Quantum-classical duality

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State of the art

It was observed by Gorsky, Zabrodin and Zotov [GZZ14], that the spectrum of the twist matrix g of the quantum inhomogeneous Heisenberg model coincides with the spectrum of the Lax matrix L of the classical rational Ruijsenaars-Schneider model under the following substitutions:

ith particle	ith spin
position y_i	inhomogeneity y_i
velocity $\dot{y}_i$	non-local Hamiltonian H_i

The non-local Hamiltonians are given by

$$H_i := \text{Res}_{z=y_i} \tau^g(z),$$

where $\tau^g(z)$ denotes the g-twisted transfer matrix. We call this fact *quantum-classical duality*.

Functional relations

Recently, Arutyunov [Aru] formulated a system of polynomial equations for the spectrum of $H_1, ..., H_N$ in terms of functional relations between higher transfer matrices $\tau_\lambda^g(z)$, where λ is a Young diagram.
The basic functional relation is

$$\tau_{[1^{k+1}]}(z) = \tau^g(z)\tau_{[1^k]}^g(z - \eta) - \tau_{[2,1^{k-1}]}(z - \eta),$$

from which we derive the recursion relation

$$\text{Res}_{z=y_i}\,\tau_{[1^{k+1}]}(z) = e_k(g)H_i + \sum_j \underbrace{\frac{\eta H_i}{y_i - y_j - \eta}}_{\text{Lax matrix}} \text{Res}_{z=y_j}\,\tau_{[1^k]}(z).$$

We see the Lax matrix of the rational RS model appearing.

Spectral equation

Iterating the recursion relation and plugging in well-known eigenvalues of the quantum determinant

$$\det g \operatorname{qdet} T(z) = \tau_{[1^\ell]}^g(z)$$

yields a Cayley-Hamilton-like identity consistent with [GZZ14]:

$$\sum_{k=0}^{\ell} e_k(g) L^k \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix} = 0,$$

where

$$b_i := \prod_{i \neq j} \frac{y_i - y_j - \eta}{y_i - y_j}.$$

This system of N equations of order ℓ form the *spectral equations*.

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Representation theory

These facts spark the quest for a deeper reason behind the correspondence between both models.

To this end, let us look at the representation theoretic structure of the Heisenberg model: The observables of the Heisenberg model live inside the Yangian $Y(\mathfrak{gl}_\ell)$ and its Hilbert space is given by the representation

$$\mathbb{C}^\ell[y_1] \otimes \cdots \otimes \mathbb{C}^\ell[y_N]|_{y_i = u_i}.$$

This has a hidden symmetry given by the degenerate affine Hecke algebra

$$\dot{H}_N = \mathbb{C}[S_N] \otimes \mathbb{C}[y_1, ..., y_N]$$

via *Schur-Weyl duality*.

Schur-Weyl duality

Schur-Weyl duality means that there is a bimodule structure

$$Y(\mathfrak{gl}_\ell) \curvearrowright \mathbb{C}^\ell[y_1] \otimes \cdots \otimes \mathbb{C}^\ell[y_N] \curvearrowleft \dot{H}_N$$

that induces a functor

$$D_{\ell,N} : \dot{H}_N \mathsf{Mod} \to Y(\mathfrak{gl}_\ell)\mathsf{Mod},$$
$$U \mapsto (\mathbb{C}^\ell[y_1] \otimes \cdots \otimes \mathbb{C}^\ell[y_N]) \otimes_{\dot{H}_N} U,$$

called the *Drinfeld functor* [Dri86]. This restricts to the well-known correspondence between representations of S_N and $\mathfrak{gl}_\ell$.

Quantum rational (spin) RS model

There is a representation of $\dot{H}_N$ on polynomials $\mathbb{C}[y_1, .., y_N]$ that extends to a representation of the degenerate double affine Hecke algebra

$$\ddot{H}_N := \mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}] \otimes \mathbb{C}[S_N] \otimes \mathbb{C}[y_1, ..., y_N].$$

The elementary symmetric polynomials $e_k(X_1, ..., X_N)$ are commuting Hamiltonians for the quantum rational RS model.

We can then look at

$$D_{\ell,N}(\mathbb{C}[y_1, ..., y_N]) \cong (\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} \mathbb{C}[y_1, ..., y_N],$$

which contains both the RS and Heisenberg model. It may be thought of as the Hilbert space of the rational *spin* RS model, in analogy to [LPS22].

Extending the Drinfeld functor

We may extend and twist the Drinfeld functor, yielding

$$D^g_{\ell,N} : \ddot{H}_N \to \ddot{H}^{S_N}_N \# Y(\mathfrak{gl}_\ell)\mathsf{Mod}.$$

Theorem (Quantum-classical duality)

The element

$$\operatorname{tr} g + \sum_i \frac{\eta b_i X_i}{z - y_i} \in \ddot{H}^{S_N}_N \llbracket z^{-1} \rrbracket$$

acts on $D^g_{\ell,N}(\mathbb{C}[y_1,...,y_N])$ *in the same way as* $\tau^g(z)$ *when* $\hbar_{RS} = 0$.

Slogan

The extended Drinfeld functor maps the Hamiltonians of the rational RS model to an $\hbar_{RS}$-deformation of the Hamiltonians of the Heisenberg model.

Loop Yangian

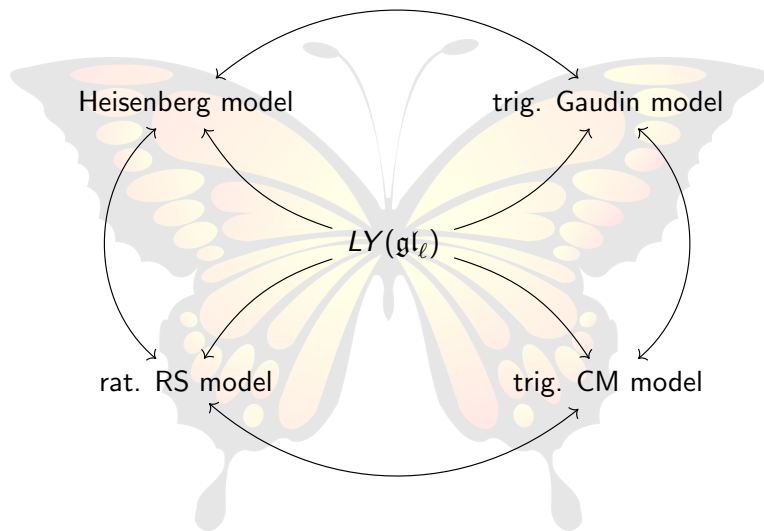What happens when $\hbar_{RS} \neq 0$? Here we resort to a result of Guay [Gua05], constructing a Schur-Weyl functor

$$\dot{D}_{\ell,N} : \ddot{H}_N\mathrm{Mod} \to LY(\mathfrak{gl}_\ell)\mathrm{Mod},$$

where $LY(\mathfrak{gl}_\ell)$ is the *loop Yangian*, which is a quotient of $L(\mathfrak{gl}_\ell)\#Y(\mathfrak{gl}_\ell)$ as well as $Y(\widehat{\mathfrak{gl}_\ell})$.

Theorem

The loop variable t living in the center of $L(\mathfrak{gl}_\ell)$ acts on $\dot{D}_{\ell,N}(\mathbb{C}[y_1,...,y_N])$ in the same way as the standard Hamiltonian of the rational RS model.

The full picture

Comparison with classical models

The loop Yangian $LY(\mathfrak{gl}_\ell)$ should be seen as the quantization of the Poisson structure found by Arutyunov and Frolov [AF98]:

They consider the rational spin RS and trigonometric spin CM models by Hamiltonian reduction of $T^*GL_N \times \mathfrak{gl}_N^*$ and derive two symmetries: The rational spin RS model has a loop symmetry and the trigonometric spin CM model has a Yangian symmetry.

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4d Chern-Simons theory

It is reasonable to ask for a geometric picture of the representation theoretic structures thus far. A natural guess where to look is 4d Chern-Simons theory, since it is known to assign the (delooping of the) category of representations of the Yangian to the formal punctured disc.

4d Chern-Simons theory is defined for a $\mathfrak{gl}_\ell$-gauge field A on a 4-manifold $C \times \Sigma$, where C is a complex curve equipped with a meromorphic 1-form ω and Σ is an oriented surface. In our case, we look at $C = \mathbb{P}^1$ with $\omega = dz$ and $\Sigma = S^1 \times [0,1]$.

It was realized by Costello [Cos13] that Wilson lines along $S^1 \times [0,1]$ with constant coordinate in $\mathbb{P}^1$ recover the Heisenberg model.

Double category of cobordisms

Such Wilson lines organize into a double category of cobordisms. Its objects are finite disjoint unions of $D^\times \times S^1_*$ and it has two types of morphisms: The first are purely topological cobordisms, such as

$(S^1 \times [1,0])^{\sqcup 3}$

the second are purely holomorphic cobordisms
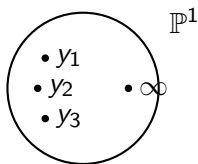
where we allow $y_1, ..., y_N \in \mathbb{P}^1 \setminus \{\infty\}$ with $y_i - y_j \neq 0, \eta$. Denote the space of such cobordisms by Y_N.

QC duality in terms of 4d CS theory

Proposition

Quasi-coherent sheaves on Y_N are equivalently modules over

$$\underline{\dot{H}}_N := \dot{H}_N[(y_i - y_j)^{-1}, (y_i - y_j - \eta)^{-1}].$$

With this identification, we can view the Drinfeld functor as a functor

$$\mathrm{Qcoh}(Y_N) \to Y(\mathfrak{gl}_\ell)\mathrm{Mod},$$

giving the part of 4d Chern-Simons theory that assigns Yangian modules to $\mathbb{P}^1 \times S^1$ with N distinguished points on which Wilson lines may end.

I expect that the full picture in terms of the loop Yangian is recovered from 5d Chern-Simons theory on $\mathbb{C} \times \mathbb{C}^\times \times [0, 1]$.

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4d CS on $E \times S^1 \times [0, 1]$

The above considerations motivate the study of 4d Chern-Simons theory on $E \times S^1 \times [0, 1]$, where $E = \mathbb{C}/\Lambda$ is an elliptic curve. We similarly look at configurations of points Y_N^{ell} and let $\underline{\dot{H}}_N^{\text{ell}}$ denote the algebra whose modules model quasi-coherent sheaves on Y_N^{ell}. I expect that there is a Schur-Weyl bimodule

$$E(\mathfrak{gl}_\ell) \curvearrowright (\mathbb{C}^\ell)^{\otimes N} \otimes \mathcal{O}(Y_N^{\text{ell}}) \curvearrowleft \underline{\dot{H}}_N^{\text{ell}}.$$

where $E(\mathfrak{gl}_\ell)$ is Belavin's elliptic quantum group. Tensoring with $\mathcal{O}(Y_N^{\text{ell}})$ over $\underline{\dot{H}}_N^{\text{ell}}$ should give the Hilbert space for the elliptic spin RS model. Its Hamiltonians will be elliptic difference operators deforming the residues of the elliptic transfer matrix.

The full picture comes from 5d Chern-Simons theory on $E \times \mathbb{C}^\times \times [0, 1]$ and an elliptic quantum group for $\widehat{\mathfrak{gl}}_\ell$.

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