$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma \quad ?$$

Let's compute $D_\mu(\underbrace{D_\nu V^\rho}_{\text{tensor}}) = \{\text{def. of covariant derivative}\}$

$$= \partial_\mu (D_\nu V^\rho) + \Gamma^\rho_{\mu\lambda} D_\nu V^\lambda - \Gamma^\lambda_{\mu\nu} D_\lambda V^\rho =$$

$$= \{-\|-\} = \partial_\mu \left[\partial_\nu V^\rho + \Gamma^\rho_{\nu\alpha} V^\alpha \right] + \Gamma^\rho_{\mu\lambda} \left[\partial_\nu V^\lambda + \Gamma^\lambda_{\nu\alpha} V^\alpha \right] -$$

$$- \Gamma^\lambda_{\mu\nu} \left[\partial_\lambda V^\rho + \Gamma^\rho_{\lambda\alpha} V^\alpha \right] = \partial_\mu \partial_\nu V^\rho + \partial_\mu \Gamma^\rho_{\nu\alpha} V^\alpha +$$

$$+ \Gamma^\rho_{\mu\lambda} \partial_\nu V^\lambda + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\alpha} V^\alpha - \Gamma^\lambda_{\mu\nu} \partial_\lambda V^\rho - \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\alpha} V^\alpha$$

We can get $D_\nu(D_\mu V^\rho)$ by $\mu \leftrightarrow \nu$

$$\Rightarrow D_\nu D_\mu V^\rho = \partial_\nu \partial_\mu V^\rho + \partial_\nu \Gamma^\rho_{\mu\alpha} V^\alpha + \Gamma^\rho_{\nu\lambda} \partial_\mu V^\lambda +$$

$$+ \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\alpha} V^\alpha - \Gamma^\lambda_{\nu\mu} \partial_\lambda V^\rho - \Gamma^\lambda_{\nu\mu} \Gamma^\rho_{\lambda\alpha} V^\alpha$$

$$\Rightarrow D_\mu D_\nu V^\rho - D_\nu D_\mu V^\rho = \cancel{\partial_\mu \partial_\nu V^\rho} + \partial_\mu \Gamma^\rho_{\nu\alpha} V^\alpha + \Gamma^\rho_{\mu\lambda} \partial_\nu V^\lambda + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\alpha} V^\alpha -$$

$$- \cancel{\Gamma^\lambda_{\mu\nu} \partial_\lambda V^\rho} - \cancel{\Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\alpha} V^\alpha} - \left(\cancel{\partial_\nu \partial_\mu V^\rho} + \partial_\nu \Gamma^\rho_{\mu\alpha} V^\alpha + \Gamma^\rho_{\nu\lambda} \partial_\mu V^\lambda + \right.$$

$$\left. + \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\alpha} V^\alpha - \cancel{\Gamma^\lambda_{\nu\mu} \partial_\lambda V^\rho} - \cancel{\Gamma^\lambda_{\nu\mu} \Gamma^\rho_{\lambda\alpha} V^\alpha} \right),$$ here we

used torsion-free condition: $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$

$$\Rightarrow [D_\mu, D_\nu] V^\rho = \partial_\mu \Gamma^\rho_{\nu\alpha} V^\alpha + \Gamma^\rho_{\mu\lambda} \partial_\nu V^\lambda + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\alpha} V^\alpha -$$

$$- \partial_\nu \Gamma^\rho_{\mu\alpha} V^\alpha - \Gamma^\rho_{\nu\lambda} \partial_\mu V^\lambda - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\alpha} V^\alpha$$

$$\partial_\mu \Gamma^\rho_{\nu\alpha} V^\alpha = (\partial_\mu \Gamma^\rho_{\nu\alpha}) V^\alpha + \Gamma^\rho_{\nu\alpha} \partial_\mu V^\alpha$$

$$\partial_\nu \Gamma^\rho_{\mu\alpha} V^\alpha = (\partial_\nu \Gamma^\rho_{\mu\alpha}) V^\alpha + \Gamma^\rho_{\mu\alpha} \partial_\nu V^\alpha$$

substitute

$$\Rightarrow [D_\mu, D_\nu] V^\rho = (\partial_\mu \Gamma^\rho_{\nu\alpha}) V^\alpha + \cancel{\Gamma^\rho_{\nu\alpha} \partial_\mu V^\alpha} + \cancel{\Gamma^\rho_{\mu\lambda} \partial_\nu V^\lambda}$$

$$+ \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\alpha} V^\alpha - (\partial_\nu \Gamma^\rho_{\mu\alpha}) V^\alpha - \cancel{\Gamma^\rho_{\mu\alpha} \partial_\nu V^\alpha} - \cancel{\Gamma^\rho_{\nu\lambda} \partial_\mu V^\lambda} -$$

$$- \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\alpha} V^\alpha =$$

$$\Rightarrow [D_\mu, D_\nu] V^\rho = \Big(\underbrace{\partial_\mu \Gamma^\rho_{\nu\alpha} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\alpha} - \partial_\nu \Gamma^\rho_{\mu\alpha} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\alpha}}_{R^\rho{}_{\alpha\mu\nu}} \Big) V^\alpha$$

$$\Rightarrow [D_\mu, D_\nu] V^\rho = R^\rho{}_{\alpha\mu\nu} V^\alpha$$

proof

$$R^\rho{}_{\alpha\mu\nu} \overset{\text{def}}{=} \partial_\mu \underbrace{\Gamma^\rho_{\alpha\nu}}_{= \Gamma^\rho_{\nu\alpha}} - \partial_\nu \underbrace{\Gamma^\rho_{\alpha\mu}}_{= \Gamma^\rho_{\mu\alpha}} + \underbrace{\Gamma^\lambda_{\alpha\nu}}_{= \Gamma^\lambda_{\nu\alpha}} \underbrace{\Gamma^\rho_{\lambda\mu}}_{= \Gamma^\rho_{\mu\lambda}} - \underbrace{\Gamma^\lambda_{\alpha\mu}}_{= \Gamma^\lambda_{\mu\alpha}} \Gamma^\rho_{\nu\lambda} =$$

$$= \partial_\mu \Gamma^\rho_{\nu\alpha} - \partial_\nu \Gamma^\rho_{\mu\alpha} + \Gamma^\lambda_{\nu\alpha} \Gamma^\rho_{\mu\lambda} - \Gamma^\lambda_{\mu\alpha} \Gamma^\rho_{\nu\lambda}$$

$\blacksquare$

1) Show that $(\partial_\rho g_{\mu\lambda}) g^{\lambda\kappa} = - g_{\mu\lambda} \partial_\rho g^{\lambda\kappa}$

We know that $g_{\mu\lambda} g^{\lambda\kappa} = \delta_\mu{}^\kappa \leftarrow$ constant tensor

$\Rightarrow \partial_\rho (\delta_\mu{}^\kappa) = 0 = \partial_\rho (g_{\mu\lambda} g^{\lambda\kappa}) = (\partial_\rho g_{\mu\lambda}) g^{\lambda\kappa} +$

$+ g_{\mu\lambda} \partial_\rho g^{\lambda\kappa} \quad \Rightarrow (\partial_\rho g_{\mu\lambda}) g^{\lambda\kappa} = - g_{\mu\lambda} \partial_\rho g^{\lambda\kappa}$ ∎

2) Show that $\partial_\rho g_{\alpha\sigma} = g_{\alpha\lambda} \Gamma^\lambda_{\sigma\rho} + g_{\sigma\lambda} \Gamma^\lambda_{\alpha\rho}$

$\Gamma^\lambda_{\sigma\rho} = \frac{1}{2} g^{\lambda\mu} \underbrace{(\partial_\rho g_{\sigma\mu} + \partial_\sigma g_{\rho\mu} - \partial_\mu g_{\rho\sigma})}_{= (\ldots)} \quad | \times g_{\alpha\lambda}$

$\Rightarrow g_{\alpha\lambda} \Gamma^\lambda_{\sigma\rho} = \frac{1}{2} g_{\alpha\lambda} g^{\lambda\mu} (\ldots) = \frac{1}{2} \delta_\alpha{}^\mu (\ldots) \Leftrightarrow$

$\Leftrightarrow \frac{1}{2} \delta_\alpha{}^\mu (\partial_\rho g_{\sigma\mu} + \partial_\sigma g_{\rho\mu} - \partial_\mu g_{\rho\sigma}) = \frac{1}{2} (\partial_\rho g_{\sigma\alpha} + \partial_\sigma g_{\rho\alpha} -$

$- \partial_\alpha g_{\rho\sigma})$

By performing $\alpha \leftrightarrow \sigma$ we can get:

$g_{\sigma\lambda} \Gamma^\lambda_{\alpha\rho} = \frac{1}{2} (\partial_\rho g_{\alpha\sigma} + \partial_\alpha g_{\rho\sigma} - \partial_\sigma g_{\rho\alpha})$

Hence:

$$g_{\alpha\lambda}\Gamma^{\lambda}_{\sigma\rho} + g_{\sigma\lambda}\Gamma^{\lambda}_{\alpha\rho} = \frac{1}{2}\Big(\partial_\rho \overbrace{g_{\sigma\alpha}}^{g_{\alpha\sigma}} + \cancel{\partial_\sigma g_{\rho\alpha}} - \cancel{\partial_\alpha g_{\rho\sigma}}\Big) +$$

$$+\frac{1}{2}\Big(\partial_\rho g_{\alpha\sigma} + \cancel{\partial_\alpha g_{\rho\sigma}} - \cancel{\partial_\sigma g_{\rho\alpha}}\Big) = \partial_\rho g_{\alpha\sigma} \quad \blacksquare$$

4) ① Let's first do it in an easy way:

Need: $R_{\mu\nu} = R_{\nu\mu}$

$$R_{\mu\nu} = g^{\rho\lambda} R_{\mu\rho\nu\lambda}$$

$R_{\mu\rho\nu\lambda} = R_{\nu\lambda\mu\rho}$ & and $g^{\rho\lambda} = g^{\lambda\rho}$

$$\Rightarrow R_{\mu\nu} = g^{\lambda\rho} R_{\nu\lambda\mu\rho} = R_{\nu\mu}$$

② Start from:

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\rho\partial_\nu g_{\mu\sigma} + \partial_\mu\partial_\sigma g_{\rho\nu} - \partial_\nu\partial_\sigma g_{\rho\mu} - \partial_\mu\partial_\rho g_{\nu\sigma}\right) +$$

$$g_{\kappa\lambda}\left(\Gamma^{\kappa}_{\mu\sigma}\Gamma^{\lambda}_{\nu\rho} - \Gamma^{\kappa}_{\mu\rho}\Gamma^{\lambda}_{\nu\sigma}\right) \qquad \Big| \times g^{\nu\sigma}$$

$$R_{\mu\rho} = g^{\nu\sigma} R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\rho \partial^\sigma g_{\mu\sigma} + \partial_\mu \partial^\nu g_{\rho\nu} - \partial^\sigma\partial_\sigma g_{\rho\mu} -\right.$$

$$\left. - g^{\nu\sigma}\partial_\mu\partial_\rho g_{\nu\sigma}\right) + g^{\nu\sigma} g_{\kappa\lambda}\left(\Gamma^{\kappa}_{\mu\sigma}\Gamma^{\lambda}_{\nu\rho} - \Gamma^{\kappa}_{\mu\rho}\Gamma^{\lambda}_{\nu\sigma}\right)$$

$$R_{\mu\rho} = \frac{1}{2}\left(\overset{①}{\partial_\rho \partial^\sigma g_{\mu\sigma}} + \overset{②}{\partial_\mu \partial^\nu g_{\rho\nu}} - \overset{③}{\partial^\sigma \partial_\sigma g_{\rho\mu}} - \overset{④}{g^{\nu\sigma} \partial_\mu \partial_\rho g_{\nu\sigma}}\right. +$$

$$+ \overset{⑤}{g^{\nu\sigma} g_{\kappa\lambda} \left(\Gamma^\kappa_{\mu\sigma} \Gamma^\lambda_{\nu\rho}\right)} - \overset{⑥}{g^{\nu\sigma} g_{\kappa\lambda} \Gamma^\kappa_{\mu\rho} \Gamma^\lambda_{\nu\sigma}}$$

① + ② is obviously symmetric

③ is symmetric becouse $g_{\rho\mu} = g_{\mu\rho}$

④ is symmetric becouse $\partial_\mu \partial_\rho = \partial_\rho \partial_\mu$

⑥ is symmetric becouse $\Gamma^\kappa_{\mu\rho} = \Gamma^\kappa_{\rho\mu}$

⑤:

$$g^{\nu\sigma} g_{\kappa\lambda} \left(\Gamma^\kappa_{\mu\sigma} \Gamma^\lambda_{\nu\rho}\right) = \frac{1}{2}\left(\overset{5.1}{g^{\nu\sigma} g_{\kappa\lambda} \Gamma^\kappa_{\mu\sigma} \Gamma^\lambda_{\nu\rho}} + \overset{5.2}{g^{\nu\sigma} g_{\kappa\lambda} \Gamma^\kappa_{\mu\sigma} \Gamma^\lambda_{\nu\rho}}\right)$$

5.2 : Let's change dummy indices: $\nu \leftrightarrow \sigma$, $\kappa \leftrightarrow \lambda$

and use symmetry of metric tensor:

5.2 : $g^{\nu\sigma} g_{\kappa\lambda} \Gamma^\lambda_{\mu\nu} \Gamma^\kappa_{\sigma\rho}$

$$\Rightarrow g^{\nu\sigma} g_{\kappa\lambda} \left(\Gamma^\kappa_{\mu\sigma} \Gamma^\lambda_{\nu\rho}\right) = \frac{1}{2} g^{\nu\sigma} g_{\kappa\lambda} \left(\Gamma^\kappa_{\mu\sigma} \Gamma^\lambda_{\nu\rho} + \Gamma^\lambda_{\mu\nu} \Gamma^\kappa_{\sigma\rho}\right) \overset{\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}}{=}$$

$$= \frac{1}{2} g^{\nu\sigma} g_{\kappa\lambda} \left(\Gamma^\kappa_{\mu\sigma} \Gamma^\lambda_{\nu\rho} + \Gamma^\lambda_{\nu\mu} \Gamma^\kappa_{\rho\sigma}\right) \leftarrow \text{symmetric}$$

■

$$ds^2 = f(r)c^2dt^2 - \frac{1}{f(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

1) since $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ we can derive components of metric tensor:

$g_{rr} = -\frac{1}{f(r)}$ $\quad g_{tt} = f(r)\cdot c^2$

$g_{\theta\theta} = -r^2$ $\quad g_{\varphi\varphi} = -r^2\sin^2\theta$

~~all~~ others are 0.

2) $L = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$ $\quad \frac{dA}{d\lambda} = \dot{A}$

$$L = g_{rr}\cdot(\dot{r})^2 + g_{tt}\cdot(\dot{t})^2 + g_{\theta\theta}\cdot(\dot{\theta})^2 + g_{\varphi\varphi}\cdot(\dot{\varphi})^2 =$$

$$= -\frac{1}{f(r)}\cdot(\dot{r})^2 + f(r)\cdot c^2\cdot(\dot{t})^2 - r^2\cdot\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\varphi}^2$$

$\theta = \frac{\pi}{2} \Rightarrow \sin\theta = 1, \quad \dot{\theta} = 0$

$$\Rightarrow L = f(r)\cdot c^2\cdot(\dot{t})^2 - \frac{(\dot{r})^2}{f(r)} - r^2\cdot(\dot{\varphi})^2$$

In case of massive particle we choose $\lambda = \tau$ (proper time)

3) $L = f(r) \cdot c^2 \dot{t}^2 - \frac{\dot{r}^2}{f(r)} - r^2 \dot{\varphi}^2$

$$\frac{\partial L}{\partial x^\alpha} - \frac{d}{d\lambda}\left[\frac{\partial L}{\partial(\partial x^\alpha / \partial \lambda)} \right] = 0$$

For "t": $0 - 2 f(r) \cdot c^2 \ddot{t} = 0$

$\Rightarrow f(r) \cdot c^2 \cdot \dot{t} = \text{const} = E \leftarrow$ energy (3.1)

(for $r \to \infty$ $f(r) \to 1$, $\dot{t} = \gamma \Rightarrow E(R \to \infty) = \gamma c^2$)

for „φ" $0 - 2r^2 \ddot{\varphi} = 0$

$\Rightarrow r^2 \dot{\varphi} = \text{const} = L_z \leftarrow$ angular momentum (3.2)

So, here L, E, L_z are conserved ~~other~~ quantities

4) We can substitute $\dot{\varphi}$ and $\dot{t}$ in L using (3.1) and (3.2)

$$\Rightarrow L = -\frac{\dot{r}^2}{f(r)} + f(r) \cdot c^2 \cdot \frac{E^2}{f^2(r) \cdot c^4} - r^2 \cdot \frac{L_z^2}{r^4} = kc^2 \quad (\times -1$$

$$\Rightarrow \frac{\dot{r}^2}{f(r)} + \frac{L_z^2}{r^2} - \frac{E^2}{f(r) \cdot c^2} = -kc^2$$

~~5) In~~

5) massless particle $\Rightarrow k=0$

$$\frac{\dot{r}^2}{f(r)} + \frac{L_z^2}{r^2} - \frac{E^2}{c^2 f(r)} = 0 \qquad (5.0)$$

In point C $(r=r_c)$ $\quad \dot{r} = \frac{dr}{d\lambda} = \underbrace{\frac{dr}{d\varphi}}_{=0} \cdot \frac{d\varphi}{d\lambda} = 0$

$$\Rightarrow \frac{L_z^2}{r_c^2} - \frac{E^2}{c^2 f(r_c)} = 0 \qquad (5.1)$$

$$\dot{r} = \frac{dr}{d\lambda} = \frac{dr}{dt} \cdot \dot{t} \overset{(3.1)}{=} \frac{dr}{dt} \cdot \frac{E}{f(r) \cdot c^2} \qquad (5.2)$$

In (5.0) substitute $\dot{r}$ using (5.2) and L_z^2 using (5.1):

$$\frac{1}{f(r)} \cdot \left(\frac{dr}{dt}\right)^2 \cdot \frac{\cancel{E^2}}{f^2(r) \cdot c^4} + \frac{\cancel{E^2}}{c^2 \cdot f(r_c)} \cdot \frac{r_c^2}{r^2} - \frac{\cancel{E^2}}{f(r) \cdot c^2} = 0$$

$$\Rightarrow \frac{1}{f^3(r)} \cdot \left(\frac{dr}{dt}\right)^2 + \frac{c^2}{f(r_c)} \cdot \frac{r_c^2}{r^2} - \frac{c^2}{f(r)} = 0$$

6)

$$\frac{dr}{dt} = \sqrt{\left(\frac{c^2}{f(r)} - \frac{c^2}{f(r_c)} \cdot \frac{r_c^2}{r^2}\right) f^3(r)} \cdot (\pm 1) =$$

$$= \pm c \sqrt{\left(1 - \frac{f(r)}{f(r_c)} \cdot \frac{r_c^2}{r^2}\right) f^2(r)}$$

in this case $\frac{dr}{dt} < 0$

$\Rightarrow$ we choose „−"

$$\Rightarrow t_{AC} = \int_{t_A}^{t_C} dt = -\frac{1}{c} \int_{r_A}^{r_C} \frac{dr}{\sqrt{\left(1 - \frac{f(r)}{f(r_c)} \frac{r_c^2}{r^2}\right) f^2(r)}} =$$

$$= \frac{1}{c} \int_{r_C}^{r_A} \frac{dr}{\sqrt{\left(1 - \frac{f(r)}{f(r_c)} \cdot \frac{r_c^2}{r^2}\right) f^2(r)}} \qquad \blacksquare$$

7) In order to use this approximation we need to have $\varepsilon, \delta << 1$. Let's evaluate them:

$$\varepsilon = \frac{r_s}{r_c} \quad , \quad r_c > R_\odot \text{ (radius of sun)}$$

$$\Rightarrow \varepsilon < \frac{r_s}{R_\odot} \approx \frac{3\,km}{7 \cdot 10^5\,km} << 1$$

$$\delta = \frac{r_s}{r} \quad , \quad r \geq r_c$$

$$\Rightarrow \delta \leq \frac{r_s}{r_c} = \varepsilon << 1$$

$\Rightarrow$ Yes, we can use this approximation

8) $$\frac{1}{\sqrt{\left(1 - \frac{f(r)}{f(r_c)}\frac{r_c^2}{r^2}\right) f^2(r)}} = \frac{r}{\sqrt{r^2 - r_c^2}}\left[1 + \frac{r_s}{r} + \frac{r_s r_c}{2r(r+r_c)}\right] =$$

$$\frac{r}{\sqrt{r^2-r_c^2}} + \frac{r_s}{\sqrt{r^2-r_c^2}} + \frac{\cancel{r}\, r_s r_c}{2\cancel{r}(r+r_c)\sqrt{r^2-r_c^2}} = \underbrace{\frac{r/r_c}{\sqrt{\left(\frac{r}{r_c}\right)^2 - 1}}}_{1} + \underbrace{\frac{r_s}{r_c\sqrt{\left(\frac{r}{r_c}\right)^2 - 1}}}_{2}$$

$$+ \underbrace{\frac{r_s r_c}{2r_c^2\left(\frac{r}{r_c}+1\right)\sqrt{\left(\frac{r}{r_c}\right)^2 - 1}}}_{3}$$

Let's separately calculate $\int_{r_c}^{r_A} dr$ for ①, ②, ③

①: $\int_{r_c}^{r_A} dr \cdot \frac{r/r_c}{\sqrt{\left(\frac{r}{r_c}\right)^2 - 1}} = \left\{ \frac{r}{r_c} = x \right\} =$

$= r_c \int_1^{r_A/r_c} \frac{x\,dx}{\sqrt{x^2-1}} = r_c \sqrt{x^2-1} \Big|_1^{r_A/r_c} = \sqrt{r_A^2 - r_c^2}$

②:
$\int_{r_c}^{r_A} dr \cdot \frac{r_s}{r_c} \frac{1}{\sqrt{\frac{r^2}{r_c^2} - 1}} = \left\{ \frac{r}{r_c} = x \right\} =$

$= r_s \int_1^{r_A/r_c} \frac{dx}{\sqrt{x^2-1}} = r_s \left[\ln \frac{x + \sqrt{x^2-1}}{1} \right] \Big|_1^{r_A/r_c} =$

$= r_s \left[\ln \frac{r_A + \sqrt{r_A^2 - r_c^2}}{r_c} \right]$

③:
$\int_{r_c}^{r_A} dr \cdot \frac{r_s}{2r_c} \cdot \frac{1}{\left(\frac{r}{r_c}+1\right)\sqrt{\left(\frac{r}{r_c}\right)^2 - 1}} = \left\{ \frac{r}{r_c} = x \right\} =$

$= \frac{r_s}{2} \sqrt{\frac{x-1}{x+1}} \Big|_1^{r_A/r_c} = \frac{r_s}{2} \sqrt{\frac{r_A - r_c}{r_A + r_c}}$

Finally

$$t_{AC} = \frac{1}{c} \int_{r_c}^{r_A} \frac{dr}{\sqrt{\left(1 - \frac{f(r)}{f(r_c)} \frac{r_c^2}{r^2}\right) f^2(r)}} \simeq \frac{1}{c} \Bigg[\sqrt{r_A^2 - r_c^2} \; +$$

$$+ \; r_s \ln \left[\frac{r_A + \sqrt{r_A^2 - r_c^2}}{r_c} \right] + \frac{r_s}{2} \sqrt{\frac{r_A - r_c}{r_A + r_c}} \Bigg] \quad (*)$$

In absence of relativistic effects we would get

$$t_{AC}^{\text{non-relativistic}} = \frac{L_{AC}}{c} = \frac{\sqrt{r_A^2 - r_c^2}}{c},$$

that we can get ~~[illegible]~~ from $(*)$ if $r_s = 0$

9) We can easily get t_{CB} by replacing $r_A \leftrightarrow r_B$

And $t_{total} = 2 \cdot t_{AB} = 2 \cdot (t_{AC} + t_{CB})$

Let's evaluate maximal time delay on AB

$$t_{AB} - t_{AB}^{\text{non-relativistic}} = \frac{r_s}{c}\left(\ln\left[\frac{r_A + \sqrt{r_A^2 - r_C^2}}{r_C} \right] + \frac{1}{2}\sqrt{\frac{r_A - r_C}{r_A + r_C}} \right) = \delta t_{AC}$$

This is a monotonically decreasing function of r_C

$\Rightarrow$ It has a maximum at $r_C = r_C^{min} = R_\odot \ll r_A, r_B$

$\Rightarrow$ We can use Taylor expansion (consider the 1^{st} non-zero term)

$$\ln\left[\frac{r_A + \sqrt{r_A^2 - R^2}}{R} \right] \simeq \ln\left[\frac{2r_A - \frac{R^2}{2r_A}}{R} \right] \simeq \ln\left[\frac{2r_A}{R} \right]$$

$$\sqrt{\frac{r_A - R}{r_A + R}} \simeq 1 - \frac{R}{r_A} \simeq 1$$

$$\Rightarrow \delta t_{AC} \simeq \frac{r_s}{c}\left[\ln\left[\frac{2r_A}{R_\odot} \right] + \frac{1}{2} \right]$$ (same for δt_{CB} $(r_A \leftrightarrow r_B)$)

$$\Rightarrow \delta t_{max} = 2(\delta t_{AC} + \delta t_{CB}) = \frac{2r_s}{c}\left[1 + \ln\left(\frac{2r_A}{R}\right) + \ln\left(\frac{2r_B}{R}\right) \right] =$$

$$= \frac{2r_s}{c}\left[1 + \ln\left(\frac{4r_A r_B}{R^2} \right) \right] \underset{\uparrow}{=} \frac{4G_N M_\odot}{c^3}\left[1 + \ln\left(\frac{4r_A r_B}{R^2} \right) \right]$$

$$r_s = \frac{2G_N M_\odot}{c^2}$$