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An Implicit Törnqvist Index of Real GDP

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Abstract

Recently much attention has been devoted to superlative indexes in the context of the national accounts. In this paper we advocate the use of the implicit Törnqvist quantity index to measure real GDP. This index, which has been proposed by Diewert and Morrison (1986), has never received serious consideration in the literature. Yet, compared to the better-known Fisher index, the implicit Törnqvist index of real GDP has a number of advantages. Thus, it can be shown to be exact for the Translog GDP function, it allows for a complete multiplicative decomposition of nominal and real GDP, and it is consistent with state-of-the-art measures of total factor productivity that typically rely on the Törnqvist aggregation. Estimates for a sample of 26 countries are reported. We find that the Laspeyres quantity index still used by the statistical agencies of most countries tends to underestimate real growth. Over the 1960–1996 period, the cumulated shortfall was as much as 13.4% of GDP in the case of Japan.

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1. Introduction

Real GDP is typically measured as a direct Laspeyres quantity index. The implicit GDP deflator, consequently, is a direct Paasche price index. Direct Laspeyres and Paasche indexes have two serious shortcomings. First, direct indexes are defined relative to a base period. They should only be used in this context, that is, to make comparisons with the reference period. Yet, runs of direct indexes are routinely used to make multilateral comparisons, that is, comparisons between arbitrary pairs of periods, an application for which they are ill-suited.[1] The resulting biases tend to be particularly significant at times of rapid changes in relative prices, and they are exacerbated by the fact that the rebasing of direct indexes is usually done rather infrequently, perhaps once per decade. This why economists generally advocate the use of bilateral (or chain) indexes to compare an arbitrary pair of years. Indeed, some countries—such as New Zealand and the United Kingdom—have recently switched to a chained Laspeyres measure of real GDP. Many other OECD members are expected to do so in the near future.

Second, it is well known that Laspeyres and Paasche indexes do not take substitution and transformation effects into account.[2] In the production theory context, Paasche price indexes tend to overstate price increases, whereas Laspeyres quantity indexes tend to underestimate quantity changes. That is, Laspeyres real GDP tends to underestimate real growth.[3] There are, in fact, a large number of other—and better—functional forms for index numbers that have been proposed in the literature. Recently, a number of countries—including the United States, Canada, and Australia—have moved to a chained Fisher measure of real GDP. The Fisher index is a superlative index,[4] but unfortunately, in the national-accounts context, it is not exact for any known GDP function, except in the restrictive case where the aggregate technology is globally separable between domestic factors, on one hand, and variable inputs and outputs, on the other. Thus, the case for the use of the Fisher index must rest on its axiomatic—rather than its economic—properties.[5]

The main purpose of this paper is to draw attention to another superlative index, the implicit Törnqvist quantity index, and to advocate its use as an index of real GDP. This index, which is obtained by deflating nominal GDP by a Törnqvist output price index, was first proposed as an index of real GDP by Diewert and Morrison (1986), but, so far, it has received little or no attention in the literature. Yet, compared to the Fisher index, the implicit Törnqvist index of real GDP has at least three important advantages. First, it can be shown to be exact for the Translog GDP function.[6] This gives it a strong economic justification. Second, the implicit Törnqvist index of real GDP makes it easy to obtain a multiplicative decomposition of real and nominal GDP growth. Third, it is fully consistent with state-of-the-art measures of total factor productivity, which typically rely on Törnqvist indexes. As far as we know, the implicit Törnqvist index of real GDP has never been used in any applied work, nor has anyone attempted to quantify the downward bias to which the Laspeyres measure might lead. A second objective of this paper is therefore to examine this question empirically. We do so on the basis of a sample of 26 OECD countries covering the period 1970–1996.

The paper proceeds as follows. In Section 2, we review a number of definitions and we define the implicit Törnqvist index of real GDP. In Section 3, we show that this index is exact for the Translog real GDP function, a description of the technology of an open economy that is often used in empirical work. An alternative interpretation of the Törnqvist index of real GDP that brings out the role of total factor productivity is given in Section 4. Section 5 reports our numerical estimates, and Section 6 concludes.

2. Measuring Real GDP

2.1. Notation and Preliminary Definitions

Let y_{it} be the quantity of GDP component $i\ (i = 1, \dots, I)$ during year $t\ (t = 0, \dots, T)$. The corresponding prices are denoted by p_{it}. The index of nominal GDP registering the change between periods s and t, $\Gamma_{t,s}\ (s < t)$, is defined as follows:

$$\Gamma_{t,s} \equiv \frac{\sum_{i=1}^{I} p_{it} y_{it}}{\sum_{i=1}^{I} p_{is} y_{is}}. \tag{1}$$

$\Gamma_{t,s}$ is equal to one plus the rate of growth in nominal GDP between periods s and t. The key question is how to divide $\Gamma_{t,s}$ into a price change and a quantity change, where the latter can be interpreted as the change in real GDP.

It is important to draw a distinction between direct indexes and chain indexes.[7] A direct index makes a direct comparison between an arbitrary period and a reference period (also called the base period), whereas a chain index is obtained by compounding indexes defined for consecutive periods. Let $P_{t,s}$ and $Y_{t,s}$ be direct price and quantity indexes, respectively. They provide a direct comparison between periods s and t $(s < t)$. The corresponding chain indexes, identified by a tilde ($\tilde{}$), are as follows (for $s < t - 1$):

$$\tilde{P}_{t,s} \equiv \prod_{h=s+1}^{t} P_{h,h-1}, \tag{2}$$

$$\tilde{Y}_{t,s} \equiv \prod_{h=s+1}^{t} Y_{h,h-1}. \tag{3}$$

Runs (or sequences) of direct indexes are often used to make multilateral comparisons. It is therefore useful to define the indirect (price or quantity) change factors that result from the comparison of direct indexes for arbitrary pairs of periods; these will be identified by a hat ($\hat{}$). For instance, let $Y_{t,0}$ be a direct quantity index at time t, defined relative to some base period 0. The indirect change factor between two arbitrary periods, s and t, is then obtained as:

$$\hat{Y}_{t,s} \equiv \frac{Y_{t,0}}{Y_{s,0}}, \tag{4}$$

and similarly for prices. As shown by Kohli (1986), runs of direct indexes have some rather undesirable properties.[8] Their use has also been denounced by Afriat (1977). Yet, they are commonly used in practice, for instance to measure inflation (in terms of the CPI or the GDP deflator) and real growth between consecutive periods.

We can now introduce the concept of an implicit price or quantity index. Implicit indexes will be denoted by a star ($\star$) and they are obtained by deflating the current value index by a price or quantity index. Thus,

$$Y_{t,s}^{\star} \equiv \frac{\Gamma_{t,s}}{P_{t,s}}, \tag{5}$$

and similarly for implicit direct price indexes. Implicit chain indexes and implicit runs of direct indexes are defined in the same way.

2.2. *The Implicit Törnqvist Index of Real GDP*

We now review the various index number formulas to which we alluded earlier. The direct Laspeyres quantity index measuring the change in aggregate quantity between periods 0 and t ($Y^L_{t,0}$) is defined as:

$$Y^L_{t,0} \equiv \frac{\sum_i p_{i0} y_{it}}{\sum_i p_{i0} y_{i0}}. \tag{6}$$

Note that if all prices are normalized to one in the base period (i.e., if $p_{i0} = 1, \forall i$), (6) simplifies to become:

$$Y^L_{t,0} = \frac{\sum_i y_{it}}{\sum_i y_{i0}}. \tag{7}$$

If, moreover, base period real GDP (Y_0) is defined as the sum of its components:

$$Y_0 \equiv \sum_i y_{i0}, \tag{8}$$

it immediately follows that the Laspeyres measure of the level of real GDP in period t (Y^L_t) is equal to:

$$\begin{aligned} Y^L_t &= Y^L_{t,0} \cdot Y_0 \\ &= \sum_i y_{it}. \end{aligned} \tag{9}$$

This last expression is of course well known to anybody familiar with national accounting: real—or constant dollar—GDP is simply equal to the sum of its components. However, there is nothing magical about adding-up, even though it cannot be denied that the Laspeyres measure of real GDP is particularly intuitive. This undoubtedly explains the popularity of this measure. Nevertheless, the precision of adding-up does not necessarily make the Laspeyres an accurate index. It is merely a convenient property, but one should not be dazzled by it.[9] On the contrary, one should be ready to forgo it if it becomes an obstacle to good measurement.[10]

We now turn to the direct Paasche price index. Relative to period 0, it is defined as follows:

$$P^P_{t,0} \equiv \frac{\sum_i p_{it} y_{it}}{\sum_i p_{i0} y_{it}}. \tag{10}$$

Making use of (1), (6), and (10), the following well-known result becomes immediately apparent:[11]

$$\Gamma_{t,0} = Y^L_{t,0} \cdot P^P_{t,0}. \tag{11}$$

Thus, given that real GDP is typically measured as a direct Laspeyres quantity index, the implicit GDP deflator is a direct Paasche price index.

The Törnqvist price index ($P^T_{t,s}$) defined for two arbitrary periods s and t ($s < t$) is as follows:[12]

$$P^T_{t,s} \equiv \prod_{i=1}^{I} \left(\frac{p_{it}}{p_{is}}\right)^{1/2(\mu_{is}+\mu_{it})}, \tag{12}$$

where μ_{it} is the GDP share of the ith component at time t:

$$\mu_{it} \equiv \frac{p_{it}y_{it}}{\sum_i p_{it}y_{it}}, \qquad t = 0, 1, \ldots, T. \tag{13}$$

The Törnqvist chain price index ($\tilde{P}^T_{t,s}$)($s < t-1$) can be defined by compounding the links for consecutive periods:

$$\tilde{P}^T_{t,s} \equiv \prod_{h=s+1}^{t} P^T_{h,h-1}. \tag{14}$$

Following in the footsteps of Diewert and Morrison (1986), we now define the implicit Törnqvist index of real GDP ($Y^{T\star}_{t,s}$):[13]

$$Y^{T\star}_{t,s} \equiv \frac{\Gamma_{t,s}}{P^T_{t,s}}. \tag{15}$$

The corresponding chain index ($\tilde{Y}^{T\star}_{t,s}$) is then obtained as usual by compounding:

$$\tilde{Y}^{T\star}_{t,s} \equiv \prod_{h=s+1}^{t} Y^{T\star}_{h,h-1}. \tag{16}$$

Equivalently, it can be obtained by deflating the nominal GDP index by the Törnqvist chain price index:

$$\tilde{Y}^{T\star}_{t,s} = \frac{\Gamma_{t,s}}{\tilde{P}^T_{t,s}}. \tag{17}$$

3. The Implicit Törnqvist Index of Real GDP and the Translog GDP Function

It is well known that the Törnqvist price index is exact for the Translog cost and revenue functions, just like the Törnqvist quantity index is exact for the Translog production and factor requirements functions; see Diewert (1976). However, these aggregator functions all four describe single-output or single-input technologies. In what follows, we show that the Törnqvist index can also be exact in the multiple-input, multiple-output case. Specifically, we show that the implicit Törnqvist index of real GDP is exact for the Translog GDP function. The GDP function approach to modeling the technology of an open economy treats domestic factor endowments as fixed in the short run, and considers the prices of imports and outputs as given (imports are treated as a negative output). Perfect competition and optimization are assumed. Let there be J domestic factors, and I import and output components (netputs). The vector of domestic factor quantities, measured at time t, is denoted $\mathbf{x}_t \equiv [x_{jt}], j = 1, \ldots, J$. The vector of the variable netput quantities is $\mathbf{y}_t \equiv [y_{it}], i = 1, \ldots, I$. The corresponding price vectors are $\mathbf{w}_t \equiv [w_{jt}]$ and $\mathbf{p}_t \equiv [p_{it}]$. We assume that the technology is convex, and that it exhibits constant returns to scale and free disposals. It can be represented by the following GDP function:[14]

$$\pi = \pi(\mathbf{p}_t, \mathbf{x}_t, t) \equiv \max_{\mathbf{y}} \{\mathbf{p}_t'\mathbf{y} : (\mathbf{y}, \mathbf{x}_t) \in T_t\}, \tag{18}$$

where T_t is the production possibilities set at time t.

The Translog functional form is well suited to describe (18). It is as follows:[15]

$$\begin{aligned} \ln \pi_t = {} & \alpha_0 + \sum_i \alpha_i \ln p_{it} + \sum_j \beta_j \ln x_{jt} + \frac{1}{2} \sum_i \sum_h \gamma_{ih} \ln p_{it} \ln p_{ht} \\ & + \frac{1}{2} \sum_j \sum_k \phi_{jk} \ln x_{jt} \ln x_{kt} + \sum_i \sum_j \delta_{ij} \ln p_{it} \ln x_{jt} \\ & + \sum_i \delta_{iT} t \ln p_{it} + \sum_j \phi_{jT} t \ln x_{jt} + \beta_T t + \frac{1}{2} \phi_{TT} t^2, \\ & i, h = 1, \ldots, I;\ j, k = 1, \ldots, J, \end{aligned} \tag{19}$$

where $\sum \alpha_i = 1$, $\sum \beta_j = 1$, $\gamma_{ih} = \gamma_{hi}$, $\phi_{jk} = \phi_{kj}$, $\sum \gamma_{ih} = 0$, $\sum \phi_{jk} = 0$, $\sum_i \delta_{ij} = 0$, $\sum_j \delta_{ij} = 0$, $\sum \delta_{iT} = 0$, and $\sum \phi_{jT} = 0$. The Translog functional form is flexible, in that it provides a second-order approximation to an arbitrary GDP function. What this means, in economic terms, is that it does not restrict the sign or the size of the various elasticities of transformation, intensity, and substitution.

Differentiation of the GDP function with respect to output prices yields the import demand and output supply functions, while differentiation with respect to factor quantities produces the inverse input demand functions.[16] In terms of shares:

$$\mu_{it} = \alpha_i + \sum_h \gamma_{ih} \ln p_{ht} + \sum_j \delta_{ij} \ln x_{jt} + \delta_{iT} t, \qquad i = 1, \ldots, I, \tag{20}$$

$$\mu_{jt} = \beta_j + \sum_i \delta_{ij} \ln p_{it} + \sum_k \phi_{jk} \ln x_{kt} + \phi_{jT} t, \qquad j = 1, \ldots, J, \tag{21}$$

where $\mu_{it} \equiv p_{it} y_{it} / \pi_t$ and $\mu_{jt} \equiv w_{jt} x_{jt} / \pi_t$ are again the GDP shares of output i and factor j, respectively.

The change in real GDP between periods $t - 1$ and t can be defined as the change in nominal GDP that would occur if factor endowments and/or the technology changed, but if variable netput prices remained the same. To measure the change in real GDP so defined, we can proceed along the same lines as Diewert and Morrison (1986). Thus, we first consider the following Laspeyres-type index ($G^L_{t,t-1}$):

$$G^L_{t,t-1} \equiv \frac{\pi(\mathbf{p}_{t-1}, \mathbf{x}_t, t)}{\pi(\mathbf{p}_{t-1}, \mathbf{x}_{t-1}, t-1)}. \tag{22}$$

This index measures the change in nominal GDP that would have taken place between periods $t-1$ and t, following the observed change in factor endowments and the technology, had output prices remained at their $t-1$ levels. Of course, one can argue that it would be equally reasonable to use period t prices (instead of period $t-1$ prices) as reference. This suggests the use of the following Paasche-type index of real GDP change ($G^P_{t,t-1}$):

$$G^P_{t,t-1} \equiv \frac{\pi(\mathbf{p}_t, \mathbf{x}_t, t)}{\pi(\mathbf{p}_t, \mathbf{x}_{t-1}, t-1)}. \tag{23}$$

Proceeding along the lines of Diewert and Morrison (1986), we can take the geometric mean of these two indexes:

$$G_{t,t-1} \equiv \sqrt{G^L_{t,t-1} \cdot G^P_{t,t-1}} = \sqrt{\frac{\pi(\mathbf{p}_{t-1}, \mathbf{x}_t, t)}{\pi(\mathbf{p}_{t-1}, \mathbf{x}_{t-1}, t-1)} \cdot \frac{\pi(\mathbf{p}_t, \mathbf{x}_t, t)}{\pi(\mathbf{p}_t, \mathbf{x}_{t-1}, t-1)}}. \tag{24}$$

This index, which has the Fisher form, has the attractive feature of taking into account the prices of both periods t and $t-1$, and it therefore seems preferable to either $G^L_{t,t-1}$ or $G^P_{t,t-1}$.

Assume that the technology is given by Translog GDP function (19). If one knew its parameters, one could easily compute the change in real GDP as indicated by (24). This would pave the way to an econometric approach to measuring changes in real GDP. Indeed, from the definition of $G^L_{t,t-1}$, we have:

$$\ln G^L_{t,t-1} = \ln \pi(\mathbf{p}_{t-1}, \mathbf{x}_t, t) - \ln \pi(\mathbf{p}_{t-1}, \mathbf{x}_{t-1}, t-1). \tag{25}$$

Substituting (19) into (25), we obtain:

$$\begin{aligned} \ln G^L_{t,t-1} = & \sum_j \left[\beta_j + \sum_i \delta_{ij} \ln p_{it-1} + \phi_{jT} t\right] (\ln x_{jt} - \ln x_{jt-1}) \\ & + \frac{1}{2} \sum_j \sum_k \phi_{jk} (\ln x_{jt} \ln x_{kt} - \ln x_{jt-1} \ln x_{kt-1}) \\ & + \sum_i \delta_{iT} \ln p_{it-1} + \sum_j \phi_{jT} \ln x_{jt-1} + \beta_T + \frac{1}{2} \phi_{TT} (2t-1). \end{aligned} \tag{26}$$

Similarly, the Paasche index can be derived as:

$$\ln G^P_{t,t-1} = \sum_j \left[\beta_j + \sum_i \delta_{ij} \ln p_{it} + \phi_{jT} t\right] (\ln x_{jt} - \ln x_{jt-1})$$
$$+ \frac{1}{2} \sum_j \sum_k \phi_{jk} (\ln x_{jt} \ln x_{kt} - \ln x_{jt-1} \ln x_{kt-1})$$
$$+ \sum_i \delta_{iT} \ln p_{it} + \sum_j \phi_{jT} \ln x_{jt-1} + \beta_T + \frac{1}{2} \phi_{TT}(2t - 1). \tag{27}$$

From (24), it can be seen that:

$$\ln G_{t,t-1} = \frac{1}{2} (\ln G^L_{t,t-1} + \ln G^P_{t,t-1}). \tag{28}$$

Hence, combining (26) and (27), we finally get:

$$\ln G_{t,t-1} = \sum_j \left[\beta_j + \frac{1}{2} \sum_i \delta_{ij} (\ln p_{it} + \ln p_{it-1}) + \phi_{jT} t\right] (\ln x_{jt} - \ln x_{jt-1})$$
$$+ \frac{1}{2} \sum_j \sum_k \phi_{jk} (\ln x_{jt} \ln x_{kt} - \ln x_{jt-1} \ln x_{kt-1})$$
$$+ \frac{1}{2} \sum_i \delta_{iT} (\ln p_{it} + \ln p_{it-1}) + \sum_j \phi_{jT} \ln x_{jt-1}$$
$$+ \beta_T + \frac{1}{2} \phi_{TT}(2t - 1). \tag{29}$$

The estimation of GDP functions such as (19) can be controversial, however, since it raises issues such as estimation technique and stochastic specification. Moreover, it requires data on domestic factor endowments (e.g., employment and the capital stock), data which are generally not available within the national accounts, and the construction of which may be subject to controversy as well. We therefore prefer to opt for a more straightforward index number approach. Fortunately, it can be shown that, as long as one knows that the GDP function is indeed Translog, one can compute (24) based on knowledge of the data alone; that is, without needing to know the parameters of (19). Moreover, it turns out that $G_{t,t-1}$ is actually identical to $Y^{T\star}_{t,t-1}$.

Consider the nominal GDP index over periods $t - 1$ and $t, \Gamma_{t,t-1}$. In terms of the GDP function, $\Gamma_{t,t-1}$ can be expressed as:

$$\ln \Gamma_{t,t-1} = \ln \pi(\mathbf{p}_t, \mathbf{x}_t, t) - \ln \pi(\mathbf{p}_{t-1}, \mathbf{x}_{t-1}, t - 1). \tag{30}$$

Introducing (19) into (30), we get:

$$\begin{aligned}
\ln \Gamma_{t,t-1} = {} & \sum_i \alpha_i(\ln p_{it} - \ln p_{it-1}) + \sum_j \beta_j(\ln x_{jt} - \ln x_{jt-1}) \\
& + \frac{1}{2}\sum_i \sum_h \gamma_{ih}(\ln p_{it} \ln p_{ht} - \ln p_{it-1} \ln p_{ht-1}) \\
& + \frac{1}{2}\sum_j \sum_k \phi_{jk}(\ln x_{jt} \ln x_{kt} - \ln x_{jt-1} \ln x_{kt-1}) \\
& + \sum_i \sum_j \delta_{ij}(\ln p_{it} \ln x_{jt} - \ln p_{it-1} \ln x_{jt-1}) \\
& + \sum_i \delta_{iT} t(\ln p_{it} - \ln p_{it-1}) + \sum_j \phi_{jT} t(\ln x_{jt} - \ln x_{jt-1}) \\
& + \sum_i \delta_{iT} \ln p_{it-1} + \sum_j \phi_{jT} \ln x_{jt-1} + \beta_T + \frac{1}{2}\phi_{TT}(2t-1). \qquad (31)
\end{aligned}$$

Consider now the difference between $\ln \Gamma_{t,t-1}$ and $\ln G_{t,t-1}$:

$$\begin{aligned}
\ln \Gamma_{t,t-1} - \ln G_{t,t-1} = {} & \sum_i \alpha_i(\ln p_{it} - \ln p_{it-1}) \\
& + \frac{1}{2}\sum_i \sum_h \gamma_{ih}(\ln p_{it} \ln p_{ht} - \ln p_{it-1} \ln p_{ht-1}) \\
& + \frac{1}{2}\sum_i \sum_j \delta_{ij}(\ln p_{it} - \ln p_{it-1})(\ln x_{jt} + \ln x_{jt-1}) \\
& + \frac{1}{2}\sum_i \delta_{iT}(\ln p_{it} - \ln p_{it-1})(2t-1) \\
= {} & \sum_i \left[\alpha_i + \frac{1}{2}\sum_h \gamma_{ih}(\ln p_{ht} + \ln p_{ht-1}) \right. \\
& + \frac{1}{2}\sum_j \delta_{ij}(\ln x_{jt} + \ln x_{jt-1}) \\
& \left. + \frac{1}{2}\delta_{iT}(2t-1)\right](\ln p_{it} - \ln p_{it-1}), \qquad (32)
\end{aligned}$$

where we have exploited the symmetry of the γ_{ih}'s. Making use of (20), (32) can be rewritten as:

$$\ln \Gamma_{t,t-1} - \ln G_{t,t-1} = \sum_i \frac{1}{2}(\mu_{it} + \mu_{it-1})(\ln p_{it} - \ln p_{it-1}). \qquad (33)$$

The right-hand side of (33) is equivalent to (the logarithm of) $P^T_{t,s}$ as defined by (12), for $s = t - 1$.[17] Thus, it is a simple matter now to calculate $G_{t,t-1}$:

$$G_{t,t-1} = \frac{\Gamma_{t,t-1}}{P^T_{t,t-1}}, \tag{34}$$

which is identical to $Y^{T\star}_{t,t-1}$ as defined by (15) for $s = t - 1$. This shows that the implicit Törnqvist real GDP index is exact for the Translog GDP function.[18] Given that the Translog GDP function is flexible,[19] this makes the implicit Törnqvist real GDP index a superlative index.[20]

The GDP function provides a convenient way to model the aggregate technology. Since constant returns to scale, optimization and competitive markets are assumed, aggregation is carried out without difficulty, with all domestic intermediate inputs canceling out.[21] If different industries use different types or vintages of capital, or different categories of labor, aggregation becomes more taxing, but it can nonetheless be carried out by considering the full array of inputs and outputs. Naturally, to keep the model manageable, it is necessary to limit the number of arguments, so that some prior aggregation will need to be carried out, but the same is true for all production—and indeed all economic—models.

If sectoral net outputs are computed using Laspeyres quantity indexes, and if relative output prices change over time, one would normally expect real products to be underestimated in most sectors. Since the Törnqvist index is not consistent over aggregation (the same is true for the Fisher index), the GDP bias will not be exactly equal to the sum of the sectoral biases, but it is generally admitted that this type of discrepancy is numerically unimportant.

4. The Implicit Törnqvist Index of Real GDP and Total Factor Productivity

The above analysis is closely linked to the measurement of total factor productivity. One state-of-the-art measure of total factor productivity is given by the following productivity index ($R_{t,t-1}$) proposed by Diewert and Morrison (1986):[22]

$$R_{t,t-1} = \frac{\Gamma_{t,t-1}}{P^T_{t,t-1} \cdot X^T_{t,t-1}}, \tag{35}$$

where $\Gamma_{t,t-1}$ and $P^T_{t,t-1}$ have already been defined, and $X^T_{t,t-1}$ is a Törnqvist index of domestic factor quantities:

$$X^T_{t,t-1} \equiv \prod_{j=1}^{J} \left(\frac{x_{jt}}{x_{jt-1}} \right)^{1/2(\mu_{jt-1}+\mu_{jt})}. \tag{36}$$

Diewert and Morrison (1986) have shown that, as long as the GDP function has the Translog form, $R_{t,t-1}$ can be interpreted as the change in nominal GDP that results from the improvement in the technology between time $t - 1$ and time t, for given output prices and given domestic factor endowments:[23]

$$R_{t,t-1} \equiv \sqrt{\frac{\pi(\mathbf{p}_{t-1}, \mathbf{x}_{t-1}, t)}{\pi(\mathbf{p}_{t-1}, \mathbf{x}_{t-1}, t-1)} \cdot \frac{\pi(\mathbf{p}_t, \mathbf{x}_t, t)}{\pi(\mathbf{p}_t, \mathbf{x}_t, t-1)}}. \tag{37}$$

$R_{t,t-1}$ can be interpreted as the geometric mean of Laspeyres-type and Paasche-type productivity indexes.[24] Making use of (15), it immediately follows from (35) that the implicit Törnqvist real GDP index can be thought of as the product of the economy's productivity index and the Törnqvist index of domestic factor endowments:

$$Y_{t,t-1}^{T\star} = R_{t,t-1} \cdot X_{t,t-1}^{T}. \tag{38}$$

This interpretation of the implicit Törnqvist real GDP index is fairly intuitive, and it precisely corresponds to our definition of the change in real GDP ("the change in nominal GDP that would occur if factor endowments and/or the technology changed, but if variable netput prices remained the same"). However, as we argued earlier, it is often difficult to get good data on domestic factor endowments, and this is why the alternative route provided by (15) should prove to be more useful.

A useful feature of the implicit Törnqvist quantity index is that it makes it easy to get a multiplicative decomposition of nominal and real GDP growth. Thus, if input data are available, $X_{t,t-1}^{T}$ can be decomposed according to its labor and capital components. Similarly, $P_{t,t-1}^{T}$ can be expressed as the product of the GDP price effects of the individual netput prices. This, together with (35) and (38), yields a complete and exact decomposition of GDP growth, with the contributions of domestic factor endowments, output and import prices, and technology clearly identified.[25]

5. Estimates for a Sample of 26 OECD Countries

According to official figures covering the period 1960 to 1996, Japanese real GDP growth has averaged about 3.5% annually.[26] Our own calculations based on (16) suggest that the true rate of growth is closer to 3.8%. That is, the direct Laspeyres quantity index has underestimated real growth by about 0.3% annually in the case of Japan. Over a 36 year period, the shortfall amounts to about 13.4% of GDP, an amount that is far from trivial. The path of the direct Laspeyres and of the chain implicit Törnqvist indexes are illustrated in Figure 1 (both indexes are set to unity in 1960).

The extent to which the Laspeyres index underestimates real GDP growth varies from country to country, depending on the changes in their relative prices and their output mix over time. It is trivial for some countries, while it appears to be quite substantial for others. Table 1 summarizes the result of our calculations for 26 OECD members. The figures in the table indicate the percentage by which the implicit Törnqvist real GDP index exceeds its direct Laspeyres counterpart by 1996, using 1960 as the base year.[27]

For 15 of the 26 countries in the sample, the shortfall reaches or exceeds 1% of GDP. Besides Japan, it is particularly large for Ireland, South Korea, and Austria.

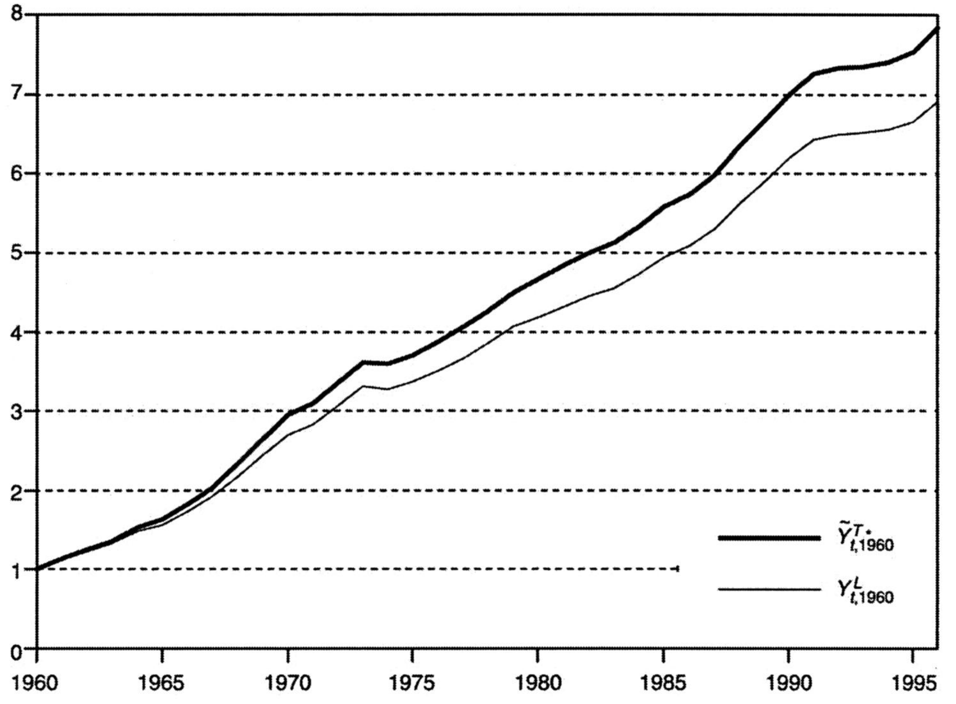

Figure 1. Laspeyres (direct) and implicit Törnqvist (chain) indexes of real GDP, Japan, 1960–1996.

For seven countries, the bias seems trivial, being less than 1% in absolute value. There are four countries for which the bias is quite large and negative: Greece, Iceland, Spain, and Switzerland. Taken at face value, this result contradicts the assumption of convexity of the production possibility set. It may be due to poor data quality.[28] Nonetheless, the bias goes in the expected direction for 20 out of 26 countries, including all the largest ones. For the entire sample, the (unweighted) mean bias amounts to about 1.7% of GDP.

While the average bias may seem small in most cases, this should not mask the fact that the runs of direct Laspeyres indexes can severely underestimate real growth in those years when changes in relative prices or quantities have been particularly large. Thus, in the Japanese case, the bias has exceeded 1% point on several occasions, and it has been as high as 1.8% in 1968. The same observation holds for other countries in the sample.

6. Conclusion

Real GDP is an abstract concept, and any measure of it is necessarily imprecise. There are many problems linked to the measurement of economic aggregates—not least GDP—and here we are just addressing one of them, namely the choice of a

Table 1. Real GDP growth, 1960–1996.

Cumulated shortfall $\tilde{Y}^{T\star}_{1996,1960}/Y^{L}_{1996,1960} - 1(\%)$			
Australia	0.6	Luxembourg	3.1
Austria	4.2	Mexico	1.2
Belgium	−0.2	Netherlands	1.0
Canada	−0.1	New Zealand	2.3
Denmark	0.4	Norway	1.6
Finland	0.8	Portugal	3.1
France	1.3	South Korea	4.4
Germany	1.8	Spain	−2.3
Greece	−2.8	Sweden	0.8
Iceland	−1.5	Switzerland	−3.8
Ireland	7.4	Turkey	2.8
Italy	1.9	United Kingdom	0.6
Japan	13.4	United States	2.0

functional form for index numbers. While all measures of real GDP are necessarily wrong to some extent, it seems sensible to select one that avoids unnecessary biases. Our point is that the official measures of real GDP in most countries contain more flaws than needed, and that superlative indexes tend to point to higher growth. Since real growth is often an argument in the policymakers reaction function, this bias can lead to policy mistakes. In the same vein, real GDP is an argument in many macroeconomic relations estimated by econometric techniques, and measurement errors can easily become a source of estimation biases. The problem can be all the more easily fixed that the data necessary to compute state-of-the-art superlative indexes are readily available in the national accounts.

If real GDP is measured by a Laspeyres quantity index—and thus under-estimated—productivity may well be underestimated as well. This will be particularly true in the case of average labor productivity, for example, GDP per capita or GDP per hour. As far as total factor productivity is concerned, the sign of the bias depends on how input quantities are aggregated. In the unlikely event that a Paasche quantity index is used (i.e., if domestic factor rental prices were aggregated using a Laspeyres price index), the quantity of aggregate input would be underestimated as well,[29] so that the total factor productivity index could be biased either way. If inputs were aggregated using a Laspeyres quantity index instead, aggregate input would be overestimated, and total factor productivity would be severely underestimated. Many empirical studies rely on a rather unsophisticated measure of total factor productivity where a Laspeyres index of real GDP is divided by a Cobb–Douglas index of the quantities of labor and capital.[30] Depending on the Hicksian elasticity of complementarity between the two factors of production, this input quantity index could be biased either way. Given that real GDP will tend to be underestimated, total factor productivity is likely to be underestimated as well. These considerations strongly suggest that it is best to rely on a superlative measure of total factor productivity such as (35).

The Paasche price index tends to overestimate price increases relative to the base period in the context of production theory. Moreover, runs of direct Paasche price indexes over more than two periods may fail to be monotonically increasing in prices. This makes their use as a measure of the price of GDP particularly inadequate.[31] Given the flaws of this price measure, the implicit quantity measure (i.e., runs of direct Laspeyres indexes of real GDP) can hardly be expected to be any more reliable. Any chained superlative index should be able to do a better job. As argued in this paper, the implicit Törnqvist quantity index is a particularly strong candidate that deserves to be given very serious consideration.

As mentioned in the Introduction, a number of countries have recently switched—or are about to do so—to chained Laspeyres quantity indexes for real GDP. Although this is undoubtedly a step in the right direction, one can only regret that they stopped short from adopting a superlative functional form, such as the Fisher or the implicit Törnqvist. Chaining takes care of the monotonicity issue, but chained indexes are not path independent, unless they happen to be exact for the underlying aggregator function. This militates in favor of superlative indexes, that is, indexes that are exact for flexible functional forms that provide a second-order approximation to an arbitrary aggregator function. The failure to opt for superlative indexes is all the more regrettable that the data required to compute them are exactly the same as those necessary to calculate Laspeyres quantity and Paasche price indexes.[32] Luckily, a handful of countries, starting with the United States, have shown the way by selecting the chained Fisher index to compute real GDP. Although we do have a preference for the implicit Törnqvist measure on economic grounds, the move to superlative indexes has to be applauded. In any case, the difference between the Fisher and the implicit Törnqvist indexes should be numerically very small. One can only hope that the example set by the U.S. Bureau of Economic Analysis will be followed by many other statistical agencies in the years to come.

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Notes

1. See Afriat (1977) and Kohli (1986), for instance.
2. They are exact for very restrictive aggregator functions only, namely the linear and the Leontief functions.
3. Landefeld and Parker (1995) argue that real GDP is overstated by the "fixed-weight" GDP index since it does not take substitution effects into account; their argument would only be valid if the composition of output were dominated by demand considerations.
4. The concept of superlative indexes has been introduced by Diewert (1976).

5. See Diewert (1992).
6. See Kohli (1978, 1991) and Woodland (1981) for a discussion of GNP/GDP functions.
7. We adopt here the terminology of Allen (1975) who uses the term "direct" in contrast to "chain" indexes. Diewert (1976), on the other hand, uses the term "direct" indexes to distinguish them from "implicit" indexes, and he uses the term "fixed-base" to designate what we call direct indexes. Direct indexes are sometimes also called "fixed-weight" indexes, but this is misleading since many direct indexes (e.g., the direct Paasche index) do not rely on fixed weights.
8. Kohli (1986) showed that runs of direct Paasche and direct Fisher price indexes fail a simple monotonicity test; that is, the price index can register an increase even if none of the disaggregated prices has increased, and some have actually fallen.
9. It is ironic that Landefeld and Parker (1997) are almost apologetic when admitting that their chained constant dollar estimates "are not precisely additive". They then attempt to reassure the reader by arguing that for observations close to the base year, the "residual" is small. Similarly, Eurostat (1995), in its European system of accounts manual (ESA 1995), states that "An aggregate is defined as the *sum* of its components", and further adds: "When base year values are extrapolated by chain volume indices, it will have to be explained to users why there is no additivity in the tables. The non-additive 'constant price' data is published without any adjustment. This method is transparent and indicates to users the extent of the *problem*. This does not preclude the possibility that there may be circumstances in which compilers may judge it preferable to eliminate the *discrepancy* in order to improve the overall *consistency* of the data" (page 240, our emphasis). In fact, a large "residual" or "discrepancy" should not be viewed as a limitation of the chain index, but rather as an indication of the poor performance of the direct Laspeyres index.
10. In any case, this property is lost as soon as the reference period is changed, assuming that relative prices are not the same as in the base period. The OECD, for instance, publishes national accounts data supplied by its member states, routinely renormalizing price and quantity for a common reference year. Consequently, adding-up of real magnitudes is no longer satisfied at the outset, and the OECD has to include a "residual".
11. This is an illustration of the Fisher factor-reversal test; see Diewert (1993), for instance.
12. Diewert (1976) has shown that the Törnqvist price index is exact for the Translog cost (or revenue) function, and hence it is a superlative index.
13. Another possible candidate for measuring real GDP would be the Törnqvist quantity index, $Y^T_{t,s}$. There are two reasons, however, why we did not retain this option. First, the change in inventories is sometimes positive, sometimes negative, which makes computation of a Tönqvist quantity index rather problematic. Second, as we shall see below, the implicit Törnqvist real GDP index is exact for the Translog GDP function, a description of the technology often used in empirical work, whereas the Törnqvist quantity index is exact for the less well-known Translog national income function; see Kohli (1991, 2003) for a definition of the national income function.
14. See Kohli (1978, 1991) and Woodland (1982) for details.
15. See Christensen et al. (1973) and Diewert (1974).
16. See Kohli (1978), for instance.
17. $P^T_{t,t-1}$ can also be interpreted as the effect on nominal GDP of the change in prices between time $t-1$ and time t, for given factor endowments and given technology:

$$P^T_{t,t-1} = \sqrt{\frac{\pi(\mathbf{p}_t, \mathbf{x}_{t-1}, t-1)}{\pi(\mathbf{p}_{t-1}, \mathbf{x}_{t-1}, t-1)} \cdot \frac{\pi(\mathbf{p}_t, \mathbf{x}_t, t)}{\pi(\mathbf{p}_{t-1}, \mathbf{x}_t, t)}}.$$

See Diewert (1983) for additional details.
18. If the true GDP function is Translog, the implicit Törnqvist index of real GDP will be path independent; this implies that $\tilde{Y}^{T\star}_{t,s} = Y^{T\star}_{t,s} (s < t-1)$ in that case.
19. See Diewert (1974).
20. Alternatively, the proof that $G_{t,t-1}$ is in fact equivalent to $Y^{T\star}_{t,t-1}$ could be derived using the Quadratic Approximation Lemma; see Caves et al. (1982).
21. See Woodland (1982), for instance.

22. This index has been used in empirical work by Morrison and Diewert (1990) and Kohli (1990, 1991) among others. It is different from the measure $Y^T_{t,t-1}/X^T_{t,t-1}$ used by Jorgenson and Griliches (1967) or, in the national accounts context, by Christensen and Cummings (1981). That measure is exact for a Translog transformation function, but not for the Translog GDP function.
23. See Diewert and Morrison (1986, Theorem 1).
24. Naturally, $R_{t,t-1}$ can also be expressed in terms of the parameters of the GDP function by substituting (19) into (38); see Kohli (1990, 1991) for econometric estimates of $R_{t,t-1}$. One advantage of the econometric approach is that it becomes possible to decompose $R_{t,t-1}$ into a secular component and a random term.
25. See Kohli (1990, 1991) for details.
26. All the data in this section are drawn from the OECD National Accounts, Main Aggregates. They are annual figures for the period 1960 to 1996, except for South Korea where they are for the years 1970–1996. The data for Germany are obtained by chaining those for West Germany prior to 1992 with those of the entire country thereafter.
27. The base year is 1970 in the case of South Korea.
28. Swiss national accounts data have long been considered as being of dubious quality. Thus, according to Abrahamsen et al. (2003), "reasonably reliable GDP data do not go back further than 1980". Switzerland has recently proceeded to a massive revision of its national account data for the period 1990–1999. Based on these new figures, we find a positive bias of about 0.8%.
29. This follows from the concavity of the GDP function with respect to the fixed input quantities.
30. See Prescott (2002), for instance.
31. Yet, it is precisely the direct Paasche GDP deflator that Taylor (1993) recommended using as a measure of inflation in his famous rule for monetary policy.
32. Eurostat (1995) expresses a preference for the Fisher formula. Nonetheless, it holds the view that "chain indices that use Laspeyres volume indices to measure changes in volume and Paasche price indices to measure year to year price movements provide acceptable alternatives to Fisher indices" (page 240).

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