

## Simplifying Equations

Nicolas's "initial" equations are

$$\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial z} - \frac{iv'}{2} \frac{\partial^2}{\partial z^2} + i\bar{\omega} \right) \langle \psi \rangle = i\zeta |\langle \psi \rangle|^2 \langle \psi \rangle, \quad (1)$$

$$\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial z} - \frac{iv'}{2} \frac{\partial^2}{\partial z^2} + i\bar{\omega} \right) \delta \psi = i\zeta \langle \psi \rangle^2 \delta \psi^\dagger + 2i\zeta |\langle \psi \rangle|^2 \delta \psi. \quad (2)$$

In what follows, we will put them more in the form of Agrawal, making connections with the nonlinear parameter  $\gamma$ , the GVD parameter  $\beta_2$ , and writing operators with units of the square root of power. We first put

$$\begin{aligned} \langle \psi \rangle &= \langle \tilde{\psi} \rangle e^{-i\bar{\omega}t}, \\ \delta \psi &= \delta \tilde{\psi} e^{-i\bar{\omega}t}, \end{aligned} \quad (3)$$

to find

$$\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial z} - \frac{iv'}{2} \frac{\partial^2}{\partial z^2} \right) \langle \tilde{\psi} \rangle = i\zeta |\langle \tilde{\psi} \rangle|^2 \langle \tilde{\psi} \rangle, \quad (4)$$

$$\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial z} - \frac{iv'}{2} \frac{\partial^2}{\partial z^2} \right) \delta \tilde{\psi} = i\zeta \langle \tilde{\psi} \rangle^2 \delta \tilde{\psi}^\dagger + 2i\zeta |\langle \tilde{\psi} \rangle|^2 \delta \tilde{\psi}. \quad (5)$$

We then introduce

$$\begin{aligned} \langle A \rangle &= \sqrt{\hbar\omega_P v} \langle \tilde{\psi} \rangle, \\ \langle \delta A \rangle &= \sqrt{\hbar\omega_P v} \delta \tilde{\psi}, \end{aligned} \quad (6)$$

and find

$$\left( \frac{1}{v} \frac{\partial}{\partial t} + \frac{\partial}{\partial z} - \frac{iv'}{2v} \frac{\partial^2}{\partial z^2} \right) \langle A \rangle = i \frac{\zeta}{\hbar\omega_P v^2} |\langle A \rangle|^2 \langle A \rangle, \quad (7)$$

$$\left( \frac{1}{v} \frac{\partial}{\partial t} + \frac{\partial}{\partial z} - \frac{iv'}{2v} \frac{\partial^2}{\partial z^2} \right) \delta A = i \frac{\zeta}{\hbar\omega_P v^2} \langle A \rangle^2 \delta A^\dagger + 2i \frac{\zeta}{\hbar\omega_P v^2} |\langle A \rangle|^2 \delta A. \quad (8)$$

From John's notes [Eq. (38)] we have that

$$\zeta = \frac{3}{\varepsilon_0 \hbar} \left( \frac{\hbar\omega_P}{2} \right)^2 \int \Gamma_3^{ijlm}(x, y) [d^i(x, y)]^* [d^j(x, y)]^* d^l(x, y) d^m(x, y) dx dy. \quad (9)$$

We can then use

$$\Gamma_3^{ijlm}(x, y) = \frac{\chi_3^{ijlm}(x, y)}{\varepsilon_0^2 n^8(x, y; \omega_P)}, \quad (10)$$

and, defining

$$\frac{1}{\mathcal{A}} = \int \frac{\bar{n}^4}{\bar{\chi}_3} \frac{\chi_3^{ijlm}(x, y)}{\varepsilon_0^2 n^8(x, y; \omega_P)} [d^i(x, y)]^* [d^j(x, y)]^* d^l(x, y) d^m(x, y) dx dy, \quad (11)$$

write our nonlinearity as

$$\frac{\zeta}{\hbar\omega_P v^3} = \frac{3\omega_P \bar{\chi}_3}{4\varepsilon_0 v^2 \bar{n}^4 \mathcal{A}} = \gamma, \quad (12)$$

making our equations

$$\left( \frac{1}{v} \frac{\partial}{\partial t} + \frac{\partial}{\partial z} - \frac{iv'}{2v} \frac{\partial^2}{\partial z^2} \right) \langle A \rangle = i\gamma |\langle A \rangle|^2 \langle A \rangle, \quad (13)$$

$$\left( \frac{1}{v} \frac{\partial}{\partial t} + \frac{\partial}{\partial z} - \frac{iv'}{2v} \frac{\partial^2}{\partial z^2} \right) \delta A = i\gamma \langle A \rangle^2 \delta A^\dagger + 2i\gamma |\langle A \rangle|^2 \delta A. \quad (14)$$

Finally, with

$$\begin{aligned} v &= \frac{1}{\beta_1}, \\ v' &\approx -\beta_2 v^3, \end{aligned} \quad (15)$$

we find

$$\left( \beta_1 \frac{\partial}{\partial t} + \frac{\partial}{\partial z} + \frac{i\beta_2}{2} v^2 \frac{\partial^2}{\partial z^2} \right) \langle A \rangle = i\gamma |\langle A \rangle|^2 \langle A \rangle, \quad (16)$$

$$\left( \beta_1 \frac{\partial}{\partial t} + \frac{\partial}{\partial z} + \frac{i\beta_2}{2} v^2 \frac{\partial^2}{\partial z^2} \right) \delta A = i\gamma \langle A \rangle^2 \delta A^\dagger + 2i\gamma |\langle A \rangle|^2 \delta A, \quad (17)$$

the first much as in Agrawal's Eq. (2.3.28) (with  $\alpha = 0$  there). Making the change of variables

$$\begin{aligned} Z &= z - t/\beta_1, \\ T &= t, \end{aligned} \quad (18)$$

implying

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial T}{\partial t} \frac{\partial}{\partial T} + \frac{\partial Z}{\partial t} \frac{\partial}{\partial Z} = \frac{\partial}{\partial T} - \frac{1}{\beta_1} \frac{\partial}{\partial Z} \\ \frac{\partial}{\partial z} &= \frac{\partial T}{\partial z} \frac{\partial}{\partial T} + \frac{\partial Z}{\partial z} \frac{\partial}{\partial Z} = \frac{\partial}{\partial Z}, \end{aligned} \quad (19)$$

we can write

$$\left( \frac{\partial}{\partial T} + \frac{i\beta_2}{2} v^3 \frac{\partial^2}{\partial Z^2} \right) \langle A \rangle = i\gamma v |\langle A \rangle|^2 \langle A \rangle, \quad (20)$$

$$\left( \frac{\partial}{\partial T} + \frac{i\beta_2}{2} v^3 \frac{\partial^2}{\partial Z^2} \right) \delta A = i\gamma v \langle A \rangle^2 \delta A^\dagger + 2i\gamma v |\langle A \rangle|^2 \delta A. \quad (21)$$

Following Agrawal, we introduce a normalized field operator

$$\begin{aligned} \langle A \rangle &= \sqrt{P_0} \langle U \rangle, \\ \delta A &= \sqrt{P_0} \delta U, \end{aligned} \quad (22)$$

a length scale normalised to the input pulse width  $Z_0$

$$\mathcal{Z} = \frac{Z}{Z_0}, \quad (23)$$

as well as the dispersion time

$$T_D = \frac{Z_0^2}{|\beta_2 v^3|}, \quad (24)$$

and nonlinear time

$$T_{NL} = \frac{1}{\gamma P_0 v}, \quad (25)$$

to arrive at

$$\left( \frac{\partial}{\partial T} + \frac{i \operatorname{sgn}(\beta_2 v^3)}{2T_D} \frac{\partial^2}{\partial \mathcal{Z}^2} \right) \langle U \rangle = \frac{i}{T_{NL}} |\langle U \rangle|^2 \langle U \rangle, \quad (26)$$

$$\left( \frac{\partial}{\partial T} + \frac{i \operatorname{sgn}(\beta_2 v^3)}{2T_D} \frac{\partial^2}{\partial \mathcal{Z}^2} \right) \delta U = \frac{i}{T_{NL}} \langle U \rangle^2 \delta U^\dagger + 2 \frac{i}{T_{NL}} |\langle U \rangle|^2 \delta U. \quad (27)$$

## Solving Equations

### Mean Field

We solve for  $\langle U \rangle$  using a split-step Fourier approach. Writing

$$\frac{\partial \langle U(T, \mathcal{Z}) \rangle}{\partial T} = [D(\mathcal{Z}) + N(T)] \langle U(T, \mathcal{Z}) \rangle, \quad (28)$$

where

$$D(\mathcal{Z}) = -\frac{i \operatorname{sgn}(\beta_2 v^3)}{2T_D} \frac{\partial^2}{\partial \mathcal{Z}^2}, \quad (29)$$

$$N(T) = \frac{i}{T_{NL}} |\langle U(T, \mathcal{Z}) \rangle|^2, \quad (30)$$

we approximate that

$$\langle U(T+h, \mathcal{Z}) \rangle \approx \exp \left[ \frac{h}{2} D(\mathcal{Z}) \right] \exp \left[ \int_T^{T+h} N(T') dT' \right] \exp \left[ \frac{h}{2} D(\mathcal{Z}) \right] \langle U(T, \mathcal{Z}) \rangle. \quad (31)$$

Furthermore, the dispersion is best dealt with in Fourier space

$$\langle U(T, \mathcal{Z}) \rangle = \int \frac{d\kappa}{\sqrt{2\pi}} \langle a_{\bar{\kappa}+\kappa}(T) \rangle e^{i\kappa \mathcal{Z}}, \quad (32)$$

where  $\hat{D}$  becomes

$$\tilde{D}(i\kappa) = \frac{i \operatorname{sgn}(\beta_2 v^3)}{2T_D} \kappa^2, \quad (33)$$

and, for small enough  $h$ , we can approximate

$$\int_T^{T+h} N(T') dT' \approx hN(T), \quad (34)$$

such that, ultimately

$$\langle U(T+h, \mathcal{Z}) \rangle \approx F_{\mathcal{Z}}^{-1} \exp \left[ \frac{h}{2} D(i\kappa) \right] F_{\mathcal{Z}} \exp [hN(T)] F_{\mathcal{Z}}^{-1} \exp \left[ \frac{h}{2} D(i\kappa) \right] F_{\mathcal{Z}} \langle U(T, \mathcal{Z}) \rangle, \quad (35)$$

where the operator  $F_{\mathcal{Z}}$  is the Fourier transform operator.

As checks on our results, we can artificially set  $N = 0$  such that the equation of motion becomes

$$\frac{\partial}{\partial T} \langle U(T, \mathcal{Z}) \rangle = \frac{i \operatorname{sgn}(\beta_2 v^3)}{2T_D} \frac{\partial^2}{\partial \mathcal{Z}^2} \langle U(T, \mathcal{Z}) \rangle, \quad (36)$$

or

$$\frac{\partial}{\partial T} \langle a_{\bar{k}+\kappa}(T) \rangle = -\frac{i \operatorname{sgn}(\beta_2 v^3)}{2T_D} \kappa^2 \langle a_{\bar{k}+\kappa}(T) \rangle, \quad (37)$$

with solution

$$\langle a_{\bar{k}+\kappa}(T) \rangle = \langle a_{\bar{k}+\kappa}(0) \rangle \exp \left( -\frac{i \operatorname{sgn}(\beta_2 v^3)}{2T_D} \kappa^2 T \right). \quad (38)$$

Substituting this in Eq. (32) we find

$$\langle U(T, \mathcal{Z}) \rangle = \int \frac{d\kappa}{\sqrt{2\pi}} \langle a_{\bar{k}+\kappa}(0) \rangle \exp \left( i\kappa \mathcal{Z} - \frac{i \operatorname{sgn}(\beta_2 v^3)}{2T_D} \kappa^2 T \right). \quad (39)$$

Thus, for

$$\langle U(0, \mathcal{Z}) \rangle = \frac{1}{(\pi)^{1/4}} \exp \left( -\frac{\mathcal{Z}^2}{2} \right), \quad (40)$$

with a FWHM of  $2\sqrt{2 \ln(2)}$ , after a time  $T$  the FWHM has increased by a factor of  $\sqrt{1 + (T/T_D)^2}$ . Alternately, when  $D = 0$ , the equation of motion becomes

$$\frac{\partial \langle U(T, \mathcal{Z}) \rangle}{\partial T} = \frac{i}{T_{NL}} |\langle U(T, \mathcal{Z}) \rangle|^2 \langle U(T, \mathcal{Z}) \rangle. \quad (41)$$

Writing

$$\frac{\partial |\langle U(T, \mathcal{Z}) \rangle|^2}{\partial T} = \frac{\partial \langle U(T, \mathcal{Z}) \rangle}{\partial T} \langle U^*(T, \mathcal{Z}) \rangle + \frac{\partial \langle U^*(T, \mathcal{Z}) \rangle}{\partial T} \langle U(T, \mathcal{Z}) \rangle, \quad (42)$$

and using Eq. (41), it is relatively easy to show that

$$\frac{\partial |\langle U(T, \mathcal{Z}) \rangle|^2}{\partial T} = 0, \quad (43)$$

and thus we can write the equation of motion as

$$\frac{\partial \langle U(T, \mathcal{Z}) \rangle}{\partial T} = \frac{i}{T_{NL}} |\langle U(0, \mathcal{Z}) \rangle|^2 \langle U(T, \mathcal{Z}) \rangle, \quad (44)$$

with solution

$$\langle U(T, \mathcal{Z}) \rangle = e^{\frac{i}{T_{NL}} |\langle U(0, \mathcal{Z}) \rangle|^2 T} \langle U(0, \mathcal{Z}) \rangle. \quad (45)$$

Thus

$$|F_{\mathcal{Z}} \langle U(T, \mathcal{Z}) \rangle|^2 = \left| \int \frac{d\mathcal{Z}}{\sqrt{2\pi}} \langle U(0, \mathcal{Z}) \rangle \int e^{\frac{i}{T_{NL}} |\langle U(0, \mathcal{Z}) \rangle|^2 T - i\kappa \mathcal{Z}} \right|^2. \quad (46)$$

## Fluctuations

We write the coupled fluctuation equations as

$$\begin{aligned} \frac{\partial \delta U(T, \mathcal{Z})}{\partial T} = & -\frac{i \operatorname{sgn}(\beta_2 v^3)}{2T_D} \frac{\partial^2}{\partial \mathcal{Z}^2} \delta U(T, \mathcal{Z}) + \frac{i}{T_{NL}} \langle U(T, \mathcal{Z}) \rangle^2 \delta U^\dagger(T, \mathcal{Z}) \\ & + 2 \frac{i}{T_{NL}} |\langle U(T, \mathcal{Z}) \rangle|^2 \delta U(T, \mathcal{Z}), \end{aligned} \quad (47)$$

$$\begin{aligned} \frac{\partial \delta U^\dagger(T, \mathcal{Z})}{\partial T} = & \frac{i \operatorname{sgn}(\beta_2 v^3)}{2T_D} \frac{\partial^2}{\partial \mathcal{Z}^2} \delta U^\dagger(T, \mathcal{Z}) - \frac{i}{T_{NL}} \langle U^\dagger(T, \mathcal{Z}) \rangle^2 \delta U(T, \mathcal{Z}) \\ & - 2 \frac{i}{T_{NL}} |\langle U(T, \mathcal{Z}) \rangle|^2 \delta U^\dagger(T, \mathcal{Z}), \end{aligned} \quad (48)$$

or, Fourier transforming

$$\begin{aligned} \frac{\partial \delta a_{\bar{k}+\kappa}(T)}{\partial T} = & \frac{i \operatorname{sgn}(\beta_2 v^3)}{2T_D} \kappa^2 \delta a_{\bar{k}+\kappa}(T) + i \int \frac{d\kappa'}{\sqrt{2\pi}} \mathcal{S}(\kappa + \kappa', T) \delta a_{\bar{k}+\kappa'}^\dagger(T) \\ & + 2i \int \frac{d\kappa'}{\sqrt{2\pi}} \mathcal{M}(\kappa - \kappa', T) \delta a_{\bar{k}+\kappa'}(T), \end{aligned} \quad (49)$$

$$\begin{aligned} \frac{\partial \delta a_{\bar{k}+\kappa}^\dagger(T)}{\partial T} = & -\frac{i \operatorname{sgn}(\beta_2 v^3)}{2T_D} \kappa^2 \delta a_{\bar{k}+\kappa}^\dagger(T) - i \int \frac{d\kappa'}{\sqrt{2\pi}} \mathcal{S}^\dagger(\kappa + \kappa', T) \delta a_{\bar{k}+\kappa'}(T) \\ & - 2i \int \frac{d\kappa'}{\sqrt{2\pi}} \mathcal{M}^*(\kappa - \kappa', T) \delta a_{\bar{k}+\kappa'}^\dagger(T), \end{aligned} \quad (50)$$

where we have introduced

$$\mathcal{S}(\kappa, T) = \int \frac{d\mathcal{Z}}{\sqrt{2\pi}} \frac{1}{T_{NL}} \langle U(T, \mathcal{Z}) \rangle^2 e^{-i\kappa \mathcal{Z}}, \quad (51)$$

$$\mathcal{M}(\kappa, T) = \int \frac{d\mathcal{Z}}{\sqrt{2\pi}} \frac{1}{T_{NL}} |\langle U(T, \mathcal{Z}) \rangle|^2 e^{-i\kappa \mathcal{Z}}. \quad (52)$$

Discretizing

$$\kappa_j = j \Delta \kappa, \quad (53)$$

such that

$$\int \frac{d\kappa'}{\sqrt{2\pi}} \mathcal{S}(\kappa + \kappa', T) \delta a_{\bar{k}+\kappa'}^\dagger(T) \approx \sum_{j'} \frac{\Delta\kappa}{\sqrt{2\pi}} \mathcal{S}(\kappa_j + \kappa_{j'}, T) \delta a_{\bar{k}+\kappa_{j'}}^\dagger(T),$$

$$\int \frac{d\kappa'}{\sqrt{2\pi}} \mathcal{M}(\kappa - \kappa', T) \delta a_{\bar{k}+\kappa'}(T) \approx \sum_{j'} \frac{\Delta\kappa}{\sqrt{2\pi}} \mathcal{M}(\kappa_j - \kappa_{j'}, T) \delta a_{\bar{k}+\kappa_{j'}}(T),$$

in vector notation, we can write

$$\frac{\partial}{\partial T} \begin{pmatrix} \delta a_{\bar{k}+\kappa_j}(T) \\ \delta a_{\bar{k}+\kappa_j}^\dagger(T) \end{pmatrix} = iQ_{jj'}(T) \begin{pmatrix} \delta a_{\bar{k}+\kappa_{j'}}(T) \\ \delta a_{\bar{k}+\kappa_{j'}}^\dagger(T) \end{pmatrix}, \quad (54)$$

where

$$Q_{jj'}(T) = \begin{pmatrix} A_{jj'}(T) & B_{jj'}(T) \\ -B_{jj'}^\dagger(T) & -A_{jj'}^*(T) \end{pmatrix}, \quad (55)$$

$$A_{jj'}(T) = \frac{\text{sgn}(\beta_2 v^3)}{2T_D} \kappa_j^2 \delta_{jj'} + 2 \frac{\Delta\kappa}{\sqrt{2\pi}} \mathcal{M}(\kappa_j - \kappa_{j'}, T), \quad (56)$$

$$B_{jj'}(T) = \frac{\Delta\kappa}{\sqrt{2\pi}} \mathcal{S}(\kappa_j + \kappa_{j'}, T). \quad (57)$$

For a small enough propagation forward in time  $\Delta T$  this has solution

$$\begin{pmatrix} \delta a_{\bar{k}+\kappa_j}(T + \Delta T) \\ \delta a_{\bar{k}+\kappa_j}^\dagger(T + \Delta T) \end{pmatrix} = \exp[i\Delta T Q_{jj'}(\Delta T)] \begin{pmatrix} \delta a_{\bar{k}+\kappa_{j'}}(T) \\ \delta a_{\bar{k}+\kappa_{j'}}^\dagger(T) \end{pmatrix}$$

$$\equiv \begin{pmatrix} X_{jj'}(\Delta T) & W_{jj'}(\Delta T) \\ W_{jj'}^*(\Delta T) & X_{jj'}^*(\Delta T) \end{pmatrix} \begin{pmatrix} \delta a_{\bar{k}+\kappa_{j'}}(T) \\ \delta a_{\bar{k}+\kappa_{j'}}^\dagger(T) \end{pmatrix}. \quad (58)$$

## Moments

Ultimately, what we want to calculate are objects such as

$$M_{jj'}(T) = \left\langle \delta a_{\bar{k}+\kappa_j}(T) \delta a_{\bar{k}+\kappa_{j'}}(T) \right\rangle, \quad (59)$$

$$N_{jj'}(T) = \left\langle \delta a_{\bar{k}+\kappa_j}^\dagger(T) \delta a_{\bar{k}+\kappa_{j'}}(T) \right\rangle. \quad (60)$$

Thus, using the solution above, in the absence of any loss, we find

$$\begin{aligned} M_{jj'}(T + \Delta T) = & X_{jj''}(\Delta T) M_{j''j'''}(T) X_{j''j'}^T(\Delta T) + W_{jj''}(\Delta T) M_{j''j'''}^*(T) W_{j''j'}^T(\Delta T) \\ & + W_{jj''}(\Delta T) N_{j''j'''}(T) X_{j''j'}^T(\Delta T) + X_{jj''}(\Delta T) N_{j''j'''}^*(T) W_{j''j'}^T(\Delta T) \\ & + X_{jj''}(\Delta T) W_{j''j'}^T(\Delta T), \end{aligned} \quad (61)$$

$$\begin{aligned}
N_{jj'}(T + \Delta T) = & W_{jj''}^*(\Delta T) M_{j''j'''}(T) X_{j'''j'}^T(\Delta T) + X_{jj''}^*(\Delta T) M_{j''j'''}^*(T) W_{j'''j'}^T(\Delta T) \\
& + X_{jj''}^*(\Delta T) N_{j''j'''}(T) X_{j'''j'}^T(\Delta T) + W_{jj''}^*(\Delta T) N_{j''j'''}^T(T) W_{j'''j'}^T(\Delta T) \\
& + W_{jj''}^*(\Delta T) W_{j'''j'}^T(\Delta T).
\end{aligned} \tag{62}$$

We can include loss by alternating the time steps above with

$$M_{jj'}(T + \Delta T) = (1 - \Gamma \Delta T) M_{jj'}(T), \tag{63}$$

$$N_{jj'}(T + \Delta T) = (1 - \Gamma \Delta T) N_{jj'}(T), \tag{64}$$

where  $\Gamma$  is the loss rate.