Algorithms: Assignment #2

Due on March 11, 2016

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Contents

Problem 1

Solve planer case of the Closest pair of points problem, in $\Theta(n \lg n)$ time.

Implemantetion

Both an $\Theta(n \lg n)$ divide-and-conquer algorithm and an $\Theta(n^2)$ brute-force algorithm are implemented, in C++.
For code, please refer to `/code/Closest-PairProblem/`.
Correctness is verified by running both of the implemented algorithms against randomly generated sets of points, and checking if their results contradicts. I got 100% correctness.
Test[a] suggests that for $n = 10^6$, the $\Theta(n \lg n)$ algorithm takes less than 1 seconds, while the $\Theta(n^2)$ algorithm seems to take forever.

[a] on an Intel Core i5 4120U Processor, 4GB DDR3 Memory, Windows 10 Pro. Hereinafter the same.

Interactive GUI

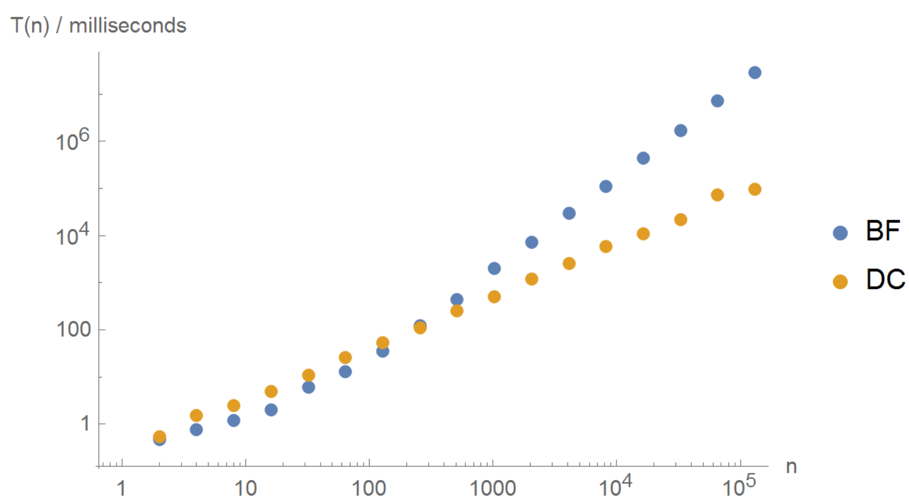
Written in C# WPF, calling pre-compiled DLL where the algorithm is implemented in C++. To be built with Visual Studio 2012.
For code generating the DLL, see `/code/Library`. For code building GUI, see `/code/Gui`.
Please execute `/code/Release/Gui.exe` in Windows to bring up the GUI. Click to add a point. To clear the canvas, restart the program.

Timing Comparison

CPU Time Consumed (microseconds)

Size of the set	$\Theta(n \lg n)$	$\Theta(n^2)$
2	0.54	0.47
4	1.53	0.76
8	2.43	1.18
16	5	2
32	11	6
64	26	13
128	54	35
256	111	123
512	253	440
1024	500	2000
2048	1200	7200
4096	2600	30400
8192	5900	110300
16384	11000	438000
32768	22000	1.72e6
65536	73000	7.41e6
131072	96000	2.9e7
...	...	...
1048576	919000	?

Seems that the cross-over point is somewhere between 128 and 256.
A double-logarithmic chart visualizing the data above:



Problem 2

CLRS Ex 5.3-5:
Prove that in the array P in procedure PERMUTE-BY-SORTING, the probability that all elements are unique is at least $1 - \frac{1}{n}$.

Proof:
Let A = {all elements are unique}.

$$n(\Omega) = (n^3)^n$$

$$n(A) = A_{n^3}^n$$

Applying classical probability model, possibility of A

$$\begin{aligned}
P(A) &= \frac{n(A)}{n(\Omega)} \\
&= \frac{A_{n^3}^n}{(n^3)^n} \\
&= \frac{(n^3)!}{(n^3-n)!n^{3n}} \\
&= \prod_{i=0}^{n-1} \frac{n^3-i}{n^3} \\
&> \prod_{i=0}^{n-1} \frac{n^3-n}{n^3} \\
&= \prod_{i=0}^{n-1} \left(1-\frac{1}{n^2}\right) \\
&= \left(1-\frac{1}{n^2}\right)^n
\end{aligned}$$

Applying binomial theorem where $a = 1, b = -\frac{1}{n^2}$,

$$\begin{aligned}
P(A) &> \left(1-\frac{1}{n^2}\right)^n \\
&= \sum_{i=0}^{n} C_n^i \left(-\frac{1}{n^2}\right)^i \\
&= 1-\frac{1}{n}+\sum_{i=2}^{n} C_n^i \left(-\frac{1}{n^2}\right)^i \\
&\geq 1-\frac{1}{n}+\sum_{j=1}^{\lfloor\frac{n-1}{2}\rfloor} C_n^{2j} \left(\left(-\frac{1}{n^2}\right)^{2j}+\left(-\frac{1}{n^2}\right)^{2j+1}\right) \\
&= 1-\frac{1}{n}+\sum_{j=1}^{\lfloor\frac{n-1}{2}\rfloor} C_n^{2j} \left(\frac{1}{n^{4j}}-\frac{1}{n^{4j+2}}\right) \\
&\geq 1-\frac{1}{n}
\end{aligned}$$

QED.