Modelling of Stiffness

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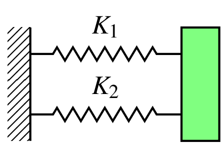
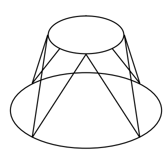
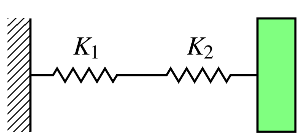
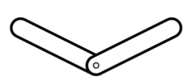
1 Intro

We’re going to talk about how to model the flexibility of a joint. Real mechanisms are never infinitely stiff, although with many robotic mechanisms the stiffness is high enough that we can mostly ignore it, or at least design suitable control laws such that we don’t excite flex modes. In some cases, however, that isn’t possible. Very long robot arms, such as the SRMS and SSRMS, have nontrivial flex dynamics that need to be modeled and managed, either via appropriate control laws or via operational procedures. Also, there is emerging research in soft robotics where the flexibility of a manipulator is in fact one of its defining features.

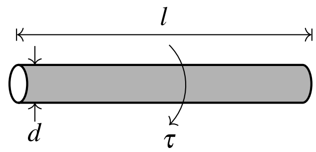
That being said, this isn’t going to be a lecture so much as it is a summary of the types of flexible structures and mechanisms you might encounter in the wild and accepted equations for modeling them. We aren’t going to derive or even really motivate these models; the modeling of deformable structures is at least an entire graduate–level class.

One introductory note: if you have two or more structural elements in combination, you can find the flexibility of the assembly as follows.

Type	Diagram	Example	Equation
Parallel		 Stiff	$K_{\text{par}} = K_1 + K_2$
Series		 Not stiff	$\frac{1}{K_{\text{ser}}} = \frac{1}{K_1} + \frac{1}{K_2}$

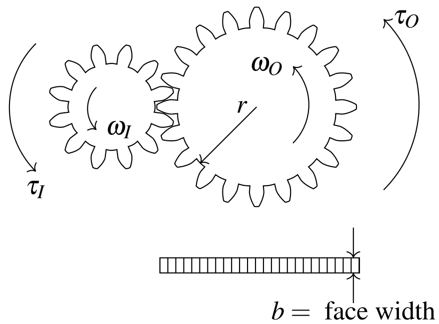
2 Flexible Elements

Shaft

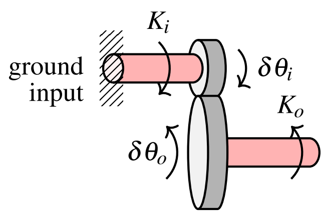


$K = \frac{G\pi d^4}{32l}$
where G is the "shear modulus"
(steel = 7.5×10^{10} N/m^2)

Gear



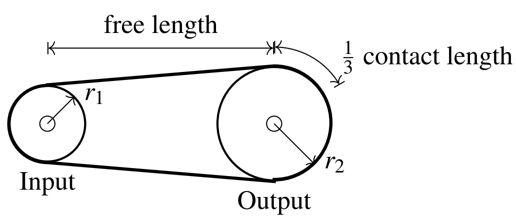
$$
\begin{aligned}
r &= \text{radius of output gear} \\
\eta &= \text{gear ratio} \\
\omega_O &= \frac{\omega_I}{\eta} \\
\tau_O &= \tau_I \eta \\
K &= C_g b r^2 \text{ (locked input gear)} \\
C_g &= 1.34 \times 10^{10} \text{ N/m}^2 \text{ (steel)}
\end{aligned}
$$



Gear reduction increases stiffness
by square of gear ratio

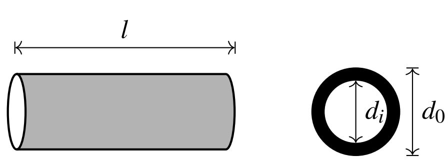
$$
\begin{aligned}
\tau_i &= K_i \delta\theta_i \\
\tau_o &= K_o \delta\theta_o \\
K_o &= \frac{\tau_o}{\delta\theta_o} = \frac{\eta \tau_i}{\delta\theta_i / \eta} \\
&= \frac{\eta K_i \cancel{\delta\theta_i}}{\cancel{\delta\theta_i}/\eta} = \eta^2 K_i
\end{aligned}
$$

Belt

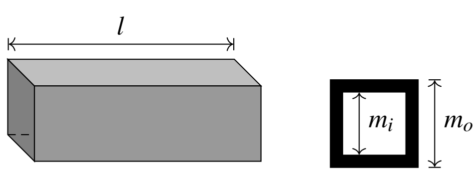


$$
\begin{aligned}
K &= \frac{AE}{l} \\
A &= \text{cross sectional area of belt} \\
E &= \text{modulus of elasticity} \\
l &= \text{free length} + \frac{1}{3} \text{ contact length}
\end{aligned}
$$

Link



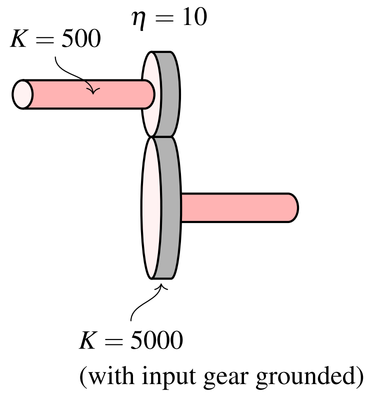
$$
\begin{aligned}
K &= \frac{3\pi E (d_o^4 - d_i^4)}{64 l^3} \\
E &= 2 \times 10^{11} \text{ N/m}^2 \text{ (steel)} \\
&= \frac{2}{3} \times 10^{11} \text{ N/m}^2 \text{ (aluminum)}
\end{aligned}
$$



$$
K = \frac{E (w_o^4 - w_i^4)}{4 l^3}
$$

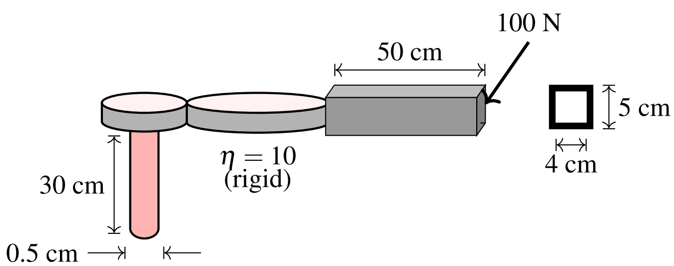
3 Example: Gear Stiffness

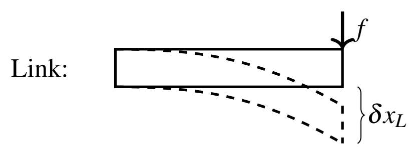
What is the output stiffness of the drive system below?



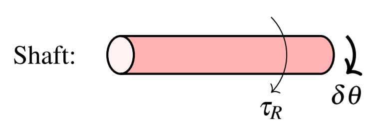
$$
\begin{aligned}
\frac{1}{K_{sys}} &= \frac{1}{K_{gear}} + \frac{1}{K_{shaft}} \\
&= \underbrace{\frac{1}{500}}_{\text{at output}} + \underbrace{\frac{1}{(10)^2(500)}}_{50000} \\
K_{sys} &= 4545 \;\; \text{N-m/rad}
\end{aligned}
$$

4 Example





$\delta x_L = F/K_L$



$$
\begin{aligned}
\delta\theta_{out} &= \tau_R / K_{S_{out}} \\
\delta x_S &= L\,\delta\theta_{out}
\end{aligned}
$$

$$K_{shaft} = \frac{G\pi d^4}{32l} = \frac{(7.5\times 16^{10})(\pi)(5\times 10^{-3})^4}{32(0.3)} = 15.3 \text{ N-m/rad} \rightarrow K_{S_{out}} = 15.3(10)^2 = 1530 \text{ N/m}^2$$

$$K_{link} = \frac{E(w_o^4 - w_i^4)}{4l^3} = \frac{(2\times 10^{11})(0.05^4 - 0.04^4)}{4(0.5)^3} = 1.476\times 10^{-5} \text{ N/m}$$

Link: $\delta x_L = 100 \text{ N}/1.476\times 10^6 \text{ N-m} = 6.78\times 10^{-5}$ m
Shaft: $\delta x_S = (0.50 \text{ m})(0.5 \text{ m}\times 100 \text{ N})/1530 \text{ N/m}^2 = 0.0163$ m

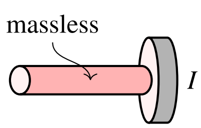
5 Estimating ω_r

$$\omega_n = \sqrt{\frac{k}{m}}$$

stiffness

effective mass or inertia

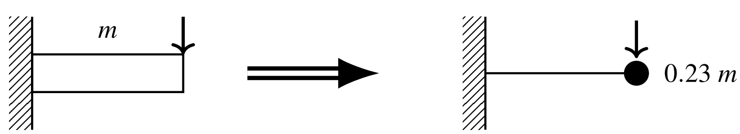
5.1 Example: Shaft



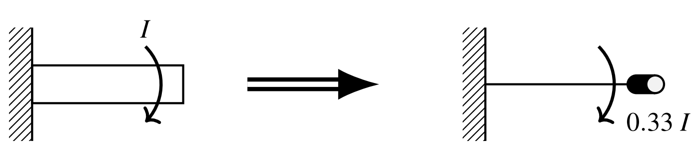
$$\begin{aligned} k &= 400 \text{ N-m/rad} \\ I &= 1 \text{ kg-m}^2 \\ \omega_r &= \sqrt{\frac{400}{m}} = 20 \text{ rad/sec} = 3.2 \text{ Hz} \end{aligned}$$

5.2 Lumped mass model

LATERAL



TORSIONAL



5.3 Example: 9.9: Link

$$\begin{aligned} m &= 4.347 \text{ kg} \\ k &= 3600 \text{ N/m} \end{aligned}$$

What is ω_r?
Assume mass is distributed. Then,

$$\begin{aligned} m_{\text{eff}} &= 0.23(4.345 \text{ kg}) = 1 \text{ kg} \\ \omega_r &= \sqrt{\frac{k}{m_{\text{eff}}}} = \sqrt{\frac{3600}{1}} = 60 \text{ rad/sec} = 9.6 \text{ Hz} \end{aligned}$$