Algorithms: Assignment #1

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目 录

Problem 1

(1)

Prove that $2n + \Theta(n^2) = \Theta(n^2)$.

Proof:
Let $f(n) = \Theta(n^2)$. By definition, there exists positive constants c_1, c_2 such that

$$c_1 n^2 \leq f(n) \leq c_2 n^2$$

Let $g(n) = f(n) + 2n$. We have

$$c_1 n^2 + 2n \leq g(n) \leq c_2 n^2 + 2n$$

For all $n > \frac{2}{c_2}$, $2n < c_2 n^2$. $c_2 n^2 + 2n < 2c_2 n^2$. Meanwhile, $c_1 n^2 + 2n > c_1 n^2$,

$$c_1 n^2 \leq g(n) \leq 2c_2 n^2$$

Choosing $c_1' = c_1, c_2' = 2c_2$, by definition we have $g(n) = \Theta(n^2)$. QED.

(2)

Prove that $\Theta(g(n)) \cup o(g(n)) = \emptyset$.

Proof:
Assume that there exists an f such that $f(n) \in \Theta(g(n)) \cup o(g(n))$.
Since $f(n) \in \Theta(g(n))$, there exists positive constants c_1, c_2, n_0 such that for all $n > n_0$,

$$c_1 g(n) \leq f(n) \leq c_2 g(n)$$

Since $f(n) \in o(g(n))$, for all $c > 0$, there exists $n_1 > 0$, such that for all $n > n_1$, $0 \leq f(n) < cg(n)$. Let c be c_1, we have

$$f(n) < c_1 g(n)$$

Let $n_2 = \max(n_0, n_1)$. For all $n > n_2$, we have both $f(n) < c_1 g(n)$ and $f(n) \geq c_1 g(n)$ in contradiction.
So $\Theta(g(n)) \cup o(g(n)) = \emptyset$. QED.

(3)

Prove that $\Theta(g(n)) \cap o(g(n)) \neq O(g(n))$.

Proof:
Let $g(n) = n^2$, $f(n) = n^(1 + \sin n)$. For all $n > 0$, $0 < f(n) \leq g(n)$, so $f(n) \in O(g(n))$.
However, $\forall k \in \mathbb{N}^*, f(k\pi + \frac{3}{2}\pi) = 0$. Thus, there does not exist such $c > 0$ that $\forall n > n_0, cn^2 < f(n)$, meaning that $f(n) \notin \Theta(g(n))$.
Meanwhile, $\forall n = k\pi + \frac{1}{2}\pi (k \in \mathbb{N}^*), f(n) = g(n)$. Thus, $\forall n_0 > 0$, there exists n s.t. $f(n) = g(n)$. Thus, $f(n) \notin o(g(n))$.
QED.

(4)

Prove that $\max((f(n), g(n))) = \Theta(f(n) + g(n))$.

Problem 1 [(4)] continued on next page...

Proof:
Let $h(n) = \max((f(n), g(n)))$.

$$\frac{1}{2}(f(n) + g(n)) \leq h(n) \leq f(n) + g(n)$$

Choosing $c_1 = \frac{1}{2}, c_2 = 1$, by definition we have $h(n) = \Theta(f(n) + g(n))$. QED.

(5)

Solve the recurrence $T(n) = 2T(\sqrt{n}) + 1$.

Let $m = \log n$, then $T(2^m) = 2T(2^{\frac{m}{2}}) + 1$.
Let $S(m) = T(2^m)$, then $S(m) = 2S(\frac{m}{2}) + 1$.
We have $a = 2, b = 2, F(m) = m^0$, and we have that $m^{\log_b a} = m^1$.
Since m^1 is polynomially larger than m, applying case 1 in the Master Theorem,

$$S(m) = \Theta(m)$$

Thus,

$$T(n) = \Theta(\log n)$$

QED.

(6)

Solve the recurrence $nT(n) = (n-2)T(n-1) + 2$.

Multiply by $(n-1)$,

$$n(n-1)T(n) = (n-1)(n-2)T(n-1) + 2(n-1)$$

Let $g(n) = n(n-1)T(n)$. Thus,

$$g(n) = g(n-1) + 2(n-1)$$

$$g(n) = \sum_{i=1}^{n} 2(n-1) + g(1) = n(n-1) + g(1)$$

$$T(n) = \frac{g(n)}{n(n-1)} = 1 + \frac{g(1)}{n(n-1)} = \Theta(1)$$

QED.

Problem 2

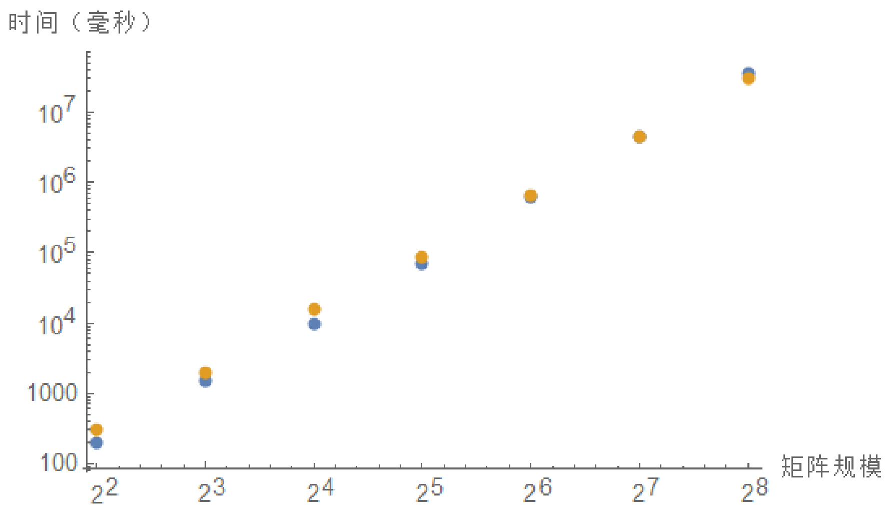
Compare naive matrix multiplication and the Strassen's algorithm.

Problem 2 continued on next page...

Both implemented in an recursive way in C++. To be compiled with Visual Studio 2012.
For code and run-time screenshots, please refer to the attachments.
Results[a] are as follows:

CPU Time Consumed (microseconds)		
Size of matrix	Naive	Strassen's
1	0	0
2	0	100
4	200	300
8	1500	2000
16	9600	16100
32	69600	88500
64	609200	643500
128	4418300	4537000
256	35934400	30109700

Seems that the cross-over point is somewhere between 128 and 256.
A double-logarithmic chart visualizing the data above:



Perfect linearity, verifying polynomial complexity.

[a] on an Intel Core i5 4120U Processor, 4GB DDR3 Memory, Windows 10 Pro