Spacecraft Thruster Control Allocation Problems

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***Abstract*—We consider the control allocation problem in a spacecraft thruster configuration. It consists on the determination of the force command to be sent to each thruster in order to point the total torque and/or thrust vectors. Here, we state four possible practical problems and propose a control allocation algorithm based on a subgradient optimization, which is shown to be faster than existing allocators.**

***Index Terms*—Control allocation, spacecraft, thruster.**

I. INTRODUCTION

Spacecraft control systems usually require multiple redundant actuators capable of achieving particular objectives for one or more control loops and stages under consideration. The task which links the control law with the particular commands to be sent to each individual actuator in the configuration is called *control allocation*. Here we consider the spacecraft control allocation problem when the actuator is a set of thrusters, and for the following controller requirements:

- **Case** T_A: Exact pointing of the torque vector in $\mathbb{R}^3$, used for attitude control. The thrust vector requirement determines two subcases: 1) Maximum thrust projection in a certain direction, considered in [1] and [2], which is useful for the attitude and orbit control subsystem (AOCS); and 2) Torque vector pointing is independent of the thrust vector (free), which is useful for the attitude control subsystem (ACS), and has been considered in [3].
- **Case** T_B: Exact thrust vector pointing in $\mathbb{R}^3$. The torque modulus is minimized. Useful for the orbit control subsystem (OCS).
- **Case** T_C: Exact torque vector pointing in $\mathbb{R}^3$, with exact thrust vector pointing in a cone of feasible directions. Useful for AOCS.
- **Case** T_D: Exact torque and thrust vectors pointing in any direction of $\mathbb{R}^6$. Useful for AOCS in certain applications (see [4]) in which attitude changes must be independent of position control.

The control allocator returns the force magnitude to be commanded to each of the n thrusters in order to achieve each desired torque and/or thrust vector pointing. A special case stated in [5], called *direct* control allocation, preserves the direction of an m-dimensional physical vector to be pointed, being optimal the allocation which attains the maximum feasible modulus. Based on [3], it can be proved that the optimal direct control allocation has desirable robustness properties. A direct control allocator based in facet search was proposed in [5] for $m = 3$. A faster version, the bisecting edge search algorithm, was presented in [6], but its implementation did not guarantee its efficiency for $m > 3$, as shown in [7]. In [8], the direct control allocation is stated as a linear programming problem in n variables. All these implementations assume independent bounds for each command and were proved efficient for $n \gg m$. When considering $m > 3$, no hierarchy is assumed between the partitions of the complete vector of physical moments.

These considerations do not necessarily apply to spacecraft thruster control applications. Therefore, we found convenient to state the direct control allocation as a continuous selection from a certain set-valued map determined by the desired vector pointing and the actuator constraints, which is presented in Section II-B. Our implementation of this selection is based on a subgradient algorithm proposed in Section II-C, and is compared with previous allocators in Sections II-D and E.

A. Notation

We assume a configuration of n thrusters located at certain positions and orientations with respect to the spacecraft's frame, fixed at its center of mass. When considering each of the problems stated, it will be useful to define a configuration matrix where the ith actuator is described by the column vector $\underline{x}^i \in \mathbb{R}^m$ and a scalar $F_i \geq 0$, for $i = 1, \ldots, n$. The particular control action made by this actuator is $Z^i = \underline{x}^i F_i$. For each case, the configuration matrix will be $X = [\underline{x}^1, \ldots, \underline{x}^n] \in \mathbb{R}^{m\times n}$ and the command vector $F = [F_1, \ldots, F_n]^T \in \mathcal{F}$, where $\mathcal{F} \subset \{F \in \mathbb{R}^n : F \geq \underline{0}\}$ is the set of feasible forces. The vectors $\underline{x}^i$ will depend on the particular control action to be considered (torque and/or thrust). The resulting control action $Z \in \mathbb{R}^m$ is $Z = XF$.

B. Controllability, Failures, and Robustness

The controllability restriction to be satisfied by the configuration of actuators is as follows: for a given set of feasible command values $\mathcal{F}$, the set $X\mathcal{F}$ must contain vectors in all directions of $\mathbb{R}^m$. To obtain a necessary condition, assume an unbounded set $\mathcal{F} = \{F \in \mathbb{R}^n : F \geq \underline{0}\}$.

Lemma I.1: ([1]) Given a matrix $X \in \mathbb{R}^{m\times n}$, the following conditions are equivalent.

1) For all $Z \in \mathbb{R}^m$, there exists $F \geq \underline{0}$ such that $XF = Z$.
2) X is full rank and its kernel has sign definite vectors, i.e., $\exists \underline{w} > \underline{0}$ such that $X\underline{w} = \underline{0}$.

For $n = m + 1$, $\underline{w}$ may be computed as follows.

Lemma I.2: [2] Given a matrix $X \in \mathbb{R}^{m\times(m+1)}$ that is full rank, the vector $\underline{w} \in \mathbb{R}^{m+1}$, $w_i = (-1)^i|\tilde{X}_i|$, generates $\ker(X)$.

As the configuration matrix after the ith actuator failure is $\tilde{X}_i$, the next lemma is useful.

Lemma I.3: [1], [2] There is no matrix $X \in \mathbb{R}^{m\times(m+2)}$, such that all the $\tilde{X}_i$ have full rank and $\ker(\tilde{X}_i)$ has sign definite vectors.

With these results, we consider the controllability in each case. The previous lemmas can be used to determine the minimum feasible number of thrusters for a configuration that does not support failures (n_0), and for a configuration that supports a single point of failure (n_1).

- **Cases** T_A **and** T_B: The vector $s \in \mathbb{R}^3$ (torque or thrust) has to be pointed, and there is complete freedom to determine the configuration matrices (A or B). For $m = 3$, we obtain $n_0 = 4$ and $n_1 = 6$ from previous results (see [1]).
- **Case** T_C: It is required to point the torque vector and two components of the thrust vector. If the cone of thrusts is around the z axis, it is necessary to point the vector $s = [T_x, T_y, T_z, U_x, U_y]^T$, which has $m = 5$ and we obtain $n_0 = 6$ (which may be found from Lemma I.2) and $n_1 = 8$ ($n_1 = 7$ is not possible due to Lemma I.3). In this case, we *impose* the positiveness of the elements in the third row of B to guarantee a positive thrust projection onto the z axis for all $F > \underline{0}$. Note that the rank of $[A^T\ B^T]^T$ must be six because we may choose three linearly independent thrusts for each torque.

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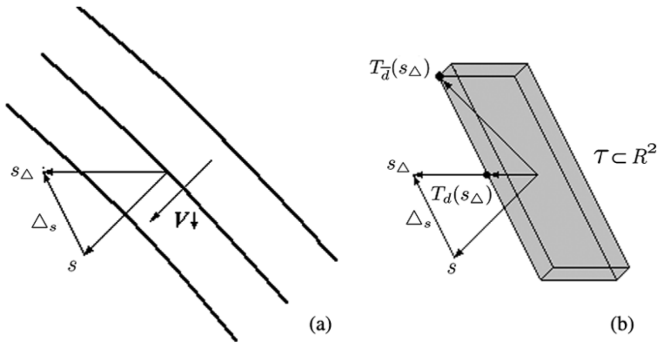

Fig. 1. (a) Level surfaces of the *clf* V. The allocator input is the direction of $s_\triangle$. (b) The optimal direct allocator generates a torque $T_d(s_\triangle)$ more robust under uncertainties $\triangle_s$ than the torque $T_{\overline{d}}(s_\triangle)$ obtained by the optimal nondirect allocator.

- **Case** T_D: We need to point the vector $s = [T_x, T_y, T_z, U_x, U_y, U_z]^T$, which has $m = 6$ and, hence, by similar arguments as in the previous case, we obtain $n_0 = 7$. There are T_D configurations that support failures with $n_1 = 10$.

For each vector to be pointed, assume that there is a control-affine subsystem which has to be asymptotically stabilized, and an associated smooth control Lyapunov function V (or *clf* in [9]) that has to be decreased. Without loss of generality, consider the case T_A for a particular maneuver. Let s be a vector perpendicular to the level surface of the smooth *clf* and pointing toward its decreasing values, as shown in Fig. 1(a). This vector has to be computed using model parameters and measured state variables, hence it will have an error. Let $s_\triangle = s + \triangle_s$ be the uncertain vector computed by the control law.

A *direct* allocator $F_d(s_\triangle)$ produces a torque $T_d(s_\triangle) = AF_d(s_\triangle)$ that preserves the direction of $s_\triangle$. We may define the optimal direct allocator for $s_\triangle \neq \underline{0}$ as the continuous force selection function $\mathrm{sel}_{\mathcal{F}}(s_\triangle) := F_d(s_\triangle)$ such that $T_d(s_\triangle) = A\,\mathrm{sel}_{\mathcal{F}}(s_\triangle)$ is the torque with maximum feasible modulus in the direction of $s_\triangle$. The optimal nondirect allocator may be defined as the forces $F_{\overline{d}}(s_\triangle)$ such that the resulting torque $T_{\overline{d}}(s_\triangle) = AF_{\overline{d}}(s_\triangle)$ attains the maximum feasible projection onto the direction computed by the control law $s_\triangle$.

Without uncertainties, i.e., $\triangle_s = \underline{0}$, the optimal nondirect allocation produces the maximum feasible decrement of the *clf*. However, when considering the uncertainty in s, Fig. 1(b) shows for a simplified torque set in $\mathbb{R}^2$ that the optimal direct allocator is better than the optimal nondirect one. This is because under the same uncertainty $\triangle_s$ the latter leads to a resulting torque perpendicular to the desired direction, which means no decrease of the associated *clf*.

II. Direct Control Allocation as a Continuous Selection

A. Set Valued Maps and Continuous Selections

A set valued map $\mathcal{M} : S_a \rightsquigarrow S_b$, also called a multifunction, is a map from one nonempty set S_a to subsets of another nonempty set S_b, both assumed finite dimensional. A selection from a set-valued map $\mathcal{M}$ is a single valued map m such that $m(x) \in \mathcal{M}(x)$ for all $x \in S_a$.

The set K is a cone if $x \in K$ implies $\alpha x \in K$ for all $\alpha \in \mathbb{R}$. If $x^1, x^2 \in K$ implies $x^1 + x^2 \in K$, then K is a convex cone. The interior of a set S and its boundary are denoted as $\mathrm{int}(S)$ and ∂S, respectively. S is absorbing if $\underline{0} \in \mathrm{int}(S)$. The set $S_a \setminus S_b$ contains the elements in S_a, not in S_b.

A convex bounded absorbing set K defines a distance to the origin:

$$d_K(x) := \inf\{a > 0 : x \in aK\} \tag{1}$$

called the Minkowski functional. We define for a given convex bounded absorbing set $K \subset S$ the *vector signum* $\mathrm{sgn}_K(x)$ such that $d_K(x)\,\mathrm{sgn}_K(x) = x$ for all $x \in S$. The minimal selection of $\mathcal{M}(x)$ with the distance d_K is

$$m(x) = \arg\min\{d_K(z) : z \in \mathcal{M}(x)\} \tag{2}$$

where the vector $z \in \mathcal{M}(x)$ that realizes the minimum of $d_K(z)$ is *assumed* unique for all x. In this case, if $\mathcal{M}(x)$ is continuous and takes closed convex values, $m(x)$ is continuous. This is proved in [10, Prop. 2.19], where the previous uniqueness assumption is guaranteed by taking K such that d_K is a norm induced by an inner product. The same result holds for a general convex bounded absorbing K if we assume $\mathcal{M}$ to be such that

$$\{v \in \mathcal{M}(x) : d_K(v) = \min\{d_K(z) : z \in \mathcal{M}(x)\}\} \tag{3}$$

has a single element for each $x \in S_a$.

B. Optimal Direct Control Allocation

Let S be the set of all possible desired vectors to be pointed. The function $\psi(s, F) : S \times \mathbb{R}^n \to \mathbb{R}$ determines the objective to be maximized, which is linear in F; for instance $\psi(s, F) = s^T XF$.

The set of feasible forces $\mathcal{F} \subset \mathbb{R}^n$ is defined as

$$\mathcal{F} = K^\ell \cap K^u \tag{4}$$

where K^ℓ and K^u are convex sets that specify the lower and upper bounds of $\mathcal{F}$. In particular, if $0 \le F^\ell < F^u \in \mathbb{R}$ are the lower and upper bounds of each F_i, we define the set of forces $\mathcal{F} = \mathcal{F}^{\ell u} = K_1^\ell \cap K_1^u$ with $K_1^\ell := \{F \in \mathbb{R}^n : F_i \ge F^\ell \ge 0,\ i = 1, \ldots, n\}$ and $K_1^u := \{F \in \mathbb{R}^n : \|F\|_\infty \le F^u\}$. In general, K^u should be absorbing and compact; while $K^\ell \subseteq \{F \in \mathbb{R}^n : F \ge \underline{0}\}$ should be such that all its supporting planes have normal vectors $\check{n}$ that satisfy $\check{n} \le \underline{0}$. This last requirement on K^ℓ guarantees that every element in $\mathcal{F}$ may be scaled to obtain an element in ∂K^u, which assures that the optimal forces vector lies on ∂K^u due to the linearity of $\psi(s, F)$ on F.

Let $\mathcal{F}_\infty \subset \mathbb{R}^n$ be the convex cone of forces, obtained by scaling $\mathcal{F}$

$$\mathcal{F}_\infty := \bigcup_{\alpha \ge 0} \alpha \mathcal{F}. \tag{5}$$

The continuous vector function $L : S \to \mathbb{R}^n$ is such that for all $s \in S$, the physical vectors determined by $Z(s) = XL(s)$ preserve the desired directions. As $L(s)$ does not necessarily belong to $\mathcal{F}$, we require a linear set valued map $\mathcal{W}(s)$ such that for all $\underline{w} \in \mathcal{W}(s)$, $Z(s) = X(L(s) + \underline{w})$ also fulfills the pointing requirements, the objective function verifies $\psi(s, L(s) + \underline{w}) = \psi(s, L(s))$, and

$$\mathrm{int}(\mathcal{F}) \cap \mathcal{W}(s) \neq \emptyset \qquad \forall\, s \in S \tag{6}$$

which means that a forces vector in $\mathcal{F}$ may achieve the desired physical directions. Note that the latter is equivalent to $\mathrm{int}(\mathcal{F}_\infty) \cap \mathcal{W}(s) \neq \emptyset$, which is a specific version of the requirement in Lemma I.1.

The set-valued map $\mathcal{M} : S \rightsquigarrow \mathbb{R}^n$ is defined as

$$\mathcal{M}(s) = \{L(s) + \underline{w} : \underline{w} \in \mathcal{W}(s)\} \cap \mathcal{F}_\infty \tag{7}$$

which takes closed convex values and is continuous on S. Let $m(s)$ be the minimal selection of $\mathcal{M}(s)$ in the sense of d_{K^u}, defined as in (2). Also, condition (3) must be assumed.

Assumption II.1: For each $s \in S$ there exists a unique $z \in \mathcal{M}(s)$ that minimizes $d_{K^u}(z)$.

The uniqueness and continuity condition ona the command vector is necessary for actuators which can not operate in a discontinuous way. This is not the case of thrusters under pulse modulation firing (PMF), but even in this case this condition is used to guarantee a continuous convergence of the trajectories generated by PMF (the command vector is translated into duty cycles) toward those generated by continuous forces (see [3]).

The control allocator may be simplified under the following assumption.

Assumption II.2: For all $z \in \mathrm{int}(K^\ell) \cap \partial K^u$ and $s \in S$ there exist vectors $\underline{w}^0 \in \mathcal{W}(s)$ such that $z - \underline{w}^0 \in \mathrm{int}(\mathcal{F})$.

Lemma II.1: Under Assumption II.1, the optimal direct control allocation is realized for all $s \in S \setminus \{\underline{0}\}$, a given $L(s)$ and $\psi(s, F)$, by the continuous selection of forces $\mathrm{sel}_\mathcal{F}(s) = \mathrm{sgn}_{K^u}(m(s))$. Under Assumption II.2, $\mathrm{sel}_\mathcal{F}(s) \in \partial K^\ell \cap \partial K^u$ for $s \in S \setminus \{\underline{0}\}$.

Proof: Due to Assumption II.1, the minimal selection may be written as the intersection $m(s) = \mathcal{M}(s) \cap d_{K^u}(m(s))K^u$ which is continuous on S (see Section II-A). The optimal selection of forces for $\psi(s, F)$ and $L(s)$ may be obtained by scaling the minimal selection $m(s)$ in the sense of d_{K^u} by the largest value $a \in \mathbb{R}$ that guarantees a forces vector in $\mathcal{F}$

$$\begin{aligned}\mathrm{sel}_\mathcal{F}(s) &= \arg\max\{\psi(s, F) : F = am(s) \in \mathcal{F}\} \\ &= \frac{m(s)}{d_{K^u}(m(s))} = \mathrm{sgn}_{K^u}(m(s)).\end{aligned}$$

This is continuous on $S \setminus \{\underline{0}\}$, where $\mathrm{sel}_\mathcal{F}(s) \in \partial K^u \cap K^\ell$. As $m(\underline{0}) = \underline{0}$, it is discontinuous at $s = \underline{0}$. Under Assumption II.2, as for any $z \in \partial K^u \cap \mathrm{int}(K^\ell)$ we have $\psi(s, \mathrm{sgn}_{K^u}(z - \underline{w}^0)) > \psi(s, z)$, the optimum is not reached on $\mathrm{int}(K^\ell) \cap \partial K^u$, which implies $\mathrm{sel}_\mathcal{F}(s) \in \partial K^\ell \cap \partial K^u$ on $S \setminus \{\underline{0}\}$. □

C. Implementation

Next, we define the search coordinates to span $\mathcal{W}(s)$. Let $\{\underline{w}^i(s)\}$, $i = 1, \ldots, n_w \leq n - m$, be a basis of $\mathcal{W}(s)$ and $W(s) = [\underline{w}^1(s), \ldots, \underline{w}^{n_w}(s)]$

$$\mathcal{M}(s) = \{L(s) + W(s)\underline{\gamma} : \underline{\gamma} \in \mathbb{R}^{n_w}\} \bigcap \mathcal{F}_\infty.$$

The low redundancy of thruster configurations makes more attractive the search over the $n - m$ coordinates γ_i than over the n components of the command vector F. The search for the minimal selection $m(s)$ in terms of the coordinates γ_i, is a nonlinear programming problem. For $n_w = 1$ the minimal selection might not require a search procedure, as in the case considered in Section II-CI. The case $n_w = 1$ is obtained for configurations with $n = m + 1$ or when the set $\mathcal{W}(s)$ is previously partitioned in lower order sets to achieve a mixed optimization. When the search procedure cannot be eliminated, and more than one coordinate γ_i has to be determined simultaneously, the minimal selection will determine in general a nonsmooth optimization problem depending on the regularity of the function $d_{K^u}(\cdot)$. In nonsmooth optimization, the gradient of the objective function is replaced by the *subdifferential* set, composed by *subgradient* elements.

1) Particular Case $\mathcal{F} = \mathcal{F}^{\ell U}$: Consider the particular forces set $\mathcal{F}^{\ell u}$. The cone of forces is determined as

$$\mathcal{F}_\infty = \left\{F \in \mathbb{R}^n : \frac{F^\ell}{F^u} \leq \frac{\min_{i=1,\ldots n}\{F_i\}}{\max_{i=1,\ldots n}\{F_i\}}\right\}.$$

Condition (6) holds if $\forall\, s \in S$ and there is a vector $\underline{w} \in \mathrm{int}(\mathcal{F}_\infty) \cap \mathcal{W}(s)$, i.e,

$$\frac{F^\ell}{F^u} < \frac{\min_{i=1,\ldots n}\{w_i\}}{\max_{i=1,\ldots n}\{w_i\}}. \tag{8}$$

In this case, $m(s) = L(s) + \arg\min_{\underline{w} \in \mathcal{W}(s)} \max_{i=1,\ldots,n}\{(L(s))_i + \gamma(\underline{w}, s)w_i\}$ and, due to Assumption II.2, $\gamma(\underline{w}, s)$ satisfies

$$\frac{F^\ell}{F^u} = \frac{\min_{i=1,\ldots,n}\{(L(s))_i + \gamma(\underline{w}, s)w_i\}}{\max_{i=1,\ldots,n}\{(L(s))_i + \gamma(\underline{w}, s)w_i\}}.$$

The optimal selection is then $\mathrm{sel}_\mathcal{F}(s) = (m(s)/\|m(s)\|_\infty)F^u$. To obtain $\gamma(\underline{w}, s)$, note that for $F^\ell = 0$ and a feasible[1] $\underline{w}$

$$\gamma(\underline{w}, s) = -\min_{i=1,\ldots,n, w_i > 0}\left\{\frac{(L(s))_i}{w_i}\right\} \tag{9}$$

which may be used to determine the optimal selection for $F^\ell > 0$ in a second step. The torque objective $\psi(s, F)$ decreases as $F^\ell \to F^u$, being the maximum torque for $F^\ell = 0$.

The following result is useful to guarantee the uniqueness Assumption II.1 and to characterize the image of $\mathrm{sel}_\mathcal{F}(s)$ for $\mathcal{F} = \mathcal{F}^{\ell u}$.

Lemma II.2: Given $\mathcal{F} = \mathcal{F}^{\ell u}$, Assumption II.1 holds if all the $(n - m) \times (n - m)$ partitions of a full-rank matrix $W(s)$ such that $\mathrm{col}(W(s)) = \mathcal{W}(s)$ are nonsingular. Moreover, $\max_{i=1,\ldots,n} F_i^* = F^u$, $\min_{i=1,\ldots,n} F_i^* = F^\ell$ and $n - m + 1$ components take extremal values.

Proof: This is a simple generalization of [3, Lemma III.4]. □

Remark II.1: In the particular case, $L(s) = X^+ s$ and $\mathcal{W}(s) = \ker(X)$. Assumption II.1 is usually guaranteed (see [5] and [8]) by evaluating the nonsingularity of all the $m \times m$ partitions of X. Our previous lemma evaluates the nonsingularity of all the $(n - m) \times (n - m)$ partitions of W, but for this particular case both conditions are equivalent, which was proved in [3].

2) A Subgradient Implementation: Let $m(\underline{\gamma}, s) := L(s) + W(s)\underline{\gamma}$ and the function $\phi_s(\underline{\gamma})$ to be minimized

$$\phi_s(\underline{\gamma}) := d_{K^u}(m(\underline{\gamma}, s)) \tag{10}$$

on the convex set $\Gamma(s) = \{\underline{\gamma} \in \mathbb{R}^{n-m} | m(\underline{\gamma}, s) \in \mathcal{F}_\infty\}$. This is achieved by the vector $\underline{\gamma}^*$ such that $m(s) = m(\underline{\gamma}^*, s)$. For the set $\mathcal{F}^{lu}$, $\phi_s(\underline{\gamma}) = \max_{j=1,\ldots,n}\{(m(\underline{\gamma}, s))_j\}/F^u$.

Next, we prove that $\phi_s(\underline{\gamma})$ is a convex function when evaluated on $\Gamma(s)$. Take $\underline{\gamma}^1, \underline{\gamma}^2 \in \Gamma(s)$, hence for $0 < \lambda < 1$, $\underline{\gamma}^3 = \lambda\underline{\gamma}^1 + (1 - \lambda)\underline{\gamma}^2 \in \Gamma(s)$. Taking $m^i = m(\underline{\gamma}^i, s)$ for $i = 1, 2, 3$, we obtain $m^3 = \lambda m^1 + (1 - \lambda)m^2 \in \mathcal{F}_\infty$. It is a known result that the Minkowski functional d_{K^u} is convex, therefore $\phi_s(\underline{\gamma}^3) \leq \lambda\phi_s(\underline{\gamma}^1) + (1 - \lambda)\phi_s(\underline{\gamma}^2)$.

This convex property of ϕ_s on $\Gamma(s)$ guarantees the convergence of the subgradient optimization algorithm (see [11]). Let $\partial\phi_s(\underline{\gamma})$ be the subdifferential and $\zeta(\underline{\gamma}) \in \partial\phi_s(\underline{\gamma})$ a subgradient of ϕ_s at $\underline{\gamma}$. The subgradient algorithm is similar to the steepest descent but the elements of $\partial\phi_s(\underline{\gamma})$ take the place of the usual gradient $\nabla\phi_s(\underline{\gamma})$

$$\underline{\gamma}(k + 1) = \underline{\gamma}(k) - \lambda_k \frac{\zeta(\underline{\gamma}(k)}{\|\zeta(\underline{\gamma}(k)\|_2} \tag{11}$$

where $\lambda_k > 0$ defines the step size sequence. In our particular implementation, the step size λ_k is attenuated if $\phi_s(\underline{\gamma}(k))$ does not decrease, and is amplified when $\phi_s(\underline{\gamma}(k))$ decreases too slow. To guarantee that $\underline{\gamma} \in \Gamma(s)$ and to improve the search, we use the fact that, under Assumption II.2, the optimal $\underline{\gamma}^*$ must satisfy $\mathrm{sel}_\mathcal{F}(s) \in \partial K^\ell \cap \partial K^u$. For the set $\mathcal{F}^{\ell u}$ with $F^\ell = 0$, it is sufficient to replace $\underline{\gamma}$ by $\hat{\underline{\gamma}} = \gamma(W(s)\underline{\gamma}, s)\underline{\gamma}$ such that $\min_{i=1,\ldots,n} m_i(\underline{\gamma}, s) = 0$ [see (9)]. This implies $\hat{\underline{\gamma}} \in \partial\Gamma(s)$, which is convenient because $\underline{\gamma}^* \in \partial\Gamma(s)$.

D. Spacecraft Thruster Control Allocators

- **Cases** T_A, T_B, **and** T_D: All these cases use the Moore–Penrose pseudoinverse of the respective configuration matrix to define the vector function $L(s) = X^+ s$, where s is defined by the particular physical vectors to be pointed, the set $\mathcal{W}(s)$ is determined by the kernel of X, and the direction of s is preserved. The function to be maximized is $\psi(s, F) = s^T XF$.

Simulation results: We have tested the subgradient-based control allocator with a six-thruster configuration of type T_{A1}. To compute the subgradient, we have used a centered cell with six nodes and a fixed

[1] It should satisfy $L(s) + \gamma(\underline{w}, s)\underline{w} \geq \underline{0}$ for some $\gamma(\underline{w}, s) \in \mathbb{R}$.

number of iterations (eight). We have obtained, with the same precision bounds as in [6], a computing time 30% better (1724 versus 2229 flops) than the fast bisecting edge search allocator proposed in [6], and 4.7 times better (1724 versus 8084 flops) than the exact facet search algorithm in [5]. For these cases, the basis of $\ker(A)$ and the pseudoinverse of A are precomputed at a design stage (see [1] and [2]); otherwise, it should demand 327 and 156 flops more, respectively. Note also that the efficiency of the bisecting edge algorithm is not maintained for $m > 3$ (see [7]). The LP formulation for the direct control allocation in [8] was implemented using the solver $lp()$ of Matlab 5 for comparison.[2] Our algorithm resulted 3.7 times faster (1724 versus 6342 flops, the latter in average). Moreover, we have observed that similar precision bounds are achieved by taking a noncentered cell with four nodes, also for eight iterations, resulting 1340 flops (30% faster).

- **Case** T_C: Let s_τ be the unit vector that points in the same direction as the desired torque, anywhere in $\mathbb{R}^3$; and s_v the unit vector that points in the same direction as the desired thrust, and takes values in a cone around the z axis. The orthogonal matrix function $C(s)$ represents the rotation from the spacecraft frame to another spacecraft fixed frame whose z axis points toward the direction of the required thrust vector, and is continuous on S. Let $\overline{B}(s)$ be the matrix composed by the first two rows of $B(s) := C(s)B$ and $b_z(s)$ its third row. We assume that the configuration is designed in such a way that $b_z(s)F > 0$ for all $s \in S$ and $F \in \mathcal{F}$. As this holds for every thrust in a solid cone and torques in all directions in $\mathbb{R}^3$, $[A^T \ B^T]^T$ must have rank six . The forces vector $F \in \mathcal{F}$ performs a direct allocation if

$$AF = \psi(s,F)s_\tau \text{ and } \overline{B}(s)F = \underline{0} \tag{12}$$

where $\psi(s,F) = s_\tau^T AF$ is the objective to be maximized. The reason for this torque priority in the optimization is that the torque also determines, indirectly through the dynamics, the inertial orientation of the thrust cone. This case may be solved as before considering

$$L(s) = X^+(s)\begin{bmatrix} s_\tau \\ \underline{0} \end{bmatrix} \quad \mathcal{W}(s) = \ker(X(s))$$
$$X(s) = \begin{bmatrix} A \\ \overline{B}(s) \end{bmatrix}. \tag{13}$$

As $X(s)$ is full-row rank, its pseudoinverse is $X^+(s) = X^T(s)(X(s)X^T(s))^{-1}$. Only the first three columns of $X^+(s)$ are necessary, hence, the number of operations involved may be reduced by 2/5, i.e., 40%.

To compute a basis of $\mathcal{W}(s)$, note that $\ker(B(s)) \equiv \ker(B)$ due to the nonsingularity of $C(s)$; hence, $\ker(A) \cap \ker(B) \subset \ker(A) \cap \ker(\overline{B}(s))$. A basis $\{\underline{w}^1, \ldots, \underline{w}^{n-6}\}$ of $\ker(A) \cap \ker(B)$ may be precomputed, but an additional vector is needed to complete the basis of $\ker(A) \cap \ker(\overline{B}(s))$. It may be obtained by using Lemma I.2 to compute the kernel of

$$[A^T \overline{B}^T(s)\underline{w}^1 \cdots \underline{w}^{n-6}]^T \in \mathbb{R}^{(n-1)\times n}. \tag{14}$$

The following example presents some simulation results.

E. Example With Five Moments and Eight Actuators

An eight thruster configuration composed by two four-thruster configurations of type T_{A1} is usually considered for satellite AOCS. We prove that the simultaneous operation of the eight thrusters qualify as a configuration of type T_C which can not work as a T_C configuration after a failure. Finally, the proposed allocator is evaluated for the eight-thruster T_C configuration.

The locations are $d_x^i = r\cos(\theta_i)$, $d_y^i = r\sin(\theta_i)$ and $d_z^i = h$, while the orientations are $e_x^i = \sin(\eta)\sin(\beta_i)$, $e_y^i = \sin(\eta)\cos(\beta_i)$ and $e_z^i = \cos(\eta)$, where $\underline{\theta} = (\pi/4)[1\ 1\ 3\ 3\ 5\ 5\ 7\ 7]^T$, $\eta = -0.436\,33$, $h = -2.5r$, $r = 0.2$ and $\underline{\beta} = (\pi/4)[7\ 3\ 5\ 1\ 3\ 7\ 1\ 5]^T$. The thruster sets $(1,2,5,6)$, $(3,4,7,8)$, $(1,4,5,8)$ and $(2,3,6,7)$ may be used as configurations of type T_{A1} because their configuration matrices are full-rank and contain $\underline{w} = [1\ 1\ 1\ 1]^T$ in their kernel.

Now, we consider the eight thrusters as a T_C configuration, hence $X(s)$ and $\mathcal{W}(s)$ are given by (13). Suppose the thrust direction objective is the z axis. Vector $\underline{w}^* = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]^T$ does not generate torque but generates a thrust in the direction of the z axis, i.e., it fulfills requirement (6), while matrix $[A^T \ B^T]^T$ has rank six. Considering $\mathcal{F} = \mathcal{F}^{\ell u}$ with $F^\ell = 0$ and $F^u = 1$ we obtain a configuration of type T_C that fulfills the particular requirement (8). Excluding the first thruster, the resulting configuration matrix $\tilde{X}_1$ has a kernel generated by the following vectors: $\underline{w}^1 = [-2\ -1\ 1\ -2\ -1\ 0\ 1]^T$ and $\underline{w}^2 = [0\ 1\ 1\ 0\ 1\ 0\ 1]^T$, which shares a zero in the sixth column. However, the $(1,2,3,4,5,7)$th columns of $[\tilde{A}_1^T \ \tilde{B}_1^T]^T$ have rank five and, hence, the requirements for a T_C configuration are not fulfilled. Hence, this eight-thruster configuration, as T_C, does not support one failure,[3] although it could still work as T_{A1}.

This configuration does not satisfy the condition in Lemma II.2[4] to guarantee the uniqueness Assumption II.1, because there are two similar sets of four thruster subconfigurations, which determines singular 3×3 partitions on $W(s)$. However, uniqueness may be achieved by selecting, for a given s, a convenient basis $\{\underline{w}^i(s)\}$ of $\mathcal{W}(s)$ taking into account $L(s)$. In particular, we take a basis composed by $\underline{w}^1(s) = [1\ 1\ 0\ 0\ 1\ 1\ 0\ 0]^T$, $\underline{w}^2(s) = [1\ 0\ 1\ 0\ 1\ 0\ 1\ 0]^T$ and $\underline{w}^3(s) = [0\ 1\ 0\ 1\ 0\ 1\ 0\ 1]^T$ if $\{i : (L(s))_i = \max(L(s))\} \subset \{1,2,5,6\}$; otherwise, we take $\underline{w}^1(s) = [0\ 0\ 1\ 1\ 0\ 0\ 1\ 1]^T$. Using the subgradient optimization algorithm in [11] with eight iterations (fixed) and computing the subgradient with a centered cell of six nodes, we obtain a worst case error lower than 3% for the thrust and 1% for the torque, similar to the results in [6] and [8]. In the particular case where the thrust direction objective is fixed in body frame, $L(s)$ and a basis of $\mathcal{W}(s)$ may be precomputed, and the allocation would demand only 1954 flops, associated to eight search iterations over $\mathcal{W}(s)$. The thrust modulus for all required torque directions is between 3.55–3.65 N. The torque set is shown in Fig. 2(a).

In [8], a linear programming approach is proposed to implement the direct control allocation. The approach may be directly applied to consider the case T_C by taking a five-dimensional vector objective composed by the three elements of the torque vector and two elements at zero. Our matrix X should be taken as the matrix CB in [8]. We used the Matlab function $lp()$, which was started with a feasible suboptimal solution (explained later) and using the torque/thrust objectives as before. The Matlab flop count measured 14 010 flops in average. Therefore, our algorithm is seven times faster than the linear programming approach.

When considering the pointing of the thrust vector inside a solid cone around the z axis, the pseudoinverse of $X(s)$ and the basis of $\mathcal{W}(s)$ have to be computed, as explained in Section II-D. Hence, our algorithm would demand 3285 flops, which is 4.25 times faster than the LP implementation.

[2]This LP library function is not optimized for a "real-time" implementation, and the Matlab $flops()$ output should be taken as an estimate.

[3]We have found eight-thruster T_C configurations which support a failure, but with a very low thrust modulus.

[4]This problem also appears in [8] for the tailless C-17 aircraft.

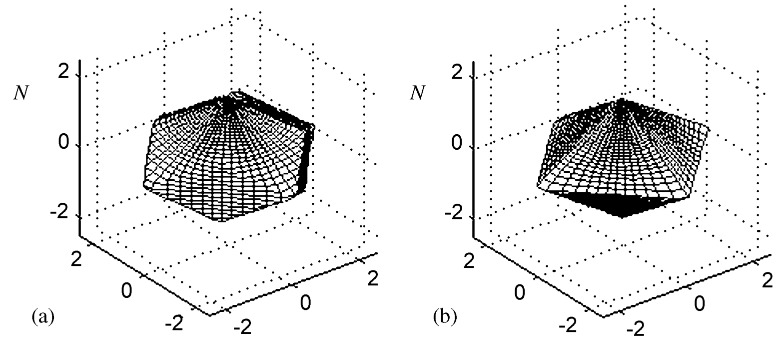

Fig. 2. Eight thrusters T_C configuration example. (a) Attainable torque set of the torque optimal selection. (b) Attainable torque set of the torque suboptimal selection.

TABLE I
NOTATION

$\times$	cross product in $\mathbb{R}^3$
$\ker(X), \mathrm{col}(X)$	kernel and column subspaces of $X \in \mathbb{R}^{m\times n}$
$\tilde{X}_i$	X without its i-th column
$X\mathcal{H}$	$\{Xh \,:\, h \in \mathcal{H}\}$, for a set $\mathcal{H}$
F_i	force magnitude of i-th thruster
$F = [F_1, \ldots, F_n]^T$	forces vector in $\mathbb{R}^n$
$T = AF$	torque vector in body frame
$U = BF$	thrust vector in body frame
A	$= [\underline{a}^1, \ldots, \underline{a}^n] \in \mathbb{R}^{3\times n}$
B	$= [\underline{e}^1, \ldots, \underline{e}^n] \in \mathbb{R}^{3\times n}$
$\underline{a}^i$	$= \underline{d}^i \times \underline{e}^i$
$\underline{e}^i$	unit vector along the i–thruster force axis in body frame
$\underline{d}^i$	i–thruster position in body frame

A simpler, faster, but torque-suboptimal selection is obtained if only one sign positive vector $\underline{w}(s) \in \mathcal{W}(s)$ is used. Considering the unit vector $\check{z}$ as the thrust objective and $\underline{w} = [1\;1\;1\;1\;1\;1\;1\;1]^T$, this allocator produces a torque that is 14% lower than the previous one [see Fig. 2(b)] with a constant thrust of 3.63 N. It demands only 32 flops to determine $\gamma(s, \underline{w})$ in (9).

Let $\mathrm{sel}^0_{\mathcal{F}}(s)$ be the previous torque-suboptimal selection, which takes the optimal selection over the partition $\mathcal{W}^0(s) := \mathrm{span}\{\underline{w}\}$. Taking as secondary objective the thrust, define

$$\mathrm{sel}^-_{\mathcal{F}}(s) = \arg\min_{F\in\mathcal{F}} \left\{ \underline{b}_z(s)F \mid F = \mathrm{sel}^0_{\mathcal{F}}(s) + \underline{w} \right.$$
$$\left. \underline{w} \in \mathcal{W}(s), w_{i\in I_p} = 0 \right\}$$
$$\mathrm{sel}^+_{\mathcal{F}}(s) = \arg\max_{F\in\mathcal{F}} \left\{ \underline{b}_z(s)F \mid F = \mathrm{sel}^0_{\mathcal{F}}(s) + \underline{w} \right.$$
$$\left. \underline{w} \in \mathcal{W}(s), w_{i\in I_p} = 0 \right\}$$

where I_p is the index set where $\mathrm{sel}^0(\cdot)$ achieves its extremal values. In particular, the implementation for $n = 8$ only requires to solve two linear equations. The average thrust modulus was computed, being 4.3 N for $\mathrm{sel}^+_{\mathcal{F}}(s)$ and 3.1 N for $\mathrm{sel}^-_{\mathcal{F}}(s)$. We might also define $\mathrm{sel}_{\mathcal{F}}(s,\lambda) := \lambda\ \mathrm{sel}^+_{\mathcal{F}}(s) + (1-\lambda)\ \mathrm{sel}^-_{\mathcal{F}}(s)$, to determine a thrust modulus regulation.

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