Title: Global Economy Equilibrium Scenario Builder

Global Economy Equilibrium Scenario Builder

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Technical requirements

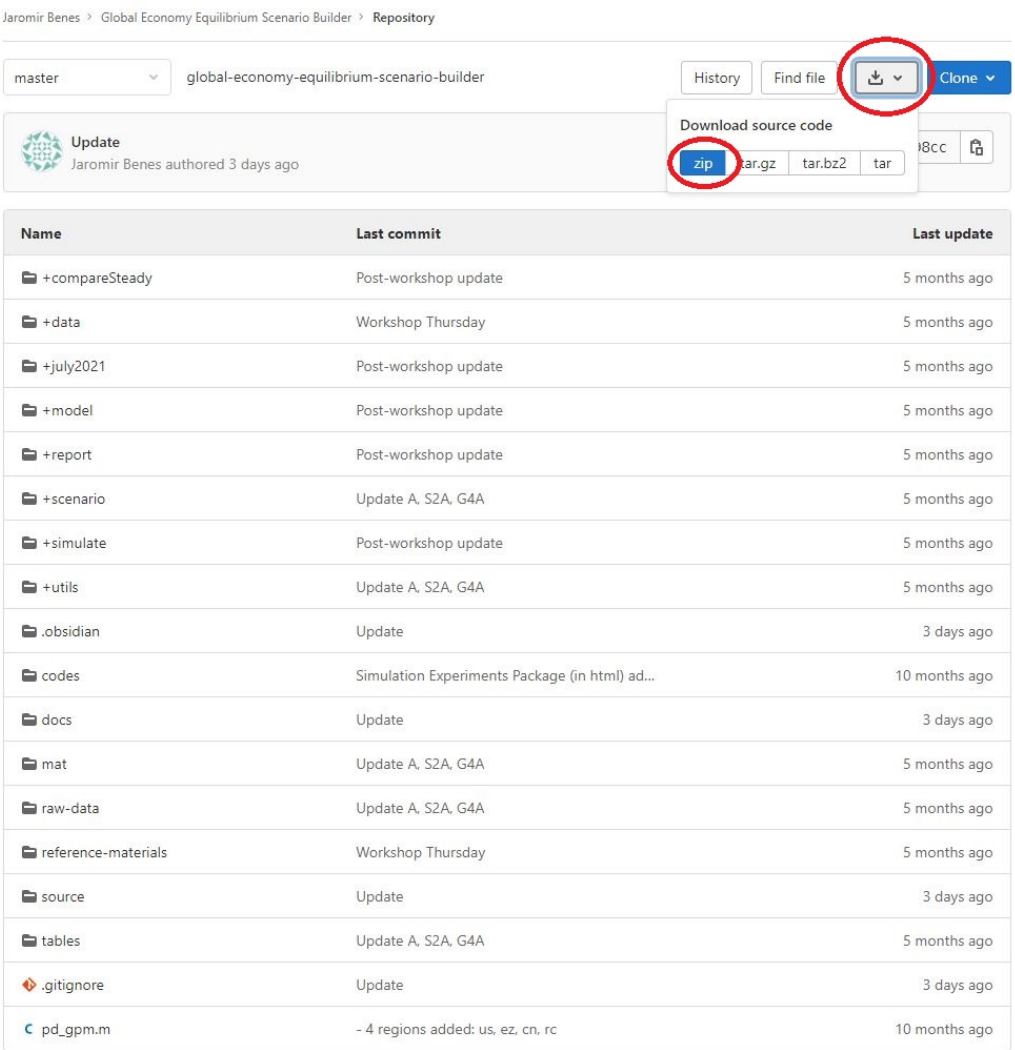
GEES Model folders are available for download **here**.

GEES Model runs on **Matlab version 2019b** or higher.

Iris Toolbox required for the GEES Model is available for download **here**.

How to download model files

Visit Github website repository with model files. Here click on the item **Download** and the option *zip*.



Alternatively, you can download the GEES Model files from the official GPM Network **website**.

Infrastructure of GEES Model

Structure of Model Folders

Project folder	Description
`source/`	Model source files (text files written in IrisT model language)
`docs/`	Documentation folder
`+model/`	Subpackage folder to contain different variants/setups of the model
`+model/xxx`	Subpackage to create and store variant xxx of the model
`+simulate`	Subpackage folder to contain simulation exercises
`+simulate/xxx`	Subpackage to contain simulation and reporting files for exercise xxx
`+compareSteady`	Subpackage folder to contain comparative static exercises
`+compareSteady/xxx`	Subpackage to contain comparative static and reporting files for exercise xxx
`+scenario`	Subpackage folder to contain scenarios
`+scenario/+xxx`	Subpackage folder to contain scenario and reporting files for exercise xxx
`+utils`	Subpackage with utility functions

Explanation of Essential GEES Model Folders

+ sign before the folder name

- It provides a flexible way of structuring folders and subfolders

`+model/`

- includes (so far) three particular versions of the model
  - +autarky – one model for the whole world/closed economy
  - +global4A – baseline version (global 4 area: US, EZ, CN, Rest of the world)
  - +symmetric2A – 2 country model version (everything is perfectly symmetric)
- *In order to create a new model version, create a new +subfolder*

`+simulate`

- Includes set of simulation experiments
- To provide intuition about the main transmission mechanisms and convergence properties
  - +areaDisinflation
  - +areaGovDebt
  - +areaMonetaryShock
  - +areaPopulationIncrease
  - +areaProductivityImpovements
  - +areaTariffs
  - +areaWAP
  - +disinflation
  - +globalPopulation
  - +globalProductivity
  - +interestFloor
    - Each of these folders has function “run.m”, which runs the experiment
- Simulation experiments are written in a way that they can run at any version of the model (e.g. you can run autarky model with productivity increase or global 4 area model with global productivity increase).
- Produces HTML report with the results

`+compareSteady`

- Stores another set of experiments – comparative steady state
- Calculates and reports how the long-run of the model changes in response to changes in some parameters or other assumptions
    - +areaGovDebt
    - +areaMonetaryNeutrality
    - +areaPriceLevel
    - +areaProductivity
    - +areaRiskAppetiteCapital
    - +areaWAP
    - +globalGovDebt
    - +globalPopulation
    - +globalProductivity
    - +globalRiskAppetiteCapital
        - Each of these folders has function “run.m”, which runs the experiment
- Produces table with results and comparative Excel sheet to see the change between the two steady states and evaluation of differences

`+scenario`

- Real event-related simulations linked to recent developments

`source/`

- Stores all model source files
- Due to flexibility the model is split into small models describing different part of the economy
    - commodity.model
    - demography.model
    - finance.model
    - fiscal.model
    - globals.model
    - local.model –local households, producers
    - open.model – exporters, balance of payments
    - trade.model – international trade linkages
    - *wrapper-autarky.model* – special file that creates a necessary closure for autarky models (a single losed economy)

    - *wrapper-multiarea.model* - special file that creates a necessary closure for multiarea model version

`docs/`

- documentation of the model is split into sections
- documentation is written in a Markup text language
    - you can use for example *Obsidian* – software package, which transforms and shows Markdown files directly in a document-readable format.

What Are We Developing?

- A **modular framework** rather than a single model
  - Modular framework allows for a great flexibility to extend/shrink the model according to user's needs and preferences
  - Various modules (sectors) can be added/removed
  - It is a multi-country framework, which allows simply to add/delete some regions
- **GEES Scenario builder** is based on a DSGE core
  - Covers a wide spectrum of real and financial economy aspects, less emphasis on deep disaggregation (top-down framework)
  - Some pragmatic features are applied, if needed
  - Easy to operate and possible to include judgmental adjustments

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Current Stage

- Work in progress, under continuous development
- Robust core engine is available and allows to build a number of prototypes (e.g. multi-country, closed economy)

Key GEES Modules

- Demography
- Households
- Production
- Commodities
- Tariffs
- Non-Commodity Exports
- Monetary Policy
- Fiscal Policy

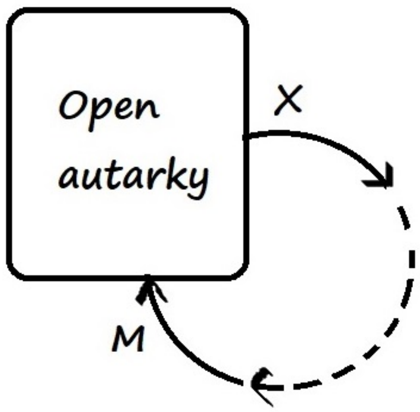
Structure of GEES Modules

There are two model options available:

- (open) autarky economy
- multi-area world

Building Multi-Area World

Open Autarky



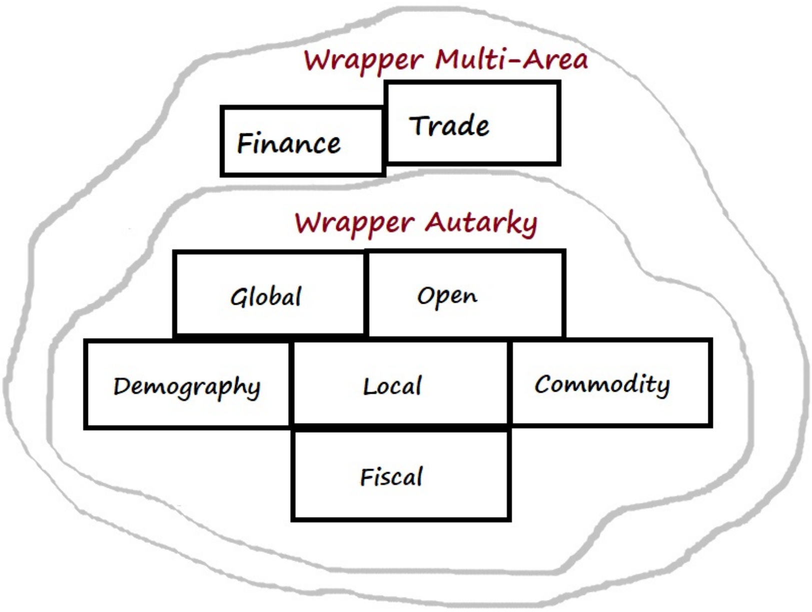
In order to build a multi-country world with the trade linkages, finance etc., it is good to start with a **one country**, which will serve **as a template** for other areas/countries in the world. This template country will be an open economy (having trade structure), however, it will be self-sufficient, which means that it will satisfy the condition

$$Country^1_{exports} = Country^1_{imports}$$

Multiply Autarky

Once this template (open autarky) economy is built, then it can be replicated multiple times (depending how big world we want). The size (economic, demographic), parametrization and general characteristics of these economies will be the same as for the template economy. The economies will be connected via trade linkages, which are symmetric.

These features of the economies together with symmetric trade condition ensure that the equilibrium for the autarky economy holds also for the multi-area economy/the world. Then individual economies can be altered, for example for different population proportion or productivity growth.



Demography

There are four categories, which describe population/labor supply in the model.

- Total population
- Working age population
- Labor force (labor participation)

Outside the demography module:

- Per-worker labor supply (e.g. per-worker hours worked)

Total Population

Global population trend component, which is driving population in all regions, is defined as

$$\Delta \log nn_t^{gg} = \rho_{nn^{gg}} \Delta \log nn_{t-1}^{gg} + (1 - \rho_{nn^{gg}}) \Delta \log \kappa_{nn}$$

$$\kappa_{nn} = \frac{nn_{\mathrm{ss}}^{gg}}{nn_{\mathrm{ss}-1}^{gg}}$$

where nn_t stand for total population and upperscript gg means global (or a feature shared by all regions).

Total population nn_t is an exogenous process, which is growing over time (possibly with some shocks). In the long run, the global population trend will be growing at a constant rate κ_{nn}. Total population trend can change up/down, which may have a permanent level effects. However, in the long run, the level of population is not determined, just the population growth rate.

Besides the total population, there are also area/country-specific assumptions. Area/country population is derived from the global population trend and some area/country-specific relative component nr_t.

Area/Country total population

$$nn_t = nr_t \cdot nn_t^{gg}$$

$$\log(nr_t) = \rho_{nr} \log nr_{t-1} + (1 - \rho_{nr}) \log nr_{\mathrm{ss}}$$

If there are no shocks to the relative component nr_t, then area/country-specific population follows the global population trend. However, a shock to a relative component nr_t may shift the area/country population even without changes in the underlying global population trend.

Labor Market Population

We assume that particular share of a working population is in its working age.

Working age population

$$\frac{nw_t}{nn_t} = \rho_{nw} \frac{nw_{t-1}}{nn_{t-1}} + (1 - \rho_{nw})\, \kappa_{nw} + \epsilon_{nw,t}$$

$$\kappa_{nw} = \frac{nw_{\mathrm{ss}}}{nn_{\mathrm{ss}}}$$

Labor force (participation rate)

$$\frac{nf_t}{nw_t} = \rho_{nf} \frac{nf_{t-1}}{nw_{t-1}} + (1 - \rho_{nf})\, \kappa_{nf} + \epsilon_{nf,t}$$

$$\kappa_{nf} = \frac{nf_{ss}}{nw_{ss}}$$

Long-run changes in these shares will induce changes in consumption path and per capita GDP.

Households

Each area's household sector is modeled as a single representative household with an exogenous time-varying number of household members, nn_t. The household enters a net position in debt instruments (e.g.
loans, deposits, fixed-income securities, etc.) with the local financial sector, bh_t, and holds a portfolio of claims on production capital in all areas (including the local area), $\sum_a s_{a,t} \, ex_{a,t} \, pk_t^a \, k_t^a$; the latter is our way to mimic corporate equity holdings with cross-border exposures.

During each period, the household purchases consumption goods, ch_t, supplies per-worker hours worked, nh_t, rents
production capital, k_t^a, out to producers in the respective area, chooses the utilization rate of local production capital, u_t, invests in creating addition local capital, i_t, pays lump-sum taxes (or receives lump-sum transfers) of two types, $txls1_t$ and $txls2_t$, and collects period profits from local producers, local exporters, and the local financial sectors (of whom all the household is the ultimate owner).

The household chooses the following quantities

- total consumption, ch_t,
- per-capita hours worked, nh_t,
- shares of claims on production capital possibly from all areas, $s_{a,t} \in [0,\, 1]$, $a \in A$,
- the utilization rate of local production capital, u_t,
- investment in local production capital, i_t,
- net financial position with the local financial sector, bh_t,

to maximize its infinite lifetime utility function subject to a dynamic budget constraint. The household derives utility from consumption, disutility from work, and utility from its wealth (net worth).

Household preferences

The household preferences are described by a time-separable utility function over an infinite life horizon, $t = 0, \ldots, \infty$. The period utility function consists of a consumption utility component, $\mathbf{U}_t^{ch}$, a work disutility component, $\mathbf{U}_t^{nh}$, and a current wealth (net worth) utility component, $\mathbf{U}_t^{netw}$. The individual utility function components are each evaluated on a per-capita basis, and the overall period utility is multiplied by the
total number of household members

$$\mathbf{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\mathbf{U}_t^{ch} - \mathbf{U}_t^{nh} + \mathbf{U}_t^{netw} \right) nn_t$$

The respective components of the utility function related to consumption, work and wealth, respectively, are given as follows

$$\mathbf{U}_t^{ch} \equiv \kappa_{ch} \, \mathbf{log} \, \frac{ch_t - \mathbf{ch}_t^{\mathrm{ref}}}{nn_t}$$

$$\mathbf{U}_t^{nh} \equiv \frac{1}{1+\eta} \, {nh_t}^{1+\eta}$$

$$\mathbf{U}_t^{netw} \equiv \nu_1 \left(\mathbf{log} \, \frac{netw_t}{pc_t \, \mathbf{ch}_t} - \nu_0 \frac{netw_t}{pc_t \, \mathbf{ch}_t} \right)$$

where

- ch_t^{ref} is the reference point in household consumption proportional to the level of real current labor income net of type 2 lump-sum taxes (or transfers) and externalized from the household optimization

$$ch_t^{\mathrm{ref}} \equiv \chi_{curr} \, \frac{curr_t}{pc_t} + \chi_c \, c_{t-1}$$

- $\kappa_{ch} \equiv 1 - ch_{\mathrm{ss}}^{\mathrm{ref}} \, {ch_{\mathrm{ss}}}^{-1}$ is a steady-state correction constant ensuring that the marginal utility of consumption equals $nn_t \, {ch_{ss}}^{-1}$ in steady state, a feature of modeling convenience,
- $curr_t$ is current labor income net of type 2 lump sum taxes (or transfers)

$$curr_t \equiv w_t \, nh_t \, nl_t - txls1_t$$

- $netw_t$ is the nominal net worth given by the sum of the value of the production capital portfolio, the net financial position of the household to the local financial sector, bh_t (a positive balance means net claims of the financial sector on the household), and the net worth of the local financial sector (whose ultimate owner the household is), bb_t,

$$netw_t \equiv \sum_a s_{a,t}\, ex_{a,t}\, pk_{a,t}\, k_{a,t} - bh_t + bb_t$$

- $ex_{a,t}$ is the cross rate between local currency and area a's currency (with movements up meaning depreciation of local currency)

$$ex_{a,t} = \frac{e_{\mathrm{local},t}}{e_{a,t}}, \quad ex_{\mathrm{local},t} = 1$$

Dynamic budget constraint

The dynamic budget constraint facing the household sector describes a stock-flow relationship between the household assets and liabilities (stocks) on the one hand, and current receipts and current outlays (flows) on the other hand. The household assets and liabilities consist of

- a net position with the local financial sector, $-bh_t$ (a positive balance means net lending by the household from the financial sector, a
negative balance means net lending by the financial sector from the household),
- claims on production capital (local and cross-border capital), $\sum_a s_{a,t}\, ex_{a,t}\, pk^a_t\, k^a_t$, and

The change in the household assets and liabilities is equal to the revaluation
of capital claims, and the total amount of current receipts and outlays:

- revaluation of claims on production capital (both from the nominal exchange rate and the capital price), $\sum_a s_{a,t-1}\, \boldsymbol{\Delta}(ex_{a,t}\, pk^a_t)\, k^a_{t-1}$,
- interest receipts or outlays on the net position with the local financial sector, $(rh_{t-1} - 1) bh_{t-1}$

- current receipts from capital rentals net of capital utilization costs, $\sum_a s_{a,t}\, ex_{a,t}\, pu_t^a\, k_t^a - \Xi_{u,t}$,
- current receipts from labor income, $w_t\, nh_t\, nl_t$,
- current receipts from selling newly installed capital, $pk_t\, i_t$,
- profits from loccal producers, $\Pi_{y,t}$, exporters, $\Pi_{x,t}$, and the financial sector, $\Pi_{b,t}$,
- current outlays on consumption goods, $-pc_t\, ch_t$,
- current outlays on investment goods, $-pi_t\, i_t$,

$$
\begin{gathered}
\sum_a s_{a,t}\, ex_{a,t}\, pk_t^a\, k_t^a - bh_t \;\cdots \\
= \;\sum_a s_{a,t-1}\, ex_{a,t}\, (1-\delta^a) pk_t^a\, k_{t-1}^a - bh_{t-1} \;\cdots \\
+ \sum_a s_{a,t}\, ex_{a,t}\, pu_t^a\, u_t^a k_t^a - (rh_{t-1}-1) bh_{t-1} \;\cdots \\
+ w_t\, nh_t\, nl_t - pc_t\, ch_t + (pk_t - pi_t) i_t - txls1_t - txls2_t \;\cdots \\
+ \mathbf{\Pi}_{y,t} + \mathbf{\Pi}_{x,t} + \mathbf{\Pi}_{b,t} - \Xi_{i,t} - \Xi_{k,t} - \Xi_{u,t} + \mathbf{\Xi}_{h,t}
\end{gathered}
$$

Lagrange multiplier associated with the budget constraint is denoted by vh_t (shadow value of nominal household wealth)

Real wage rigidities

The labor market exhibits real wage rigidities. These rigidities do not derive from explicit microfoundations in our model; they are introduced as an ad-hoc correction to the law of motion for the real wage rate in the following way. The household makes its choices as though the wage rate was fully flexible; we denote this hypothetical level of the nominal wage rate by ww_t, and use this hypothetical wage in the household Lagrangian, in
place of the actual wage rate. Once the hypothetical flexible optimum wage rate is determined, the actual wage rate follows an autoregressive process with asymptotic convergence to the flexible optimum

$$
\log \frac{w_t}{pc_t} = \rho_w \log \frac{\kappa_w\, w_{t-1}}{pc_{t-1}} + (1-\rho_w) \log \frac{ww_t}{pc_t} + \epsilon_{w,t}
$$

where the past real wage is indexed by a steady-state adjustment constant, κ_w, given by the gross rate of change in the steady-state real wage rate

$$\kappa_w \equiv \hat{w}_{ss}\, \hat{pc}_{ss}{}^{-1}$$

and $\rho_w \in [0,\, 1)$ is an auto-regression parameter.

Costs of short-term adjustment processes

The optimizing behavior of the representative household is subjected to two types of costly short-term adjustment processes:

- an investment adjustment/installation cost
- a capital utilization cost.

The investment adjustment/installation cost comprises two components: departures from a steady-state investment-to-capital ratio, and departures from a steady-state rate of change in investment

$$\Xi_{i,t} \equiv \tfrac{1}{2}\, \xi_{i1}\, pi_t\, \mathbf{i}_t \left(\mathbf{log}\, i_t - \mathbf{log}\, \mathbf{i}_t^{\mathrm{ref}}\right)^2 + \tfrac{1}{2}\, \xi_{i2}\, pi_t\, \mathbf{i}_t \left(\mathbf{\Delta log}\, i_t - \mathbf{log}\, \kappa_i\right)^2$$

where i_t^{ref} is a point of reference derived from the steady-state investment-to-capital ratio applied to the stock of capital last period,

$$i_t^{\mathrm{ref}} \equiv \frac{i_{\mathrm{ss}}}{k_{\mathrm{ss}}}\, k_{t-1}\, \hat{\imath}_{\mathrm{ss}}$$

and $\kappa_i \equiv \hat{\imath}_{\mathrm{ss}}$ is a steady-state adjustment constant ensuring that the cost term disappears in steady-state.

The cost of capital utilization give rise to a cyclical response in the rate of utilization of the existing stock of capital. The cost function is given by

$$\Xi_{u,t} \equiv s_{\mathrm{local},t}\, py_t\, k_t\, \frac{v_0}{1+v_1}\, u_t{}^{1+v_1}$$

Capital accumulation

The household purchases investment goods, converts them to newly installed production capital (paying the adjustment/installation cost in the process) and adds these to the existing stock of capital

$$k_t = (1-\delta)\; k_{t-1} + i_t$$

Lagrangian for the household optimization problem

The Lagrangian for the constrained optimization problem facing the representative household consists of the lifetime utility function and a sequence of dynamic budget constraints for each time from now until infinity, $t = 0, \ldots, \infty$. Note that we use ww_t in place of w_t in the Lagrangian.

$$\mathbf{max}\;_{\{ch_t, bh_t, s_{a,t}, i_t, nh_t, u_t\}}$$

$$\sum\nolimits_t \beta^t \left[\kappa_{ch} \,\mathbf{log}\, \frac{ch_t - \mathbf{ch}_t^{\mathrm{ref}}}{nn_t} + \frac{1}{1+\eta}\; {nh_t}^{1+\eta} + \nu_1 \left(\mathbf{log}\, \frac{netw_t}{pc_t\, \mathbf{ch}_t} - \nu_0 \frac{netw_t}{pc_t\, \mathbf{ch}_t} \right) \right] nn_t \;\cdot$$

$$+ \sum\nolimits_t \beta^t vh_t \Big\{ - \sum\nolimits_a s_{a,t}\, ex_{a,t}\, pk_t^a\, k_t^a + bh_t \;\cdots$$

$$+ \;\sum\nolimits_a s_{a,t-1}\, ex_{a,t}\, (1-\delta^a) pk_t^a\, k_{t-1}^a - bh_{t-1} \;\cdots$$

$$+ \;(\beta_{k\,t-1})^t \sum\nolimits_a s_{a,t}\, ex_{a,t}\, pu_t^a\, u_t^a k_t^a - (rh_{t-1} - 1) bh_{t-1} \;\cdots$$

$$+ \;ww_t\, nh_t\, nl_t - pc_t\, ch_t + (pk_t - pi_t) i_t - txls1_t - txls2_t. \;\cdots$$

$$+ \mathbf{\Pi}_{y,t} + \mathbf{\Pi}_{x,t} + \mathbf{\Pi}_{b,t} - \;\Xi_{i,t} - \Xi_{k,t} - \Xi_{u,t} + \mathbf{\Xi}_{h,t} \Big\}$$

where vh_t is the Lagrange multiplier on time-t budget constraint, and β_k is the additional discount factor applied to the value of corporate equity holdings to compensate for the risk aversion of households.

Optimality conditions

The optimal (utility maximizing) choices of the representative household are described by the following first-order conditions.

- Consumption, ch_t

$$vh_t \, ph_t = \kappa_{ch} \, \frac{1}{ch_t - \mathbf{ch}_t^{\mathrm{ref}}} \, nn_t$$

- Per-worker hours worked depending on the hypothetical flexible wage rate,
  ww_t

$$vh_t \, ww_t = {nh_t}^{\eta}$$

- Net position with the financial sector, bh_t (an intertemporal no-arbitrage condition)

$$vh_t = \beta \, vh_{t+1} \, rh_t + \nu_1 \frac{1}{pc_t \, \mathbf{ch}_t} \left(\frac{pc_t \, \mathbf{ch}_t}{netw_t} - \nu_0 \right)$$

- Utilization rate of production capital, u_t

$$\upsilon_0 \, {u_t}^{\upsilon_1} \, py_t = pu_t$$

- Investment in local production capital, i_t

$$pk_t = pi_t \Big[1 + \xi_{i1} \big(\log i_t - \log \mathbf{i}_t^{\mathrm{ref}} \big) + \xi_{i2} \big(\boldsymbol{\Delta} \log i_t - \kappa_i \big) - \xi_{i2} \, \beta \, zk_t \, \big(\boldsymbol{\Delta} \log i_{t+1} - \kappa_i \big) \Big]$$

- Claims on area a's production capital, $s_{a,t}$, $\forall a \in A$

$$\beta \, vh_{t+1} \, rh_t \, pk_t^a \, ex_t^a = vh_t \, pu_t^a \, ex_t^a \, u_t^a + \beta \, \beta_{kt} \, vh_{t+1} \, (1 - \delta^a) pk_{t+1}^a \, ex_{t+1}^a$$

The last set of equations defines arbitrage-free conditions (AFCs) for a corporate equity portfolio choice. We need to further address the following two characteristics of these NACs:

1. As is common in macro models, the AFCs themselves do not determine the
   actual portfolio shares, $s_{a,t}$, only the relationship between the price of production capital, the cash flows it generates, and the

household discount factor. The actual shares are then calibrated and kept fixed in the baseline version of the model.

2. Since we allow for cross-border holdings, each area's capital is subject to muliple AFCs, each relating to the household residing in a different area and exhibiting, in general, different preferences. We therefore create aggregate AFCs by taking the weighted average with the weights equal the portfolio shares. The aggregate AFCs for the capital markets are described in the Global equilibrium section.

Demography

Area	Population	Pop'n Growth	Working Age	Labor Part'n
World	7,794,799	1.09%	68%	66%
US	331,003	0.65%	65%	72%
EU	515,105	0.12%	64%	74%
RW	7,123,815	1.10%	.	.

PPP Output Per Worker (2011 PPP US$)

Area	2010-2020E Growth	2019 Level
World	2.3%	37,782
US	0.9%	116,385
EU	1.0%	84,501

Oil and Gas (2019 EIA)

Area	Oil (Mb)	
World	100,608	
US	19,471	
EU	<2,000	

Production

- Several pairwise stages of production
- Input factors
    - Labor
    - Intermediate imports
    - Commodity inputs
    - Capital
- Real flexibilities to flatten the marginal cost schedule
    - Variable utilization of capital
    - Roundabout production
- Sticky prices

Productivity

Productivity process is modeled using the same principle as population.

Global productivity trend component

$$\mathbf{\Delta log}\ a^{gg}_t = \rho_a\ \mathbf{\Delta log}\ a^{gg}_{t-1} + (1 - \rho_a)\mathbf{log}\ \hat{a}^{gg}_{\mathrm{ss}}$$

Area-specific relative productivity component

$$\mathbf{log}\ ar_t = \rho_{ar}\ \mathbf{log}\ ar_{t-1} + (1 - \rho_a r)\ \mathbf{log}\ ar_{\mathrm{ss}}$$

Total area productivity

$$a_t = a^{gg}_t\, ar_t$$

Production technology with time-varying elasticity

The firms determine the technology choice alongside the technology frontier. We differentiate between the short-run and long-run elasticity of substitution in the production sector. In particular, the response in the input factor demand to the changes in the input factor prices should be relatively inelastic in the very short run (one year), but much more elastic in the longer run (five-years).

The model uses the Leontief-Cobb-Douglas technology combination. In the short run, the function, which describes how firms can combine the production inputs, is the *Leontief function* (with zero elasticity of substitution). The technology constraint, which is applied to the choice of technology, is the *Cobb-Douglas*.

This optimization problem is perfectly equivalent (up to a scale constant) to the Cobb-Douglas, if there are no adjustment costs. However, the situation changes once we introduce the cost of adjustment to the technology levels. If the cost of changing the proportion of technology levels is infinite, then we have zero elasticity substitution environment (Leontief case). If we allow for some finite adjustment costs, we can move alongside the technology frontier and eventually get to the Cobb-Douglas unit elasticity of substitution case - by changing the weight of adjustment costs, we can move from the Leontief to the Cobb-Douglas.

Production technology based on a unit-elasticity (Cobb-Douglas) production function

$$y_t = F(a_t, b_t) = {a_t}^{\gamma} \, {b_t}^{1-\gamma}$$

Period profits are given by

$$\Pi_t \equiv py_t \, y_t - pa_t \, a_t - pb_t \, b_t - \Xi_{y,t}$$

and include the cost of changing the input factor proportions

$$\Xi_{y,t} \equiv \tfrac{1}{2} \xi_y \left[pa_t \, \mathbf{a}_t \left(\mathbf{\Delta log} \, \frac{a_t}{\mathbf{y}_t} \right)^2 + pb_t \, \mathbf{b}_t \left(\mathbf{\Delta log} \, \frac{b_t}{\mathbf{y}_t} \right)^2 \right]$$

Optimization problem with a possibly heavier discounting, $\beta_y \in [0,1]$, to incorporate higher uncertainty of future profit flows

$$\mathbf{max}\ _{\{a_t, b_t\}}\ \mathbf{E}_0 \sum_{t=0}^{\infty} (\beta\, \beta_y)^t\, vh_t\, \Pi_{y,t}$$

Optimal choice of input factors (omitting higher-order terms from the adjustment costs)

$$\gamma\, py_t\, y_t \approx pa_t\, a_t \left[1 + \xi_t \left(\mathbf{\Delta log}\ \frac{a_t}{y_t} - \beta\, \beta_y\, \mathbf{\Delta log}\ \frac{a_{t+1}}{y_{t+1}} \right) \right]$$

$$(1-\gamma)\, py_t\, y_t \approx pb_t\, b_t \left[1 + \xi_t \left(\mathbf{\Delta log}\ \frac{b_t}{y_t} - \beta\, \beta_y\, \mathbf{\Delta log}\ \frac{b_{t+1}}{y_{t+1}} \right) \right]$$

Production Stages

The whole production process is divided into stages (pair-wise) and assumption of time-varying elasticity is applied.

T-4: Combine (non-commodity) imports from other areas

$$mm_t = F_4 \left(mm_t^1, \ldots, mm_t^A \right)$$

$$mm = my_t + mx_t$$

where

- my_t is the intermediate import inputs into local production
- mx_t is the intermediate import inputs into export production (re-exports)

T-3: Combine non-commodity variable factors

$$y_{3,t} = F_3 \big(mm_t, nv_t \big)$$

$$nv_t \equiv (nh_t - \gamma_{nv} nh_{\mathrm{ss}})\, nl_t$$

where

- nv_t is the variable labor input with $\gamma_{nv} nh_{\mathrm{ss}}$ being the overhead labor needed to maintain production regardless of the output actually produced

T-2: Combine variable factors with capital

$$y_{2,t} = F_2\big(uk_t \, y_{3,t}\big)$$

$$uk_t = u_t \, k_t$$

T-1: Add dependence on commodity inputs

$$y_{1,t} = F_1\big(y_{2,t}, \; mq_t\big)$$

T-0: Add a roundabout production layer and sticky prices

$$y_t = F_0\big(y_{1,t}, \; yz_t\big) + yz_t$$

Sticky Prices

Downward sloping demand curve

$$y_t = \mathbf{y}_t \left(\frac{py_t}{\mathbf{py}_t} \right)^{-\mu_{py}/(\mu_{py}-1)}$$

where

- mu_{py} is the monopoly power of the representative producer in its own market, and $\mu_{py}/(\mu_{py} - 1)$ is the underlying elasticity of substitution of demand for the producer's
  output (which gives rise to the monopoly power)

Period profits

$$\Pi_{y0,t} \equiv (py_t - py_{0,t}) \, y_t - \Xi_{py,t}$$

with the price adjustment costs given by

$$\Xi_{py,t} \equiv \tfrac{1}{2} \, \xi_{py} \, \mathbf{py}_t \, \mathbf{y_t} \, \big(\mathbf{\Delta log} \, py_t - j_t\big)^2$$

where j_t is a price indexation factor given by

$$j \equiv \zeta_{py} \, \mathbf{log} \, \hat{\mathbf{py}}_{t-1} + (1 - \zeta_{py}) \, \mathbf{log} \, \hat{py}_{\mathrm{ss}}$$

Maximization problem

$$\max_{\{y_t, py_t\}} \mathbf{E}_t \sum_t \left(\beta\, \beta_{y_t}\right)^t vh_t\, \Pi_{y0,t}$$

where

- β_y is an additional discount factor to compensate for the uncertainty of cash flows generated by real economic activity

Optimal price setting with no adjustment cost (steady state) is a markup over the marginal cost

$$p_{y,t} = \mu_{py}\, p_{y0,t}$$

Optimal price setting with adjustment cost is a Phillips curve

$$p_{y,t} \left\{1 + (\mu_{py} - 1)\, \xi_{py} \Big[(\mathbf{\Delta log}\, p_{y,t} - j_t) - \beta\, \beta_y\, (\mathbf{\Delta log}\, p_{y,t+1} - j_{t+1})\Big]\right\} = \mu_{py}\, p_{y0,t}$$

Production sector total profits

$$\begin{gathered}\Pi_{y,t} \equiv py_t\, y_{0,t} - pmm_t\, my_t - w_t\, nh_t\, nl_t - pu_t\, u_t\, k_t \\ - \Xi_{y4,t} - \Xi_{y3,t} - \Xi_{y2,t} - \Xi_{y1,t} - \Xi py, t\end{gathered}$$

Final Goods Assembly

Domestic production and directly imported goods are combined into the final goods

$$y_t = ch_t + cg_t + ih_t + yx_t$$

where

- ch_t is private consumption (by households)
- cg_t is government consumption
- ih_t is private investment (by households)
- yx_t is the local component in the export production

Tariffs

Assembly of Non-Commodity Imports

Production process has several stages (as mentioned in the *Production chapter*). The initial stage is an assembly of non-commodity imports that flow into the country. The aggregated function has a time-varying elasticity of substitution. Specifically, we assume a very low elasticity of substitution in the short run, but a unit elasticity of substitution in the long run.

Production function for assembling the imports

$$mm_t = F(mm_t^1,\ \ldots,\ mm_t^K)$$

Imports in a country mm_t are assembled from the goods originating in the remaining K regions of the world (such as $mm_t^1 ... mm_t^K$), with certain weights calibrated based on population size, trade patterns etc. The technology choice decides how the imports are assembled.

Demand for imports, ignoring adjustment costs, is defined as

$$\frac{mm_t^k}{mm_t} \propto \left(\frac{\frac{e_t}{e_t^k}\, pxx_t^k}{pmm_t} \right)^{-1}$$

where pxx_t^k is a price from the region K quoted in country's currency and pmm_t is a marginal cost of assembling the imports. Proportions of imports from individual areas/countries are price elastic.

Import Tariffs

The model accounts for area/country-specific tariffs. The overall tariffs on imports from area K consist of a common component applied across all regions trm and and area/country-specific component trm^k. The impact of the area/country-specific tariffs depends on the size of area/country.

Demand for imports with tariffs

$$\frac{mm_t^k}{mm_t} \propto \left[\frac{pmm_t^{\mathrm{fob},k}\left(1 + trm_t + trm_t^k\right)}{pmm_t} \right]^{-1}$$

Import price FOB destination

$$pmm_t^{\mathrm{fob},k} = \frac{e_t}{e_t^k}\, pxx_t^k$$

(Final) import price including tariffs

$$pmm_t^k = \frac{e_t}{e_t^k}\, pxx_t^k\left(1 + trm_t + trm_t^k\right)$$

Import Price Indicators

Price	Where placed in the model
$pmm_t^{\mathrm{fob},k}$	Balance of payments
pmm_t^k	Input price facing local agents
$pmm_t^{\mathrm{fob},k} - pmm_t^k$	Government revenues

Non-Commodity Exports

- Simple transformation of domestic output
- Productivity improvements faster in exportable sectors
- Assembled with reexports

Assembly of Non-Commodity Exports

Function to assemble the exports

$$xx_t = F_{xx}\left({ar_t}^{\gamma_{xx}} \cdot yxx_t,\ mxx_t \right)$$

Monetary Policy

Monetary policy depends on what type of regions are included and/or what episodes are more closely analyzed. It can follow either of these rules:

- Inflation targeting reaction function
- Exchange rate peg
- Inflation targeting with exchange rate management

Fiscal

In the basic model version, there are few simplifying assumptions regarding:

- Government consumption and investment
- Government debt placed locally
- Lump-sum taxes
- Crowding in/out effects

Fiscal Budget Constraint

Government constraint is very simple

$$dg_t = rg_{t-1} \cdot dg_{t-1} + pcg_t \cdot cg_t + pig_t \cdot ig_t - trl1_t - trl2_t$$

accounting for the government debt dg_t, two types of expenditures (consumption cg_t and public investment ig_t) and two types of lump-sum taxies levied on households ($txl1_t$ and $txl2_t$).

The difference between these two types of lump-sum taxes is that $txl2_t$ affects also the current income of the household, while $txl1_t$ does not. However, both $txl1_t$ and $txl2_t$ enter the household budget constraint. It could be interpreted as that the lump-sum tax of type 1 is affecting rather wealthier (or financially unconstrainted) people (as it does not affect their current income). However. changing $txl2_t$ will affect poorer (or financially constrained) people and drive their current-income-linked part of the consumption.

Public Capital

Public capital is included in a simplest possible way, mostly in order to allow to run the scenarios for example the country's government spending money on improving the infrastructure and increasing the debt.

Public capital

$$kg_t = (1 - \delta)\, kg_{t-1} + ig_t$$

Target level for the public capital is a proportion to the private capital

$$kg_t^{\mathrm{tar}} = \psi \cdot kh_t$$

with the investment rule

$$\Delta \log ig_t = \log \kappa_{ig} + \tau_{ig} \left(\log kg_t - kg_t^{\mathrm{tar}} \right)$$

$$\kappa_{ig} = \frac{ig_{\mathrm{ss}}}{ig_{\mathrm{ss}}}$$

Government Consumption and Taxes

The model also includes stabilizing mechanism in order to keep debt at a target level as a ratio to nominal GDP

$$\frac{dg_t}{ngdp_t}$$

- A wide range of mechanisms to stabilize debt

Crowding in/out effects

Crowding in effect occurs in the short run, crowding out in the longer run.

Crowding in:

- If government expenditures go up, the economy expands. The mechanism, through which this is achieved, is the current income channel - if the government spends more, it increases the production, the current income, leading to households spending more.

Crowding out:

- If government expenditures lead to a permanent increase in government debt (as a ratio to GDP), then (through the net worth channel) it leads to an increase in the real interest rates and decrease of the levels of private investment, consumption and production capacities.

Local Financial Sector

The local financial sector holds net claims (net positions) on three sectors, all in the form of debt (i.e. non-equity) instruments:

- the local representative household, bh_t,
- the local government, bg_t,
- the rest of the world, bf_t.

Furthermore, we denote the value (net worth) of the local financial sector by bb_t; this value is part of the net worth of the local household (who are the ultimate owners of the local financial sector). The balance sheet identity (defining effectively the net worth of the financial sector) is given by

$$bb_t \equiv bh_t + bg_t + bf_t$$

Any of the net balances occurring in this equation can take a positive or a negative value. A positive value of bh_t, bg_t or bf_t means a positive net asset position (net claim) on the respective sector, and vice versa.

The period profit or loss of the local financial sector, $\Pi_{b,t}$, is given by the sum of the net interest receivables/payables

$$\Pi_{b,t} = rh_{t-1}\, bh_{t-1} + rg_{t-1}\, bg_{t-1} + rf_{t-1}\, bf_{t-1}$$

This profit/loss is transferred to the local household budget every period.

Aggregation and Market Clearing

Private adjustment costs

$$\Xi_{h,t} = \Xi_{ih,t} + \Xi_{kh,t} + \Xi_{u,t} + \Xi_{py,t} + \Xi_{x,t}$$

Gross domestic product

Nominal GDP (the area's value added at current prices) can be easily calculated from the expenditure side components:

$$ngdp_t \equiv pch_t \, ch_t + pih_t \, ih_t$$

Real GDP, on the other hand, is a more ambiguous concept in an economy where different components are supplied at different prices. We use a discrete time Tornqvist index number to construct an approximate measure of
the chain-linked rate of change in real GDP, $g\hat{d}p_t$.

The Tornqvist index is a weighted average of rates of change in the individual components

$$g\hat{d}p_t \equiv w_c \, c\hat{h}_t$$

where the weights are determined by the nominal expenditure share of the respective component in nominal GDP averaged over the current and previous periods.

Nonlinear Simulations

System of Nonlinear Equations with Model-Consistent Expectations

$$
\begin{gathered}
\mathbf{E}_t\left[f_1\left(x_{t-1}, x_t, x_{t+1}, \epsilon_t \mid \theta\right)\right]=0 \\
\vdots \\
\mathbf{E}_t\left[f_k\left(x_{t-1}, x_t, x_{t+1}, \epsilon_t \mid \theta\right)\right]=0
\end{gathered}
$$

$$
\mathrm{E}\ \epsilon_t \epsilon_t^{\prime}=\Omega
$$

Methods for Nonlinear Simulations

Characteristics	Local Approximation	Global Approximation	Stacked Time
Solution form	Function	Function	Sequence of Numbers
Terminal condition problem	No	No	Yes
Global nonlinearity	No	Yes	Yes
Stochastic uncertainty	Yes	Yes	No
Fully automated design	Yes	No	Yes
Large scale models	Yes	No	Yes

Local Approximation

Deviations from non-stochastic steady state

$$\hat{x}_t = x_t - \bar{x} \qquad \text{or} \qquad \hat{x}_t = \log x_t - \log \bar{x}$$

Find a function approximated around the nonstochastic steady state by terms
up to a desired order, with coefficient matrices (solution matrices) A_1^k, A_2^k, $\ldots$,
B^k

$$\hat{x}_t^k = A_1^k \, \hat{x}_{t-1} + \hat{x}_{t-1}' \, A_2^k \, \hat{x}_{t-1} + \ \cdots \ + B_1^k \, \epsilon_t + \epsilon_t' \, B_2^k \, \epsilon_t$$

that are consistent with the original system of equations up to a desired order

The coefficient matrices $A_1^k, \ A_2^k, \ A_3^k, \ \ldots, B_1^k, B_2^k, \ \ldots$ dependent on

- the Taylor expansions of the original functions $f_1, \ \ldots, \ f_k$
- model parameters θ

and the higher-order coefficient matrices $A_2^k, A_3^k, \ldots, B_2^k, B_3^k \ \ldots$ also dependent on

- the covariance matrix of shocks, Ω

Global Approximation

Find a parametric function g

$$x_t = g\left(x_{t-1}, \epsilon_t \mid \theta, \Omega\right)$$

that is consistent with the original system taking into account the expectations operator

The function g is a parameterized global approximation of the true function, e.g.

- parameterized sum of polynominals
- function over a discrete grid of points

Stacked Time

Find a sequence of numbers, $x_1, \ldots, x_T$ that comply with the original system of equations stacked T times underneath each other, dropping the expectations operator

$$
\begin{gathered}
f_1\left(x_{-1}, x_1, x_2, \epsilon_1 \mid \theta\right)=0 \\
\vdots \\
f_k\left(x_{t-1}, x_t, x_2, \epsilon_1 \mid \theta\right)=0 \\
\vdots \\
\vdots \\
f_1\left(x_{T-1}, x_T, x_{T+1}, \epsilon_T \mid \theta\right)=0 \\
\vdots \\
f_k\left(x_{T-1}, x_T, x_{T+1}, \epsilon_T \mid \theta\right)=0
\end{gathered}
$$

Initial condition x_{-1}

Terminal condition x_{T+1}

Combining Anticipated and Unanticipated Shocks in Stacked Time

By design, all shocks included within one particular simulation run are known/seen/anticipated throughout the simulation range

Simulating a combination of anticipated and unanticipated shocks means

- split the simulation range into sub-ranges by the occurrence of unanticipated shocks
- run each sub-range as a separate simulation, taking the end-points of the previous sub-range simulation as initial condition
- make sure you run a sufficient number of periods in each sub-simulation

---

IrisT Approach to Stacked Time

- Sequential block pre-analyzer (Blazer)
- (Quasi) Newton method

* Terminal condition derived from the first-order solution
* Analytical Jacobian
* Detection of sparse Jacobian pattern
* Detection of invariant Jacobian elements
* Frames: Automated handling of combined anticipated and unanticipated shocks (or exogenized data points)

Curse of Dimensionality

Dimension of the problem: $K =$ number of equations $\times$ number of periods

Dimensions of the Jacobian: $K \times K$

The conventional ways of handling terminal condition require a larger number of
periods to be simulated (to discount the effect of the "wrong" terminal condition)

Reduce the actual dimensionality and accelerate:

* Base terminal condition upon the first order solution dramatically reduces the number of periods needed
* Detect the sparse Jacobian pattern to avoid zero points
* Detect the Jacobian elements that need to be evaluated only once (at the beginning)

(Quasi) Newton Method of Solving Nonlinear Equations

Plain-vanilla Newton with variable step size
for exactly determined systems (nonlinear simulations)

$$x_k = x_{k-1} - s_k \, J_{k-1}^{-1} \, F_{k-1}$$

Quasi-Newton where the Jacobian is regularized using the steepest descent
method for underdetermined systems (steady state solver for growth

models
with "independent" degrees of freedeom, or steady state solver with the `"fixLevel"` option)

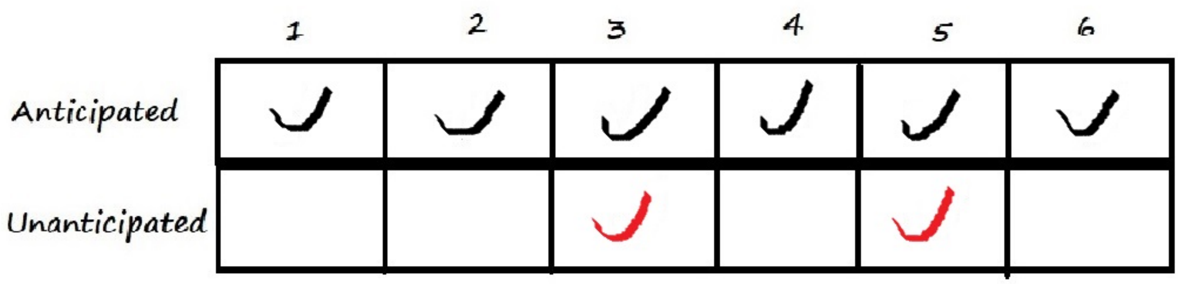
$$x_k = x_{k-1} - s_k \left(J_{k-1}^T J_{k-1} + \lambda_k\right)^{-1} J_{k-1} F_{k-1}$$

where

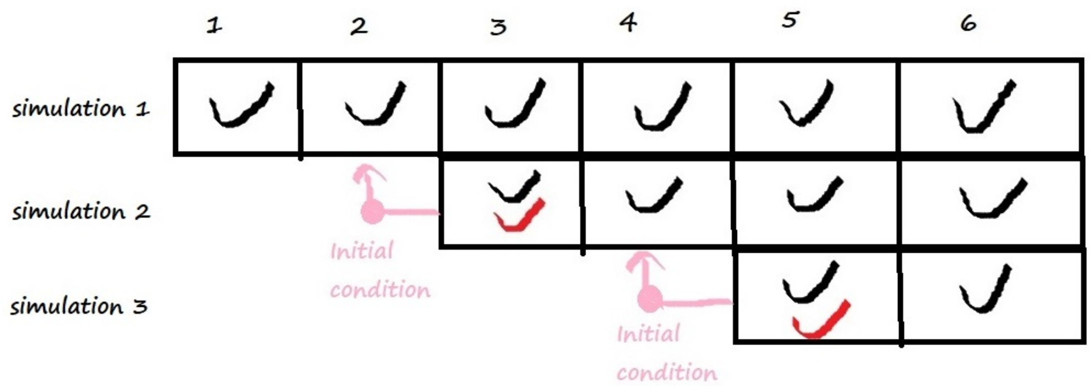
- function evaluation $F_k = F(x_k)$
- Jacobian evaluation $J_k = J(x_k)$

Frames

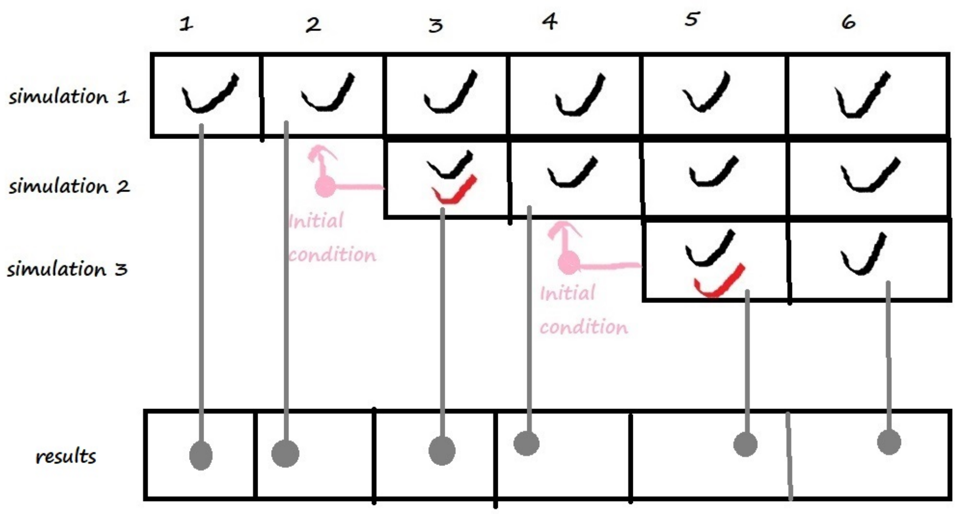
The simulation frames work in a following principle. Assume that you have a simulation for 6 periods combining both the anticipated and unanticipated shocks such that unanticipated shocks occur in periods 3 and 5.



In this case you need to run 3 simulations and final results will be assembled from these 3 simulations. Each additional simulation enriches the information set (by additional shocks).



Initial conditions for simulation n are taken from the simulation $n-1$. In this case, initial conditions for simulation 2 (which enriches the information set for an unexpected shock in period 3) are taken from the simulation 1 in period 2. Similarly, simulation 3 (which enriches the information set for unexpected shock in period 5) will use initial conditions from the period 4 in simulation 2. The final result is then an assembly from all three simulations such that



The `simulate` Function

```
[s, info, frameDb] = simulate( ...
    m, d, range, ...
    "deviation",    true -or- false, ...
    "anticipate",   true -or false, ...
    "plan",         empty -or- Plan, ...
    "method",       "stacked", ...
    "blocks",       true -or- false, ...
    "terminal",     "firstOrder" -or- "data", ...
    "startIter",    "firstOrder" -or- "data", ...
    "successOnly",  false -or- true, ...
    "window",       @auto -or- numeric, ...
    "solver",       @auto -or- {"iris-newton", "optionName", value, "optionName"
);
```

The `solver` options:

```
{ "iris-newton", ...
    "skipJacobUpdate",        0 -or- numeric, ...
    "lastJacobUpdate",        Inf -or- numeric, ...
    "functionNorm",           2 -or- Inf, ...
    "maxIterations",          5000 -or- numeric, ...
```

```
    "maxFunctionEvaluation",    @auto -or- numeric, ...
    "functionTolerance",        1e-12 -or- numeric, ...
    "stepTolerance",            1e-12 -or- numeric, ...
}
```

Practical Tricks

False slope (occasionally binding constraints)

- discontinuities pose relatively little difficulties
- the zero slope of the `max` or `min` functions is a more serious problem

Pressure relief valves (PRVs)

- mechanical
- structural

Notational Conventions

We use the following conventions for naming and denoting variables, parameters and function:

Notation	Description	Meaning
x_t	Lowercase latin	Model variables
$\hat{x}_t$	Hat accent	Gross rate of change, $\hat{x}_t = x_t / x_{t-1}$
x_{ss}	An ss subscript	Steady state of a model variable
x_t^a	Lowercase latin	Model variables with an explicit area reference
$\mathbf{x_t}$	Lowercase bold upright	Model variables externalized in the respective optimization problem
$\mathbf{log}$	Bold	Functions and function components
α	Lowercase Greek	Parameters
Π_t	Uppercase Greek	Some of model nominal flows

We denote by A the set of all areas included in the model. Currently, $A = \{\mathrm{us}, \mathrm{ea}, \mathrm{ch}, \mathrm{rw}\}$. An additional
code, GG, is used to index global common trends, such as the global productivity trend or the global population trend.

Area code	Description
us	United States (global reference area)
ea	Euro Area
ch	China
rw	Rest of world
gg	Common global trends

In most of the text, we do not explicitly include thee reference area in the names of variables for the ease of notation. Absent an explicit area reference, the variable or parameter belongs simply to the respective local area.

In several places, we use the concept of a so-called global reference area (GRA); for instance, the local nominal exchange rates are defined as the rates between the respective local area's currency and the GRA's currency. The convention is that the global reference area is always ordered first in the list of areas. In the baseline setup of the model, the United States is used as the GRA.

Glossary of Model Quantities

Glossary of variables

Variable	Source code name	Description
a_t^{gg}	gg_a	Global productivity trend component
ar_t	ar	Area-specific productivity component
bh_t	bh	Net claims of the financial sector on the household
ch_t	ch	Household consumption
$\frac{ch}{nn}$	ch_to_nn	Per-capita private consumption
$\mathbf{ch_t^{ref}}$	ref_ch	Point of reference in household consumption
$curr_t$	curr	Nominal current income of households
dg_t	dg	Government debt
$\frac{dg}{ngdp}$	dg_to_ngdp	Government debt to GDP ratio
i_t	ih	Investment in production capital
k_t	k	Private production capital
nn_t^{gg}	gg_nt	Global population trend component
$netw_t$	netw	Nominal net worth of households
nh_t	nh	Per-worker labor supply (e.g. per-worker hours worked)
nn_t	nn	Total population
nr_t	nr	Total population relative to global population component
nw_t	nw	Working age population
nf_t	nf	Labor force

Variable	Source code name	Description
pc_t	`pc`	Price of consumption goods
pi_t	`pih`	Price of investment goods
pu_t	`pu`	Rental price (user cost) of capital services
pk_t	`pk`	Price of production capital
py_t	`py`	Price of domestic production
$\frac{pq^{gg}}{pxx}$	`gg_pq_to_pxx`	Global real price of commodities
qq^{gg}	`gg_qq`	Long-run level of global commodity production
q^{gg}	`gg_q`	Global commodity demand
$qexc^{gg}$	`gg_qexc`	Excess demand in global commodity market
qq^{gg}	`gg_pq`	Global price of commodities, Reference currency
rdf	`rdf`	Real discount factor
rh_t	`rh`	Nominal household rate, LCY
rg_t	`rg`	Government debt interest rate
u_t	`u`	Rate of production capital utilization
uk	`uk`	Production capital services
$s_{a,t}$		Share of claims on private production capital in area a
$txls1_t$	`txls1`	Type 1 net lump-sum taxes+/transfers–on wealthy people
$txls2_t$	`txls2`	Type 2 net lump-sum taxes+/transfers–on poorer people
trm	`trm`	Import tariff rate
vh_t	`vh`	Shadow value of household budget constraint
w_t	`w`	Nominal wage rate

Variable	Source code name	Description
β_{y_t}	`zy`	Uncertainty discount factor on production cash flows
β_{kt}	`zk`	Uncertainty discount factor on capital
ww_t	`ww`	Hypothetical nominal wage rate absent labor market rigidities
e_t	`e`	Nominal exchange rate against the global reference area's currency
$ex_{a,t}$		Nominal cross rate between local currency and area a's currency
rh_t	`rh`	Gross rate of interest on financial claims on the household sector
rg_t	`rg`	Gross rate of interest on financial claims on the government
$\Pi_{y,t}$		Profits from local producers
$\Pi_{x,t}$		Profits from local exporters
$\Pi_{b,t}$		Profits from the local financial sector
$\Xi_{i,t}$		Investment adjustment cost
$\Xi_{k,t}$		Cost of deviations from capital reference point
$\Xi_{u,t}$		Capital utilization cost
$\Xi_{h,t}$		All private costs paid to the household

Glossary of steady-state parameters

Parameter	Source code name	Description
β	`beta`	Household discount factor
δ	`delta`	Depreciation rate of production capital
ν_0	`nu_0`	Level parameter in utility from net worth
ν_1	`nu_1`	Elasticity parameter in utility from net worth
υ_0	`upsilon_0`	Level parameter in capital utilization cost function
υ_1	`upsilon_1`	Elasticity parameter in capital utilization cost function

Glossary of transitory parameters

Parameter	Source code name	Description
χ	`chi`	Parameter in point of reference in consumption
ρ_w	`rho_w`	Auto-regression in real wage