies so you can compute it, but they don't actually define it), I'm going to try to "black-box" tensors and only reproduce here the relevant tools that we will need to use. You can then always refer to the appendix if you like to look up the details.

To aid in understanding, we will reproduce here special cases of corresponding definitions and results in the appendix.

> If you don't understand the details of this section your first time through, that's fine. Learn what you can, and

[2]This isn't literally true, but avoiding tensors in this context makes things so messy that it's not particularly feasible.

then come back as you need to when the concepts come up in the actual differentiation part of the chapter.

6.1.2 The dual-space

We first discuss the *dual-space*.

Definition 6.1.2.1 — Dual-space Let V be a real vector space. Then, the ***dual-space*** of V is the real vector space

$$V^\dagger := \mathrm{Mor}_{\mathbf{Vect}_{\mathbb{R}}}(V, \mathbb{R}). \tag{6.1.2.2}$$

R The elements of $V^\dagger$ are ***linear functionals*** or ***covectors***.

R $\mathbf{Vect}_F$, for F a field, is the category whose objects are vector spaces over F and whose morphisms are linear-transformations—see Example D.1.9. Thus, $\mathrm{Mor}_{\mathbf{Vect}_{\mathbb{R}}}(V, \mathbb{R})$ is our fancy-schmancy notation for the vector space of linear functions from V into $\mathbb{R}$.

R $V^\dagger$ has the structure of a vector space by defining addition and scalar multiplication pointwise.

R In other words, the elements of $V^\dagger$ take in elements of V and spit out numbers. For example, as explained at the beginning of this section, the derivative takes in a vector (the direction in which to differentiate) and spits out a number (the directional derivative in that direction).

R See Definition D.3.1.1 for the 'official' version.

Of critical importance is that the dual of the dual is the original vector space.[3]

[3]Careful: It will frequently be the case that $V^\dagger$ *is* isomorphic to V, but in a noncanonical way. On the other hand $(V^\dagger)^\dagger$ and V are *naturally isomorphic*. The

Proposition 6.1.2.3 — $[V^\dagger]^\dagger \cong V$ Let V be a finite-dimensional real vector space. Then, the map Then, the map $V \to [V^\dagger]^\dagger$ defined by

$$v \mapsto (\phi \mapsto \phi(v)) \tag{6.1.2.4}$$

is a natural isomorphism.

R

For this reason, in this context, we do not distinguish between $[V^\dagger]^\dagger$ and V.

Note that one often is careless and identifies any two things that are isomorphic, but sometimes this is inappropriate. For example, we know that $(-1,1)$ is homeomorphic to $\mathbb{R}$, but of course for many purposes identifying these two would be silly, even sometimes just in the context of topological spaces.

However, if two things are naturally isomorphic or "unique up to unique isomorphic", then they're not just 'the same', but they're *the same*. Of course, they're in general not literally equal, but for all intents and purposes, you can get away with identifying them.

As a concrete example of this, for example, the dual of $\mathbb{R}^2$ is isomorphic to $\mathbb{R}^2$, but yet we do not identify $\mathbb{R}^2$ with $[\mathbb{R}^2]^\dagger$; however, we *do* identify $\mathbb{R}^2$ with $[[\mathbb{R}^2]^\dagger]^\dagger$.

R See Theorem D.3.1.8 for the 'official' version.

6.1.3 Tensors and the tensor product

If all we needed to understand was the dual, I probably wouldn't be black-boxing things. However, the tensor product is quite tricky—see

way to make this intuition precise requires more category theory than is probably helpful, but if you're curious, see Appendix B.3.2. In brief, the idea is to show that the 'constructions' (read "functors" (Definition B.3.1.1)) are isomorphic, not just the objects themselves.

Theorem D.3.2.6 for the "official" version. For us, it will be useful to think of tensor products in the following way.

Let V and W be real vector spaces, and let $v \in V$ and $w \in W$. Then, there is a 'thing' $v \otimes w$, the *tensor product* of v and w, and $V \otimes W$ is the vector space spanned by these things. Students often ask "Okay, but what actually *is* $v \otimes w$?". The answer is that $V \otimes W$ is uniquely defined by a property it satisfies (just as was the case with $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$), and that space is spanned by elements of the form $v \otimes w$. For example, if you asked me "But what actually *is* a real number?", I could tell you "A real number is a Dedekind cut.", but not only is this a borderline lie, it's also just not useful. We used Dedekind cuts to construct the real numbers, and, several hundred pages of theory later, we haven't needed to use them since. Dedekind cuts served the purpose of proving that there was an object $\mathbb{R}$ that satisfied the properties we claimed it did. Similarly, a more explicit construction of $v \otimes w$ really only serves to prove that an object $V \otimes W$ exists that satisfies the properties we claim it does. After that, however, this explicit construction doesn't matter, and the only thing used are those properties which uniquely define $V \otimes W$.

As it is difficult to reproduce the definition of the tensor product in a way that is easy to digest, we refrain from reproducing the definition here. You should consider $V \otimes W$ and $v \otimes w$ to be "officially" defined as in Theorem D.3.2.6, and here we will only focus on the facts that you need to know.

> Given real vector spaces V and W, there is a vector space $V \otimes W$, the *tensor product* of V and W, and a function $V \times W \to V \otimes W$, written $\langle v, w \rangle \mapsto v \otimes w \in V \otimes W$. $v \otimes w$ is the *tensor product* of v and w.

Proposition 6.1.3.1 Let V and W be real vector spaces.

(i).

$$(\alpha v) \otimes w = \alpha(v \otimes w) + v \otimes (\alpha w) \qquad (6.1.3.2)$$

for all $v \in V$, $w \in W$, and $\alpha \in \mathbb{R}$.

(ii).

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2 \tag{6.1.3.3}$$

for all $v \in V$ and $w_1, w_2 \in W$.

(iii).

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w \tag{6.1.3.4}$$

for all $v_1, v_2 \in V$ and $w \in W$.

R This is a consequence of the definition Theorem D.3.2.6.

Proposition 6.1.3.5 Let V and W be real vector spaces. Then,

$$V \otimes W = \operatorname{Span}\{v \otimes w : v \in V,\ w \in W\}. \tag{6.1.3.6}$$

R See Proposition D.3.2.16 for the 'official' version.

Having defined the tensor product, we are now able to define tensors themselves.

Definition 6.1.3.7 — Tensor Let V be a real vector space. Then, a ***tensor*** of ***rank*** $\langle k, l\rangle$ over V is an element of

$$\bigotimes\nolimits_l^k V := \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(\underbrace{V \otimes \cdots \otimes V}_{l}, \underbrace{V \otimes \cdots \otimes V}_{k}). \tag{6.1.3.8}$$

R k is the ***contravariant rank*** and l is the ***covariant rank***. If $l = 0$, then the tensor is ***contravariant***, and if $k = 0$, then the tensor is ***covariant***.

R Thus, by the definition of the tensor product (Definition D.3.2.41), a tensor of rank $\langle k, l\rangle$ is 'the same as' a multilinear map from $\underbrace{V \times \cdots \times V}_{l}$ to $\underbrace{V \otimes \cdots \otimes V}_{k}$.

Thus, a tensor of rank $\langle k, l\rangle$ is a thing that takes in l vectors and 'spits out' 'k vectors'[a] in a multilinear manner.

R In finite dimensions, Theorem 6.1.3.27 will tell us that $\bigotimes_l^k V$ is naturally isomorphic to

$$\begin{aligned} V^{k\otimes, l\otimes\dagger} &:= V^{k\otimes} \otimes [V^{\dagger}]^{l\otimes} \\ &:= \underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^{\dagger} \otimes \cdots \otimes V^{\dagger}}_{l}. \end{aligned} \tag{6.1.3.9}$$

R See Definition D.3.2.41 for the 'official' version.

[a]More accurately, a contravariant tensor of rank k.

By now you're well aware of the fact that people often abuse notation and write "$f(x)$" for the function f. Strictly speaking, $f(x)$ is the value of f at x, but it is very useful to use this notation as it gives you some information about the domain. For instance, if context is clear, you can infer that the domain of $f(x, y)$ is $\mathbb{R}^2$ but the domain of $g(x, y, z)$ is $\mathbb{R}^3$.

Similarly for a tensor T, we are going to 'decorate' "T" with symbols that tell us what the domain of T is.[4] In fact, we'll also want to "decorate" "T" with symbols to indicate the *codomain* as well. This might make more sense to you if you use Theorem 6.1.3.27(iii) to view a rank $\langle k, l\rangle$ tensor as a function with domain

$$\underbrace{V^{\dagger} \times \cdots \times V^{\dagger}}_{k} \times \underbrace{V \times \cdots \times V}_{l}, \tag{6.1.3.10}$$

so that we need two types of decorations to specify the domain—we can't just write $T(x, y, z)$, because this notation won't distinguish

[4]Is it V, $V \otimes V$, $V \otimes V \otimes V$, etc.?

between functions on, for example, $V \times V \times V$ and functions on $V \times V \times V^{\dagger}$. We resolve this ambiguity by instead writing these 'dummy variables' as subscripts and superscripts.

Notation 6.1.3.11 — Index notation Let V be a real vector space and let

$$T \in \operatorname{Mor}\big(\underbrace{V \otimes \cdots \otimes V}_{k}, \underbrace{V \otimes \cdots \otimes V}_{l}\big) \tag{6.1.3.12}$$

be a tensor of rank $\langle k, l \rangle$. To indicate the rank of T, we shall write

$$T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}\,. \tag{6.1.3.13}$$

R If we are dealing with tensors over two distinct vector spaces, we will tend to use distinct scripts for the different vector spaces. For example, we will probably write something like $T^{a}{}_{\alpha}$ for an element of $V \otimes W^{\dagger}$ instead of $T^{a}{}_{b}$. We might also write capitals for one space, or perhaps start at different place in the same alphabet (e.g. α, β, γ, etc. for V vs. μ, ν, ρ, etc. for W).

R This is called ***abstract index notation***, ***Penrose index notation***, or just ***index notation***. This is similar in form, but conceptually distinct, from *Einstein index notation*. In Einstein index notation, one has chosen a basis, and then the indices indicate the coordinates with respect to that basis. Note, however, that abstract index notation was designed so that one could do computations as if one was using Einstein index notation without actually picking a basis. Roughly speaking, abstract index notation is to Einstein index notation as linear-transformations are to matrices, though the distinction matters even less because the notation is designed to work so similarly. For this reason, the distinction is one that matters much more in theory than it does in practice.

R *Do not be sloppy by not staggering your indices!* If you do, you will eventually make a mistake. For example, later we will be raising and lowering indices. Suppose I start with T^{ab}, I lower to obtain T^a_b, and then I raise again to obtain T^{ba}—I should obtain the same thing, but in general $T^{ab} \neq T^{ba}$, and so I have made an error. It may seem obvious to the point of being silly when I point it out like this, but this is a mistake that is easy to make if there is a big long computation in between the raising and lowering (especially if it's more than just a and b floating around). And of course, you will never have this problem if you stagger: T^{ab} goes to $T^a{}_b$ goes back to T^{ab}.

R We emphasize that this is all "coordinate-free"—no need to pick bases—despite what the notation might superficially suggest.

R This is conceptually different, but mechanically very similar to Einstein's index notation. You might say that abstract index notation is choice-free Einstein index notation (the choice of course being a choice of basis).[a]

[a] If you don't know what Einstein index notation is, don't worry—this remark is only to be helpful to those who might already have some familiarity.

The tensor product we have already done in Theorem D.3.2.6, and so we simply explain the how to write the tensor product in index notation.

Notation 6.1.3.14 — Tensor product in index notation Let V be a $\mathbb{K}$-$\mathbb{K}$-bimodule and let T_1 and T_2 respectively be tensors over V of rank $\langle k_1, l_1\rangle$ and $\langle k_2, l_2\rangle$. The *tensor product* of $[T_1]^{a_1\ldots a_{k_1}}{}_{b_1\ldots b_{l_1}} \in \bigotimes^{k_1}_{l_1} V$ and $[T_2]^{a_1\ldots a_{k_2}}{}_{b_1\ldots b_{l_2}} \in \bigotimes^{k_2}_{l_2} V$

is denoted

$$[T_1 \otimes T_2]^{a_1\cdots a_{k_1}c_1\cdots c_{k_2}}{}_{b_1\cdots b_{l_1}d_1\cdots d_{l_2}} =: [T_1]^{a_1\ldots a_{k_1}}{}_{b_1\ldots b_{l_1}}[T_2]^{c_1\ldots c_{k_2}}{}_{d_1\ldots d_{l_2}}. \tag{6.1.3.15}$$

R To avoid obfuscating things even further, we omit the definition in case there are indices are the left, but it works exactly as one would expect—that is, as here, you literally just juxtapose them.

R In particular, if you ever see "⊗" used explicitly, this should be taken as an indication that we are *not* using index notation (and so subscripts and superscripts should not be interpreted as such).

R Note that *everything commutes with everything* in index notation. For example,

$$T^a{}_b v^c = v^c T^a{}_b. \tag{6.1.3.16}$$

The letters keep track of what goes where—you don't need to use the order in which the symbols are written to do the same job.

We now turn to *contraction*.

Definition 6.1.3.17 — Contraction Let V be real vector space. Then, for $k, l \in \mathbb{N}$, and $1 \leq i \leq k$ and $1 \leq j \leq l$, the $\langle i, j\rangle$ ***contraction*** of $\langle k, l\rangle$ tensors is the unique map $V^{k\otimes,l\otimes\dagger} \to V^{(k-1)\otimes,(l-1)\otimes\dagger}$ such that

$$\bigotimes_m v_m \otimes \bigotimes_n \phi_n \mapsto (\phi_j \mid v_i) \bigotimes_{m\neq i} v_m \otimes \bigotimes_{n\neq j} \phi_n. \tag{6.1.3.18}$$

R Warning: While this definition makes sense in general, we need V to be a finite-dimensional vector space over a field for $V^{k\otimes,l\otimes}$ be actually 'be' the space of $\langle k, l\rangle$ tensors—see Theorem 6.1.3.27.

R If this doesn't yet make sense, don't worry until you've read the upcoming Notation 6.1.3.19 and the remarks contained therein.

Notation 6.1.3.19 — Contraction in index notation Let V be a real vector space, let $k, l \in \mathbb{N}$, and $1 \le i \le k$ and $1 \le j \le l$. Then, the $\langle i, j\rangle$ *contraction* of the tensor $T^{a_1\ldots a_k}{}_{b_1\ldots b_l} \in V^{k\otimes, l\otimes\dagger}$ is denoted

$$T^{a_1\ldots a_{i-1}ca_{i+1}\ldots a_k}{}_{b_1\ldots b_{j-1}cb_{j+1}\ldots b_l} \tag{6.1.3.20}$$

R All these indices might make this seem unapproachable, but it's actually quite simple. Covectors take in vectors and spit out numbers, and so the contraction of a tensor product in its a_i and b_j index is formed by plugging in the i^{th} vector into the j^{th} covector, which is denoted by using the same letter for both the i^{th} index upstairs and the j^{th} index downstairs.

R Keep in mind that you can *only* contract upper-indices (contravariant) with lower (covariant) ones.

R Note that the letter you use for contraction doesn't matter—it just needs to be the same upstairs as it is downstairs and not conflict with other indices. This is analogous to the fact that

$$\int \mathrm{d}x\, f(x) = \int \mathrm{d}t\, f(t) : \tag{6.1.3.21}$$

it doesn't matter whether you use x or t, only that the letter in "d–" agree with the letter in "$f(-)$".

R Note that contraction of indices reduces both the contravariant and covariant rank by 1. For this reason, contracted indices are usually ignored when it comes to determine the type of the tensor. For example, people will say things like "The left-hand side of the following equation has only an a index upstairs.

$$T^{ab}{}_{b} = v^{a}\text{''}. \tag{6.1.3.22}$$

This is analogous to how $\int \mathrm{d}x\, f(x,y)$ is only a function of one variable.

We mentioned some of the following examples before, but let's now do them 'officially'.

Example 6.1.3.23

(i). Vectors (written v^a) themselves are tensors of type $\langle 1, 0\rangle$.

(ii). Covectors (or linear functionals) (written ω_a) are of type $\langle 0, 1\rangle$. For ω a linear functional and v a vector, $\omega(v)$ is written as $\omega_a v^a$.

(iii). The dot product (written temporarily as g_{ab}) is an example of a tensor of type $\langle 0, 2\rangle$—it takes in two vectors and spits out a number, written $v\dot{w} = g_{ab}v^a w^b$.

(iv). Linear-transformations ($T^a{}_b$) are tensors of type $\langle 1, 1\rangle$—it takes in a single vector and spits out another vector (written $v^a \mapsto T^a{}_b v^b$).

Example 6.1.3.24 — Linear-functionals in index notation Let V be a real vector space, let $v^a \in V$, and let $\phi_a \in V^\dagger$.

We can take their tensor product $v^a \phi_b$, which is a tensor of rank $\langle 1, 1\rangle$.

There is only one possible contraction of this tensor: $v^a\phi_a$, which is by definition equal to $(\phi \mid v) := \phi(v)$. Compare what this would look like in coordinates: $\phi(v) = \sum_{k=1}^d v^k \phi_k$.

■ **Example 6.1.3.25 — Linear-transformations in index notation** Let V and W be be real vector spaces, let $T^{\alpha}{}_{a} \in V^{\dagger} \otimes W$,[a], and let $v^{a} \in V$.

We can take their tensor product $T^{A}{}_{a} v^{b}$, which is a tensor of rank $\langle 2, 1 \rangle$.[b] There is only one possible contraction of this tensor:[c] $T^{A}{}_{a} v^{a}$, which is index notation for $T(v)$.[d]

[a] Note that this is 'the same' as $\mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(V, W)$ in the case of finite-dimensional vector spaces—see Corollary D.3.2.38.

[b] Again, we technically haven't defined tensors that 'live in' more than module, but it should be clear what is meant.

[c] We cannot contract A and a as they are for different modules (hence the different scripts). Of course, if it so happens that $W = V$, then we could contract, and one finds that $T^{b}{}_{b} v^{a} = \mathrm{tr}(T)v$.

[d] This equation gives a formula for Av, A a matrix and v a column vector: $[Av]^{i} = \sum_{j=1}^{n} A^{i}{}_{j} v^{j}$.

■ **Example 6.1.3.26 — Composition in index notation** Let U, V, and W be real vector spaces, $\mathbb{K}$ a cring, and let $S^{A}{}_{a} \in U^{\dagger} \otimes V$ and $T^{\alpha}{}_{A} \in V^{\dagger} \otimes W$.[a]

We can take their tensor product $T^{\alpha}{}_{A} S^{B}{}_{a}$, which is a tensor of rank $\langle 2, 2 \rangle$. There is only one possible contradiction of this tensor: $T^{\alpha}{}_{A} S^{A}{}_{a}$, which is index notation for $T \circ S$.

[a] Again, by Corollary D.3.2.38, these are respectively the same as $\mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(U, V)$ and $\mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(V, W)$ for finite-dimensional vector spaces.

We mentioned previously that there are several ways to think about tensors of a fixed rank. We could have made this statement precise before, but as I find this easier to explain using index notation, we waited to present this until now.

Theorem 6.1.3.27. Let V be a finite-dimensional real vector space and let $k, l \in \mathbb{N}$. Then, the following are naturally isomorphic.

(i).

$$\bigotimes\nolimits^{k}_{l} V. \tag{6.1.3.28}$$

(ii).

$$\underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^{\dagger} \otimes \cdots \otimes \otimes V^{\dagger}}_{l}. \tag{6.1.3.29}$$

(iii). The vector space of multilinear maps

$$\underbrace{V^{\dagger} \times \cdots \times V^{\dagger}}_{k} \times \underbrace{V \times \cdots \times V}_{l} \to \mathbb{K}. \tag{6.1.3.30}$$

R It is easiest to understand the meaning of this result with an example. Consider the $\langle 1, 2\rangle$ tensor $T^{a}{}_{bc}$.

Effectively, this result says that I can think of $T^{a}{}_{bc}$ as an element of $\bigotimes^{1}_{2} V := \mathrm{Mor}_{\mathbf{Vect}_{\mathbb{R}}}(V \otimes V, V)$ ((i)), as an element of $V \otimes V^{\dagger} \otimes V^{\dagger}$ ((ii)), or as a trilinear map $V^{\dagger} \otimes V \otimes V \to \mathbb{R}$.

In index notation, (i) corresponds to thinking of the tensor as the map

$$V^{\dagger} \otimes V^{\dagger} \ni v^{a} w^{b} \mapsto T^{a}{}_{xy}\, v^{x} w^{x} \in V, \tag{6.1.3.31}$$

(ii) corresponds to thinking of the tensor as just

$$T^{a}{}_{bc} \tag{6.1.3.32}$$

itself, and (iii) corresponds to thinking of the tensor as the map

$$V^{\dagger} \otimes V \otimes V \ni \omega_{a} v^{b} w^{c} \mapsto T^{x}{}_{yz}\, \omega_{x} v^{y} w^{z}. \tag{6.1.3.33}$$

The entire point of this result is that you should be able to freely change your perspective of what $T^{a}{}_{bc}$ is acting as. Does it take in two vectors and spit out a vector, or does it two vectors and a covector and spit out a number? The point is, it doesn't really matter—they're both difference perspectives of the same thing.

R These are not the only three ways to think about things either.[a] For example, I can think of $T^a{}_{bc}$ as a function that takes in vectors and spits out linear-transformations:

$$v^a \mapsto T^a{}_{bx}\, v^x. \tag{6.1.3.34}$$

In fact, that isn't the only way I can do this:

$$v^a \mapsto T^a{}_{xb}\, v^x. \tag{6.1.3.35}$$

R All this might seem complicated, but it's actually nothing fancy. For example, consider the vector-valued function $f(x,y)$. I can consider this as a function of two variables (analogous to (i)) or I can consider this as thing that is a function of x for every fixed y:

$$y \mapsto f(\cdot, y), \tag{6.1.3.36}$$

where $f(\cdot, y)$ is the function $x \mapsto f(x,y)$, which is exactly analogous to the perspective explained in the previous remark. Note that, just as there, I have another way in which I can do this:

$$x \mapsto f(x, \cdot). \tag{6.1.3.37}$$

In brief, I can think of $f(x,y)$ as a function of two variables, or, I can think of it as a function of the single variable x for every y (and the other way around).

[a] Writing an exhaustive list of all possibilities would not be useful. Once this "clicks", everything else not listed will be obvious.

Proof. This follows from Corollary D.3.2.39. ■

We need one more ingredient before we can get to actual differentiation, namely that of a *metric*.

Definition 6.1.3.38 — Metric (on a vector space) Let V be a real vector space. Then, a ***metric*** g^a on V is a covariant tensor of rank 2 such that

(i). (Symmetry) $g(v_1, v_2) = g(v_2, v_1)$; and
(ii). (Nonsingularity) the map from V to $V^\dagger$ defined by $v \mapsto g(v, \cdot)$, where $g(v, \cdot)$ is the linear functional which sends w to $g(v, w)$, is an isomorphism of vectors spaces.

R If v^a is a vector, then we write $v_a := g_{ab}v^b$. v_a is the ***dual vector*** (which itself is not a vector—it's a covector) of v^a. *Nonsingularity* is key because it allows us to reverse this process. If ω_a is a covector, then because the map $v^a \mapsto v_a$ is an *isomorphism*, there is a unique vector, written ω^a, that is equal to ω_a under this map.

R (i) can be written $g_{ab} = g_{ba}$. Also note that $g(v_1, v_2) = [v_1]^a[v_2]^b g_{ab}$.

R The idea of a notion of a metric on a vector space and a metric on a set (in the context of uniform space theory) have little to nothing to do with each other. It is merely a coincidence of terminology that is so ingrained that even I dare not go against it.

R The term "metric" in this sense of the word should really not be thought of as a sort of distance, but rather as a sort of dot product. Indeed, you can verify that the dot product is a metric, and furthermore, in a sense that we don't bother to make precise, every positive-definite metric (on a vector space) is equivalent to the usual Euclidean dot product. There is *some* connection with the other notion of metric, however—positive-definite metrics give us norms (the square-root $g(v, v)$), which in turn gives us a metric (in the other sense).

> **R** Nonsingularity is usually replaced with the requirement that $g(v, w) = 0$ for all w implies that $v = 0$ (called ***nondegeneracy***. In finite dimensions, this is equivalent to nonsingularity (by the Rank-Nullity Theorem). In infinite dimensions, however, they are not equivalent, and it is nonsingularity that we want (so that we can raise and lower indices).
>
> [a]The "g" is for gravity (in general relativity, gravity is modeled as a metric on space-time).

It's worth noting that, everything *except* raising and lowering indices we can do without a metric. To raise and lower indices, we do need that *extra* structure. In particular, if you pick a different metric, then your meaning of v_a will change even though the metric does not appear explicitly in this notation.

In summary:

(i). The tensor product of two vectors v^a and w^a, written $v^a w^b$, is defined to be the bilinear map that sends the pair of covectors $\langle \omega_a, \eta_a \rangle$ to $(\omega_a v^a)(\eta_a w^a)$. In practice, it's not particularly helpful to think of what this is[5]—in practice what matters is can you manipulate them.

(ii). A general tensor of rank $\langle k, l \rangle$ is an element in the tensor product of k copies of V with l copies of $V^\dagger$.

(iii). The definition of the tensor product of vectors can be extended to the tensor product of any tensors. In index notation, this is denoted simply by juxtaposition.

(iv). We can contract indices.

(v). If we have a metric, we can also raise and lower indices.

6.2 The definition

One thing that I personally found conceptually confusing with differentiation in $\mathbb{R}^d$ itself that was elucidated for me when passing to the study of more general manifolds was the distinction between a *vector*

[5]When you add 2 to 3 do you think about 2 being an equivalence class of sets with respect to the equivalence relation of isomorphism in the category of sets? Here, I'll help you out: No, you do not.

and a *point*. The problem in $\mathbb{R}^d$ is that the space of points and the space of vectors are effectively the same thing: they are both given by a d-tuple of real numbers, when in fact, they are really playing quite different roles. The points tell you *"where"* we are and the vectors tell you *"what direction"* to go in. In a general manifold, the points that tell you "where" form the points of the space itself and the vectors do *not* live in the entire space itself, but rather the *tangent spaces*.

Thus, while we have no intention of doing manifold theory in general,[6] we will make use of some suggestive notation that comes from the theory.

> Throughout this chapter, the symbol $\mathbb{R}^d$ will be used to denote d-dimensional Euclidean space *as a metric space*. For each $x \in \mathbb{R}^d$, we define $\mathrm{T}_x(\mathbb{R}^d)$, the ***tangent space at*** x ***in*** $\mathbb{R}^d$ to be the *metric vector space* $\mathbb{R}^d$ (with metric being the dot product).[a] Furthermore, we declare that $\mathrm{T}_{x_1}(\mathbb{R}^d) \neq \mathrm{T}_{x_2}(\mathbb{R}^d)$ for $x_1 \neq x_2$.[b] We will often, but not always, use abstract index notation for vectors $v^a \in \mathrm{T}_x(\mathbb{R}^d)$ to help remind us that they are to be thought of as vectors instead of points.
>
> [a]Careful: the word "metric" here is being used in two totally different senses. I know the terminology is perverse, but don't look at me! I'm not the one that came-up with it.
>
> [b]There are many ways to do this, but one way, for example, is to take $\mathrm{T}_x(\mathbb{R}^d) := \mathbb{R}^d \times \{x\}$.

In particular, as sets, we might have that $\mathbb{R}^d = \mathrm{T}_x(\mathbb{R}^d)$, but that's it—the two objects don't even live in the same category, and so it doesn't even make sense to ask whether there is some isomorphism between them (unless you forget some of the structure, in which case you're actually changing the object). For example, $\mathbb{R}^d$ is just a metric space—you cannot add any two of its elements. Tangent vectors, elements of $\mathrm{T}_x(\mathbb{R}^d)$, on the other hand, we can add just fine.

[6]In contrast to topology, for example, you do have to prove essentially all of your results in $\mathbb{R}^d$ first and *then* extend them to arbitrary manifolds, whereas in principle you can prove all the results about topology you ever wanted without even mentioning $\mathbb{R}$. This is one reason among others why we do general topology but not manifold theory.

To clarify, this is not actually how the definition goes in general. The general definition of the tangent space requires us to first be able to talk about manifolds, which in turn requires us to know how differentiation in $\mathbb{R}^d$ works, which of course we have not done yet. Thus, we are making use of this notation only to help clarify the study of differentiation in $\mathbb{R}^d$. In principle, once enough of this theory has been developed so that we can talk about tangent spaces in general, we would replace the above with the 'actual' definition.

This speak of tangent spaces allows us to make an important definition.

Definition 6.2.1 — Tensor field A ***tensor field*** of rank $\langle k, l\rangle$ is a function on $\mathbb{R}^d$ whose value at x is a rank $\langle k, l\rangle$ tensor on $\mathrm{T}_x(\mathbb{R}^d)$. A tensor field $T^{a_1\ldots a_k}{}_{b_1\ldots b_l}$ is ***continuous*** iff $[v_1]^{b_1}\cdots[v_l]^{b_l}[\omega_1]_{a_1}\cdots[\omega_k]_{a_k}T^{a_1\ldots a_k}{}_{b_1\ldots b_l}$ is continuous for all vectors $[v_1]^a,\ldots,[v_l]^a$ and covectors $[\omega_1]_a,\ldots,[\omega_k]_a$.

R It's just an assignment of a tensor to every point in $\mathbb{R}^d$. For example, a vector field is an assignment of a vector to every point.

6.2.1 The definition itself

Finally, with this (hopefully elucidating) notation in hand, we can define the derivative.

Definition 6.2.1.1 — Directional derivative Let $D\subseteq\mathbb{R}^d$,[a] let $f\colon D\to\mathbb{R}$, let $x\in\operatorname{Int}(D)$, and let $v\in\mathrm{T}_x(\mathbb{R}^d)$. Then, the ***directional derivative*** of f at x in the direction v, $\mathrm{D}_v f(x)$, is defined by

$$\mathrm{D}_v f(x) := \lim_{\varepsilon\to 0^+}\frac{f(x+\varepsilon v)-f(x)}{\varepsilon}. \tag{6.2.1.2}$$

If this limit exists, then f is ***differentiable at x in the direction v^a***.

The expression inside the limit on the right-hand side of (6.2.1.2) is the ***difference quotient***.

R Note that the limit a taken as $\varepsilon \to 0^+$, and not as $\varepsilon \to 0$. This has the advantage that it allows us to talk about the directional derivative of functions like $x \mapsto |x|$: its directional derivative in the direction of $+1$ is 1 and its directional derivative in the direction of -1 is (also) 1. If we had written "$\varepsilon \to 0$", neither of these directional derivatives would exist. Furthermore, for differentiable functions (Meta-definition 6.2.1.10), this won't actually matter (that is, we get the same answer as if we had used $\varepsilon \to 0$ instead).

R Note that we *only* define the directional derivative at points on the interior of the domain. The reason we do this is because otherwise we cannot guarantee that $x + \varepsilon v$ is in the domain of f for even a single $\varepsilon > 0$, so that $f(x + \varepsilon v)$ does not make sense, in which case it is nonsensical to ask whether the limit of the difference quotient exists or not.

R For some reason, people tend to write $\lim_{h \to 0^+}$[b] in this definition instead of $\lim_{\varepsilon \to 0^+}$. Not sure why. ε is used for 'small' numbers everywhere else in analysis. Moreover, what happens if your function is called h (not that uncommon of a name for a function)? What are you going to do? Write $h(x + hv)$? Ew.

[a]"D" is for "domain".

[b]Well, actually they probably write $\lim_{h \to 0}$, with the $^+$, but that is not the difference I mean to point out at the moment.

Our next issue is to define "differentiability". Naively, it might seem like the definition is obvious: "differentiable" should mean that the derivative exists, or more precisely, that all directional derivatives exist. Certainly you can make this definition,[7] but it's not particularly useful. The problem is that there exists functions f for which $\mathrm{D}_v f(x)$ exists for all $x \in \mathbb{R}^d$ and $v \in \mathrm{T}_x(\mathbb{R}^d)$, but yet the map $v \mapsto \mathrm{D}_v f(x)$ is not linear. That is, the derivative won't be a tensor, which is problematic if you want to take higher derivatives.

[7]This is called Gâteaux-differentiable.

Okay then, so "differentiable" should mean that the derivative exists and is linear, right? Again, one can make this definition,[8] and this *is* useful, but it's not the standard term. Instead, the conventional definition of *differentiable*, or *Fréchet-differentiable* for emphasis, is not as straightforward and given in Definition 6.2.1.5. Before we get there, however, we introduce the other two notions of differentiability.

Definition 6.2.1.3 — Gâteaux-differentiable Let $D \subseteq \mathbb{R}^d$, let $f\colon D \to \mathbb{R}$ and let $x \in \operatorname{Int}(D)$. Then, f is ***Gâteaux-differentiable*** at x iff $\mathrm{D}_v f(x)$ exists for all $v \in \mathrm{T}(T)\mathbb{R}^d[x]$.

W For f to be *Gâteaux-differentiable* on all of D does *not* mean that it be differentiable at every $x \in \operatorname{Int}(D)$—this is addressed in Meta-definition 6.2.1.10. The same remark applies to linear-differentiability and Fréchet-differentiability.

Definition 6.2.1.4 — Linearly-differentiable Let $D \subseteq \mathbb{R}^d$, let $f\colon D \to \mathbb{R}$ and let $x \in \operatorname{Int}(D)$. Then, f is ***linearly-differentiable*** at x iff

(i). $\mathrm{D}_v f(x)$ exists for all $v \in \mathrm{T}_x(\mathbb{R}^d)$;
(ii). the map $\mathrm{T}_x(\mathbb{R}^d) \ni v \mapsto \mathrm{D}_v f(x)$ is linear; and
(iii). the map $\mathrm{T}_x(\mathbb{R}^d) \ni v \mapsto \mathrm{D}_v f(x)$ is continuous.[a]

R In many ways, this is a more natural definition of differentiability than the more standard Fréchet-differentiability. As such, we will only assume linearly-differentiable when we can get away with it, but there are a couple of big pathologies with this condition which explain why it is almost never taken as 'the' definition of differentiability.

IMHO, the biggest pathology is that the composition of two linearly-differentiable functions need not be linearly-differentiable (Example 6.5.45), which means that they cannot be made to form a category.

[8]This is sometimes ([DM07, pg. 117]) also called Gâteaux-differentiable, but to avoid confusion we shall use the nonstandard term Linearly-differentiable.

But also there are differentiable functions that are not continuous—see Example 6.5.1. Fortunately, this cannot happen in one dimension—see Proposition 6.5.8.

[a] Actually, in finite dimensions, continuity follows from linearity for free (though we will not check this). We state this explicitly because you will want to do so when you generalize to infinite dimensions, where you won't get continuity for free.

Definition 6.2.1.5 — Fréchet differentiable Let $D \subseteq \mathbb{R}^d$, let $f\colon D \to \mathbb{R}$ and let $x \in \operatorname{Int}(D)$. Then, f is ***Fréchet-differentiable*** at x iff there exists a unique continuous linear-transformation $\mathrm{d}f_x\colon \mathrm{T}_x(\mathbb{R}^d) \to \mathbb{R}$ such that

$$\lim_{v\to 0} \frac{|f(x+v) - f(x) - \mathrm{d}f_x(v)|}{\|v\|} = 0. \tag{6.2.1.6}$$

R We will usually be specific to clarify which notion of differentiability we're using, but you should be aware that, for other authors, "differentiable" will almost certainly mean "Fréchet-differentiable".

R Note that $\mathrm{d}f_x(v) = \mathrm{D}_v f(x)$. Thus, Fréchet-differentiability implies not only that the directional derivative exist and define a continuous linear-transformation, but furthermore, that continuous linear-transformation approximates $f(x+v) - f(x)$ sufficiently fast. More accurately, it means that

$$f(x+v) - f(x) = \mathrm{d}f_x(v) + \|v\|^2 \cdots, \tag{6.2.1.7}$$

that is, $f(x+v) - f(x) = \|v\|\, \mathrm{d}f_x(v)$ up to order $\|v\|^2$.

First of all, note that we have Fréchet implies linearly implies Gâteaux, with the first two being equivalent in one-dimension—see Proposition 6.2.1.11. Despite the fact that $v \mapsto \mathrm{D}_v f(x)$ need not be linear, fortunately, this map is nonnegative homogeneous, a fact particularly relevant in one-dimension—see one of the remarks in Proposition 6.2.1.14.

Exercise 6.2.1.8 Let $D \subseteq \mathbb{R}^d$, let $f\colon D \to \mathbb{R}$, let $x \in D$, let $v \in \mathrm{T}_x(\mathbb{R}^d)$, and let $\alpha \in \mathbb{R}_0^+$. Show that, if $\mathrm{D}_v f(x)$ exists, then $\mathrm{D}_{\alpha v} f(x)$ exists and furthermore that

$$\mathrm{D}_{\alpha v} f(x) = \alpha \, \mathrm{D}_v f(x). \tag{6.2.1.9}$$

R Note that this is *false* in general if $\alpha < 0$. For example, consider what happens to $\mathrm{D}_{\alpha\cdot 1}|x|$ at $x = 0$ for $\alpha = -1$.

We now say what it means for a function to be differentiable on its domain.

Meta-definition 6.2.1.10 — Differentiable Let XYZ be either "Gâteaux", "linearly", or "Fréchet", let $D \subseteq \mathbb{R}^d$, and let $f\colon D \to \mathbb{R}$. Then, f is ***XYZ-differentiable*** (on D) iff there is an open neighborhood $U \supseteq D$ and a $g\colon U \to \mathbb{R}$ such that g is XYZ-differentiable at x for all $x \in U$ and $f = g|_D$.

R In brief, we say that "f extends to an XYZ-differentiable function on an open neighborhood of D.", or sometimes even less precisely, "f is XYZ-differentiable on a neighborhood of D.".

W Warning: "differentiable on D" does not mean "differentiable at every $x \in \mathrm{Int}(D)$". Instead, this is only true if D is open. To see why we take this as the definition, for example, if $D := \{\langle x, 0\rangle \in \mathbb{R}^2 : x \in \mathbb{R}\}$, then $\mathrm{Int}(D) = \emptyset$, but yet we don't want it to be the case that *every* function on D is differentiable. This is just suggestive, however. The real reason I took this definition is because now $C^\infty(D)$ (see Definition 6.3.1) should agree with what diffeology says it should be.[a]

R As the directional derivative is only defined for points on the interior of a set, it is immediate that we had to reduce the definition of "differentiable" to the case of when the domain is an open set. As discussed in the previous warning, the way we chose to do this is to take differentiable to mean "differentiable in an open neighborhood". An alternative, perhaps more obvious choice was to require that f be differentiable at all $x \in \operatorname{Int}(D)$, though, as briefly mentioned before, this is not desirable. In any case, see Example 6.5.14 for a 'naturally-occurring' example of where these two definitions differ.

R Perhaps to guide your intuition in the case that D is not open, you should check in Exercise 6.2.1.21 that a function on a closed interval $[a, b]$ is differentiable iff it is is differentiable in the interior and the one-sided derivatives at a and b exist.

---

[a]Very roughly speaking, topological manifolds are to topology as smooth manifolds are to diffeology. The only point of relevance here is that diffeology gives a way of defining smooth functions (see a remark in Meta-definition 6.2.1.19) on *any* subset of $\mathbb{R}^d$, not must submanifolds.

We now check that Fréchet is stronger than linearly is stronger than Gâteaux.

Proposition 6.2.1.11 Let $D \subseteq \mathbb{R}^d$, let $f\colon D \to \mathbb{R}$, and let $x \in \operatorname{Int}(D)$.

(i). If f is Fréchet-differentiable at x (resp. on D), then it is linearly-differentiable at x (resp. on D).

(ii). If f is linearly-differentiable at x (resp. on D), then it is Gâteaux-differentiable at x (resp. on D).

(iii). For $d = 1$, f is linearly-differentiable at x (resp. on D) iff it is Fréchet differentiable at x (resp. on D).

W Warning: The converses of both (i) and (ii) fail—see Examples 6.2.1.22 and 6.5.1.

R In view of (iii), in one-dimension, we will more often avoid our normal specificity and simply write ***differentiable*** for linearly-differentiable/Fréchet-differentiable.

Proof. (i) We leave this as an exercise.

Exercise 6.2.1.12 Prove the result.

(ii) Immediate from the definitions.

(iii) We leave this as an exercise.

Exercise 6.2.1.13 Prove the result.

■

Before moving on, I want to mention that, for all the fuss we're making about different types of differentiability, in practice, one doesn't need to worry. The reason for this is that, usually, either you are interested in continuous things or you are interested in *smooth* (Meta-definition 6.2.1.19) things—functions which are differentiable and just differentiable are not as commonly used.

We've talked awhile about directional derivatives and differentiability, but we still have yet to define the derivative itself! It's about time we do this.

Proposition 6.2.1.14 — Derivative (of a function) Let $D \subseteq \mathbb{R}^d$ and let $f\colon D \to \mathbb{R}$. Then, if f is linearly-differentiable, then there is a unique covector field on D, $\nabla_a f$, the ***derivative*** of f, such that $v^a \nabla_a f(x) = \mathrm{D}_v\, g(x)$ for all linearly-differentiable extensions g of f, all $x \in D$, and all $v \in \mathrm{T}_x(D)$.

f is ***continuously-differentiable*** iff it is linearly-differentiable and $\nabla_a f$ is continuous.[a]

R Thus far, we have only defined $\mathrm{T}_x(\mathbb{R}^d)$, and not $\mathrm{T}_x(D)$ for $D \subset \mathbb{R}^d$. Here, $\mathrm{T}_x(D)$ is defined by

$$\mathrm{T}_x(D) := \big\{v \in \mathrm{T}_x(\mathbb{R}^d) : x + \varepsilon v \in D \\ \text{for all } \varepsilon > 0 \text{ sufficiently small.}\big\} .^b$$

Note that this will *not* be a vector space in general (e.g. $\mathrm{T}_0([0,1]) = [0,\infty)$).

R In particular, if D is open, then we have that $v^a \nabla_a f(x) = \mathrm{D}_v f(x)$ for all $x \in D$ and all $v \in \mathrm{T}_x(\mathbb{R}^d)$. In other words, the derivative of f at a point $x \in D$ is the continuous linear map that sends a tangent vector to the directional derivative of f in the direction of that tangent vector. In order to make sense of this, however, we need to be working in an open set, and this statement here says that the answer we get is independent of how we extend to an open neighborhood of the domain.

R The notation $v^a \nabla_a f$ is obviously suggestive. Hopefully it reminds you of the fact from multivariable calculus that the directional derivative of a function f in the direction $\vec{v}$ is given by the dot product of $\vec{v}$ with the gradient of f: $\vec{v} \cdot \vec{\nabla} f$.[c] For fixed f and x, the map $v^a \mapsto [v^a \nabla_a f](x)$ is linear in v^a, and so defines a *linear functional*, that is to say, $\nabla_a f(x) \in \mathrm{T}_x(\mathbb{R}^d)^\dagger$, or equivalently, that $\nabla_a f$ is a covector field on $\mathbb{R}^d$. This covector field is the *derivative*[d] or ***gradient*** of f at x.

R If our function is differentiable in one dimension, we *always* take $v^a = 1 \in \mathrm{T}_x(\mathbb{R}) \cong_{\mathbf{Vect}_{\mathbb{R}}} \mathbb{R}$ and write

$$\tfrac{\mathrm{d}}{\mathrm{d}x} f(x) := \nabla f(x) := v^a \nabla_a f(x), \qquad (6.2.1.15)$$

for every other directional derivative may be obtained from this single number by scaling (e.g. the directional derivative for $v = -3$ will be $-3 \cdot \frac{\mathrm{d}}{\mathrm{d}x} f(x)$). If we want to use the symbol $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$ to refer to the function $x \mapsto \frac{\mathrm{d}}{\mathrm{d}x} f(x)$, then we may write $\frac{\mathrm{d}}{\mathrm{d}x}\big|_{x=c} f(x)$ to refer to the value of this function at c.

R Perhaps more accurate notation would have been $[v^a\nabla_a f](x)$, that is, $v^a\nabla_a f$ is a function (in the case that f is differentiable anyways), and so $v^a\nabla_a f(x)$ is the value of $v^a\nabla_a f$ at x, as opposed to, the derivative of the function $f(x)$, which is just 0 of course.[e] The point is: you must compute the derivative, and *then* plug-in x. It might seem silly in such a simple context, but in much more complicated contexts I myself have made this very mistake. For example, suppose you are computing a functional derivative (to find a Noether charge, say) of some action functional in physics, and you want to see that this quantity is conserved 'on-shell' (i.e. when the equations of motion hold)—you cannot use the equations of motion before you finish computing the Noether charge 'off-shell': that's cheating (and more importantly, possibly just plain wrong)![f]

[a] Recall that (Definition 6.2.1) this means that $v^a\nabla_a f$ is continuous for all vectors v^a.

[b] That is, $v \in \mathrm{T}_x(D)$ iff there is some $\varepsilon_0 > 0$ such that $x + \varepsilon v \in D$ for all $\varepsilon_0 \geq \varepsilon > 0$.

[c] Contraction of indices should always be thought of as a sort of dot product.

[d] That is to say, $\nabla_a f$ is the derivative and $v^a\nabla_a f$ is the derivative in the direction of v^a (or the directional derivative in the direction of v^a).

[e] I know to some this may seem pedantic, but f is the function and $f(x)$ is the value at x of the function, so that $f(x)$ is just a number.

[f] Once again, if the physics analogy is meaningless to you, just ignore it.

Proof. We leave this as an exercise.

Exercise 6.2.1.16 Prove the result yourself.

■

Proposition 6.2.1.17 — Continuously-differentiable implies Fréchet-differentiable Let $D \subseteq \mathbb{R}^d$ and let $f\colon D \to \mathbb{R}$. Then, if f is continuously-differentiable, it is Fréchet-differentiable.

R By definition, continuously-differentiable means that it is linearly-differentiable (so that $\nabla_a f$ makes sense) and $\nabla_a f$ is continuous. In the definition, we could have assumed Fréchet-differentiability instead. This result essentially says that, if the derivative exists and is continuous, one need not worry about the distinction between linearly-differentiable and Fréchet-differentiable.

Proof. We leave this as an exercise.

Exercise 6.2.1.18 Prove the result.

■

We can use the definition of the derivative of a function to define the derivative for *all* tensor fields. The idea is that, by plugging in enough vectors and covectors, all tensor fields reduce to just a function.

Meta-definition 6.2.1.19 — Derivative (of a tensor) Let XYZ be either "Gâteaux", "linearly", or "Fréchet", let $D \subseteq \mathbb{R}^d$, let $T^{a_1\dots a_k}{}_{b_1\dots b_l}$ be a tensor field of rank $\langle k, l\rangle$ on D, and let $x \in \operatorname{Int}(D)$. Then, $T^{a_1\dots a_k}{}_{b_1\dots b_l}$ is ***XYZ-differentiable*** at x iff for all covectors $\omega_1, \dots, \omega_k$ and all vectors $v_1, \dots, v_l$, the function $[\omega_1]_{a_1} \cdots [\omega_k]_{a_k} [v_1]^{b_1} \cdots [v_l]^{b_l} T^{a_1\dots a_k}{}_{b_1\dots b_l} : D \to \mathbb{R}$ is XYZ-differentiable at x.

$T^{a_1\dots a_k}{}_{b_1\dots b_l}$ is ***XYZ-differentiable*** (on D) iff for all covectors $\omega_1, \dots, \omega_k$ and all vectors $v_1, \dots, v_l$, the function $[\omega_1]_{a_1} \cdots [\omega_k]_{a_k} [v_1]^{b_1} \cdots [v_l]^{b_l} T^{a_1\dots a_k}{}_{b_1\dots b_l} : D \to \mathbb{R}$ is XYZ-differentiable.

For T linearly-differentiable, the ***derivative***,

$$\nabla_a T^{a_1\dots a_k}{}_{b_1\dots b_l}, \tag{6.2.1.20}$$

of $T^{a_1\ldots a_k}{}_{b_1\ldots b_l}$ is defined by

$$v^a[\omega_1]_{a_1}\cdots[\omega_k]_{a_k}[v_1]^{b_1}\cdots[v_l]^{b_l}\nabla_a T^{a_1\ldots a_k}{}_{b_1\ldots b_l} = v^a\nabla_a\left([\omega_1]_{a_1}\cdots[\omega_k]_{a_k}[v_1]^{b_1}\cdots[v_l]^{b_l}T^{a_1\ldots a_k}{}_{b_1\ldots b_l}\right)$$

for all covectors $\omega_1,\ldots,\omega_k$ and all vectors $v, v_1,\ldots,v_l$.

R This looks a lot more atrocious than it is. All we are doing is reducing tensor fields to functions, and then applying the definitions we know for functions. So, for example, consider a tensor field $T^{ab}{}_c$ of rank $\langle 2,1\rangle$. For every choice of vector $[v_1]^a$ and choice of two covectors $[\omega_1]_a, [\omega_2]_a$, we obtain a *function*, $[v_1]^a[\omega_1]_b[\omega_2]_c T^{bc}{}_a$. The tensor $T^{ab}{}_c$ is then *XYZ-differentiable* if *every* one of these functions is XYZ-differentiable, and in this case, the derivative is a tensor field that now takes in *two* vectors $v^a, [v_1]^a$ and two covectors $[\omega_1]_a, [\omega_2]_a$, and spits out the function $v^a\nabla_b\left([v_1]^b[\omega_1]_c[\omega_2]_d T^{cd}{}_b\right)$. If this still seems like a bit hard to swallow, see Example 6.2.1.26 for some examples with which you are likely already familiar.

R Thus, this in particular shifts the covariant rank of a tensor up by 1, that is, the derivative of a tensor of rank $\langle k,l\rangle$ is of rank $\langle k+1,l\rangle$ (for example, as functions are just $\langle 0,0\rangle$ tensors, it takes functions to covector fields).

R Perhaps the most significant aspect of this definition is that we now know how to take higher derivatives of a function f, e.g. $\nabla_a\nabla_b f$. If all higher derivatives of f exist, then f is ***infinitely-differentiable***. If all higher derivatives of f exist *and* they are continuous, then f is ***smooth***.[a] The issue here is that infinite-differentiability is really infinite-linear-differentiability, whereas smooth is infinite-Fréchet-differentiability.[b]

[a] Yes, there is a difference. In fact, infinitely-differentiable functions need not even be continuous—see Example 6.5.1.

[b] There is no analogue for infinite-Gâteaux-differentiability as the (first) derivative is then not necessarily a tensor, in which case the higher derivatives are not defined.

Exercise 6.2.1.21 Let $a, b \in \mathbb{R}$ and let $f\colon [a,b] \to \mathbb{R}$. Show that f is linearly-differentiable iff $f|_{(a,b)}$ is differentiable, $\mathrm{D}_1 f(a)$ exists, and $\mathrm{D}_{-1} f(b)$ exists.

■ **Example 6.2.1.22 — A function which is Gâteaux differentiable but not differentiable** [a] In other words, we seek a function all of whose directional derivatives exist at a point, but for which the map $v \mapsto \mathrm{D}_v f(x)$ is not linear.[b]

Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(\langle x,y\rangle) := \begin{cases} 0 & \text{if } \langle x,y\rangle = \langle 0,0\rangle \\ \frac{x^2y}{x^2+y^2} & \text{otherwise.} \end{cases} \tag{6.2.1.23}$$

Let $v := \langle v_x, v_y\rangle \in \mathrm{T}_{\langle 0,0\rangle}(\mathbb{R}^d)$ be nonzero (the directional derivative in the direction of the zero vector is always—gasp—zero). Then,

$$\begin{aligned} \frac{f(\langle 0,0\rangle + \varepsilon v) - f(\langle 0,0\rangle)}{\varepsilon} &= \frac{\frac{(\varepsilon v_x)^2(\varepsilon v_y)}{(\varepsilon v_x)^2+(\varepsilon v_y)^2}}{\varepsilon} \\ &= \frac{v_x^2 v_y}{v_x^2 + v_y^2}. \end{aligned} \tag{6.2.1.24}$$

In particular, the limit of this as $\varepsilon \to 0^+$ always exist. On the other hand, the map

$$v \mapsto \frac{v_x^2 v_y}{v_x^2 + v_y^2} =: \mathrm{D}_v f(\langle 0,0\rangle) \tag{6.2.1.25}$$

is certainly not linear. For example,

$$\begin{aligned} D_{\langle 1,1\rangle} f(\langle 0,0\rangle) &= \frac{1}{2} \neq 0+0 \\ &= D_{\langle 1,0\rangle} f(\langle 0,0\rangle) + D_{\langle 0,1\rangle} f(\langle 0,0\rangle). \end{aligned}$$

---

[a]This example comes from Ted Shifrin's math.stackexchange answer.

[b]Actually, what we really want to 'break' is $D_{v_1+v_2} f(x) = D_{v_1} f(x) + D_{v_2} f(x)$. For one, $D_{\alpha v} f(x) = \alpha D_v f(x)$ *is* true for $\alpha > 0$, and it's relatively easy to find a counter-example of $\alpha < 0$—see Exercise 6.2.1.8.

In fact, things that are arguably even worse than this can happen—it can happen that all directional derivatives exist *and* furthermore the map that seconds a vector to the directional derivative in that direction is linear, but yet the function not even be continuous. Unfortunately, we will have to postpone such a counter-example until after having defined the exponential function—see Example 6.5.1

And now we present some familiar examples of derivatives of tensors.

▪ Example 6.2.1.26 — Gradient, divergence, curl, and Laplacian The gradient, divergence, curl, and Laplacian from multivariable calculus are all just special cases of derivatives of tensors (along with other tensorial constructions such as contraction).

Throughout this example, let f be a smooth function on $\mathbb{R}^d$ and let v^a be a vector field on $\mathbb{R}^d$.

The *gradient* of f is simply $\nabla_a f$. You've already seen this guy.

The ***divergence*** of v^a is $\nabla_a v^a$. Note how in multivariable calculus you might write this as $\vec{\nabla} \cdot \vec{v}$, the 'dot product' of the gradient with the vector field $\vec{v}$. The index contraction here hopefully makes obvious how this is the divergence that you know and (maybe) love.

The *curl* is trickier, but this is not surprising, as it was trickier than the rest in multivariable calculus as well. First of all, the curl of a vector field is in general *not* a vector field—this is very specific to three-dimensions, essentially because the cross-product is specific to three-dimensions. In general, it is also more convenient to define the curl of a *covector* field instead of a vector field.[a] For us, the *curl* of a covector field ω_a will be defined to be $\nabla_a\omega_b - \nabla_b\omega_a$.[b][c] For what it's worth, in higher dimensions, mathematicians don't call this the curl—they call it the *exterior derivative*.

The ***Laplacian*** of a function is the divergence of the gradient—literally (modulo the raising of an index), that is, the Laplacian of f is $\nabla_a\nabla^a f$.

[a] They are essentially equivalent, however, as you can translate back and forth by raising and lowering indices.

[b] If you are familiar with the wedge product, this is essentially just $[\nabla \wedge \omega]_{ab}$.

[c] To get back a vector field *in three-dimensions* you have to in turn contract this with ϵ^{abc}, where ϵ^{abc} is the unique completely antisymmetric contravariant tensor of rank 3 of trace 1. "of trace 1" here means $\epsilon_{abc}\epsilon^{abc} = 1$. The reasons you don't get back a vector in higher dimensions is because in higher dimensions this becomes $\epsilon^{a_1\cdots a_d}$, and then you will have instead a contravariant tensor of rank $d-2$.

6.3 A smooth version of $\mathrm{Mor}_{\mathbf{Top}}(\mathbb{R}^d, \mathbb{R})$: $C^\infty(\mathbb{R}^d)$

Our goal in this section is to find a notion of convergence (i.e. a topology) that has the property that if a net of *smooth* functions converges to another function, then the limit function is necessarily smooth. The statement that $\mathrm{Mor}_{\mathbf{Top}}(\mathbb{R}^d, \mathbb{R})$ is complete from the last chapter (Theorem 4.4.1.1) says that if a net of continuous functions converges with respect to the topology on $\mathrm{Mor}_{\mathbf{Top}}(\mathbb{R}^d, \mathbb{R})$ (i.e. converges uniformly on quasicompact subsets), the limit function is likewise necessarily continuous function.[9] Unfortunately, this fails *miserably* if we replace "continuous" with the word "smooth". In fact, we can have a sequence of *smooth* functions which converge

[9] In fact, it says more than this—if your net is Cauchy, then it has a limit and that limit is continuous.

uniformly to a function that is *differentiable-nowhere*.[10] To construct such a counter-example, we will have to wait until we have access to trigonometric functions[11]—see Example 6.5.38. There is a 'simple' solution however—if your derivatives are continuous, and you want them to converge to something continuous, just force them to converge uniformly as well.

Definition 6.3.1 — $C^\infty(\mathbb{R}^d)$ $C^\infty(\mathbb{R}^d)$ is the seminormed algebra whose underlying set is the collection of all smooth real-valued functions, whose algebra structure[a] is given by the pointwise operations, and for each quasicompact $K \subseteq \mathbb{R}^d$, $m \in \mathbb{N}$, and $[v_1]^a, \ldots, [v_m]^a \in \mathbb{R}^d$ there is a seminorm $\|\cdot\|_{K,v_1,\ldots,v_m}$ defined by

$$\|f\|_{K,v_1,\ldots,v_m} := \sup_{x \in K} \left|[v_1]^{a_1} \cdots [v_m]^{a_m} \nabla_{a_1} \cdots \nabla_{a_m} f(x)\right|.$$

R This can be generalized in the obvious way to the case where the domain is an arbitrary open subset U. *However*, the proper definition of the topology on $C^\infty(S)$ for S arbitrary is tricky, and as we won't make use of it, we leave this topology undefined.

R $C^\infty(\mathbb{R}^d, \mathbb{R}^m)$ is the set of all smooth functions from $\mathbb{R}^d$ into $\mathbb{R}^m$. The differences: (i) there is now no longer any multiplication, and (ii) in the definition of the seminorm you interpret the absolute value as the Euclidean norm in $\mathbb{R}^m$.

R We have not defined it, but in the category of manifolds, **Man**, it will turn out that $C^\infty(\mathbb{R}^d, \mathbb{R}^m) := \mathrm{Mor}_{\mathbf{Man}}(\mathbb{R}^d, \mathbb{R}^m)$.[b]

[10] Though it is of course necessarily continuous by completeness of $\mathrm{Mor}_{\mathbf{Top}}(\mathbb{R}^d, \mathbb{R})$.

[11] Perhaps one can construct an example without them, but the standard example does use them.

R For what it's worth, the "∞" here is referring to the existence of infinitely-many derivatives, and the "C" is for "continuous" (the derivatives are required to be continuous).

R This is why smooth functions are sometimes referred to as C^∞ *functions*—see Definition 6.3.8.

[a] Recall that an algebra (Definition 4.2.3.13) is a vector space that is also a ring subject to a compatibility condition. In other words, you can add things, you can multiply things, and you can scale things.

[b] The objects in **Man** are manifolds, not men. Don't objectify people. It's wrong.

Proposition 6.3.2 Let $\lambda \mapsto f_\lambda \in C^\infty(\mathbb{R}^d)$ be a net and let $f_\infty \in C^\infty(\mathbb{R}^d)$. Then, $\lambda \mapsto f_\lambda$ converges to f_∞ (resp. is Cauchy) in $C^\infty(\mathbb{R}^d)$ iff $\lambda \mapsto [v_1]^{a_1} \cdots [v_m]^{a_m} \nabla_{a_1} \cdots \nabla_{a_m} f_\lambda$ converges to $[v_1]^{a_1} \cdots [v_m]^{a_m} \nabla_{a_1} \cdots \nabla_{a_m} f_\infty$ (resp. is Cauchy) in $\mathrm{Mor}_{\mathbf{Top}}(\mathbb{R}^d, \mathbb{R})$ for all $m \in \mathbb{N}$ and $[v_1]^a, \ldots, [v_m]^a \in \mathbb{R}^d$.

R You might say that the definition of the seminorms was made the way they were so that this is the case.

Proof. We leave this as an exercise.

Exercise 6.3.3 Prove this yourself.

■

Theorem 6.3.4. $C^\infty(\mathbb{R}^d)$ is complete.

Proof. Let $\lambda \mapsto f_\lambda \in C^\infty(\mathbb{R}^d)$ be Cauchy. By the previous result,

$$\lambda \mapsto [v_1]^{a_1} \cdots [v_m]^{a_m} \nabla_{a_1} \cdots \nabla_{a_m} f_\lambda \tag{6.3.5}$$

is Cauchy in $\mathrm{Mor}_{\mathbf{Top}}(\mathbb{R}^d,\mathbb{R})$ for all $m\in\mathbb{N}$ and $v_1,\ldots,v_m\in\mathbb{R}^d$, and so, for each $m\in\mathbb{N}$ and $v_1,\ldots,v_m\in\mathbb{R}^d$, there is some $g_{v_1,\ldots,v_m}\in\mathrm{Mor}_{\mathbf{Top}}(\mathbb{R}^d,\mathbb{R})$ such that $\lambda\mapsto[v_1]^{a_1}\cdots[v_m]^{a_m}\nabla_{a_1}\cdots\nabla_{a_m}f_\lambda$ converges to $g_{v_1,\ldots,v_m}$ in $\mathrm{Mor}_{\mathbf{Top}}(\mathbb{R}^d,\mathbb{R})$.

Exercise 6.3.6 Complete the proof by showing that

$$g_{v_1,\ldots,v_m}=[v_1]^{a_1}\cdots[v_m]^{a_m}\nabla_{a_1}\cdots\nabla_{a_m}g. \tag{6.3.7}$$

■

We defined above in Definition 6.3.1 $C^\infty(\mathbb{R}^d)$. In fact, there is a $C^k(\mathbb{R}^d)$ for all $k\in\mathbb{N}\cup\{\infty\}$ (and even something special generally written as $m=\omega$). We won't really be using this terminology much ourselves, but it's quite common, and so you should be aware of what it means.

Definition 6.3.8 — C^k function Let $D\subseteq\mathbb{R}^d$ and let $f\colon D\to\mathbb{R}$. Then, for $k\in\mathbb{N}$, f is C^k iff the first k derivatives of f exist and extend to continuous functions on D. f is C^∞ iff f is smooth. f is C^ω iff f is analytic.

R Note that *analytic* has not yet been defined—see Definition 6.4.5.12. In brief, a function is analytic iff it is equal to its Taylor series (Definition 6.4.5.10) in a neighborhood of each interior point.

R I am not 100% positive about this, but I believe that the symbol ω is used here as this is generally the symbol used to denote the smallest infinite ordinal.[a]

[a] We of course haven't defined ordinals, and we have no plans to do so, but in brief, ordinals are to well-ordered sets as cardinals are to sets. That is, ordinals are equivalence classes of well-ordered sets with respect to the equivalence relation of isomorphism in the category of well-ordered sets.

6.4 Calculus

6.4.1 Tools for calculation

And now we come to the results which you are probably most familiar with from calculus.

Proposition 6.4.1.1 — Algebraic Derivative Theorems Let $f, g\colon \mathbb{R}^d \to \mathbb{R}$ be linearly-differentiable.[a] and let $\alpha \in \mathbb{R}$. Then,

(i). (Linearity) $\nabla_a(f+g) = \nabla_a f + \nabla_a g$;
(ii). (Homogeneity) $\nabla_a(\alpha f) = \alpha \nabla_a f$;
(iii). (Product Rule)$\nabla_a(fg) = (\nabla_a f)g + f(\nabla_a g)$;[b] and
(iv). (Quotient Rule) $\nabla_a\left(\frac{f}{g}\right) = \frac{(\nabla_a f)g - f(\nabla_a g)}{g^2}$ whenever $g \neq 0$.

R The first three are true just as well for f and g arbitrary tensor fields, with essentially the same exact proofs (the juxtaposition denotes the tensor product of course). The Quotient Rule does not make sense, however, as in general you cannot invert tensors.

[a] Everything works just as well (with the same proof) if you just assume it is differentiable in some direction at some point, but the notation is more tedious.

[b] I would get in the habit of not mixing-up the order of f and g (e.g. by writing $(\nabla_a f)g + (\nabla_a g)f$ or something of the like). It won't matter for us, but it can and will latter when you're working with things that are not commutative (the cross-product of vectors is probably the most elementary example).

Proof. (i) and (ii) follows straight from the corresponding results above limits—see Proposition 2.4.3.12.(i) and Proposition 2.4.3.12.(ii).

As for (iii), we have

$$\frac{f(x+hv)g(x+hv)-f(x)g(x)}{h} =^{a} \left(\frac{f(x+hv)-f(x)}{h}\right) g(x+hv) + f(x)\left(\frac{g(x+hv)-g(x)}{h}\right),$$

and so taking limits gives us the Product Rule.

Similarly, the proof of the Quotient Rule amounts to just algebraic manipulation of the difference quotient:

$$\frac{\frac{f(x+hv)}{g(x+hv)} - \frac{f(x)}{g(x)}}{h} = \frac{\left(\frac{f(x+hv)-f(x)}{h}\right) g(x) - f(x)\left(\frac{g(x+hv)-g(x)}{h}\right)}{g(x+hv)g(x)}.$$

■

[a] We added and subtracted $f(x)g(x+hv)$.

Proposition 6.4.1.2 — Chain Rule Let $f^\alpha\colon \mathbb{R}^d \to \mathbb{R}^m$ and $g^\mu\colon \mathbb{R}^m \to \mathbb{R}^n$ be Fréchet-differentiable.[a] Then,

$$\nabla_a[g \circ f]^\mu(x) = \nabla_\alpha g^\mu(f(x))\nabla_a f^\alpha(x). \tag{6.4.1.3}$$

R One of the things I really love about index notation is that it almost dictates what the answer has to be—how many ways can you? construct a tensor with one $\mathbb{R}^d$ covariant index and one $\mathbb{R}^n$ contravariant index using only f^α, g^μ, and their derivatives?

W Warning: This analogous statement for linearly-differentiability couldn't possibly be true as there $g \circ f$ might not be linearly-differentiable—see Example 6.5.45. Thus, it's not that the statement is false, it's that it wouldn't even make sense!

[a] The μ index is used to remind us that g^μ lives in a $\mathbb{R}^n$ (as opposed to $\mathbb{R}^m$ or $\mathbb{R}^d$).

Proof. To prove this, by the definition of the derivative of tensors, we need to show that

$$\omega_\mu \nabla_a [g \circ f]^\mu(x) = \omega_\mu \nabla_\alpha g^\mu(f(x)) \nabla_a f^\alpha(x) \quad (6.4.1.4)$$

for all covectors (living in $\mathbb{R}^n$) ω_μ. In particular, it suffices to prove the result for $g\colon \mathbb{R}^m \to \mathbb{R}$ (because now $\omega_\mu g^\mu : \mathbb{R}^m \to \mathbb{R}$).

Let v^a be a constant vector field on $\mathbb{R}^d$. Then, what we actually want to show is

$$v^a \nabla_a [g \circ f](x) = (\nabla_\alpha g(f(x)))\,(v^a \nabla_a f^\alpha(x)), \quad (6.4.1.5)$$

as v^a is arbitrary. On one hand

$$v^a \nabla_a [g \circ f](x) = \lim_{h \to 0} \frac{g\,(f(x+hv)) - g(f(x))}{h} \quad (6.4.1.6)$$

and on the other hand

$$\begin{aligned}&(\nabla_\alpha g(f(x)))\,(v^a \nabla_a f^\alpha(x)) \\ &\quad = \lim_{h \to 0} \frac{1}{h} \left[g\,(f(x) + hv^a \nabla_a f(x)) - g(f(x)) \right]\end{aligned} \quad (6.4.1.7)$$

Thus, we want to show that

$$f(x+hv) = f(x) + hv^a \nabla_a f(x) \text{ as } h \to 0, \quad (6.4.1.8)$$

but of course this is just the very definition of Fréchet-differentiability. ∎

6.4.2 The Mean Value Theorem

One of the most important theorems of traditional calculus is of course the *Mean Value Theorem*. We first prove a couple of results, important in their own right, leading up to it.

Definition 6.4.2.1 — Local extrema Let $f: X \to \mathbb{R}$ be a function from a topological space into $\mathbb{R}$ and let $x_0 \in X$. Then, x_0 is a ***local maximum*** of f iff there is some open neighborhood U of x_0 such that (i) $f(x_0) = \sup_{x \in U}\{f(x)\}$ and (ii) x_0 is the *unique* such point in U.[a] x_0 is a ***local minimum*** of f iff there is some open neighborhood U of x_0 such that (i) $f(x_0) = \inf_{x \in U}\{f(x)\}$ and (ii) x_0 is the *unique* such point in U. x_0 is a ***local extremum*** of f iff it is either a local maximum or a local minimum.

[a] So for example, constant functions have no local maxima.

Proposition 6.4.2.2 — First Derivative Test Let $D \subseteq \mathbb{R}^d$, let $x_0 \in \mathrm{Int}(D)$, and let $f: D \to \mathbb{R}$ be Fréchet-differentiable at x_0. Then, if x_0 is a local extremum of f, then $\nabla_a f(x_0) = 0$.

R Sometimes this is also referred to as ***Fermat's theorem***. Evidently, it was not the last result of his life.

R The first derivative alone, however, cannot detect the difference between local minima and maxima. For that we need the *Higher Derivative Test*, a stronger version of Second Derivative Test, with which you are likely more familiar—see Theorem 6.5.59.

W This should fail for linear-differentiability—see Example 6.5.1.

Proof. Suppose that f is Fréchet-differentiable at x_0 and x_0 is a local extremum of f. Let U be an open neighborhood about x_0 so that x_0 is either a maximum or minimum in U. We proceed by contradiction: suppose that $\nabla_a f(x_0) \neq 0$. Then, there is some $v^a \in \mathrm{T}_{x_0}(\mathbb{R}^d)$ such that $v^a \nabla_a f(x_0) \neq 0$. Without loss of generality, suppose that $v^a \nabla_a f(x_0) > 0$. Then, there is some $h > 0$ such that $x_0 + hv, x_0 - hv \in U$ and

$$\frac{f(x_0 + hv) - f(x_0)}{h} =: K > 0 \text{ and } \frac{f(x_0 - hv) - f(x_0)}{-h} =: L > 0, \quad (6.4.2.3)$$

and so

$$f(x_0+hv) = f(x_0)+hK > f(x_0) \text{ and } f(x_0-hv) = f(x_0)-hL < f(x_0), \quad (6.4.2.4)$$

so that $f(x_0)$ can be neither a maximum nor a minimum in U: a contradiction. ■

In fact, there is a generalization of the First Derivative Test, which is essentially just the precise statement that allows one to use the method of *Lagrange multipliers*.

Theorem 6.4.2.5 — Lagrange Multiplier Theorem. Let $D \subseteq \mathbb{R}^d$, let $x_0 \in \mathrm{Int}(D)$, and let $f, g\colon D \to \mathbb{R}$ be Fréchet-differentiable at x_0 with $\nabla_a g(x_0) \neq 0$. Then, if $x_0 \in g^{-1}(0)$ is a local extremum of $f|_{g^{-1}(0)}$, then there is a unique $\lambda \in \mathbb{R}$ such that $\nabla_a f(x_0) = \lambda \nabla_a g(x_0)$.

R λ is the ***Lagrange multiplier***.

R f and g should be thought of as playing different roles here. In this situation, we are thinking of g as a 'constraint', and f itself as the thing we are maximizing or minimizing.[a] For example, if $g\colon \mathbb{R}^3 \to \mathbb{R}$ is defined by $g(x, y, z) := x^2 + y^2 + z^2 - 1$, then the "constraint" $g(x, y, z) = 0$ in this case is the condition that $x^2 + y^2 + z^2 = 1$, that is, the condition that $\langle x, y, z\rangle$ be on the unit sphere. This would be applicable if, for example, you are trying to find the maximum of f not on all of $\mathbb{R}^3$, but instead only on the unit sphere.

R When you turn to study manifolds, you will learn that the condition here that $\nabla_a g(x_0) \neq 0$ amounts to the statement that 0 is a "regular value" of g, whence it follows that the preimage of 0, $g^{-1}(0)$, is a submanifold.

R The basic idea here is that the conditions imply that $\nabla_a f(x_0)$ and $\nabla_a g(x_0)$ are both orthogonal to $\mathrm{T}_{x_0}(g^{-1}(0))$, and as the space of vectors orthogonal to this tangent space is only one-dimensional, $\nabla_a f(x_0)$ and $\nabla_a g(x_0)$ must be scalar multiples of one another.

[a]Well, of course the statement of the theorem doesn't care what you're trying to do, but in practice this is used when you're trying to maximize f subject to the constraint g.

Proof. We leave the proof as an exercise.

Exercise 6.4.2.6 Prove the result yourself.

Hint: Probably shouldn't be an exercise—see [Jr13, Theorem 5.8.3].

∎

Example 6.4.2.7 Suppose we want to maximize the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) \coloneqq x+y$ on the unit circle in $\mathbb{R}^2$. The relevant constraint function is $g\colon \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) \coloneqq x^2+y^2-1$.

The Lagrange Multiplier Theorem says that if $\langle x,y\rangle$ is to be a local maximum on the unit circle, then there must be a unique real number λ such that

$$\langle 1,1\rangle_a = \nabla_a f(x,y) = \lambda \nabla_a g(x,y) = \langle 2\lambda x, 2\lambda y\rangle_a. \tag{6.4.2.8}$$

Thus, we must have that $2\lambda x = 1 = 2\lambda y$. As $x^2+y^2=1$, it follows that $\frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}=1$, and so $\lambda = \pm\frac{1}{\sqrt{2}}$.[a] Plugging this back into the equations $2\lambda x = 1 = 2\lambda y$ gives us

$$\langle x,y\rangle = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle \text{ or } \langle x,y\rangle = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle. \tag{6.4.2.9}$$

[a]If you're confused about the lack of uniqueness here, this is because we don't yet know what $\langle x,y\rangle$ is. For a *given* $\langle x,y\rangle$, there is going to be a unique λ. However, at the moment we are *solving* for $\langle x,y\rangle$, and there is going to be more than one solution, and hence more than one Lagrange multiplier (one for each solution of $\langle x,y\rangle$).

Theorem 6.4.2.10 — Mean Value Theorem

. Let $f, g, h\colon [a,b] \to \mathbb{R}$ be differentiable,[a] and define $D\colon [a,b] \to \mathbb{R}$ by

$$D(x) := \det\left(\begin{bmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{bmatrix}\right). \tag{6.4.2.11}$$

Then, there is some $c \in (a,b)$ such that $\nabla D(c) = 0$.

R There are three successive special cases of this, listed in the following three remarks.

R Taking $h(x) := 1$ yields the following special case:

Let $f, g\colon [a,b] \to \mathbb{R}$ be differentiable. Then, there is some $c \in (a,b)$ such that

$$\nabla f(c)(g(b)-g(a)) = \nabla g(c)(f(b)-f(a)). \tag{6.4.2.12}$$

This is sometimes referred to as the ***Ratio Mean Value Theorem*** or ***Cauchy's Mean Value Theorem***.

R Taking $g(x) := x$ in Cauchy's Mean Value Theorem yields the following special case:

Let $f\colon [a,b] \to \mathbb{R}$ be differentiable Then, there is some $c \in (a,b)$ such that

$$\nabla f(c) = \frac{f(b)-f(a)}{b-a}. \tag{6.4.2.13}$$

This is usually referred to itself as just the ***Mean Value Theorem***, in which case the most general version is referred to as the ***Determinant Mean Value Theorem***. If we want to distinguish (and it's not immediately obvious from context), to clarify, we will refer to this case as the ***Classical Mean Value Theorem***.

R If $f(a) = f(b)$ in the Classical Mean Value Theorem, then we obtain the following special case:

Let $f\colon [a,b] \to \mathbb{R}$ be differentiable. Then, if $f(a) = f(b)$, then there is some $c \in (a,b)$ such that

$$\nabla f(c) = 0. \tag{6.4.2.14}$$

This special case is known as ***Rolle's Theorem***.

R In the Ratio Mean Value Theorem, if $g(a) \neq g(b)$ and $\nabla g(c) \neq 0$, then we can rearrange (6.4.2.12) to read

$$\frac{\nabla f(c))}{\nabla g(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \tag{6.4.2.15}$$

R The right-hand side of (6.4.2.13) is called the ***average slope*** of f on $[a, b]$. Hence the name *Mean* Value Theorem.

R Intuitively, the Classical Mean Value Theorems says that if over a given period of time your *average speed* was v, then in fact at some point your *instantaneous* speed must have been v. Rolle's Theorem can of course be understood as the special case of this in which $v = 0$.

You can likewise interpret the Ratio Mean Value Theorem as saying something similar for the curve $t \mapsto \langle f(t), g(t)\rangle$. On the other hand, I am not aware of any intuition (or use, for that matter) of the Determinant Mean Value Theorem.

R This essentially does not generalize at all to higher dimensions. It just fails outright if you increase the dimension of the codomain—see Example 6.4.2.16. On the other hand, there is the question of what should the statement of the Mean Value Theorem even be if you increase the dimension of the domain—what is the average slope over a square, for example? What you can do is state the result in terms of curves between points in $\mathbb{R}^d$, but this is really just the one-dimensional statement here for the composition of the function with the curve (which of course itself is a function from a closed interval in $\mathbb{R}$ to $\mathbb{R}^d$).

---

[a] In your calculus class, you may have seen it required that f, g, h are continuous on $[a, b]$ and differentiable on (a, b). However, recall that (see the definition, Meta-definition 6.2.1.10) a function being differentiable means that it is differentiable on a neighborhood of the domain. In particular, by Proposition 6.5.8, our hypothesis implies that f, g, h are all continuous on $[a, b]$.

Proof. D is continuous on $[a, b]$, and therefore, by the Extreme Value Theorem, attains a maximum and a minimum somewhere on $[a, b]$. Suppose that both the maximum and minimum were achieved at the endpoints. As $D(a) = 0 = D(b)$, this would imply that 0 was both a minimum and a maximum for D on $[a, b]$, which in turn would imply that $D(x) = 0$ for all $x \in [a, b]$. In particular, $\nabla D(c) = 0$ for any $c \in (a, b)$. On the other hand, if at least one of the maximum and the minimum where attained at $c \in (a, b)$, then it would be a local extremum, and so by the First Derivative Test, we would have that $\nabla D(c) = 0$. ∎

▪ Example 6.4.2.16 — A function differentiable on $[a, b]$ into $\mathbb{R}^2$ for which the Mean Value Theorem fails Define $f_1, f_2 : [-1, 1] \to \mathbb{R}$ by

$$f_1(x) := x^2 \text{ and } f_2(x) := x^3 \tag{6.4.2.17}$$

and $f : [-1, 1] \to \mathbb{R}^2$ by

$$f(x) := \langle f_1(x), f_2(x) \rangle . \tag{6.4.2.18}$$

We show there is no $c \in (-1, 1)$ such that

$$\nabla f(c) = \frac{f(1) - f(-1)}{1 - (-1)} . \tag{6.4.2.19}$$

$f_1(-1) = 1$ and $f_1(1) = 1$, and on the other hand, $f_2(-1) = -1$ and $f_2(1) = 1$. Therefore, the right-hand side of this equation becomes

$$\frac{f(1) - f(-1)}{1 - (-1)} = \langle 0, 1 \rangle . \tag{6.4.2.20}$$

If (6.4.2.19) were to hold, then we must have that $f_1'(c) = 0$, which forces $c = 0$. But then, $f_2'(c) = 0 \neq 1$.

Exercise 6.4.2.21 Let $f\colon [a,b] \to \mathbb{R}$ be differentiable. Show that the following are true.

(i). If $\nabla f(c) = 0$ for all $c \in (a,b)$, then f is constant on $[a,b]$.
(ii). if $\nabla f(c) > 0$ for all $c \in (a,b)$, then f is increasing on $[a,b]$.
(iii). If $\nabla f(c) < 0$ for all $c \in (a,b)$, then f is decreasing on $[a,b]$.

Be careful though: we can deduce nothing if the derivative is positive at a point alone.

Example 6.4.2.22 — A point at which the derivative is positive but has no neighborhood in which the function is increasing Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} x - \frac{1}{4} & \text{if } x \in \mathbb{Q} \\ x^2 & \text{if } x \in \mathbb{Q}^{\mathrm{C}}. \end{cases} \tag{6.4.2.23}$$

We first show that the derivative at $\frac{1}{2}$ is 1.

$$\begin{aligned} \frac{f(\frac{1}{2}+\varepsilon) - f(\frac{1}{2})}{\varepsilon} &= \begin{cases} \frac{\left((\frac{1}{2}+\varepsilon)-\frac{1}{4}\right)-\frac{1}{4}}{\varepsilon} & \text{if } \varepsilon \in \mathbb{Q} \\ \frac{\left(\frac{1}{2}+\varepsilon\right)^2-\frac{1}{4}}{\varepsilon} & \text{if } x \in \mathbb{Q}^{\mathrm{C}} \end{cases} \\ &= \begin{cases} 1 & \text{if } \varepsilon \in \mathbb{Q} \\ 1+\varepsilon & \text{if } \varepsilon \in \mathbb{Q}^{\mathrm{C}}. \end{cases} \end{aligned} \tag{6.4.2.24}$$

Hence, we do indeed have that

$$\lim_{\varepsilon \to 0^+} \frac{f(\frac{1}{2}+\varepsilon) - f(\frac{1}{2})}{\varepsilon} = 1 > 0. \tag{6.4.2.25}$$

On the other hand, let U be any open neighborhood of $\frac{1}{2}$ and let $\varepsilon > 0$ be *rational* and such that $\varepsilon + \frac{1}{2} \in U$. Now choose $0 < \delta < \varepsilon$ *irrational* and such that $\delta + \delta^2 > \varepsilon$.[a] Then, despite the fact that $\frac{1}{2} + \delta < \frac{1}{2} + \varepsilon$, we nevertheless have that

$$f(\tfrac{1}{2}+\delta) = (\tfrac{1}{2}+\delta)^2 = \tfrac{1}{4}+\delta+\delta^2 > \tfrac{1}{4}+\varepsilon = f(\tfrac{1}{2}+\varepsilon). \quad (6.4.2.26)$$

[a] Why does such a δ exist?

The (Ratio) Mean Value Theorem has an important 'corollary'[12] that you probably recall from elementary calculus: ***L'Hôpital's Rule***.

Theorem 6.4.2.27 — L'Hôpital's Rule. Let $f, g\colon (a,b) \to \mathbb{R}$ be differentiable. Then, if (i) $g(x), \nabla g(x) \neq 0$ for all x in some neighborhood of b, (ii) if $\lim_{x\to b}|f(x)| = 0 = \lim_{x\to b}|g(x)|$ or $\lim_{x\to b}|g(x)| = \infty$, and (iii) the limit $\lim_{x\to b}\frac{\nabla f(x)}{\nabla g(x)}$ exists, then

$$\lim_{x\to b}\frac{f(x)}{g(x)} = \lim_{x\to b}\frac{\nabla f(x)}{\nabla g(x)}. \quad (6.4.2.28)$$

R Note that L'Hôpital's Rule is applicable even if you only have $\lim_{x\to b}|g(x)| = \infty$, that is, you do *not* need $\lim_{x\to b}|f(x)| = \infty$. It's just that, in this case, L'Hôpital's Rule is not necessary as we already know the limit in this case is 0.

Proof. Suppose that (i) $g(x), \nabla g(x) \neq 0$ for all x in some neighborhood of b, (ii) if $\lim_{x\to b}|f(x)| = 0 = \lim_{x\to b}|g(x)|$ or $\lim_{x\to b}|g(x)| = \infty$, and (iii) the limit $\lim_{x\to b}\frac{\nabla f(x)}{\nabla g(x)}$ exists.

For the remainder of the proof, let us replace a with a value sufficiently close to b such that $g(x), \nabla g(x) \neq 0$ for all $x \in (a,b)$. It follows that, if $x, y \in (a,b)$ are distinct, then $g(x) - g(y) \neq 0$, because otherwise the Mean Value Theorem would imply that $\nabla g(x)$ vanishes somewhere.

[12] In quotes because it's not *that* immediate.

Let us write

$$L := \lim_{x \to b} \frac{\nabla f(x)}{\nabla g(x)}. \tag{6.4.2.29}$$

Now, for every $x \in (a, b)$, define

$$m(x) := \inf_{t \in (x,b)} \frac{\nabla f(t)}{\nabla g(t)} \in [-\infty, \infty) \tag{6.4.2.30}$$

$$M(x) := \inf_{t \in (x,b)} \frac{\nabla f(t)}{\nabla g(t)} \in (-\infty, \infty] \tag{6.4.2.31}$$

Note that $\lim_{x \to b} m(x) = L = \lim_{x \to b} M(x)$ by Proposition 2.4.3.31 ($\limsup$ and $\liminf$ agree with the limit if it exists). Thus, by the Squeeze Theorem (for functions) (Exercise 2.5.1.9), it suffices to show that $m(x) \leq \frac{f(x)}{g(x)} \leq M(x)$ for all $x \in (a, b)$.

For every $y \in (x, b)$, the Ratio Mean Value Theorem says that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{\nabla f(c)}{\nabla g(c)} \tag{6.4.2.32}$$

for some $c \in (x, y)$. It follows that

$$m(x) \leq \frac{f(x) - f(y)}{g(x) - g(y)} \leq M(x) \tag{6.4.2.33}$$

for all $y \in (x, b)$.[a]

If $\lim_{y \to b} |f(y)| = 0 = \lim_{y \to b} |g(y)|$, then taking the limit of (6.4.2.32) as $y \to b$ gives $m(x) \leq \frac{f(x)}{g(x)} \leq M(x)$, as desired.

So, suppose that $\lim_{y \to b} |g(y)| = \infty$. We have that

$$m(x) \leq \frac{f(x) - f(y)}{g(x) - g(y)} = \frac{\frac{f(y)}{g(y)} - \frac{f(x)}{g(y)}}{1 - \frac{g(x)}{g(y)}} \leq M(x), \tag{6.4.2.34}$$

and so in this case we have

$$L = \limsup_{x \to b} m(x) \leq \limsup_{x \to b} \frac{f(x)}{g(x)} \leq \limsup_{x \to b} M(x) = L.^{b}$$

It follows that $\limsup_{x\to b}\frac{f(x)}{g(x)} = L$. Similarly, $\liminf_{x\to b}\frac{f(x)}{g(x)} = L$, and hence $\lim_{x\to b}\frac{f(x)}{g(x)}$ exists and is equal to L by Proposition 2.4.3.31. ■

[a] The basic idea of the entire proof was that (i) it suffices to show that $m(x) \leq \frac{f(x)}{g(x)} \leq M(x)$ and (ii) this holds by the Ratio Mean Value Theorem and the fact that $f(y)$ and $g(y)$ are 'negligible'. Thus, the remainder of the proof boils down to showing that $f(y)$ and $g(y)$ are 'negligible' in an appropriate sense.

[b] We must use lim sup becasue we do not know a priroi that this limit exists.

We mentioned in a remark of the Intermediate Value Theorem (Theorem 3.8.1.28) that functions which are derivatives also possess the intermediate value property, even though they need not be continuous. This is another 'corollary' of the Mean Value Theorem, and is known as ***Darboux's Theorem***.

Theorem 6.4.2.35 — Darboux's Theorem. . Let $D \subseteq \mathbb{R}$ be connected and let $f\colon D \to \mathbb{R}$ be differentiable. Then, $\nabla f\colon \operatorname{Int}(D) \to \mathbb{R}$ has the intermediate value property.

R Because of this result, sometimes functions which possess the intermediate value property are called ***Darboux continuous***. However, Darboux continuous functions need not actually be continuous (Exercise 6.4.2.37), and so I recommend to just use "intermediate value property" instead (this term is also just more descriptive).

Proof. We leave this as an exercise.

Exercise 6.4.2.36 Prove this yourself.

R Hint: Another exercise that probably should not be. Check out [Pug02, pg. 144].

■

Exercise 6.4.2.37 Find an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ that has the intermediate value property but is not continuous.

R You probably did something similar in Exercise 3.8.1.32. The difference is that now you are being asked to find such a function in $\mathbb{R}$. Before, you could 'cheat' by using other topological spaces.

Exercise 6.4.2.38 Is this true in $\mathbb{R}^d$ for $d \geq 2$? More precisely, is it the case that $\mathbb{R}^d \ni x \mapsto v(x)^a \nabla_a f(x)$ has the intermediate value property for all smooth vector fields $x \mapsto v(x)^a$?

6.4.3 The Fundamental Theorem of Calculus

So there is this theorem. It's kind of important. Maybe you've even heard of it. It's called the ***Fundamental Theorem of Calculus*** and it asserts that

$$f(x) = \int_a^x \mathrm{d}x\, \tfrac{\mathrm{d}}{\mathrm{d}x} f(x). \tag{6.4.3.1}$$

Well, sort of. If we're not careful about what we mean, this can completely fail.. That's right—your calculus teachers *lied to you*. You should sue them.[13]

▪ Example 6.4.3.2 — A uniformly-continuous function on $[0,1]$ that is 0 at 0, 1 at 1, but whose derivative is 0 almost-everywhere—the Devil's Staircase We've seen this bad boy before—see Example 5.1.5.54.[a] The construction is relatively long, and so we don't repeat it here, but if it helps you remember, it's that function which is constant on the components of the complement of the Cantor Set in $[0,1]$. In fact, as the Cantor Set has measure 0, it follows from this fact that the derivative of the Devil's Staircase is 0 almost-everywhere. Hence, on

[13] I'm exaggerating for dramatic effect. (So dramatic, isn't it?) When you first took calculus, the idea that a derivative would be defined only almost-everywhere was probably just not a thing.

one hand we have that $\int_0^1 \mathrm{d}x \, \frac{\mathrm{d}}{\mathrm{d}x} f(x) = 0$, but on the other hand we have $f(1) - f(0) = 1$. Turns out that a lot can happen on a set of measure 0!

In fact, you can modify this by adding x to obtain a *uniform-homeomorphism* (the Cantor Function—see the same example, Example 5.1.5.54) which has derivative 1 almost-everywhere, but yet goes from 0 to *not* 1, but 2, on $[0, 1]$. The point is, you can have *ridiculously nice* functions for which the Fundamental Theorem of Calculus fails if the derivative is only required to exist almost-everywhere.

[a] There, it was used to generate an example of a uniform-homeomorphism of $\mathbb{R}$ that preserves neither measurability not measure 0.

Actually, this shouldn't come as that much of a surprise.[14] For one thing, pretty much everything you integrate in standard calculus classes is a polynomial, a trig function, an exponential function, an inverse of one of these, or a function formed from these by addition multiplication etc.. That is, you only deal with incredibly nice functions all the time. While the Devil's Staircase is actually relatively nice as far as all continuous functions go (it's uniformly-continuous), it certainly isn't smooth everywhere.

Also, it's worth mentioning that we cheated a bit for these counter-examples: the Devil's Staircase is *not* differentiable on $[0, 1]$—it is only differentiable almost-everywhere. It's important that we allow this, however. Remember, we want to go *both* ways: given an integrable function, we want to use the integral to (hopefully) generate a function that is an antiderivative[15]; and conversely given a differentiable function, we want to use the integral to (hopefully) recover the original function as the integral of the derivative. The point is that, in one of these cases, the input is an integrable function which gets stuck inside an integral, in which case things only matter *up to measure zero*.

[14] By now, you should pretty much expect many things that seem to be completely reasonable to be utterly false.

[15] That is, a function that has the property that when you differentiate it, you get back the original function.

To clarify: if $F(x) \coloneqq \int_a^x \mathrm{d}t\, f(t)$, we could never hope to have that $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$ for *all* x[16]—instead, we can only hope this to be true almost-everywhere. Fortunately enough, this much is true.

Theorem 6.4.3.3 — Fundamental Theorem of Calculus—Part I. Let $f\colon [a,b] \to \mathbb{R}$ be integrable. Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \mathrm{d}t\, f(t) = f(x) \tag{6.4.3.4}$$

for almost-all $x \in [a,b]$.

R In other words, "The derivative of the integral is the original function.".

Proof. We leave this an an exercise for the reader.

Exercise 6.4.3.5 Prove this yourself.

R Hint: This is definitely in the category of "This really shouldn't be an exercise, but it is anyways because Jonny ran out of time.". Pugh ([Pug02]) actually proves something stronger than this on pg. 396, so your hint is to look there.

■

In the direction in which your ''input' is a differentiable function, of course, there is (seemingly) nothing to worry about because we are a priori assuming that the function is differentiable *everywhere* (not just almost-everywhere), and in this case, we have the following.

Theorem 6.4.3.6 — Fundamental Theorem of Calculus—Part II. Let $f\colon [a,b] \to \mathbb{R}$ be differentiable and let $x \in [a,b]$.

[16] Take any function f, any point $x \in \mathbb{R}$, and any 'value' $y \in \mathbb{R}$—you can redefine f to be equal to y at x, and you will have no effect on its integral, so that $F(x)$ will not change, but $f(x)$ will, and so we will no longer have that $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$.

Then, if $t \mapsto \frac{\mathrm{d}}{\mathrm{d}t} f(t)$ is integrable on (a, b), then

$$f(x) - f(a) = \int_a^x \mathrm{d}t\, \tfrac{\mathrm{d}}{\mathrm{d}t} f(t). \tag{6.4.3.7}$$

R In other words, "The integral of the derivative is the original function.".

R In particular, the following is true: Given $f\colon (a, b) \to \mathbb{R}$ integrable and $F\colon [a, b] \to \mathbb{R}$ *any* differentiable function that satisfies $\frac{\mathrm{d}}{\mathrm{d}x} F(x) = f(x)$ for all $x \in (a, b)$, then $\int_a^b \mathrm{d}t\, f(t) = F(b) - F(a)$. This is how it is most often used in calculus—you find some such F and then compute $F(b) - F(a)$ to compute the integral.

R If $\frac{\mathrm{d}}{\mathrm{d}x} F(x) = f(x)$, then F is an ***antiderivative*** or ***primitive*** of f. The practical use of this result, as described in the previous remark, is that you can compute integrals by finding an antiderivative of the integrand.

Proof. [a] Suppose that $x \mapsto \frac{\mathrm{d}}{\mathrm{d}x} f(x)$ is integrable on (a, b). Without loss of generality, assume that $x = b$. Let $\varepsilon > 0$. Then, by Carathéodory-Vitali Theorem (Theorem 5.2.3.95), there is a lower-semicontinuous function $g\colon (a, b) \to \mathbb{R}$ such that (i) $\frac{\mathrm{d}}{\mathrm{d}t} f \leq g$ and (ii)

$$\int_a^b \mathrm{d}t\, g(t) < \int_a^b \mathrm{d}t\, \tfrac{\mathrm{d}}{\mathrm{d}t} f(t) + \varepsilon. \tag{6.4.3.8}$$

By replacing g with $g + \delta$ for sufficiently small δ, we may without loss of generality assume that $\frac{\mathrm{d}}{\mathrm{d}t} f < g$. For $\eta > 0$, define $F_\eta : [a, b] \to \mathbb{R}$ by

$$F_\eta(x) := \int_a^x \mathrm{d}t\, g(t) - f(x) + f(a) + \eta(x - a). \tag{6.4.3.9}$$

By lower-semicontinuity and the definition of the derivative, for each $x \in [a, b)$, there is some $\delta_x > 0$ such that

$$g(t) > \tfrac{\mathrm{d}}{\mathrm{d}t} f(x) \text{ and } \frac{f(t) - f(x)}{t - x} < \tfrac{\mathrm{d}}{\mathrm{d}t} f(x) + \varepsilon ta \quad (6.4.3.10)$$

for all $t \in (x, x + \delta_x)$. Thus, for $t \in (x, x + \delta_x)$, we have

$$\begin{aligned} F_\eta(t) - F_\eta(x) &= \int_x^t \mathrm{d}s\, g(s) - [f(t) - f(x)] + \eta(t - x) \\ &> (t - x)\tfrac{\mathrm{d}}{\mathrm{d}t} f(x) - (t - x)\left[\tfrac{\mathrm{d}}{\mathrm{d}t} f(x) + \eta\right] + \eta(t - x) \\ &= 0. \end{aligned}$$

Define

$$x_\eta := \sup\{t \in [a, b] : F_\eta(t) = 0\}. \quad (6.4.3.11)$$

By continuity, $F_\eta(x_\eta) = 0$, It then follows from the previous inequality that $F_\eta(b) \geq 0$. As η is arbitrary, it follows (6.4.3.9) that

$$f(b) - f(a) \leq \int_a^b \mathrm{d}t\, g(t) < \int_a^b \mathrm{d}t \tfrac{\mathrm{d}}{\mathrm{d}t} f(t) + \varepsilon, \quad (6.4.3.12)$$

and so

$$f(b) - f(a) \leq \int_a^b \mathrm{d}t\, \tfrac{\mathrm{d}}{\mathrm{d}t} f(t). \quad (6.4.3.13)$$

Replacing everywhere in this argument f with $-f$ gives us

$$-f(b) - (-f(a)) \leq -\int_a^b \mathrm{d}t\, \tfrac{\mathrm{d}}{\mathrm{d}t} f(t). \quad (6.4.3.14)$$

Combining these two inequalities, we obtain

$$f(b) - f(a) = \int_a^b \mathrm{d}t,\, \tfrac{\mathrm{d}}{\mathrm{d}t} f(t). \quad (6.4.3.15)$$

■

[a] Proof adapted from [Rud87, pg. 149].

Though we obviously can't yield the full computational power of this theorem you probably remember as we don't yet have in hand most of the functions that likely pervaded your calculus course, we do have access to one type of function that we can show an example with: polynomials.

Example 6.4.3.16 Define

$$D := \{\langle x, y\rangle \in [0, 1] \times [0, 1] : y \leq x\} \tag{6.4.3.17}$$

and $f : D \to \mathbb{R}$ by $f(x, y) := x^3 y$. Let us attempt to compute $\int_D \mathrm{d}\langle x, y\rangle\, f(x, y)$.

The Fundamental Theorem of Calculus is fantastic, but it really is a one-dimensional result. To reduce the computation of this integral to one-dimension, we apply Fubini's Theorem (Theorem 5.2.3.27).

We first verify the hypotheses of Fubini's Theorem. As $D \subseteq [0, 1] \times [0, 1]$, f is continuous, and $[0, 1] \times [0, 1]$ is quasi-compact, by the Extreme Value Theorem (Theorem 3.8.2.2), f is bounded on D, and so, as D has finite measure, f is integrable on D.

One problem still remains: Fubini's Theorem only tells us how to integrate over *rectangles* (i.e. sets of the form $S_1 \times S_2$). To get around this, we extend f to all of $[0, 1] \times [0, 1]$ so that it is 0 on $([0, 1] \times [0, 1]) \setminus D$. Then, we use the fact that $\int_D \mathrm{d}\langle x, y\rangle\, f(x, y) = \int_{[0,1]\times[0,1]} \mathrm{d}\langle x, y\rangle\, \chi_D(x, y) f(x, y)$. Note that, for a fixed $x \in [0, 1]$, $\{y \in [0, 1] : \langle x, y\rangle \in D\} = [0, x]$.

Hence,

$$
\begin{aligned}
\int_D \mathrm{d}\langle x,y\rangle\, f(x,y) \\
&= \int_{[0,1]\times[0,1]} \mathrm{d}\langle x,y\rangle\, \chi_D(x,y) f(x,y) \\
&= \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, \chi_{[0,x]}(y) f(y) \\
&=: \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\, x^3 y \\
&= \int_0^1 \mathrm{d}x\, x^3 \left(\tfrac{1}{2} y^2\big|_0^x\right) \\
&= \tfrac{1}{2}\int_0^1 \mathrm{d}x\, x^5 \\
&= \tfrac{1}{2}\left(\tfrac{1}{6}x^6\big|_0^1\right) \\
&= \tfrac{1}{12}.
\end{aligned}
\tag{6.4.3.18}
$$

R One thing to take note of that is not stressed in elementary multivariable calculus courses is the distinction between the "double integral" and the "iterated integrals". The "double integral", $\int_D \mathrm{d}\langle x,y\rangle\, f(x,y)$, is (usually) the thing you are ultimately interested in computing—it is of course just a *single* integral, just generally in higher dimensions. To compute such integrals, you apply Fubini to reduce the problem to the computation of several single-variable integrals, that is, the "iterated integrals". In this case, the "iterated integrals" are *tools* used in the calculation of the thing you ultimately care about, the "double integral".

6.4.4 Change of variables

I think it's fair to say that the Fundamental Theorem of Calculus is the chief tool of computation in elementary calculus. It does not work alone, however. No doubt you'll recall many tricks for 'massaging' integrals into a form that is easier, or just plain possible, to compute. One of the first such tools one learns to aid in the computation of integrals you probably know as *u-substitution*. This is just the one-dimensional version of the more general Change of Variables Theorem you are likewise probably familiar with.

Theorem 6.4.4.1 — Change of Variables Theorem. Let $U \subseteq \mathbb{R}^d$ be open, let $\phi \colon U \to \phi(U)$ be a C^1 diffeomorphism onto its image, and let $f \colon \phi(U) \to \mathbb{R}$ be integrable. Then,

$$\int_{\phi(U)} \mathrm{d}x\, f(x) = \int_U \mathrm{d}x \left|\det\left(\nabla_a \phi(x)^b\right)\right| f(\phi(x)). \quad (6.4.4.2)$$

R A C^k ***diffeomorphism*** is a function f such that (i) f is bijective, (ii) f is C^k, and (iii) f^{-1} is C^k—see Definition 6.3.8. Diffeomorphisms wind-up being the isomorphisms in the category of manifolds, but as we don't touch the category of manifolds ourselves, we leave this definition 'unofficial' and only in a remark.

R The derivative $\mathrm{T}_x(\mathbb{R}^d) \ni v^a \mapsto v^b \nabla_b f(x)^a \in \mathrm{T}_{f(x)}(\mathbb{R}^d)$ is a linear-transformation, and so it makes sense to take its determinant. In fact, the determinant of the derivative has a name—the ***Jacobian***.

R Recall that (Exercise 5.1.5.17) $\mathrm{m}(T(S)) = |\det(T)|\,\mathrm{m}(S)$ for $S \subseteq \mathbb{R}^d$ and $T \colon \mathbb{R}^d \to \mathbb{R}^d$ a linear-transformation. In this sense, the absolute value of the determinant is a measure[a] of how much a linear-transformation scales measure. It thus makes sense that the absolute value of the Jacobian would make an appearance here.

R Note that you need ϕ to be *bijective* (onto its image). My impression is that this point is not typically stressed in elementary calculus courses, especially in single variable calculus. You have to be careful however. For example, note what happens if you try to naively substitute $u := x^2$ into the integral $\int_{-1}^{1} \mathrm{d}x\, x^2$—even if you notice that it is wrong to change both limits to $1^2 = 1 = (-1)^2$, you should still get the wrong answer.

[a] No pun intended. Seriously—the term "measure" here is not being used in a technical sense.

Proof. We leave this as an exercise.

Exercise 6.4.4.3 Prove this yourself.

R Hint: This should not be an exercise, but see [Jr13, Theorem 6.7.1].

■

6.4.5 Functions defined by their derivatives: exp, sin, and cos

So, we've proven several properties about how to manipulate derivatives, but what is there to differentiate? Polynomials? That's no fun. Let's find a function even easier to differentiate. In fact, let's see if we can find a function that is equal to its own derivative

$$\tfrac{\mathrm{d}}{\mathrm{d}x} f(x) = f(x). \tag{6.4.5.1}$$

Of course, you already know the answer—or so you think you do. Can you define $\exp(x)$? For what it's worth, you do have the tools to do so at this point, but it's quite likely that someone told you once upon a time that $\mathrm{e} \approx 2.718\ldots$ and then $\exp(x) := \mathrm{e}^x$. If it's not clear to you at this point that this is just complete and utter nonsense, then apparently I'm not very good at writing mathematical exposition.[17] What you could do, however, is define exp by its power-series, $\exp(x) := \sum_{m\in M} \frac{x^m}{m!}$, but where did that formula come from? The proper way to define the exponential function is that it is the

[17] Or maybe you're just stupid. (In case it's not obvious, I'm not being serious. . . .)

unique function from $\mathbb{R}$ to $\mathbb{R}$ that (i) is equal to its own derivative and (ii) is 1 at 0.[18] Of course, as always, we can't just go around asserting things like this exist willy-nilly. Who can even comprehend the chaos that might ensue? We must *prove* that such a thing exists, and that only one such thing exists. The tool that will allow us to do this is a theorem (Theorem 6.4.5.15) concerning the existence and uniqueness of (ordinary) differential equations. This result doesn't just assert the existence of any old functions, however. It asserts the existence of *analytic* functions, and so before we get to the result about solutions to differential equations, we first briefly investigate analytic functions.

Analytic functions

.

Definition 6.4.5.2 — Radius of convergence Let $x_0 \in \mathbb{R}^d$ and for $m \in \mathbb{N}$, let $[c_m]_{a_1 \cdots a_m}$ be a rank m covariant tensor. Then, the ***radius of convergence*** of the ***power series***

$$\sum_{m\in\mathbb{N}} [c_m]_{a_1\cdots a_m}(x-x_0)^{a_1}\cdots(x-x_0)^{a_m}, \tag{6.4.5.3}$$

is

$$\sup\left\{|x-x_0| : \text{the series converges absolutely for } x\in\mathbb{R}^d.\right\}.$$

In one dimension, we can say a bit more about the radius of convergence.

Proposition 6.4.5.4 — Cauchy-Hadamard Theorem The radius of convergence of the power series

$$\sum_{m\in\mathbb{N}} c_m(x-x_0)^m \tag{6.4.5.5}$$

[18]The unique function that that is equal to its own derivative and is equal to 0 at 0 is the function that is everywhere 0. 1 is the next most obvious choice, as opposed to, say $\sqrt{\pi}$.

is the unique real number r_0 such that (i) the series converges absolutely for all $x \in \mathbb{R}$ with $|x - x_0|$ and (ii) diverges for all $x \in \mathbb{R}^d$ with $|x - x_0| > r_0$.

Furthermore,

$$r_0 = \frac{1}{\limsup_m |c_m|^{1/m}} \tag{6.4.5.6}$$

R Note that in a lot of examples, the sequence $m \mapsto |c_m|^{1/m}$ will just simply converge, in which case this lim sup just comes a lim (Proposition 2.4.3.31):

$$r_0 = \frac{1}{\lim_m |c_m|^{1/m}}. \tag{6.4.5.7}$$

The point is, if you're the type of person that gets terrified at the sight of a lim sup (or lim inf), don't.

W Warning: Anything may happen if $|x - x_0| = r$—see the following exercise (Exercise 6.4.5.9).

Proof. We leave this as an exercise.

Exercise 6.4.5.8 Prove this yourself. Hint: Use the Root Test.

∎

Exercise 6.4.5.9 In the all of the following three examples, show that the radius of convergence is 1. Furthermore, show that

(i). $\sum_{m \in \mathbb{N}} x^m$ doesn't converge if $|x| = 1$;

(ii). $\sum_{m \in \mathbb{N}} \frac{x^m}{m}$ diverges for $x = 1$ and converges for $x = -1$; and

(iii). $\sum_{m \in \mathbb{N}} \frac{x^m}{m^2}$ converges absolutely for $|x| = 1$.

Definition 6.4.5.10 — Taylor series Let $D \subseteq \mathbb{R}^d$, let $x_0 \in \mathrm{Int}(D)$, and let $f\colon D \to \mathbb{R}$ be infinitely-differentiable at x_0. Then, the ***Taylor series*** of f at x_0 is the function $B_r(x_0) \to \mathbb{R}$ defined by

$$x \mapsto \sum_{m\in\mathbb{N}} \frac{\nabla_{a_1}\cdots\nabla_{a_m} f(x_0)}{m!}(x-x_0)^{a_1}\cdots(x-x_0)^{a_m}, \quad (6.4.5.11)$$

where r is the radius of convergence.

R If $x_0 = 0$, then this is the ***Maclaurin series*** of f.

R Every smooth function has a Taylor series. Not every smooth function is the limit of its Taylor series—see Exercise 6.4.5.42.

As mentioned in the remark, not every smooth function is the limit of its Taylor series. In fact, there is a name for such functions which *do* possess this property.

Definition 6.4.5.12 — Analytic Let $D \subseteq \mathbb{R}^d$, let $x \in \mathrm{Int}(D)$, and let $f\colon D \to \mathbb{R}$. Then, f is ***analytic*** at x_0 iff (i) f is infinitely-differentiable at x_0 and (ii) there is a neighborhood U of x_0 on which f is the limit of its Taylor series at x in $C^\infty(U)$.

f is ***analytic*** (on D) iff there is an open neighborhood $U \supseteq D$ and a $g\colon U \to \mathbb{R}$ such that g is analytic at x for all $x \in U$ and $f = g|_D$.

f is ***globally-analytic*** iff (i) f is analytic; and (ii) for every analytic extension g of f to an open neighborhood $U \supseteq D$ of D and for every $x_0 \in \mathrm{Int}(D)$, g is the limit of its Taylor series at x_0 in $C^\infty(U)$.

R The difference between analytic and globally-analytic is that, in order to be globally-analytic, the Taylor series at every point has to converge to the function on the *entire domain*. For example, the function $\mathbb{R} \setminus \{1\} \ni x \mapsto \frac{1}{1-x}$ is analytic, but not globally-analytic: its Taylor series at 0 is $1 + x + x^2 + \cdots$, which does not converge on all of $\mathbb{R} \setminus \{1\}$. On the other hand, we will see that exp *is* globally-analytic.

R Recall that there is a difference between smooth and infinitely-differentiable (see one of the remarks in the definition of differentiability, Meta-definition 6.2.1.19). Analytic functions are in fact smooth, because polynomials are smooth and analytic functions are limits of polynomials[a] in C^∞.

R As was mentioned above in the definition of Taylor series themselves, not all smooth functions are analytic—see Exercise 6.4.5.42.

R This is actually one *huge* difference between real and complex analysis. When you study complex analysis, you will find that *all* (complex) differentiable functions are analytic.

R The first condition (infinite-differentiability) is really just imposed so that the Taylor series itself makes sense.

[a] Namely the partial sums of their Taylor series.

You'll recall from calculus that a lot of the functions you worked with (e.g. ln, arctan, etc.) were defined as inverses to other functions. It will thus be of interest to us in order to know that inverses of analytic functions are analytic.

Proposition 6.4.5.13 Let $D \subseteq \mathbb{R}$, let $x \in \mathrm{Int}(D)$, and let $f\colon D \to \mathbb{R}$ be injective. Then, if f is analytic at x and $\nabla_a f(x) \neq 0$, then $f^{-1}\colon f(D) \to \mathbb{R}$ is analytic at $f(x)$.

W Warning: This will fail if $\nabla_a f(x) = 0$. For example, $\mathbb{R} \ni x \mapsto x^3 \in \mathbb{R}$ is analytic and bijective but its inverse is not even differentiable at 0.

Warning: Even if f is bijective with $\nabla_a f(x) \neq 0$ for all $x \in \mathbb{R}$, f globally-analytic does not imply that f^{-1} is globally-analytic. For example, we will see that exp is bijective, $\frac{\mathrm{d}}{\mathrm{d}x}\exp(x) \neq 0$ for all $x \in \mathbb{R}$, and exp is globally-analytic, but yet its inverse, ln, is not globally-analytic, because, for example, its Taylor series at $x = 1$ is $(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots$, which does not converge on all of $\mathbb{R}^+$.

Proof. We leave this an an exercise.

Exercise 6.4.5.14 Prove the result yourself.

■

And now we return to the result concerning the existence and uniqueness of solutions to differential equations.

Theorem 6.4.5.15 — Existence and uniqueness of linear constant coefficient ODEs. Let $A^a{}_b : \mathbb{R}^d \to \mathbb{R}^d$ be a nonzero linear map, let $t_0 \in \mathbb{R}$, and let $[\gamma_0]^a \in \mathbb{R}^d$. Then, there exists a unique differentiable function $\gamma^a : \mathbb{R} \to \mathbb{R}^d$ such that

(i). $\gamma^a(t_0) = [\gamma_0]^a$; and

(ii).

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)^a = A^a{}_b\gamma(t)^b. \tag{6.4.5.16}$$

Furthermore, this function γ^a is globally-analytic.

Proof. Step 1: Reduce the proof to the case $t_0 = 0$

Suppose we have proven the result for the case in which $t_0 = 0$. Then, if γ^a is the unique solution with $\gamma(0)^a = [\gamma_0]^a$, then $\delta(t)^a := \gamma(t-t_0)^a$ will be the unique solution with $\delta(t_0)^a = [\gamma_0]^a$.

STEP 2: SHOW SMOOTHNESS GIVEN EXISTENCE
For the proof, we do not make use of index notation.

STEP 3: SHOW EXISTENCE
Define $\gamma_m : \mathbb{R} \to \mathbb{R}^d$ by

$$\gamma_m(t) := \sum_{k=0}^{m} \frac{A^k t^k}{k!} \gamma_0. \tag{6.4.5.17}$$

Exercise 6.4.5.18 Show that $m \mapsto \gamma_m$ is Cauchy in $C^\infty(\mathbb{R}, \mathbb{R}^d)$.

By completeness (Theorem 6.3.4), we may then define $\gamma := \lim_m \gamma_m \in C^\infty(\mathbb{R}, \mathbb{R}^m)$. By the definition of convergence in C^∞ or rather, a corollary of it—see Proposition 6.3.2), we have that

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\gamma &= \lim_m \tfrac{\mathrm{d}}{\mathrm{d}t}\gamma_m = \lim_m \sum_{k=1}^{m} \frac{A^k}{(k-1)!} t^{k-1} \\ &= A \lim_m \sum_{k=1}^{m-1} \frac{A^{k-1}}{(k-1)!} t^{k-1} = A \lim_m \sum_{k=0}^{m} \frac{A^k}{k!} t^k \\ &= A\gamma. \end{aligned} \tag{6.4.5.19}$$

And the other condition follows easily from the fact that $\gamma_m(0) = \gamma_0$ for all $m \in \mathbb{N}$. By construction,[a] γ is the limit of its Taylor series in $C^\infty(\mathbb{R})$, and therefore, by the previous exercise, is globally-analytic.

STEP 4: SHOW UNIQUENESS

Let $c : \mathbb{R} \to \mathbb{R}^d$ be some other differentiable function that satisfies that satisfies (i) $c(0) = \gamma_0$ and (ii)

$$\frac{\mathrm{d}}{\mathrm{d}t} c = Ac. \tag{6.4.5.20}$$

Define

$$L := \sup_{\|v\|=1} \|Av\|. \tag{6.4.5.21}$$

As A is nonzero, this is positive. We wish to show that $c = \gamma$ on an interval of length $\frac{1}{L}$ centered at 0. We will then have that they satisfy the same differential equation and same initial condition at $\frac{1}{2L}$, and so by the same argument we can extend the interval on which they agree from an interval of length $\frac{1}{L}$ to an interval of length $\frac{1}{L} + \frac{1}{2L}$. Extending in other direction as well shows that they agree on an interval of length $\frac{2}{L}$ centered at 0. Continuing this process inductively, we show that they agree on all of $\mathbb{R}$.

Define

$$T\colon \operatorname{Mor}_{\mathbf{Top}}\left(\left[-\tfrac{1}{2L}, \tfrac{1}{2L}\right], \mathbb{R}^d\right) \to \operatorname{Mor}_{\mathbf{Top}}\left(\left[-\tfrac{1}{2L}, \tfrac{1}{2L}\right], \mathbb{R}^d\right)$$

by

$$[T(f)](t) := \gamma_0 + A \int_0^t \mathrm{d}s\, f(s). \tag{6.4.5.22}$$

We show that T is a contraction mapping on $\operatorname{Mor}_{\mathbf{Top}}\left(\left[-\frac{1}{2L}, \frac{1}{2L}\right], \mathbb{R}^d\right)$. It will then follow from the Banach Fixed-Point Theorem that T has a unique fixed point, that is, there is a *unique* continuous function γ on $\left[-\frac{1}{2L}, \frac{1}{2L}\right]$

$$\gamma(t) = \gamma_0 + A \int_0^t \mathrm{d}s\, \gamma(s). \tag{6.4.5.23}$$

By the Fundamental Theorem of Calculus, this is equivalent to $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = A\gamma(t)$, which will complete the proof.

To show that it is a contraction mapping, note that

$$
\begin{aligned}
\|T(f)-T(g)\| &= \sup_{t\in\left[-\frac{1}{2L},\frac{1}{2L}\right]}\left|A\int_0^t \mathrm{d}s\, f(s)-g(s)\right| \\
&\leq L \sup_{t\in\left[-\frac{1}{2L},\frac{1}{2L}\right]}\int_0^t \mathrm{d}s\, |f(s)-g(s)| \\
&\leq L\cdot\frac{L}{2}\cdot\|f-g\| = \tfrac{1}{2}\|f-g\|,
\end{aligned}
$$

so that this is indeed a contraction mapping. ■

[a] Almost—you have to check that this is actually the Taylor series of the function.

We get as an easy corollary the following result.

Corollary 6.4.5.24 Let $m\in\mathbb{N}$, $a_0,\ldots,a_m\in\mathbb{R}$ with $a_m\neq 0$, $t_0\in\mathbb{R}$, and $f_0^0,\ldots,f_0^{m-1}\in\mathbb{R}$. Then, there exists a unique smooth map $f\colon\mathbb{R}\to\mathbb{R}$ such that

(i). $\frac{\mathrm{d}^k}{\mathrm{d}x^k}f^{(k)}(t_0)=f_0^k$ for $0\leq k\leq m-1$; and
(ii).

$$a_m\frac{\mathrm{d}^m}{\mathrm{d}x^m}f+\cdots+a_0f=0. \tag{6.4.5.25}$$

Proof. For $0\leq k\leq m-1$, define $g_k:=\frac{\mathrm{d}^k}{\mathrm{d}x^k}f$. Then, (6.4.5.25) together with these definitions gives us a system of m *first-order* differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} g_0\\ \vdots\\ g_{m-1}\end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -\frac{a_0}{a_m} & -\frac{a_1}{a_m} & -\frac{a_2}{a_m} & \cdots & -\frac{a_{m-1}}{a_m}\end{pmatrix}\begin{pmatrix} g_0\\ \vdots\\ g_{m-1}\end{pmatrix} \tag{6.4.5.26}$$

So the general notation makes this look obtuse, but I promise, everything going on here is absolutely trivial. For example, consider the differential equation $\frac{\mathrm{d}^2}{\mathrm{d}x^2}f-3\frac{\mathrm{d}}{\mathrm{d}x}f+2f=0$. Then, upon defining $g_0 := f$ and $g_1 := \frac{\mathrm{d}}{\mathrm{d}x} f = \frac{\mathrm{d}}{\mathrm{d}x} g_0$, then our system would read

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}x} g_0 &= & &+ 1 \cdot g_1 \\ \tfrac{\mathrm{d}}{\mathrm{d}x} g_1 &= & -2 \cdot g_0 \quad &+ 3 \cdot g_1 . \end{aligned} \tag{6.4.5.27}$$

By the previous theorem, there is a unique smooth solution to (6.4.5.26) that satisfies $g_k(t_0) = f_0^k$ for $0 \leq k \leq m-1$. This is just the same as saying, however, that there is a unique smooth map $f\colon \mathbb{R} \to \mathbb{R}$ that satisfies (6.4.5.25) and $\frac{\mathrm{d}}{\mathrm{d}x^k} f(t_0) = f_0^k$ for $0 \leq k \leq m-1$. ■

The definitions

Definition 6.4.5.28 — Exponential function The ***exponential function***, exp, is the unique function $\exp : \mathbb{R} \to \mathbb{R}$ such that (i) $\frac{\mathrm{d}}{\mathrm{d}x} \exp(x) = \exp(x)$ and (ii) $\exp(0) = 1$.

Definition 6.4.5.29 — Euler's Number ***Euler's Number*** is $\mathrm{e} := \exp(1)$.

Cool, so now we have a function that is its own derivative. What about a function that is *minus* its own derivative. Well, now that we have exp in hand, this is actually really easy.

Exercise 6.4.5.30 Show that $x \mapsto \exp(-x)$ is the unique differentiable function from $\mathbb{R}$ to $\mathbb{R}$ such that (i) is equal to minus its derivative and (ii) is equal to 1 at 0.

What about second derivatives? Well, of course, exp is equal to its own second derivative. That's easy. But what about a function that is equal to *minus* its own second derivative? If we wanted to play the same trick as before, we would need to consider the function $x \mapsto \exp(\alpha x)$ for some number α with $\alpha^2 = -1$, but that's just craziness. Only looney-toons would ever think about such nonsense.[19]

[19]But seriously: why is there no *real* number whose square is -1?

Looks like we're just going to have to come up with a new name for such a function. "Cosine" sounds rad. Let's call it that.

Definition 6.4.5.31 — Cosine and sine The ***cosine function***, cos, is the unique function $\cos \colon \mathbb{R} \to \mathbb{R}$ such that (i) $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(x) = -\cos(x)$, (ii) $\cos(0) = 1$, and (iii) $\frac{\mathrm{d}}{\mathrm{d}x}\cos(0) = 0$.

The ***sine function***, sin, is the unique function $\sin \colon \mathbb{R} \to \mathbb{R}$ such that (i) $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin(x) = -\sin(x)$, (ii) $\sin(0) = 0$, and (iii) $\frac{\mathrm{d}}{\mathrm{d}x}\sin(0) = 1$.

R In the case of exp, because it was defined by a differential equation of *first*-order, only one initial condition was required to uniquely specify it. In contrast, however, there is *more* than one function equal to minus its own second-derivative (and is 1 at 0). To uniquely specify the function, you must also specify its first derivative. sin and cos are "essentially everything" in the sense that any other function that is equal to minus its own second derivative can be written as a linear combination of these (and furthermore, you can figure out what that linear combination should be by looking at initial conditions at 0).

In terms of finding functions that are equal to their own derivatives or minus their own derivatives, we are essentially done. exp always works for finding functions equal to its own n^{th} derivative, and if you want a function equal to *minus* its own n^{th} derivative, $x \mapsto \exp(-x)$ will work for n odd, and cos and sin will work for n even. For the time being then, we simply return to the study of cos and sin.

Their properties

Exercise 6.4.5.32 — exp is an exponential function Show that $\exp(x) = \mathrm{e}^x$.

R exp is the unique function that is equal to its own derivative and is 1 at 0. On the other hand, $x \mapsto \mathrm{e}^x$ is an exponential function in the usual sense, that is, in the sense of Theorem 2.5.1.20. These are not a priori the same thing.

Corollary 6.4.5.33

(i). $\exp(x+y) = \exp(x)\exp(y)$ for all $x, y \in \mathbb{R}$.
(ii). $\exp(x)^y = \exp(xy)$ for all $x, y \in \mathbb{R}$.

Proof. Theorem 2.5.1.20 says that $x \mapsto e^x$ is the unique continuous function that satisfies (i) $e^{x+y} = e^x e^y$ and (ii) $e^0 = 1$. This, together with the result of the previous exercise, give $\exp(x+y) = \exp(x)\exp(y)$ for all $x, y \in \mathbb{R}$. Similarly, the second result follows from Exercise 2.5.1.27. ■

Exercise 6.4.5.34 Show that

$$\lim_{x\to\infty} \exp(x) = \infty \text{ and } \lim_{x\to\infty} x^m \exp(-x) = 0 \quad (6.4.5.35)$$

for all $m \in \mathbb{N}$.

Exercise 6.4.5.36 Show that the image of exp is $\mathbb{R}^+$.

In particular, as $\frac{d}{dx}\exp = \exp > 0$, exp is increasing, and in particular, injective. This allows us to make the following definition.

Definition 6.4.5.37 — Natural logarithm The ***natural logarithm***, $\ln : \mathbb{R}^+ \to \mathbb{R}$, is defined by $\ln := \exp^{-1}$.

Exercise 6.4.5.38

(i). Show that $\ln(xy) = \ln(x) + \ln(y)$ for all $x, y \in \mathbb{R}^+$.
(ii). Show that $x\ln(y) = \ln(y^x)$ for all $x \in \mathbb{R}$ and $y \in \mathbb{R}^+$.

Exercise 6.4.5.39 Show that $\frac{d}{dx}\ln(x) = \frac{1}{x}$.

Exercise 6.4.5.40 Proposition 6.4.5.13 implies that ln is analytic. Compute the Taylor series of ln at $x = 1$ and compute its radius of convergence using Proposition 6.4.5.4. Conclude that

$$\ln(2) = \sum_{m \in \mathbb{Z}^+} \frac{(-1)^{m+1}}{m}, \tag{6.4.5.41}$$

finally wrapping up a loose end from way back in Example 2.4.4.32.

Exercise 6.4.5.42 — Taylor series need not converge to the original function Define $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} \exp\left(-\frac{1}{x}\right) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases} \tag{6.4.5.43}$$

Show that $\frac{\mathrm{d}^m}{\mathrm{d}x^m} f(0) = 0$ for all $m \in \mathbb{N}$.

R If you are familiar with Taylor series from calculus, this example is meant to show that there are functions which are not equal to their Taylor series—this function has Taylor series identically 0 at 0, but of course the function itself is not 0. Thus, there is no general theorem concerning existence of Taylor series—sometimes they just do plain not exist (or at least, do not converge to what you want them to).

Exercise 6.4.5.44 Show that $\frac{\mathrm{d}}{\mathrm{d}x} \sin = \cos$ and $\frac{\mathrm{d}}{\mathrm{d}x} \cos = -\sin$.

Exercise 6.4.5.45 Show that

$$\begin{aligned} \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\ \sin(x+y) &= \cos(x)\sin(y) + \cos(y)\sin(x) \end{aligned} \tag{6.4.5.46}$$

for all $x, y \in \mathbb{R}$.

Exercise 6.4.5.47 Show that $\sin^2 + \cos^2 = 1$.

Proposition 6.4.5.48 — τ, π, and the periodicity of $\sin$ and $\cos$ There is a positive real number $\tau \in \mathbb{R}^+$ such that

(i). τ is the smallest positive real number such that $\cos(x + \tau) = \cos(x)$ for all $x \in \mathbb{R}$; and

(ii). τ is the smallest positive real number such that $\sin(x + \tau) = \sin(x)$ for all $x \in \mathbb{R}$.

Furthermore, it satisfies

$$\begin{aligned} \cos\left(x + \tfrac{\tau}{4}\right) &= -\sin(x) \\ \sin\left(x + \tfrac{\tau}{4}\right) &= \cos(x). \end{aligned} \tag{6.4.5.49}$$

R For historical reasons, it is customary to use the half-period $\pi := \frac{1}{2}\tau$ instead.

Proof. It follows from the definition that $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(0) = -\cos(0) = -1$. We first show that $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos$ is not everywhere negative. If this were the case that $\frac{\mathrm{d}}{\mathrm{d}x}\cos$ would be (strictly) decreasing. As $\frac{\mathrm{d}}{\mathrm{d}x}\cos(0) = 0$, there would then be some $x_0 > 0$ with $\frac{\mathrm{d}}{\mathrm{d}x}\cos(x_0) =: y_0 < 0$. Then, because the derivative is decreasing again, we would have that $\frac{\mathrm{d}}{\mathrm{d}x}\cos(t) \leq y_0$ for all $y \geq x_0$. Hence,

$$\begin{aligned} \cos(x) &= \int_0^x \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t}\cos(t) \\ &= \int_{x_0}^x \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t}\cos(t) + \int_0^{x_0} \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t}\cos(t) \\ &\leq y_0 \cdot (x - x_0) + \int_0^{x_0} \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t}\cos(t). \end{aligned} \tag{6.4.5.50}$$

By taking x sufficiently large, we would obtain the existence of some point $x \in \mathbb{R}$ with $\cos(x) < -1$: a contradiction.

Therefore, the set $\left\{x \in \mathbb{R}^+ : \frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(x) = 0\right\}$ is nonempty, and so we may define

$$\theta := \inf\left\{x \in \mathbb{R}^+ : \frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(x) = 0\right\}. \tag{6.4.5.51}$$

By continuity, we have that $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(\theta) = 0$, and so $\cos(\theta) = 0$ (from the differential equation). As $\cos^2 + \sin^2 = 1$, it follows that $\sin(\theta) = \pm 1$.

We check that in fact $\sin(\theta) = 1$. As $\sin(0) = 0$, it suffices to show that $\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) > 0$ for $x \in (0, \theta)$. However, $\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$, so it suffices to show that $\cos(x) > 0$ for $x \in (0, \theta)$. By the Intermediate Value Theorem, it suffices to show that cos does not vanish on $(0, \theta)$. However, if cos were to vanish on $(0, \theta)$, then $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos$ would vanish on $(0, \theta)$, contradicting the definition of θ.

We then have that

$$\begin{aligned}
&\cos(x + 4\theta) \\
&\quad = \cos(x + 3\theta)\cdot\cos(\theta) - \sin(x + 3\theta)\cdot\sin(\theta) \\
&\quad = -\sin(x + 3\theta) \\
&\quad = -\cos(x + 2\theta)\sin(\theta) - \sin(x + 2\theta)\cos(\theta) \\
&\quad = -\cos(x + 2\theta) \\
&\quad = -\cos(x + \theta)\cos(\theta) + \sin(x + \theta)\sin(\theta) \\
&\quad = \sin(x + \theta) = \cos(x)\sin(\theta) + \sin(x)\cos(\theta) \\
&\quad = \cos(x).
\end{aligned}$$

We wish to show that 4θ is the smallest such positive real number. So, let $T \in \mathbb{R}^+$ be some other positive real number such that $\cos(x + T) = \cos(x)$ for all $x \in \mathbb{R}$. We wish to show that $4\theta \leq T$. To show this, by the definition of θ, it suffices to show that $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(\frac{T}{4}) = 0$. We compute again using the 'angle addition formula'

$$\begin{aligned}1 &= \cos(0) = \cos(T) = \cos\left(\tfrac{T}{4}+\tfrac{T}{4}+\tfrac{T}{4}+\tfrac{T}{4}\right)\\ &=^{a} \cos\left(\tfrac{T}{4}\right)^4 - 6\cos\left(\tfrac{T}{4}\right)^2\sin\left(\tfrac{T}{4}\right)^2 + \sin\left(\tfrac{T}{4}\right)^4\\ &= C^4 - 6C^2(1-C^2) + (1-C^2)^2 = 8C^4 - 8C^2 + 1,\end{aligned}$$

where of course we have defined $C := \cos\left(\frac{T}{4}\right)$. From this equation, it follows that either $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos\left(\frac{T}{4}\right) = C = 0$, in which case we are done, from the definition of θ, we have that $\frac{T}{4} \geq \theta$, or

$$C^2 - 1 = 0, \tag{6.4.5.52}$$

from which it follows that $\cos\left(\frac{T}{4}\right) = \pm 1$. If it were -1, that would imply that cos, and hence $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos$ vanishes in $(0, \frac{T}{4})$, which implies that $\frac{T}{4} > \theta$. If it were 1, then as cos is less than 1 in a neighborhood of 0, there would have to be some point in $x_0 \in (0, \frac{T}{4})$ for which the derivative was positive. As $\frac{\mathrm{d}}{\mathrm{d}\cos(0)} = 0$, this implies that there must be some point in $x_1 \in (0, x_0)$ on which the second derivative is positive. However, as the second derivative is negative in a neighborhood of 0, this implies (by the Intermediate Value Theorem again), that the second derivative must vanish in $(0, x_1)$. Then, $\theta < x_1 < x_0 < \frac{T}{4}$. And so, in all cases, we do indeed have that $\theta \leq \frac{T}{4}$, as desired.

Exercise 6.4.5.53 Finish the theorem by showing that $\cos(x+\pi) = -\sin(x)$ and $\sin(x+\pi) = \cos(x)$.

■

[a] Algebra omitted. It's tedious.

Exercise 6.4.5.54 Show that

(i). cos is nonvanishing on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; and (ii). sin is nonvanishing on $(0, \pi)$.

This enables us to make the following definitions.

Definition 6.4.5.55 — tan, cot, sec, and csc Define $\tan, \sec : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ and $\cot, \csc : (0, \pi) \to \mathbb{R}$ by

(i). $\tan := \frac{\sin}{\cos}$;
(ii). $\cot := \frac{\cos}{\sin}$;
(iii). $\sec := \frac{1}{\cos}$; and
(iv). $\csc := \frac{1}{\sin}$.

Exercise 6.4.5.56 Show that

(i). $\frac{\mathrm{d}}{\mathrm{d}x} \tan = \sec^2$;
(ii). $\frac{\mathrm{d}}{\mathrm{d}x} \cot = -\csc^2$;
(iii). $\frac{\mathrm{d}}{\mathrm{d}x} \sec = \sec\tan$; and
(iv). $\frac{\mathrm{d}}{\mathrm{d}x} \csc = -\csc\cot$.

Thus, tan is increasing and cot is decreasing. In particular, they are both injective.

Exercise 6.4.5.57 Show that the image of both tan and cot is $\mathbb{R}$.

This allows us to make the following definition.

Definition 6.4.5.58 — Arctangent and arccotangent The ***arctangent*** and ***arccotangent***, $\arctan, \operatorname{arccot} : \mathbb{R} \to \mathbb{R}$, are defined by $\arctan := \tan^{-1}$ and $\operatorname{arccot} := \cot^{-1}$.

R You can define inverses for the rest of these so-called 'trigonometric' functions as well, but in general you must restrict the domain of definition (for example, cos is not injective on $\mathbb{R}$).

Exercise 6.4.5.59 Show that

(i). $\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) = \frac{1}{x^2+1}$; and
(ii). $\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arccot}(x) = -\frac{1}{x^2+1}$.

Exercise 6.4.5.60 Show that $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\operatorname{arccot} : \mathbb{R} \to (0, \pi)$ are homeomorphisms.

R Recall that they cannot be *uniform*-homeomorphisms because $\mathbb{R}$ is complete, and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $(0, \pi)$ are not.

The Multinomial Theorem and the Power Rule

Despite all that we have done, there is one rule for differentiation you are likely familiar with that is still noticeably absent from the calculation tools we have developed thus far: the *power rule*.

Oddly, polynomials were the one type of function that we could have defined a long time ago, but yet we have waited to compute their derivatives until last. The reason for this is perhaps the most efficient way to prove the power rule uses the *Binomial Theorem*, which, in its most general form (the form that we *will* need), required some knowledge of exponentials and logarithms.

The classical statement of the Binomial Theorem gives an expression for $(x+y)^m$ for $x, y \in \mathbb{R}$ and $m \in \mathbb{N}$. We will generalize this in two ways. First,[20] we will prove this for $m \in \mathbb{R}$, that is, we will allow m to be *any* real number. Secondly, we will prove an analogous expression for $(x_1 + \cdots + x_n)^m$, $n \in \mathbb{N}$. This expression is usually known as the *Multinomial Theorem*. We actually don't have a direct use for this result, but it's something you should know, and if it's going to be taught anywhere, it should probably be taught alongside the Binomial Theorem. I actually don't know if you can generalize the Multinomial Theorem to allow for $m \in \mathbb{R}$—I will say more about this in remarks of the statement itself.

[20]Well, actually, second, but no one ever said mathematicians were good at counting.

Definition 6.4.5.61 — Multinomial coefficient Let $m \in \mathbb{N}$ and $k_1, \ldots, k_n \in \mathbb{N}$ be such that $k_1 + \cdots + k_n = m$. Then, the $\langle m, \langle k_1, \ldots, k_n \rangle\rangle$ ***multinomial coefficient***, $\binom{m}{k_1, \ldots, k_n}$, is defined by

$$\binom{m}{k_1, \ldots, k_n} := \frac{m!}{k_1! \cdots k_n!} \tag{6.4.5.62}$$

R Combinatorically, $\binom{m}{k_1, \ldots, k_n}$ represents the number of ways in which you can place m objects into n boxes, with k_1 objects in the first box, k_2 objects in the second box, etc..

R The term *multinomial coefficient* itself comes from the statement of the Multinomial Theorem—the statement of that result should make it obvious why they are called what they are called.

R I suppose the case $m = 2$ is classically referred to as *binomial coefficients*, however, the case $n = 2$ is special in that you can generalize the definition to allow more than just natural numbers, and so we state the definition separately—see Definition 6.4.5.70.

The most elegant way to state the Multinomial Theorem involves what are known as *multiindices*.

Definition 6.4.5.63 — Multiindex A ***multiindex*** of degree n is an element of $\mathbb{N}^n$.

For $\alpha \in \mathbb{N}^n$, we write

$$|\alpha| := \alpha_1 + \cdots + \alpha_n \text{ and } \alpha! := \alpha_1! \cdots \alpha_n! \tag{6.4.5.64}$$

.

For $\alpha \in \mathbb{N}^n$ and $x \in \mathbb{R}^n$, we write

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \tag{6.4.5.65}$$

.

For $\alpha \in \mathbb{N}^n$ and $m \in \mathbb{N}$, we write

$$\binom{m}{\alpha} := \binom{m}{\alpha_1, \ldots, \alpha_n}. \tag{6.4.5.66}$$

R So it's a bit silly to talk about multiindices themselves—what's so special about elements of $\mathbb{N}^n$? The point of multiindices is not the things themselves, but rather the simplification of notation they allow. For example, compare the statements of the Multinomial Theorem with and without multiindices.

Theorem 6.4.5.67 — Multinomial Theorem. Let $x \in \mathbb{R}^n$ and let $m \in \mathbb{N}$. Then,

$$(x_1 + \cdots + x_n)^m = \sum_{|\alpha|=m} \binom{m}{\alpha} x^\alpha. \tag{6.4.5.68}$$

R Explicitly, not using multiindex notation, this reads

$$(x_1 + \cdots + x_n)^m = \sum_{k_1+\cdots+k_n=m} \binom{m}{k_1, \ldots, k_n} x_1^{k_1} \cdots x_n^{k_n}.$$

R The issue with generalizing this for $m \in \mathbb{R}$ is that it is not clear to me how to define $\binom{m}{\alpha}$ for $m \in \mathbb{R}$ and α a multiindes.

Proof. We leave this as an exercise.

Exercise 6.4.5.69 Prove this yourself.

■

We now turn to the Binomial Theorem. As mentioned in a remark of the definition of multinomial coefficients (Definition 6.4.5.61), we wish to define binomial coefficients separately.

Definition 6.4.5.70 — Binomial coefficient Let $a \in \mathbb{R}$ and $k \in \mathbb{N}$. Then, the $\langle a, k \rangle$ ***binomial coefficient***, $\binom{a}{k}$, is defined by

$$\binom{a}{k} := \frac{a(a-1)(a-2)\cdots(a-(k+1))}{k!}. \qquad (6.4.5.71)$$

R Note that in the case $a \in \mathbb{N}$, this does indeed agree with (6.4.5.62) (the definition of the multinomial coefficients with $n = 2$). The significance of this is that this makes sense for *any* real number $a \in \mathbb{R}$ (not just natural numbers).

This now allows us to state the Binomial Theorem.

Theorem 6.4.5.72 — Binomial Theorem. Let $x, y \in \mathbb{R}_0^+$ and $a \in \mathbb{R}$. Then, if $|x| < |y|$, then

$$(x+y)^a = \sum_{k \in \mathbb{N}} \binom{a}{k} x^k y^{a-k}. \qquad (6.4.5.73)$$

R Note that, unlike in the case of the Multinomial Theorem, we do require that $x, y \geq 0$. For example, take $x = 0$, $y = -1$, and $a = \frac{1}{2}$. The left-hand side of this then becomes $(-1)^{\frac{1}{2}}$—sheer lunacy.[a]

R Note the condition that $|x| < |y|$ is nonissue. If you happen to have $|x| > |y|$, rename your variables, et voilà, you have your result (and of course if $|x| = |y|$ you don't need to work this hard to begin with). To see why you might need this, write the expression the left-hand side instead as $|x|^a(1 + \frac{y}{x})^a$. For example, if $a = -1$, we need $\left|\frac{y}{x}\right| < 1$ for the resulting series to converge.

[a] That said, it's worth noting that if you do allow the complex numbers, this can be made to work for all $x, y \in \mathbb{R}$ (and all $a \in \mathbb{C}$). This is why we have included the absolute values here, even though they are not strictly necessary—they will be necessary when you go to generalize.

Proof. We leave this as an exercise.

Exercise 6.4.5.74 Prove this yourself.

■

We now can finally use this to prove the power rule.

Exercise 6.4.5.75 — Power Rule Let $a \in \mathbb{R}$. Show that $\frac{\mathrm{d}}{\mathrm{d}x} x^a = ax^{a-1}$.

R By x^a, I mean the function $(0, \infty) \ni x \mapsto x^a$. It is necessary to restrict the domain as exponential functions are only a priori defined for a positive base—see Theorem 2.5.1.20. That said, if $a \in \mathbb{N}$, then the result of holds just as well for the function $\mathbb{R} \ni x \mapsto x^a$, which is now defined on all of $\mathbb{R}$ (and your proof should work basically the same for this case).

The p-series test

Way back in Subsection 2.4.4 Series, we mentioned the p-series Test, though we decided to postpone the proof at the time as the most natural proof (IMHO) required the use of the integral, specifically application of the Integral Test (Proposition 5.2.3.15). While that was enough to do it in principle, we needed the Fundamental Theorem of Calculus to actually compute the integrals we wished to compare to. We needed one final ingredient, however, namely, an antiderivative of $\frac{1}{x}$.

Proposition 6.4.5.76 — p-series Test Let $p \in \mathbb{R}$. Then, $\sum_{n\in\mathbb{Z}^+} \frac{1}{n^p}$ iff $p > 1$.

Proof. If $p \leq 0$, $\lim_n \frac{1}{n^p} \neq 0$, and so it diverges.

If $p > 0$, $[1, \infty) \ni x \mapsto \frac{1}{x^p} \in [0, \infty)$ is nonincreasing, and so we may apply the Integral Test, which implies that $\sum_{n\in\mathbb{Z}^+} \frac{1}{n^p}$ converges iff $\int_1^\infty \mathrm{d}x\, \frac{1}{x^p}$ converges. The Fundamental Theorem of Calculus—Part II and the Power Rule give us that (for $p > 1$)

$$\int_1^\infty \mathrm{d}x\, \frac{1}{x^p} = \left.\frac{1}{1-p}x^{1-p}\right|_1^\infty = 0 - \frac{1}{1-p} = \frac{1}{p-1}, \tag{6.4.5.77}$$

so that the series converges if $p > 1$. The same computation shows that the series diverges for $0 < p < 1$. As for $p = 1$, we have

$$\int_1^\infty \mathrm{d}x\, \frac{1}{x} = \ln(x)|_1^\infty = \infty - 0 = \infty, \tag{6.4.5.78}$$

which shows that the series diverges for $p = 1$. ■

The irrationality of e

We mentioned a long time ago way back in Chapter 2 that e was irrational. Now that we (finally!) know what e is, it is time to return to this issue.

Theorem 6.4.5.79. e is irrational.

R In fact, much more is true than this. First of all, as you're probably aware, π is irrational as well. Even more is true, however: both e and π are *transcendental*, that is, not algebraic.[a] Unfortunately, however, the most elegant way to see these results that I know of[b] requires the use of complex numbers, at least to be applied to π as well. I am confident this method can be 'hacked' so as to not technically require direct reference to $\mathbb{C}$, but it would be just that, a hack, and doing this result is really probably best saved until after one has introduced the complex numbers.

[a]Recall (Definition 2.1.13) that an algebraic number is a number that is the root of a polynomial with integer coefficients. Also recall that every rational number $\frac{m}{n} \in \mathbb{Q}$ is the root of $nx - m$, and so being transcendental is indeed (much) stronger than being irrational.

[b]Namely, by using the Lindemann-Weierstrass Theorem. A special case says that if x is a nonzero algebraic number, then e^x is transcendental. In particular, $\mathrm{e} = \mathrm{e}^1$ is transcendental. Furthermore, if π were algebraic, then $\mathrm{e}^{\mathrm{i}\pi} = -1$ would be transcendental: a contradiction.

Proof. Step 1: Express e as a series

As $\exp(x)$ is globally-analytic and $\frac{\mathrm{d}}{\mathrm{d}x}\exp(x) = \exp(x)$, the Taylor series of exp centered at 0 gives in particular

$$\mathrm{e} \coloneqq \exp(1) = \sum_{m\in\mathbb{N}} \frac{1}{m!}. \tag{6.4.5.80}$$

Step 2: Show that $\mathrm{e} \notin \mathbb{Z}$

We first check that e is not an integer. To do that, we show that it is strictly larger than 2 and strictly less than 3. That is is strictly greater than 2 is obvious: $\mathrm{e} = 1 + 1 + \text{positive stuff} > 2$. On the other hand, to see that $\mathrm{e} < 3$, note that $2^m \leq m!$ for

$m \geq 4$, so that

$$\begin{aligned} \mathrm{e} &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \sum_{m=4}^{\infty} \frac{1}{m!} \\ &\leq 1 + 1 + \frac{1}{2} + \frac{1}{6} + \sum_{m=4}^{\infty} \frac{1}{2^m} \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + 2^{-4} \sum_{m=0}^{\infty} \frac{1}{2^m} \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{2^{-4}}{1 - \frac{1}{2}} \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + 2^{-3} = 1 + 1 + \frac{7}{24} < 3. \end{aligned} \tag{6.4.5.81}$$

STEP 3: PROCEED BY CONTRADICTION
Suppose that $\mathrm{e} = \frac{m}{n}$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}^+$, and $\gcd(m, n) = 1$. Note that, as we now know that e is not an integer, $n \geq 2$.

STEP 4: DEFINE $x \in \mathbb{R}$
Define

$$x := \sum_{k=n+1}^{\infty} \frac{n!}{k!}. \tag{6.4.5.82}$$

STEP 5: SHOW THAT $x \in \mathbb{Z}$

Note that

$$\begin{aligned} x &:= n! \sum_{k=n+1}^{\infty} \frac{1}{k!} \\ &= n!\left(\sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!}\right) = n!\left(\mathrm{e} - \sum_{k=0}^{n} \frac{1}{k!}\right) \\ &= n!\left(\frac{m}{n} - \sum_{k=0}^{n} \frac{1}{k!}\right) \\ &= m(n-1)! - \sum_{k=0}^{n} \frac{n!}{k!} \in \mathbb{Z}. \end{aligned} \tag{6.4.5.83}$$

STEP 6: SHOW THAT $0 < x < 1$

That $x > 0$ is obvious: it is an infinite sum of positive numbers. For the other inequality, first note that, for $k > n$,

$$\frac{n!}{k!} = \frac{1}{(n+1)\cdots(n+(k-n))} \leq \frac{1}{(n+1)^{k-n}}. \tag{6.4.5.84}$$

Hence,

$$\begin{aligned} x &:= \sum_{k=n+1}^{\infty} \frac{n!}{k!} \leq \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-n}} \\ &= \sum_{k=0}^{\infty} \frac{1}{(n+1)^{k+1}} = \frac{1}{n+1} \frac{1}{1-\frac{1}{n+1}} = \frac{1}{n} < 1. \end{aligned} \tag{6.4.5.85}$$

STEP 7: FINISH THE PROOF

The last two steps give us our contradiction, and so it must have been that $\mathrm{e} \notin \mathbb{Q}$. ■

6.5 Differentiability and continuity

Having defined the exponential function, we can now tie up a loose end from before.

Example 6.5.1 — A function which is infinitely-differentiable but not continuous [a] Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(\langle x,y\rangle) := \begin{cases} 0 & \text{if } x=0 \\ \frac{\exp\left(-\frac{1}{x^2}\right)y}{\exp\left(-\frac{2}{x^2}\right)+y^2} & \text{otherwise.} \end{cases} \tag{6.5.2}$$

Order $\mathbb{R}^+$ by the usual order and consider the net $\mathbb{R}^+ \ni t \mapsto x_t := \langle t, \exp\left(-\frac{1}{t^2}\right)\rangle \in \mathbb{R}^2$. Then, $\lim_{t\to 0^+} x_t = \langle 0,0\rangle$, but

$$\begin{aligned} f(x_t) = f(\langle t, \exp(-\tfrac{1}{t^2})\rangle) &= \frac{\exp(-\frac{2}{t^2})}{\exp(-\frac{2}{t^2})+\exp(-\frac{2}{t^2})} \\ &= \tfrac{1}{2} \neq 0 = f(\langle 0,0\rangle), \end{aligned} \tag{6.5.3}$$

and so f is not continuous at $\langle 0,0\rangle$.

On the other hand, it is smooth away from $x=0$, so we need only show that it is infinitely-differentiable on the set $\{\langle x,y\rangle : x=0\}$. For $v := \langle v_x, v_y\rangle \in \mathrm{T}_{\langle 0,y\rangle}(\mathbb{R}^2)$, $v_x \neq 0$,[b] we have

$$\begin{aligned} \frac{f(\langle 0,y\rangle+\varepsilon v) - f(\langle 0,y\rangle)}{\varepsilon} &= \frac{\frac{\exp\left(-\frac{1}{(\varepsilon v_x)^2}\right)(y+\varepsilon v_y)}{\exp\left(-\frac{2}{(\varepsilon v_x)^2}\right)+(y+\varepsilon v_y)^2}}{\varepsilon} \\ &= \frac{1}{\varepsilon}\frac{\exp\left(-\frac{1}{(\varepsilon v_x)^2}\right)(y+\varepsilon v_y)}{\exp\left(-\frac{2}{(\varepsilon v_x)^2}\right)+(y+\varepsilon v_y)^2} \\ &= \frac{1}{\varepsilon}\frac{y+\varepsilon v_y}{\exp\left(-\frac{1}{(\varepsilon v_x)^2}\right)+(y+\varepsilon v_y)^2\exp\left(\frac{1}{(\varepsilon v_x)^2}\right)}. \end{aligned} \tag{6.5.4}$$

Hence,

$$\begin{aligned} \mathrm{D}_v f(\langle 0,y\rangle) &:= \lim_{\varepsilon\to 0^+} = \frac{f(\langle 0,y\rangle+\varepsilon v)-f(\langle 0,y\rangle)}{\varepsilon} \\ &= 0. \end{aligned} \tag{6.5.5}$$

Of course, the map $v \mapsto 0$ is linear.

Thus, $\mathrm{D}_v f(\langle 0,y\rangle)$ exists for all $v \in \mathrm{T}_{\langle 0,y\rangle}(\mathbb{R}^d)$ *and* the map $v \mapsto D_v f(\langle 0,0\rangle)$ is linear, but nevertheless f is not continuous at $\langle 0,0\rangle$.

Exercise 6.5.6 Show that in fact f is infinitely-differentiable.

R Hint: Note that there is nothing to worry about ever away from $x=0$. As for $x=0$, try using an induction argument to show that the power of $\exp\left(\frac{1}{(\varepsilon v_x)^2}\right)$ will always be greater than the power of the same thing in the numerator of every term of every component of the higher derivatives.

[a] This comes from [GO92, pg. 116].

[b] If $v_x=0$, then the $f(\langle 0,y\rangle+\varepsilon v)=0$, and so of course the directional derivative in this direction is 0.

Exercise 6.5.7 Does there exist an infinitely-differentiable function which is nowhere continuous?

R Disclaimer: I do not know the answer.

Fortunately, however, this cannot happen in one-dimension (thank god!).

Proposition 6.5.8 Let $f\colon \mathbb{R}\to\mathbb{R}$ be a function and let $x_0\in\mathbb{R}$. Then, if f is differentiable at x_0, then it is continuous at x_0.

Proof. Consider

$$f(x) - f(x_0) = \left(\frac{f(x) - f(x_0)}{x - x_0} \right) (x - x_0). \qquad (6.5.9)$$

Taking the limit of both sides as $x \to x_0$, because $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \frac{\mathrm{d}}{\mathrm{d}x} f(x_0)$ exists (and is finite), we find that $\lim_{x \to x_0}(f(x) - f(x_0)) = 0$, so that f is continuous at x_0. ■

Be careful though: f need not be continuous in a neighborhood of x_0.

Example 6.5.10 — A function on $\mathbb{R}$ differentiable at a point and not continuous in a neighborhood of that point Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q}^{\mathrm{C}}. \end{cases} \qquad (6.5.11)$$

Exercise 6.5.12 Show that f is continuous at 0 and only at 0.

Exercise 6.5.13 Show that f is differentiable at 0 with derivative 0.

R You might be tempted to ask "Well, what if it is twice differentiable at a point?". The problem with this is, in order to even define the second derivative at the point, the derivative has to exist in a neighborhood of that point,[a] in which case it is then automatically continuous on that same neighborhood by the previous result.

[a] Otherwise, the limit as $h \to 0$ of the difference quotient of the derivative (the definition of the second derivative) does not make sense.

Recall that (Meta-definition 6.2.1.10) for a function to be differentiable means that it extends to a differentiable function on a neighborhood of the domain; it does *not* mean that it is differentiable on the interior. What follows is a 'naturally occurring' example of how these may differ.[21]

Example 6.5.14 — A function differentiable on (a, b) but not on $[a, b]$ Define $f\colon [0, 1] \to \mathbb{R}$ by

$$f(x) := \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases} \tag{6.5.15}$$

Then, f is differentiable on $[0, 1]$ with derivative[a]

$$\tfrac{\mathrm{d}}{\mathrm{d}x} f(x) = \sin(\tfrac{1}{x}) - \tfrac{1}{x}\cos(\tfrac{1}{x}). \tag{6.5.16}$$

Exercise 6.5.17 From here, show that f is not Lipschitz-continuous on $[0, 1]$.

[a] Remember, we only define the derivative at *interior* points of domain, and f is in fact *smooth* on $(0, 1)$.

What follows is a list of exercises that deal with conditions that are *not* sufficient for differentiability.

Exercise 6.5.18 — A function differentiable in one but not all directions Find a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ that is differentiable along the x-axis at the origin, but not along the y-axis.

Exercise 6.5.19 — A function differentiable on the standard basis but not differentiable Find a function $f\colon \mathbb{R}^2 \to$

[21] As opposed to stupid examples in which $\mathrm{Int}(D) = \emptyset$.

$\mathbb{R}$ that is differentiable along both the x-axis *and* the y-axis at the origin, but not in the direction of the vector $\langle 1, 1\rangle$.

R In particular, it is quite possible for a the partial derivatives[a] of a function to exist without that function being differentiable.

[a]The ***partial derivatives*** of a function are the directional derivatives in the direction of a standard orthonormal basis vector.

Exercise 6.5.20 — A continuous function that is not differentiable Find an example a function that is continuous but not differentiable.

As a matter of fact, you can even find continuous functions which *aren't differentiable anywhere*!

Example 6.5.21 — A continuous function that is nowhere-differentiable [a] Define $f_0 : [0, 1] \to \mathbb{R}$ by

$$f_0(x) := \begin{cases} x & \text{if } x \in [0, \frac{1}{2}] \\ 1 - x & \text{if } x \in [\frac{1}{2}, 1]. \end{cases} \tag{6.5.22}$$

and extend $f_0 : \mathbb{R} \to \mathbb{R}$ periodically with period 1. For $k \in \mathbb{N}$, now define $f_k : \mathbb{R} \to \mathbb{R}$ by

$$f_k(x) := \tfrac{1}{4^k} f_0(4^k x). \tag{6.5.23}$$

Then define $s_m : \mathbb{R} \to \mathbb{R}$ by

$$s_m(x) := \sum_{k=0}^{m} f_k(x). \tag{6.5.24}$$

We wish to show that $m \mapsto s_m$ is Cauchy in $\mathrm{Mor}_{\mathbf{Top}}(\mathbb{R}, \mathbb{R})$. Then, because $\mathrm{Mor}_{\mathbf{Top}}(\mathbb{R}, \mathbb{R})$ is complete (Theorem 4.4.1.1), we may define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \sum_{k=0}^{\infty} f_k(x) \tag{6.5.25}$$

and will automatically have that $s \in \mathrm{Mor}_{\mathbf{Top}}(\mathbb{R},\mathbb{R})$ (that is, that f is continuous). We will then show that f is not differentiable anywhere.

We thus verify Cauchyness. For $n > m$, we have

$$\begin{aligned} |s_n(x) - s_m(x)| &= \sum_{k=m+1}^{n} f_k(x) < \sum_{k=m+1}^{n} \frac{1}{4^k} \\ &\leq \sum_{k=m+1}^{\infty} \frac{1}{4^k} = \frac{1}{1-\frac{1}{4}} - \sum_{k=0}^{m} \frac{1}{4^k} \\ &= \frac{1}{1-\frac{1}{4}} - \frac{1-(\frac{1}{4})^{m+1}}{1-\frac{1}{4}} = \tfrac{4}{3}\frac{1}{4^{m+1}}. \end{aligned} \tag{6.5.26}$$

In fact, for $K \subseteq \mathbb{R}$ quasicompact,

$$\|s_n - s_m\|_K \leq \|s_n - s_m\|_{\mathbb{R}} \leq \frac{4}{3}\frac{1}{4^{m+1}}, \tag{6.5.27}$$

and hence the sequence $m \mapsto s_m$ is Cauchy in $\mathrm{Mor}_{\mathbf{Top}}(\mathbb{R},\mathbb{R})$, and hence converges to f.

We now turn to the proof that f is differentiable nowhere. So, fix $x \in \mathbb{R}$. We want to show that f is not differentiable at $\mathbb{R}$. To prove this, we first prove a small lemma: we show that there exists a sequence $m \mapsto \varepsilon_m$ such that $\varepsilon_m = \frac{1}{4^{m+1}}$ and

$$\left|f_m(x+\varepsilon'_m) - f_m(x)\right| = \left|\varepsilon'_m\right| \tag{6.5.28}$$

whenever $\varepsilon'_m \leq \varepsilon_m$. Define $r_m := 4^m x - \lfloor 4^m x \rfloor$, so that $r_m \in [0,1)$. We define

$$\varepsilon_m := \tfrac{1}{4^{m+1}} \cdot \begin{cases} 1 & \text{if } r \in [0,\frac{1}{4}) \text{ or } r \in [\frac{1}{2},\frac{3}{4}) \\ -1 & \text{if } r \in [\frac{1}{4},\frac{1}{2}) \text{ or } r \in [\frac{3}{4},1). \end{cases} \tag{6.5.29}$$

Obviously $\epsilon_m = \frac{1}{4^{m+1}}$. For the other fact, notice that

$$f_0(4^m x) = \begin{cases} r & \text{if } r_m \in [0,\frac{1}{2}) \\ 1-r & \text{if } x \in [\frac{1}{2},1), \end{cases} \tag{6.5.30}$$

Thus,

$$\begin{aligned}
&f_0(4^m(x+\varepsilon_m)) - f_0(4^m x) \\
&\qquad = f_0(r_m + 4^m \varepsilon_m) - f_0(r_m) \\
&\qquad = f_0(r_m \pm \tfrac{1}{4}) - f_0(r_m).
\end{aligned} \tag{6.5.31}$$

The sign was purposefully chosen so that both r_m and $r_m \pm \frac{1}{4}$ lie in the same 'half' of the interval $[0, 1]$, so that the same 'rule' in the definition of f_0 applies. In fact, the same is of course true if $\pm\frac{1}{4}$ replaced by any $4^m \varepsilon'_m$ with $\varepsilon'_m \leq \varepsilon_m$. Thus, in either case, for such an ε'_m, we have that

$$f_0(4^m(x+\varepsilon'_m)) - f_0(4^m x) = 4^m \varepsilon'_m. \tag{6.5.32}$$

From this it follows that

$$\begin{aligned}
\left| f_m(x+\varepsilon'_m) - f_m(x) \right| &= \tfrac{1}{4^m} \left| f_0(4^m(x+\varepsilon'_m)) - f_0(4^m x) \right| \\
&= \tfrac{1}{4^m} \cdot 4^m \varepsilon'_m = \varepsilon'_m
\end{aligned}$$

Take $n > m$. Then,

$$\begin{aligned}
f_n(x+\varepsilon_m) &= \tfrac{1}{4^n} f_0(4^n(x+\varepsilon_m)) \\
&= \tfrac{1}{4^n} f_0(4^n x \pm 4^{n-(m+1)}) \\
&= {}^b \tfrac{1}{4^n} f_0(4^n x) = f_n(x).
\end{aligned} \tag{6.5.33}$$

Thus,

$$\begin{aligned}
f(x+\varepsilon_m) - f(x) &= \sum_{k=0}^{\infty} [f_k(x+\varepsilon_m) - f_k(x)] \\
&= \sum_{k=0}^{m} [f_k(x+\varepsilon_m) - f_k(x)].
\end{aligned} \tag{6.5.34}$$

On the other hand, for $n \leq m$,

$$\begin{aligned}
f_n(x) &= \tfrac{1}{4^n} f_0(f^n x) = \tfrac{4^{m-n}}{4^n} f_0\left(4^m \frac{x}{4^{m-n}}\right) \\
&= 4^{m-n} f_m(\tfrac{1}{4^{m-n}} x),
\end{aligned} \tag{6.5.35}$$

and so

$$
\begin{aligned}
&|f_n(x+\varepsilon_m) - f_n(x)| \\
&\qquad = 4^{m-n}\left|f_n\left(\frac{x}{4^{m-n}} + \frac{h_m}{4^{m-n}}\right) - f_n\left(\tfrac{x}{4^{m-n}}\right)\right| \\
&\qquad = {}^{c}4^{m-n}\cdot\frac{\varepsilon_m}{4^{m-n}} = \varepsilon_m
\end{aligned} \tag{6.5.36}
$$

Thus,

$$
\begin{aligned}
\Delta_m &:= \frac{f(x+\varepsilon_m) - f(x)}{\varepsilon_m} \\
&= \sum_{k=0}^{m} \frac{f_k(x+\varepsilon_m) - f_k(x)}{\varepsilon_k}.
\end{aligned} \tag{6.5.37}
$$

By (6.5.36), each term in this sum is ± 1, so that the difference quotient $\frac{f(x+\varepsilon_m)-f(x)}{\varepsilon_m}$ is an integer. Let n_m^+ denote the number of $+1$s that appear in this finite sum and similarly let n_m^- denote the number of -1s that appear in this sum, so that $\Delta_m = n_m^+ - n_m^-$ and $m = n_m^+ + n_m^-$. Hence, $\Delta_m = m - 2n_m^-$. Thus, the parity of Δ_m is constant, and so cannot converge. Thus, this limit, the derivative, does not exist at x.

[a] Proof adapted from [Col10].

[b] Because f_0 has period 1 and $n \geq m+1$.

[c] Because, for $n \leq m$, $\frac{\varepsilon_m}{4^{m-n}} \leq \varepsilon_m$.

In fact, if you thought this was bad, we can do even worse than this. (This also wraps up a loose end when we motivated the introduction of the topology on $C^\infty(\mathbb{R}^d)$.)

Example 6.5.38 — A sequence of smooth functions which converges in $\mathrm{Mor}_{\mathbf{Top}}(\mathbb{R},\mathbb{R})$ to a function which is nowhere-differentiable We leave this as an exercise.

Exercise 6.5.39 Find a sequence of smooth functions which converges to a continuous function in $\mathrm{Mor}_{\mathbf{Top}}(\mathbb{R}, \mathbb{R})$ that is nowhere-differentiable.

R Hint: Look up the *Weierstrass Function*.

On the other hand, if the derivative *extends* to a continuous function on the entire domain, then we are good to go.

Exercise 6.5.40 Let $K \subseteq \mathbb{R}^d$ be quasicompact and convex, and let $f\colon K \to \mathbb{R}$. Show that if f is continuously-differentiable, then f is Lipschitz-continuous.

W Warning: This will fail for a more general K—see the following exercise (Exercise 6.5.43).

R ***Convex*** means that

$$\{(1-t)x + ty : t \in [0,1]\} \subseteq K \qquad (6.5.41)$$

for every $x, y \in K$. Note that

$$\{(1-t)x + ty : t \in [0,1]\} \qquad (6.5.42)$$

is just the 'line segment' from x to y. Thus, a set is convex iff it contains the line segment between any two of its points.

R Recall that (Proposition 6.2.1.14) "f is continuously-differentiable" means that f is differentiable and that its derivative $\nabla_a f$ is continuous on D (recall that (Definition 6.2.1) this in turn means that $v^a \nabla_a f$ is continuous for all vectors v^a).

Exercise 6.5.43 Find an example of a continuously differentiable function $f\colon D \to \mathbb{R}$ which is not Lipschitz-continuous.

Can you find an example with D bounded? Compact? Connected? Bounded and connected? Compact and connected?

The derivative (as a function on the tangent space) is itself continuous (hence uniformly-continuous because continuous homomorphisms of topological groups are uniformly-continuous (Proposition 4.2.2.22).

Exercise 6.5.44 Show that the map $\mathrm{T}_x(\mathbb{R}^d) \ni v^a \mapsto v^a \nabla_a f(x) \in \mathbb{R}$ is linearly-continuous[a] if f is differentiable at x.

[a] Recall that $\mathrm{T}_x(\mathbb{R}^d)$ is a *metric* vector space, with the metric being the dot product. This metric induces a norm (the usual Euclidean norm), which in turn induces a metric, which in turn induces a uniformity, which in turn induces a topology.

While the next example doesn't have to do with continuity per se, I think it makes most sense to place here, along with a series of other counter-examples.

Example 6.5.45 — A composition of linearly-differentiable functions that is not linearly-differentiable [a] Define $g\colon \mathbb{R} \to \mathbb{R}^2$ by

$$g(t) := \langle t, 2t^2 \rangle \tag{6.5.46}$$

and define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) := \begin{cases} \sin\left(\frac{1}{y^2}\right) & \text{if } x > 0 \text{ and } x^2 < y < 3x^2 \\ 0 & \text{otherwise.} \end{cases} \tag{6.5.47}$$

g is certainly Fréchet-differentiable.

Exercise 6.5.48 Check that $\mathrm{D}_v f(0,0) = 0$ for all $v \in \mathrm{T}_{\langle 0,0\rangle}(\mathbb{R}^2)$.

Thus, f is differentiable at $\langle 0,0\rangle = g(0)$. However,

$$[f\circ g](t) = \begin{cases} \sin\left(\frac{1}{t^2}\right) & \text{if } t>0 \\ 0 & \text{otherwise} \end{cases} \tag{6.5.49}$$

is not differentiable at $t=0$.

R In fact, this shows that $f\circ g$ need not be linearly-differentiable even if g is Fréchet-differentiable.

[a] This was influenced by math.stackexchange.

Exercise 6.5.50 Let $U\subseteq \mathbb{R}^d$ and $V\subseteq \mathbb{R}^e$ be open, and let $f\colon U\to V$, and $g\colon V\to \mathbb{R}$. If g and f are linearly-differentiable, need $g\circ f$ be linearly-differentiable?

R The difference between this and the previous example is that the previous example had to do with *pointwise* differentiability, whereas here we are interested in differentiability on the entire domain.

And now we come to the classical "Do the partial derivatives commute?" question.

Theorem 6.5.51 — Clairut's Theorem. Let $D\subseteq \mathbb{R}^d$, let $x\in D$, and let $f\colon D\to \mathbb{R}$. Then, if f has a second derivative that is continuous at x, then, $\nabla_a\nabla_b(x)f = \nabla_b\nabla_a f(x)$.

R If f is smooth, by applying this result inductively, it follows that all mixed partials agree.

R This applies equally well for arbitrary tensors of course, with the usual proof or just reducing it to this case.

W Warning: This fails in general, even for smooth functions, if there is curvature on a Riemannian manifold! As a matter of fact, the curvature tensor is *defined* by this lack of commutativity.

Proof. Let $v^a, w^a \in \mathbb{R}^d$. Then,

$$\begin{aligned}[v^a\nabla_a(w^b\nabla_b f)](x) &= \lim_{\varepsilon_v\to 0^+}\frac{w^b\nabla_b f(x+\varepsilon_v v)-w^b\nabla_b f(x)}{\varepsilon_v}\\ &= \lim_{\varepsilon_v\to 0^+}\lim_{\varepsilon_w\to 0^+}\frac{f(x+\varepsilon_v v+\varepsilon_w w)-f(x+\varepsilon_v v)-f(x+\varepsilon_w w)+f(x)}{\varepsilon_v\varepsilon_w}.\end{aligned}$$

For $\varepsilon_w > 0$, define $g_{\varepsilon_w}\colon U_{h_w}\to\mathbb{R}$ by

$$g_{\varepsilon_w}(\varepsilon) \coloneqq f(x+\varepsilon_w w+\varepsilon v)-f(x+\varepsilon v), \tag{6.5.52}$$

where $U_{\varepsilon_w}\subseteq\mathbb{R}$ is an open neighborhood of 0 chosen so that $x+\varepsilon_w+\varepsilon v\in U$ is in the domain of f for all $\varepsilon\in\operatorname{Cls}(U_{\varepsilon_w})$. In fact, by making U_{ε_w} smaller if necessary, way may without loss of generality assume that it is an open interval containing 0. Using this definition, we have

$$[v^a\nabla_a(w^b\nabla_b f)](x) = \lim_{\varepsilon_v\to 0^+}\lim_{\varepsilon_w\to 0^+}\frac{1}{\varepsilon_w}\frac{g_{\varepsilon_w}(\varepsilon_v)-g_{\varepsilon_w}(0)}{\varepsilon_v}.$$

g_{ε_w} is continuous on the closed interval $\operatorname{Cls}(U_{\varepsilon_w})$ and differentiable on its interior,[a] and so by the Mean Value Theorem there is some $c_{\varepsilon_v}\in(0,\varepsilon_v)$ such that

$$\tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}g_{\varepsilon_w}(c_{\varepsilon_v}) = \frac{g_{\varepsilon_w}(\varepsilon_v)-g_{\varepsilon_w}(0)}{\varepsilon_v}, \tag{6.5.53}$$

so that

$$\begin{aligned}[v^a\nabla_a(w^b\nabla_b f)](x) &= \lim_{\varepsilon_v\to 0^+}\lim_{\varepsilon_w\to 0^+}\frac{1}{\varepsilon_w}\tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}g_{\varepsilon_w}(c_{\varepsilon_v})\\ &= \lim_{\varepsilon_v\to 0^+}\lim_{\varepsilon_w\to 0^+}\frac{v^a\nabla_a f(x+\varepsilon_w w+c_{\varepsilon_v}v)-v^a\nabla_a f(x+c_{\varepsilon_v}v)}{\varepsilon_w}.\end{aligned}$$

Similarly as before, for $\varepsilon_v > 0$, define[b] $h_{\varepsilon_v}\colon V_{\varepsilon_v}\to\mathbb{R}$ by

$$h_{\varepsilon_v}(\varepsilon) \coloneqq v^a\nabla_a f(x+c_{\varepsilon_v}v+\varepsilon w) \tag{6.5.54}$$

for a sufficiently small neighborhood V_{ε_v} of 0. Then,

$$\left[v^a\nabla_a(w^b\nabla_b f)\right](x) = \lim_{\varepsilon_v\to 0^+}\lim_{\varepsilon_w\to 0^+}\frac{h_{\varepsilon_v}(\varepsilon_w)-h_{\varepsilon_v}(0)}{\varepsilon_w}.$$

Applying the Mean Value Theorem
again, we deduce the existence of some $c_{\varepsilon_w} \in (0, \varepsilon_w)$ such that

$$\tfrac{\mathrm{d}}{\mathrm{d}\varepsilon} h_{\varepsilon_v}(c_{\varepsilon_w}) = \frac{h_{\varepsilon_v}(\varepsilon_w) - h_{\varepsilon_v}(0)}{\varepsilon_w}, \tag{6.5.55}$$

so that

$$\begin{aligned} &\left[v^a \nabla_a (w^b \nabla_b f)\right](x) \\ &\quad = \lim_{\varepsilon_v \to 0^+} \lim_{\varepsilon_w \to 0^+} w^b \nabla_b \left(v^a \nabla_a f(x + c_{\varepsilon_v} v + c_{\varepsilon_w} w)\right). \end{aligned}$$

Continuity of the second derivative applied to this gives the desired result. ■

[a]Despite the fact that the existence of the derivative of f does not imply continuity of f, it *is* the case that the existence of directional derivatives of f implies existence of derivative of g_{ε_w}, which, because it is defined in one-dimension, *does* imply continuity.

[b]For what it's worth, this is what ultimately motivated me to change from hs to εs in the difference quotient—at first I tried writing $g^1_{h_w}$ and $g^2_{h_v}$, which works, but. . . ew. Being able to use h is so much better.

Exercise 6.5.56 Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(\langle x, y\rangle) := \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } \langle x, y\rangle \neq \langle 0, 0\rangle \\ 0 & \text{if } \langle x, y\rangle = \langle 0, 0\rangle. \end{cases} \tag{6.5.57}$$

Show that this function is differentiable everywhere with gradient (in standard coordinates) given by

$$\nabla_a f(\langle x, y\rangle) = \begin{cases} \left\langle \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}, \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2}\right\rangle & \text{if } \langle x, y\rangle \neq \langle 0, 0\rangle \\ \langle 0, 0\rangle & \text{if } \langle x, y\rangle = \langle 0, 0\rangle. \end{cases}$$

In other words, for a vector $v^a := \langle v_x, v_y\rangle \in \mathbb{R}^2$,

$$\begin{aligned} &v^a \nabla_a f(\langle x, y\rangle) \\ &\quad = \begin{cases} v_x \left(\frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}\right) + v_y \left(\frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2}\right) & \text{if } \langle x, y\rangle \neq \langle 0, 0\rangle \\ 0 & \text{if } \langle x, y\rangle = \langle 0, 0\rangle. \end{cases} \end{aligned}$$

Show that

$$\mathbb{R}^2 \ni w \mapsto \mathrm{D}_w\, (v^a f)\, (\langle 0, 0\rangle) \tag{6.5.58}$$

is *not* linear, so that f is *not* twice-differentiable at $\langle 0, 0\rangle$.

R On the other hand, if you plug-in $v = \langle 1, 0\rangle$ and $w = \langle 0, 1\rangle$, and then plug-in $v = \langle 0, 1\rangle$ and $w = \langle 1, 0\rangle$, you should get different answers. Thus, the second partial derivatives do *not* commute, but it is also not twice-differentiable. Thus, this example does *not show the necessity of the continuity assumption* in the previous theorem. Can you find an example of a function second differentiable at a point that is not symmetric there?

Clairut's Theorem finally allows us to address what is probably one of the most familiar (and useful) results from elementary calculus: the second derivative test.[22]

Theorem 6.5.59 — Higher Derivative Test. Let $D \subseteq \mathbb{R}^d$, let $x_0 \in \mathrm{Int}(D)$, $f\colon D \to \mathbb{R}$ be Fréchet-differentiable at x_0, and let $m \in \mathbb{Z}^+ \cup \{\infty\}$ be the least for which $\nabla_{a_1} \cdots \nabla_{a_m} f(x_0) \neq 0$.[a] Then, if $m < \infty$, then x_0 is a local maximum (resp. local minimum) of f iff (i) m is even and (ii) $\nabla_{a_1} \cdots \nabla_{a_m} f(x_0)$ is negative-definite (resp. positive-definite).

R A tensor $T_{a_1 \cdots a_m}$ is ***positive-definite*** iff $v^{a_1} \cdots v^{a_m} T_{a_1 \cdots a_m} \geq 0$ for all v and vanishes iff $v = 0$. Likewise for ***negative-definite***.

W Warning: There are smooth functions with local extrema and all derivatives vanishing at this point (i.e. $m = \infty$), in which case the test is not applicable—see the following exercise.

[22]Though we present a stronger result, what we call simply the ***Highest Derivative Test***.

R For example, this will tell us that $\mathbb{R} \ni x \mapsto x^4 \in \mathbb{R}$ has a local minimum at $x = 0$, something which the Second Derivative Test is unable to do (it is inconclusive).

[a] With $m := \infty$ if all derivatives which exist vanish.

Proof. We leave this as an exercise.

Exercise 6.5.60 Prove this yourself.

∎

Exercise 6.5.61 Find a smooth function $f : \mathbb{R} \to \mathbb{R}$ for which $x = 0$ is a local maximum and $\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(0) = 0$ for all $n \in \mathbb{Z}^+$.

6.6 The Inverse Function Theorem

We have talked about functions which take values in $\mathbb{R}^m$ before, $f : \mathbb{R}^d \to \mathbb{R}^m$, and we have interpreted them as vector fields on $\mathbb{R}^d$. I don't want to do this now. Now, the codomain is going to play what will be the role of a general manifold when you generalize, so for the time being I will write $M := \mathbb{R}^m$ to be suggestive of this.

When you generalize to manifolds and you have a smooth function $f: M \to N$ between manifolds, then for each $x \in X$ the derivative becomes a linear map at the tangent space $\mathrm{T}_x(M) \ni v^a \mapsto v^b \nabla_b f(x)^\mu \in \mathrm{T}_{f(x)}(N)$. (Once again, the μ index indicates that this vector lives in a different vector space than v^a does.) Unfortunately, in the very special case of $f: \mathbb{R}^d \to \mathbb{R}^m$, because $\mathrm{T}_x(M) \cong_{\mathbf{Vect}_{\mathbb{R}}} \mathbb{R}^d$ is 'the same as'[a] M and $\mathrm{T}_{f(x)}(N) \cong_{\mathbf{Vect}_{\mathbb{R}}} \mathbb{R}^m$ is 'the same as' N, the 'same' thing gets treated in very different ways in different, and this can make things very confusing. For example, $f(x) \in \mathbb{R}^m$ the metric space[b], but $v^a \nabla_a f(x)^\mu \in \mathbb{R}^m$ the vector space (namely, the tangent space of $\mathbb{R}^m$ at $f(x)$).

---

[a] Or so the symbols would suggest.

[b] If we could, we should regard it as a manifold, but we don't know what manifolds are, and a metric space is the next best thing (that we have at our disposal).

Thus, for a smooth map $f: \mathbb{R}^d \to M$, at each point, $\nabla_a f(x)^\mu \in \mathrm{Mor}_{\mathbf{Vect}_{\mathbb{R}}}(\mathrm{T}_x(\mathbb{R}^d), \mathrm{T}_{f(x)}(M))$, the linear map being defined as $v^a \mapsto v^b \nabla_b f(x)^\mu$.

For the special case $d = m$, something a little bit special happens: we can then ask whether the derivative is invertible at each point or not (regarding as a linear map between tangent spaces *of the same dimension*). An incredibly important fact, known as the ***Inverse Function Theorem***, is that, if a function is smooth and the derivative is invertible at a point, then the function itself is *invertible with smooth inverse* in a neighborhood of that point. You might say that infinitesimal invertibility implies local invertibility.

Theorem 6.6.1 — Inverse Function Theorem. Let $S \subseteq \mathbb{R}^d$, let $f^a: D \to \mathbb{R}^d$ be smooth, and let $x \in \mathrm{Int}(D)$. Then, if $\nabla_a f(x)^b : \mathrm{T}_x(\mathbb{R}^d) \to \mathrm{T}_{f(x)}(\mathbb{R}^d)$ is invertible, then there exists an open neighborhood U of x such that $f|_U$ is invertible with smooth inverse.

R Bijective functions which are smooth and have smooth inverse will be called ***diffeomorphisms*** when you pass to the category of manifolds (they wind-up being of course the isomorphisms in the category of manifolds). Thus, we say that if a smooth map has a derivative that is invertible at a point, then it is a ***local diffeomorphism***.

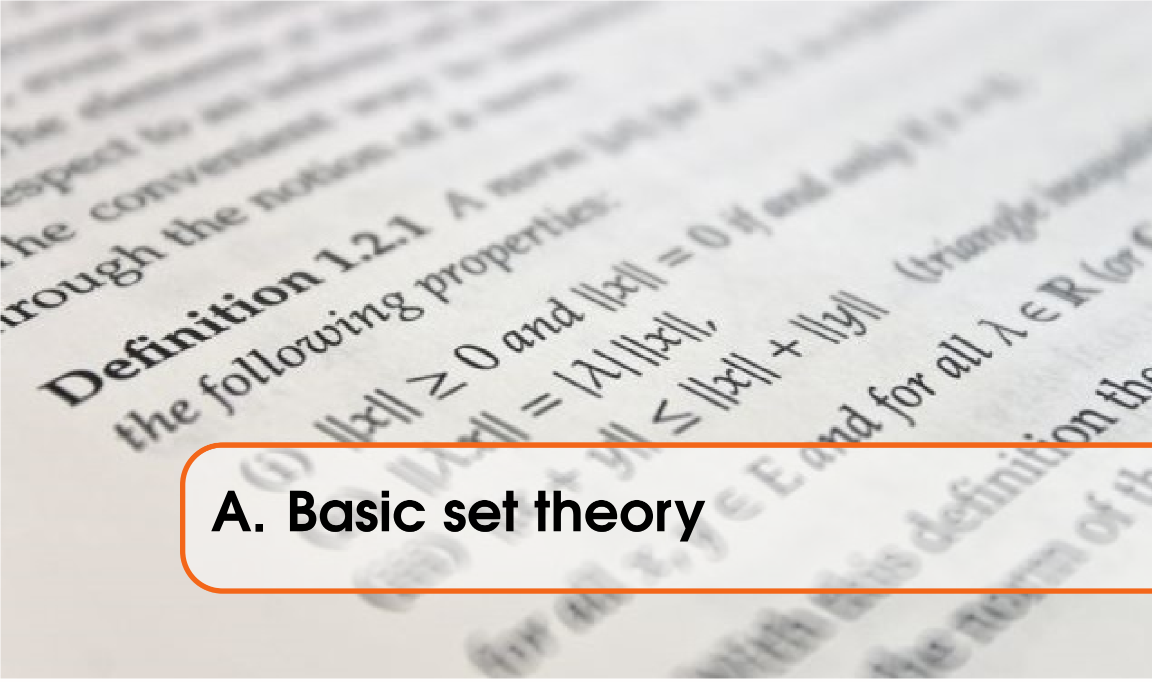
Proof. We leave this as an exercise.

Exercise 6.6.2 Prove this yourself.

R Hint: Yet another exercise that should not be—see [Rud76, pg. 221].

∎

Appendices



A. Basic set theory

A.1 What is a set?

When writing the text, it was important to me to be able to reduce all of the content here to essentially 'nothing'. By this, I mean that (i) if you grant me that naive logical deduction that your brain tells you is should be true is valid; and (ii) if you allow me to make use of the naive concept of a 'set', then you can prove everything in these notes.

When I say "naive concept of a 'set'", I am referring to the idea that, if you have a collection of things, whatever they may be, you are allowed to give that collection of things a name (e.g. "X"), and now X is a new thing that we may talk about, the *set* of the aforementioned things. From this perspective, you might say that the naive notion of a set is more a linguistic tool than anything else. Indeed, because of this, for the most part, we will completely ignore any set-theoretic concerns in these notes, but before we do just blatantly ignore any potential issues, we should first probably (attempt to) justify this dismissal.

Intuitively, a set is just a thing that 'contains' a bunch of other things, but this itself is of course not a precise mathematical definition. Ultimately, I claim there there is no need to have such a precise definition, but let's suppose for the moment that we would like to define what a set is. One way to do this would be to attempt to develop an axiomatic set theory, but there is a certain 'circularity' problem in doing this.

The term "axiomatic set theory" here refers to any collection of axioms which attempt to make precise the intuitive idea of a set. In a given theory, however, the symbols which we make use of to write down the axioms themselves form a *set*. The point is that, in attempting to write down a mathematically precise definition of a set, one must make use of the naive notion of a set.

Of course this example might not be very convincing. Why not just not think of all the symbols together and just think of them individually? It is true that if you fudge things around a bit you may be able to convince yourself that you're not really making use of the naive notion of a set here. That being said, even if you can convince yourself that you can get around the problem of first requiring a 'set' of symbols, sooner or later, in attempting to make sense out of an axiomatic set theory, you will need to make use of the naive notion of a set.

Because of this, we consider the idea of a set to be so fundamental as to be undefinable, and we simply assume that we can freely work with this intuitive idea of a collection of things all thought of as one thing, namely a set.

One has to be careful however. Naive set theory has paradoxes, a famous example of which is Russel's Paradox. Consider for example the set[1]

$$X \coloneqq \{Y : Y \notin Y\} . \tag{A.1.1}$$

Is $X \in X$? One resolution of this paradox is that it is nonsensical to construct the set of *all* things satisfying a certain property. Whenever you construct a set in this manner, your objects have to be already 'living inside' some other set. For example, we can write

$$X \coloneqq \{Y \in U : Y \notin Y\} \tag{A.1.2}$$

for some fixed set U.[2] Russel's Paradox now becomes the statement that $X \notin U$.

[1]Hopefully you have seen notation like this before. If not, really quickly skip ahead to Appendix A.2 The absolute basics to look up the meaning of this notation.

[2]"U" is for "universe".

This is still somehow not enough. For example, if you turn to Example B.1.3, the category of sets, you'll see that we do need to make use of the notion of the collection of "all sets", and we've just said that we are not allowed to quantify over *everything*, but only over things that are elements of a fixed set. One way to do this is to fix a set U which is closed under all the usual operations of set theory,[3] and then to interpret statements that refer to something like "All sets such that. . . " as in fact meaning "All elements of U such that. . . ". Upon doing this, the construction involved in Russel's Paradox is perfectly valid, and indeed, does give us a new set, and the 'paradox' itself now simply becomes the argument that this new set is not an element of U.[4]

The content of this section was meant only to convince you that (i) there is no way of getting around the fact that the idea of collecting things together is undefinably fundamental, and that (ii) ultimately this naive idea is not paradoxical.

Disclaimer: I am neither a logician nor a set-theorist, so take what I say with a grain of salt.

A.2 The absolute basics

A.2.1 Some comments on logical implication

For us, the term ***statement*** will refer to something that is either true or false. The word ***iff*** is short-hand for the phrase *if and only if*. So, for example, if A and B are statements, then the sentence "A iff B." is logically equivalent to the two sentences "A if B." and "A only if B.". In symbols, we write $B \Rightarrow A$ and $A \Rightarrow B$ respectively. The former logical implication is perhaps more obvious; the other might be slightly trickier to translate from the English to the mathematics. The way you might think about it is this: if A is true, then, because A is true *only if* B is true, it must have been the case that B was true too.

[3] One way in which to make this precise is what is called a *Grothendieck universe*. The details of this will not matter for us, but if you're curious feel free to Google the term.

[4] One nice thing about this approach of avoiding paradoxes is that *everything* is still a set, that is, there is no need to make this awkward distinction between 'actual' sets and what would be referred to as *proper classes* .

Thus, "A only if B." is logically equivalent to "A implies B.". We then write "$A \Leftrightarrow B$ as alternative notation for the English "A iff B".

If A and B are statements, then $A \Rightarrow B$ is a statement: True $\Rightarrow$ True is considered true, True $\Rightarrow$ False is considered false, False $\Rightarrow$ True is considered true, and False $\Rightarrow$ False is considered true. Hopefully the first two of these make sense, but how does one understand why it should be the case that False $\Rightarrow$ True is true? To see this, I think it helps to first note the following.[5]

$$\text{"}\forall x \in X,\ \mathcal{P}(x)\text{." is logically equivalent to "}x \in X \Rightarrow \mathcal{P}(x)\text{.",} \tag{A.2.1.1}$$

where $\mathcal{P}(x)$ is a statement that depends on x.

Now consider the following example in English.

$$\text{Every pig on Mars owns a shotgun.} \tag{A.2.1.2}$$

Is this statement true or false? Under the (hopefully legitimate assumption) that there is no pig on Mars at all, my best guess is that most native English speakers would say that this is a true statement. In any case, this is mathematics, not linguistics, and for the sake of definiteness, we simply declare a statement such as this to be *vacuously true* (unless of course there are pigs on Mars, in which case we would need to determine if they all owned shotguns). This example is meant to convince you that, in the case that X is empty, it is reasonable to declare the statement $\forall x \in X, P(x)$ to be true for tautological reasons.

Now, appealing back to (A.2.1.1), hopefully it now also seems reasonable to declare statements of the form False $\Rightarrow B$ to be true (where B is any statement), likewise for tautological reasons.

If we know a certain statement to be true, there are several other statements that we know automatically to be true. For example, if A is true, then $\neg\neg A$ is automatically true.[6] Another important example of this is given by the *contrapositive*.

[5]The symbol "$\forall$" in English reads "for all". Similarly, the symbol "$\exists$" is read as "there exists".

[6]$\neg A$, read "not A", is a statement which is false if A is true and false if A is true.

Definition A.2.1.3 — Converse, inverse, and contrapositive Let A and B be statements. Then,

(i). the ***converse*** of the statement $A \Rightarrow B$ is the statement $B \Rightarrow A$;
(ii). the ***inverse*** of the statement $A \Rightarrow B$ is $\neg A \rightarrow \neg B$; and
(iii). the ***contrapositive*** of the statement $A \Rightarrow B$ is $\neg B \rightarrow \neg A$.

R Referring back to our earlier comments on the phrase "iff", if ever you want to prove "A iff B", you must prove $A \Rightarrow B$ (i.e. "A only if B.") as well as its *converse*, $B \Rightarrow A$ (i.e. A if B).

Proposition A.2.1.4 Let A and B be statements. Then, $A \Rightarrow B$ is true iff its contrapositive $\neg B \Rightarrow \neg A$ is true.

Proof. ($\Rightarrow$) Suppose that $A \Rightarrow B$ is true. We would like to show that $\neg B \Rightarrow \neg A$. So, suppose that $\neg B$ is true. We would then like to prove $\neg A$. We proceed by contradiction: suppose that A is true. Then, as $A \Rightarrow B$, it must be that B is true: a contradiction of the fact that we have assumed that $\neg B$ is true. Therefore, our assumption that A is true must have been false. Thus, it must be that $\neg A$ is true.

($\Leftarrow$) As the contrapositive of the contrapositive is the original statement, this follows from ($\Rightarrow$). ■

■ **Example A.2.1.5** Let P be the statement "If it is raining, then it is wet.".

The converse of P is "If it is wet, then it is raining.".

The inverse of P is "If it is not raining, then it is not wet.".

The contrapositive of P is "If it is not wet, then it is not raining.

Hopefully this makes it clear how the converse can be false even if the original statement is true. Also be sure to understand in this example how the contrapositive is indeed equivalent to the original statement.

Given that a statement is true iff its contrapositive is true, it is important to know how to correctly negate statements (and of course this is important to know for other reasons as well).

■ **Example A.2.1.6** Let $\mathcal{P}(x)$ be a statement that depends on x.

(i). "$\neg(\forall x,\ \mathcal{P}(x))$" is equivalent to "$\exists x,\ \neg\mathcal{P}(x)$".
(ii). "$\neg(\exists x,\ \mathcal{P}(x))$" is equivalent to "$\forall x,\ \neg\mathcal{P}(x)$".
(iii). "$\neg(A$ and $B)$" is equivalent to "$\neg A$ or $\neg B$".
(iv). "$\neg(A$ or $B)$" is equivalent to "$\neg A$ and $\neg B$".

R For example, suppose you want to prove the statement "Every positive integer is even." is *false*. To do this, you want to exhibit a positive integer which is not even. Explicitly, the original statement is "$\forall x \in \mathbb{Z}^+,\ x$ is even.", and so its negation is "$\exists x \in \mathbb{Z}^+,\ x$ is not even.". For some reason, this tends to trip students up when I ask them to show that a statement is false: to prove that statements of this form[a] are false, you *must* exhibit a counter-example—explaining why a counter-example should exist, without *proving*[b] one exists, is not enough. For example, don't say "The statement "Every partially-ordered set is totally-ordered." is false because there is an extra condition in the definition of totally-ordered."—in this case, you *must* give an example of a partially-ordered set which is not totally-ordered.[c]

[a]That is, of the form "$\forall x,\ \mathcal{P}(x)$". Of course, not every statement is of this form, and so proving a statement is false doesn't necessarily mean you have to give a counter-example (for example, if I ask you to prove that $|N| = |\mathbb{R}|$ is false, it would not make sense to give a counter-example).

[b]It is almost always the case that the easiest way to prove a counter-example exists is simply to write one down.

[c]See Definitions A.3.3.6 and A.3.3.13.

A.2.2 A bit about proofs

Proofs are absolutely fundamental to mathematics. Indeed, you might say that mathematics *is* the study of those truths which are provable.[7] But what actually *is* a proof?

A proof is essentially just a particularly detailed argument that a statement is true. The question then is "How much detail?". Well, an extremist might say that a proof should be detailed enough so as to be verifiable by a computer—if a computer can verify it using axioms alone, then there can be no doubt at all as to the truth of the statement. Doing this in practice, however, well, would be a little bit insane—no one (or almost no one) writes proofs in this amount of detail.

The objective then I would say it to provide enough detail so as to convince *your target audience* that enough detail could be filled in, at least *in principle*, so as to be verified by a computer, if a member of your target audience really wanted to take (waste?) their time doing so. This is why two different proofs of the same statement, one several pages long and another a paragraph long, can both be considered equally valid proofs: one proof could have been written to be accessible to undergraduates and the other to be accessible to professional mathematicians. As a student, however, I would recommend you consider your target audience to be *yourself*. You should put down enough detail so that, if you came back to your proof after a year of not thinking about it, you should be able to follow your work no problem. In particular, if you're ever writing a proof and you wonder "Is this valid?", the answer is "No, it's not valid."—you need to add more detail until there is *no doubt* whatsoever that your argument is correct. Tricking me (or yourself) into thinking you know the details when in fact you do not is not the way to go about learning mathematics.

Okay, so enough with this wishy-washy philosophical BS. I should probably at least give you some *concrete* advice about proof-writing.

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[7]In contrast to those truths are which true by observation. For example, while the statements "$x \in \mathbb{R}$ implies $x^2 \geq 0$." and "The mass of the electron is $9.109\,383\,56(11) \times 10^{-31}$ kg." are both true, they are true in fundamentally different ways—the former is true because we can prove it and the latter is true because we measure it.

I think probably most of proof-writing should be learned by doing, but I suppose I can say at least a couple of things.[8][9]

Iff

We mentioned the meaning of the word "iff" in the previous section, and we wound up giving an example of a proof which involved the phrase (Proposition A.2.1.4). Allow us to elaborate.

> If ever asked to prove a statement of the form "A iff B", you need to prove *two things*: first, assuming A, you prove B; then, assuming B, you prove A.

See Proposition A.2.1.4 for a concrete example of this.

The following are equivalent

The phrase "The following are equivalent." is similar to the phrase "iff", but is used when dealing with more than two statements. For example, Exercise 1.2.26 reads

Let $m, n \in \mathbb{Z}$. Then, the following are equivalent.

(i). $m < n$.
(ii). $m \leq n - 1$.
(iii). $m + 1 \leq n$.

To prove this, you need to prove that (i) iff (ii), (i) iff (iii), and (ii) iff (iii)—this is exactly what it means for all the three statements to be logically-equivalent to one another. On the face of it, it seems like this would mean we would have to do $2 \times 3 = 6$ proofs. Not so. In fact, it is enough to prove (i) implies (ii), (ii) implies (iii), and (iii) implies (i). Using these three implications alone, you can go from any one statement to any other. For example, (ii) implies (i) because, if (ii), then (iii), and hence (i).

[8]Keep in mind that in the following subsubsections we will often make use of examples to illustrate concepts that we technically have not yet developed the mathematics for yet. First of all, you needn't worry, as because we are just using the examples for the purposes of illustration, this doesn't make our development circular. Secondly, if you can't follow an example because you haven't seen it before, don't worry—just get what you can out of it and move on.

[9]If you are fine with proofs, you can probably safely skip to the next subsection, Appendix A.2.3 Sets.

For all...

If the statement you are trying to prove is of the form "$\forall X \in X,\ \mathcal{P}(x)$", you should almost certainly start your proof with something like "Let $x \in X$ be arbitrary." You then prove $\mathcal{P}(x)$ itself. Pretty self-explanatory.

The contrapositive and proof by contradiction

Proof by contradiction and *proof by contraposition* are two closely related proof techniques. In fact, in a sense to be explained below, they're the *same* proof technique. Before we get there, however, let us first explain what these two techniques refer to.

First, we explain "contradiction". Assume you want to prove the statement "A implies B.". Of course, you first assume that A is true. You now try to prove that B is true. Sometimes doing this directly can prove difficult, and in such cases, you can try what is referred to as ***proof by contradiction***: Suppose that $\neg B$ is true. Now, using A *and* $\neg B$, try to prove something you already know to be false. As the *only* assumption you made was $\neg B$, that assumption must have been incorrect, and therefore $\neg\neg B$ is true, and hence B is true.[10]

On the other hand, ***proof by contraposition*** refers to nothing more than an application of Proposition A.2.1.4. That is, if you would like to prove that "A implies B", you instead prove that "$\neg B$ implies $\neg A$".

All that remains in this subsubsection is an explanation of the relationship between proof by contraposition and proof by contradiction. As this relationship is not particularly important, feel free to skip to the next subsubsection.

Superficially, proof by contradiction and proof by contraposition appear to be distinct, but related techniques. On the other hand, they are equivalent in a sense to be described as follows.[11] First of all, we have to be precise about what we mean by "proof by

[10]This logic implicitly uses what is called the ***Principle of the Excluded Middle***, which says that, if A is a statement, then A is true or A is false. Some mathematicians reject this as valid (or so Wikipedia claims). They are crazy. Such crazy mathematicians thus cannot use proofs by contradiction. Pro-tip: don't be crazy.

[11]The explanation of exactly in what sense these two proof techniques are equivalent is not particularly useful. Certainly, I find it highly unlikely that what follows in this subsubsection will be of significant use in actual proof writing. Thus, feel free to skip to the end of the following proof unless you are particularly curious.

contradiction" and "proof by contraposition". The precise statement of "proof by contraposition" is given in Proposition A.2.1.4: "A implies B" is equivalent to "$\neg B$ implies $\neg A$". On the other hand, the precise statement of "proof by contradiction" is given in the following statement.

Proposition A.2.2.1 Let A and B be statements. Then, $A \Rightarrow B$ is true iff (A and $\neg B$) $\Rightarrow$ False is true.

Proof. ($\Rightarrow$) Suppose that $A \Rightarrow B$ is true. We would like to show that (A and $\neg B$) $\Rightarrow$ False. So, suppose that A and $\neg B$ are true. As $A \Rightarrow B$ is true, it follows that B is true. But then, B and $\neg B$ are true, and hence False.

($\Leftarrow$) Suppose that (A and $\neg B$) $\Rightarrow$ False is true. Taking the contrapositive, it follows that $\neg A$ or B is true. We would like to show that $A \Rightarrow B$ is true. Taking the contrapositive again, it suffices to show that $\neg B \Rightarrow \neg A$. So, suppose $\neg B$. We wish to prove $\neg A$. However, as we know that $\neg A$ or B is true, it in fact must be the case that $\neg A$ is true. ■

If you examine the proofs of Proposition A.2.1.4 and Proposition A.2.2.1,[12] you will find respectively that the former makes use of proof by contradiction and that the latter makes use of proof by contraposition. It is in this sense that they are equivalent proof techniques.

Okay, so up until now, this has all be pretty precise, but I would like to give an intuitive explanation as to the difference between the two. Every proof by contraposition and be reduced to a proof by contradiction in the following way: assume your hypotheses A, proceed by contradiction and assume $\neg B$, and proceed to prove $\neg A$: contradiction. The key is that in a proof by contraposition your contradiction is of the form A and $\neg A$, whereas with proof by contradiction you obtain more general contradictions. Thus, while superficially proof by contradiction might seem much stronger, it is in

[12]The precise statements of "proof by contraposition" and "proof by contradiction" respectively.

fact not actually so. This is similar to how "the Principle of Strong Induction seems stronger than (just) the Principle of Induction, but in fact they are equivalent statements—see below.

Finally, we end with a quick comment on usage. First of all, it is very easy to rephrase a proof by contraposition as a proof by contradiction (as explained above), and so, if you like, you needn't worry about proof by contraposition at all. Furthermore, proof by contradiction tends to be most useful when "B" in "A implies B." is somehow 'already negated'. For example, in the proof of Cantor's Cardinality Theorem (Theorem 2.1.15), we wish to show that there is no surjection from X to 2^X. Here, we could proceed by contradiction, and say "Suppose there exists a surjection $f: X \to 2^X$.", and then use this to produce a contradiction.[13]

Without loss of generality...

You will often see the phrase "Without loss of generality..." used in proofs. It is easiest to demonstrate what this means, and how to use it yourself, with an example.

The definition of integral (Definition A.4.22) reads

> A rg $\langle X, +, 0, \cdot\rangle$ is *integral* iff it has the property that, whenever $x \cdot y = 0$, it follows that either $x = 0$ or $y = 0$.

So, imagine you were doing a proof, and you know that $\langle X, +, 0, \cdot\rangle$ was an integral rg,[14] and that $x \cdot y = 0$ for $x, y \in X$. The definition of integral implies that $x = 0$ or $y = 0$. *At this point, you could say*, "Without loss of generality, suppose that $x = 0$.". You then continue the proof using the fact that $x = 0$.

Logically, you would first have to assume that $x = 0$, finish the proof in that case, and then go back to the case where $y = 0$, and finish the proof again. However, if the proofs are essentially identical[15] in these two cases, you are 'allowed' to cut your work in half with the phrase "Without loss of generality..."—there is no point in repeating the same logic with a different letter twice.

[13] We don't quite phrase it like that, but this is essentially what we do.

[14] You shouldn't need to know what a rg is to understand the explanation of "Without loss of generality...".

[15] Obviously, if the proof needed to do the case $x = 0$ is significantly different from the proof needed to do the case $y = 0$ (i.e. is more than just a letter swap $x \leftrightarrow y$), you should not use this phrase and instead write up the proofs of both cases individually.

If XYZ we are done, so suppose that ¬XYZ

As with "Without loss of generality...", the use of this phrase is easiest to demonstrate with an example.

The definition of prime (Definition C.5) reads

> Let $p \in \mathbb{Z}$. Then, p is *prime* iff ... whenever $p \mid (mn)$, it follows that $p \mid m$ or $p \mid n$.[16]

So, let $p \in \mathbb{Z}$ and suppose that you want to prove that p is prime. To prove this condition, you would say "Let $m, n \in \mathbb{Z}$ and suppose that $p \mid (mn)$." From here, you now want to prove that $p \mid m$ or $p \mid n$. At this point, you can say "If $p \mid m$ we are done, so suppose that $p \nmid m$.".

Hopefully, the logic of this is pretty self-explanatory. If it helps, however, you can view this as essentially the same as proof by contradiction. If we were to proceed by contradiction, instead we would say "Suppose that $p \nmid m$ and $p \nmid n$.", and from that deduce a contradiction. Instead, in this case, we shall assume $p \nmid m$, and use that to prove that $p \mid m$.

Proving two sets are equal

A lot of proofs require you to show that two different sets are in fact one in the same. The way to prove this is made precise with Exercise A.2.3.4. In the meantime, however, we can say the following.

> Let S and T be sets and suppose that you want to prove that $S = T$. You then need to prove *two* things: if $s \in S$, then $s \in T$; and if $t \in T$, then $t \in S$.

This logic is very much identical to that used for proving "iff" statements.

Induction

Induction

Let $\mathcal{P}_m$ be a statement for $m \in \mathbb{N}$. The ***Principle of Induction*** says that

[16]We have omitted part of the definition in the "..." that is irrelevant for us at the moment.

> If (i) $\mathcal{P}_0$ is true and (ii) $\mathcal{P}_m \Rightarrow \mathcal{P}_{m+1}$ is true for all $m \in \mathbb{N}$, then $\mathcal{P}_m$ is true for all $m \in \mathbb{N}$.[17]

The logic is as follows. Suppose we want to prove $\mathcal{P}_3$. Then, because $\mathcal{P}_0$ and $\mathcal{P}_0 \Rightarrow \mathcal{P}_1$, it follows that $\mathcal{P}_1$.[18] Then, because $\mathcal{P}_1$ and $\mathcal{P}_1 \Rightarrow \mathcal{P}_2$, it follows that $\mathcal{P}_2$. Then, because $\mathcal{P}_2$ and $\mathcal{P}_2 \Rightarrow \mathcal{P}_3$, it follows that $\mathcal{P}_3$. Of course, nothing is special about $m = 3$, and so this logic can be used to prove $\mathcal{P}_m$ for all $m \in \mathbb{N}$.[19]

You can also use similar logic to define things. For example, you might define a bijection $f\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by

$$\begin{aligned} f(0) &:= \langle 0, 0\rangle \\ f(m+1) &:= \begin{cases} \langle f(m)_x - 1, f(m)_y + 1\rangle & \text{if } f(m)_x \geq 1 \\ \langle f(m)_x + f(m)_y + 1, 0\rangle & \text{otherwise.}^{20} \end{cases} \end{aligned} \tag{A.2.2.2}$$

Strong Induction

Equally as valid, for the exact same reason, is what is sometimes referred to the ***Principle of Strong Induction***, which says that

> If (i) $\mathcal{P}_0$ is true and (ii) $(\forall 0 \leq k \leq m,\ \mathcal{P}_k) \Rightarrow \mathcal{P}_{m+1}$ is true for all $m \in \mathbb{N}$, then $\mathcal{P}_m$ is true for all $m \in \mathbb{N}$.

The key difference between this and 'regular' induction is that, in the induction step, you don't just assume $\mathcal{P}_m$, but instead, you assume $\mathcal{P}_0$, and $\mathcal{P}_1$, and $\mathcal{P}_2$, and $\ldots\, \mathcal{P}_m$. Superficially, this does indeed seem stronger,[23] but in fact 'regular' induction and strong induction are

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[17](i) is referred to as the ***initial case*** or the ***initial step***, and (ii) is referred to as the ***inductive step***.

[18]Incidentally, the passage from "A and $A \Rightarrow B$" to B is called ***modus ponens***, which itself is short for "modus pōnendō pōnēns", which is Latin for (literally) "the method to be affirmed, affirms".

[19]Though it won't be able to prove $\mathcal{P}_\infty$! For example, a common fake proof that π is rational essentially proves by induction that, for all $m \in \mathbb{Z}^+$, the decimal approximation of π with m digits is rational, and 'therefore' π is rational. Sorry, but that's not how induction works.

[22]The picture is that you go 'down the diagonal', unless you 'hit the edge', in which case you 'hop to the top of the next diagonal'.

[23]Because during the inductive step, you don't get to assume just the single statement $\mathcal{P}_m$, but rather the $m + 1$ statements $\mathcal{P}_0$, and $\mathcal{P}_1$, and $\mathcal{P}_2$, and $\ldots\, \mathcal{P}_m$.

equivalent, though sometimes it can be quite convenient to be make use of all of $\mathcal{P}_0, \dots, \mathcal{P}_m$ instead of just $\mathcal{P}_m$.

For example, suppose that you want to prove that every positive integer greater-than-or-equal to 2 is divisible by some prime. In this case, the initial step is simply to prove that "2 is divisible by some prime.". This is of course trivial: 2 is divisible by 2, which is prime. Here's where things get a bit different, however: for the inductive step, assume that $2, 3, 4, \dots, m$ are all divisible by some prime. Using this, we want to show that $m + 1$ is divisible by some prime. Well, either $m + 1$ is prime itself, in which case $m + 1$ is divisible by $m + 1$, or it is not prime, in which case $m + 1$ is divisible by some integer k with $2 \leq k \leq m$. By the induction hypothesis, k is divisible by some prime, and hence $m + 1$ is in turn divisible by some prime.

Note that we will not usually make a distinction between 'regular' induction and strong induction in these notes. When using either method, we shall simply say something like "We proceed by induction. . . ".

Well-founded induction

The most powerful form of induction which subsumes all other 'types' of induction is known as *well-founded induction*. As it is less elementary than what we have discussed thus far, we leave a detailed discussion of this until Appendix A.3.4. We can, however, at least say a little for the time being..

The basic idea is to replace the set with which you index your statements (previously $\mathbb{N}$) with a more general type of set X. It turns out that the only structure on $\mathbb{N}$ relevant to induction is the ordering, and so you don't just replace $\mathbb{N}$ with a set X, you replace $\langle \mathbb{N}, \leq \rangle$ with a pair $\langle X, \leq \rangle$, where X is a set and $\leq$ is a *relation* (Definition A.3.1) on X. It turns out that (Theorem A.3.4.3) the only property of $\leq$ on $\mathbb{N}$ that was necessary for induction to work is what is called *well-foundedness* (Definition A.3.5.3), which states that every nonempty subset has a minimal element (Definition A.3.5.3).

The ***Principle of Well-Founded Induction*** then says that

> If for every $x_0 \in X$, $(\forall x < x_0, \mathcal{P}_x) \Rightarrow \mathcal{P}_{x_0}$, then $\mathcal{P}_x$ is true for all $x \in X$.

If you ever want to perform an induction-like argument, but it's not working because you have more than countably-infinite many statements, try well-founded induction.

Finally, we mention that there is a method of proof called *transfinite induction*. It is technically a special case of well-founded induction, but much more common.[24] Roughly speaking, transfinite induction is to well-ordered as well-founded induction is to well-founded. We refrain from discussing it because (i) we don't need to—well-founded induction is stronger and (ii) the usual way it's stated requires the development of what are called *ordinals*), which would take us quite astray. Still, you should be aware of it so you can go and learn it if you ever feel this will be useful to you (if you become a mathematician, it almost certainly will be at some point).

A.2.3 Sets

The idea of a set is something that contains other things.

> If X is a *set* which contains an *element* x, then we write $x \in X$.[26]Two sets are equal iff they contain the same elements. (A.2.3.1)

Definition A.2.3.2 — Empty-set The ***empty-set***, $\emptyset$, is the set $\emptyset := \{\}$.

> **R** That is, $\emptyset$ is the set which contains no elements.

> **R** If ever you see an equals sign with a colon in front of it (e.g. in "$\emptyset := \{\}$"), it means that the equality is true *by definition*. This is used in definitions themselves, but also outside of definitions to serve as a reminder as to why the equality holds.

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[24]For some reason, not too many people seem to know of well-founded induction.

[30]Sometimes we will also write $X \ni x$ if it happens to be more convenient to write it in that order (for example, in $\mathbb{R} \ni x \mapsto x^2$).

Definition A.2.3.3 — Subset Let X and Y be sets. Then, X is a ***subset*** of Y, written $X \subseteq Y$, iff whenever $x \in X$ it is also the case that $x \in Y$.

R You should note that many authors use the notation "$X \subset Y$" simply to indicate that X is a (*not-necessarily-proper*) subset of Y.

R Generally speaking we put slashes through symbols to indicate that the statement that would have been conveyed without the slash is false. For example, $x \notin X$ means that x is not an element of X, the statement that $X \not\subseteq Y$ means that X is not a subset of Y, etc..

Exercise A.2.3.4 Let X and Y be sets. Show that $X = Y$ iff $X \subseteq Y$ and $Y \subseteq X$.

Definition A.2.3.5 — Proper subset Let X be a subset of Y. Then, X is a ***proper subset*** of Y, written $X \subset Y$, iff there is some $y \in Y$ that is not also in X.

R Let X be a set, let $\mathcal{P}$ be a property that an element in X may or may not satisfy, and let us write $\mathcal{P}(x)$ iff x satisfies the property $\mathcal{P}$. Then, the notation

$$\{x \in X : \mathcal{P}(x)\}$$

is read "The set of all elements in X such that $\mathcal{P}(x)$." and represents a set whose elements are precisely those elements of X for which $\mathcal{P}$ is true. Sometimes this is also written as

$$\{x \in X | \mathcal{P}(x)\},$$

but my personal opinion is that this can look ugly (or even slightly confusing) if, for example, $\mathcal{P}(x)$ contains an absolute value in it, e.g.

$$\{x \in \mathbb{R} | |x| < 1\}.$$

Definition A.2.3.6 — Complement Let X and Y be sets. Then, the ***complement*** of Y in X, $X \setminus Y$, is defined by

$$X \setminus Y := \{x \in X : x \notin Y\}. \tag{A.2.3.7}$$

If X is clear from context, sometimes we write $Y^{\mathsf{C}} := X \setminus Y$.

Definition A.2.3.8 — Union and intersection Let A, B be subsets of a set X. Then, the ***union*** of A and B, $A \cup B$, is defined by

$$A \cup B := \{x \in X : x \in A \text{ or } x \in B\}\,. \tag{A.2.3.9}$$

The ***intersection*** of A and B, $A \cap B$, is defined by

$$A \cap B := \{x \in X : x \in A \text{ and } x \in B\}\,. \tag{A.2.3.10}$$

R More generally, if $\mathscr{S}$ is a collection[a] of subsets of X, then the *union* and *intersection* of all sets in $\mathscr{S}$ are defined by

$$\bigcup_{S \in \mathscr{S}} S := \{x \in X : \exists S \in \mathscr{S} \text{ such that } x \in S.\}$$

and

$$\bigcap_{S \in \mathscr{S}} S := \{x \in X : \forall S \in \mathscr{S},\ x \in S.\}\,.$$

[a] Technically, the term ***collection*** is just synonymous with the term "set", though it tends to be used in cases when the elements of the set itself are to be thought of as other sets (e.g. here where the elements of $\mathscr{S}$ are subsets of X).

Definition A.2.3.11 — Disjoint and intersecting Let A, B be subsets of a set X. Then, A and B are ***disjoint*** iff $A \cap B = \emptyset$. A and B ***intersect*** (or ***meet***) iff $A \cap B \neq \emptyset$.

Exercise A.2.3.12 — De Morgan's Laws Let $\mathscr{S}$ be a collection of subsets of a set X. Show that

$$\left(\bigcup_{S \in \mathscr{S}} S\right)^{\mathsf{C}} = \bigcap_{S \in \mathscr{S}} S^{\mathsf{C}} \text{ and } \left(\bigcap_{S \in \mathscr{S}} S\right)^{\mathsf{C}} = \bigcup_{S \in \mathscr{S}} S^{\mathsf{C}}. \quad \text{(A.2.3.13)}$$

Exercise A.2.3.14 Let X be a set and let $S, T \subseteq X$. Show that $S \setminus T = S \cap T^{\mathsf{C}}$.

Definition A.2.3.15 — Symmetric-difference Let A, B be subsets of a set X. Then, the ***symmetric-difference*** of A and B, $A \bigtriangleup B$, is defined by

$$A \bigtriangleup B := (A \cap B^{\mathsf{C}}) \cup (A^{\mathsf{C}} \cap B). \quad \text{(A.2.3.16)}$$

R If you draw a "Venn diagram", you break up X into four disjoint pieces: everything outside A and B, things inside both A and B, things inside A but not B, and things inside B but not A. The symmetric difference is the union of the last two regions.

Put another way, the symmetric difference is the elements in $A \cup B$ that A and B do *not* have in common.

The union and intersection of two sets are ways of constructing new sets, but one important thing to keep in mind is that, a priori, the two sets A and B are assumed to be contained within another set X. But how do we get entirely new sets without already 'living' inside another? There are several ways to do this.

Definition A.2.3.17 — Cartesian-product Let X and Y be sets. Then, the ***Cartesian-product*** of X and Y, $X \times Y$, is

$$X \times Y := \{\langle x, y\rangle : x \in X, y \in Y\} . \tag{A.2.3.18}$$

R If you really insist upon everything being defined in terms of sets we can take

$$\langle x, y\rangle := \{x, \{x, y\}\} . \tag{A.2.3.19}$$

The reason we use the notation $\langle x, y\rangle$ as opposed to the probably more common notation (x, y) is to avoid confusion with the notation for open intervals.

R If $Y = X$, then it is common to write $X^2 := X \times X$, and similarly for products of more than two sets (e.g. $X^3 := X \times X \times X$). Elements in finite products are called ***tuples*** or sometimes ***lists***. For example, the elements of X^2 are 2-tuples (or just ***ordered pairs***), the elements in X^3 are 3-tuples, etc..[a]

[a]If you really want to be pedantic about things, you might complain "OMG what is this crazy new symbol '3'!? We haven't defined the naturals yet!". In this case, you should merely interpret X^3 as short-hand for $X \times X \times X$. Similar comments apply throughout this appendix.

Definition A.2.3.20 — Disjoint-union Let X and Y be sets. Then, the ***disjoint-union*** of X and Y, $X \sqcup Y$, is

$$\begin{aligned} &X \sqcup Y \\ &\quad := \{\langle a, m\rangle : m \in \{0, 1\},\ a \in X \text{ if } m = 0,\ a \in Y \text{ if } m = 1\} . \end{aligned}$$

R Intuitively, this is supposed to be a copy of X together with a copy of Y. a can come from either set, and the 0 or 1 tells us which set a is supposed to come from. Thus, we think of $X \subseteq X \sqcup Y$ as $X = \{\langle a, 0\rangle : a \in X\}$ and $Y \subseteq X \sqcup Y$ as $Y \{\langle a, 1\rangle : a \in Y\}$.

The key difference between the union and disjoint-union is that, in the case of the union of A and B, an element that x is both in A and in B is a *single* element in $A \cup B$, whereas in the disjoint-union there will be two copies of it: one in A and one in B. Hopefully the next example will help clarify this.

■ **Example A.2.3.21 — Union vs. disjoint-union** Define $A := \{a, b, c\}$ and $B := \{c, d, e, f\}$. Then, $A \cup B = \{a, b, c, d, e, f\}$. On the other hand, $A \sqcup B = \{a, b, c_A, c_B, d, e, f\}$, where $A \sqcup B \supseteq A = \{a, b, c_A\}$ and $A \sqcup B \supseteq B = \{c_B, d, e, f\}$.

Definition A.2.3.22 — Power set Let X be a set. Then, the ***power set*** of X, 2^X, is the set of all subsets of X,

$$2^X := \{A : A \subseteq X\}. \tag{A.2.3.23}$$

R We will discuss the motivation for this notation in the next subsection (see Exercise A.3.31).

A.3 Relations, functions, and orders

Having defined Cartesian products, we can now make the following definition.

Definition A.3.1 — Relation A ***relation*** between two sets X and Y is a subset R of $X \times Y$.

R For a given relation R, we write $x \sim_R y$, or just $x \sim y$ if R is clear from context, iff $\langle x, y\rangle \in R$. Often we will simply refer to the relation by the symbol $\sim$ instead of R.

R It is important to be able to understand how to translate between the two different notations for writing a relation. In one direction, if you know $R \subseteq X \times Y$, then $x \sim y$ iff $\langle x, y\rangle \in R$, as already mentioned. In the other direction, if you know $\sim$, then $R = \{\langle x, y\rangle \in X \times Y : x \sim y\}$.

R If $X = Y$, we will say that $\sim$ is a relation *on* X.

Definition A.3.2 — Composition Let X, Y, and Z be sets, and let R be a relation on X and Y, and let S be a relation on Y and Z. Then, the ***composition***, $S \circ R$, of R and S is the relation on X and Z defined by

$$\begin{aligned} S \circ R := \{&\langle x, z\rangle \in X \times Z : \\ &\exists y \in Y \text{ such that } \langle x, y\rangle \in R \text{ and } \langle y, z\rangle \in S.\} . \end{aligned} \tag{A.3.3}$$

R If R is a relation on X (so that $R \circ R$ makes sense), for $k \in \mathbb{N}$, we shall abbreviate $R^k := \underbrace{R \circ \cdots \circ R}_{k}$, with $R^0 := \{\langle x, x\rangle \in X \times X : x \in X\}$.[a]

R You will see in the next definition that a function is in fact just a very special type of relation, in which case, this composition is exactly the composition that you (hopefully) know and love.

[a] R^0 is of course the identity function on X—see Example A.3.12.

Exercise A.3.4 Let R, S, and T be relations between X and Y, Y and Z, and Z and W respectively. Show that

$$(T \circ S) \circ R = T \circ (S \circ R). \tag{A.3.5}$$

While the appropriate generality in which to make the next definitions of restriction and corestriction is for arbitrary relations, the intuition you should use to understand the concepts will almost certainly come from your understanding of functions (and we will have little to no need to make use of these concepts in this amount of generality), so feel free to first read the definition of a function (Definition A.3.9) and related concepts and then come back to this if at first this seems confusing.

Definition A.3.6 — Restriction and corestriction Let f be a relation between two sets X and Y, let $S \subseteq X$, and let $T \subseteq Y$. Then, the ***restriction*** of f to S, $f|_S$, is a relation between S and Y defined by

$$f|_S \coloneqq f \circ \{\langle s, x\rangle \in S \times X : s = x\}\,.^a \tag{A.3.7}$$

The ***corestriction*** of f to T, $f|^T$, is a relation between X and T defined by

$$f|^T \coloneqq \{\langle y, t\rangle \in Y \times T : y = t\} \circ f.^b \tag{A.3.8}$$

If $g = f|_S$, then we will also say that f ***extends*** g. If $g = f|^T$, then we will also say that f ***coextends*** g.

R As mentioned before, to understand this, it probably helps to think of what these concepts mean in the case that the relation is in fact a function (note that a function is just a special type of relation—see Definition A.3.9). For example, if we have the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) \coloneqq x^2$, then we can obtain a new function $f|_{(-1,1)}: (-1,1) \to \mathbb{R}$ by restricting to $(-1,1) \subseteq \mathbb{R}$, and that new function is still given by the same 'rule', $f(x) \coloneqq x^2$—the only thing that has changed is the domain.[c]

Corestriction, on the other hand, is when we change the codomain of the relation. For example, we can corestrict this same function to $\mathbb{R}_0^+$ to obtain the function $f|^{\mathbb{R}_0^+}: \mathbb{R} \to \mathbb{R}_0^+$, once again, still with the 'rule' $f(x) \coloneqq x^2$.[d]

R These concepts will almost always arise in the case where the relation f is in fact a function. One reason we state the definitions in this more general case, besides just to be more general, is that *the* *corestriction of a function is not always going to be a new function*. For example, if we again define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := x^2$, $f|^{(-1,1)}$ is a relation that is no longer a function—the reason is that, for example, $\langle 2, y\rangle \notin f|^{(-1,1)}$ for any $y \in (-1,1)$. This is easy, but quite subtle, and not super important, so don't worry if this doesn't make sense to you at the moment. In fact, the corestriction of a function to $T \subseteq Y$ is another function iff T is contained in the image of f.

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[a] Here, $\{\langle s, x\rangle \in S \times X : s = x\}$ is of course to be interpreted as a relation between S and X.

[b] Similarly as before, $\{\langle y, t\rangle \in Y \times T : y = t\}$ is to be interpreted as a relation between Y and T.

[c] I realize we are making use of notation we have not yet technically. This is not a problem from a mathematical perspective as I am only trying to explain. From a pedagogical perspective, hopefully I'm not making use of anything unfamiliar to you—if so, flip ahead.

[d] The term "corestriction" is incredibly uncommon, as one often simply changes the codomain of the function without explicitly mentioning so. This is technically sloppy, but almost never actually causes problems. Still, it is important to realize that functions with different codomains are always different functions.

There are several different important types of relations. Perhaps the most important is the notion of a function.

Definition A.3.9 — Function A ***function*** from a set X to a set Y is a relation $\sim_f$ that has the property that for each $x \in X$ there is exactly one $y \in Y$ such that $x \sim_f y$. For a given function $\sim_f$, we denote by $f(x)$ that unique element of Y such that $x \sim_f f(x)$. X is the ***domain*** of f and Y is the ***codomain*** of f. The notation $f : X \to Y$ means "f is a function from X to Y.". The set of all functions from X to Y is denoted Y^X.

R The motivation for this notation is that, if X and Y are finite sets, then the cardinality (see Chapter 1 What is a number?) of the set of all functions from X to Y is $|Y|^{|X|}$.

R The arrow "$\mapsto$" can be used to define a function with necessarily giving it a name. For example, one can write

$$\mathbb{R} \ni x \mapsto 3x^2 - 5 \in \mathbb{R} \tag{A.3.10}$$

as notation for the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) := 3x^2 - 5$. Of course, this is convenient if it is unnecessary to give a specific name.

R The "x" in "$f(x)$" is sometimes referred to as the ***argument*** of the function.

Notation A.3.11 — Placeholders for arguments Let f be a function. A lot of the time, if we want to refer to f, we just say, well, "f". Besides this, however, we also frequently write "$f(x)$". Strictly speaking, this is incorrect—the function itself is $x \mapsto f(x)$, whereas $f(x)$ is the value of the function f at x in the domain. In practice, however, it is very common to write "$f(x)$" to denote the function itself, and not just a particular value.

We may also use the symbols "$\cdot$" or "$-$" to indicate the argument of a function. For example, we might denote a function using the notation $\langle\cdot,\cdot\rangle: V \times V \to \mathbb{K}$ or $\langle -,-\rangle: V \times V \to \mathbb{K}$. This notation means that "$\langle\cdot,\cdot\rangle$" is the name of a function, and furthermore, we denote its value at the element $\langle v_1, v_2\rangle \in V \times V$ as "$\langle v_1, v_2\rangle$". Thus, the dots tell you where to 'plug in' the variables. Similarly for "$-$".

■ **Example A.3.12 — Identity function** For every set X, there is a function, $\mathrm{id}_X : X \to X$, the ***identity function***, defined by

$$\mathrm{id}_X(x) := x. \tag{A.3.13}$$

Definition A.3.14 — Inverse function Let $f\colon X \to Y$ and $g\colon Y \to X$ be functions. Then, g is a ***left-inverse*** of f iff $g \circ f = \mathrm{id}_X$; g is a ***right-inverse*** of f iff $f \circ g = \mathrm{id}_Y$; g is a ***two-sided-inverse***, or just ***inverse***, iff g is both a left- and right-inverse of f.

Exercise A.3.15 Let g and h be two (two-sided)-inverses of f. Show that $g = h$.

Because of the uniqueness of two-sided-inverses, we may write f^{-1} for the unique two-sided-inverse of f.

Exercise A.3.16 Provide examples to show that left-inverses and right-inverses need not be unique.

Exercise A.3.17 Let X be a nonempty set.

(i). Explain why there is *no* function $f\colon X \to \emptyset$.
(ii). Explain why there is *exactly one* function $f\colon \emptyset \to X$.
(iii). How many functions are there $f\colon \emptyset \to \emptyset$?

Definition A.3.18 — Image Let $f\colon X \to Y$ be a function and let $S \subseteq X$. Then, the ***image*** of S under f, $f(S)$, is

$$f(S) := \{f(x) : x \in S\}. \tag{A.3.19}$$

The ***range*** of f, $f(X)$, is the image of X under f.

R We may also write $\mathrm{Im}(f) := f(X)$ for the range of f. If we simply say "image of f", you should interpret this to mean "image of X under f", i.e., the range $f(X)$.

R Note the difference between range and codomain. For example, consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2$. Then, the codomain is $\mathbb{R}$ but the range is just $[0, \infty)$. In fact the range and codomain are the same precisely when f is surjective (see Exercise A.3.24.(ii)).

Definition A.3.20 — Preimage Let $f : X \to Y$ be a function and let $T \subseteq Y$. Then, the ***preimage*** of T under f, $f^{-1}(T)$, is

$$f^{-1}(T) := \{x \in X : f(x) \in T\} . \tag{A.3.21}$$

Exercise A.3.22 Let $f : X \to Y$ be a function and let $T \subseteq Y$. Show that $f^{-1}(T^{\mathsf{C}}) = f^{-1}(T)^{\mathsf{C}}$. For $S \subseteq X$, find examples to show that we need not have either $f(S^{\mathsf{C}}) \subseteq f(S)^{\mathsf{C}}$ nor $f(S)^{\mathsf{C}} \subseteq f(S^{\mathsf{C}})$.

Definition A.3.23 — Injectivity, surjectivity, and bijectivity Let $f : X \to Y$ be a function. Then,

(i). (Injective) f is ***injective*** iff for every $y \in Y$ there is at most one $x \in X$ such that $f(x) = y$.
(ii). (Surjective) f is ***surjective*** iff for every $y \in Y$ there is at least one $x \in X$ such that $f(x) = y$.
(iii). (Bijective) f is ***bijective*** iff for every $y \in Y$ there is exactly one $x \in X$ such that $f(x) = y$.

R It follows immediately from the definitions that a function $f : X \to Y$ is bijective iff it is both injective and surjective.

Exercise A.3.24 Let $f : X \to Y$ be a function.

(i). Show that f is injective iff whenever $f(x_1) = f(x_2)$ it follows that $x_1 = x_2$.
(ii). Show that f is surjective iff $f(X) = Y$.

▪ Example A.3.25 — The domain and codomain matter Consider the 'function' $f(x) := x^2$. Is this 'function' injective or surjective? Defining functions like this may have been kosher back when you were doing mathematics that wasn't actually mathematics, but no longer. The question does not make sense because you have not specified the domain or codomain. For example, $f\colon \mathbb{R} \to \mathbb{R}$ is neither injective nor surjective, $f\colon \mathbb{R}_0^+ \to \mathbb{R}$ is injective but not surjective, $f\colon \mathbb{R} \to \mathbb{R}_0^+$ is surjective but not injective, and $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is both injective and surjective. Hopefully this example serves to illustrate: functions are not (just) 'rules'—if you have not specified the domain and codomain, then *you have not specified the function*.

Exercise A.3.26 Let $f\colon X \to Y$ be a function between nonempty sets. Show that

(i). f is injective iff it has a left inverse;
(ii). f is surjective iff it has a right inverse; and
(iii). f is bijective iff it has a (two-sided) inverse.

R By Exercise A.3.17.(ii), there *is* exactly one function from $\emptyset$ to $\{\emptyset\}$. This function is definitely injective as every element in the codomain has *at most one* preimage. On the other hand, there is *no* function from $\{\emptyset\}$ to $\emptyset$ (by Exercise A.3.17.(i)), and so certainly no left-inverse to the function from $\emptyset$ to $\{\emptyset\}$. This is why we require the sets to be nonempty.

Exercise A.3.27 Let $f\colon X \to Y$ be a function, and let $\mathscr{S}$ and $\mathscr{T}$ be a collection of subsets of X and Y respectively. Show that the following statements are true.

(i). $f^{-1}\left(\bigcup_{T\in\mathscr{T}} T\right) = \bigcup_{T\in\mathscr{T}} f^{-1}(T)$.
(ii). $f^{-1}\left(\bigcap_{T\in\mathscr{T}} T\right) = \bigcap_{T\in\mathscr{T}} f^{-1}(T)$.
(iii). $f\left(\bigcup_{S\in\mathscr{S}} S\right) = \bigcup_{S\in\mathscr{S}} f(S)$
(iv). $f\left(\bigcap_{S\in\mathscr{S}} S\right) \subseteq \bigcap_{S\in\mathscr{S}} f(S)$.

Find an example to show that we need not have equality in (iv). On the other hand, show that (iv) is true if f is injective.

Exercise A.3.28 Show that

(i). the composition of two injections is an injection;
(ii). the composition of two surjections is a surjection; and
(iii). the composition of two bijections is a bijection.

Exercise A.3.29 Let $f\colon X \to Y$ be a function, let $S \subseteq X$, and let $T \subseteq Y$. Show that the following statements are true.

(i). $f\left(f^{-1}(T)\right) \subseteq T$, with equality for all T iff f is surjective.
(ii). $f^{-1}\left(f(S)\right) \supseteq S$, with equality for all S iff f is injective.

Find examples to show that we need not have equality in general.

R Maybe this is a bit silly, but I remember which one is which as follows. First of all, write these both using $\subseteq$, not $\supseteq$, that is, $S \subseteq f^{-1}(f(S))$ and $f(f^{-1}(S)) \subseteq S$. Then, the "-1" is always closest to the symbol that represents being 'smaller' (that is "$\subseteq$"). It is easy to remember which conditions imply equality if you remember that surjective functions have right-inverses and injective functions have left-inverse.[a]

[a]Modulo the stupid case when the domain is the empty-set—see the remark in Equation (A.3.26).

Exercise A.3.30 Let X and Y be sets, and let $x_0 \in X$ and $y_0 \in Y$. If there is some bijection from X to Y, show that in fact there is a bijection from X to Y which sends x_0 to y_0.

Exercise A.3.31 Let X be a set. Construct a bijection from 2^X, the power set of X, to $\{0,1\}^X$, the set of functions from X into $\{0,1\}$.

R This is the motivation for the notation 2^X to denote the power set.

A.3.1 Arbitrary disjoint-unions and products

Definition A.3.1.1 — Disjoint-union (of a collection) Let $\mathscr{X}$ be an indexed collection[a] of sets. Then, the ***disjoint-union*** over all $X \in \mathscr{X}$, $\coprod_{X \in \mathscr{X}} X$, is

$$\coprod_{X \in \mathscr{X}} X := \{\langle x, X\rangle : X \in \mathscr{X}\ x \in X\}. \tag{A.3.1.2}$$

R The intuition and way to think of notation is just the same as it was in the simpler case of the disjoint-union of two sets (Definition A.2.3.20).

[a] By ***indexed collection*** we mean a set in which elements are allowed to be repeated. So, for example, $\mathscr{X}$ is allowed to contain two copies of $\mathbb{N}$. The reason for the term "*indexed* collection" is that indices are often used to distinguish between the two identical copies, e.g., $\mathscr{Y} = \{\mathbb{N}_1, \mathbb{N}_2\}$—as sets are not allowed to 'repeat' elements, we add the indices so that, strictly speaking, $\mathbb{N}_1 \neq \mathbb{N}_2$ as elements of $\mathscr{X}$, even though they represent the same set. (If this is confusing, don't think about it too hard—it's just a set where elements are allowed to be repeated.)

Definition A.3.1.3 — Restrictions (of functions defined on a disjoint-union) Let $\mathscr{X}$ be an indexed collection of sets, let Y be a set, and let $f\colon \coprod_{X \in \mathscr{X}} X \to Y$ be a function. Then, the ***restriction of f to X***, $f|_X : X \to Y$, is defined by

$$f|_X(x) := f(\langle x, X\rangle). \tag{A.3.1.4}$$

In particular, the ***inclusion*** is defined to be

$$\iota_X := [\mathrm{id}_{\coprod_{X \in \mathscr{X}}}]|_X, \tag{A.3.1.5}$$

that is, the restriction of the identity $\mathrm{id}_{\coprod_{X \in \mathscr{X}} X} : \coprod_{X \in \mathscr{X}} X \to \coprod_{X \in \mathscr{X}} X$.

R While from a set-theoretic perspective, this is just a special case of restriction (see Definition A.3.6), we state it separately because we wish to draw an analogy with projections (see Definition A.3.1.9), a concept which is not a special case of something we have seen before.

Definition A.3.1.6 — Cartesian-product (of a collection) Let $\mathscr{X}$ be an indexed collection of sets. Then, the ***Cartesian-product*** over all $X \in \mathscr{X}$, $\prod_{X \in \mathscr{X}} X$, is

$$\prod_{X \in \mathscr{X}} X := \left\{ f \colon \mathscr{X} \to \coprod_{X \in \mathscr{X}} X : f(X) \in X \right\}. \quad \text{(A.3.1.7)}$$

R Admittedly this notation is a bit obtuse. The Cartesian-product is still supposed to be thought of a collection of ordered-'pairs', except now the pairs aren't just pairs, but can be 3, 4, or even infinitely many 'coordinates'. The coordinates are indexed by elements of $\mathscr{X}$, and the X-coordinate for $X \in \mathscr{X}$ must lie in X itself. Thus, for example, $X_1 \times X_2 = \prod_{X \in \mathscr{X}} X$ for $\mathscr{X} = \{X_1, X_2\}$. The key that is probably potentially the most confusing is that the elements of $\mathscr{X}$ are playing more than one role: on one hand, they index the coordinates, and on the other hand, they are the set in which the coordinates take their values. Hopefully keeping in mind the case $\mathscr{X} = \{X_1, X_2\}$ helps this make sense. So, for example, in the statement "$f(X) \in X$", on the left-hand side, X is being thought of as an 'index', and on the right-hand side it is being thought of as the 'space' in which a coordinate 'lives'. This is thus literally just the statement that the X-coordinate of $f \in \prod_{X \in \mathscr{X}} X$ must be an element of the set X.

R For $x \in \prod_{X \in \mathscr{X}} X$, we write $x_X := x(X)$ for the ***X-component*** or ***X-coordinate***.

R For $x \in \mathscr{I}$, we may also suggestively write

$$\langle x_i : i \in \mathscr{I} \langle := x, \quad \text{(A.3.1.8)}$$

analogous to how one writes $\langle x, y\rangle \in X \times Y$ for elements in a Cartesian product of two sets. (We have only changed the letter of our indexing set[a] for legibility.)

R For a function defined on a Cartesian product, say $f\colon X \times Y \to Z$, we shall write $f(x, y) := f(\langle x, y\rangle)$.

[a]Like the term "collection", the term ***indexing set*** is technically just synonymous with the word "set". There is nothing mathematically different about it. This term is only used to clarify to the human readers out there how one should intuitively think of the set, specifically that it is a set whose elements are being used to "index" other things.

Definition A.3.1.9 — Components (of functions into a product) Let $\mathscr{X}$ be an indexed collection of sets, let Y be a set, and let $f\colon Y \to \prod_{X\in\mathscr{X}} X$ be a function. Then, the ***X-component***, $f_X : Y \to X$, is defined by

$$f_X(y) := f(y)_X. \tag{A.3.1.10}$$

In particular, the ***projection***, π_X, is defined to be

$$\pi_X := [\mathrm{id}_{\prod_{X\in\mathscr{X}} X}]_X, \tag{A.3.1.11}$$

that is, it is the X-component of the identity $\mathrm{id}_{\prod_{X\in\mathscr{X}} X} : \prod_{X\in\mathscr{X}} X \to \prod_{X\in\mathscr{X}} X$.

R For example, in the case $f\colon Y \to X_1 \times X_2$, then $f(y) = \langle f_1(y), f_2(y)\rangle$.

Exercise A.3.1.12 Let $\mathscr{I}$ and X be sets.

(i). Find a bijection

$$\mathscr{I} \times X \to \coprod_{i\in\mathscr{I}} X. \tag{A.3.1.13}$$

(ii). Find a bijection

$$X^{\mathscr{I}} \to \prod_{i \in \mathscr{I}} X. \tag{A.3.1.14}$$

R (i) says that if all the sets appearing in a disjoint-union are the same, then that disjoint-union is 'the same as' the Cartesian product of the indexing set and the single set appearing in the disjoint union.

Similarly, (ii) says that if all the sets appearing in are Cartesian-product are the same, then it is 'the same as' the set of all functions from the indexing set to the single set appearing in the product.

Before introducing other important special cases of relations, we must first introduce several properties of relations.

Definition A.3.1.15 Let $\sim$ be a relation on a set X.

(i). (Reflexive) $\sim$ is ***reflexive*** iff $x \sim x$ for all $x \in X$.

(ii). (Symmetric) $\sim$ is ***symmetric*** iff $x_1 \sim x_2$ is equivalent to $x_2 \sim x_1$ for all $x_1, x_2 \in X$.

(iii). (Transitive) $\sim$ is ***transitive*** iff $x_1 \sim x_2$ and $x_2 \sim x_3$ implies $x_1 \sim x_3$.

(iv). (Antisymmetric) $\sim$ is ***antisymmetric*** iff $x_1 \sim x_2$ and $x_2 \sim x_1$ implies $x_1 = x_2$.[a]

(v). (Total) $\sim$ is ***total*** iff for every $x_1, x_2 \in X$, $x_1 \sim x_2$ or $x_2 \sim x_1$.

[a] Admittedly the terminology here with "symmetric" and "antisymmetric" is a bit unfortunate.

A.3.2 Equivalence relations

Definition A.3.2.1 — Equivalence relation An ***equivalence relation*** on a set X is a relation on X that is reflexive, symmetric, and transitive.

Example A.3.2.2 — Integers modulo m Let $m \in \mathbb{Z}^+$ and let $x, y \in \mathbb{Z}$. Then, x and y are ***congruent modulo*** m, written $x \cong y \pmod{m}$, iff $x - y$ is divisible by m.

Exercise A.3.2.3 Check that $\cong \pmod{m}$ is an equivalence relation.

For example, 3 and 10 are congruent modulo 7, 1 and -3 are congruent modulo 4, -2 and 6 are congruent modulo 8, etc..

R We will see a 'better' way of viewing the integers modulo m in Example A.4.1.18. It is better in the sense that it is much more elegant and concise, but requires a bit of machinery and will probably not be as transparent if you have never seen it before. Thus, it is probably more enlightening, at least the first time, to see things spelled out in explicit detail.

Definition A.3.2.4 — Equivalence class Let $\sim$ be an equivalence relation on a set X and let $x_0 \in X$. Then, the ***equivalence class*** of x_0, $[x_0]_\sim$, is

$$[x_0]_\sim := \{x \in X : x \sim x_0\} = \{x \in X : x_0 \sim x\} . \quad \text{(A.3.2.5)}$$

R If $\sim$ is clear from context, we may simply write $[x_0] := [x_0]_\sim$.

R We may also on occasion write $x_0/\sim := [x_0]_\sim$ for the equivalence class.

R In words, the equivalence class of x_0 is the set of elements equivalent to x.

R Note that the second equation of (A.3.2.5) uses the symmetry of the relation.

■ **Example A.3.2.6 — Integers modulo** m This is a continuation of Example A.3.2.2. For example, the equivalence class of 5 modulo 6 is

$$[5]_{\cong\,(\mathrm{mod}\ 6)} = \{\ldots, -1, 5, 11, 17, \ldots\}, \tag{A.3.2.7}$$

the equivalence class of -1 modulo 8 is

$$[1]_{\cong\,(\mathrm{mod}\ 8)} = \{\ldots, -17, -9, -1, 7, 15, \ldots\}, \tag{A.3.2.8}$$

etc..

An incredibly important property of equivalence classes is that they form a partition of the set.

Definition A.3.2.9 — Partition Let X be a set. Then, a ***partition*** of X is a collection $\mathscr{X}$ of subsets of X such that

(i). $X = \bigcup_{U \in \mathscr{X}} U$; and
(ii). for $U_1, U_2 \in \mathscr{X}$ either $U_1 = U_2$ or U_1 is disjoint from U_2.

Proposition A.3.2.10 Let $\sim$ be an equivalence relation on a set X and let $x_1, x_2 \in X$. Then, either (i) $x_1 \sim x_2$ or (ii) $[x_1]_\sim$ is disjoint from $[x_2]_\sim$.

Proof. If $x_1 \sim x_2$ we are done, so suppose that this is not the case. We wish to show that $[x_1]_\sim$ is disjoint from $[x_2]_\sim$, so suppose that this is not the case. Then, there is some $x_3 \in X$ with $x_1 \sim x_3$ and $x_3 \sim x_2$. Then, by transitivity $x_1 \sim x_2$: a contradiction. Thus, it must be the case that $[x_1]_\sim$ is disjoint from $[x_2]_\sim$. ■

Corollary A.3.2.11 Let X be a set and let $\sim$ be an equivalence relation on X. Then, the collection $\mathscr{X} := \{[x]_\sim : x \in X\}$ is a partition of X.

Proof. The previous proposition, Proposition A.3.2.10, tells us that $\mathscr{X}$ has property (ii) of the definition of a partition, Definition A.3.2.9. Property (i) follows from the fact that $x \in [x]_\sim$, so that indeed

$$X = \bigcup_{x \in X} [x]_\sim = \bigcup_{U \in \mathscr{X}} U. \tag{A.3.2.12}$$

■

Conversely, a partition of a set defines an equivalence relation.

Exercise A.3.2.13 Let X be a set, let $\mathscr{X}$ be a partition of X, and define $x_1 \sim x_2$ iff there is some $U \in \mathscr{X}$ such that $x_1, x_2 \in U$. Show that $\sim$ is an equivalence relation.

▪ **Example A.3.2.14 — Integers modulo m** This in turn is a continuation of Example A.3.2.6. The equivalence classes modulo 4 are

$$\begin{aligned}
[0]_{\cong \pmod 4} &= \{\ldots, -8, -4, 0, 4, 8, \ldots\} \\
[1]_{\cong \pmod 4} &= \{\ldots, -7, -3, 1, 5, 9, \ldots\} \\
[2]_{\cong \pmod 4} &= \{\ldots, -6, -2, 2, 6, 10, \ldots\} \\
[3]_{\cong \pmod 4} &= \{\ldots, -5, -1, 3, 7, 11, \ldots\}.
\end{aligned} \tag{A.3.2.15}$$

You can verify directly that (i) each integer appears in at least one of these equivalence classes and (ii) that no integer appears in more than one. Thus, indeed, the set $\left\{[0]_{\cong \pmod 4}, [1]_{\cong \pmod 4}, [2]_{\cong \pmod 4}, [3]_{\cong \pmod 4}\right\}$ is a partition of $\mathbb{Z}$.

Given a set X with an equivalence relation $\sim$, we obtain a new set $X/\sim$, the collection of all equivalence classes of elements in X with respect to $\sim$.

Definition A.3.2.16 — Quotient set Let $\sim$ be an equivalence relation on a set X. Then, the ***quotient of X with respect to*** $\sim$, $X/\sim$, is defined by

$$X/\sim := \{[x]_\sim : x \in X\}\,. \tag{A.3.2.17}$$

The function $\mathrm{q} : X \to X/\sim$ defined by $\mathrm{q}(x) := [x]_\sim$ is the ***quotient function***.

> R If one wants to define a function f on $X/\sim$, often times one will define $f([x]_\sim)$ in terms of x itself. This is dubious, however, as how do we know that our definition gives the same result if $x_1 \sim x_2$? (We can't have $f([x_1]_\sim) \neq f([x_2]_\sim)$ if $[x_1]_\sim = [x_2]_\sim$, now can we?) Thus, if ever we do want to make a definition like this we must first prove that our definition does not depend on the "representative" of the equivalence class $[x]_\sim$ we have chosen. More precisely, we must show that if $x_1 \sim x_2$, then $f(x_1) = f(x_2)$. If this is the case, we say that our definition f is ***well-defined***. (We elaborate on this below.)

Of course the quotient function is surjective. What's perhaps a bit more surprising is that *every* surjective function can be viewed as the quotient function with respect to some equivalence relation.

Exercise A.3.2.18 Let $\mathrm{q} : X \to Y$ be surjective and for $x_1, x_2 \in X$, define $x_1 \sim_{\mathrm{q}} x_2$ iff $x_1, x_2 \in \mathrm{q}^{-1}(y)$ for some $y \in Y$. Show that (i) $\sim_{\mathrm{q}}$ is an equivalence relation on X and (ii) that $\mathrm{q}(x) = [x]_{\sim_{\mathrm{q}}}$.

■ **Example A.3.2.19 — Integers modulo m** This in turn is a continuation of Example A.3.2.14. For example, the quotient set mod 5 is

$$\mathbb{Z}/\cong \ (\text{mod} 5) = \{[0]_{\cong \ (\text{mod } 5)}, [1]_{\cong \ (\text{mod } 5)}, [2]_{\cong \ (\text{mod } 5)}, [3]_{\cong \ (\text{mod } 5)}, [4]_{\cong \ (\text{mod } 5)}\} .$$

It is quite common for us, after having defined the quotient set, to want to define operations on the quotient set itself. For example, we would like to be able to add integers modulo 24 (we do this when telling time). In this example, we could make the following definition.

$$[x]_{\cong \ (\text{mod } 24)} + [y]_{\cong \ (\text{mod } 24)} := [x+y]_{\cong \ (\text{mod } 24)}. \qquad \text{(A.3.2.20)}$$

This is okay, but before we proceed, we have to check that this definition is *well-defined*. That is, there is a potential problem here, and we have to check that this potential problem doesn't actually happen. I will try to explain what the potential problem is.

Suppose we want to add 3 and 5 modulo 7. On one hand, we could just do the obvious thing $3 + 5 = 8$. But because we are working with *equivalence classes*, I should just as well be able to add 10 and 5 and get the same answer. In this case, I get $10 + 5 = 15$. At first glance, it might seem we got different answers, but, alas, while 8 and 15 are not the same integer, they 'are' the same *equivalence class* modulo 7.

In symbols, if I take two integers x_1 and x_2 and add them, and you take two integers y_1 and y_2 *with* y_1 *equivalent to* x_1 *and* y_2 *equivalent to* x_2, it had better be the case that $x_1 + x_2$ is equivalent to $y_1 + y_2$. That is, the answer should not depend on the "representative" of the equivalence class we chose to do the addition with.

■ Example A.3.2.21 — Integers modulo m This in turn is a continuation of Example A.3.2.19. Let $m \in \mathbb{Z}^+$, let $x_1, x_2 \in \mathbb{Z}$, and define

$$[x_1]_{\cong \ (\text{mod } m)} + [x_2]_{\cong \ (\text{mod } m)} := [x_1 + x_2]_{\cong \ (\text{mod } m)}. \qquad \text{(A.3.2.22)}$$

We check that this is well-defined. Suppose that $y_1 \cong x_1 \ (\text{mod } m)$ and $y_2 \cong x_2 \ (\text{mod} m)$. We must show that $x_1 + x_2 \cong y_1 + y_2 \ (\text{mod} m)$. Because $y_k \cong x_k \ (\text{mod} m)$, we know that

$y_k - x_k$ is divisible by m, and hence $(y_1 - x_1) + (y_2 - x_2) = (y_1 + y_2) - (x_1 + x_2)$ is divisible by m. But this is just the statement that $x_1 + x_2 \cong y_1 + y_2 \pmod{m}$, exactly what we wanted to prove.

Exercise A.3.2.23 Define multiplication modulo m and show that is is well-defined.

A.3.3 Preorders

Definition A.3.3.1 — Preorder A ***preorder*** on a set X is a relation $\leq$ on X that is reflexive and transitive. A set equipped with a preorder is a ***preordered set***.

R The notation $x_1 < x_2$ is shorthand for "$x_1 \leq x_2$ and $x_1 \neq x_2$".

R Note that an equivalence relation is just a very special type of preorder.

Exercise A.3.3.2 Find an example of

(i). a relation that is both reflexive and transitive (i.e. a preorder);
(ii). a relation that is reflexive but not transitive;
(iii). a relation that is not reflexive but transitive; and
(iv). a relation that is neither reflexive nor transitive.

The notion of an *interval* is obviously important in mathematics and you almost have certainly encountered them before in calculus. We give here the abstract definition (see Proposition 2.4.3.62 to see that this agrees with what you are probably familiar with).

Definition A.3.3.3 — Interval Let $\langle X, \leq \rangle$ be a preordered set and let $I \subseteq X$. Then, I is an ***interval*** iff for all $x_1, x_2 \in I$ with $x_1 \leq x_2$, whenever $x_1 \leq x \leq x_2$, it follows that $x \in I$.

R In other words, I is an interval iff everything in-between two elements of I is also in I.

R As you are probably aware, the following notation is common.

$$\begin{aligned}
[x_1, x_2] &:= \{x \in X : x_1 \leq x \leq x_2\} \\
(x_1, x_2) &:= \{x \in X : x_1 < x < x_2\} \\
[x_1, x_2) &:= \{x \in X : x_1 \leq x < x_2\} \\
(x_1, x_2] &:= \{x \in X : x_1 < x \leq x_2\}.
\end{aligned}$$

The first and second are called respectively ***closed intervals*** and ***open intervals***. Terminology for the third and fourth is less common, but you might call them respectively the ***half-closed-open intervals*** and ***half-open-closed intervals***.

Feel free to check that these sets are all in fact intervals.[a]

[a] Warning: Though there can be intervals not of this form—see Example 2.4.3.63.

Definition A.3.3.5 — Monotone Let X and Y be preordered sets and let $f : X \to Y$ be a function. Then, f is ***nondecreasing*** iff $x_1 \leq x_2$ implies that $f(x_1) \leq f(x_2)$. If the second inequality is strict for distinct x_1 and x_2, i.e. if $x_1 < x_2$ implies $f(x_1) < f(x_2)$, then f is ***increasing***. If the inequality is in the other direction, i.e. if $x_1 \leq x_2$ implies $f(x_1) \geq f(x_2)$, then f is ***nonincreasing***. If it is both strict and reversed, i.e. if $x_1 < x_2$ implies $f(x_1) > f(x_2)$, then f is ***decreasing***. f is ***monotone*** iff it is either nondecreasing or nonincreasing and f is ***strictly monotone*** iff it is either increasing or decreasing.

R Note that the $\leq$ that appears in $x_1 \leq x_2$ is *different* than the $\leq$ that appears in $f(x_1) \leq f(x_2)$: the former is the preorder on X and the latter is the preorder on Y. We will often abuse notation in this manner.

In this course, we will almost always be dealing with preordered sets whose preorder is in addition antisymmetric (or are equivalence relations).

Definition A.3.3.6 — Partial-order A ***partial-order*** is an antisymmetric preorder. A set equipped with a partial-order is a ***partially-ordered set*** or a ***poset***.

There are two preorders that you can define on any set. They are not terribly useful, except perhaps for producing counter-examples.

■ Example A.3.3.7 — Discrete and indiscrete (orders Let X be a set. Declare $x_1 \leq_{\mathrm{D}} x_2$ iff $x_1 = x_2$. That is, $x \leq_{\mathrm{D}} x$ is true, and nothing else.

Exercise A.3.3.8 Show that $\langle X, \leq_{\mathrm{D}} \rangle$ is a partial-order.

R $\leq_{\mathrm{D}}$ is the ***discrete-order*** on X.

Now declare $x_1 \leq_{\mathrm{I}} x_2$ for all $x_1, x_2 \in X$. That is, $x_1 \leq_{\mathrm{I}} x_2$ is *always* true.

Exercise A.3.3.9 Show that $\langle X, \leq_{\mathrm{I}} \rangle$ is a total preorder that is not antisymmetric in general.

R In particular, this shows that there are total preorders which are not partial-orders (Definition A.3.3.6).

R $\leq_{\mathrm{I}}$ is the ***indiscrete-order*** on X.

R For the etymology of the terminology, see Definitions 3.1.2.9 and 3.1.2.13.

A much more useful collection of examples of partially-ordered sets is that are those exhibited as power-sets.

■ **Example A.3.3.10 — Power set** The archetypal example of a partially-ordered set is given by the power set. Let X be a set and for $U, V \in 2^X$, define $U \le V$ iff $U \subseteq V$.

Exercise A.3.3.11 Check that $\langle 2^X, \le \rangle$ is in fact a partially-ordered set.

Exercise A.3.3.12 What is an example of a preorder that is not a partial-order?

While we will certainly be dealing with nontotal partially-ordered sets, totality of an ordering is another property we will commonly come across.

Definition A.3.3.13 — Total-order A ***total-order*** is a total partial-order. A set equipped with a total-order is a ***totally-ordered set***.

Exercise A.3.3.14 What is an example of a partially-ordered set that is not a totally-ordered set.

And finally we come to the notion of well-ordering, which is an incredibly important property of the natural numbers.

Definition A.3.3.15 — Well-order A ***well-order*** on a set X is a total-order that has the property that every nonempty subset of X has a smallest element. A set equipped with a well-order is a ***well-ordered set***.

In fact, we do not need to assume a priori that the order is a total-order. This follows simply from the fact that every nonempty subset has a smallest element.

Proposition A.3.3.16 Let X be a partially-ordered set that has the property that every nonempty subset of X has a smallest element. Then, X is totally-ordered (and hence well-ordered).

Proof. Let $x_1, x_2 \in X$. Then, the set $\{x_1, x_2\}$ has a smallest element. If this element is x_1, then $x_1 \preceq x_2$. If this element is x_2, then $x_2 \preceq x_1$. Thus, the order is total, and so X is totally-ordered. ■

Exercise A.3.3.17 What is an example of a totally-ordered set that is not a well-ordered set?

A.3.4 Well-founded induction

In the very beginning of this chapter when discussing proof techniques (Appendix A.2.2), we mentioned ***well-founded induction***. It is time we return to this and make the statement precise.

Definition A.3.4.1 — Well-founded Let X be a set and let $\preceq$ be a relation on X. Then, $\langle X, \preceq\rangle$ is ***well-founded*** iff every nonempty subset of X has a minimal element.

The relationship between being well-ordered and well-founded is as follows.

Proposition A.3.4.2 Let X be a set and let $\preceq$ be a relation on X. Then, $\langle X, \preceq\rangle$ is a well-ordered set iff $\preceq$ is a well-founded total-order.

Proof. ($\Rightarrow$) Suppose that $\langle X, \preceq\rangle$ is a well-ordered set. Then, $\preceq$ is well-founded because minima are minimal. Let $x_1, x_2 \in X$. Then, $\{x_1, x_2\}$ is nonempty, and so has a minimum, say x_1. Then, $x_1 \preceq x_2$, and so $\preceq$ is total.

($\Leftarrow$) Suppose that $\preceq$ is a well-founded total-order. Let $S \subseteq X$ be nonempty. Then, S has a minimal element $s_0 \in S$. We wish to show that s_0 is a minimum of S. So, let $s \in S$. We wish to show that $s_0 \preceq s$. By totality, either $s_0 \preceq s$ or $s \preceq s_0$. In the former case, we are done. In the latter case, by minimality, we have $s = s_0$. By reflexivity (total-orders are reflexive), this would imply $s_0 \preceq s$, and we are done. ■

We are ultimately interested in well-foundedness itself because of its relevance to the most powerful form of induction (I know of).

Theorem A.3.4.3 — Well-founded Induction. Let X be a set, let $\preceq$ be a well-founded relation on X, and let $\mathcal{P}\colon X \to \{0, 1\}$. Then, if $\mathcal{P}(y) = 1$ for all $y \preceq x$, $y \neq x$, implies that $\mathcal{P}(x) = 1$, it follows that $\mathcal{P}(x) = 1$ for all $x \in X$.

R Of course, we are thinking of $\mathcal{P}(x)$ as a statement that may or may not be true for a given x, and $\mathcal{P}(x) = 1$ corresponds to it being true and $\mathcal{P}(x) = 0$ corresponds to it being false.

R Note how this gives us 'normal' induction in case $X := \mathbb{N}$ and $\preceq := \leq$. Indeed, it similarly generalizes transfinite induction (whatever that is).

Proof. Suppose that $\mathcal{P}(y) = 1$ for all $y \preceq x$, $y \neq x$, implies that $\mathcal{P}(x) = 1$. Define $S := \{x \in X : \mathcal{P}(x) = 0\}$. We wish to show that S is empty. We proceed by contradiction: suppose that S is nonempty. Then, S has a minimal element $s_0 \in S$. Let $y \in X$ be such that $y \preceq s_0$ and $y \neq s_0$. If $y \in S$, then by minimality, we would have $y = s_0$, which is not the case. Thus, $y \notin S$, and hence $\mathcal{P}(y) = 1$. Thus, by hypotheses, $\mathcal{P}(s) = 1$: a contradiction. ■

A.3.5 Zorn's Lemma

We end this subsection with an incredibly important result known as *Zorn's Lemma*. At the moment, it's importance might not seem obvious, and perhaps one must see it in action in order to appreciate its significance. For the time being at least, let me say this: if ever you are trying to produce something maximal by adding things to a set one-by-one (e.g. if you are trying to construct a basis by picking linearly-independent vectors one-by-one), but you are running into trouble because, somehow, this process will never stop, not even if you 'go on forever': give Zorn's Lemma a try.

Definition A.3.5.1 — Upper-bound and lower-bound Let $\langle X, \leq \rangle$ be a preordered set, let $S \subseteq X$, and let $x \in X$. Then, x is an ***upper-bound*** iff $s \leq x$ for all $s \in S$. x is a ***lower-bound*** iff $x \leq s$ for all $s \in S$.

Definition A.3.5.2 — Maximum and minimum Let $\langle X, \leq \rangle$ be a preordered set and let $x \in X$. Then, x is a ***maximum*** of X iff x is an upper-bound of all of X. x is a ***minimum*** of X iff x is a lower-bound of all of X.

Definition A.3.5.3 — Maximal and minimal Let $\langle X, \leq \rangle$ be a preordered set, let $S \subseteq X$, and let $x \in S$. Then, x is ***maximal*** in S iff whenever $y \in S$ and $y \geq x$, it follows that $x = y$. x is ***minimal*** in S iff whenever $y \in S$ and $y \leq x$ it follows that $x = y$.

R In other words, maximal means that there is no element in S strictly greater than x (and similarly for minimal). Contrast this with maxi*mum* and mini*mum*: if x is a maximum of S it means that $y \leq x$ for all $y \in S$ (and analogously for minimum).

R Note that, in a *partially*-ordered set anyways, maximum elements are always maximal (see Exercise A.3.5.6, but not conversely (and similarly for minimum and minimal) (see Example A.3.5.4).

▪ **Example A.3.5.4 — Maximum vs. maximal** To understand the difference between maximal and maximum, consider the following diagram.[a]

$$\begin{tikzcd}
A & & B \\
C \arrow[u] \arrow[urr] & & D \arrow[u] \arrow[ull] \\
& E \arrow[ul] \arrow[ur] &
\end{tikzcd} \tag{A.3.5.5}$$

Then, A and B are both *maximal*, because nothing is strictly larger than them. On the other hand, neither of them are *maximum* (and in fact, there is no maximum), because neither of them is larger than everything (A is not larger than B and B is not larger than A). Of course, the difference between minimal and minimum is exactly analogous.

[a] This diagram is meant to define a poset in which $E \leq C$, $E \leq D$, $C \leq A$, $C \leq B$, $D \leq A$, and $D \leq B$ (of course, these aren't the only relations—for example, we also mean to imply that $C \leq C$, that $E \leq A$, etc.).

Exercise A.3.5.6 Let X be a *partially*-ordered set and let $S \subseteq X$.

(i). Show that every maximum of S is maximal in S.
(ii). Show that S has at most one maximum element.
(iii). Come up with an example of X and S where S has two distinct maximal elements.

Definition A.3.5.7 — Downward-closed and upward-closed Let X be a preordered set and let $S \subseteq X$. Then, S is ***downward-closed*** in X iff whenever $x \leq s \in S$ it follows that $x \in S$. S is ***upward-closed*** in X iff whenever $x \geq s \in S$ it follows that $x \in S$.

Proposition A.3.5.8 Let X be a well-ordered set and let $S \subset X$ be downward-closed in X. Then, there is some $s_0 \in X$ such that $S = \{x \in X : x < s_0\}$.

Proof. As S is a proper subset of X, S^{C} is nonempty. As X is well-ordered, it follows that S^{C} has a smallest element s_0. We claim that $S = \{x \in X : x < s_0\}$. First of all, let $x \in X$ and suppose that $x < s_0$. If it were *not* the case that $x \in S$, then s_0 would no longer be the smallest element in S^{C}. Hence, we must have that $x \in S$. Conversely, let $x \in S$. By totality, either $x \leq s_0$ or $s_0 \leq x$. As $x \in S$ and $s_0 \in S^{\mathsf{C}}$, we cannot have that $x = s_0$, so in fact, in the former case, we would have $x < s_0$, and we are done, so it suffices to show that $s_0 \leq x$ cannot happen. If $s_0 \leq x$, then because S is downward-closed in X and $x \in S$, it would follows that $s_0 \in S$: a contradiction. Therefore, it cannot be the case that $s_0 \leq x$. ■

Theorem A.3.5.9 — Zorn's Lemma. Let X be a partially-ordered set. Then, if every well-ordered subset has an upper-bound, then X has a maximal element.

Proof. [a] Step 1: Make hypotheses
Suppose that every well-ordered subset has an upper bound. We proceed by contradiction: suppose that X has no maximal element.

Step 2: Show that every well-ordered subset has an upper-bound *not* contained in it
Let $S \subseteq X$ be a well-ordered subset, and let u be some upper-bound of S. If there were no element in X strictly greater than u, then u would be a maximal element of X. Thus, there is some $u' > u$. It cannot be the case that $u' \in S$ because then we would have $u' \leq u$ because u is an upper-bound of S. But then the fact that $u' \leq u$ and $u \leq u'$ would imply that $u = u'$: a

contradiction. Thus, $u' \notin S$, and so constitutes an upper-bound not contained in S.

STEP 3: DEFINE $u(S)$
For each well-ordered subset $S \subseteq X$, denote by $u(S)$ some upper-bound of S not contained in S.

STEP 4: DEFINE THE NOTION OF A u-SET
We will say that a well-ordered subset $S \subseteq X$ is a u-set iff $x_0 = u(\{x \in S : x < x_0\})$ for all $x_0 \in S$.

STEP 5: SHOW THAT FOR u-SETS S AND T, EITHER S IS DOWNWARD-CLOSED IN S OR T IS DOWNWARD CLOSED IN S
Define

$$D := \bigcup_{\substack{A \subseteq X \\ A \text{ is downward-closed in } S \\ A \text{ is downward-closed in } T}} A. \tag{A.3.5.10}$$

That is, D is the union of all sets that are downward-closed in both S and T.

We first check that D itself is downward-closed in both S and T. Let $d \in D$, let $s \in S$, and suppose that $s \leq d$. As $d \in D$, it follows that $d \in A$ for some $A \subseteq X$ downward-closed in both S and T. As A is in particular downward-closed in S, it follows that $s \in D$, and so D is downward-closed in S. Similarly it is downward-closed in T.

If $D = S$, then $S = D$ is downward-closed in T, and we are done. Likewise if $D = T$. Thus, we may as well assume that D is a proper subset of both S and T. Then, by Proposition A.3.5.8, there are $s_0 \in S$ and $t_0 \in T$ such that $\{s \in S : s < s_0\} = D = \{t \in T : t < t_0\}$. Because S and T are u-sets, it follows that

$$s_0 = u(\{s \in S : s < s_0\}) = u(\{t \in T : t < t_0\}) = t_0.$$

Define $D \cup \{s_0\} =: D' := D \cup \{t_0\}$. Let $d \in D'$, let $s \in S$, and suppose that $s \leq d$. Either $d = s_0$ or $d \in D$. In the latter case, $d < s_0$. Either way, $d \leq s_0$, and so we have that $s \leq d \leq s_0$, and so either $s = s_0$ or $s < s_0$; either way, $s \in D'$. The conclusion is that D' is downward-closed in S. It is similarly downward-closed in T. By the definition of D, we must have that $D' \subseteq D$: a contradiction. Thus, it could not have been the case that D was a proper subset of both S and T.

STEP 6: DEFINE U

Define

$$U := \bigcup_{\substack{S \subseteq X \\ S \text{ is a } u\text{-set}}} S. \tag{A.3.5.11}$$

We show that U is a u-set in the next step. Here, we argue that this is sufficient to complete the proof.

Define $U' := U \cup \{u(U)\}$. We wish to check that U' is likewise a u-set. First note that U' is still a well-ordered subset of X. Now let $x_0 \in U'$. We wish to show that $x_0 = u(\{x \in U' : x < x_0\})$. Note that $\{x \in U' : x < x_0\} = \{x \in U : x < x_0\}$. Hence, because U is a u-set, $u(\{x \in U' : x < x_0\}) = x_0$, as desired.

Thus, as U' is a u-set, from the definition of U, we will have $U' \subseteq U$: a contradiction, which will complete the proof. Thus, it does indeed suffice to show that U is a u-set.

STEP 7: FINISH THE PROOF BY SHOWING THAT U IS A u-SET

We first need to check that U is well-ordered. Let $A \subseteq U$ be nonempty. For $S \subseteq X$ a u-set, define $A_S := A \cap S$. For each A_S that is nonempty, denote by a_S the smallest element in A_S (which exists as S is in particular well-ordered). Let $T \subseteq X$ be some other u-set. Then, by Step 5, without loss of generality, S is downward-closed in T. In particular, $S \subseteq T$ so that $a_S \in T$. Hence, $a_T \leq a_S$. Then, because S is downward-closed in T,

$a_T \in S$, and hence $a_T \leq a_S$, and hence $a_T = a_S$. We claim that this unique element is a smallest element of A.

To see this, let $a \in A$. a is then in particular an element of U, and there is some u-set S such that $a \in S$. Then, $a \in A_S := A \cap A$, and hence $a_S \leq a$.

Let $u_0 \in U$. All that remains to be shown is that $u_0 = u(\{x \in U : x < u_0\})$. To do this, we first show that every u-set is downward-closed in U.

Let $S \subseteq X$ be a u-set, let $s \in S$, let $x \in U$, and suppose that $x \leq s$. As $x \in U$, there is some u-set T such that $x \in T$. Then, by Step 5 again, either S is downward-closed in T or T is downward-closed in S. If the former case, then we have that $x \in S$ because $x \leq s$. On the other hand, in the latter case, we have that $x \in S$ because $x \in T \subseteq S$.

Now we finally return to showing that $u_0 = u(\{x \in U : x < u_0\})$. By definition of U, $u_0 \in S$ for some u-set S, and therefore, $u_0 = u(\{x \in S : x < u_0\})$. Therefore, it suffices to show that if $x \in U$ is less than u_0, then it is in S (because then $\{x \in S : x < u_0\} = \{x \in U : x < u_0\}$) u. This, however, follows from the fact that S is downward-closed in U. ■

[a] Proof adapted from [Gra07].

A.4 Sets with algebraic structure

Definition A.4.1 — Binary operation A ***binary operation*** $\cdot$ on a set X is a function $\cdot : X \times X \to X$. It is customary to write $x_1 \cdot x_2 := \cdot(x_1, x_2)$ for binary operations.

R Sometimes people say that *closure* is an axiom. This is not necessary. That a binary operation on X takes values *in* X implicitly says that the operation is closed. That doesn't mean that you never have to check closure, however. For example, in order to verify that the even integers $2\mathbb{Z}$ are a subrng (see Definition A.4.13) of $\mathbb{Z}$, you do have to check closure—you need to check this in order that $+\colon 2\mathbb{Z}\times 2\mathbb{Z} \to 2\mathbb{Z}$ be a binary operation on $2\mathbb{Z}$ (and similarly for $\cdot$).

Definition A.4.2 Let $\cdot$ be a binary relation on a set X.

(i). (Associative) $\cdot$ is ***associative*** iff $(x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3)$ for all $x_1, x_2, x_3 \in X$.

(ii). (Commutative) $\cdot$ is ***commutative*** if $x_1 \cdot x_2 = x_2 \cdot x_1$ for all $x_1, x_2 \in X$.

(iii). (Identity) An ***identity*** of $\cdot$ is an element $1 \in X$ such that $1 \cdot x = x = x \cdot 1$ for all $x \in X$.

(iv). (Inverse) If $\cdot$ has an identity and $x \in X$, then an ***inverse*** of x is an element $x^{-1} \in X$ such that $x \cdot x^{-1} = 1 = x^{-1} \cdot x$.

We first consider sets equipped just a single binary operation.

Definition A.4.3 — Magma A ***magma*** is a set equipped with a binary operation.

Exercise A.4.4 Let $\langle X, \cdot \rangle$ be a magma and let $x_1, x_2, x_3 \in X$. Show that $x_1 = x_2$ implies $x_1 \cdot x_3 = x_2 \cdot x_3$.

R My hint is that the solution is so trivial that it is easy to overlook.

R This is what justifies the 'trick' (if you can call it that) of doing the same thing to both sides of an equation that is so common in algebra.

R Note that the converse is not true in general. That is, we can have $x_1 \cdot x_2 = x_1 \cdot x_3$ with $x_2 \neq x_3$.

Definition A.4.5 — Semigroup A ***semigroup*** is a magma $\langle X, \cdot \rangle$ such that $\cdot$ is associative.

Definition A.4.6 — Monoid A ***monoid*** $\langle X, \cdot, 1\rangle$ is a semigroup $\langle X, \cdot\rangle$ equipped with an identity $1 \in X$.

Exercise A.4.7 — Identities are unique Let X be a monoid and let $1, 1' \in X$ be such that $1 \cdot x = x = x \cdot 1$ and $1' \cdot x = x = x \cdot 1'$ for all $x \in X$. Show that $1 = 1'$.

Definition A.4.8 — Group A ***group*** is a monoid $\langle X, \cdot, 1\rangle$ equipped with a function $-^{-1} : X \to X$ so that x^{-1} is an inverse of x for all $x \in X$.

R Usually this is just stated as "X has inverses.". This isn't wrong, but this way of thinking about things doesn't generalize to universal algebra or category theory quite as well. The way to think about this is that, inverses, like the binary operation (as well as the identity) is *additional structure*. This is in contrast to the axiom of associativity which should be thought of as a *property* satisfied by an *already existing* structure (the binary operation).

R Usually we write $\langle X, \cdot, 1, -^{-1}\rangle$ to denote a group X with binary operation $\cdot$ with identity 1 and with the inverse of $x \in X$ being given by x^{-1}. However, especially if the group is commutative, it is also common to write $\langle X, +, 0, -\rangle$ to denote the same thing. In this case, what we previously would have written as x^3, would now be written as $3x$. It is important to realize that, even though the symbols being used are different, the axioms they are required to satisfy are exactly the same—the change in notation serves no other purpose other than to be suggestive.

Exercise A.4.9 — Inverses are unique Let X be a group, let $x \in X$, and let $y, z \in X$ both be inverses of x. Show that $y = z$.

Exercise A.4.10 Let $\langle X, \cdot, 1, -^{-1} \rangle$ be a group and let $x_1, x_2, x_3 \in X$. Show that if $x_1 \cdot x_2 = x_1 \cdot x_3$, then $x_2 = x_3$.

R Thus, the converse to Exercise A.4.4 holds in the case of a group.

Definition A.4.11 — Homomorphism (of magmas) Let X and Y be magmas and let $f : X \to Y$ be a function. Then, f is a ***homomorphism*** iff $f(x_1 \cdot x_2) = f(x_1) \cdot f(x_2)$ for all $x_1, x_2 \in X$.

R Informally, we say that "f *preserves*" the binary operation.

R Note that, once again, the $\cdot$ in $f(x_1 \cdot x_2)$ is *not* the same as the $\cdot$ in $f(x_1) \cdot f(x_2)$. Confer the remark following the definition of a nondecreasing function, Definition A.3.3.5.

R There are similar definitions for monoids and groups, with extra conditions because of the extra structure. For monoids,[a] we additionally require that $\phi(1) = 1$. For groups, in turn additionally require that $\phi(x^{-1}) = \phi(x)^{-1}$. This is why we might say "homomorphism of monoids" instead of just "homomorphism"—we are clarifying that we are additionally requiring this extra condition.

[a] And more generally any magma with identity.

We now move on to the study of sets equipped with *two* binary operations.

Definition A.4.12 — Rg A ***rg*** is a set equipped with two binary operations $\langle X, +, \cdot \rangle$ such that

(i). $\langle X, + \rangle$ is a commutative monoid,
(ii). $\langle X, \cdot \rangle$ is a semigroup, and

(iii). $\cdot$ ***distributes*** over $+$, that is, $x_1 \cdot (x_2 + x_3) = x_1 \cdot x_2 + x_1 \cdot x_3$ and $(x_1 + x_2) \cdot x_3 = x_1 \cdot x_3 + x_2 \cdot x_3$ for all $x_1, x_2, x_3 \in X$.

R In other words, writing out what it means for $\langle X, +\rangle$ to be a commutative monoid and for $\langle X, \cdot\rangle$ to be a a semigroup, these three properties are equivalent to

(i). $+$ is associative,
(ii). $+$ is commutative,
(iii). $+$ has an identity,
(iv). $\cdot$ is associative,
(v). $\cdot$ distributes over $+$.

R For $x \in X$ and $m \in \mathbb{Z}^+$, we write $m \cdot x := \underbrace{x + \cdots + x}_{m}$.

Note that we do *not* make this definition for $m = 0 \in \mathbb{N}$. An empty-sum is *always* 0 (by definition), but $0 \cdot x$ need not be 0 in a general rg (see the tropical integers in Example 1.3.2).

R Whenever we say that a rg is commutative, we mean that the *multiplication* is commutative (this should be obvious—addition is always commutative). Instead of saying referring to things as "commutative rgs" etc. we will often shorten this to "crg" etc..

As commutativity is such a nice property to have, elements which commute with everything have a special name: $x \in X$ is ***central*** iff $x \cdot y = y \cdot x$ for all $y \in X$.

R I have actually never seen the term rg used before. That being said, I haven't seen *any* term to describe such an algebraic object before. Nevertheless, I have seen both the terms rig and rng before (see below), and, well, given those terms, "rg" is pretty much the only reasonable term to give to such an algebraic object. We don't have a need to work with rgs directly, but we will work with both rigs and rngs, and so it is nice to have an object of which both rigs and rngs are special cases.

Definition A.4.13 — Rng A ***rng*** is a rg such that $\langle X, +, 0, - \rangle$ is a commutative group, that is, a rg that has additive inverses.

Exercise A.4.14 Let $\langle X, +, 0 -, \cdot \rangle$ be a rng and let $x_1, x_2 \in X$. Prove the following properties.

(i). $0 \cdot x_1 = 0$ for all $x_1 \in X$.
(ii). $(-x_1) \cdot x_2 = -(x_1 \cdot x_2)$ for all $x_1, x_2 \in X$.

▪ **Example A.4.15 — A rg that is not a rng** The even natural numbers $2\mathbb{N}$ with their usual addition and multiplication is also an example of a rg that is not a rng.

Definition A.4.16 — Rig A ***rig*** is a rg such that $\langle X, \cdot, 1 \rangle$ is a monoid, that is, a rg that has a multiplicative identity.

R In a rig R, we write

$$R^{\times} := \{ r \in R : r \text{ has a multiplicative inverse.} \} . \quad \text{(A.4.17)}$$

$R^{\times}$ is a group with respect to the ring multiplication and is known as the ***group of units*** in R.

R Just as the empty sum is defined to be 0 in a rg (see the remark in Definition A.4.12), the empty product is defined to be 1 in a rig.[a]

R I believe it is more common to refer to rigs as ***semirings***. I dislike this terminology because it suggests an analogy with semigroups, of which there is none. The term rig is also arguably more descriptive—even if you didn't know what the term meant, you might have a good chance of guessing, especially if you had seen the term rng before.

[a] Of course, similar conventions apply for all types of algebraic objects (in particular, for monoids), but we shall not keep repeating this.

Definition A.4.18 — Characteristic Let $\langle X, +, 0, \cdot, 1\rangle$ be a rig. Then, either (i) there is some $m \in \mathbb{Z}^+$ such that $m \cdot 1 = 0 \in X$ or (ii) there is no such m. In the former case, the smallest positive integer such that $\underbrace{1 + \cdots + 1}_{m} = 0 \in X$ is the ***characteristic***, and in the latter case the ***characteristic*** is 0. We denote the characteristic by $\mathrm{Char}(X)$.

> R For example, the characteristic of $\mathbb{Z}/m\mathbb{Z}$ is m, whereas the characteristic of $\mathbb{Z}$ is 0.

▪ Example A.4.19 — A rg that is not a rig The even natural numbers $2\mathbb{N}$ with their usual addition and multiplication is a rg that is not a rig.

Definition A.4.20 — Ring A ***ring*** is a rg that is both a rig and a rng.

> R The motivation for the terminology is as follows. Historically, the term "ring" was the first to be used. It is not uncommon for authors to use the term ring to mean both our definition and our definition minus the requirement of having a multiplicative identity. To remove this ambiguity in terminology, we take the term "ring" to imply the existence of the identity and the removal of the "i" from the word is the term used for objects which do not necessarily have an identity. Similarly, thinking of the "n" in "ring" as standing for "negatives", a rig is just a ring that does not necessarily posses additive inverses.

> R Note that it follows from Exercise A.4.14 that $-1 \cdot x = -x$ for all $x \in X$, X a ring.

Exercise A.4.21 Let X be a ring and suppose that $0 = 1$. Show that $X = \{0\}$.

R This is called the ***zero cring***.

Definition A.4.22 — Integral A rg $\langle X, +, 0, \cdot \rangle$ is ***integral*** iff it has the property that, whenever $x \cdot y = 0$, it follows that either $x = 0$ or $y = 0$.

R Usually the adjective "integral" is applied only to crings, in which case people refer to this as an ***integral domain*** instead of an integral cring. As the natural numbers have this property (i.e. $xy = 0 \Rightarrow x = 0$ or $y = 0$) I wanted an adjective that would describe rgs with this property and "integral" was an obvious choice because of common use of the term "integral domain".[a] It is then just more systematic to refer to them as integral crings instead of integral domains. This is usually not an issue because it is not very common to work with rigs or rgs.

[a]The adjective "integral" itself also appears in the context of schemes, and the usage there is consistent with the usage here (in a sense that will be obvious on the off-chance you know what a scheme is).

Definition A.4.23 — Division ring A ***division ring*** is a ring $\langle X, +, 0, -\cdot, 1 \rangle$ such that $\langle X \setminus \{0\}, \cdot, 1, -^{-1} \rangle$ is a group.

R In other words, a division ring is a ring in which every nonzero element has a multiplicative inverse.

R This condition makes just as much sense for rigs as it does for rings, however, to the best of my knowledge there is no accepted term for rigs in which every nonzero element has a multiplicative inverse (and as we shall have no need for such objects, we refrain from introducing a term ourselves).

R Sometimes people use the term ***skew-field*** instead of division ring.

Exercise A.4.24 Show that all division rings are integral.

Definition A.4.25 — Field A ***field*** is a commutative division ring.

Exercise A.4.26 Let F be a field with positive characteristic p. Show that p is prime.

Definition A.4.27 — Homomorphism (of rgs) Let $\langle X, +, \cdot\rangle$ and $\langle Y, +, \cdot\rangle$ be rgs and let $f\colon X \to Y$ be a function. Then, f is a ***homomorphism*** iff f is both a homomorphism (of magmas) from $\langle X, +\rangle$ to $\langle Y, +\rangle$ and from $\langle X, \cdot\rangle$ to $\langle Y, \cdot\rangle$.

R Explicitly, this means that

$$f(x+y) = f(x)+f(y), f(0) = 0, \text{ and } f(xy) = f(x)f(y).$$

R Similarly as in the definition of monoid homomorphisms Definition A.4.11, we add corresponding extra conditions about preserving identities and inverses for rigs, rngs, and rings.[a]

[a] But *not* fields. This is why the definitions are stated in such a way that the additive inverses for rings are regarded as *structure*, whereas the multiplicative inverses for fields are regarded as *properties*—homomorphisms should preserve all "structure". This is a subtle and, for now, unimportant point, and so if this doesn't make sense, you can ignore it for the time being.

A.4.1 Quotient groups and quotient rngs

It is probably worth noting that this subsubsection is of relatively low priority. We present this information here essentially because it gives a more unified, systematic, sophisticated, and elegant way to view things presented in other places in the notes, but it is also not really strictly required to understand these examples.

If you have never seen quotient rngs before, it may help to keep in the back of your mind a concrete example as you work through the definitions. We recommend you keep in mind the example $R := \mathbb{Z}$ and $I := m\mathbb{Z}$ (all multiples of m) for some $m \in \mathbb{Z}^+$. In this case, the quotient R/I is (supposed to be, and in fact will turn-out to be) the integers modulo m. While this is a quotient rng, it is also of course a quotient group (just forget about the multiplication), so this example may also help you think about quotient groups as well.

Before we get started with the precise mathematics, let's talk about the intuition.[31] At a naive level, if you ask yourself "How does one obtain $\mathbb{Z}/m\mathbb{Z}$ from $\mathbb{Z}$?", while I suppose you might come up with other answers, the 'correct' one is that "You obtain $\mathbb{Z}/m\mathbb{Z}$ from $\mathbb{Z}$ by 'setting $m = 0$'.". The intuition and motivation for quotient rings is *how to make precise the intuition of 'setting things equal to zero'*.

For reasons of 'consistency', you'll see that you can't *just* set $m = 0$. If you set $m = 0$, you must also set $m + m = 2m = 0$, and so on. Thus, if you want to set $m = 0$, in fact you must set all multiples of m equal to zero. In general, the sets of objects which are you 'allowed' to set equal to zero at once are called *ideals*. Thus, $\{m\}$ itself is not an ideal because it would be 'inconsistent' to only set $m = 0$. Instead, you take the 'ideal generated by m', which turns out to be $m\mathbb{Z}$, and set all elements of $m\mathbb{Z}$ equal to zero. If R is a rng and $I \subseteq R$ is an ideal, then R/I is the notation we use to represent the rng obtained from R by 'setting' every element of I equal to 0.

As we shall use quotient groups to define quotient rngs, we do them first. The first thing to notice is that every subgroup of a group induces an equivalence relation.

Proposition A.4.1.1 — Cosets (in groups) Let G be a group, let $H \subseteq G$, and define

$$g_1 \cong g_2 \ (\mathrm{mod}\, H) \text{ iff } g_2^{-1} g_1 \in H \text{ for } g_1, g_2 \in G. \quad \text{(A.4.1.2)}$$

Then, $\cong \pmod{H}$ is an equivalence relation iff H is a subgroup of G.

[31] I think it's fair to say that quotient algebraic structures are among the most difficult things students encounter when first beginning algebra, and so it is worthwhile to take some extra time to step back and think about what one is actually trying to accomplish.

Furthermore, in the case this is an equivalence relation,

$$[g]_{\cong\,(\mathrm{mod}\ H)} = gH. \tag{A.4.1.3}$$

R To clarify, $[g]_{\cong\,(\mathrm{mod}\ H)}$ is the equivalence class of g with respect to $\cong \pmod{H}$ and $gH := \{gh : h \in H\}$.

R The equivalence class of g with respect $\cong \pmod{H}$ is the ***left H-coset***. The set of all left H-cosets is denoted by $G/H := G/\sim_{\cong\,(\mathrm{mod}\ H)} = \{gH : g \in G\}$.

R By changing the definition of the equivalent relation to "... iff $g_1 g_2^{-1} \in H$", then we obtain the corresponding definition of ***right H-cosets***, given explicitly by Hg. The set of all right H-cosets is denoted by $H\backslash G$.[a] Of course, in general if the binary operation is not commutative, then $gH \neq Hg$.

[a]This notation is technically ambiguous with the notation used for relative set complementation, however, in practice there will never be any confusion. Furthermore, if you pay extra special attention to the spacing, this uses the symbol `\backslash` where set complementation uses the symbol `\setminus`.

Proof. ($\Rightarrow$) Suppose that $\cong \pmod{H}$ is an equivalence relation. Let $g_1, g_2 \in S$. As $g_i \cong g_i \pmod{H}$, we have that $g_i^{-1} g_i = 1 \in H$. Then, $1^{-1} g_i = g_i \in H$, and so $g_i \cong 1 \pmod{H}$. By symmetry, $1 \cong g_i \pmod{S}$, and so $g_i^{-1} 1 = g_i^{-1} \in H$. We then have that $g_1 \cong 1 \pmod{H}$ and $1 \cong g_2 \pmod{H}$, and hence, by transitivity,

$$g_1 \cong g_2 \pmod{H}, \tag{A.4.1.4}$$

and hence $g_2^{-1} g_1 \in H$. Thus, H is indeed a subgroup of G.

($\Leftarrow$) Suppose that H is a subgroup of G. Then, $1 \in H$, and so $g^{-1} g = 1 \in H$, and so $g \cong g \pmod{H}$. That is,

$\cong \pmod{H}$ is reflexive. If $g_1 \cong g_2 \pmod{S}$, then $g_2^{-1}g_1 \in H$, then $g_1^{-1}g_2 = (g_2^{-1}g_1)^{-1} \in H^{-1} = H$, and so $g_2 \cong g_1 \pmod{H}$. Thus, $\cong \pmod{S}$ is symmetric. If $g_1 \cong g_2 \pmod{S}$ and $g_2 \cong g_3 \pmod{H}$, then $g_2^{-1}g_1, g_3^{-1}g_2 \in H$, and so $g_3^{-1}g_1 = (g_3^{-1}g_2)(g_2^{-1}g_1) \in HH \subseteq H$, and so $g_1 \cong g_3 \pmod{H}$. Thus, $\cong \pmod{H}$ is transitive, hence an equivalence relation.

We now prove the "Furthermore..." part. Certainly, as $(gh)^{-1}g = h^{-1}g^{-1}g = h^{-1} \in H$, $g \cong gh \pmod{H}$ for all $h \in H$. On the other hand, if $g_1 \cong g_2 \pmod{H}$, then $g_2^{-1}g_1 \in H$, and so $g_2^{-1}g_1 = h$ for some $h \in H$, so that $g_1 = g_2h$. Thus, $[g]_{\cong \pmod{H}} = gH$. ■

For a subgroup H of G, G/H will always be a set. However, in good cases, it will be more than just a set—it will be a group in its own right.

Definition A.4.1.5 — Ideals and quotient groups Let G be a group, let $H \subseteq G$ be a subgroup, and let $g_1, g_2 \in G$. Define

$$(g_1H) \cdot (g_2H) := (g_1g_2)H. \tag{A.4.1.6}$$

H is an ***ideal*** iff this is well-defined on the quotient set G/H. In this case, G/H is itself a group, the ***quotient group*** of G modulo H.

R Recall that (Proposition A.4.1.1) gH is the equivalent class of g modulo H, and so, in particular, these definitions involve picking representatives of equivalence classes. Thus, in order for these operations to make sense, they must be well-defined. In general, they will not be well-defined, and we call H an *ideal* precisely in the 'good' case where these operations make sense.

R In the spirit of Proposition A.4.1.1, you should really be thinking of H as a *subset* that has the property that $\cong \pmod{H}$ (defined by (A.4.1.2)) is an equivalence relation. Of course, this is perfectly equivalent to being a subgroup, but that's not the reason we care—we care because it gives us an equivalence relation. This distinction will be more important for rings.

R In the context of groups, it is *much* more common to refer to ideals as ***normal subgroups***. As always, we choose the terminology we do because it is more universally consistent, even if less common.

There is an easy condition to check that in order to determine whether a given subgroup is in fact an ideal that does not require checking the well-definedness directly.

Exercise A.4.1.7 Let G be a group and let $H \subseteq G$ be a subset. Show that H is an ideal iff (i) it is a subgroup and (ii) $gHg^{-1} \subseteq H$ for all $g \in G$.

And now we turn to quotient rngs, whose development is completely analogous.

Proposition A.4.1.8 — Cosets (in rngs) Let R be a group, let $S \subseteq R$, and define

$$r_1 \cong r_2 \,(\operatorname{mod} S) \text{ iff } -r_2 + r_1 \in S \text{ for } r_1, r_2 \in R. \quad (\text{A.4.1.9})$$

Then, $\cong \,(\operatorname{mod} S)$ is an equivalence relation iff S is a subgroup of $\langle R, +, 0, -\rangle$.

Furthermore, in the case this is an equivalence relation,

$$[r]_{\cong\,(\operatorname{mod}\, S)} = r + S. \quad (\text{A.4.1.10})$$

R To clarify, $[r]_{\cong\,(\operatorname{mod}\, S)}$ is the equivalence class of r with respect to $\cong \,(\operatorname{mod} S)$ and $r + S := \{r + s : s \in S\}$.

R The equivalence class of r with respect $\cong \,(\operatorname{mod} S)$ is the ***left S-coset***. The set of all left S-cosets is denoted by $R/S := R/\sim_{\cong\,(\operatorname{mod}\, S)} = \{r + S : r \in R\}$.

R By changing the definition of the equivalent relation to "... iff $r_1 - r_2 \in S$", then we obtain the corresponding definition of ***right** S**-cosets***, given explicitly by $S + r$. In this case, however, the binary operation in question ($+$) is commutative, and so $r + S = S + r$, that is, the left and right cosets coincide, and so we can simply say ***coset***. In particular, there is no need to talk about the set of right S-cosets, which would have been denoted $S \backslash R$.

Proof. We leave this as an exercise.

Exercise A.4.1.11 Prove this yourself.

R Hint: Use the proof of Proposition A.4.1.1 as a guide.

■

You can check that $m\mathbb{Z}$ is indeed a subrng of $\mathbb{Z}$ and that $\mathbb{Z}/m\mathbb{Z}$ consists of just m cosets:

$$0 + m\mathbb{Z}, 1 + m\mathbb{Z}, \ldots, (m-1) + m\mathbb{Z}, \tag{A.4.1.12}$$

though you are probably more familiar just writing this as

$$0 \pmod{m}, 1 \pmod{m}, \ldots, m-1 \pmod{m}. \tag{A.4.1.13}$$

Of course, however, $\mathbb{Z}/m\mathbb{Z}$ is more than just a set, it has a ring structure of its own, and in good cases, R/S will obtain a canonical ring structure of its own as well.

Definition A.4.1.14 — Ideals and quotient rngs Let R be a rng, let $S \subseteq \langle R, +, 0, - \rangle$ be a subgroup, and let $r_1, r_2 \in R$. Define

$$(r_1 + S) + (r_2 + S) := (r_1 + r_2) + S \tag{A.4.1.15}$$

and

$$(r_1 + S) \cdot (r_2 + S) := (r_1 \cdot r_2) + S. \tag{A.4.1.16}$$

S is an ***ideal*** iff both of these operations are well-defined. In this case, R/S is the ***quotient rng*** of R modulo S.

R I mentioned in a remark of the definition of quotient groups Definition A.4.1.5 that you should really be thinking of the condition there that "$H \subseteq G$ a subgroup" as the condition that $\cong \pmod{H}$ be well-defined. This shows its relevance here as, in the spirit of Proposition A.4.1.8, the appropriate condition is *not* "$S \subseteq R$ a subrng" but instead that "$S \subseteq \langle R, 0, +, - \rangle$ be a subgroup". This is particularly important if you're working with rings, as in this case, the 'correct' definition of subring requires that subrings include 1, however, if an ideal I contains 1, then $I = R$.[a] Thus, if you write "$S \subseteq R$ a subring" instead of "$S \subseteq \langle R, +, 0, - \rangle$ a subgroup", your definition would imply that the only ideal in R is R itself![b]

[a]Because, by the absorption property, $r = r \cdot 1 \in I$ for all $r \in R$.

[b]In case it's not obvious, that would constitute a particularly shitty definition.

Just as before, we have an easy way of checking whether a given subring is in fact an ideal.

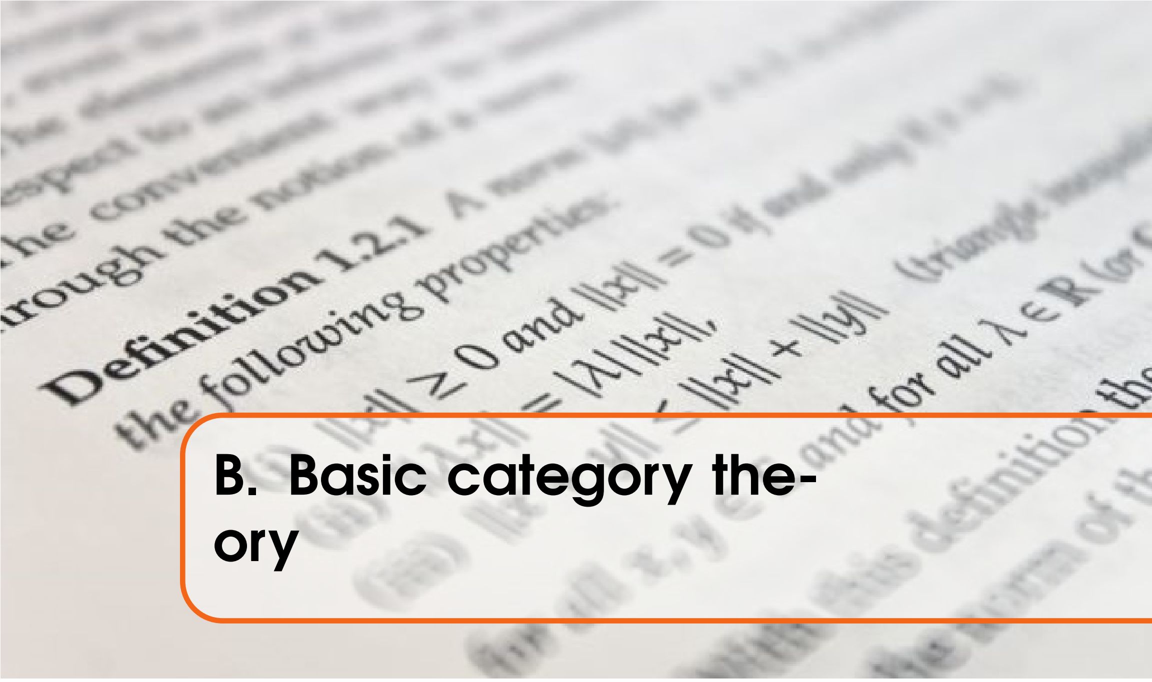
Exercise A.4.1.17 Let R be a rng and let $S \subseteq R$ be a subset. Show that S is an ideal iff (i) it is a subrng and (ii) $r \in R$ and $s \in S$ implies that $r \cdot s, s \cdot r \in S$.

R The second property is sometimes called "absorbing", because elements in the ideal 'absorb' things into the ideal when you multiply them.

■ **Example A.4.1.18 — Integers modulo m** Let $m \in \mathbb{Z}^+$.

Exercise A.4.1.19 Show that $m\mathbb{Z}$ is an ideal in $\mathbb{Z}$.

Then, the *integers modulo m* are defined to be the quotient cring $\mathbb{Z}/m\mathbb{Z}$.



B. Basic category theory

First of all, a disclaimer: it is probably not best pedagogically speaking to start with even the very basics of category theory. While in principle anyone who has the prerequisites for these notes knows everything they need to know to understand category theory, it may be difficult to understand the motivation for things without a collection of examples to work with in the back of your mind. Thus, if anything in this section does not make sense the first time you read through it, you should not worry—it will only be a problem if you do not understand ideas here as they occur in the text. In fact, it is probably perfectly okay to completely skip this section and reference back to it as needed. In any case, our main motivation for introducing category theory in a subject like this is simply that we would like to have more systematic language and notation.

B.1 What is a category?

In mathematics, we study many different types of objects: sets, preordered sets, monoids, rngs, topological spaces, schemes, etc..[1] In all of these cases, however, we are not only concerned with the

[1] No, you are not expected to know what all of these are.

objects themselves, but also with maps between them that 'preserve' the relevant structure. In the case of a set, there is no extra structure to preserve, and so the relevant maps are *all* the functions. In contrast, however, for topological spaces, we will see that the relevant maps are not all the functions, but instead all *continuous* functions.[2] Similarly, the relevant maps between monoids are not all the functions but rather the *homomorphisms*. The idea then is to come up with a definition that deals with both the objects and the relevant maps, or morphisms, simultaneously. This is the motivating idea of the definition of a category.

Definition B.1.1 — Category A ***category*** $\mathbf{C}$ is

(I). a collection $\mathrm{Obj}(\mathbf{C})$, the ***objects*** of $\mathbf{C}$; together with
(II). for each $A, B \in \mathrm{Obj}(\mathbf{C})$, a collection $\mathrm{Mor}_{\mathbf{C}}(A, B)$, the ***morphisms*** from A to B in $\mathbf{C}$;[a]
(III). for each $A, B, C \in \mathrm{Obj}(\mathbf{C})$, a function $\circ : \mathrm{Mor}_{\mathbf{C}}(B, C) \times \mathrm{Mor}_{\mathbf{C}}(A, B) \to \mathrm{Mor}_{\mathbf{C}}(A, C)$ called ***composition***;
(IV). and for each $A \in \mathrm{Obj}(\mathbf{C})$ a distinguished element $\mathrm{id}_A \in \mathrm{Mor}_{\mathbf{C}}(A, A)$, the ***identity*** of A;

such that

(i). $\circ$ is 'associative', that is, $f \circ (g \circ h) = (f \circ g) \circ h$ for all morphisms f, g, h for which these composition make sense,[b] and
(ii). $f \circ \mathrm{id}_A = f = \mathrm{id}_A \circ f$ for all $A \in \mathrm{Obj}(\mathbf{C})$.

R We write

$$\mathrm{Mor}_{\mathbf{C}} := \bigsqcup_{A,B \in \mathrm{Obj}(\mathbf{C})} \mathrm{Mor}_{\mathbf{C}}(A, B) \tag{B.1.2}$$

for the collection of all morphisms in $\mathbf{C}$.

R The term ***map*** is often used synonymously with the term "morphism", though perhaps in a more casual

[2]You might say that the entire point of the notion of a topological space is it is one of the most general contexts in which the notion of continuity makes sense. We will see exactly how this works later in the body of the text.

manner. For example, it is not uncommon to see people say "linear map" instead of "map of vector spaces" or "map in the category of vector spaces".

R If the category $\mathbf{C}$ is clear from context, we may simply write $\mathrm{Mor}(A, B)$.

R We mentioned above that the morphisms relevant to topological spaces are the continuous functions. Of course, nothing about the definition of a category *requires* this be the case—you could just as well consider the category whose objects are vector spaces and whose morphisms are *all* functions—it just turns out that these weird examples aren't particularly useful.

[a] No, we do not require that $\mathrm{Mor}_{\mathbf{C}}(A, B)$ be a (small) set. (This comment is really intended for those who have seen this definition elsewhere—often times authors fix a universe U, whose elements are referred to as the *small sets*, and in the definition of a category they require that the morphisms form small sets—we make no such requirement.)

[b] In case you're wondering, the quotes around "associative" are used because usually the word "associative" refers to a property that a binary operation has. A binary operation on a set S is, by definition, a function from $X \times X$ into X. Composition however in general is a function from $X \times Y$ into Z for $X := \mathrm{Mor}_{\mathbf{C}}(B, C)$, $Y := \mathrm{Mor}_{\mathbf{C}}(A, B)$ and $Z := \mathrm{Mor}_{\mathbf{C}}(A, C)$, and hence not a binary operation.

The intuition here is that the objects $\mathrm{Obj}(\mathbf{C})$ are the objects you are interested in studying (for example, topological spaces), and for objects $A, B \in \mathrm{Obj}(\mathbf{C})$, the morphisms $\mathrm{Mor}_{\mathbf{C}}(A, B)$ are the maps relevant to the study of the objects in $\mathbf{C}$ (for example, continuous functions from A to B). For us, it will usually be the case that every element of $\mathrm{Obj}(\mathbf{C})$ is a set equipped with extra structure (e.g. a binary operation) and the morphisms are just the functions that 'preserve' this structure (e.g. homomorphisms). In fact, there is a term for such categories—see Definition B.1.7.

At the moment, this might seem a bit abstract because of the lack of examples. As you continue through the main text, you will encounter more examples of categories, which will likely elucidate this abstract definition. However, even already we have a couple basic examples of categories.

■ Example B.1.3 — The category of sets The category of sets is the category **Set**

(i). whose collection of objects Obj(**Set**) is the collection of all sets,[a];
(ii). with morphism set $\mathrm{Mor}_{\mathbf{Set}}(X, Y)$ precisely the set of all functions from X to Y;
(iii). whose composition is given by ordinary function composition; and
(iv). whose the identities are given by the identity functions.

[a] See Appendix A.1 for clarification as to what we actually mean by the phrase "all sets".

We also have another example at our disposal, namely the category of preordered sets.

■ Example B.1.4 — The category of preordered sets The category of preordered sets is the category **Pre**

(i). whose collection of objects Obj(**Pre**) is the collection of all preordered sets;
(ii). with morphism set $\mathrm{Mor}_{\mathbf{Pre}}(X, Y)$ precisely the set of all nondecreasing functions from X to Y;
(iii). whose composition is given by ordinary function composition; and
(iv). whose identities are given by the identity functions.

The idea here is that the only structure on a preordered set is the preorder, and that the precise notion of what it means to 'preserve' this structure is to be nondecreasing. Of course, we could everywhere replace the word "preorder" (or its obvious derivatives) with "partial-order" or "total-order" and everything would make just as much sense. Upon doing so, we would obtain the category of partially-ordered sets and the category of totally-ordered sets respectively.

We also have the category of magmas.

■ **Example B.1.5 — The category of magmas** The category of magmas is the category **Mag**

(i). whose collection of objects $\mathrm{Obj}(\mathbf{Mag})$ is the collection of all magmas;
(ii). with morphism set $\mathrm{Mor}_{\mathbf{Mag}}(X,Y)$ precisely the set of all homomorphisms from X to Y;
(iii). whose composition is given by ordinary function composition; and
(iv). whose identities are given by the identity functions.

Similarly, the idea here is that the only structure here is that of the binary operation (and possibly an identity element) and that it is the homomorphisms which preserve this structure. Of course, we could everywhere here replace the word "magma" with "semigroup", "monoid", "group", etc. and everything would make just as much sense. Upon doing so, we would obtain the categories of semigroups, the category of monoids, and the category of groups respectively.

Finally we have the category of rgs.

■ **Example B.1.6 — The category of rgs** The category of rgs is the category **Rg**

(i). whose collection of objects $\mathrm{Obj}(\mathbf{Rg})$ is the collection of all rgs;
(ii). with morphism set $\mathrm{Mor}_{\mathbf{Rg}}(X,Y)$ is precisely the set of all homomorphisms from X to Y;
(iii). whose composition is given by ordinary function composition; and
(iv). and whose identities are given by the identity functions.

R The same as before, we could have everywhere replaced the word "rg" with "rig", "rng", or "ring". These categories are denoted **Rig**, **Rng**, and **Ring** respectively.

As mentioned previously, it should almost always be the case that the examples of categories we encounter are of this form, that is, in which the objects are "sets equipped with extra structure" and the morphisms are "functions which 'preserve' this structure". The term for such categories is *concrete*.

Definition B.1.7 — Concrete category Let $\mathbf{C}$ be a category. Then, $\mathbf{C}$ is ***concrete*** iff for all $A, B \in \mathrm{Obj}(\mathbf{C})$, $\mathrm{Mor}_{\mathbf{C}}(A, B) \subseteq \mathrm{Mor}_{\mathbf{Set}}(A, B)$.

R Warning: Strictly speaking, this doesn't actually make sense as A and B are not actually sets. Implicit in this is that we are additionally given a way of regarding objects of $\mathbf{C}$ as sets. For example, in the case of the category of vector spaces, we regard a vector space as a set simply by "forgetting" the addition and scaling operations. To better understand this, it might help to see an example of a nonconcrete category—see the following example.

While not terribly important for us, as you might now be wondering "What could a nonconcrete category possibly look like?", we present the following example.

■ Example B.1.8 — A category that is not concrete Let $\langle X, \leq\rangle$ be a preordered set and define $\mathbf{C}_X$ to be the category

(i). with collection of objects $\mathrm{Obj}(\mathbf{C}_X) \coloneqq X$;
(ii). with morphism set $\mathrm{Mor}_{\mathbf{C}_X}(x, y)$ taken to be a singleton iff $x \leq y$ and empty otherwise—in the case that $x \leq y$, let us write $x \to y$ for the unique element of $\mathrm{Mor}_{\mathbf{C}_X}(x, y)$;
(iii). with composition defined by $(y \to z) \circ (x \to y) \coloneqq x \to z$; and
(iv). with identity $\mathrm{id}_x \coloneqq x \to x$.

Exercise B.1.9 Check that $\mathbf{C}_X$ is in fact a category.

R Note how the axiom of reflexivity corresponds to the identities and the axiom of transitivity corresponds to composition.

B.2 Some basic concepts

The real reason we introduce the definition of a category in notes like these is that it allows us to introduce consistent notation and terminology throughout the text. Had we forgone even the very basics of categories, we would still be able to do the same mathematics, but the notation and terminology would be much more ad hoc.

Definition B.2.1 — Domain and codomain Let $f: A \to B$ be a morphism in a category. Then, the ***domain*** of f is A and the ***codomain*** of f is B.

R Of course, these terms generalize the notions of domain and codomain for sets.

Definition B.2.2 — Endomorphism Let $\mathbf{C}$ be a category and let $A \in \mathrm{Obj}(\mathbf{C})$. Then, an ***endomorphism*** is a morphism $f \in \mathrm{Mor}_{\mathbf{C}}(A, A)$. We write $\mathrm{End}_{\mathbf{C}}(A) := \mathrm{Mor}_{\mathbf{C}}(A, A)$ for the collection of endomorphisms on A.

R In other words, "endomorphism" is just a fancy name for a morphism with the same domain and codomain.

Definition B.2.3 — Isomorphism Let $f: A \to B$ be a morphism in a category. Then, f is an ***isomorphism*** iff it is invertible, i.e., iff there is a morphism $g: B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. In this case, g is an ***inverse*** of

f. The collection of all isomorphisms from A to B is denoted by $\mathrm{Iso}_{\mathbf{C}}(A, B)$.

Exercise B.2.4 Let $f: A \to B$ be a morphism in a category and let $g, h: B \to A$ be two inverses of f. Show that $g = h$.

> **R** As a result of this exercise, we may denote *the* inverse of f by f^{-1}.[a]

[a] If inverses were not unique, then the notation f^{-1} would be ambiguous: what inverse would we be referring to?

Exercise B.2.5 Show that a morphism in **Set** is an isomorphism iff it is bijective.

Exercise B.2.6 Show that a morphism in **Mag** is an isomorphism iff (i) it is bijective, (ii) it is a homomorphism, and (iii) its inverse is a homomorphism.

Exercise B.2.7 Show that the inverse of a bijective homomorphism of magmas is itself a homomorphism.

> **R** Thus, if you want to show a function is an isomorphism of magmas, in fact you only need to check (i) and (ii) of the previous exercise, because then you get (iii) for free. (Of course, essentially the very same thing happens in **Rg** as well.)

Definition B.2.8 — Isomorphic Let $A, B \in \mathrm{Obj}(\mathbf{C})$ be objects in a category. Then, A and B are ***isomorphic*** iff there is an isomorphism from A to B. In this case, we write $A \cong_{\mathbf{C}} B$, or just $A \cong B$ if the category $\mathbf{C}$ is clear from context.

Exercise B.2.9 Let $\mathbf{C}$ be a category. Show that $\cong_{\mathbf{C}}$ is an equivalence relation on $\mathrm{Obj}(\mathbf{C})$.

Definition B.2.10 — Automorphisms Let $\mathbf{C}$ be a category and let $A \in \mathrm{Obj}(\mathbf{C})$. Then, an ***automorphism*** $f\colon A \to A$ is a morphism which is both an endomorphism and an isomorphism. We write $\mathrm{Aut}_{\mathbf{C}}(A) := \mathrm{Iso}_{\mathbf{C}}(A, A)$ for the collection of automorphisms of A.

R The automorphisms of A are often thought of as the *symmetries* of A.

The following result can be seen as a reason why the concepts of monoid and group are so ubiquitous in mathematics.

Proposition B.2.11 Let $\mathbf{C}$ be a category and let $A \in \mathrm{Obj}(\mathbf{C})$. Then,

(i). $\langle \mathrm{End}_{\mathbf{C}}(A), \circ, \mathrm{id}_A \rangle$ is a monoid; and
(ii). $\langle \mathrm{Aut}_{\mathbf{C}}(A), \circ, \mathrm{id}_A, -^{-1} \rangle$ is a group.

Proof. We leave this as an exercise.

Exercise B.2.12 Prove this yourself.

■

Finally, we end this section with a concrete example of isomorphism.

■ **Example B.2.13** The category we work in is $\mathbf{Grp}$. Thus, we are going to present an example of two different groups which are isomorphic in $\mathbf{Grp}$.

On one hand, we have $\langle \mathbb{Z}/2\mathbb{Z}, +, 0, - \rangle$, which if you have been reading along in the appendix, should be relatively familiar to you by now.[a]. Regardless, however, we list the addition table for $\mathbb{Z}/2\mathbb{Z}$ for absolute concreteness.

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \tag{B.2.14}$$

On the other hand, let's define a group you haven't seen before, $C_2 := \{1, -1\}$ with binary operation defined by

$$\begin{array}{c|cc} \cdot & 1 & -1 \\ \hline 1 & 1 & -1 \\ -1 & -1 & 1 \end{array}\;. \tag{B.2.15}$$

(Feel free to check that this does indeed satisfy the axioms of a group (in fact, commutative group) if you like, but this is not so crucial .)

Now, the key to notice is the following: aside from a relabeling of symbols, *the tables in* (B.2.14) *and* (B.2.15) *are identical*. Explicitly, the relabeling is $0 \mapsto 1$, $1 \mapsto -1$, and $+ \mapsto \cdot$. The precise way of saying this is: the function $\phi\colon \mathbb{Z}/2\mathbb{Z} \to C_2$ defined by $\phi(0) := 1$ and $\phi(1) := -1$ *is an isomorphism in* **Grp** (or, to say the same thing in slightly different language, *is an isomorphism of groups*).

While not always literally true, depending on your category, I think at an intuitive level it is safe to think of two objects that are isomorphic as being 'the same' up to a relabeling of the elements. This is why, in mathematics, it is very common to not distinguish between objects which are isomorphic. This would be like making a distinction between the two equations $x^2 + 5x - 3 = 0$ and $y^2 + 5y - 3 = 0$ in elementary algebra: the name of the variable in question doesn't have any serious effect on the mathematics—it's just a name.

---

[a]Note that, we can regard $\mathbb{Z}/2\mathbb{Z}$ as a ring, explicitly $\langle \mathbb{Z}/2\mathbb{Z}, +, 0, -, \cdot \rangle$, but we don't. We're *forgetting* about the extra binary operation, and upon doing so, we obtain the group $\langle \mathbb{Z}/2\mathbb{Z}, +, 0, - \rangle$.

We end this subsection with relatively tricky concepts, that of *embedding* and *quotient*. Roughly speaking, you might say that "embedding" is the categorical generalization of the concept of a subset. In a general category with objects A and B, the statement $A \subseteq B$ just doesn't make sense—we need A and B to be sets to even posit the question "Is A a subset of B?". But even in concrete categories, where $A \subseteq B$ *does* make sense, simply being a subset of an object is the 'wrong' notion—see Example B.2.17. The basic idea which we want to make precise then is that, in addition to A being a subset of B, the structure on A is somehow the structure 'inherited' from B.

The first thing to realize is that if we are to be categorical about things (by which I mean that the *morphisms* are to play a central role), is that we shouldn't try to generalize the concept of "subset" to objects, but rather, to morphisms. That is to say, the objective should be to figure out what it means for a *morphism* to be an embedding. So, let $f\colon A \to B$ be a morphism in a concrete category.[3] In order to be an embedding, f has to be at the very least an embedding of the underlying sets, that is, f should be injective. We need more than this however:[4] if C is another object 'contained' in A, by which I mean there is a morphism $g\colon C \to A$, then, just as I can consider a subset of a subset as a subset,[5] if f is to be an embedding, I should be able to consider C directly as being 'contained' in B, that is, $f \circ g$ should be a morphism as well. This is made precise with the following definition (as well as the 'dual' concept of *quotient*).

Definition B.2.16 — Embedding and quotient Let $\mathbf{C}$ be a concrete category, let $A, B \in \mathrm{Obj}(\mathbf{C})$, and let $f \in \mathrm{Mor}_{\mathbf{C}}(A, B)$.

[3]We take our category to be concrete because, to the best of my knowledge, there is no definition of embedding/quotient that is satisfactory for all (not-necessarily-concrete) categories.

[4]See Example 3.1.3.12.

[5]That is, if X is a subset of Y and Y is a subset of Z, then of course I can consider X as a subset of Z as well.

(i). f is an ***embedding*** iff f is injective and whenever a function $g\colon C \to A$ is such $f \circ g \in \mathrm{Mor}_{\mathbf{C}}(C, B)$ (with $C \in \mathrm{Obj}(\mathbf{C})$), it follows that $g \in \mathrm{Mor}_{\mathbf{C}}(C, A)$.

(ii). f is a ***quotient*** iff f is surjective and whenever a function $g\colon B \to C$ is such that $g \circ f \in \mathrm{Mor}_{\mathbf{C}}(A, C)$ (with $C \in \mathrm{Obj}(\mathbf{C})$), it follows that $g \in \mathrm{Mor}_{\mathbf{C}}(B, C)$.

Now that we have the precise definition in hand, we can *prove* that being an injective morphism is not enough.

Example B.2.17 — A nondecreasing injective function that is not an embedding Define $X := \{A, B\}$ equipped with the trivial partial-order[a] and define $Y := \{1, 2\}$ with the only nontrivial relation being $1 \leq 2$.

Define $f\colon X \to Y$ by $f(A) := 1$ and $f(B) := 2$. If $x_1 \leq x_2$ in X, then in fact we must have that $x_1 = x_2$ (because it's the trivial order), and so of course $f(x_1) \leq f(x_2)$ (in fact, we have equality. Thus, f is nondecreasing. It is also certainly injective (in fact, bijective).

We wish to show that f is not an embedding. Define $g\colon Y \to X$ by $g(A) := 1$ and $g(B) := 2$. Then, $f \circ g = \mathrm{id}_Y$ is certainly nondecreasing (i.e. a morphism in **Pre**), but yet g is not nondecreasing. Hence, f is not an embedding.

R Though you may be able to follow the proof, it's also important to understand why f *shouldn't* be an embedding. That is to say, while it may be true that our definition has the property that f is not an embedding, furthermore, any definition we might have come up with should have this property. The reason is that, if you consider X as a subset of Y (via f), then the order on Y would dictate that $A \leq B$ (because $f(A) \leq f(B)$), which is not the case. In this case, the 'structure' on X is *not* that inherited from Y via f.

[a] That is, $A \leq A$, $B \leq B$, and nothing else.

On the other hand, we do have the following.

Exercise B.2.18 Let **C** be either the category **Set**, **Rg**, or **Mag**.

(i). Show that a morphism in **C** is an embedding iff it is injective.
(ii). Show that a morphism in **C** is a quotient iff it is surjective.

Exercise B.2.19

(i). Show that a morphism f in **Pre** is an embedding iff it is injective and has the property that $f(x_1) \leq f(x_2)$ iff $x_1 \leq x_2$.
(ii). Show that a morphism f in **Pre** is a quotient iff it is surjective and has the property that $f(x_1) \leq f(x_2)$ iff $x_1 \leq x_2$.

B.3 Functors and natural-transformations

B.3.1 Functors

The motivating idea behind the definition of a category is we wanted a definition that would contain the data of both the objects under study and the *morphisms* which 'preserve' the relevant structure.[6] We've just now introduced a new type of object, namely, a category, and so now what might ask "What is the 'right' notion of morphisms between categories?". Well, looking back at the definition (Definition B.1.1), we see that there are four pieces of data (I) the objects, (II) the morphisms, (III) the composition, and (IV) the identity morphisms. One thus comes up with the following definition of a 'morphism of categories', what is called a *functor*.

[6]Of course, any choice of morphisms can work so long as they satisfy the axioms, but most of the examples we're interested in the morphisms will be taken to be the ones which "preserve" the structure, and furthermore, when dealing with general abstract categories, I find this to be my guiding intuition.

Definition B.3.1.1 — Functor Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A ***functor*** is

(I). a function $\mathbf{f}\colon \mathrm{Obj}(\mathbf{C}) \to \mathrm{Obj}(\mathbf{D})$; together with
(II). for every $A, B \in \mathrm{Obj}(\mathbf{C})$, a function[a]

$$\mathbf{f}\colon \mathrm{Mor}_{\mathbf{C}}(A,B) \to \mathrm{Mor}_{\mathbf{D}}(\mathbf{f}(A), \mathbf{f}(B)) \qquad \text{(B.3.1.2)}$$

such that

(i). $\mathbf{f}(g \circ f) = \mathbf{f}(g) \circ \mathbf{f}(f)$ for all composable morphisms $f, g \in \mathrm{Mor}_{\mathbf{C}}$; and
(ii). $\mathbf{f}(\mathrm{id}_A) = \mathrm{id}_{\mathbf{f}(A)}$ for all $A \in \mathrm{Obj}(\mathbf{C})$.

R Thus, a functor between two categories maps the objects to objects and the morphisms to the morphisms, and does so in such a way so as to "preserve" composition and the identities.

[a] Of course, this function depends on A and B, but by abuse of notation we omit this dependence.

■ **Example B.3.1.3 — The category of categories** The category of categories is the category $\mathbf{Cat}$

(i). whose collection of objects $\mathrm{Obj}(\mathbf{Cat})$ is the collection of all categories;
(ii). with morphism set $\mathrm{Mor}_{\mathbf{Cat}}(\mathbf{C}, \mathbf{D})$ precisely the set of all functors from $\mathbf{C}$ to $\mathbf{D}$;
(iii). whose composition is given by ordinary function composition of both the functions on objects and morphisms;
(iv). and whose identities are given by the functors for which both the functions on objects and morphisms is the identity function.

R Okay, so I won't blame you if you think this is total nonsense. "Category of categories"? That's just asking for some sort of paradox. The answer to this of course is that we're being sloppy.

You'll recall during our discussion of Russell's Paradox (around (A.1.1)), we explained that our method of resolving the paradox was to fix some set U, the "universe", that satisfied certain properties which made it reasonable to do "all of mathematics" inside U. Russel's Paradox was then resolved by concluding that the set in question simply was just not an element of U.

Something nearly identity is going on here. Implicitly, we pick a smallest "universe" we wish to work in U_0. Then, whenever we say something like "the category of all rgs", it is implicit that all those rgs are coming from U_0. Doing this will usually result in categories that are themselves not contained in U_0, but rather, a larger universe U_1. Then, when we form "the category of all categories", it is implicit that all our categories are coming from U_1. And again, the resulting category will not be in U_1. But so what. 99% of the time, that doesn't matter. Yes, yes, technically we should be saying "The category of all categories in U_1.", but you can see how that would get really annoying really fast, and as it will never matter for us, we are sloppy.[a]

[a] And also give this remark because we feel guilty about being sloppy.

When I think of functors intuitively, I think of them as some sort of mathematical "constructions", something that takes in objects of $\mathbf{C}$ and spits out objects of $\mathbf{D}$. For example, the entire subject of algebraic topology originated as studying "constructions" that associate algebraic objects (like groups) to "geometric" objects (like topological spaces). Unfortunately, our limited background gives us a limited supply of examples, but we do have some things.

▪ **Example B.3.1.4 — The power-set functor** Consider the functor $\mathbf{Set} \to \mathbf{Pos}$ from the category of sets to the category of partially-ordered sets defined by

$$\mathrm{Obj}(\mathbf{Set}) \ni X \mapsto \langle 2^X, \subseteq \rangle \in \mathrm{Obj}(\mathbf{Pos}) \qquad \text{(B.3.1.5a)}$$

$$\mathrm{Mor}_{\mathbf{Set}}(X, Y) \ni f \mapsto f \in \mathrm{Mor}_{\mathbf{Pos}}(2^X, 2^Y), \qquad \text{(B.3.1.5b)}$$

where $f \in \mathrm{Mor}_{\mathbf{Pos}}(2^X, 2^Y)$ is the function that sends $S \subseteq X$ to $f(Y) \subseteq Y$.

Exercise B.3.1.6 Check that this defines a functor.

The next example we would like to consider is the *dual-space* functor $V \mapsto V^\dagger$ on the category of vector spaces. Unfortunately, however, we run into a bit of a problem when trying to do this naively. This functor on objects acts as $V \mapsto V^\dagger$, and so naturally on morphisms this functor acts as $T \mapsto T^\dagger$. The definition of a functor, however, requires that, if $T \in \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(V, W)$, then $T^\dagger \in \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(V^\dagger, W^\dagger)$. But it's not. $T^\dagger$ is not a map from $V^\dagger$ to $W^\dagger$, but rather, a map from $W^\dagger$ to $V^\dagger$. We get around this little hiccup by defining the *opposite category*.

Definition B.3.1.7 — Opposite category Let $\mathbf{C}$ be a category. Then, the ***opposite category*** of $\mathbf{C}$, $\mathbf{C}^{\mathrm{op}}$, is defined by

(I). $\mathrm{Obj}(\mathbf{C}^{\mathrm{op}}) := \mathrm{Obj}(\mathbf{C})$;

(II). for $A, B \in \mathrm{Obj}(\mathbf{C}^{\mathrm{op}})$,

$$\mathrm{Mor}_{\mathbf{C}^{\mathrm{op}}}(A, B) := \mathrm{Mor}_{\mathbf{C}^{\mathrm{op}}}(B, A); \tag{B.3.1.8}$$

(III). for $f, g \in \mathrm{Mor}_{\mathbf{C}^{\mathrm{op}}}$ composable,

$$f \circ_{\mathrm{op}} g := g \circ f; \tag{B.3.1.9}$$

and

(IV). for $A \in \mathrm{Obj}(\mathbf{C}^{\mathrm{op}})$, $\mathrm{id}_A := \mathrm{id}_A$.

R In brief, $\mathbf{C}^{\mathrm{op}}$ has the same objects as $\mathbf{C}$, but the morphisms go in the *opposite* direction.

Such a construction might seem a bit silly, but it allows us to make the following convenient definition.

Definition B.3.1.10 — Cofunctor Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Then, a ***cofunctor*** from $\mathbf{C}$ to $\mathbf{D}$ is a functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{D}$.

R We will often write things like "Let $\mathbf{f}\colon \mathbf{C} \to \mathbf{D}$ be a cofunctor...". Strictly speaking, the domain category is $\mathbf{C}$, not $\mathbf{C}^{\mathrm{op}}$, but this notation is convenient for what will eventually be obvious reasons.

R This is more commonly referred to as a ***contravariant functor*** (in which case an 'ordinary' functor is called a ***covariant functor*** for contrast). I actually think this is pretty terrible terminology as in pretty much every other context in mathematics something with the prefix "co" is the "dual" thing, not the 'primary' thing.

Proposition B.3.1.11 Let $\mathbf{C}$ and $\mathbf{D}$ be categories, let $\mathbf{f}\colon \mathrm{Obj}(\mathbf{C}) \to \mathrm{Obj}(\mathbf{D})$ be a function, and for every $A, B \in \mathrm{Obj}(\mathbf{C})$ let $\mathbf{f}\colon \mathrm{Mor}_{\mathbf{C}}(A, B) \to \mathrm{Mor}_{\mathbf{D}}(\mathbf{f}(B), \mathbf{f}(A))$ be a function. Then, this defines a cofunctor iff

(i). $\mathbf{f}(g \circ f) = \mathbf{f}(f) \circ \mathbf{f}(g)$ for all composable morphisms $f, g \in \mathrm{Mor}_{\mathbf{C}}$; and

(ii). $\mathbf{f}(\mathrm{id}_A) = \mathrm{id}_{\mathbf{f}(A)}$.

R Note that this is exactly the same as the definition of a functor (Definition B.3.1.1), except here the requirement is $\mathbf{f}(g \circ f) = \mathbf{f}(f) \circ \mathbf{f}(g)$ (instead of $\mathbf{f}(g \circ f) = \mathbf{f}(g) \circ \mathbf{f}(f)$). In this sense, cofunctors are just functors which 'flip' the order of composition.

Proof. We leave this as an exercise.

Exercise B.3.1.12 Prove the result.

■

With this language in hand, we can now return to what is probably the most relevant example for us.

■ **Example B.3.1.13 — Dual-space** Let $\mathbb{K}$ be a cring and consider the *co*functor $\mathbb{K}\text{-}\mathbf{Mod} \to \mathbb{K}\text{-}\mathbf{Mod}$ defined by

$$\begin{aligned}\mathrm{Obj}(\mathbb{K}\text{-}\mathbf{Mod}) \ni V &\mapsto V^\dagger \in \mathrm{Obj}(\mathbb{K}\text{-}\mathbf{Mod})\\ \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(V,W) \ni T &\mapsto T^\dagger \in \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(W^\dagger, V^\dagger)\end{aligned}$$

Exercise B.3.1.15 Check that this defines a functor.

B.3.2 Natural-transformations

We can compose the dual-space functor with itself to obtain the "double-dual-space" functor $V \mapsto [V^\dagger]^\dagger$. Recall that (Theorem D.3.1.8), for every $\mathbb{K}$-module V, we actually have a linear-transformation $V \to [V^\dagger]^\dagger$, and in fact, this is an isomorphism in the context of finite-dimensional vector spaces.

On the other hand, if V is a finite-dimensional vector space, then so is $V^\dagger$. Furthermore, they have the same dimension, and hence are isomorphic.[7] The isomorphisms $V \cong V^\dagger$ and $V \cong [V^\dagger]^\dagger$ are fundamentally different, however. The latter is what is called a *natural isomorphism*. To see more clearly how these isomorphisms are different, let us recall more explicitly what they are.

First of all, the map $V \to [V^\dagger]^\dagger$ is given by

$$v \mapsto \langle \cdot, v\rangle. \tag{B.3.2.1}$$

On the other hand, the isomorphism $V \to V^\dagger$ is more complicated. Let $\{b_1, \ldots, b_d\}$ be a basis for V, so that the dual basis $\{b_1^\dagger, \ldots, b_d^\dagger\}$ is a basis for $V^\dagger$. Then,

$$V \ni \alpha_1 \cdot b_1 + \cdots + \alpha_d \cdot b_d \mapsto \alpha_1 \cdot b_1^\dagger + \cdots + \alpha_d \cdot b_d^\dagger \in V^\dagger \tag{B.3.2.2}$$

is an isomorphism. The significant thing to note here is that the definition of this morphism *depended on an arbitrary choice* of a basis for V.

[7] Because they are both isomorphic to $\mathbb{K}^d$ by taking coordinates (after choosing bases).

Thus, in order to define the morphism $V \to V^\dagger$ for every vector space V, you'll have to pick a basis for every single vector space, and there is no single "natural" way to do so. The isomorphisms that I happen to use are almost certainly going to be different than the ones you choose to use. On the other hand, there is *no choice* when it comes to the isomorphism $V \to [V^\dagger]^\dagger$. Intuitively, this isomorphism doesn't depend on V, the same definition works for every vector space, whereas the isomorphisms $V \to V^\dagger$ do depend on V.

The precise notion which distinguishes these two is that of a *natural transformation*: $V \to [V^\dagger]^\dagger$ will be a natural isomorphism, whereas $V \to V^\dagger$ will not be.

Definition B.3.2.3 — Natural-transformation Let $\mathbf{C}$ and $\mathbf{D}$ be functors, and let $\mathbf{f}, \mathbf{g}\colon \mathbf{C} \to \mathbf{D}$ be functors. Then, a ***natural transformation*** from $\mathbf{f}$ to $\mathbf{g}$ is, for every $A \in \mathrm{Obj}(\mathbf{C})$, a morphism $\eta_A\colon \mathbf{f}(A) \to \mathbf{f}(B)$, such that

$$\begin{tikzcd} \mathbf{f}(A) \arrow[r, "\mathbf{f}(f)"] \arrow[d, "\eta_A"'] & \mathbf{f}(B) \arrow[d, "\eta_B"] \\ \mathbf{g}(A) \arrow[r, "\mathbf{g}(f)"'] & \mathbf{g}(B) \end{tikzcd} \tag{B.3.2.4}$$

commutes for every morphism $f \in \mathrm{Mor}_{\mathbf{C}}(A, B)$ and $A, B \in \mathrm{Obj}(\mathbf{C})$.

R A *diagram* in this sense of the word refers to a set of objects and morphisms between them indicated by drawing arrows for each morphism (from the domain to the codomain). A path from one object to another (in the direction indicated by the arrows) then corresponds to the composition of the corresponding morphisms. If you select any two objects in a given diagram, in general, there will be more than one path from the first to the second; the diagram is said to *commute* iff all corresponding compositions agree.

For example, in this case, the phrase "the following diagram commutes" should be understood as shorthand for the statement "$\mathbf{g}(f) \circ \eta_A = \eta_B \circ \mathbf{f}(f)$".

R The entire natural-transformation is usually just denoted "η", in which case η_A is referred to as the ***component*** of η at A.

R Admittedly, this definition is difficult to understand at first sight. For one thing, it's not clear how this captures "doesn't depend on A/B". I will do the best I can to explain.

The idea is that, if I can define the morphisms η_A in such a way that doesn't make use of anything special to this particular A, then it shouldn't matter whether I 'go to a different object' and then apply the morphism, or if I first apply the morphism and then "go to a different object". (Here, "going to a different object" corresponds to $\mathbf{f}(f)$ and $\mathbf{g}(f)$, and "the morphism" refers to η_A and η_B.)

It turns out that, in the following sense, natural-transformations should themselves be thought of as *morphisms of functors*.

Definition B.3.2.5 — Category of functors Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Then, the ***category of functors*** from $\mathbf{C}$ to $\mathbf{D}$, $\mathrm{Mor}_{\mathbf{Cat}}(\mathbf{C},\mathbf{D})$, is defined by

(I). $\mathrm{Obj}\left(\mathrm{Mor}_{\mathbf{Cat}}(\mathbf{C},\mathbf{D})\right) := \mathrm{Mor}_{\mathbf{Cat}}(\mathbf{C},\mathbf{D})$;

(II). $\mathrm{Mor}_{\mathrm{Mor}_{\mathbf{Cat}}(\mathbf{C},\mathbf{D})}(\mathbf{f},\mathbf{g})$ is the collection of natural-transformations from $\mathbf{f}$ to $\mathbf{g}$;

(III). composition of natural-transformations is defined componentwise; and

(IV). $[\mathrm{id}_{\mathbf{f}}]_A := \mathrm{id}_A$.

R Thus, the objects are functors, the morphisms are the natural-transformations, composition is done the way you would expect (componentwise), and the identity on $\mathbf{f} \in \mathrm{Obj}\left(\mathrm{Mor}_{\mathbf{Cat}}(\mathbf{C},\mathbf{D})\right)$ is the natural-transformation whose component as the object $A \in \mathbf{C}$ is id_A.

R We are abusing notation and using $\mathrm{Mor}_{\mathbf{Cat}}(\mathbf{C},\mathbf{D})$ to denote the collection of functors from $\mathbf{C}$ to $\mathbf{D}$ as well as the corresponding category.

As in every category, we have a notion of isomorphism. In the case of categories of functors, the following term is used.

Definition B.3.2.6 — Natural-isomorphism Let $\eta\colon \mathbf{f} \to \mathbf{g}$ be a natural-transformation of functors $\mathbf{f}, \mathbf{g}\colon \mathbf{C} \to \mathbf{D}$. Then, η is a ***natural-isomorphism*** iff η is an isomorphism in $\mathrm{Mor}_{\mathbf{Cat}}(\mathbf{C}, \mathbf{G})$.

> R If we're thinking of functors as some sort of mathematical 'constructions' (e.g. taking the dual of a vector space), then, intuitively speaking, saying that "$\mathbf{f}$ is naturally-isomorphic to $\mathbf{g}$" is saying more than just "$\mathbf{f}(A)$ is isomorphic to $\mathbf{g}(A)$ for all $A \in \mathrm{Obj}(\mathbf{C})$", but rather that the entire "constructions" $A \mapsto \mathbf{f}(A)$ and $A \mapsto \mathbf{g}(A)$ are isomorphic.

Fortunately, this is equivalent to what you would hope.

Proposition B.3.2.7 Let $\eta\colon \mathbf{f} \to \mathbf{g}$ be a natural-transformation of functors $\mathbf{f}, \mathbf{g}\colon \mathbf{C} \to \mathbf{D}$. Then, η is a natural-isomorphism iff $\eta_A\colon \mathbf{f}(A) \to \mathbf{g}(A)$ is an isomorphism for all $A \in \mathrm{Obj}(\mathbf{C})$.

Proof. We leave this as an exercise.

Exercise B.3.2.8 Prove the result.

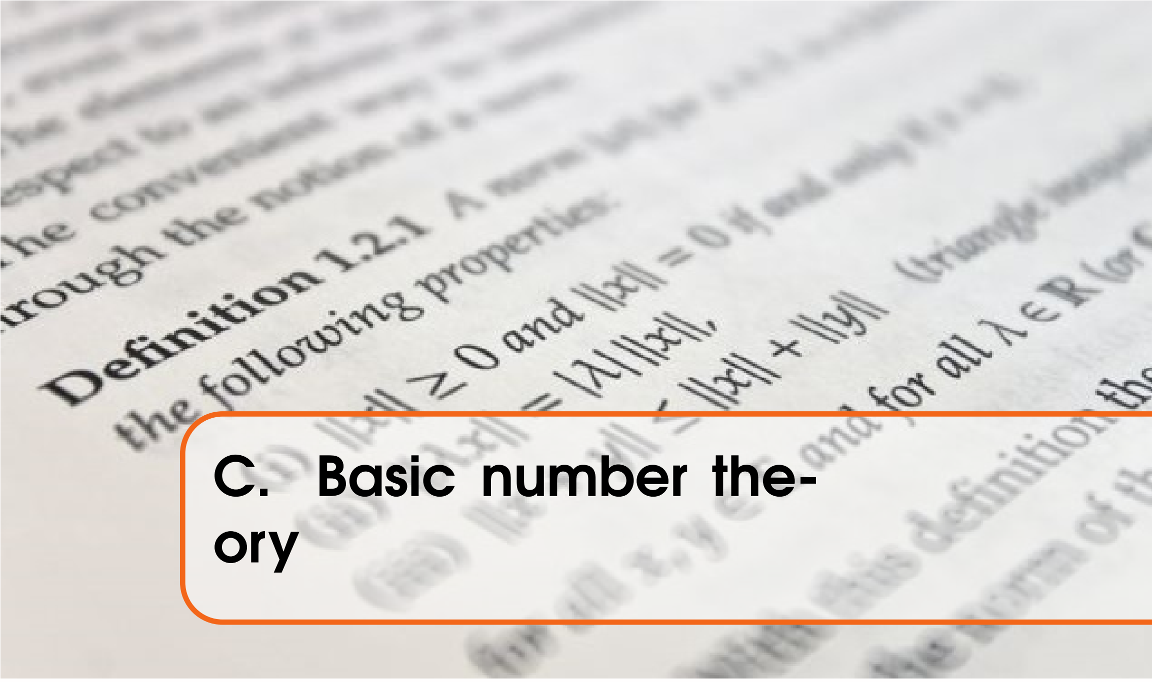
■

We now finally return to the example of primary interest.

■ **Example B.3.2.9 — $V \to [V^\dagger]^\dagger$** Let $\mathbb{K}$ be a cring and consider the linear-transformation $\eta_V\colon V \to [V^\dagger]^\dagger$ defined by $v \mapsto \langle \cdot, v\rangle$.

Exercise B.3.2.10 Show that this is a natural-transformation.

Thus, by Theorem D.3.1.8, it is in fact a natural-isomorphism in the category of finite-dimensional vector spaces.



C. Basic number theory

Note that, unlike the previous two appendices, this one requires some prerequisites from the main text. In particular, you should have technically read through the section defining the integers.[1]

On one hand, these are note on *analysis*, and so it might seem a bit odd to include an appendix on number theory. On the other hand, there are places throughout the notes where we will need to make use of elementary number theoretic concepts (e.g, prime numbers, divisibility, greatest common divisors, etc.). There's a good chance you probably know a lot of this already, but in the spirit of building *everything* from the ground-up, these basic concepts deserve to be treated somewhere.

The first thing we must do is what is called the ***Division Algorithm***, and it is simply a precise statement of how division (with remainder) works.

Theorem C.1 — Division Algorithm. Let $m \in \mathbb{Z}$ and let $n \in \mathbb{Z}^+$. Then, there are unique integers $q, r \in \mathbb{Z}$ such that

[1] Though I realize that, in practice, it's quite likely you'll be able to understand the contents of this appendix without having seen a formal development of the integers.

(i). $m = qn + r$; and
(ii). $0 \leq r < n$.

R m is the ***dividend***, n is the ***divisor***, q is the ***quotient***, and r is the ***remainder***.

Proof. Define $S := \{m - nx \in \mathbb{N} : x \in \mathbb{Z}\}$.

Exercise C.2 Show that S is nonempty.

W Warning: It is not as simply as just 'plugging-in' $x = 0$—m itself may not be a natural number.

As S is nonempty, because $\mathbb{N}$ is well-ordered, it must have a smallest element. Call that element $r = m - nq \in \mathbb{N}$. We thus immediately have that $m = nq + r$. As $r \in \mathbb{N}$, $r \geq 0$. We must check that $r < n$.

We proceed by contradiction: suppose that $r \geq n$. Then, $r - n = m - n(q + 1) \in S$ is smaller than r because $n > 0$, a contradiction of the fact that r was the *smallest* element of S. Therefore, $r < n$.

We now must prove uniqueness. Let $q', r' \in \mathbb{Z}$ be such that (i) $m = nq' + r'$ and (ii) $0 \leq r' < n$. Without loss of generality, suppose that $q' \geq q$. Rearranging $nq + r = m = nq' + r'$ gives $r - r' = n(q' - q)$. If $q' = q$, then from this it follows likewise that $r' = r$, and we are done, so suppose that $q' \neq q$. As we have assumed that $q' \geq q$, we must have that $q' > q$, and so $q' - q \geq 1$. It follows that $r - r' = n(q' - q) \geq n$, a contradiction of the fact that $r < n$. Thus, it must have been the case that $q' = q$ and $r' = r$. ∎

The next thing we must investigate are the fundamental concepts of divides, irreducibility, and prime.

Definition C.3 — Divides Let $m, n \in \mathbb{Z}$. Then, m ***divides*** n, written $m \mid n$, iff there is some $k \in \mathbb{Z}$ such that $n = mk$.

R We also say that n is a ***multiple*** of m or that m is a ***factor*** of n. Note that the set of all multiples of n is just $n\mathbb{Z} := \{nk : k \in \mathbb{Z}\}$, so alternatively this definition can be written as "m divides n iff $m \in n\mathbb{Z}$.". This will be important when you go to generalize this concept to other rings (or even other rgs, I suppose, though I've never seen this done).

R Though we technically don't know what division is yet, rearranging this suggestively as $\frac{n}{m} = k$, we see that this definition is just the statement that n divided by m is an integer.

This actually gives us our first 'natural' example of a preorder that is not a partial-order.

Exercise C.4 Show that $\langle \mathbb{Z}, \mid \rangle$ is a preorder that is not a partial-order.

Among other reasons, the concept of divisibility is important because it allows us to define the notion of *primality*

Definition C.5 — Prime Let $p \in \mathbb{Z}$. Then, p is ***prime*** iff

(i). $p \neq 0$;
(ii). $p \notin \mathbb{Z}^{\times}$; and
(iii). whenever $p \mid (mn)$, it follows that $p \mid m$ or $p \mid n$.

R Recall (Definition A.4.16) that $R^{\times}$ is the group of units in the rig R, that is, the elements which have a multiplicative inverse. For $\mathbb{Z}$, we have $\mathbb{Z}^{\times} = \{1, -1\}$, and so this is just a fancy way of saying "p is irreducible iff $p \neq \pm 1$ and whenever $p = mn$, it follows that$m = \pm 1$ or $n = \pm 1$.". The important condition here is thus the second one—the first serves only to rule out the 'stupid' case of ± 1 being irreducible.

R Using the remark of the previous definition (Definition C.3), note that this can be written as "p is prime iff whenever $mn \in p\mathbb{Z}$, it follows that $m \in p\mathbb{Z}$ or $n \in p\mathbb{Z}$.". Same as before, this will be important when generalizing this concept to other rings.

R This is probably not the definition you are used to. The definition you are used to is what is called *irreducible* (see the next definition (Definition C.6)). It turns out that they are equivalent for $\mathbb{Z}$[a] (Proposition C.16), though *this will fail for general rings*. We make our terminology the way we do so that it agrees with the more general case (where the concepts are indeed distinct ones).

[a] And I suppose, more generally, for what are called *GCD domains*.

Definition C.6 — Irreducible Let $p \in \mathbb{Z}$. Then, p is ***irreducible*** iff

(i). $p \neq 0$;
(ii). $p \notin \mathbb{Z}^{\times}$; and
(iii). whenever $p = mn$, it follows that $m \in \mathbb{Z}^{\times}$ or $n \in \mathbb{Z}^{\times}$.

R The first two conditions are simply to rule out 'stupid' cases. The important condition is the last one and, in English, says that (a nonzero nonunit) is irreducible iff it cannot be factored as the product of two nonunits.

Our first objective is to show that these two definitions are equivalent. Before we do so, however, we're going to need a couple of extra tools, one of which is the *greatest common divisor*.

Definition C.7 — Greatest common divisor Let $m, n \in \mathbb{Z}$, not both 0, and let $d \in \mathbb{Z}$. Then, d is a ***greatest common divisor*** of m and n iff

(i). $d \mid m$;

(ii). $d \mid n$; and
(iii). if $k \mid m, n$, then $k \mid d$.

If $m = 0 = n$, then we declare the greatest common divisor to be 0.[a]

R That is to say, a greatest common divisor of m and n is a maximum (that is, the *greatest*) of the set $\{k \in \mathbb{Z} : k \mid m, n\}$ (that is, the set of *common divisors*), with respect to the preorder defined by $\mid$ (Exercise C.4).

R Note that the word "greatest" here really refers to the preorder defined by $\mid$, *not* the preorder defined by $\leq$. The reason for this is that the relation $\mid$ makes sense in any crg, whereas the 'normal' $\leq$ constitutes extra structure that is lacking in many important examples of crgs.

R Note that great common divisors are *not* unique. For example, both $+6$ and -6 are greatest common divisors of 12 and 30. In the special case of $\mathbb{Z}$, greatest common divisors do come in pairs, and so there is a canonical one, namely, the positive one (in the example, $+6$). Some authors, especially if they intend to work mostly or exclusively with $\mathbb{Z}$, would only consider 6 as a greatest common divisor or 12 and 30. As per usual, however, we prefer to keep our definitions in a form that generalizes as easy as possible.

[a] In this case, everything is a common divisor, and in particular, there is no greatest one. For the sake of definiteness the, we simply define the only greatest common divisor of 0 and 0 to be 0.

We similarly have a 'dual' concept.

Definition C.8 — Least common multiple Let $m, n \in \mathbb{Z}$ and let $l \in \mathbb{Z}$. Then, l is a ***least common multiple*** of m and n iff

(i). $m \mid l$;
(ii). $n \mid l$; and
(iii). if $m, n \mid k$, then $l \mid k$.

It turns out that any two integers have a greatest common divisor, though this is not immediate.

Theorem C.9 — Bézout's Identity. Let $m, n \in \mathbb{Z}$. Then, there exists a unique positive greatest common divisor $d \in \mathbb{Z}^+$ of m and n. Furthermore, d is the smallest positive integer which can be written as an integral linear combination of m and n, that is, it is the smallest positive element of the set

$$\{xm + yn : x, y \in \mathbb{Z}\}. \tag{C.10}$$

R As previously mentioned, greatest common divisors are not actually unique. Thus, to get uniqueness, we need to impose the extra condition of "positive". This won't make sense in general of course, but it makes sense in $\mathbb{Z}$, which is our primary cring of interest at the moment. We shall thus write $d = \gcd(m, n)$ for the unique positive greatest common divisor when working in $\mathbb{Z}$. Similarly, there is a unique nonnegative least common multiple of m and n, which we denote by $\operatorname{lcm}(m, n)$.

W Warning: Existence of greatest common divisors fails in general integral crings. In particular, this statement is not immediate—some proving is most definitely required.

R The reason this is sometimes referred to as an "identity" is because the second part of the statement says in particular that $d = xm + yn$ for some $x, y \in \mathbb{Z}$.

R The intuition for the second part is that, any common divisor of $m, n \in \mathbb{Z}$ must divide $xm + yn$ for $x, y \in \mathbb{Z}$, and so in particular the greatest common divisor cannot be strictly 'larger' than any element of this form. One might thus conjecture that the smallest element of this form is a greatest common divisor, and in fact, this is precisely the case.

Proof. Step 1: Define d
Define $D := \{xm + yn \in \mathbb{Z}^+ : x, y \in \mathbb{Z}\}$.

Exercise C.11 Show that D is nonempty.

As D is nonempty, it in particular has a smallest element, $d = x_d m + y_d n \in D$. We wish to show that d is the greatest common divisor of m and n.

Step 2: Show that every element of D is a multiple of d
Let $a \in D$. By the Division Algorithm, there are unique $q, r \in \mathbb{Z}$ such that $a = qd + r$ and $0 \leq r < d$. It suffices to show that $r = 0$. We proceed by contradiction: suppose that $r > 0$. Note that, a is an integral linear combination of m and n and so is qd, and hence $r = a - qd$ is likewise an integral linear combination of m and n. Hence, as $r > 0$, $r \in D$. As $r < d$, this is a contradiction of the fact that d was the smallest element of D.

Step 3: Deduce that d is a common divisor of m and n
By $m \leftrightarrow n$ symmetry, it suffices to prove that d is a divisor of m. If $m = 0$, then of course d is a divisor of m ($m = 0 = d \cdot 0$). On the other hand, if $m > 0$, then $m = 1 \cdot m + 0 \cdot n \in D$, and if $m < 0$, then $-m = -1 \cdot m + 0 \cdot n \in D$. In either case, by the previous step, $\pm m$ id a multiple of d, and so d is a divisor of m.

Step 4: Show that d is the *greatest* common divisor of m and n
Let $d' \in \mathbb{Z}$ be another common divisor of m and n. As $d = x_d m + y_d n$, it follows that d' in turn divides d, as desired.

Step 5: Show uniqueness

Redefine notation and let $d' \in \mathbb{Z}$ be another positive greatest common divisor of m and n. As d is a *greatest* common divisor and d' is a common divisor, $d' \mid d$. Similarly, as d' is a *greatest* common divisor and d is a divisor, $d \mid d'$.

Exercise C.12 Show that this implies that $d' = \pm d$.

As d and d' are both positive, we must then have that $d' = d$, as desired. ■

The case when $\gcd(m,n) = 1$ is particularly important, and this even has a name of its own.

Definition C.13 — Relatively prime Let $m, n \in \mathbb{Z}$. Then, m and n are ***relatively prime*** iff $\gcd(m,n) = 1$.

R Put another way, this means that m and n have no common factors. For us, this comes up for example in the uniqueness of the fraction representing a rational number—see Proposition 1.3.8.

One reason this is important is the following.

Proposition C.14 — Euclid's Lemma Let $m, n, o \in \mathbb{Z}$. Then, if m and n are relatively prime and $m \mid (no)$, then $m \mid o$.

Proof. Suppose that m and n are relatively prime and $m \mid (no)$. By Bézout's Identity, there are $x, y \in \mathbb{Z}$ such that $1 = xm + yn$. From the definition of divisibility, there is some $k \in \mathbb{Z}$ such that $no = mk$. Multiplying through by y and using the previous equation, we get

$$ymk = yno = (1 - xm)o, \tag{C.15}$$

and so $o = m(yk + ox)$. Thus, $m \mid o$, as desired. ■

Now we finally return to the issue of the equivalence of irreducibility and primality in $\mathbb{R}$.

Proposition C.16 Let $p \in \mathbb{Z}$. Then, p is prime iff it is irreducible.

W Warning: As was previously mentioned, this result *does not generalize* to other rings. On the other hand, prime *always*[a] implies irreducible, and the converse holds if the integral cring is something called a *GCD domain*.[b]

[a]"Always" meaning "in integral crings.". You definitely need integrality–for example, $3 \in \mathbb{Z}/4\mathbb{Z}$ is irreducible, but, as 3 divides $0 = 2 \cdot 2$ and 3 doesn't divide 2, 3 is not prime.

[b]Though it seems that irreducible iff prime holds even in some rings that are not GCD domains—see http://math.stackexchange.com/questions/1244890/.

Proof. ($\Rightarrow$) Suppose p is prime. p is by definition not 0 and not a unit, and so we only need to check (iii) of Definition C.6. So, let $m, n \in \mathbb{Z}$ be such that $p = mn$. Then, in particular, $p \mid (mn)$,[a] and so as p is prime, $p \mid m$ or $p \mid n$. Without loss of generality, suppose that $p \mid m$. We can then write $m = pk$ for some $k \in \mathbb{Z}$, and so $p = mn$ reads $p = pkn$. Rearranging gives $p(1 - kn) = 0$, and so as $\mathbb{Z}$ is integral and $p \neq 0$, $1 - kn = 0$, that is, $kn = 1$, or in other words, n is a unit.

($\Leftarrow$) Suppose that p is irreducible. p is by definition not 0 and not a unit, and so we only need to check (iii) of Definition C.5. So, let $m, n \in \mathbb{Z}$ and suppose that $p \mid (mn)$. Define $d := \gcd(pn, mn)$. Of course n is a common divisor of pn and mn. By hypothesis, however, p is likewise a common divisor of both pn and mn. Hence, $p \mid d$ and $n \mid d$, so that $pk = d = nl$ for some $k, l \in \mathbb{Z}$. As $d = pk$ divides pn, it follows that $k \mid n$. The equation $pk = nl$ then gives[b] $p = \frac{n}{k} l$.[c] As p is irreducible, we must then have that either $\frac{n}{k}$ or l is a unit, i.e. ± 1. In the first case, $l = \pm p$, and so, as $d = nl$ divides mn, l divides m, and hence $p = \pm l$ divides m. On the other hand, in the second case, $\frac{n}{k} = \pm p$, and so $n = p \cdot (\pm k)$, that is, $p \mid n$.

If $p \mid m$, we are done, so suppose this is not the case. As p is irreducible, This means that there is some $k \in \mathbb{Z}$ such that $mn = pk$. ■

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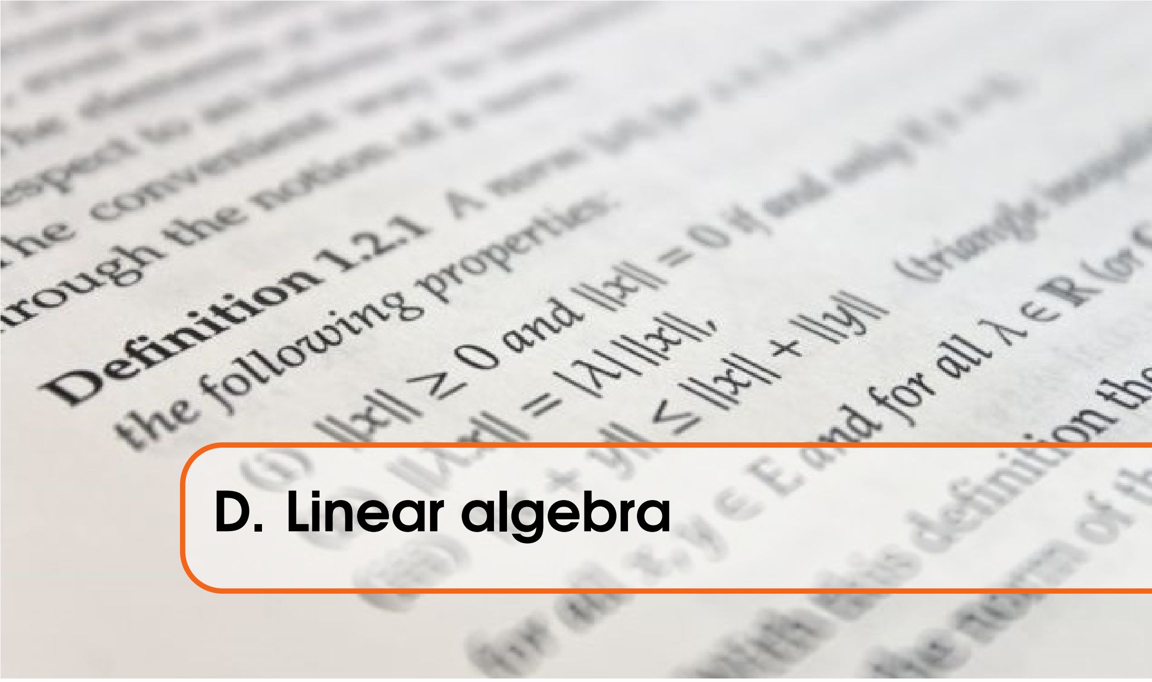
[a] $mn = p \cdot 1$.

[b] We need $k \neq 0$ to 'cancel' it. If $k = 0$, then $d = 0$, which forces $pn = 0 = mn$, which forces $n = 0$ (because $p \neq 0$). Then, $p \mid n$, and we are done. Thus, we can assume that $k \neq 0$.

[c] As we don't technically have multiplicative inverses yet, you should regard $\frac{n}{k}$ as short-hand for the unique integer which satisfies $p = k\frac{n}{k}$.



D. Linear algebra

As explained at the beginning of this appendix and elsewhere, when all was said and done, I very much wanted to be able to say "We did everything from scratch.". Linear algebra, while not analysis proper, does make an appearance throughout the main text, especially in the chapter on differentiation (Chapter 6). Thus, in the spirit of reductionism ad infinitum, it is necessary to include a treatment of linear algebra somewhere in these notes, and I felt as if an appendix was an appropriate location to place such relevant facts.

That said, in practice, while linear algebra is not technically a prerequisite for these notes, the overwhelming majority of students reading these notes will have at least a passing familiarity with the basic concepts of linear algebra. Thus, it doesn't make sense to go into an incredible amount of depth. The purpose of this appendix then is to really more provide a reference to the relevant facts (and show that everything can in fact be done from scratch), rather than to teach linear algebra per se.

In any case, let us begin.

D.1 Basic definitions

Linear algebra is the study of vector spaces.[1] If V is a vector space, the elements of V are typically referred to as *vectors*. This is slightly unfortunate terminology, which, while second nature to mathematicians, can be a bit confusing for beginning students because *the elements of V are not necessarily 'actual vectors'*. For example, the set of real-valued functions on a set can be made into a vector space, and certainly functions themselves are not traditionally thought of as "vectors". Thus, to avoid any possible confusion or ambiguity, I will be careful to make the following distinction.

> Elements of a general vector space will simply be referred to as ***vectors***. Elements of the specific vector space $\mathbb{R}^d$ will be referred to as ***column vectors***.[a]
>
> [a] More precisely, we will use this terminology when regarding $\mathbb{R}^d$ as an object in $\mathbf{Vect}_{\mathbb{R}}$, the category of vector spaces over $\mathbb{R}$—see Example D.1.9.

Okay, now that that caveat is out of the way, let us return to the definition of a vector space. A vector space is (i) a set, the elements of which are called *vectors*[2]; together with (ii) rules for adding and scaling the vectors.[3] This is the intuition anyways. In this way, the definition of vector spaces in general is an abstraction of the set of column vectors $\mathbb{R}^d$ that you know and love. This is not unlike how we 'abstracted' the definition of a general topological space (Definition 3.1.1) from the properties of open sets in $\mathbb{R}^d$ (Exercises 2.5.2.3 and 2.5.2.4 and Theorems 2.5.2.28 and 2.5.2.30).

To specify the scaling operation, however, we first need to say what the "scalars" are. For vector spaces, the scalars are taken to be a field. That said, the axioms of a vector spaces make no references to the existence of multiplicative inverses, and so in fact the definition makes just as much sense if you allow your field to be a more general

[1] Or possibly *finite-dimensional* vector spaces if you prefer—it's not as if there is a strict definition of what the term "linear algebra" refers to.

[2] Though are most definitely not necessarily "actual" column vectors—we just spent the last couple of paragraphs harping on this point.

[3] Subject to various axioms of course.

ring. This is what is called an *R-module*. As there is no reason to needlessly throw away this extra generality,[4] we present the definition in this form.

Definition D.1.1 — R-module Let R be a ring. Then, an ***R*-module** is

(I). a commutative group $\langle M, +, 0, - \rangle$; together with
(II). a function $\cdot \colon R \times M \to M$;

such that

(i). $(\alpha_1 \alpha_2) \cdot v = \alpha_1 \cdot (\alpha_2 \cdot v)$;
(ii). $1 \cdot v = v$;
(iii). $(\alpha_1 + \alpha_2) \cdot v = \alpha_1 \cdot v + \alpha_2 \cdot v$; and
(iv). $\alpha \cdot (v_1 + v_2) = \alpha \cdot v_1 + \alpha \cdot v_2$;

for all $\alpha, \alpha_1, \alpha_2 \in R$ and $v, v_1, v_2 \in M$.

R In elementary linear algebra textbooks, this is often listed as eight (or more) axioms. This is quite a bit, and I think it is easy to remember them as follows: four of those axioms are simply equivalent to the statement that $\langle M, +, 0, - \rangle$ is a commutative group, two of them are distributive axioms, and the other two are natural "compatibility" axioms between the multiplication in R and $\cdot$.

R This is sometimes referred to as a *left* R-module because the "scalars" appear on the left. Of course, rewriting things a bit, it is easy enough to write down the definition of a *right* R-module as well. There really isn't any serious difference between the two definitions, but one convention can be more convenient than the other in certain contexts.[a]

R In case you're wondering why we're not using the letter m for elements of M (like we would use r for elements of a ring R), it's because I really would like to reserve the symbol "m" for elements of $\mathbb{N}$ (or $\mathbb{Z}$).

[a] This is not unlike how it is sometimes convenient to write composition in ***postfix notation***: $f \cdot g := g \circ f$.

[4] Except for perhaps, you know, pedagogy.

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Exercise D.1.2 Let R be a ring and let M be an R-module.

(i). Show that $0 = 0 \cdot v$ for all $v \in M$.
(ii). Show that $-v = (-1) \cdot v$ for all $v \in M$.

Example D.1.3 — Commutative groups are $\mathbb{Z}$-modules Let $\langle V, +, 0, -\rangle$ be a commutative group, and for $n \in \mathbb{Z}$, define

$$n \cdot v := \operatorname{sgn}(n)(\underbrace{v + \cdots + v}_{|n|}). \tag{D.1.4}$$

Exercise D.1.5 Check that indeed $\langle V, +, 0, -, \mathbb{Z}, \cdot\rangle$ is a $\mathbb{Z}$-module.

R In fact, every $\mathbb{Z}$-module also determines a commutative group, simply by "forgetting" about the scaling operation. These constructions, from commutative groups to $\mathbb{Z}$-modules and from $\mathbb{Z}$-modules to commutative groups, are inverse to one another. (The intuition is that the group structure determines the scaling operation by integers, and so adding the additional structure of a $\mathbb{Z}$-module gives nothing new.) For this reason, one often does not distinguish between commutative groups and $\mathbb{Z}$-modules, and you may freely change how you think about things depending on what is most convenient to the objective at hand.

Having defined R-modules, we are now able to present the definition of a vector space.

Definition D.1.6 — Vector space A ***vector space*** over F is an F-module where F is a field.

R Elements of F are called ***scalars***. Elements of V are called ***vectors***.

R I can't guarantee that I won't use the terms "scalar" and "vector" in the context of R-modules, R not-necessarily-a-field, though this is not standard.

Having defined a type of mathematical object, we are now morally obligated to specify what the relevant notion of morphism is.

Definition D.1.7 — Linear transformation Let R be a ring, let M and N be R-modules, and let $T: M \to N$ be a function. Then, T is an ***R-linear transformation*** iff

(i). $T: M \to N$ is a group homomorphism; and
(ii). $T(\alpha \cdot v) = \alpha \cdot T(v)$ for all $\alpha \in R$ and $v \in V$.

R Explicitly, (i) means that $T(v_1 + v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in M$.

R If R is clear from context as it often is, we shall simply say ***linear transformation***.

R A synonym for R-linear transformation is ***R-module homomorphism***.

With a working notion of morphism in hand, we obtain corresponding categories.

■ **Example D.1.8 — The category of R-modules** Let R be a ring. Then, the category of R-modules is the category R-**Mod**

(i). whose collection of objects $\mathrm{Obj}(R\text{-}\mathbf{Mod})$ is the collection of all R-modules;
(ii). with morphism set $\mathrm{Mor}_{R\text{-}\mathbf{Mod}}(M, N)$ precisely the set of all linear transformations from M to N;

(iii). whose composition is given by ordinary function composition; and
(iv). whose the identities are given by the identity functions.

▪ **Example D.1.9 — The category of vector spaces** Let F be a field. Then, the category of vector spaces over F is the category $\mathbf{Vect}_F$

(i). whose collection of objects $\mathrm{Obj}(\mathbf{Vect}_F)$ is the collection of all vector spaces over F;
(ii). with morphism set $\mathrm{Mor}_{\mathbf{Vect}_F}(V,W)$ precisely the set of all linear transformations from V to W;
(iii). whose composition is given by ordinary function composition; and
(iv). whose the identities are given by the identity functions.

D.1.1 Bimodules

In fact, we would like the morphism sets $\mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(V,W)$ themselves to furnish examples of $\mathbb{K}$-modules. Unfortunately, if $\mathbb{K}$ is not commutative, we can't quite do this. To see this, we would need to be able to define a notion of scaling of linear-transformations, and essentially the only thing one could write down is

$$[\alpha\cdot T](v)\coloneqq \alpha\cdot T(v). \tag{D.1.1.1}$$

Unfortunately, however, this is not linear in general:

$$\begin{aligned}[\alpha\cdot T](\beta\cdot v)&\coloneqq \alpha\cdot T(\beta\cdot v)=\alpha\beta\cdot T(v)\neq\beta\cdot(\alpha\cdot T(v))\\ &\eqqcolon \beta\cdot[\alpha\cdot T](v).\end{aligned} \tag{D.1.1.2}$$

In order for what we have written as an inequality to be an equality, we would need to know that $\mathbb{K}$ is commutative. There is a silly way to fix this, however—instead, let us try scaling T *on the right*:

$$[T\cdot\alpha](\beta\cdot v)\coloneqq T(\beta\cdot v)\cdot\alpha=\beta\cdot T(v)\cdot\alpha=\beta\cdot[T\cdot\alpha](v), \tag{D.1.1.3}$$

and so $T\cdot\alpha$ is again $\mathbb{K}$-linear (on the left).[5]

[5]Note that the scaling in W has to be written on the right in order for this to make sense.

Definition D.1.1.4 — $\mathbb{K}$-$\mathbb{L}$-bimodule Let $\mathbb{K}$ and $\mathbb{L}$ be rings. Then, a $\mathbb{K}$-$\mathbb{L}$-***bimodule*** is

(I). a left $\mathbb{K}$-module $\langle V, +, 0, -, \mathbb{K}, \cdot \rangle$; and
(II). a right $\mathbb{L}$-module $\langle V, +, 0, -, \mathbb{L}, \cdot \rangle$[a]

such that

$$(\alpha \cdot v) \cdot \zeta = \alpha \cdot (v \cdot \zeta) \tag{D.1.1.5}$$

for all $v \in V$, $\alpha \in \mathbb{K}$, and $\zeta \in \mathbb{K}$.

R Similarly as we may view commutative groups as $\mathbb{Z}$-modules (Example D.1.3), we may view any left $\mathbb{K}$-module as a $\mathbb{K}$-$\mathbb{Z}$ bimodule, and likewise, we may view any right $\mathbb{L}$-module as a $\mathbb{Z}$-$\mathbb{L}$ bimodule.

[a] Of course, the two scaling operations here are not the same, even those we abuse notation and simply write "$\cdot$" for both.

■ **Example D.1.1.6 — The R-R-bimodule R** Let R be a ring. Writing $V := R$ to keep conceptually straight which Rs are being thought of as rings and which ones are being thought of as bimodules, the left scaling $R \times V \to V$ is defined by $\langle r, v \rangle \mapsto r \cdot v$ and the right scaling $V \times R \to V$ is defined by $\langle v, r \rangle \mapsto v \cdot r$. That is, left (right) scaling is just given by multiplication on the left (right).

■ **Example D.1.1.7 — $\mathbb{K}$-modules over crings** Let $\mathbb{K}$ be a cring and let V be a $\mathbb{K}$-module. Then in fact we can view V as a $\mathbb{K}$-$\mathbb{K}$-bimodule by defining

$$v \cdot \alpha := \alpha \cdot v. \tag{D.1.1.8}$$

Exercise D.1.1.9 Check that this does indeed make V into a $\mathbb{K}$-$\mathbb{K}$-bimodule.

R Note how this requires commutativity.

Thus, if $\mathbb{K}$ is commutative, left $\mathbb{K}$-modules are 'the same as' $\mathbb{K}$-$\mathbb{K}$-bimodules are 'the same as' right $\mathbb{K}$-modules, and so in this case, there is no need to really distinguish.

Now that we have bimodules, however, we are again morally obligated to introduce the new relevant of morphism.

Definition D.1.1.10 — Linear-transformation (of bimodules) Let $\mathbb{K}$ and $\mathbb{L}$ be rings, let V and W be $\mathbb{K}$-$\mathbb{L}$ bimodules, and let $T\colon V \to W$ be a function. Then, T is a ***$\mathbb{K}$-$\mathbb{L}$-linear-transformation*** iff

(i). T is a $\mathbb{K}$-linear-transformation; and
(ii). T is an $\mathbb{L}$-linear-transformation.

R Explicitly, this means that (i) $T(v_1 + v_2) = T(v_1) + T(v_2)$, (ii) $T(\alpha \cdot v) = \alpha \cdot T(v)$, and (iii) $T(v \cdot \zeta) = T(v) \cdot \zeta$.

R If $\mathbb{K}$ and $\mathbb{L}$ are clear from context, we shall simply say ***Linear-transformation***.

R If T only satisfies (i), then we may say that T is ***left linear*** or ***linear on the left***. Likewise, if T only satisfies (ii), we may say that T is ***right linear*** or ***linear on the right***. In this context, we may say that T is ***two-sided linear*** if T satisfies both (i) and (ii) for emphasis to distinguish between just "left-linear" and "right-linear".

R A synonym for "linear-transformation of bimodules" is ***bimodule-homomorphism***.

R Warning: Don't use the term "bilinear-transformation" for this. That means something else—see Definition D.3.2.1.

Again, having defined a new type of object as well as the relevant notion of morphism, we obtain a corresponding category.

▪ Example D.1.1.11 — The category of $\mathbb{K}$-$\mathbb{L}$-bimodules
Let $\mathbb{K}$ and $\mathbb{L}$ be rings. Then, the category of $\mathbb{K}$-$\mathbb{L}$-bimodules is the concrete category $\mathbb{K}$-$\mathbf{Mod}$-$\mathbb{L}$

(i). whose collection of objects $\mathrm{Obj}(\mathbb{K}\text{-}\mathbf{Mod}\text{-}\mathbb{L})$ is the collection of all $\mathbb{K}$-$\mathbb{L}$-bimodules; and
(ii). with morphism set $\mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}\text{-}\mathbb{L}}(V, W)$ precisely the set of all linear-transformations from V to W.

We now turn to the issue of equipping morphism sets of *bimodules* with another bimodule structure. As we can view any $\mathbb{K}$-module V as a $\mathbb{K}$-$\mathbb{Z}$-bimodule, this will allow us to equip $\mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(V, W)$ with the structure of a bimodule. We will find, however, that the bimodule structure on $\mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(V, W)$ is not just that of a left $\mathbb{K}$-module—if it were that simple, we wouldn't have needed to take this excursion on bimodules in the first place.

▪ Example D.1.1.12 — $\mathrm{Mor}_{R\text{-}\mathbf{Mod}}(V, W)$ is an S-T-bimodule
Let R, S, and T be rings, and let V be an R-S-bimodule and let W be an R-T-bimodule. Then, as V and W are both left R-modules, and so while we cannot speak of linear-transformations, we can speak of left linear-transformations. In fact, $\mathrm{Mor}_{R\text{-}\mathbf{Mod}}(V, W)$ can be given the structure of an S-T-bimodule:[a]

$$[s \cdot T \cdot t](v) := T(v \cdot s) \cdot t. \tag{D.1.1.13}$$

Exercise D.1.1.14 Check that $s \cdot T \cdot t$ is still left linear.

Exercise D.1.1.15 Check that $\mathrm{Mor}_{R\text{-}\mathbf{Mod}}(V, W)$ is an S-T-bimodule.

[a]Addition is defined pointwise. We don't mention this explicitly because addition is always pointwise and the sum of two linear-transformations is always again linear.

Example D.1.1.16 — $\mathrm{Mor}_{\mathbf{Mod}\text{-}S}(V,W)$ is a T-R-bimodule Let R, S, and T be rings, let V be an R-S-bimodule and let W be a T-S-bimodule.[a] Thus, in this case, we can speak of the *right* linear-transformations. In fact, $\mathrm{Mor}_{\text{-}\mathbf{Mod}\text{-}S}(V,W)$ can be given the structure of a T-R-bimodule:

$$[t\cdot T\cdot r](v) := t\cdot T(r\cdot v). \tag{D.1.1.17}$$

Exercise D.1.1.18 Check that $t\cdot T\cdot t$ is still right linear.

Exercise D.1.1.19 Check that $\mathrm{Mor}_{\mathbf{Mod}\text{-}S}(V,W)$ is an T-R-bimodule.

[a]Note that V is still an R-S-bimodule as in the previous example, but now W is a T-S-bimodule (before it was an R-T-bimodule).

While this might seem complicated, there is actually a relatively simple mnemonic to keep this straight. You can remember these respectively as

$$(R\text{-}S)^{\mathrm{co}}\times(R\text{-}T)\mapsto S\text{-}T \tag{D.1.1.20}$$

and

$$(R\text{-}S)^{\mathrm{co}}\times(T\text{-}S)\mapsto T\text{-}R. \tag{D.1.1.21}$$

(The common Rs and Ss 'annihilate' each other, similar to the mnemonic for the dimensions for multiplication of matrices.) That is, the morphism set[6] from a R-S-bimodule to a R-T-bimodule is

[6]Note how it only makes sense to speak of the left linear transformations, and so I there is no ambiguity as to what morphism set I could possibly be referring to.

and S-T-bimodule, and the morphism set from a R-S-bimodule to a T-S-bimodule is a T-R-bimodule.

D.2 Linear-ind., spanning, bases, and dimension

In the previous section, we gave two equivalent definitions of R-modules, and likewise gave the definition of a linear transformation. We now turn to the basic theory R-modules and linear transformations.

Definition D.2.1 — Linear-combination Let V be a $\mathbb{K}$-module and let $S \subseteq V$. Then, a ***linear-combination*** of elements of S is an element in V of the form

$$\sum_{v \in S} \alpha_v \cdot v \tag{D.2.2}$$

for $\alpha_v \in \mathbb{K}$.

R We will find that many concepts are most clearly illustrated in the finite case, and the definition of linear-combination is no exception: If $S = \{v_1, \ldots, v_m\}$, then a linear-combination of $v_1, \ldots, v_m$ is an element in V of the form

$$\alpha_1 \cdot v_1 + \cdots + \alpha_m \cdot v_m \tag{D.2.3}$$

for $\alpha_1, \ldots, \alpha_m \in \mathbb{K}$.

Definition D.2.4 — Linear-independence Let V be a $\mathbb{K}$-module and let $S \subseteq M$. Then, S is ***linearly-independent*** iff whenever

$$\sum_{v \in S} \alpha_v \cdot v = 0 \tag{D.2.5}$$

for $\alpha_v \in \mathbb{K}$, it follows that $\alpha_v = 0$ for all $v \in S$.

S is ***linearly-dependent*** iff it is not linearly-independent.

R In words, S is linearly-independent iff the only way to obtain 0 by taking linear-combinations of elements of S is with the trivial linear combination.

R If $S = \{v_1, \ldots, v_m\}$ is finite, this definition is equivalent to the statement that S is linearly-independent iff

$$\alpha_1 \cdot v_1 + \cdots + \alpha_m \cdot v_m \tag{D.2.6}$$

implies that $\alpha_1 = 0, \ldots, \alpha_m = 0$.

Furthermore, even if S is not necessarily finite, you may prefer to use the equivalent definition: "S is linearly-independent iff for every $m \in \mathbb{N}$ and $v_1, \ldots, v_m \in S$

$$\alpha_1 \cdot v_1 + \cdots + \alpha_m \cdot v_m = 0 \tag{D.2.7}$$

implies

$$\alpha_1 = 0, \ldots, \alpha_m = 0. \tag{D.2.8}$$

R Explicitly, S is linearly-dependent iff there are $\alpha_1, \ldots, \alpha_m \in \mathbb{K}$ *not all zero* and $v_1, \ldots, v_m \in S$ such that

$$\alpha_1 \cdot s_1 + \cdots + \alpha_m \cdot s_m = 0. \tag{D.2.9}$$

R Note that S is automatically linearly-dependent if $0 \in S$.[a]

[a] Why?

Proposition D.2.10 Let V be a $\mathbb{K}$-module, and let $S \subseteq V$. Then, S is linearly-independent iff for every $w \in \operatorname{Span}(S)$ there are unique $w^v \in \mathbb{K}$ such that

$$w = \sum_{v \in S} w^v \cdot v. \tag{D.2.11}$$

R The significant thing here is the *uniqueness*. We of course know automatically that there exist some coefficients for which this works from the definition of Span (Theorem D.2.19)—the linear-independence of S tells us that these coefficients are *unique*.

R If $S = \{v_1, \ldots, v_m\}$ is finite, then this is equivalent to the statement that for every $w \in \mathrm{Span}(S)$ there are unique $w^1, \ldots, w^m \in \mathbb{K}$ such that

$$w = w^1 \cdot v_1 + \cdots + w^m \cdot v_m. \tag{D.2.12}$$

Proof. ($\Rightarrow$) Suppose that S is linearly-independent. Let $w \in \mathrm{Span}(S)$. From Theorem D.2.19, the definition of Span, it follows that there are $w^v \in \mathbb{K}$ such that

$$w = \sum_{v \in S} w^v \cdot v. \tag{D.2.13}$$

We wish to show that this linear-combination is unique. So, suppose also that

$$w = \sum_{v \in S} \alpha^v \cdot v \tag{D.2.14}$$

for $\alpha^v \in \mathbb{K}$. Subtracting (D.2.14) from (D.2.13), we find

$$0 = \sum_{v \in S} (w^v - \alpha^v) \cdot v. \tag{D.2.15}$$

The definition of linear-independence then implies that $w^v - \alpha^v = 0$, that is, $w^v = \alpha^v$.

Suppose that for every $w \in \mathrm{Span}(S)$ there are unique $w^v \in \mathbb{K}$ such that

$$w = \sum_{v \in S} w^v \cdot v. \tag{D.2.16}$$

Suppose that

$$0 = \sum_{v \in S} \alpha_v \cdot v \tag{D.2.17}$$

for $\alpha_v \in \mathbb{K}$. As we also have

$$0 = \sum_{v \in S} 0 \cdot v, \tag{D.2.18}$$

by uniqueness, we have that $\alpha_v = 0$. Hence, by definition, S is linearly-independent. ∎

We're currently heading towards the definition of dimension, and hence of basis. You might recall that a basis is a linearly-independent *spanning* set, which brings us to the following result.

Theorem D.2.19 — Span. Let V be a $\mathbb{K}$-module and let $S \subseteq V$. Then, there is a unique subspace of V, the ***span*** of S, $\operatorname{Span}(S)$, such that

(i). $S \subseteq \operatorname{Span}(S)$; and
(ii). if $W \subseteq V$ is another subspace containing S, it follows that $\operatorname{Span}(S) \subseteq W$.

Furthermore, explicitly, $\operatorname{Span}(S)$ is the set of all linear-combinations of elements of S.

R Thus, if $S = \{v_1, \ldots, v_m\}$, we have that

$$\begin{aligned}\operatorname{Span}(v_1, \ldots, v_m) &:= \operatorname{Span}(S) \\ &= \{\alpha_1 \cdot v_1 + \cdots + \alpha_m \cdot v_m : \\ &\qquad \alpha_k \in \mathbb{K}\}\,.\end{aligned} \tag{D.2.20}$$

R In the context of R-modules, this is usually referred to as the subspace ***generated*** by S.

Proof. We leave this as an exercise.

Exercise D.2.21 Prove this yourself.

∎

Definition D.2.22 — Spanning Let V be a $\mathbb{K}$-module and let $S \subseteq V$. Then, S is ***spanning*** iff $\mathrm{Span}(S) = V$.

R Synonymously, we also say that S ***spans*** V.

R In other words, this means that every vector in V can be written as a linear combination of elements of S.

Definition D.2.23 — Basis Let V be a $\mathbb{K}$-module, and let $\mathcal{B} \subseteq V$. Then, $\mathcal{B}$ is a ***basis*** of V iff for every $v \in V$ there are unique $v^b \in \mathbb{K}$ such that

$$v = \sum_{b \in \mathcal{B}} v^b \cdot b. \tag{D.2.24}$$

R In words, $\mathcal{B}$ is a basis iff every vector can be written as a *unique* linear-combination of elements of $\mathcal{B}$.

Proposition D.2.25 Let V be a $\mathbb{K}$-module, and let $\mathcal{B} \subseteq V$. Then, $\mathcal{B}$ is a basis of V iff $\mathcal{B}$ is linearly-independent and spans V.

Proof. $(\Rightarrow)$ Suppose that $\mathcal{B}$ is a basis of V. By definition, every element of V is a linear-combination of elements of $\mathcal{B}$, and so $\mathcal{B}$ spans V. It is linearly-independent by Proposition D.2.10.

$(\Leftarrow)$ Suppose that $\mathcal{B}$ is linearly-independent and spans V. Let $v \in V$. As $\mathcal{B}$ spans V, there are $v^b \in \mathbb{K}$ such that

$$v = \sum_{b \in \mathcal{B}} v^b \cdot b. \tag{D.2.26}$$

As $\mathcal{B}$ is linearly-independent, by Proposition D.2.10, this is the unique such linear-combination, and so $\mathcal{B}$ is a basis by definition. ∎

If we start with a collection of linearly-independent vectors, adding more vectors to our collection runs of the risk of making our collection linearly-*dependent*. In the other direction, if we start with a collection of vectors which span the vector space, removing vectors from our collection runs the risk of obtaining a collection that fails to span. Likewise, adding vectors makes it easier to span and removing vectors makes it easier to be linearly-independent. A basis is an exact middle ground between spanning and linear-independence in the following sense.

Proposition D.2.27 Let V be a vector space and let $\mathcal{B} \subseteq V$. Then, the following are equivalent.

(i). $\mathcal{B}$ is a basis.
(ii). $\mathcal{B}$ is a maximal linearly-independent set.
(iii). $\mathcal{B}$ is a minimal spanning set.

R Intuitively, "maximal linearly-independent set" means that if you add any new vector at all to $\mathcal{B}$, the resulting set must fail to be linearly-independent. Similarly, "minimal spanning set" means that if you remove any vector at all from $\mathcal{B}$, the resulting set must fail to span. See Definition A.3.5.3 for the precise definitions of maximal and minimal.

R Warning: This fails over general R-modules. That said, (i) implies both (ii) and (iii) no matter what.

Proof. Denote the ground division ring by $\mathbb{F}$.

($(i) \Rightarrow (ii)$) Suppose that $\mathcal{B}$ is a basis. $\mathcal{B}$ is linearly-independent by Proposition D.2.25, so it only remains to show maximality. Let $C \subseteq V$ be linearly-independent and such that $C \supseteq \mathcal{B}$. We wish to show that $\mathcal{B} = C$, that is, that $C \subseteq \mathcal{B}$. So, let $c \in C$. If $c \in \mathcal{B}$, we're done, so suppose $c \notin \mathcal{B}$. As $\mathcal{B}$ spans by Proposition D.2.25 again, there are unique $c^b \in \mathbb{K}$ such that

$$c = \sum_{b \in \mathcal{B}} c^b \cdot b. \tag{D.2.28}$$

It follows that

$$\sum_{b \in \mathcal{B}} c^b \cdot b - 1 \cdot c = 0, \tag{D.2.29}$$

and so as each $b \in C$ and C is linearly-independent, it follows that $1 = 0$,[a] and so $\mathbb{F} = 0$. But then the only linear-combinations of elements of $\mathcal{B}$ is 0, and so $V = 0$, in which case we must have $\mathcal{B} = \emptyset = C$.

($(ii) \Rightarrow (i)$) $\mathcal{B}$ is a maximal linearly-independent set. It remains to show that $\mathcal{B}$ spans. So, let $v \in V$. If $v \in \mathcal{B}$, of course $v \in \operatorname{Span}(\mathcal{B})$. Otherwise, $\mathcal{B} \cup \{v\}$ is strictly larger than $\mathcal{B}$, and so as $\mathcal{B}$ is a maximal linearly-independent set, it must be the case that $\mathcal{B} \cup \{v\}$ is linearly-dependent, so that there are $\alpha^b, \alpha \in \mathbb{F}$, not all 0, such that

$$\sum_{b \in \mathcal{B}} \alpha^b \cdot b + \alpha \cdot v = 0. \tag{D.2.30}$$

If $\alpha = 0$, linear-independent of $\mathcal{B}$ implies that every $\alpha^b = 0$ as well, which is impossible (not all of these coefficients are 0). Therefore, $\alpha \neq 0$. Rearranging and multiplying by α^{-1} yields

$$v = -\alpha^{-1} \sum_{b \in \mathcal{B}} \alpha^b \cdot b \in \operatorname{Span}(\mathcal{B}). \tag{D.2.31}$$

($(i) \Rightarrow (iii)$) Suppose that $\mathcal{B}$ is a basis. Let $C \subseteq V$ be a spanning set such that $C \subseteq \mathcal{B}$. We wish to show that $\mathcal{B} \subseteq C$. So, let $b \in \mathcal{B}$. If $b \in C$, we're done, so suppose that $b \notin C$. As C is a spanning set, there are $b^c \in \mathbb{F}$ such that

$$b = \sum_{c \in C} b^c \cdot c. \tag{D.2.32}$$

It follows that

$$\sum_{c \in C} b^c \cdot c - b = 0, \tag{D.2.33}$$

and so, as each $c \in \mathcal{B}$ and $\mathcal{B}$ is linearly-independent, it follows in particular that $1 = 0$, so that $\mathbb{F} = 0$, and hence $V = 0$, and hence $\mathcal{B} = \emptyset = C$.

($(iii) \Rightarrow (i)$) Suppose that $\mathcal{B}$ is a minimal spanning set. We wish to show that $\mathcal{B}$ is linearly-independent. So, suppose that

$$0 = \sum_{b \in \mathcal{B}} \alpha_b \cdot b \tag{D.2.34}$$

for $\alpha_b \in \mathbb{K}$. If every $\alpha_b = 0$, we're done, so suppose this is not the case. Then, there is some $b_0 \in \mathcal{B}$ such that $\alpha_{b_0} \neq 0$. Then, we can rearrange this to find

$$b_0 = -\alpha_{b_0}^{-1} \sum_{\substack{b \in \mathcal{B} \\ b \neq b_0}} \alpha_b \cdot b \in \operatorname{Span}(\mathcal{B}). \tag{D.2.35}$$

It then follows that $\mathcal{B} \setminus \{b_0\}$ is again a spanning set, which contradicts minimality. ■

[a] This implicitly uses the fact that $c \notin \mathcal{B}$, for otherwise we might have, for example, $c = b_1$, in which case we would instead deduce that $c^1 = 1$.

We are now able to define the *dimension* of a vector space.

Theorem D.2.36 — Fundamental Theorem of Dimension. Let V be a vector space.

(i). V has a basis.

(ii). Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases of V. Then, $\mathcal{B}_1$ and $\mathcal{B}_2$ have the same cardinality.

R Warning: $\mathbb{K}$-modules in general need not have a basis.

R In fact, those $\mathbb{K}$-modules which do have a basis are called ***free modules***.[a] However, even when $\mathbb{K}$-modules do have bases, distinct bases need not have the same cardinality. On the other hand, if $\mathbb{K}$ is commutative, then distinct bases do have the same cardinality.

R "Fundamental Theorem of Dimension" is not a standard name for this result (it doesn't have a standard name), so don't expect other people to know what you're talking about if you decide to use it.

[a]Using this terminology, (i) becomes the statement that modules over division rings are free.

Proof. STEP 1: INTRODUCE NOTATION
Denote the ground division ring by $\mathbb{F}$. To simplify notation by getting rid of some subscripts, let us instead write $\mathcal{B} := \mathcal{B}_1$ and $C := \mathcal{B}_2$.

STEP 2: PROVE (I)
The idea of the proof is to show that V has a maximal linearly-independent set. This will be a basis by Proposition D.2.27. The reason this is the strategy is because we have a result that asserts the existence of maximal things—*Zorn's Lemma (Theorem A.3.5.9)*.

So, define

$$\mathscr{B} := \{S \subseteq V : S \text{ is linearly-independent.}\}\,. \tag{D.2.37}$$

We consider $\mathscr{B}$ as a partially-ordered set with respect to the relation of inclusion. As just explained, a maximal element of $\mathscr{B}$ will be a basis. Furthermore, Zorn's Lemma states that $\mathscr{B}$ will have a maximal element if every well-ordered subset of $\mathscr{B}$ has an upper-bound in $\mathscr{B}$, and so it suffices to show this.

So, let $\mathscr{S} \subseteq \mathscr{B}$ be a well-ordered subset of $\mathscr{B}$. Define

$$\mathcal{B} := \bigcup_{S\in\mathscr{S}} S \tag{D.2.38}$$

This certainly contains every element of $\mathscr{S}$, and so it only remains to check that $\mathcal{B} \in \mathscr{B}$, that is, we need to check that $\mathcal{B}$ is linearly-independent. So, let $b_1, \ldots, b_m \in \mathcal{B}$ and suppose that

$$\alpha_1 \cdot b_1 + \cdots + \alpha_m \cdot b_m = 0 \tag{D.2.39}$$

for $\alpha_k \in \mathbb{F}$. As $b_k \in \mathcal{B}$, there is some $S_k \in \mathscr{S}$ such that $b_k \in S_k$. As $\mathscr{S}$ is totally-ordered, some S_k contains all the others. Without loss of generality, suppose that $S_1 \supseteq S_1, \ldots, S_m$. It follows that $b_k \in S_1$ for all $1 \leq k \leq m$, and so as S_1 is linearly-independent, we find that each $\alpha_k = 0$, as desired.

STEP 3: PROVE THAT IF $\mathcal{B}$ IS FINITE, THEN C IS FINITE
Suppose that $\mathcal{B}$ is finite. Write $\mathcal{B} =: \{b_1, \ldots, b_d\}$. For each $b_k \in \mathcal{B}$, there is a nonempty finite subset $C_k \subseteq C$ such that $b_k \in \operatorname{Span}(C_k)$. Thus,

$$\{b_1, \ldots, b_d\} \subseteq \operatorname{Span}\left(\bigcup_{k=1}^{d} C_k\right), \tag{D.2.40}$$

and hence

$$\begin{aligned} V &= \operatorname{Span}(b_1, \ldots, b_d) \\ &\subseteq \operatorname{Span}\left(\bigcup_{k=1}^{d} C_k\right) \subseteq \operatorname{Span}(C) = V, \end{aligned} \tag{D.2.41}$$

and so all of these inclusions must be equalities. In particular,

$$\operatorname{Span}\left(\bigcup_{k=1}^{d} C_k\right) = V. \tag{D.2.42}$$

However, as C is a basis, it is a minimal spanning set, and so

$$C = \bigcup_{k=1}^{d} C_k, \tag{D.2.43}$$

and so C is finite.

STEP 4: PROVE (II) IN THE CASE ONE IS FINITE
Without loss of generality, suppose that $\mathcal{B}$ is finite. By the previous step, C must also be finite. Write $\mathcal{B} = \{b_1, \ldots, b_d\}$ and $C = \{c_1, \ldots, c_e\}$. We wish of course to show that $d = e$. Without loss of generality, suppose that $d \leq e$.

$b_1 \in \operatorname{Span}(C)$, so we may write

$$b_1 = {b^1}_1 \cdot c_1 + \cdots + {b^e}_1 \cdot c_e \tag{D.2.44}$$

for ${b^k}_1 \in \mathbb{F}$. Not all of these coefficients can be 0, so without loss of generality, suppose that ${b^1}_1 \neq 0$. Then, rearranging, we see that $c_1 \in \operatorname{Span}(b_1, c_2, \ldots, c_e)$, and hence

$$V = \operatorname{Span}(c_1, \ldots, c_e) = \operatorname{Span}(b_1, c_2, \ldots, c_e). \tag{D.2.45}$$

Exercise D.2.46 Verify that $\{b_1, c_2, \ldots, c_e\}$ is still linearly-independent.

We thus have a new basis $\{b_1, c_2, \ldots, c_e\}$ with exactly e elements. In particular, $b_2 \in \operatorname{Span}(b_1, c_2, \ldots, c_e)$, and so we can write

$$b_2 = {a^1}_2 \cdot b_1 + {b^2}_2 \cdot c_2 + \cdots + {b^e}_2 \cdot c_e. \tag{D.2.47}$$

We can't have all of the ${b^k}_2 = 0$, for then $\{b_1, b_2\}$ would be linearly-dependent. Thus, again, without loss of generality, ${b^2}_2 \neq 0$, and so $c_2 \in \operatorname{Span}(b_1, b_2, c_3, \ldots, c_e)$. Proceeding as before, we find that $V = \operatorname{Span}(b_1, b_2, c_3, \ldots, c_e)$.

Proceeding inductively, we may replace the c_ks with the corresponding b_ks to obtain bases, and eventually we find that

$$\{b_1, \ldots, b_d, c_{d+1}, \ldots, c_e\} \tag{D.2.48}$$

is a basis, and in particular, is a linearly-independent set that contains $\{b_1, \ldots, b_d\}$. As $\{b_1, \ldots, b_d\}$ is maximal linearly-independent, it follows that

$$\{b_1, \ldots, b_d\} = \{b_1, \ldots, b_d, c_{d+1}, \ldots, c_e\}, \qquad \text{(D.2.49)}$$

and hence $d = e$, as desired.

STEP 5: PROVE (II) IN THE CASE BOTH ARE INFINITE
Now suppose that the cardinalities of both $\mathcal{B}$ and $\mathcal{C}$ are infinite. We wish to show that $|\mathcal{B}| \leq |\mathcal{C}|$. If we can show this, then by $\mathcal{B} \leftrightarrow \mathcal{C}$ symmetry, the same argument can be used to show that $|\mathcal{C}| \leq |\mathcal{B}|$, whence we will have $|\mathcal{B}| = |\mathcal{C}|$ by the Bernstein-Cantor-Schröder Theorem (Theorem 1.1.3.5), as desired.

So, we would like to show that $|\mathcal{B}| \leq |\mathcal{C}|$. Let $c \in \mathcal{C}$. As $\mathcal{B}$ spans V, there are $b_1, \ldots, b_m \in \mathcal{B}$ and $c^1, \ldots, c^m \in \mathbb{F}$ nonzero such that

$$c = c^1 \cdot b_1 + \cdots + c^m \cdot b_m. \qquad \text{(D.2.50)}$$

Define $\mathcal{B}_c \coloneqq \{b_1, \ldots, b_m\}$, and for $F \subseteq \mathcal{B}$ finite, define

$$\mathcal{C}_F \coloneqq \{c \in \mathcal{C} : \mathcal{B}_c = F\}. \qquad \text{(D.2.51)}$$

We have that

$$\mathcal{C} = \bigcup_{\substack{F \subseteq \mathcal{B} \\ F \text{ finite}}} \mathcal{C}_F. \qquad \text{(D.2.52)}$$

By Proposition 1.1.1.5, the cardinality of the collection of finite subsets of $\mathcal{B}$ is just $|\mathcal{B}|$, and so the above is a union of $|\mathcal{B}|$ many nonempty finite sets, and hence $|\mathcal{C}| \leq |\mathcal{B}|$ by Proposition 1.1.1.8. ■

Definition D.2.53 — Dimension Let V be a vector space. Then, the *dimension* of V, $\dim(V)$, is the cardinality of a basis of V.

R Of course, this makes sense and is well-defined by Theorem D.2.36. Indeed, in some sense, that was the entire point of this theorem.

R Sometimes we will want to consider the same set of vectors as a vector space over different division rings. In such a case, we will write $\dim_{\mathbb{F}}(V)$ to clarify what ground division ring we mean to work over. For example, we have $\dim_{\mathbb{C}}(\mathbb{C}) = 1$ but $\dim_{\mathbb{R}}(\mathbb{C}) = 2$.

Proposition D.2.54 Let V and W be finite-dimensional vector spaces with $\dim(V) = \dim(W)$ and let $T \colon V \to W$ be linear. Then, the following are equivalent.

(i). T is injective.
(ii). T is surjective.
(iii). T is bijective.

Proof. We leave this as an exercise.

Exercise D.2.55 Prove the result.

■

D.3 The tensor product and dual space

You do not need to understand the precise definition of a tensor product to use tensors to the extent necessary in these notes. If this bothers you, think how you (probably) knew how to use integrals quite awhile before you knew how to define the integral. Similarly, here, one does not need to know the technical details[7] to work with tensors. In particular:

> Do not worry if nothing in this section makes sense. You only need to understand tensors insofar as they are used in the differentiation chapter.

Anyways, let's begin our definition-theorem-proof extravaganza.

D.3.1 The dual space

Definition D.3.1.1 — Dual-space Let V be a $\mathbb{K}$-$\mathbb{K}$-bimodule. Then, the ***dual-space*** of V is the $\mathbb{K}$-$\mathbb{K}$-bimodule

$$V^{\dagger} := \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(V, \mathbb{K}). \tag{D.3.1.2}$$

R We also say simply that $V^{\dagger}$ is the ***dual*** of V.

R Elements of $V^{\dagger}$ are ***covectors*** or ***linear-functionals***. The terms are synonymous, though "covector" is more commonly used in the context of tensors while "linear-functional" tends to be used elsewhere.

R In other words, the elements of $V^{\dagger}$ take in elements of V and spit out scalars. Recall that the motivating example of this was the derivative: this takes in a vector (the direction in which to differentiate) and spits out a number (the directional derivative in that direction).

R Recall that (Appendix D.1.1) $\mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(V, W)$ does not have the structure of a module if V and W are just modules. If we want morphism sets to have some sort of nontrivial module structure, we need to work with *bimodules* from the get-go. Requiring that V be a $\mathbb{K}$-$\mathbb{K}$-bimodule ensures that $\mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(V, \mathbb{K})$ is

[7] Of course, they're not really technical details—it's an absolutely fundamental definition! Only for our purposes is this content to be regarded as "technical details".

a $\mathbb{K}$-$\mathbb{K}$-bimodule as well, so that V and $V^\dagger$ are the same type of object.[a]

That said, recall that (Example D.1.1.7) modules over commutative rings have a canonical bimodule structure, and so if we're working over commutative rings, we can get away with just saying "$\mathbb{K}$-module" everywhere.

R The "†" is for "transpose"—we'll see why later. This is uncommon notation. More common notation includes V^* and V'. The former I choose not to use as I reserve this notation for the *conjugate*-dual, and the latter, well, V' just looks weird to me. There's also the fact that "†" kind of just looks like a "t".[b]

R If V comes with a topology, you're only going to want to look at the *continuous* linear functionals. Of course, you can look at all of them (including the discontinuous ones), but this is probably not going to be as useful.

R You might say that this is the "left dual-space" and that $\mathrm{Mor}_{\mathbf{Mod}\text{-}\mathbb{K}}(V, \mathbb{K})$ would be the "right dual-space". Just as we only worked with left modules by default (even though there was a corresponding notion of right module), we won't every worry about the right dual-space. Besides, if $\mathbb{K}$ is commutative, there isn't going to be any difference anyways.

[a] Note that $\mathbb{K}$ is a $\mathbb{K}$-$\mathbb{K}$-bimodule (Example D.1.1.6), and hence $V^\dagger := \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(V, \mathbb{K})$ is a $\mathbb{K}$-$\mathbb{K}$-bimodule by Example D.1.1.12.

[b] In English, "†" is actually read as "dagger" (the TEX command for this is "`\dagger`, though I imagine the English usage came first).

There is an analogous notion for linear-transformations, the *transpose*.

Definition D.3.1.3 — Transpose Let V_1 and V_2 be $\mathbb{K}$-$\mathbb{K}$-bimodules and let $T\colon V_1 \to V_2$. Then, the ***transpose*** of

$T, T^\dagger : V_2^\dagger \to V_1^\dagger$, is defined by

$$(T^\dagger(w_2) \mid v_1) := (w_2 \mid T(v_1)) \tag{D.3.1.4}$$

for $w_2 \in V_2^\dagger$ and $v_1 \in V_1$.

R $T^\dagger$ is also called the ***dual-map*** or just ***dual*** of T.

Proposition D.3.1.5 — The dual basis Let V be a $\mathbb{K}$-$\mathbb{K}$-bimodule, let $\mathcal{B}$ be a basis of V, for $b \in \mathcal{B}$ let $b^\dagger : V \to \mathbb{K}$ be the unique linear map such that

$$b^\dagger(c) = \begin{cases} 1 & \text{if } c = b \\ 0 & \text{otherwise,} \end{cases} \tag{D.3.1.6}$$

and define $\mathcal{B}^\dagger := \{b^\dagger : b \in \mathcal{B}\}$.

(i). If V is a vector space, then $\mathcal{B}^\dagger \subseteq V^\dagger$ is linearly-independent.

(ii). If V is a finite-dimensional vector space, then $\mathcal{B}^\dagger \subseteq V^\dagger$ is a basis.

R If $\mathcal{B}^\dagger$ is actually a basis of $V^\dagger$, then it is referred to as the ***dual basis*** of $\mathcal{B}$.

Proof. We leave this as an exercise.

Exercise D.3.1.7 Prove the result.

■

One reason the term "dual" is used is because, while on one hand, we can obviously view elements of $V^\dagger$ as linear-functionals on V, we can "dually" view elements of V as linear-functionals on $V^\dagger$.

Theorem D.3.1.8 — Duality of the dual. Let V be a $\mathbb{K}$-$\mathbb{K}$-bimodule. Then, the map $V \to [V^\dagger]^\dagger$ defined by

$$v \mapsto (\phi \mapsto \phi(v)) \tag{D.3.1.9}$$

is linear and natural.

Furthermore,

(i). if V is a vector space, this map is injective; and
(ii). if V is a finite-dimensional vector space, this map is an isomorphism.

R See Appendix B.3.2 for a discussion of what is meant here by the term "natural". You should note, however, that it's not particularly important and could require a relatively large effort to fully understand, especially if you've never seen anything like this before. Thus, you might consider just pretending the word "natural" didn't appear anywhere in the statement above—you won't be missing out on all *that* much.

R For $v \in V$ and $\phi \in V^\dagger$, we write

$$(\phi \,|\, v) := \phi(v). \tag{D.3.1.10}$$

Using this notation, the map $V \to [V^\dagger]^\dagger$ can be written as

$$v \mapsto (\cdot \,|\, v), \tag{D.3.1.11}$$

where $(\cdot \,|\, v)$ is the linear-functional on $V^\dagger$ that sends $\phi \in V^\dagger$ to $(\phi \,|\, v) := \phi(v) \in \mathbb{K}$.

This notation is used suggestively when we want to think of V and $V^\dagger$ as 'on the same footing'—on one hand, we can view elements of $V^\dagger$ as linear-functionals on V (e.g. for $\phi \in V^\dagger$, $(\phi \,|\, \cdot)$ is a linear-functional on V), but on the other hand we can also view elements of V as linear-functional on $V^\dagger$ (e.g. for $v \in V$, $(\cdot \,|\, v)$ is a linear-functional on $V^\dagger$). Thinking of things in terms of this "duality" is particularly appropriate when V is a finite-dimensional vector space, so that, up to natural isomorphism, V is 'the same as' $[V^\dagger]^\dagger$.

Warning: (i) need not hold if V is not a vector space and (ii) need not hold if V is a vector space but not finite-dimensional.

Proof. We first check that it is linear. Let $v, w \in V$ and let $\alpha, \beta \in \mathbb{K}$. We wish to show that

$$(\cdot \,|\, \alpha v + \beta w) = \alpha(\cdot \,|\, v) + \beta(\cdot \,|\, w). \tag{D.3.1.12}$$

However, this will be the case iff for all $\phi \in V^\dagger$ we have

$$(\phi \,|\, \alpha v + \beta w) = \alpha(\phi \,|\, v) + \beta(\phi \,|\, w). \tag{D.3.1.13}$$

However, by definition of the notation $(\cdot \,|\, \cdot)$, this equation is the same as

$$\phi(\alpha v + \beta w) = \alpha\phi(v) + \beta\phi(w), \tag{D.3.1.14}$$

which is of course true as ϕ is linear.

We now check that it is natural.[a] Let $T\colon V \to W$ be a linear-transformation between $\mathbb{K}$-modules. By definition (Definition B.3.2.3), $V \mapsto [V^\dagger]^\dagger$ is natural iff the following diagram commutes.

$$\begin{tikzcd} V \arrow[r] \arrow[d, "T"'] & {[V^\dagger]^\dagger} \arrow[d, "{[T^\dagger]^\dagger}"] \\ W \arrow[r] & {[W^\dagger]^\dagger} \end{tikzcd} \tag{D.3.1.15}$$

By definition, this means that we want to show that

$$(\psi \,|\, [[T^\dagger]^\dagger]((\cdot \,|\, v))) = (\psi \,|\, T(v)) \tag{D.3.1.16}$$

for al $v \in V$ and $\psi \in W^\dagger$. However, by definition of the transpose and $(\cdot \,|\, v)$,

$$\begin{aligned}(\psi \,|\, [[T^\dagger]^\dagger]((\cdot \,|\, v))) &:= ([T^\dagger](\psi) \,|\, (\cdot \,|\, v)) \\ &:= ([T^\dagger](\psi) \,|\, v) := (\psi \,|\, T(v)),\end{aligned} \tag{D.3.1.17}$$

as desired.

(i) Suppose that V is a vector space. To show that it is injective, we check that the kernel is 0. So, let $v \in V$ suppose that $(\phi \mid v) = 0$ for all $\phi \in V^\dagger$. If $\mathbb{K} = 0$, then $V = 0$, and so we are immediately done. Otherwise, if $v \neq 0$, then there is a linear-functional $\phi \colon V \to \mathbb{K}$ that sends v to 1, in which case $(\phi \mid v) = 1 \neq 0$, a contradiction. Thus, it must be the case that $v = 0$.

(ii) Suppose that V is a finite-dimensional vector space. We know from the defining result of the dual basis (Proposition D.3.1.5) that $\dim(V) = \dim(V^\dagger) = \dim([V^\dagger]^\dagger)$. Thus, we have a injective linear map $V \to [V^\dagger]^\dagger$ between two finite-dimensional vector spaces of the same dimension, and hence it must in fact be an isomorphism (Proposition D.2.54). ■

[a] This part of the proof makes uses of things we have not yet encountered. You can verify it is not circular as these new things don't make use of this result. (It doesn't make sense to move this result so drastically just to avoid this small worry about potential circularity). It makes use of the concept of a *commutative diagram*, which is explained in a remark of Theorem D.3.2.6.

D.3.2 The tensor product

Definition D.3.2.1 — Multilinear-transformation
Let $V_1, \ldots, V_m$ be respectively $\mathbb{K}_k$-$\mathbb{K}_{k+1}$-bimodules, let V be a $\mathbb{K}_1$-$\mathbb{K}_{m+1}$-bimodule, and let $T \colon V_1 \times \cdots \times V_m \to V$ be a function. Then, T is ***multilinear*** iff

(i).

$$V_k \in v \mapsto T(v_1, \ldots, v_{k-1}, v, v_{k+1}, \ldots, v_m) \in V$$

is a group homomorphism for all $1 \leq k \leq m$;

(ii).

$$T(\alpha \cdot v_1, v_2, \ldots, v_{m-1}, v_m \cdot \beta) = \alpha \cdot T(v_1, v_2 \ldots, v_{m-1}, v_m) \cdot \beta$$

for $\alpha \in \mathbb{K}_1$ and $\beta \in \mathbb{K}_{m+1}$; and

(iii).

$$\begin{aligned} & T(v_1, \ldots, v_k \cdot \alpha, v_{k+1}, \ldots, v_m) \\ & \quad = T(v_1, \ldots, v_k, \alpha \cdot v_{k+1}, \ldots, v_m) \end{aligned} \tag{D.3.2.2}$$

for $\alpha \in \mathbb{K}_k$.

R If $m = 2$, the term ***bilinear*** is more commonly used instead of "multilinear-transformation". While I can't say I've heard the term before, it would stand to reason that ***trilinear*** would be used for the $m = 3$ case, etc..

R In essence, this means that (i) each argument preserves addition and (ii) you can move scalars around as you please just so long as you don't move scalars past vectors.

If you find this confusing, I wouldn't worry. Our interest is primarily in the commutative case, in which case these conditions simplify to something more understandable—see Proposition D.3.2.3.

R At this level of generality, you might hear this concept being referred to as ***balanced***, in which case "multilinear-linear transformation" would only be used in the commutative case where the condition simplifies—see Proposition D.3.2.3.

Proposition D.3.2.3 Let $V_1, \ldots, V_m, V$ be $\mathbb{K}$-modules, $\mathbb{K}$ a cring, and let $T\colon V_1 \times \cdots \times V_m \to V$ be a function. Then, T is multilinear iff

$$V_k \ni v \mapsto T(v_1, \ldots, v_{k-1}, v, v_{k+1}, \ldots, v_m) \in V \tag{D.3.2.4}$$

is linear for all $1 \leq k \leq m$.

R In other words, over commutative rings, multilinear is equivalent to being linear in every argument.

Proof. We leave this as an exercise.

Exercise D.3.2.5 Prove the result.

■

Theorem D.3.2.6 — Tensor product (of bimodules). Let V be an R-S-bimodule and let W be an S-T-bimodule. Then, there is a unique bilinear map $- \otimes -\colon V \times W \to V \otimes_S W$ into the R-T-bimodule $V \otimes_S W$, the ***tensor product*** of V and W over S, such that if $V \times W \to U$ is any other bilinear map into an R-T-bimodule U, then there is a unique map of bimodules $V \otimes_S W \to U$ such that the following diagram commutes.

$$\begin{tikzcd} V \times W \arrow[r, "- \otimes -"] \arrow[dr] & V \otimes_S W \arrow[d, dashed] \\ & U \end{tikzcd} \tag{D.3.2.7}$$

R For $v \in V$ and $w \in W$, the image under the bilinear map $V \times W \to V \otimes_S W$ is written $v \otimes w \in V \otimes_S W$ and is the ***tensor product*** of v and w.

R There is an analogous result for not just bilinear maps, but all types of multilinear maps. Specifically, if V_k is a $\mathbb{K}_k$-$\mathbb{K}_{k+1}$-bimodules, then we have a multilinear map $V_1 \times \cdots \times V_m \to V_1 \otimes_{\mathbb{K}_1} \cdots \otimes_{\mathbb{K}_{m-1}} V_m$ into a $\mathbb{K}_1$-$\mathbb{K}_m$-bimodule that is "universal" in a sense exactly analogous to (D.3.2.7).

Additionally, the empty tensor product over $\mathbb{K}$, that is, the tensor product of no spaces, is defined to be $\mathbb{K}$ itself. In symbols:

$$\underbrace{V \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V}_{0} := \mathbb{K}. \tag{D.3.2.8}$$

R To clarify, there are tensor products of *bimodules*, and then there are tensor products of *vectors themselves*. The tensor product of two vectors 'lives in' the tensor product of the corresponding bimodules. And in fact, *everything* in $V \otimes_S W$, while *not* of the form $v \otimes w$ itself necessarily, can be written as a finite sum of elements of this form—see Proposition D.3.2.16. (Elements of the form $v \otimes w$ are sometimes called ***pure*** or ***simple***, as opposed to, e.g. , $v_1 \otimes w_1 + v_2 \otimes w_2$).

R If S is clear from context, it may be omitted: $V \otimes W := V \otimes_S W$.

R For the purpose of (hopefully) increasing clarity, we are actually being sloppy here. When we say that $V \otimes_S W$ is "unique", what we actually mean is that $V \otimes_S W$ is "unique up to unique isomorphism" in the sense that, if $V \times W \to U$ is some other bilinear map into an R-T-bimodule U that satisfies this property, then there is a unique isomorphism (of R-T-bimodules) $V \otimes_S W \to U$ such that the following diagram commutes.

$$\begin{tikzcd} V \times W \arrow[r] \arrow[dr] & V \otimes_S W \arrow[d, dashed] \\ & U \end{tikzcd} \tag{D.3.2.9}$$

Thus, while people do often say "unique up to *unique* isomorphism", the isomorphism is itself not unique—it would be more accurate to say "unique up to unique isomorphisms which commute with blah blah blah diagram". That is to say, while there might be many isomorphisms between $V \otimes_S W$ and U, there is only one which makes the above diagram commute. Given that the latter option, while more accurate, is incredibly verbose, people just stick to "unique up to unique isomorphism".

R A common question I've gotten from students is "But what actually *is* $v \otimes w$?". I'm afraid there's not a terribly good answer for that. It is what it is: the image of $\langle v, w \rangle \in V \times W$ under the canonical bilinear map $V \times W \to V \otimes_S W$. As that's probably not very satisfying, I ask you to consider the following.

What if I asked you "But what actually *is* $\sqrt{2} \cdot \pi$?". The answer is that it is what it is: it's the product of $\sqrt{2}$ and π. You can't really reduce it to something simpler without going so far out of your way so as to not be worth it. For example, what are you going to do? Try to argue that the number $\sqrt{2} \cdot \pi$ is π added to itself $\sqrt{2}$ times? Good luck with that.

Anyways, I'm sure you probably don't feel very uncomfortable talking about "$\sqrt{2} \cdot \pi$", and my claim is that if you feel comfortable working with this, then you should feel comfortable working with $v \otimes w$.

That said, in special cases, one can be a bit more explicit—see the blue box in the remark of Definition D.3.2.41. I personally don't find this perspective particularly useful, but I have found that some students to.

R Thus you can take the tensor product of an R-S-bimodule and an S-T-bimodule, the result being an R-T-bimodule. To remember this, you might note that this is exactly analogous to matrix multiplication: the 'inner' things have to be the same in which case the result has the structure coming from the 'outer' things.

This was one motivation for working with bimodules.[a] If I were working just with vector spaces, then you could take the tensor product of any two things you like, but in this context, you can only take the tensor product of an $R\times S$-bimodule and an S-T-bimodule with the result being an R-T-bimodule—this makes it clearer what roles everything is playing.

R *This is important—do not ignore.* Essentially what this result says is that, instead of working with *bilinear* maps $V \times W \to U$, instead, we can work with *linear* maps $V \otimes_S W \to U$. You'll find in time that this 'trade-off' is worth it.

In practice, this is often used in the following way. Suppose you want to define a function $T \colon V \otimes_S W \to$

U. The definition of the tensor product says that *you only need to say where elements of the form $v \otimes w$ map to*. In practice, you will say something like "Let $T(v \otimes w) := $ blah blah blah. . . ", and while superficially it doesn't look like you're defining T on all of $V \otimes_S W$ (because you're not), this is enough. As long as your "blah blah blah" is bilinear in $\langle v, w\rangle \in V \times W$, the definition of the tensor product says that this corresponds to a unique linear map $V \otimes_S W \to U$. Thus, you can define a linear-transformation on all of $V \otimes_S W$ by only specifying what happens to elements of the form $v \otimes w$.

TL;DR:

> To define linear maps $V \otimes_S W \to U$, it suffices to say where elements of the form $v \otimes w \in V \otimes_S W$ get mapped to. As long as what you write down is bilinear in $\langle v, w\rangle$, the definition of the tensor product says that this serves to define a unique linear map on all of $V \otimes_S W$.

[a] The other big motivation is that you will need to learn tensor products in this level of generality at some point in your mathematical life, so may as well learn it now.

Proof. We leave this as an exercise.

Exercise D.3.2.10 Prove the result.

■

This result says that if I every have a bilinear map $V \times W \to U$, I can 'replace' it with a linear map $V \otimes_S W \to U$. In this sense, the tensor product reduces the study of multilinear maps to linear maps.

Another way of thinking about $V \otimes W$ that might help your intuition is in terms of bases. Though this is only true for vector spaces,[8] it essentially says the following: if you have a bunch of b_ks that are a

[8] Duh. We don't have bases for general modules.

basis for V and a bunch of c_ls that are a basis for W, then the collection of all $b_k \otimes c_l$s forms a basis for $V \otimes W$. Thus, the elements of $V \otimes W$ are precisely those things that can be written (uniquely) as a linear combinations of $b_k \otimes c_l$s.

Proposition D.3.2.11 — Basis for $V \otimes W$ Let V and W be vector spaces over a field $\mathbb{F}$, and let $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W respectively. Then,

$$\{b \otimes c : b \in \mathcal{B}, c \in \mathcal{C}\} \tag{D.3.2.12}$$

is a basis for $V \otimes W$.

R In particular, $\dim(V \otimes W) = \dim(V)\dim(W)$.

R We see immediately from working in the level of generality that we did that the ground division ring need be commutative, that is, a field. If it weren't, then V would be just an $\mathbb{F}$-$\mathbb{Z}$-bimodule and W would be a $\mathbb{F}$-$\mathbb{Z}$-bimodule, in which case we could not take their tensor product!

R This result can probably be generalized to the noncommutative case, but then we will need to take V to be a $\mathbb{K}_1$-$\mathbb{L}$-bimodule and W to be an $\mathbb{L}$-$\mathbb{K}_2$-bimodule, with $\mathbb{K}_1$ and $\mathbb{K}_2$ division rings. Furthermore, the statement would require us to have a notion of "basis" for bimodules. It's easy enough to write one down, but as we have not done so, we refrain from 'officially' giving this noncommutative version.

Proof. We leave this as an exercise.

Exercise D.3.2.13 Prove the result.

∎

Corollary D.3.2.14 Let V and W be vector spaces over a field, and let $v \in V$ and $w \in W$. Then, if $v \otimes w = 0$, then $v = 0$ or $w = 0$.

R Warning: This may fail for general $\mathbb{K}$-modules.

R Just as the previous result should generalize to the noncommutative case, so to should this one.

Proof. We leave this as an exercise.

Exercise D.3.2.15 Prove the result.

■

The following result is important in that it will allow us to change our perspective on things, that is, we can think of something as a map $U \otimes V \to W$, or alternatively we can think of it as a map $U \to V^{\dagger} \otimes W$.

Proposition D.3.2.16 Let R, S, and T be rings, let V be an R-S-bimodule and let W be an S-T-bimodule. Then,

$$V \otimes_S W = \operatorname{Span}\{v \otimes w : v \in V, w \in W\}. \qquad \text{(D.3.2.17)}$$

Proof. We leave this as an exercise.

Exercise D.3.2.18 Prove the result.

■

What follows is one of the most important properties of the tensor product.

Theorem D.3.2.19 — Tensor-Hom Adjunction. Let R, S, and T be rings, and let U be an R-S bimodule, V and S-T bimodule, and W an R-T bimodule.

(i). The map

$$\begin{aligned}\mathrm{Mor}_{R\text{-}\mathbf{Mod}\text{-}T}(U \otimes_S V, W) \leftarrow \\ \mathrm{Mor}_{S\text{-}\mathbf{Mod}\text{-}T}(V, \mathrm{Mor}_{R\text{-}\mathbf{Mod}}(U, W))\end{aligned} \tag{D.3.2.20}$$

defined by

$$(u \otimes v \mapsto [\phi(v)](u)) \leftarrow\!\shortmid \phi \tag{D.3.2.21}$$

is an isomorphism of commutative groups.

(ii). The map

$$\begin{aligned}\mathrm{Mor}_{R\text{-}\mathbf{Mod}\text{-}T}(U \otimes_S V, W) \leftarrow \\ \mathrm{Mor}_{R\text{-}\mathbf{Mod}\text{-}S}(U, \mathrm{Mor}_{\mathbf{Mod}\text{-}T}(V, W))\end{aligned} \tag{D.3.2.22}$$

defined by

$$(u \otimes v \mapsto [\phi(u)](v)) \leftarrow\!\shortmid \phi \tag{D.3.2.23}$$

is an isomorphism of commutative groups.

R The R, S, and Ts everywhere clutter things up. Dropping all of the notational baggage, these become the more readable

$$\mathrm{Mor}(U \otimes V, W) \cong \mathrm{Mor}(V, \mathrm{Mor}(U, W))$$
$$\mathrm{Mor}(U \otimes V, W) \cong \mathrm{Mor}(U, \mathrm{Mor}(V, W)).$$

R To understand this, it might first help to understand an analogous result in a different category: the map defined analogously as above yields an isomorphism

$$\mathrm{Mor}_{\mathbf{Set}}(X \times Y, Z) \to \mathrm{Mor}_{\mathbf{Set}}(X, \mathrm{Mor}_{\mathbf{Set}}(Y, Z)).$$

In other words, functions from $X \times Y$ into Z are 'the same as' functions from X into $\mathrm{Mor}_{\mathbf{Set}}(Y, Z)$; given a function of two variables, we can instead think of it as a function-valued function $f \mapsto (x \mapsto (y \mapsto f(x, y)))$. In computer science, this concept is called *currying*. Thus, you could say that this result is just the linear algebraic analogue of currying.

R The "Hom" in "Tensor-Hom Adjunction" comes from the fact that "Mor" is often written as "Hom".

R Though you (probably) don't know what the term means yet, it turns out that this (by which I mean (i)) actually yields what is called an adjunction[a] between the functors $U \otimes_S -: S\text{-}\mathbf{Mod}\text{-}T \to R\text{-}\mathbf{Mod}\text{-}T$ and $\mathrm{Mor}_{R\text{-}\mathbf{Mod}}(U,-): R\text{-}\mathbf{Mod}\text{-}T \to S\text{-}\mathbf{Mod}\text{-}T$, hence "Tensor-Hom Adjunction". In this case, we say that $U \otimes_S -$ is *left adjoint* to $\mathrm{Mor}_{R\text{-}\mathbf{Mod}}(U,-)$, and the other way around, that $\mathrm{Mor}_{R\text{-}\mathbf{Mod}}(U,-)$ is *right adjoint* to $U \otimes_S -$. Thus, as the tensor product is the *left* adjoint and the "Hom" is the *right* adjoint, I recommend you say "tensor-hom adjunction" and *not* "hom-tensor adjunction".

Dually, (ii) yields an adjunction between the functors $- \otimes_S V$ and $\mathrm{Mor}_{\mathbf{Mod}\text{-}T}(V,-)$.

[a]This means that not only is (D.3.2.21) an isomorphism, but it defines an isomorphism that is natural (Definition B.3.2.3) in both V and W.

Proof. We prove (i). The proof of (ii) is essentially identical.

Given $f: U \otimes_S V \to W$ a map of R-T-bimodules, define $g_f: V \to \mathrm{Mor}_{R\text{-}\mathbf{Mod}}(U,W)$ by

$$[g_f(v)](u) := f(u \otimes v). \tag{D.3.2.24}$$

First of all, note that $g_f(v) \in \mathrm{Mor}_{R\text{-}\mathbf{Mod}}(U,W)$ as f is linear and the tensor product is bilinear.

To show that $g_f: V \to \mathrm{Mor}_{R\text{-}\mathbf{Mod}}(U,W)$ is a map of S-T-bimodules, we must show that

$$[g_f(s \cdot v \cdot t)](u) = [s \cdot [g_f(v)] \cdot t](u) \tag{D.3.2.25}$$

for all $u \in U$. However, recall from Example D.1.1.12 that the S-T-bimodule action on $\mathrm{Mor}_{R\text{-}\mathbf{Mod}}(U,W)$ is given by

$$[s \cdot T \cdot t](u) := T(u \cdot s) \cdot t, \tag{D.3.2.26}$$

and hence what we would actually like to show is that

$$[g_f(s\cdot v\cdot t)](u)=[g_f(v)](u\cdot s)\cdot t. \tag{D.3.2.27}$$

From the definition of g_f, this means we would like to show that

$$f\left(u\otimes(s\cdot v\cdot t)\right)=f\left((u\cdot s)\otimes v\right)\cdot t. \tag{D.3.2.28}$$

This is of course true because f is linear and because of properties of the tensor product.

Finally, to check that $f\mapsto g_f$ is a group homomorphism

$$\mathrm{Mor}_{R\text{-}\mathbf{Mod}\text{-}T}(U\otimes_S V,W)\to\mathrm{Mor}_{S\text{-}\mathbf{Mod}\text{-}T}(V,\mathrm{Mor}_{R\text{-}\mathbf{Mod}}(U,W)),$$

we must show that $g_{f_1+f_2}=g_{f_1}+g_{f_2}$. In other words, we must show that

$$\begin{aligned}[][f_1+f_2](u\otimes v)&=[g_{f_1+f_2}(v)](u)\\ &=[[g_{f_1}+g_{f_2}](v)](u)\\ &=f_1(u\otimes v)+f_2(u\otimes v)\end{aligned} \tag{D.3.2.29}$$

for all $u\in U$ and $v\in U$. This is of course true because of the definition of addition of functions.

To show that $f\mapsto g_f$ is an isomorphism, we construct an inverse $g\mapsto f_g$ from $\mathrm{Mor}_{S\text{-}\mathbf{Mod}\text{-}T}(V,\mathrm{Mor}_{R\text{-}\mathbf{Mod}}(U,W))$ to $\mathrm{Mor}_{R\text{-}\mathbf{Mod}\text{-}T}(U\otimes_S V,W)$. So, let $g\colon V\to\mathrm{Mor}_{R\text{-}\mathbf{Mod}}(U,W)$ be a map of S-T-bimodules and define $f_g\colon\ \mathrm{Mor}_{R\text{-}\mathbf{Mod}\text{-}T}(U\otimes_S V,W)$ by

$$f_g(u\otimes v):=[g(v)](u). \tag{D.3.2.30}$$

As this is bilinear in u and v, this serves to define a map of S-T-bimodules $U\otimes_S V\to W$—see the last remark in the definition of the tensor product (Theorem D.3.2.6). As the check that $g\mapsto f_g$ is a group homomorphism is similar to before, we omit it (it comes down to the definition of addition of functions).

It remains to check that $f \mapsto g_f$ and $g \mapsto f_g$ are inverse to each other. To do that, we must show that $g_{f_g} = g$ and $f_{g_f} = f$. For the first one, note that

$$[[g_{f_g}](v)](u) := f_g(u \otimes v) := [g(v)](u). \tag{D.3.2.31}$$

As this holds for all $u \in U$ and $v \in V$, we have $g_{f_g} = g$. For the other one, note that

$$[f_{g_f}](u \otimes v) := [g_f(v)](u) := f(u \otimes v), \tag{D.3.2.32}$$

and again we have that $f_{g_f} = f$, as desired. ■

What follows are a couple of results similar in flavor to the tensor-hom adjunction. While the tensor-hom adjunction is probably more important in mathematics in general, for us, the following three results will be more important, and you should take note of them, especially the case of finite-dimensional vector spaces.

Theorem D.3.2.33 — $\mathrm{Mor}(V_1, W_1) \otimes \mathrm{Mor}(V_2, W_2) \cong \mathrm{Mor}(V_1 \otimes V_2, W_1 \otimes W_2)$. Let V_1, W_1, V_2, and W_2 be $\mathbb{K}$-$\mathbb{K}$-bimodules. Then, the map

$$\mathrm{Mor}(V_1, W_1) \otimes \mathrm{Mor}(V_2, W_2) \to \mathrm{Mor}(V_1 \otimes V_2, W_1 \otimes W_2)$$

defined by

$$S \otimes T \mapsto (v_1 \otimes v_2 \mapsto S(v_1) \otimes T(v_2)) \tag{D.3.2.34}$$

is linear and natural.

Furthermore,

(i). if V_1, W_1, V_2, and W_2 are vector spaces, then this map is injective; and

(ii). if V_1, W_1, V_2, and W_2 are finite-dimensional vector spaces, then this map is an isomorphism.

R We will abuse notation write write $S \otimes T$ for both the element in the tensor product $\mathrm{Mor}(V_1, W_1) \otimes$

$\mathrm{Mor}(V_2, W_2)$ and the linear-transformation it defines $V_1 \otimes V_2 \to W_1 \otimes W_2$. Of course, this result says that this isn't even really an abuse of notation if everything involved is a vector space, but even if they weren't, this abuse should not cause any confusion.

R Warning: This need not be an isomorphism even for vector spaces if they are not finite-dimensional.

Proof. As this is bilinear in S and T, it defines a linear-transformation on the tensor product.

To show naturality, let $f\colon U_1 \to V_1$ be a linear-transformation. By definition (Definition B.3.2.3), this will be natural in the first space iff the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Mor}(V_1, W_1) \otimes \mathrm{Mor}(V_2, W_2) \arrow[r] \arrow[d] & \mathrm{Mor}(V_1 \otimes V_2, W_1 \otimes W_2) \arrow[d] \\
\mathrm{Mor}(U_1, W_1) \otimes \mathrm{Mor}(V_2, W_2) \arrow[r] & \mathrm{Mor}(U_1 \otimes V_2, W_1 \otimes W_2)
\end{tikzcd}$$

By definition, this commutes iff

$$S(f(u_1)) \otimes T(v_2) = S(f(u_1)) \otimes T(v_2), \tag{D.3.2.35}$$

which if tautologically true.[a] By $V_1 \leftrightarrow V_2$ symmetry, it is natural in V_2 as well. A similar check shows that it is natural in W_1, and hence W_2 as well.

(i) Suppose that V_1, W_1, V_2, and W_2 are vector spaces. Let $\sum_{k=1}^{m} \sum_{l=1}^{n} S_k \otimes T_l$ be an arbitrary nonzero element of $\mathrm{Mor}(V_1, W_1) \otimes \mathrm{Mor}(V_2, W_2)$. Without loss of generality, let $m, n \in \mathbb{Z}^+$ be the smallest such positive integers. [b]

Now suppose that this element of $\mathrm{Mor}(V_1, W_1) \otimes \mathrm{Mor}(V_2, W_2)$ is sent to 0. In other words,

$$\sum_{k=1}^{m} \sum_{l=1}^{n} S_k(v_1) \otimes T_l(v_2) = 0 \tag{D.3.2.36}$$

for all $v_1 \in V_1$ and $v_2 \in V_2$. Writing this as

$$\left(\sum_{k=1}^{m} S_k(v_1)\right)\left(\sum_{l=1}^{n} T_l(v_2)\right) = 0, \tag{D.3.2.37}$$

using Corollary D.3.2.14, we deduce that either $\sum_{k=1}^{m} S_k(v_1) = 0$ or $\sum_{l=1}^{n} T_l = 0$. In the the former case, we can replace S_m in $\sum_{k=1}^{m}\sum_{l=1}^{n} S_k \otimes T_l$ with $-\sum_{k=1}^{n-1} S_k$, thereby writing this element with only $n-1$ w_ls: a contradiction. The latter case is identical. Thus, it cannot be the case that a nonzero element of $\mathrm{Mor}(V_1, W_1) \otimes \mathrm{Mor}(V_2, W_2)$ is sent to 0, and this map is injective.

(ii) Suppose that V_1, W_1, V_2, and W_2 are finite-dimensional vector spaces. $\mathrm{Mor}(V_1, W_1) \otimes \mathrm{Mor}(V_2, W_2)$ and $\mathrm{Mor}(V_1 \otimes V_2, W_1 \otimes W_2)$ have the same dimension, namely $\dim(V_1)\dim(W_1)\dim(V_2)\dim(W_2)$, and hence the injective map $\mathrm{Mor}(V_1, W_1) \otimes \mathrm{Mor}(V_2, W_2) \to \mathrm{Mor}(V_1 \otimes V_2, W_1 \otimes W_2)$ must in fact be an isomorphism (Proposition D.2.54). ∎

[a]Going right then down essentially gives $S(v_1) \otimes T(v_2)$ and then replaces v_1 with $f(u_1)$. Going down then right replaces v_1 with $f(u_1)$ and then takes $S(f(u_1)) \otimes T(u_1)$.

[b]So, for example, if you can simplify $S_1 \otimes T_1 + S_2 \otimes T_2$ to something of the form $S \otimes T$, do that, and take $m = 1 = n$ instead of $m = 2 = n2$.

Corollary D.3.2.38 — $V^\dagger \otimes W \cong \mathrm{Mor}(V, W)$ Let V and W be $\mathbb{K}$-$\mathbb{K}$-bimodules. Then, the map

$$V^\dagger \otimes_{\mathbb{K}} W \ni \phi \otimes w \mapsto (v \mapsto \phi(v)w) \in \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(V, W)$$

is linear and natural.

Furthermore,

(i). if V and W are vector spaces, this map is injective; and
(ii). if V and W are finite-dimensional vector spaces, this map is an isomorphism.

R In particular, for finite-dimensional vector spaces, using language that we will learn shortly (Definition D.3.2.41), $\langle 1,1\rangle$ tensors are 'the same as' linear-transformations.[a]

R Warning: This need not be an isomorphism even for vector spaces if they are not finite-dimensional.

[a]We technically don't define "tensor" unless $W = V$, but that doesn't really affect what's going on here—this is just a matter of language.

Proof. Take $W_1 = \mathbb{K}$, $V_2 = \mathbb{K}$, $V_1 = V$, and $W_2 = W$ in the previous result (Theorem D.3.2.33). Using the fact that $W \cong \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(\mathbb{K}, W)$, $V \cong V \otimes_{\mathbb{K}} \mathbb{K}$, and $W \cong \mathbb{K} \otimes_{\mathbb{K}} W$ naturally, Theorem D.3.2.33 reduces to exactly the statement of this corollary. ■

Corollary D.3.2.39 — $\mathrm{Mor}(U \otimes V, W) \cong \mathrm{Mor}(U, V^{\dagger} \otimes W)$ Let U, V, and W be $\mathbb{K}$-$\mathbb{K}$-bimodules. Then, the map

$$\mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(U, V^{\dagger} \otimes_{\mathbb{K}} W) \to \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(U \otimes V, W),$$

given by the composition of the maps

$$\mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(U, V^{\dagger} \otimes_{\mathbb{K}} W) \to \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(U, \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(V, W))$$

and

$$\mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(U, \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(V, W)) \to \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(U \otimes V, W),$$

is linear and natural.

Furthermore,

(i). if V and W are vector spaces, this map is injective; and
(ii). if V and W are finite-dimensional vector space, this map is an isomorphism.

Proof. This map is a composition of the map from the previous corollary and the Tensor-Hom Adjunction (Theorem D.3.2.19), and so this corollary follows immediately from those two results. ■

We now turn to tensors themselves.

Definition D.3.2.41 — Tensor Let V be a $\mathbb{K}$-$\mathbb{K}$-bimodule. Then, a ***tensor*** of ***rank*** $\langle k, l\rangle$ over V is an element of

$$\bigotimes_l^k V := \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(\underbrace{V \otimes \cdots \otimes V}_{l}, \underbrace{V \otimes \cdots \otimes V}_{k}). \quad \text{(D.3.2.42)}$$

R k is the ***contravariant rank*** and l is the ***covariant rank***. If $l = 0$, then the tensor is ***contravariant***, and if $k = 0$, then the tensor is ***covariant***.

We write

$$\bigotimes^k V := \bigotimes_0^k V \quad \text{(D.3.2.43)}$$

and

$$\bigotimes_l V := \bigotimes_l^0 V \quad \text{(D.3.2.44)}$$

respectively for the spaces of rank k contravariant tensors and rank l covariant tensors.

R Though uncommon, I have seen the term ***valence*** used synonymously with "rank" in this context.

R Instead of saying "T is a tensor of rank $\langle k.l\rangle$", we may use the less verbose "T is a $\langle k, l\rangle$ tensor".

R Thus, by the definition of the tensor product (Definition D.3.2.41), a tensor of rank $\langle k, l\rangle$ is 'the same as' a multilinear map from $\underbrace{V \times \cdots \times V}_{l}$ to $\underbrace{V \otimes \cdots \otimes V}_{k}$.

Thus, a tensor of rank $\langle k, l\rangle$ is a thing that takes in l vectors and 'spits out' 'k vectors'[a] in a multilinear manner.

R Using the same sort of isomorphisms referenced in the previous remark, for V a finite-dimensional vector space, there is a natural isomorphism

$$\begin{aligned}\bigotimes\nolimits_l^k V &:= \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(\underbrace{V \otimes \cdots \otimes V}_{l}, \underbrace{V \otimes \cdots \otimes V}_{k}) \\ &\cong \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(\underbrace{V \otimes \cdots \otimes V}_{l} \otimes \underbrace{V^{\dagger} \otimes \cdots \otimes V^{\dagger}}_{k}, \mathbb{K}).\end{aligned}$$

Thus, in regards to the question "But what actually *is* a tensor?", this says that, for V a finite-dimensional vector space anyways:

> A tensor of rank $\langle k, l\rangle$ is a multilinear map
>
> $$\underbrace{V \times \cdots \times V}_{l} \times \underbrace{V^{\dagger} \times \cdots \times V^{\dagger}}_{k} \to \mathbb{K}.$$

R Note that the notation $\bigotimes_l^k V$ is nonstandard (though based on the standard notation $\Lambda^l V$ for something new which we will become acquainted with later on).

[a] More accurately, a contravariant tensor of rank k.

D.4 The determinant

Again, the purpose of this section is primarily just so the notes are technically self-contained—it would be difficult to learn from here without having previously encountered the determinant.

D.4.1 Permutations

Definition D.4.1.1 — Symmetric group Let S be a set. Then, the ***symmetric group*** of S is $\mathrm{Aut}_{\mathbf{Set}}(S)$.

R In this context, elements of $\mathrm{Aut}_{\mathbf{Set}}(S)$ tend to be referred to as ***permutations*** of S.

R $\mathrm{Aut}_{\mathbf{Set}}(S)$ is our fancy-schmancy category-theoretic notation for the set of all bijections $S \to S$.

Definition D.4.1.2 — Cycle notation Let $S = \{1, \ldots, m\}$ be a finite set and let $x_1, \ldots, x_n \in S$ be distinct. Then, $(x_1\ \ldots\ x_n) \in \mathrm{Aut}_{\mathbf{Set}}(S)$ is the unique bijection that sends x_k to x_{k+1} for $1 \leq k \leq n-1$, sends x_n to x_1, and fixes everything else.

R Permutations of the form $(x_1\ \ldots\ x_n)$ are ***cycles***.

R The ***length*** of $(x_1 \ldots x_n)$ is n.

R A ***transposition*** is a cycle of length 2.

R For example, $(325) \in \mathrm{Aut}_{\mathbf{Set}}(\{1,2,3,4,5\})$ is shorthand for the function $\{1,2,3,4,5\} \to \{1,2,3,4,5\}$

$$1 \mapsto 1 \tag{D.4.1.3a}$$
$$2 \mapsto 5 \tag{D.4.1.3b}$$
$$3 \mapsto 2 \tag{D.4.1.3c}$$
$$4 \mapsto 4 \tag{D.4.1.3d}$$
$$5 \mapsto 3. \tag{D.4.1.3e}$$

To discuss antisymmetry, we're going to need to discuss the *sign* of a permutation.

Theorem D.4.1.4 — Sign of a permutation. Let S be a finite set and let $\sigma \in \mathrm{Aut}_{\mathbf{Set}}(S)$.

(i). σ can be written as a product of transpositions.

(ii). If $s_1 \cdots s_m = \sigma = t_1 \cdots t_n$ with each s_k and t_k a transposition, then m and n have the same parity $\mathrm{sgn}(\sigma) \in \{1, -1\}$, the ***sign*** σ.[a]

(iii). $\mathrm{sgn}\colon \mathrm{Aut}_{\mathbf{Set}}(S) \to \{1, -1\} \cong \mathbb{Z}/2\mathbb{Z}$ is a group homomorphism.

R This says that every permutation can be written as a product of transpositions, and furthermore, the number of these transpositions is unique modulo 2.

R For $X = \{1, \ldots, m\}$ a finite set and $S = \{i_1, \ldots, i_k\} \subseteq S$, write $S^{\mathsf{C}} =: \{j_{k+1}, \ldots, j_m\}$ with $j_1 < \cdots < j_m$. Then, we shall write $\mathrm{sgn}(S) := \mathrm{sgn}(\sigma_S)$ for the unique permutation $\sigma\colon X \to X$ such that

$$\sigma(x) := \begin{cases} i_x & \text{if } x \leq k \\ j_x & \text{if } x \geq k + 1. \end{cases} \tag{D.4.1.5}$$

For example, for $S := \{2, 4, 5\} \subseteq \{1, 2, 3, 4, 5\}$, σ_S sends 1 to 2, 2 to 4, 3 to 5, 4 to 1, and 5 to 3, that is, $\sigma_S = (124)(35)$, and so

$$\mathrm{sgn}(S) = -1. \tag{D.4.1.6}$$

[a] That is, m is even/odd iff n is even/odd.

Proof. We leave this as an exercise.

Exercise D.4.1.7 Prove the result.

■

D.4.2 (Anti)symmetric tensors

Definition D.4.2.1 — (Anti)symmetric Let V be a $\mathbb{K}$-$\mathbb{K}$-bimodule and let $T^{a_1\cdots a_k} \in \bigotimes^k V$.

(i). $T^{a_1\cdots a_k}$ is ***symmetric*** iff

$$T^{a_1\cdots a_k} = T^{a_{\sigma(1)}\cdots a_{\sigma(k)}} \tag{D.4.2.2}$$

for all $\sigma \in \mathrm{Aut}_{\mathbf{Set}}(\{1,\ldots,k\})$.

(ii). $T^{a_1\cdots a_k}$ is ***antisymmetric*** iff

$$T^{a_1\cdots a_k} = \operatorname{sgn}(\sigma) T^{a_{\sigma(1)}\cdots a_{\sigma(k)}} \tag{D.4.2.3}$$

for all

$$\sigma \in \mathrm{Aut}_{\mathbf{Set}}(\{1,\ldots,k\}). \tag{D.4.2.4}$$

R The condition of being ***(anti)symmetric*** is defined for covariant tensors in an essentially identical manner. A general tensor is then ***(anti)symmetric*** iff it is (anti)symmetric in both its contravariant and covariant indices.

R The set of symmetric tensors of rank $\langle k, l\rangle$ is a subspace of $\bigotimes_l^k$ which is denoted

$$\bigvee\nolimits_l^k V. \tag{D.4.2.5}$$

The set of antisymmetric tensors of rank $\langle k, l\rangle$ is a subspace of $\bigotimes_l^k$ which is denoted

$$\bigwedge\nolimits_l^k V. \tag{D.4.2.6}$$

R We write

$$\bigvee\nolimits^k V := \bigvee\nolimits_0^k V \tag{D.4.2.7}$$

and

$$\bigvee_l V := \bigvee_l^0 V \tag{D.4.2.8}$$

respectively for the spaces of symmetric rank k contravariant tensors and symmetric rank l covariant tensors.

Similarly, we write

$$\bigwedge^k V := \bigwedge_0^k V \tag{D.4.2.9}$$

and

$$\bigwedge_l V := \bigwedge_l^0 V \tag{D.4.2.10}$$

respectively for the spaces of antisymmetric rank k contravariant tensors and antisymmetric rank l covariant tensors.

R As we had with $\bigotimes_l^k V$ (Definition D.3.2.41), we have that $\bigvee_0^0 V \cong \mathbb{K} \cong \bigwedge_0^0 V$, $\bigvee_0^1 V \cong V \cong \bigwedge_0^1 V$, and $\bigvee_1^0 V = V^\dagger = \bigwedge_1^0 V$—tensors of these ranks (along with $\bigvee_1^1 V = \mathrm{Mor}_{\mathbb{K}\text{-}\mathbf{Mod}}(V, V) = \bigwedge_1^1 V$) are vacuously (anti)symmetric.

R This is nonstandard notation. First of all, usually people only work with covariant tensors in this context, in which case they denote these respectively by $\mathrm{Sym}^l(V)$ and $\Lambda^l(V)$.

R Sometimes people will say ***totally symmetric*** and ***totally antisymmetric*** for these concepts respectively, presumably to emphasize that one is discussing *all* the indices.

R Elements of $\bigwedge_l V$ are sometimes called ***differential forms*** or just ***forms***, for reasons obviously having to do with calculus. As such, these terms are usually reserved when doing manifold theory, in which case they probably referred not to just a single tensor but a tensor *field*.[a]

[a] "Field" in this context intuitively means that you associate a different tensor to each point (e.g. "vector field").

D.4.3 The determinant itself

Theorem D.4.3.1 — Determinant. Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and let $T\colon V\to V$ be linear. Then, there is a unique scalar $\det(T)\in\mathbb{F}$, the ***determinant*** of T, such that

$$\bigwedge^d T=\det(T)\,\mathrm{id}, \tag{D.4.3.2}$$

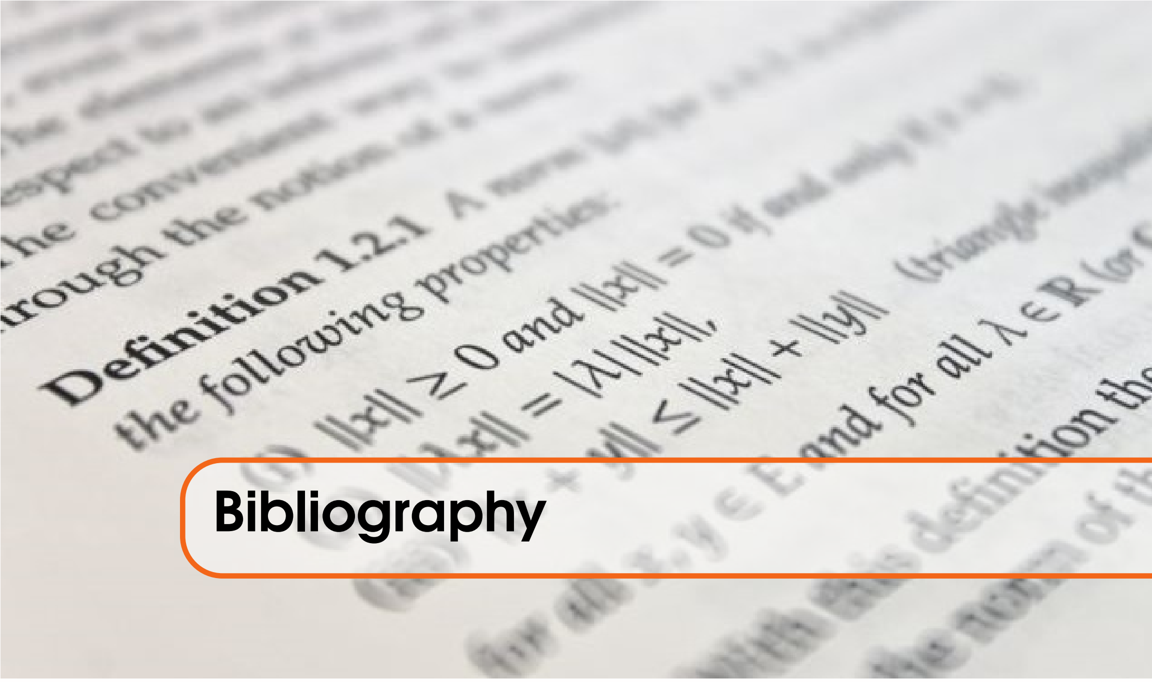
where $d:=\dim(V)$ and id is the identity on $\bigwedge^d V$.

R Some authors write $|A|$ to denote the determinant of A. We shall not make use of this notation.

Proof. We leave this as an exercise.

Exercise D.4.3.3 Prove the result.

■



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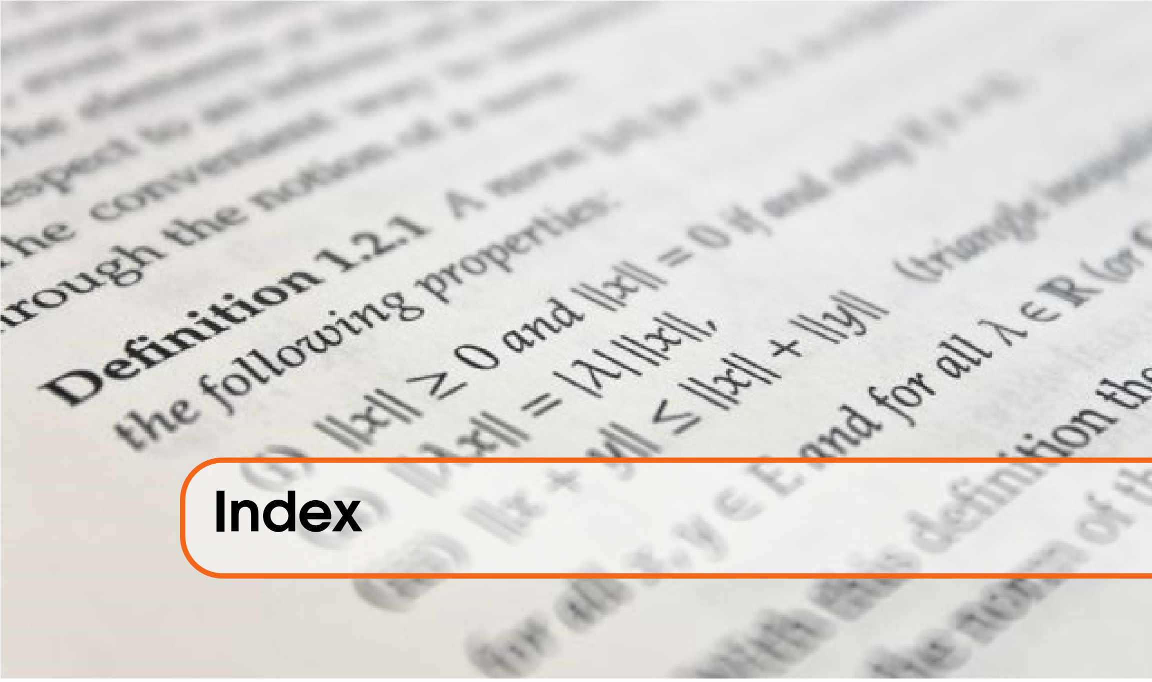

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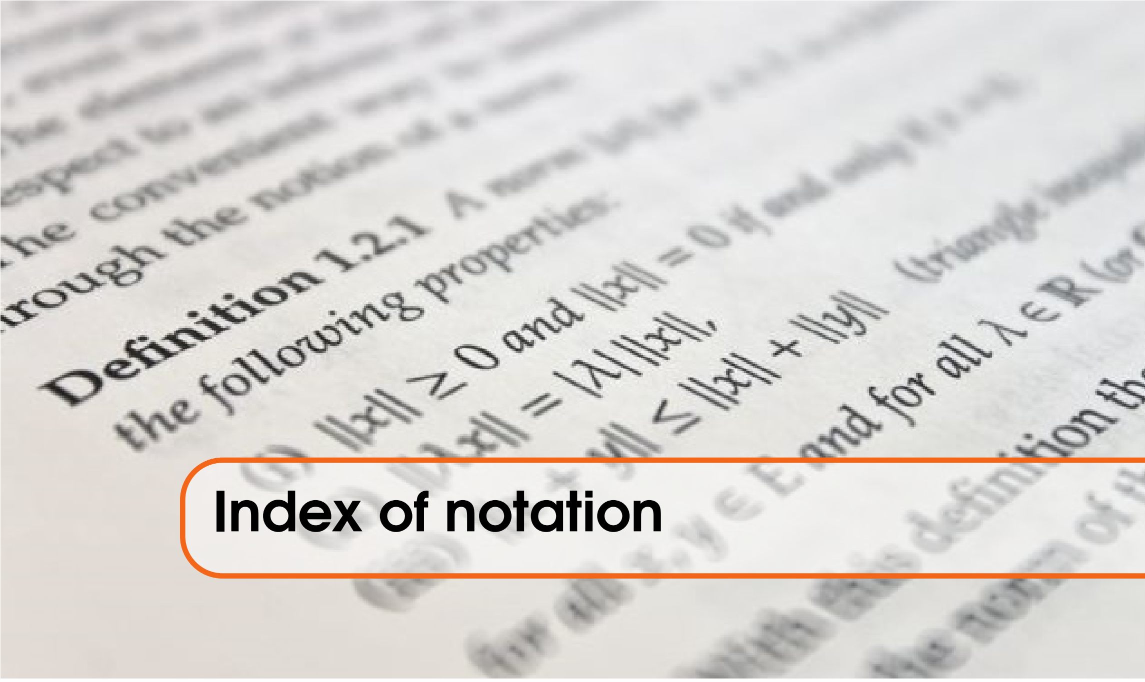

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