Kerr Squeezing Notes

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Simplifying Equations

Nicolas's "initial" equations are

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}-\frac{iv'}{2}\frac{\partial^2}{\partial z^2}+i\bar{\omega}\right)\langle\psi\rangle=i\zeta\left|\langle\psi\rangle\right|^2\langle\psi\rangle, \tag{1}$$

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}-\frac{iv'}{2}\frac{\partial^2}{\partial z^2}+i\bar{\omega}\right)\delta\psi=i\zeta\langle\psi\rangle^2\delta\psi^\dagger+2i\zeta\left|\langle\psi\rangle\right|^2\delta\psi. \tag{2}$$

In what follows, we will put them more in the form of Agrawal, making connections with the nonlinear parameter γ, the GVD parameter β_2, and writing operators with units of the square root of power. We first put

$$\begin{aligned}\langle\psi\rangle=&\left\langle\tilde{\psi}\right\rangle e^{-i\bar{\omega}t},\\ \delta\psi=&\delta\tilde{\psi}e^{-i\bar{\omega}t},\end{aligned} \tag{3}$$

to find

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}-\frac{iv'}{2}\frac{\partial^2}{\partial z^2}\right)\left\langle\tilde{\psi}\right\rangle=i\zeta\left|\left\langle\tilde{\psi}\right\rangle\right|^2\left\langle\tilde{\psi}\right\rangle, \tag{4}$$

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}-\frac{iv'}{2}\frac{\partial^2}{\partial z^2}\right)\delta\tilde{\psi}=i\zeta\left\langle\tilde{\psi}\right\rangle^2\delta\tilde{\psi}^\dagger+2i\zeta\left|\left\langle\tilde{\psi}\right\rangle\right|^2\delta\tilde{\psi}. \tag{5}$$

We then introduce

$$\begin{aligned}\langle A\rangle=&\sqrt{\hbar\omega_{\mathrm{P}}v}\left\langle\tilde{\psi}\right\rangle,\\ \langle\delta A\rangle=&\sqrt{\hbar\omega_{\mathrm{P}}v}\delta\tilde{\psi},\end{aligned} \tag{6}$$

and find

$$\left(\frac{1}{v}\frac{\partial}{\partial t}+\frac{\partial}{\partial z}-\frac{iv'}{2v}\frac{\partial^2}{\partial z^2}\right)\langle A\rangle=i\frac{\zeta}{\hbar\omega_{\mathrm{P}}v^2}\left|\langle A\rangle\right|^2\langle A\rangle, \tag{7}$$

$$\left(\frac{1}{v}\frac{\partial}{\partial t}+\frac{\partial}{\partial z}-\frac{iv'}{2v}\frac{\partial^2}{\partial z^2}\right)\delta A=i\frac{\zeta}{\hbar\omega_{\mathrm{P}}v^2}\langle A\rangle^2\delta A^\dagger+2i\frac{\zeta}{\hbar\omega_{\mathrm{P}}v^2}\left|\langle A\rangle\right|^2\delta A. \tag{8}$$

From John's notes [Eq. (38)] we have that

$$\zeta=\frac{3}{\varepsilon_0\hbar}\left(\frac{\hbar\omega_{\mathrm{P}}}{2}\right)^2\int\Gamma_3^{ijlm}(x,y)\left[d^i(x,y)\right]^*\left[d^j(x,y)\right]^*d^l(x,y)\,d^m(x,y)\,\mathrm{d}x\mathrm{d}y. \tag{9}$$

We can then use

$$\Gamma_3^{ijlm}(x,y)=\frac{\chi_3^{ijlm}(x,y)}{\varepsilon_0^2n^8(x,y;\omega_{\mathrm{P}})}, \tag{10}$$

and, defining

$$\frac{1}{\mathcal{A}}=\int\frac{\bar{n}^4}{\bar{\chi}_3}\frac{\chi_3^{ijlm}(x,y)}{\varepsilon_0^2n^8(x,y;\omega_{\mathrm{P}})}\left[d^i(x,y)\right]^*\left[d^j(x,y)\right]^*d^l(x,y)\,d^m(x,y)\,\mathrm{d}x\mathrm{d}y, \tag{11}$$

write our nonlinearity as

$$\frac{\zeta}{\hbar\omega_{\mathrm{P}}v^3} = \frac{3\omega_{\mathrm{P}}\bar{\chi}_3}{4\varepsilon_0 v^2\bar{n}^4\mathcal{A}} = \gamma, \tag{12}$$

making our equations

$$\left(\frac{1}{v}\frac{\partial}{\partial t} + \frac{\partial}{\partial z} - \frac{iv'}{2v}\frac{\partial^2}{\partial z^2}\right)\langle A\rangle = i\gamma\left|\langle A\rangle\right|^2\langle A\rangle, \tag{13}$$

$$\left(\frac{1}{v}\frac{\partial}{\partial t} + \frac{\partial}{\partial z} - \frac{iv'}{2v}\frac{\partial^2}{\partial z^2}\right)\delta A = i\gamma\langle A\rangle^2\delta A^\dagger + 2i\gamma\left|\langle A\rangle\right|^2\delta A. \tag{14}$$

Finally, with

$$\begin{aligned} v =& \frac{1}{\beta_1}, \\ v' \approx& -\beta_2 v^3, \end{aligned} \tag{15}$$

we find

$$\left(\beta_1\frac{\partial}{\partial t} + \frac{\partial}{\partial z} + \frac{i\beta_2}{2}v^2\frac{\partial^2}{\partial z^2}\right)\langle A\rangle = i\gamma\left|\langle A\rangle\right|^2\langle A\rangle, \tag{16}$$

$$\left(\beta_1\frac{\partial}{\partial t} + \frac{\partial}{\partial z} + \frac{i\beta_2}{2}v^2\frac{\partial^2}{\partial z^2}\right)\delta A = i\gamma\langle A\rangle^2\delta A^\dagger + 2i\gamma\left|\langle A\rangle\right|^2\delta A, \tag{17}$$

the first much as in Agrawal's Eq. (2.3.28) (with $\alpha = 0$ there). Making the change of variables

$$\begin{aligned} Z =& z - t/\beta_1, \\ T =& t, \end{aligned} \tag{18}$$

implying

$$\begin{aligned} \frac{\partial}{\partial t} =& \frac{\partial T}{\partial t}\frac{\partial}{\partial T} + \frac{\partial Z}{\partial t}\frac{\partial}{\partial Z} = \frac{\partial}{\partial T} - \frac{1}{\beta_1}\frac{\partial}{\partial Z} \\ \frac{\partial}{\partial z} =& \frac{\partial T}{\partial z}\frac{\partial}{\partial T} + \frac{\partial Z}{\partial z}\frac{\partial}{\partial Z} = \frac{\partial}{\partial Z}, \end{aligned} \tag{19}$$

we can write

$$\left(\frac{\partial}{\partial T} + \frac{i\beta_2}{2}v^3\frac{\partial^2}{\partial Z^2}\right)\langle A\rangle = i\gamma v\left|\langle A\rangle\right|^2\langle A\rangle, \tag{20}$$

$$\left(\frac{\partial}{\partial T} + \frac{i\beta_2}{2}v^3\frac{\partial^2}{\partial Z^2}\right)\delta A = i\gamma v\langle A\rangle^2\delta A^\dagger + 2i\gamma v\left|\langle A\rangle\right|^2\delta A. \tag{21}$$

Following Agrawal, we introduce a normalized field operator

$$\begin{aligned} \langle A\rangle =& \sqrt{P_0}\langle U\rangle, \\ \delta A =& \sqrt{P_0}\delta U, \end{aligned} \tag{22}$$

the input pulse width Z_0, as well as the dispersion time

$$T_D = \frac{Z_0^2}{\beta_2 v^3}, \tag{23}$$

and nonlinear time

$$T_{NL} = \frac{1}{\gamma P_0 v}, \tag{24}$$

to arrive at

$$\left(\frac{\partial}{\partial T} + \frac{iZ_0^2}{2T_D}\frac{\partial^2}{\partial Z^2}\right)\langle U\rangle = \frac{i}{T_{NL}}\left|\langle U\rangle\right|^2\langle U\rangle, \tag{25}$$

$$\left(\frac{\partial}{\partial T} + \frac{iZ_0^2}{2T_D}\frac{\partial^2}{\partial Z^2}\right)\delta U = \frac{i}{T_{NL}}\langle U\rangle^2\delta U^\dagger + 2\frac{i}{T_{NL}}\left|\langle U\rangle\right|^2\delta U. \tag{26}$$

Solving Equations

Mean Field

We solve for $\langle U\rangle$ using a split-step Fourier approach. Writing

$$\frac{\partial\left\langle U\left(T,Z\right)\right\rangle}{\partial T}=\left[D\left(Z\right)+N\left(T\right)\right]\left\langle U\left(T,Z\right)\right\rangle, \tag{27}$$

where

$$D\left(Z\right)=-\frac{iZ_0^2}{2T_D}\frac{\partial^2}{\partial Z^2}, \tag{28}$$

$$N\left(T\right)=\frac{i}{T_{NL}}\left|\left\langle U\left(T,Z\right)\right\rangle\right|^2, \tag{29}$$

we approximate that

$$\left\langle U\left(T+h,Z\right)\right\rangle\approx\exp\left[\frac{h}{2}D\left(Z\right)\right]\exp\left[\int_T^{T+h}N\left(T'\right)\mathrm{d}T'\right]\exp\left[\frac{h}{2}D\left(Z\right)\right]\left\langle U\left(T,Z\right)\right\rangle. \tag{30}$$

Furthermore, the dispersion is best dealt with in Fourier space

$$\left\langle U\left(T,Z\right)\right\rangle=\int\frac{\mathrm{d}\kappa}{\sqrt{2\pi}}\left\langle a_{\bar{k}+\kappa}\left(T\right)\right\rangle e^{i\kappa\mathcal{Z}}, \tag{31}$$

where $\hat{D}$ becomes

$$\tilde{D}\left(i\kappa\right)=\frac{iZ_0^2}{2T_D}\kappa^2, \tag{32}$$

and, for small enough h, we can approximate

$$\int_T^{T+h}N\left(T'\right)\mathrm{d}T'\approx hN\left(T\right), \tag{33}$$

such that, ultimately

$$\left\langle U\left(T+h,Z\right)\right\rangle\approx F_{\mathcal{Z}}^{-1}\exp\left[\frac{h}{2}D\left(i\kappa\right)\right]F_{\mathcal{Z}}\exp\left[hN\left(T\right)\right]F_{\mathcal{Z}}^{-1}\exp\left[\frac{h}{2}D\left(i\kappa\right)\right]F_{\mathcal{Z}}\left\langle U\left(T,Z\right)\right\rangle, \tag{34}$$

where the operator $F_{\mathcal{Z}}$ is the Fourier transform operator.

As checks on our results, we can artificially set $N=0$ such that the equation of motion becomes

$$\frac{\partial}{\partial T}\left\langle U\left(T,Z\right)\right\rangle=\frac{iZ_0^2}{2T_D}\frac{\partial^2}{\partial Z^2}\left\langle U\left(T,Z\right)\right\rangle, \tag{35}$$

or

$$\frac{\partial}{\partial T}\left\langle a_{\bar{k}+\kappa}\left(T\right)\right\rangle=-\frac{iZ_0^2}{2T_D}\kappa^2\left\langle a_{\bar{k}+\kappa}\left(T\right)\right\rangle, \tag{36}$$

with solution

$$\left\langle a_{\bar{k}+\kappa}\left(T\right)\right\rangle=\left\langle a_{\bar{k}+\kappa}\left(0\right)\right\rangle\exp\left(-\frac{iZ_0^2}{2T_D}\kappa^2T\right). \tag{37}$$

Substituting this in Eq. (31) we find

$$\left\langle U\left(T,Z\right)\right\rangle=\int\frac{\mathrm{d}\kappa}{\sqrt{2\pi}}\left\langle a_{\bar{k}+\kappa}\left(0\right)\right\rangle\exp\left(i\kappa Z-\frac{iZ_0^2}{2T_D}\kappa^2T\right). \tag{38}$$

Thus, for

$$\left\langle U\left(0,Z\right)\right\rangle=\exp\left(-\frac{Z^2}{2Z_0^2}\right), \tag{39}$$

with a FWHM of $2\sqrt{2\ln(2)}Z_0$, after a time T the FWHM has increased by a factor of $\sqrt{1+(T/T_D)^2}$. Alternately, when $D=0$, the equation of motion becomes

$$\frac{\partial \langle U(T,Z)\rangle}{\partial T} = \frac{i}{T_{NL}} \left|\langle U(T,Z)\rangle\right|^2 \langle U(T,Z)\rangle . \tag{40}$$

Writing

$$\frac{\partial \left|\langle U(T,Z)\rangle\right|^2}{\partial T} = \frac{\partial \langle U(T,Z)\rangle}{\partial T} \langle U^*(T,Z)\rangle + \frac{\partial \langle U^*(T,Z)\rangle}{\partial T} \langle U(T,Z)\rangle , \tag{41}$$

and using Eq. (40), it is relatively easy to show that

$$\frac{\partial \left|\langle U(T,Z)\rangle\right|^2}{\partial T} = 0, \tag{42}$$

and thus we can write the equation of motion as

$$\frac{\partial \langle U(T,Z)\rangle}{\partial T} = \frac{i}{T_{NL}} \left|\langle U(0,Z)\rangle\right|^2 \langle U(T,Z)\rangle , \tag{43}$$

with solution

$$\langle U(T,Z)\rangle = e^{\frac{i}{T_{NL}}\left|\langle U(0,Z)\rangle\right|^2 T} \langle U(0,Z)\rangle . \tag{44}$$

Thus

$$\left|F_Z \langle U(T,Z)\rangle\right|^2 = \left| \int \frac{\mathrm{d}Z}{\sqrt{2\pi}} \langle U(0,\mathcal{Z})\rangle \int e^{\frac{i}{T_{NL}}\left|\langle U(0,Z)\rangle\right|^2 T - i\kappa Z} \right|^2 . \tag{45}$$

Fluctuations

We write the coupled fluctuation equations as

$$\begin{aligned}\frac{\partial \delta U(T,Z)}{\partial T} =& -\frac{iZ_0^2}{2T_D}\frac{\partial^2}{\partial Z^2}\delta U(T,Z) + \frac{i}{T_{NL}} \langle U(T,Z)\rangle^2 \delta U^\dagger(T,Z) \\ &+ 2\frac{i}{T_{NL}} \left|\langle U(T,Z)\rangle\right|^2 \delta U(T,Z) ,\end{aligned} \tag{46}$$

$$\begin{aligned}\frac{\partial \delta U^\dagger(T,Z)}{\partial T} =& \frac{iZ_0^2}{2T_D}\frac{\partial^2}{\partial Z^2}\delta U^\dagger(T,Z) - \frac{i}{T_{NL}} \left\langle U^\dagger(T,Z)\right\rangle^2 \delta U(T,Z) \\ &- 2\frac{i}{T_{NL}} \left|\langle U(T,Z)\rangle\right|^2 \delta U^\dagger(T,Z) ,\end{aligned} \tag{47}$$

or, Fourier transforming

$$\begin{aligned}\frac{\partial \delta a_{\bar{k}+\kappa}(T)}{\partial T} =& \frac{iZ_0^2}{2T_D}\kappa^2 \delta a_{\bar{k}+\kappa}(T) + i\int \frac{\mathrm{d}\kappa'}{\sqrt{2\pi}} \mathcal{S}(\kappa+\kappa',T)\, \delta a^\dagger_{\bar{k}+\kappa'}(T) \\ &+ 2i\int \frac{\mathrm{d}\kappa'}{\sqrt{2\pi}} \mathcal{M}(\kappa-\kappa',T)\, \delta a_{\bar{k}+\kappa'}(T) ,\end{aligned} \tag{48}$$

$$\begin{aligned}\frac{\partial \delta a^\dagger_{\bar{k}+\kappa}(T)}{\partial T} =& -\frac{iZ_0^2}{2T_D}\kappa^2 \delta a^\dagger_{\bar{k}+\kappa}(T) - i\int \frac{\mathrm{d}\kappa'}{\sqrt{2\pi}} \mathcal{S}^\dagger(\kappa+\kappa',T)\, \delta a_{\bar{k}+\kappa'}(T) \\ &- 2i\int \frac{\mathrm{d}\kappa'}{\sqrt{2\pi}} \mathcal{M}^*(\kappa-\kappa',T)\, \delta a^\dagger_{\bar{k}+\kappa'}(T) ,\end{aligned} \tag{49}$$

where we have introduced

$$\mathcal{S}(\kappa,T) = \int \frac{\mathrm{d}Z}{\sqrt{2\pi}} \frac{1}{T_{NL}} \langle U(T,Z)\rangle^2 e^{-i\kappa Z}, \tag{50}$$

$$\mathcal{M}(\kappa,T) = \int \frac{\mathrm{d}Z}{\sqrt{2\pi}} \frac{1}{T_{NL}} \left|\langle U(T,Z)\rangle\right|^2 e^{-i\kappa Z}. \tag{51}$$

Discretizing

$$\kappa_j = j\Delta\kappa, \tag{52}$$

such that

$$\int \frac{\mathrm{d}\kappa'}{\sqrt{2\pi}} \mathcal{S}\left(\kappa+\kappa', T\right) \delta a_{\bar{k}+\kappa'}^{\dagger}\left(T\right) \approx \sum_{j'} \frac{\Delta\kappa}{\sqrt{2\pi}} \mathcal{S}\left(\kappa_j+\kappa_{j'}, T\right) \delta a_{\bar{k}+\kappa_{j'}}^{\dagger}\left(T\right),$$

$$\int \frac{\mathrm{d}\kappa'}{\sqrt{2\pi}} \mathcal{M}\left(\kappa-\kappa', T\right) \delta a_{\bar{k}+\kappa'}\left(T\right) \approx \sum_{j'} \frac{\Delta\kappa}{\sqrt{2\pi}} \mathcal{M}\left(\kappa_j-\kappa_{j'}, T\right) \delta a_{\bar{k}+\kappa_{j'}}\left(T\right).$$

Moving to vector notation, we can write

$$\frac{\partial}{\partial T} \begin{pmatrix} \delta a_{\bar{k}+\kappa_j}\left(T\right) \\ \delta a_{\bar{k}+\kappa_j}^{\dagger}\left(T\right) \end{pmatrix} = iQ_{jj'}\left(T\right) \begin{pmatrix} \delta a_{\bar{k}+\kappa_{j'}}\left(T\right) \\ \delta a_{\bar{k}+\kappa_{j'}}^{\dagger}\left(T\right) \end{pmatrix}, \tag{53}$$

where

$$Q_{jj'}\left(T\right) = \begin{pmatrix} A_{jj'}\left(T\right) & B_{jj'}\left(T\right) \\ -B_{jj'}^{\dagger}\left(T\right) & -A_{jj'}^{*}\left(T\right) \end{pmatrix}, \tag{54}$$

$$A_{jj'}\left(T\right) = \frac{\operatorname{sgn}\left(\beta_2 v^3\right)}{2T_D} \kappa_j^2 \delta_{jj'} + 2\frac{\Delta\kappa}{\sqrt{2\pi}} \mathcal{M}\left(\kappa_j-\kappa_{j'}, T\right), \tag{55}$$

$$B_{jj'}\left(T\right) = \frac{\Delta\kappa}{\sqrt{2\pi}} \mathcal{S}\left(\kappa_j+\kappa_{j'}, T\right). \tag{56}$$

For a small enough propagation forward in time ΔT this has solution

$$\begin{aligned} \begin{pmatrix} \delta a_{\bar{k}+\kappa_j}\left(T+\Delta T\right) \\ \delta a_{\bar{k}+\kappa_j}^{\dagger}\left(T+\Delta T\right) \end{pmatrix} &= \exp\left[i\Delta T Q_{jj'}\left(\Delta T\right)\right] \begin{pmatrix} \delta a_{\bar{k}+\kappa_{j'}}\left(T\right) \\ \delta a_{\bar{k}+\kappa_{j'}}^{\dagger}\left(T\right) \end{pmatrix} \\ &\equiv \begin{pmatrix} X_{jj'}\left(\Delta T\right) & W_{jj'}\left(\Delta T\right) \\ W_{jj'}^{*}\left(\Delta T\right) & X_{jj'}^{*}\left(\Delta T\right) \end{pmatrix} \begin{pmatrix} \delta a_{\bar{k}+\kappa_{j'}}\left(T\right) \\ \delta a_{\bar{k}+\kappa_{j'}}^{\dagger}\left(T\right) \end{pmatrix}. \end{aligned} \tag{57}$$

Moments and Measurements

Ultimately, what we want to calculate are objects such as

$$M_{jj'}\left(T\right) = \left\langle \delta a_{\bar{k}+\kappa_j}\left(T\right) \delta a_{\bar{k}+\kappa_{j'}}\left(T\right) \right\rangle, \tag{58}$$

$$N_{jj'}\left(T\right) = \left\langle \delta a_{\bar{k}+\kappa_j}^{\dagger}\left(T\right) \delta a_{\bar{k}+\kappa_{j'}}\left(T\right) \right\rangle. \tag{59}$$

Thus, using the solution above, in the absence of any loss, we find

$$\begin{aligned} M_{jj'}\left(T+\Delta T\right) =& X_{jj''}\left(\Delta T\right) M_{j''j'''}\left(T\right) X_{j'''j'}^{T}\left(\Delta T\right) + W_{jj''}\left(\Delta T\right) M_{j''j'''}^{*}\left(T\right) W_{j'''j'}^{T}\left(\Delta T\right) \\ &+ W_{jj''}\left(\Delta T\right) N_{j''j'''}\left(T\right) X_{j'''j'}^{T}\left(\Delta T\right) + X_{jj''}\left(\Delta T\right) N_{j''j'''}^{T}\left(T\right) W_{j'''j'}^{T}\left(\Delta T\right) \\ &+ X_{jj''}\left(\Delta T\right) W_{j''j'}^{T}\left(\Delta T\right), \end{aligned} \tag{60}$$

$$\begin{aligned} N_{jj'}\left(T+\Delta T\right) =& W_{jj''}^{*}\left(\Delta T\right) M_{j''j'''}\left(T\right) X_{j'''j'}^{T}\left(\Delta T\right) + X_{jj''}^{*}\left(\Delta T\right) M_{j''j'''}^{*}\left(T\right) W_{j'''j'}^{T}\left(\Delta T\right) \\ &+ X_{jj''}^{*}\left(\Delta T\right) N_{j''j'''}\left(T\right) X_{j'''j'}^{T}\left(\Delta T\right) + W_{jj''}^{*}\left(\Delta T\right) N_{j''j'''}^{T}\left(T\right) W_{j'''j'}^{T}\left(\Delta T\right) \\ &+ W_{jj''}^{*}\left(\Delta T\right) W_{j''j'}^{T}\left(\Delta T\right). \end{aligned} \tag{61}$$

We can include loss by alternating the time steps above with

$$M_{jj'}\left(T+\Delta T\right) = \left(1-\Gamma\Delta T\right) M_{jj'}\left(T\right), \tag{62}$$

$$N_{jj'}\left(T+\Delta T\right) = \left(1-\Gamma\Delta T\right) N_{jj'}\left(T\right), \tag{63}$$

where Γ is the loss rate.

In a homodyne measurement, one measures $\Delta^2 X_\phi = \left\langle X_\phi^2 \right\rangle - \left\langle X_\phi \right\rangle^2$ where

$$X_\phi = e^{i\phi} \delta A_U + e^{-i\phi} \delta A_U^{\dagger}, \tag{64}$$

with

$$\delta A_U = \int \mathrm{d}\kappa \, \langle U^* (T_f, \kappa) \rangle \, \delta a_{\bar{k}+\kappa}, \tag{65}$$

and $T_f = L/v$ the time it takes light to traverse the length of the nonlinear region L. In particular

$$\frac{\Delta^2 X_\phi}{\sum_{jj'} \langle U_j \rangle \left\langle U_{j'}^\dagger \right\rangle} = e^{2i\phi} \frac{\sum_{jj'} \langle U_j^* \rangle M_{jj'} \langle U_{j'}^* \rangle^T}{\sum_{jj'} \langle U_j \rangle \left\langle U_{j'}^\dagger \right\rangle} + \text{c.c.} + 2 \frac{\sum_{jj'} \langle U_j \rangle N_{jj'} \langle U_{j'}^* \rangle^T}{\sum_{jj'} \langle U_j \rangle \left\langle U_{j'}^\dagger \right\rangle} + 1, \tag{66}$$

where we have introduced

$$\langle U_j \rangle = \left\langle U \left(T_f, \bar{k} + \kappa_j \right) \right\rangle. \tag{67}$$

Approximate Solutions

If these were not operator equations, in the absence of dispersion, instead of (43), we would have

$$U(T, Z) = e^{\frac{i}{T_{NL}} |U(0,Z)|^2 T} U(0, Z). \tag{68}$$

Writing

$$U(T, Z) = \langle U(T, Z) \rangle + \delta U(T, Z), \tag{69}$$

at this stage, we would find

$$\begin{aligned} U(T, Z) =& e^{\frac{i}{T_{NL}} [\langle U(0,Z)\rangle + \delta U(0,Z)] \left[\left\langle U^\dagger (0,Z) \right\rangle + \delta U^\dagger (0,Z) \right] T} \left[\langle U(0, Z) \rangle + \delta U(0, Z) \right] \\ \approx& e^{\frac{i}{T_{NL}} \left[|\langle U(0,Z)\rangle|^2 + \langle U(0,Z)\rangle \delta U^\dagger (0,Z) + \left\langle U^\dagger (0,Z) \right\rangle \delta U(0,Z) \right] T} \left[\langle U(0, Z) \rangle + \delta U(0, Z) \right] \\ \approx& e^{\frac{i}{T_{NL}} |\langle U(0,Z)\rangle|^2 T} \left[\langle U(0, Z) \rangle + \mu \delta U(0, Z) + \nu \delta U^\dagger (0, Z) \right], \end{aligned} \tag{70}$$

where

$$\mu(T) = 1 + \frac{iT}{T_{NL}} \left| \langle U(0, Z) \rangle \right|^2, \tag{71}$$

$$\nu(T) = \frac{iT}{T_{NL}} \langle U(0, Z) \rangle^2. \tag{72}$$

In a homodyne measurement, one will measure

$$\begin{aligned} \Delta^2 X_\phi =& \int \mathrm{d}Z \left\langle \int \mathrm{d}\tau \left[\left(\mu^*(t) \delta U^\dagger + \nu^*(t) \delta U \right) \langle U \rangle e^{i\phi} - \left(\mu(t) \delta U + \nu(t) \delta U^\dagger \right) \left\langle U^\dagger \right\rangle e^{-i\phi} \right] \right. \\ & \left. \times \left[\left(\mu(t-\tau) \delta U + \nu(t-\tau) \delta U^\dagger \right) \left\langle U^\dagger \right\rangle e^{-i\phi} - \left(\mu^*(t-\tau) \delta U^\dagger + \nu^*(t-\tau) \delta U \right) \langle U \rangle e^{i\phi} \right] \right\rangle \\ =& \int \mathrm{d}Z \left(|\nu|^2 |\langle U \rangle|^2 - \mu^* \nu^* \langle U \rangle^2 e^{2i\phi} - \mu\nu \left\langle U^\dagger \right\rangle^2 e^{-2i\phi} + |\mu|^2 |\langle U \rangle|^2 \right). \end{aligned}$$

Treating $\langle U(0, Z) \rangle$ as real, $U(Z)$, and putting the pulse peak nonlinear phase shift $T/T_{NL} = x$ for simplicity, we write

$$\begin{aligned} \Delta^2 X_\phi =& \int \mathrm{d}Z \, U^2(Z) \left(|\nu|^2 + |\mu|^2 - 2\Re \left\{ \mu^* \nu^* e^{2i\phi} \right\} \right) \\ =& \int \mathrm{d}Z \, U^2(Z) \left[1 + 2x^2 U^4(Z) - 2\sqrt{1 + x^2 U^4(Z)} x U^2(Z) \cos(\gamma - \gamma_0) \right], \end{aligned} \tag{73}$$

where

$$\gamma = \tan^{-1} \left(\frac{1}{x^2 U^4(Z)} \right), \tag{74}$$

and we have chosen

$$2\phi = \gamma_0 = \tan^{-1} \left(\frac{1}{y^2} \right). \tag{75}$$

Further simplifying, the quadrature variance can be written

$$\Delta^2 X_\phi = \int \mathrm{d}Z \, U^2(Z) \left[1 + 2x^2 U^4(Z) - \frac{2xU^2(Z) \left(1 + yxU^2(Z) \right)}{\sqrt{1 + y^2}} \right]. \tag{76}$$

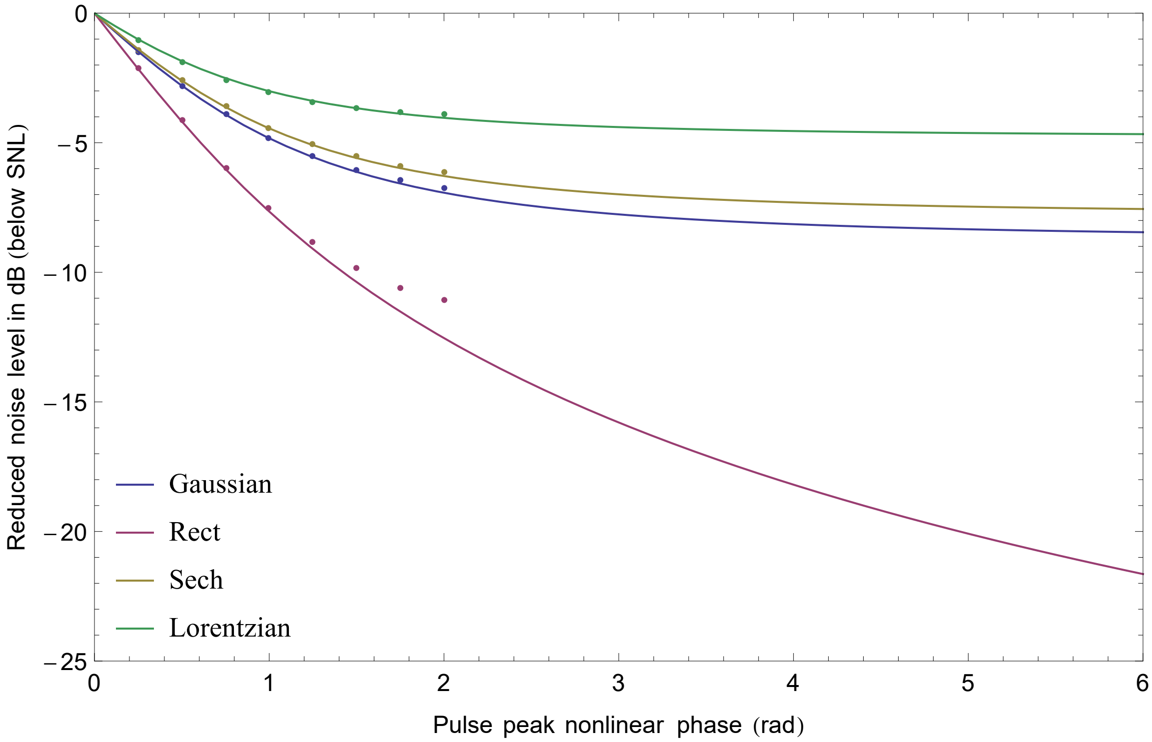

Figure 1: Squeezing below the shot noise level. The solid lines represent analytic results obtained with (76) and the individual points results obtained with (66). Neither case includes loss or group velocity dispersion.

For

$$U(Z) = \exp\left(-Z^2/2\right), \tag{77}$$

we find

$$\frac{\Delta^2 X_\phi}{\int \mathrm{d}Z\, U^2(Z)} = 1 + \frac{2x^2}{\sqrt{3}} - \frac{\frac{2x^2 y}{\sqrt{3}} + \sqrt{2}x}{\sqrt{1+y^2}}, \tag{78}$$

exactly as in J. Opt. Soc. Am. B **7**, 30 (1990) when $y = x$. We note that the optimal value for y (the one that will minimize the function above) is, in fact, $y = \sqrt{2/3}x$. As a check on our numerical results to the full problem, we we compare analytic results from (76) with (66) in the limit of $\alpha = 0$, $\beta_2 = 0$ in Fig. 1.

Introducing loss within this framework is reasonably easy, for all one needs to do is replace $U(Z)$ with

$$U(Z) = \exp\left(-Z^2/2\right)\exp\left(-\alpha T/2\right). \tag{79}$$