Simplifying Equations

Nicolas's "initial" equations are

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}-\frac{iv'}{2}\frac{\partial^2}{\partial z^2}+i\bar{\omega}\right)\langle\psi\rangle=i\zeta\left|\langle\psi\rangle\right|^2\langle\psi\rangle, \tag{1}$$

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}-\frac{iv'}{2}\frac{\partial^2}{\partial z^2}+i\bar{\omega}\right)\delta\psi=i\zeta\langle\psi\rangle^2\delta\psi^\dagger+2i\zeta\left|\langle\psi\rangle\right|^2\delta\psi. \tag{2}$$

In what follows, we will put them more in the form of Agrawal, making connections with the nonlinear parameter γ, the GVD parameter β_2, and writing operators with units of the square root of power. We first put

$$\begin{aligned}\langle\psi\rangle=&\left\langle\tilde{\psi}\right\rangle e^{-i\bar{\omega}t},\\ \delta\psi=&\delta\tilde{\psi}e^{-i\bar{\omega}t},\end{aligned} \tag{3}$$

to find

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}-\frac{iv'}{2}\frac{\partial^2}{\partial z^2}\right)\left\langle\tilde{\psi}\right\rangle=i\zeta\left|\left\langle\tilde{\psi}\right\rangle\right|^2\left\langle\tilde{\psi}\right\rangle, \tag{4}$$

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}-\frac{iv'}{2}\frac{\partial^2}{\partial z^2}\right)\delta\tilde{\psi}=i\zeta\left\langle\tilde{\psi}\right\rangle^2\delta\tilde{\psi}^\dagger+2i\zeta\left|\left\langle\tilde{\psi}\right\rangle\right|^2\delta\tilde{\psi}. \tag{5}$$

We then introduce

$$\begin{aligned}\langle A\rangle=&\sqrt{\hbar\omega_{\mathrm{P}}v}\left\langle\tilde{\psi}\right\rangle,\\ \langle\delta A\rangle=&\sqrt{\hbar\omega_{\mathrm{P}}v}\delta\tilde{\psi},\end{aligned} \tag{6}$$

and find

$$\left(\frac{1}{v}\frac{\partial}{\partial t}+\frac{\partial}{\partial z}-\frac{iv'}{2v}\frac{\partial^2}{\partial z^2}\right)\langle A\rangle=i\frac{\zeta}{\hbar\omega_{\mathrm{P}}v^2}\left|\langle A\rangle\right|^2\langle A\rangle, \tag{7}$$

$$\left(\frac{1}{v}\frac{\partial}{\partial t}+\frac{\partial}{\partial z}-\frac{iv'}{2v}\frac{\partial^2}{\partial z^2}\right)\delta A=i\frac{\zeta}{\hbar\omega_{\mathrm{P}}v^2}\langle A\rangle^2\delta A^\dagger+2i\frac{\zeta}{\hbar\omega_{\mathrm{P}}v^2}\left|\langle A\rangle\right|^2\delta A. \tag{8}$$

From John's notes [Eq. (38)] we have that

$$\zeta=\frac{3}{\varepsilon_0\hbar}\left(\frac{\hbar\omega_{\mathrm{P}}}{2}\right)^2\int\Gamma_3^{ijlm}(x,y)\left[d^i(x,y)\right]^*\left[d^j(x,y)\right]^*d^l(x,y)\,d^m(x,y)\,\mathrm{d}x\mathrm{d}y. \tag{9}$$

We can then use

$$\Gamma_3^{ijlm}(x,y)=\frac{\chi_3^{ijlm}(x,y)}{\varepsilon_0^2n^8(x,y;\omega_{\mathrm{P}})}, \tag{10}$$

and, defining

$$\frac{1}{\mathcal{A}}=\int\frac{\bar{n}^4}{\bar{\chi}_3}\frac{\chi_3^{ijlm}(x,y)}{\varepsilon_0^2n^8(x,y;\omega_{\mathrm{P}})}\left[d^i(x,y)\right]^*\left[d^j(x,y)\right]^*d^l(x,y)\,d^m(x,y)\,\mathrm{d}x\mathrm{d}y, \tag{11}$$

write our nonlinearity as

$$\frac{\zeta}{\hbar\omega_{\mathrm{P}}v^3} = \frac{3\omega_{\mathrm{P}}\bar{\chi}_3}{4\varepsilon_0 v^2\bar{n}^4\mathcal{A}} = \gamma, \tag{12}$$

making our equations

$$\left(\frac{1}{v}\frac{\partial}{\partial t} + \frac{\partial}{\partial z} - \frac{iv'}{2v}\frac{\partial^2}{\partial z^2}\right)\langle A\rangle = i\gamma\left|\langle A\rangle\right|^2\langle A\rangle\,, \tag{13}$$

$$\left(\frac{1}{v}\frac{\partial}{\partial t} + \frac{\partial}{\partial z} - \frac{iv'}{2v}\frac{\partial^2}{\partial z^2}\right)\delta A = i\gamma\langle A\rangle^2\delta A^\dagger + 2i\gamma\left|\langle A\rangle\right|^2\delta A. \tag{14}$$

Finally, with

$$\begin{aligned} v =& \frac{1}{\beta_1}, \\ v' \approx& -\beta_2 v^3, \end{aligned} \tag{15}$$

we find

$$\left(\beta_1\frac{\partial}{\partial t} + \frac{\partial}{\partial z} + \frac{i\beta_2}{2}v^2\frac{\partial^2}{\partial z^2}\right)\langle A\rangle = i\gamma\left|\langle A\rangle\right|^2\langle A\rangle\,, \tag{16}$$

$$\left(\beta_1\frac{\partial}{\partial t} + \frac{\partial}{\partial z} + \frac{i\beta_2}{2}v^2\frac{\partial^2}{\partial z^2}\right)\delta A = i\gamma\langle A\rangle^2\delta A^\dagger + 2i\gamma\left|\langle A\rangle\right|^2\delta A, \tag{17}$$

the first much as in Agrawal's Eq. (2.3.28) (with $\alpha = 0$ there). Making the change of variables

$$\begin{aligned} Z =& z - t/\beta_1, \\ T =& t, \end{aligned} \tag{18}$$

implying

$$\begin{aligned} \frac{\partial}{\partial t} =& \frac{\partial T}{\partial t}\frac{\partial}{\partial T} + \frac{\partial Z}{\partial t}\frac{\partial}{\partial Z} = \frac{\partial}{\partial T} - \frac{1}{\beta_1}\frac{\partial}{\partial Z} \\ \frac{\partial}{\partial z} =& \frac{\partial T}{\partial z}\frac{\partial}{\partial T} + \frac{\partial Z}{\partial z}\frac{\partial}{\partial Z} = \frac{\partial}{\partial Z}, \end{aligned} \tag{19}$$

we can write

$$\left(\frac{\partial}{\partial T} + \frac{i\beta_2}{2}v^3\frac{\partial^2}{\partial Z^2}\right)\langle A\rangle = i\gamma v\left|\langle A\rangle\right|^2\langle A\rangle\,, \tag{20}$$

$$\left(\frac{\partial}{\partial T} + \frac{i\beta_2}{2}v^3\frac{\partial^2}{\partial Z^2}\right)\delta A = i\gamma v\langle A\rangle^2\delta A^\dagger + 2i\gamma v\left|\langle A\rangle\right|^2\delta A. \tag{21}$$

Following Agrawal, we introduce a normalized field operator

$$\begin{aligned} \langle A\rangle =& \sqrt{P_0}\,\langle U\rangle\,, \\ \delta A =& \sqrt{P_0}\delta U, \end{aligned} \tag{22}$$

a length scale normalised to the input pulse width Z_0

$$\mathcal{Z} = \frac{Z}{Z_0}, \tag{23}$$

as well as the dispersion time

$$T_D = \frac{Z_0^2}{|\beta_2 v^3|}, \tag{24}$$

and nonlinear time

$$T_{NL} = \frac{1}{\gamma P_0 v}, \tag{25}$$

to arrive at

$$\left(\frac{\partial}{\partial T} + \frac{i\mathrm{sgn}\left(\beta_2 v^3\right)}{2T_D}\frac{\partial^2}{\partial \mathcal{Z}^2}\right)\langle U\rangle = \frac{i}{T_{NL}}\left|\langle U\rangle\right|^2\langle U\rangle\,, \tag{26}$$

$$\left(\frac{\partial}{\partial T} + \frac{i\mathrm{sgn}\left(\beta_2 v^3\right)}{2T_D}\frac{\partial^2}{\partial \mathcal{Z}^2}\right)\delta U = \frac{i}{T_{NL}}\langle U\rangle^2\delta U^\dagger + 2\frac{i}{T_{NL}}\left|\langle U\rangle\right|^2\delta U. \tag{27}$$

Solving Equations

Mean Field

We solve for $\langle U\rangle$ using a split-step Fourier approach. Writing

$$\frac{\partial\left\langle U\left(T,\mathcal{Z}\right)\right\rangle}{\partial T} = \left[D\left(\mathcal{Z}\right) + N\left(T\right)\right]\left\langle U\left(T,\mathcal{Z}\right)\right\rangle, \tag{28}$$

where

$$D\left(\mathcal{Z}\right) = -\frac{i\mathrm{sgn}\left(\beta_2 v^3\right)}{2T_D}\frac{\partial^2}{\partial \mathcal{Z}^2}, \tag{29}$$

$$N\left(T\right) = \frac{i}{T_{NL}}\left|\left\langle U\left(T,\mathcal{Z}\right)\right\rangle\right|^2, \tag{30}$$

we approximate that

$$\left\langle U\left(T+h,\mathcal{Z}\right)\right\rangle \approx \exp\left[\frac{h}{2}D\left(\mathcal{Z}\right)\right]\exp\left[\int_T^{T+h} N\left(T'\right)\mathrm{d}T'\right]\exp\left[\frac{h}{2}D\left(\mathcal{Z}\right)\right]\left\langle U\left(T,\mathcal{Z}\right)\right\rangle. \tag{31}$$

Furthermore, the dispersion is best dealt with in Fourier space

$$\left\langle U\left(T,\mathcal{Z}\right)\right\rangle = \int\frac{\mathrm{d}\kappa}{\sqrt{2\pi}}\left\langle a_{\bar{k}+\kappa}\left(T\right)\right\rangle e^{i\kappa\mathcal{Z}}, \tag{32}$$

where $\hat{D}$ becomes

$$\tilde{D}\left(i\kappa\right) = \frac{i\mathrm{sgn}\left(\beta_2 v^3\right)}{2T_D}\kappa^2, \tag{33}$$

and, for small enough h, we can approximate

$$\int_{T}^{T+h} N\left(T'\right) \mathrm{d}T' \approx hN\left(T\right), \tag{34}$$

such that, ultimately

$$\left\langle U\left(T+h,\mathcal{Z}\right)\right\rangle \approx F_{\mathcal{Z}}^{-1} \exp\left[\frac{h}{2}D\left(i\kappa\right)\right] F_{\mathcal{Z}} \exp\left[hN\left(T\right)\right] F_{\mathcal{Z}}^{-1} \exp\left[\frac{h}{2}D\left(i\kappa\right)\right] F_{\mathcal{Z}} \left\langle U\left(T,\mathcal{Z}\right)\right\rangle, \tag{35}$$

where the operator $F_{\mathcal{Z}}$ is the Fourier transform operator.

As checks on our results, we can artificially set $N = 0$ such that the equation of motion becomes

$$\frac{\partial}{\partial T} \left\langle U\left(T,\mathcal{Z}\right)\right\rangle = \frac{i\mathrm{sgn}\left(\beta_2 v^3\right)}{2T_D} \frac{\partial^2}{\partial \mathcal{Z}^2} \left\langle U\left(T,\mathcal{Z}\right)\right\rangle, \tag{36}$$

or

$$\frac{\partial}{\partial T} \left\langle a_{\bar{k}+\kappa}\left(T\right)\right\rangle = -\frac{i\mathrm{sgn}\left(\beta_2 v^3\right)}{2T_D} \kappa^2 \left\langle a_{\bar{k}+\kappa}\left(T\right)\right\rangle, \tag{37}$$

with solution

$$\left\langle a_{\bar{k}+\kappa}\left(T\right)\right\rangle = \left\langle a_{\bar{k}+\kappa}\left(0\right)\right\rangle \exp\left(-\frac{i\mathrm{sgn}\left(\beta_2 v^3\right)}{2T_D}\kappa^2 T\right). \tag{38}$$

Substituting this in Eq. (32) we find

$$\left\langle U\left(T,\mathcal{Z}\right)\right\rangle = \int \frac{\mathrm{d}\kappa}{\sqrt{2\pi}} \left\langle a_{\bar{k}+\kappa}\left(0\right)\right\rangle \exp\left(i\kappa\mathcal{Z} - \frac{i\mathrm{sgn}\left(\beta_2 v^3\right)}{2T_D}\kappa^2 T\right). \tag{39}$$

Thus, for

$$\left\langle U\left(0,\mathcal{Z}\right)\right\rangle = \frac{1}{\left(\pi\right)^{1/4}} \exp\left(-\frac{\mathcal{Z}^2}{2}\right), \tag{40}$$

with a FWHM of $2\sqrt{2\ln\left(2\right)}$, after a time T the FWHM has increased by a factor of $\sqrt{1+\left(T/T_D\right)^2}$. Alternately, when $D = 0$, the equation of motion becomes

$$\frac{\partial \left\langle U\left(T,\mathcal{Z}\right)\right\rangle}{\partial T} = \frac{i}{T_{NL}} \left|\left\langle U\left(T,\mathcal{Z}\right)\right\rangle\right|^2 \left\langle U\left(T,\mathcal{Z}\right)\right\rangle. \tag{41}$$

Writing

$$\frac{\partial \left|\left\langle U\left(T,\mathcal{Z}\right)\right\rangle\right|^2}{\partial T} = \frac{\partial \left\langle U\left(T,\mathcal{Z}\right)\right\rangle}{\partial T} \left\langle U^*\left(T,\mathcal{Z}\right)\right\rangle + \frac{\partial \left\langle U^*\left(T,\mathcal{Z}\right)\right\rangle}{\partial T} \left\langle U\left(T,\mathcal{Z}\right)\right\rangle, \tag{42}$$

and using Eq. (41), it is relatively easy to show that

$$\frac{\partial \left|\left\langle U\left(T,\mathcal{Z}\right)\right\rangle\right|^2}{\partial T} = 0, \tag{43}$$

and thus we can write the equation of motion as

$$\frac{\partial\left\langle U\left(T,\mathcal{Z}\right)\right\rangle}{\partial T}=\frac{i}{T_{NL}}\left|\left\langle U\left(0,\mathcal{Z}\right)\right\rangle\right|^{2}\left\langle U\left(T,\mathcal{Z}\right)\right\rangle, \tag{44}$$

with solution

$$\left\langle U\left(T,\mathcal{Z}\right)\right\rangle=e^{\frac{i}{T_{NL}}\left|\left\langle U\left(0,\mathcal{Z}\right)\right\rangle\right|^{2}T}\left\langle U\left(0,\mathcal{Z}\right)\right\rangle. \tag{45}$$

Thus

$$\left|F_{\mathcal{Z}}\left\langle U\left(T,\mathcal{Z}\right)\right\rangle\right|^{2}=\left|\int\frac{\mathrm{d}\mathcal{Z}}{\sqrt{2\pi}}\left\langle U\left(0,\mathcal{Z}\right)\right\rangle\int e^{\frac{i}{T_{NL}}\left|\left\langle U\left(0,\mathcal{Z}\right)\right\rangle\right|^{2}T-i\kappa\mathcal{Z}}\right|^{2}. \tag{46}$$

Fluctuations

We write the coupled fluctuation equations as

$$\begin{aligned}\frac{\partial\delta U\left(T,\mathcal{Z}\right)}{\partial T}=&-\frac{i\mathrm{sgn}\left(\beta_{2}v^{3}\right)}{2T_{D}}\frac{\partial^{2}}{\partial\mathcal{Z}^{2}}\delta U\left(T,\mathcal{Z}\right)+\frac{i}{T_{NL}}\left\langle U\left(T,\mathcal{Z}\right)\right\rangle^{2}\delta U^{\dagger}\left(T,\mathcal{Z}\right)\\&+2\frac{i}{T_{NL}}\left|\left\langle U\left(T,\mathcal{Z}\right)\right\rangle\right|^{2}\delta U\left(T,\mathcal{Z}\right),\end{aligned} \tag{47}$$

$$\begin{aligned}\frac{\partial\delta U^{\dagger}\left(T,\mathcal{Z}\right)}{\partial T}=&\frac{i\mathrm{sgn}\left(\beta_{2}v^{3}\right)}{2T_{D}}\frac{\partial^{2}}{\partial\mathcal{Z}^{2}}\delta U^{\dagger}\left(T,\mathcal{Z}\right)-\frac{i}{T_{NL}}\left\langle U^{\dagger}\left(T,\mathcal{Z}\right)\right\rangle^{2}\delta U\left(T,\mathcal{Z}\right)\\&-2\frac{i}{T_{NL}}\left|\left\langle U\left(T,\mathcal{Z}\right)\right\rangle\right|^{2}\delta U^{\dagger}\left(T,\mathcal{Z}\right),\end{aligned} \tag{48}$$

or, Fourier transforming

$$\begin{aligned}\frac{\partial\delta a_{\bar{k}+\kappa}\left(T\right)}{\partial T}=&\frac{i\mathrm{sgn}\left(\beta_{2}v^{3}\right)}{2T_{D}}\kappa^{2}\delta a_{\bar{k}+\kappa}\left(T\right)+i\int\frac{\mathrm{d}\kappa'}{\sqrt{2\pi}}\mathcal{S}\left(\kappa+\kappa',T\right)\delta a^{\dagger}_{\bar{k}+\kappa'}\left(T\right)\\&+2i\int\frac{\mathrm{d}\kappa'}{\sqrt{2\pi}}\mathcal{M}\left(\kappa-\kappa',T\right)\delta a_{\bar{k}+\kappa'}\left(T\right),\end{aligned} \tag{49}$$

$$\begin{aligned}\frac{\partial\delta a^{\dagger}_{\bar{k}+\kappa}\left(T\right)}{\partial T}=&-\frac{i\mathrm{sgn}\left(\beta_{2}v^{3}\right)}{2T_{D}}\kappa^{2}\delta a^{\dagger}_{\bar{k}+\kappa}\left(T\right)-i\int\frac{\mathrm{d}\kappa'}{\sqrt{2\pi}}\mathcal{S}^{\dagger}\left(\kappa+\kappa',T\right)\delta a_{\bar{k}+\kappa'}\left(T\right)\\&-2i\int\frac{\mathrm{d}\kappa'}{\sqrt{2\pi}}\mathcal{M}^{*}\left(\kappa-\kappa',T\right)\delta a^{\dagger}_{\bar{k}+\kappa'}\left(T\right),\end{aligned} \tag{50}$$

where we have introduced

$$\mathcal{S}\left(\kappa,T\right)=\int\frac{\mathrm{d}\mathcal{Z}}{\sqrt{2\pi}}\frac{1}{T_{NL}}\left\langle U\left(T,\mathcal{Z}\right)\right\rangle^{2}e^{-i\kappa\mathcal{Z}}, \tag{51}$$

$$\mathcal{M}\left(\kappa,T\right)=\int\frac{\mathrm{d}\mathcal{Z}}{\sqrt{2\pi}}\frac{1}{T_{NL}}\left|\left\langle U\left(T,\mathcal{Z}\right)\right\rangle\right|^{2}e^{-i\kappa\mathcal{Z}}. \tag{52}$$

Discretizing

$$\kappa_{j}=j\Delta\kappa, \tag{53}$$

such that

$$\int \frac{\mathrm{d}\kappa'}{\sqrt{2\pi}} \mathcal{S}\left(\kappa+\kappa', T\right) \delta a_{\bar{k}+\kappa'}^{\dagger}(T) \approx \sum_{j'} \frac{\Delta\kappa}{\sqrt{2\pi}} \mathcal{S}\left(\kappa_j+\kappa_{j'}, T\right) \delta a_{\bar{k}+\kappa_{j'}}^{\dagger}(T),$$

$$\int \frac{\mathrm{d}\kappa'}{\sqrt{2\pi}} \mathcal{M}\left(\kappa-\kappa', T\right) \delta a_{\bar{k}+\kappa'}(T) \approx \sum_{j'} \frac{\Delta\kappa}{\sqrt{2\pi}} \mathcal{M}\left(\kappa_j-\kappa_{j'}, T\right) \delta a_{\bar{k}+\kappa_{j'}}(T),$$

in vector notation, we can write

$$\frac{\partial}{\partial T}\begin{pmatrix} \delta a_{\bar{k}+\kappa_j}(T) \\ \delta a_{\bar{k}+\kappa_j}^{\dagger}(T) \end{pmatrix} = i Q_{jj'}(T) \begin{pmatrix} \delta a_{\bar{k}+\kappa_{j'}}(T) \\ \delta a_{\bar{k}+\kappa_{j'}}^{\dagger}(T) \end{pmatrix}, \tag{54}$$

where

$$Q_{jj'}(T) = \begin{pmatrix} A_{jj'}(T) & B_{jj'}(T) \\ -B_{jj'}^{\dagger}(T) & -A_{jj'}^{*}(T) \end{pmatrix}, \tag{55}$$

$$A_{jj'}(T) = \frac{\operatorname{sgn}\left(\beta_2 v^3\right)}{2T_D} \kappa_j^2 \delta_{jj'} + 2\frac{\Delta\kappa}{\sqrt{2\pi}} \mathcal{M}\left(\kappa_j-\kappa_{j'}, T\right), \tag{56}$$

$$B_{jj'}(T) = \frac{\Delta\kappa}{\sqrt{2\pi}} \mathcal{S}\left(\kappa_j+\kappa_{j'}, T\right). \tag{57}$$

For a small enough propagation forward in time ΔT this has solution

$$\begin{aligned} \begin{pmatrix} \delta a_{\bar{k}+\kappa_j}(T+\Delta T) \\ \delta a_{\bar{k}+\kappa_j}^{\dagger}(T+\Delta T) \end{pmatrix} &= \exp\left[i\Delta T Q_{jj'}(\Delta T)\right] \begin{pmatrix} \delta a_{\bar{k}+\kappa_{j'}}(T) \\ \delta a_{\bar{k}+\kappa_{j'}}^{\dagger}(T) \end{pmatrix} \\ &\equiv \begin{pmatrix} X_{jj'}(\Delta T) & W_{jj'}(\Delta T) \\ W_{jj'}^{*}(\Delta T) & X_{jj'}^{*}(\Delta T) \end{pmatrix} \begin{pmatrix} \delta a_{\bar{k}+\kappa_{j'}}(T) \\ \delta a_{\bar{k}+\kappa_{j'}}^{\dagger}(T) \end{pmatrix}. \end{aligned} \tag{58}$$

Moments

Ultimately, what we want to calculate are objects such as

$$M_{jj'}(T) = \left\langle \delta a_{\bar{k}+\kappa_j}(T)\, \delta a_{\bar{k}+\kappa_{j'}}(T) \right\rangle, \tag{59}$$

$$N_{jj'}(T) = \left\langle \delta a_{\bar{k}+\kappa_j}^{\dagger}(T)\, \delta a_{\bar{k}+\kappa_{j'}}(T) \right\rangle. \tag{60}$$

Thus, using the solution above, in the absence of any loss, we find

$$\begin{aligned} M_{jj'}(T+\Delta T) =& X_{jj''}(\Delta T) M_{j''j'''}(T) X_{j'''j'}^{T}(\Delta T) + W_{jj''}(\Delta T) M_{j''j'''}^{*}(T) W_{j'''j'}^{T}(\Delta T) \\ &+ 2W_{jj''}(\Delta T) N_{j''j'''}(T) X_{j'''j'}^{T}(\Delta T) + X_{jj''}(\Delta T) W_{j''j'}^{T}(\Delta T), \end{aligned} \tag{61}$$

$$\begin{aligned}
N_{jj'}\left(T+\Delta T\right) =& W^*_{jj''}\left(\Delta T\right) M_{j''j'''}\left(T\right) X^T_{j'''j'}\left(\Delta T\right) + X^*_{jj''}\left(\Delta T\right) M^*_{j''j'''}\left(T\right) W^T_{j'''j'}\left(\Delta T\right) \\
&+ X^*_{jj''}\left(\Delta T\right) N_{j''j'''}\left(T\right) X^T_{j'''j'}\left(\Delta T\right) + W^*_{jj''}\left(\Delta T\right) N_{j'''j''}\left(T\right) W_{j'j'''}\left(\Delta T\right) \\
&+ W^*_{jj''}\left(\Delta T\right) W^T_{j''j'}\left(\Delta T\right) .
\end{aligned} \tag{62}$$

We can include loss by alternating the time steps above with

$$M_{jj'}\left(T+\Delta T\right) = \left(1-\Gamma\Delta T\right) M_{jj'}\left(T\right), \tag{63}$$

$$N_{jj'}\left(T+\Delta T\right) = \left(1-\Gamma\Delta T\right) N_{jj'}\left(T\right), \tag{64}$$

where Γ is the loss rate.