Topological Groups

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Introduction

Group Theory and Topology are two of the great areas of Mathematics, emerged in the 19th century, and that study different objects: Groups and Topological spaces, respectively. A topological group is a set on which a binary operation is defined and a topology is imposed such that the operation and inversion are continuous functions. This interaction between the algebraic structure of groups and the topological properties of spaces makes topological groups important objects of study, with applications in many areas of mathematics and beyond.

Definition and examples

Definition 1.1. Let $(G, \cdot)$ be a group and τ_G a topology on G. The trio $(G, \cdot, \tau_G)$ is said to be a **topological group** if the maps

$$\begin{aligned} i\colon G &\longrightarrow G \\ x &\longmapsto x^{-1} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \cdot\colon G\times G &\longrightarrow G \\ (x,y) &\longmapsto x\cdot y, \end{aligned}$$

called **inversion** and **group operation**, respectively, are continuous with respect to τ_G and its induced product topology.

Example 1.2. Consider the group $(\mathbb{K}_4, \cdot)$ and $\tau_{\mathbb{K}_4} = \{\emptyset, \{1, \mathrm{ab}\}, \{\mathrm{a}, \mathrm{b}\}, \mathbb{K}_4\}$. Then the trio $(\mathbb{K}_4, \cdot, \tau_{\mathbb{K}_4})$ is a topological group.

Example 1.3. Consider the additive group of integers $(\mathbb{Z}, +)$ and an arbitrary prime p. Let $\mathcal{V}_p := \{U \subset \mathbb{Z} \mid \text{exists } n \in \mathbb{N} \text{ such that } p^n\mathbb{Z} \subset U\}$ and the topology

$$\tau := \{V \subset \mathbb{Z} \mid \text{for each } v \in V, \text{exists } U \in \mathcal{V}_p, \text{such that } v + U \subset V\}$$

.

Then, the trio $(\mathbb{Z}, +, \tau)$ is a topological group.

Example 1.4. Consider the circle group $(\mathbb{S}^1, \cdot)$. The torus $\mathbb{T}^2$ is a topological group, because is the direct product $\mathbb{S}^1 \times \mathbb{S}^1$, and $\mathbb{S}^1$ is a topological group with respect to the subespace topology induced by $\mathbb{C}^*$.

Example 1.5. Consider the general linear group over $\mathbb{R}$, $(GL_n(\mathbb{R}), \cdot)$. Then $(GL_n(\mathbb{R}), \cdot)$ is a topological group with respect to topology induced by $\mathbb{R}^{(n^2)}$.

General properties of topological groups

Definition 1.6. Let $(G, \cdot)$ be a group and $\mathcal{F}$ be a filter of G. We say that $\mathcal{F}$ is **viable** when

I. For each $U \in \mathcal{F}$, exists $V \in \mathcal{F}$ such that $V \cdot V \subset U$.

II. For each $U \in \mathcal{F}$, exists $V \in \mathcal{F}$ such that $V^{-1} \subset U$.

III. For each $U \in \mathcal{F}$, exists $V \in \mathcal{F}$ such that $V \cdot V^{-1} \subset U$.

IV. For each $U \in \mathcal{F}$ and $a \in G$, $aUa^{-1} \in \mathcal{F}$.

Theorem 1.7. Let $(G, \cdot, \tau_G)$ be a topological group and $\mathcal{V}(1_G)$ be the filter of all neighborhoods of 1_G in that same topology. So $\mathcal{V}(1_G)$ is a viable filter.

Theorem 1.8. Let $(G, \cdot)$ be a group and $\mathcal{V}$ be a viable filter. Then there exists a unique topology τ in G that makes $(G, \cdot, \tau)$ a topological group and that makes $\mathcal{V}$ coincide with $\mathcal{V}(1_G)$, the filter of all neighborhoods of 1_G in this topology.

Theorem 1.9. Let $(G, \cdot, \tau_G)$ be a topological group and let H be a subgroup of G. Then

I. H is open if and only if $int(H) \neq \emptyset$.

II. If H is open, then H is closed.

III. If H is closed and has finite index, then H is open.

Theorem 1.10. Let $(G, \cdot, \tau_G)$ be a topological group. Let H be a subset of G and $\mathcal{V}(1_G)$, the filter of all neighborhoods of the neutral element of G. Then

I. $\overline{H} = \bigcap_{U\in\mathcal{V}(1_G)} (UH) = \bigcap_{U\in\mathcal{V}(1_G)} (HU) = \bigcap_{U,V\in\mathcal{V}(1_G)} (UHV)$.

II. If H is a subgroup of G, then $\overline{H}$ is a subgroup of G. If H is a normal subgroup, then also $\overline{H}$ is a normal subgroup.

III. $N = \overline{\{1\}}$ is a closed normal subgroup.

We will now state a series of results about separation axioms and connected and compact topological groups.

Separation Axioms

Theorem 1.11. For a topological group $(G, \cdot, \tau_G)$ the following are equivalent:

I. G is $\mathbb{T}_3$ (Regular and $\mathbf{T}_1$).

II. G is Hausdorff.

III. G is $\mathbf{T}_1$.

IV. G is $\mathbf{T}_0$.

Theorem 1.12. Let $(G, \cdot, \tau_G)$ be a topological group and let H be a normal subgroup of G. Then

I. The quotient G/H is discrete if and only if H is open.

II. The quotient G/H is Hausdorff if and only if H is closed.

Connectedness

Theorem 1.13. The connected component of the neutral element is a closed normal subgroup of G.

Theorem 1.14. Let $(G, \cdot, \tau_G)$ be a topological group and let N be a closed normal subgroup of G.

I. If both N and G/N are connected, then G is also connected.

II. If both N and G/N are totally disconnected, then G is also totally disconnected.

Theorem 1.15. Let $(G, \cdot, \tau_G)$ be a topological group and N, the connected component of neutral element of G. The group G/N is totally disconnected.

Compactness

Theorem 1.16. Let $(G, \cdot, \tau_G)$ be a topological group and let C and K closed subsets of G.

I. If K is compact, then CK and KC are closed.

II . If both C and K are compact, then CK and KC are compact.

III . If K is contained in an open subset U of G, then there exists an open neighborhood V of 1_G such that $KV \subseteq U$.

Theorem 1.17. Let $(G, \cdot, \tau_G)$ be a topological group and let K a compact subgroup of G. Then the canonical projection $\pi : G \longrightarrow G/K$ is closed.

Theorem 1.18. Let $(G, \cdot, \tau_G)$ be a topological group and let H be a closed subgroup of G.

I. If G is compact, then G/H is compact.

II. . If H and G/H are compact, then G is compact.

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