Inverse Kinematics I

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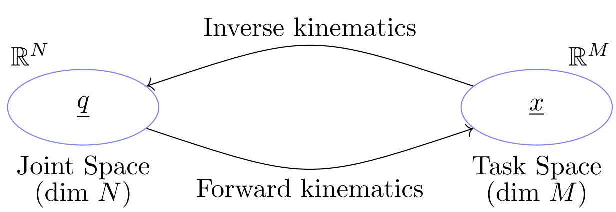
1 Overview

Inverse kinematics is the problem of starting with a desired end effector pose ${}^0_N T$ and finding the joint angles $\underline{q}$ that cause the end effector to assume that pose. It is often the case that you will have not a single desired pose but a Cartesian curve or trajectory ${}^0_N T(t)$ and will want to find a trajectory in joint space $\underline{q}(t)$ that causes the end effector to follow the trajectory.

Note that while describing translational trajectories is pretty straightforward, using cubic splines or something equivalent, describing trajectories in rotation is a bit more complicated. It isn't that hard to use Euler of fixed angles, as long as you avoid the orientations where these descriptions become singular, but the fact that they do become singular at certain orientations means that 4–value descriptions such as quaternions is normally how it's done in practice. The math required to describe trajectories using quaternions is a bit beyond this course. For the following we'll assume that you have a single desired end effector pose.



Formally, the problem is

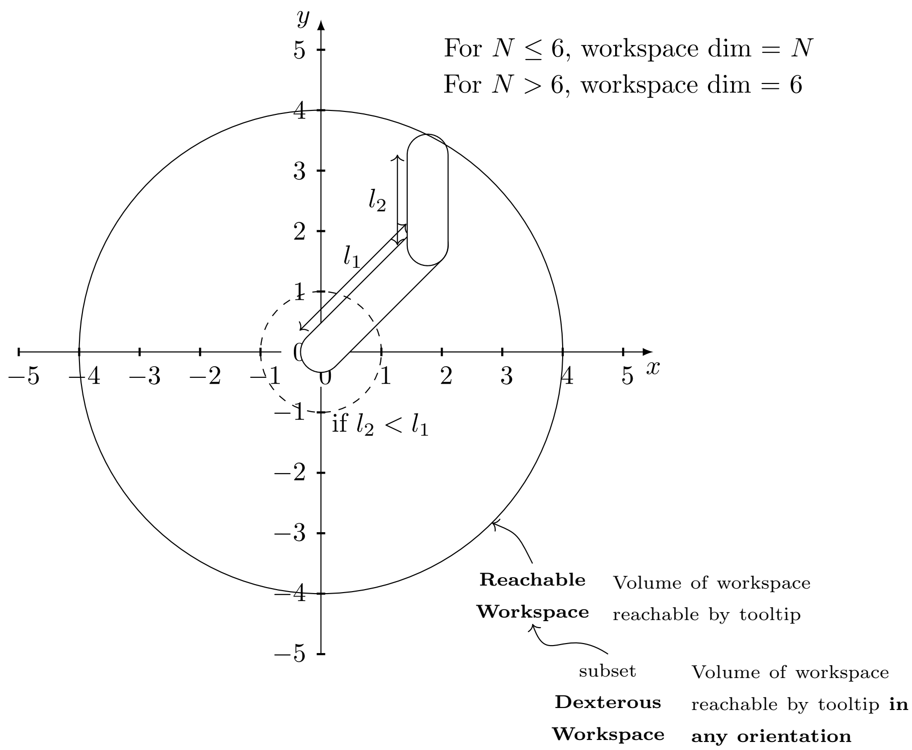
$$\begin{aligned} \text{GIVEN:} &\quad {}^0_N T \\ \text{FIND:} &\quad \underline{q} \end{aligned}$$

where

$${}^0_N R,\ {}^0\underline{p}_N \Rightarrow \begin{cases} \text{6 independent equations} \\ \text{nonlinear} \\ \text{transcendental} \end{cases}$$

2 Workspace

The *workspace* is the Cartesian position and orientation subspace spanned by the manipulator, e.g. the set of all $\{x, y, z, \phi, \theta, \psi\}$ such that $\exists\ \underline{q} : {}^B_T T(\underline{q}) = {}^B_T T(x, y, z, \phi, \theta, \psi)$:



3 Closed–form Solutions

In some case you can find a closed–form solution for the inverse kinematics of a manipulator.

3.1 Example: 3–link planar manipulator

For this particular manipulator we can find a close–form solution for $\theta_1, \theta_2, \theta_3$. In fact there are two ways to go about it, an algebraic approach and a geometric approach. You can see the details of the geometric approach in section 4.4 of the textbook. I'll do the algebraic one here.

Set ${}^0_3T = {}^0_W T_{goal} \cdots$

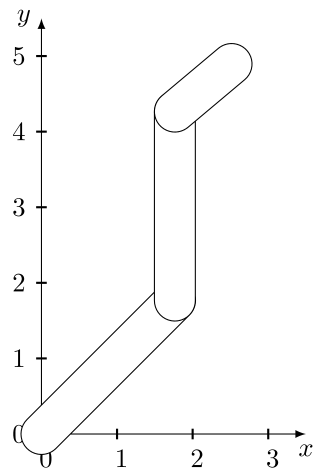


Figure 1: Three–link planar manipulator

$$
{}^0_3T = \left[\begin{array}{ccc|c} c_{123} & -s_{123} & 0 & l_1c_1 + l_2c_{12} \\ s_{123} & c_{123} & 0 & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

and

$$
{}^B_WT = \left[\begin{array}{ccc|c} c_\phi & -s_\phi & 0 & x \\ s_\phi & c_\phi & 0 & y \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

So we can, by inspection, get four equations that have to be solved for $\theta_1, \theta_2, \theta_3$:

$$
\begin{aligned}
c_\phi &= c_{123} && (1) \\
s_\phi &= s_{123} && (2) \\
x &= l_1c_1 + l_2c_{12} && (3) \\
y &= l_1s_1 + l_2s_{12} && (4)
\end{aligned}
$$

We can square 3 and 4 and add them:

$$
x^2 + y^2 = l_1^2 + l_2^2 + 2l_1l_2c_2
$$

and solve for c_2:

$$
c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}
$$

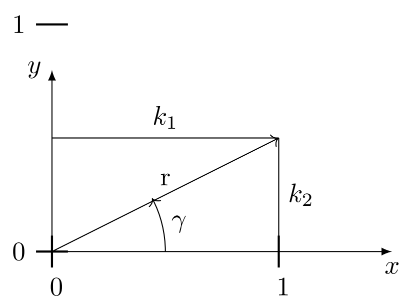
You can then get $s_2 = \pm\sqrt{1 - c_2^2}$ and then $\theta_2 = \text{atan2}(s_2, c_2)$.

We can then solve for θ_1. This is a bit involved, but it's a pattern that occurs a lot in inverse kinematics.

First, substitute $k_1 = l_1 + l_2 c_2$ and $k_2 = l_2 s_2$ into Eqs. 1 and 2. This makes the structure of the equations a bit clearer:

$$x = k_1 c_1 - k_2 s_1 \tag{5}$$
$$y = k_1 s_1 + k_2 c_1 \tag{6}$$

Now we can use a change of variables. Specifically, we're going to change from Cartesian to polar coordinates, as shown in the diagram:



Define $r = \sqrt{k_1^2 + k_2^2}$ and $\gamma = \text{atan2}(k_2, k_1)$. Then

$$k_1 = r\cos\gamma \tag{7}$$
$$k_2 = r\sin\gamma \tag{8}$$

We can now write Eqs. 5 and 6 as

$$\frac{x}{r} = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1$$
$$\frac{y}{r} = \cos\gamma\sin\theta_1 + \sin\gamma\cos\theta_1$$

or

$$\cos\gamma + \theta_1 = \frac{x}{r} \tag{9}$$
$$\sin\gamma + \theta_1 = \frac{y}{r} \tag{10}$$

We can then find

$$\gamma + \theta_1 = \text{atan2}(\frac{y}{r}, \frac{x}{r}) = \text{atan2}(y, x)$$

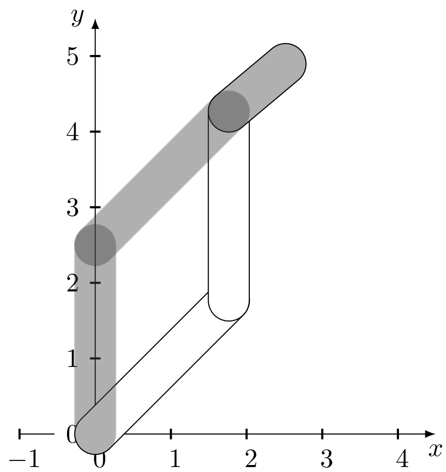
Therefore

$$\theta_1 = \text{atan2}(y, x) - \text{atan2}(k_2, k_1)$$

And then using Eqs. 1 and 2 you can solve for θ_3:

$$\theta_1 + \theta_2 + \theta_3 = \text{atan2}(s_\phi, c_\phi) = \phi \rightarrow \theta_3 = \phi - \theta_1 - \theta_2 \tag{11}$$

Note that there are in fact two solutions to the inverse kinematics problem in this case, as illustrated in the diagram below:



And you can also see this by virtue of the fact that in the derivation above we chose $s_2 = \pm\sqrt{1 - c_2^2}$. You get to choose the sign — one choice corresponds to the elbow–up configuration and the other to the elbow–down configuration.

Closed–form solutions are great when you can find them. They are numerically stable (they always return exactly the same result, to the limit of the precision of the computer you are using) and can be calculated quickly. Unfortunately, in general there is no straightforward way to find a closed–form expression for the kinematics of a given robotic manipulator. There isn't even a guarantee that a closed–form solution exists.

3.2 Pieper's Method

There is, however, one important case in which a closed–form solution can be found: when

- you have a 6 DOF arm; and
- three consecutive axes intersect at a point (usually the first three or the last three).

This requirement means that the problem can be decomposed into two successive 3–DOF problems. By far the most common case is a revolute manipulator where the last three rotational axes intersect; this is sometimes known as a *spherical wrist*. It implies that the point of intersection is at the origin of all three frames.

This technique is called "Pieper's Method".

The key observation is that ${}^0\underline{p}_4 = {}^0\underline{p}_5 = {}^0\underline{p}_6$.

Steps:

- Solve for ${}^0\underline{p}_4$:

$$ {}^0\underline{p}_T = {}^0\underline{p}_4 + {}^0_4R\ \cancelto{0}{{}^4\underline{p}_6} + \underbrace{{}^0_6R}_{={}^0_TR}\ \underbrace{{}^6\underline{p}_T}_{\text{given}} $$

- Solve for θ_1, θ_2, and θ_3:

$$ {}^{0}\underline{p}_4 = {}^{0}_{1}T(\theta_1)\, {}^{1}_{2}T(\theta_2)\, {}^{2}_{3}T(\theta_3)\, \underbrace{{}^{3}\underline{p}_4}_{\text{given}} $$

  Note that this gives three equations in three unknowns.

- Solve for θ_4, θ_5, and θ_6:

$$ {}^{0}_{T}R = \underbrace{{}^{0}_{3}R}_{\substack{\text{for} \\ \theta_1 - \theta_3}}\ \underbrace{{}^{3}_{6}R}_{\substack{\text{for} \\ \theta_4 - \theta_6}}\ \underbrace{{}^{6}_{T}R}_{=I} $$

$$ {}^{3}_{6}R = {}^{0}_{3}R(\theta_1, \theta_2, \theta_3)^T\, {}^{0}_{T}R $$

  Note that this also gives three equations in three unknowns.