Quantum-classical Duality between Heisenberg and Ruijsenaars-Schneider Models

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Abstract

We examine a quantum-classical duality between inhomogeneous Heisenberg models and rational Ruijsenaars-Schneider models building on observations of [GZZ14] and [Aru]. We show how generalized Schur-Weyl duality between the Yangian and the degenerate affine Hecke algebra provides a clear category theoretic reason for the existence of this duality, also giving a generalization to spins in non-fundamental representations. Employing the theory of degenerate double affine Hecke algebras, we extend this point of view to a quantum-classical duality between trigonometric Gaudin and Calogero-Moser models, showing how all four models arise from two S-dual representations of the loop Yangian. Finally, we give a geometric picture of generalized Schur-Weyl duality that makes apparent how the rational Ruijsenaars-Schneider and inhomogeneous Heisenberg models emerge naturally in four-dimensional Chern-Simons theory when constructed as a category valued functorial quantum field theory on surfaces. We then use this functorial quantum field theory to extrapolate our results to the case of the elliptic Ruijsenaars-Schneider model.

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Chapter 0

Introduction

Integrability [Aru20] is a key phenomenon in many physical models that allows for exact solutions. Nonetheless, solving integrable models often necessitates surprisingly non-trivial methods. Abstractly, integrability can be understood as the presence of some highly organized structure. Though this can happen in many different ways, a common theme among integrable models is the existence of a large number of commuting conserved quantities. An important tool in diskovering such conserved quantities are certain algebras that act on states and observables. Famous representatives in this class of algebras are (double) affine Hecke algebras and affine quantum groups as well as Yangians and their various degenerations, which allow for a reduction of many phenomena in the study of integrable models to phenomena in the representation theory of these algebras. In particular, sharing the same representation theory gives rise to many coincidences in the mathematical descriptions of integrable models, even when these models look very different on the surface. This has led to the diskovery of many dualities between integrable models.

In [GZZ14], a *quantum-classical* duality between the quantum twisted inhomogeneous Heisenberg $\mathfrak{gl}_\ell$-spin chain and the classical rational Ruijsenaars-Schneider model was first worked out. For our purposes, we will simply refer to this duality as *the* quantum-classical duality. The first model describes a chain of N atoms, labeled by inhomogeneities $y_1, ..., y_N \in \mathbb{C}$, whose local Hilbert spaces[1] are given by the vector representation of $\mathfrak{gl}_\ell$. The Hamiltonian imposes nearest-neighbor interactions with boundary conditions that are periodic up to a twist matrix. The second model describes N relativistic point particles acting on each other by mutual centrifugal forces. In this model, the conserved quantities can be neatly described via eigenvalues of a Lax matrix. *Loc. cit.* then describes the quantum-classical duality in terms of a coincidence of spectra of the twist and Lax matrices on the quantum and classical side respectively. This involves the following substitutions: The inhomogeneities of the Heisenberg spin chain model become the positions of the particles in the Ruijsenaars-Schneider model and the eigenvalues of

[1]We use the term *Hilbert space* in the loose sense of describing the state space of a quantum system, *i.e.* it does not necessarily come equipped with an inner product. In accordance, we will generally not be careful about Hermiticity.

certain non-local spin chain Hamiltonians correspond to the momenta of these particles.

Recently, a novel angle on this duality has appeared in [Aru], where the Lax matrix of the rational Ruijsenaars-Schneider model miraculously appears in functional relations among higher transfer matrices of the spin chain. These encode the spectrum of the inhomogeneous Heisenberg model. Such hints spark the quest for a more conceptual reason behind the coincidence.

To this end, our first step will be to identify the relevant algebras at play. For the Heisenberg $\mathfrak{gl}_\ell$-spin chain, it is well known that its Hilbert space is a representation of the Yangian $Y(\mathfrak{gl}_\ell)$, which in turn contains all relevant observables. For the classical rational Ruijsenaars-Schneider model, it is at first glance more mysterious which algebra we are supposed to consider. However, working with the *quantum* rational Ruijsenaars-Schneider model first, we can identify the degenerate affine Hecke algebra

$$\dot{H}_N = \mathbb{C}[S_N] \otimes \mathbb{C}[y_1, ..., y_N]$$

contained in the degenerate *double* affine Hecke algebra

$$\ddot{H}_N = \mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}] \otimes \mathbb{C}[S_N] \otimes \mathbb{C}[y_1, ..., y_N],$$

which possesses a representation on $\mathbb{C}[y_1, ..., y_N]$ via Macdonald difference operators that yields the relevant wave functions, *i.e.* the generators y_i become the position operators of the particles and the generators X_i correspond to momentum operators. We will show how the coincidence in the descriptions of the Heisenberg and Ruijsenaars-Schneider models can be summarized by twisting an old result of Drinfeld [Dri86]: There exists a generalized Schur-Weyl functor from the category of $\dot{H}_N$-modules to the category of $Y(\mathfrak{gl}_\ell)$-modules, which is fully faithful when $\ell > N$, giving an equivalence between $\dot{H}_N$-modules and $Y(\mathfrak{gl}_\ell)$ of weight N. Applying this functor to the wave function representation $\mathbb{C}[y_1, ..., y_N]$ reduces to the sought-after result, yielding the representation

$$(\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} \mathbb{C}[y_1, ..., y_N]$$

of the Yangian, on which the momentum-like operators X_i act exactly as the non-local Hamiltonians of the spin chain when the Planck constant $\hbar$ of the rational Ruijsenaars-Schneider model goes to zero.

This is however not the full picture. The fact that we are suddenly dealing with the Laurent generators X_i from the degenerate *double* affine Hecke algebra points to a missing piece. Recall that *non*-degenerate double affine Hecke algebras have an S-duality automorphism by way of interchanging their two sets of Laurent generators. As a remnant of this S-duality, the representation of $\ddot{H}_N$ on $\mathbb{C}[y_1, ..., y_N]$ above has an S-dual representation on $\mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}]$ via Dunkl differential operators. This representation describes the quantum trigonometric Calogero-Moser model. In a similar fashion as above, we can view it as a representation of the affine symmetric group $\dot{S}_N = S_N \ltimes \mathbb{Z}^N$ which sits inside $\ddot{H}_N$. Using Schur-Weyl duality for affine symmetric groups, we obtain a representation of the loop algebra $L(\mathfrak{gl}_\ell)$. We then combine the action of

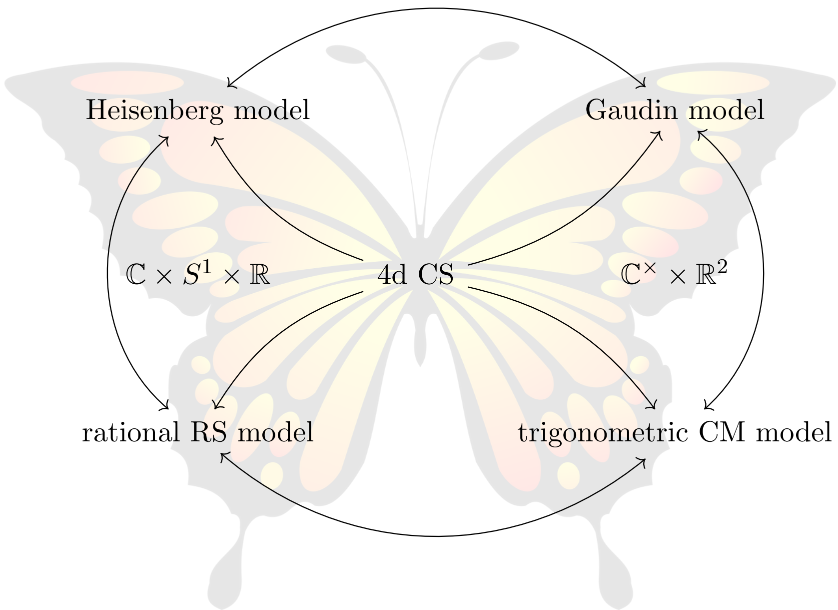


Figure 1: A pictorial summary of the results. The reflection symmetry around the horizontal axis comes from generalized Schur-Weyl duality, while the reflection symmetry around the vertical axis comes from S-duality. Here, we respectively write $\mathbb{P}^1$ without the double pole at infinity as $\mathbb{C}$ and $\mathbb{P}^1$ without the single poles at zero and infinity as $\mathbb{C}^\times$.

the loop algebra and the action of the Yangian to an action of the loop Yangian $LY(\mathfrak{gl}_\ell)$, unifying all four models using generalized Schur-Weyl duality between the degenerate double affine Hecke algebra and the loop Yangian [Gua05].

To finish our discussion, we will reframe this representation theoretic picture in a more geometric setting, thus connecting it to gauge theory. It will turn out that the whole story can be retold in terms of four-dimensional Chern-Simons theory, which was first described in [Cos13] and subsequently expanded on in a series of papers starting with [CWY18]. Four-dimensional Chern-Simons theory is a semi-topological quantum field theory or, more precisely, holomorphic in two directions with complex coordinate z and topological in the other two directions. Thus, it lives on a four-manifold $C \times \Sigma$, where C is a complex curve and Σ an oriented surface. Its main dynamical variable is a $\mathfrak{gl}_\ell$-valued connection 1-form $A(z)$ on Σ, which is meromorphic in $z \in C$. One also fixes a meromorphic 1-form ω on C, which is wedged with the Chern-Simons 3-form of A to obtain the Langrangian density of the theory. Its poles and zeros respectively give rise to order and disorder defects of the gauge field A.

Specializing four-dimensional Chern-Simons theory to the four-manifold $C \times \Sigma$ with $C = \mathbb{P}^1$ and the 1-form

$$\omega = \frac{(z - y_1 - \eta) \cdots (z - y_N - \eta)}{(z - y_1) \cdots (z - y_N)} dz,$$

where η is the Planck constant of the theory, as well as $\Sigma = S^1 \times \mathbb{R}$ a cylinder, we will see how the rational Ruijsenaars-Schneider model describes the boundary conditions at the order defects

y_i of the gauge field A while Wilson loops around the cylinder describe the transfer matrices of the Heisenberg model. S-Dually, we may also specialize to $C = \mathbb{P}^1$ with

$$\omega = (?)\frac{dz}{z}$$

and $\Sigma = \mathbb{R}^2$. (tba) The entire situation is summarized by the mock commutative diagram in figure 1.

In short, the novel contributions of this thesis are the following:

- We explicitly show that the quantum trigonometric Calogero-Moser model and rational Ruijsenaars-Schneider model are realized as S-dual representations of the degenerate double affine Hecke algebra $\ddot{H}_N$ with commuting Hamiltonians living in the spherical subalgebra $S\ddot{H}_N$. This is largely parallel to existing literature

- We twist the Drinfeld functor $D_{\ell,N} : \dot{H}_N\mathsf{Mod} \to Y(\mathfrak{gl}_\ell)\mathsf{Mod}$ to give the *preaffine Drinfeld functor* $D^g_{\ell,N} : \ddot{H}_N\mathsf{Mod} \to S\ddot{H}_N\#Y(\mathfrak{gl}_\ell)\mathsf{Mod}$ and show that it maps the Hilbert space and Hamiltonian operators of the quantum rational Ruijsenaars-Schneider model to the Hilbert space and Hamiltonian operators of the twisted inhomogeneous Heisenberg model in the limit where the Plack constant for the rational Ruijsenaars-Schneider model $\hbar$ goes to zero.

- We show the same for the quantum trigonometric Calogero-Moser model, mapping to the trigonometric Gaudin model when $\hbar \to 0$.

- We construct a 2-functorial quantum field theory representing four-dimensional Chern-Simons theory on $C \times S^1 \times \mathbb{R}$ with C a punctured complex curve and show how the genus zero case $C = \mathbb{P}^1$ yields the rational Ruijsenaars-Schneider and Heisenberg models upon dimensional reduction by S^1 or $\mathbb{P}^1$, respectively.

- We compute the genus one case where C is an elliptic curve and connect it to the elliptic Ruijsenaars-Schneider model.

The remaining chapters of this thesis are structured as follows. Chapter 1 gives a full review from the basics of integrability to the state of the art of the pertinent models and establishes the setup that frames our discussion. Chapter 2 begins with a detailed discussion of the recent appearance of the Lax matrix of the rational Ruijsenaars-Schneider model in the functional relations for transfer matrices of the Heisenberg model used to derive the spectral equation whose solutions yield the energy spectrum of the Heisenberg model. We then move on to describe the mathematical underpinnings of generalized Schur-Weyl duality and how it gives rise to quantum-classical duality. The end of chapter 2 is dedicated to explicating S-duality, showing how the S-dual models, *i.e.* the trigonometric Gaudin model and the trigonometric Calogero-Moser model are also related by quantum-classical duality. Chapter 3 then continues with a description of the geometry behind the mathematical structures of chapter 2, constructing two functorial field theories: One encapsulating Heisenberg-Ruijsenaars-Schneider duality and the other S-dual theory encapsulating Gaudin-Calogero-Moser duality. Each are shown to arise from four-dimensional Chern-Simons theory. After the conclusion (chapter 4), we give supplementary material on Young diagrams (appendix A) as well as results on Dunkl operators for the trigonometric Calogero-Moser model (appendix B).

– *Enjoy!* –

Chapter 1

Integrability

1.1 Classical integrability

1.1.1 Liouville theorem

Physical models in classical mechanics are described by phase spaces that are $2N$-dimensional symplectic manifolds together with a choice of Hamiltonian H. In this setting, the definitive definition of integrability is given by *Liouville integrability*, which comes from the basic idea that conserved quantities reduce the effective dimensionality of the phase space. To see this, assume that we are handed a conserved quantity f, in other words $\dot{f} = \{H, f\} = 0$. By definition, f will be constant along the Hamiltonian flow generated by the Hamiltonian vector field of H, say $f \equiv c \in \mathbb{R}$, so we may narrow our phase space to individual level sets of f, which generically have codimension one. Continuing this argument inductively by adding more conserved quantities while making sure that they all Poisson-commute, one will eventually arrive at a half-dimensional submanifold on which the equations of motion simplify greatly. This requires a full set of N independent Poisson-commuting observables, including the Hamiltonian. The precise statement is given by the Liouville theorem:

Theorem 1.1.1 (Liouville theorem [Arn89])**.** *Let M be a $2N$-dimensional symplectic manifold and suppose there exist N smooth functions $f_1, ..., f_N \in C^\infty(M)$ such that all pairwise Poisson brackets vanish,* i.e. *$\{f_i, f_j\} = 0$ for all $1 \leq i, j \leq N$ and the Hamiltonian H is a function of the f_i. Given $c = (c_1, ..., c_N) \in \mathbb{R}^N$, consider the level set*

$$M_c := \{p \in M \mid f_i(p) = c_i\}.$$

If the 1-forms df_i are linearly independent on M_c, then:

(i) M_c is a smooth submanifold invariant under the Hamiltonian flow.

(ii) If M_c is compact and connected, then M_c is diffeomorphic to $(S^1)^N$. In this case, the Hamiltonian flow for H is linearly periodic, i.e.

$$\dot{\varphi}_i = \omega_i$$

for φ_i the ith angular coordinate and ω_i a frequency dependent only on c and H.

Proof. Part (i) is an application of the Frobenius theorem. To show (ii), observe that the Hamiltonian flow of the commuting conserved quantities $f_1, ..., f_N$ generate an action of the N-dimensional commutative Lie algebra $\mathbb{R}^N$ on M_c. The 1-forms df_i were assumed to be linearly independent, which implies that this action is locally free and hence is transitive and has diskrete stabilizer, which must then be of the form $\mathbb{Z}^k \subseteq \mathbb{R}^N$. Compactness of M_c implies $k = N$ and $M_c \cong \mathbb{R}^N/\mathbb{Z}^N \cong (S^1)^N$. □

Example. The simplest example of a Liouville integrable model is the classical harmonic oscillator. Its phase space is $(\mathbb{R}^2, dp \wedge dq)$ with Hamiltonian

$$H(p, q) = \tfrac{1}{2}(p^2 + q^2).$$

The Hamiltonian itself trivially provides enough conserved quantities for Liouville integrability to hold. This directly manifests in the time evolution of the harmonic oscillator: For a fixed energy $E = \frac{\alpha^2}{2}$, the equations of motion reduce to $\dot{\varphi} = 1$, where we have introduced the angular variable φ parameterizing p and q via

$$p(\varphi) = \alpha \cos(\varphi), \quad q(\varphi) = \alpha \sin(\varphi).$$

This is easily generalized to N uncoupled, possibly anisotropic harmonic oscillators with phase space $(\mathbb{R}^{2N}, \sum_i dp_i \wedge dq_i)$, conserved quantities

$$f_i = \tfrac{1}{2}(p_i^2 + \omega_i^2 q_i^2), \quad \omega_i \in \mathbb{R},$$

and Hamiltonian $H = \sum_i f_i$. The level sets for these integrals of motion are of the form $(S^1)^N$ with angular variables $\varphi_1, ..., \varphi_N$ evolving linearly as $\dot{\varphi}_i = \omega_i$. Note that the time evolution is only truly periodic when the $\omega_1, ..., \omega_N$ are all integer multiples of a fundamental frequency.

1.1.2 Lax pairs

The Liouville theorem leaves open the following question: How do we find enough conserved quantities? This is generally a very hard task, but there are some structures that can help us. One important such structure is the existence of a *Lax pair*:

Definition 1.1.2. A pair (L, M) of $n \times n$ matrices of observables is a *Lax pair* if the *Lax equation*

$$\dot{L} = [M, L].$$

holds. We then call L a *Lax matrix.*

Remark. Lax pairs are not unique. In fact, any Lax pair may be twisted by an invertible matrix of observables g via the gauge-like transformation

$$L \mapsto gLg^{-1}, \quad M \mapsto gMg^{-1} + \dot{g}g^{-1}$$

We may also add any polynomial in L to M without changing the Lax equation.

Proposition 1.1.3. *Given a Lax pair (L, M), the spectral invariants $I_k := \operatorname{tr} L^k$ for $k \in \mathbb{Z}$ constitute a family of conserved quantities.*

Proof. Notice that L^{-1} is also a Lax matrix:

$$\dot{L}^{-1} = -L^{-1}\dot{L}L^{-1} = -L^{-1}[M, L]L^{-1} = [M, L^{-1}].$$

Hence, for $k \geq 0$ we obtain

$$\dot{I}_{\pm k} = k \operatorname{tr}(L^{\pm 1})^{k-1}\dot{L}^{\pm 1} = k \operatorname{tr}(L^{\pm 1})^{k-1}[M, L^{\pm 1}] = k \operatorname{tr}[M, L^{\pm k}] = 0,$$

making use of the cyclic property of the trace. □

Example. Let us again consider the harmonic oscillator and introduce the matrices

$$L = \frac{1}{2}\begin{pmatrix} p & q \\ q & -p \end{pmatrix}, \quad M = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We check

$$\dot{L} = \frac{1}{2}\begin{pmatrix} -q & p \\ p & q \end{pmatrix} = \frac{1}{4}\begin{pmatrix} p & q \\ q & -p \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p & q \\ q & -p \end{pmatrix} = [M, L]$$

and see that the conserved quantity $\operatorname{tr} L^2 = \frac{1}{2}(p^2 + q^2)$ is the Hamiltonian.

We quickly remark that the existence of a Lax pair is by itself insufficient to guarantee Liouville integrability. Firstly, it might fail to produce enough *independent* conserved quantities, and secondly, it might fail to produce *Poisson-commuting* conserved quantities. However, there is a way to guarantee that the spectral invariants of the Lax matrix Poisson-commute. We briefly state this result here:

Theorem 1.1.4 (Babelon-Viallet [Aru20])**.** *The eigenvalues of a matrix L of observables on a phase space (M, ω) Poisson-commute if and only if there exists a so-called* dynamical r-matrix $r \in \operatorname{Mat}_n(C^\infty(M))^{\otimes 2}$ *with*

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2],$$

where L_1, L_2 respectively denote $L \otimes 1, 1 \otimes L$ and r_{21}, r_{12} respectively denote r with and without the tensor factors swapped.

1.1.3 Lax connections

The definition of a Lax pair arguably looks ad hoc. One way to see how this structure makes sense is to look at how Lax pairs arise from the flatness equation of so called *Lax connections* in two-dimensional field theories. For later purposes, we will consider Lax connections with a *spectral parameter.*

Definition 1.1.5. Let Σ be an oriented surface and $\mathfrak{g}$ a semi-simple Lie algebra. A *Lax connection with spectral parameter* is a $\mathfrak{g}$-valued 1-form $A(z)$ on Σ meromorphic in z such that $A(z)$ is flat when z is not a pole. Writing

$$A(z) = A_t(z)dt + A_x(z)dx$$

in local coordinates t, x, this means

$$\partial_t A_x(z) - \partial_x A_t(z) = [A_t(z), A_x(z)].$$

Proposition 1.1.6 ([Bei14])**.** *Let $\Sigma = \mathbb{R} \times S^1$ be a cylinder with axial coordinate t and angular coordinate x and $A(z) = A_t(z)dt + A_x(z)dx$ be a Lax connection. Let z not be a pole and*

$$L(z) := \mathrm{Hol}_\gamma(A(z)), \quad M(z) := A_t(z)|_{x=0},$$

where γ is a loop winding once around the cylinder and $\mathrm{Hol}_\gamma(A)$ denotes the holonomy of A around γ, which is invariant under homotopies due to flatness. Then $(L(z), M(z))$ is a Lax pair.

Proof. Let γ be a curve starting at $(t_0, x_0) \in \Sigma$ and ending at $(t_1, x_1) \in \Sigma$. The defining partial differential equations for the holonomy give

$$\frac{\partial}{\partial t_0} \mathrm{Hol}_\gamma(A(z)) = -A_t(z)|_{x=x_0} \mathrm{Hol}_\gamma(A(z)), \quad \frac{\partial}{\partial t_1} \mathrm{Hol}_\gamma(A(z)) = A_t(z)|_{x=x_1} \mathrm{Hol}_\gamma(A(z)).$$

Choosing $x_0 = x_1 = 0$ and $t_0 = t_1$ and letting γ wind once around the cylinder implies

$$\begin{aligned} \dot{L}(z) &= \frac{\partial}{\partial t_0} \mathrm{Hol}_\gamma(A(z)) + \frac{\partial}{\partial t_1} \mathrm{Hol}_\gamma(A(z)) \\ &= A_t(z)|_{x=0} \mathrm{Hol}_\gamma(A(z)) - A_t(z)|_{x=0} \mathrm{Hol}_\gamma(A(z)) = [M(z), L(z)], \end{aligned}$$

which shows that we indeed have a Lax pair. $\square$

Remark. The conserved quantities $I_k(z) = \operatorname{tr} L(z)^k$ arising from Lax connections on a cylinder $\Sigma = \mathbb{R} \times S^1$ are exactly the traces of holonomies of loops with winding number k around the cylinder. We may expand $I_k(z)$ around $z = 0$ to obtain an infinite tower of conserved quantities.

1.1.4 Ruijsenaars-Schneider and Calogero-Moser models

We now come to two very important classes of examples of classical integrable models: The Ruijsenaars-Schneider and Calogero-Moser models.

Definition 1.1.7. The *classical Ruijsenaars-Schneider models*, originally constructed in [Rui87], for N particles of positions y_i and momenta p_i, speed of light c, and coupling constant η have the Hamiltonians

$$H^{\mathrm{RS}} := \sum_i \cosh(p_i/c) \prod_{i \neq j} \sqrt{1 + \frac{u(y_i - y_j)}{u(\eta/c)}} \tag{1.1}$$

with $u(y) := 1/y^2$ for the rational and $u(y) := 1/(4\sinh^2(y/2))$ for the trigonometric case.

For the sequel, we will set $c = 1$, though we first note that setting

$$H^{\mathrm{CM}} := \frac{1}{2}\sum_i p_i^2 + \frac{\eta^2}{2}\sum_{i\neq j} u(y_i - y_j), \tag{1.2}$$

and expanding H^{RS} around $c = \infty$ yields

Proposition 1.1.8.

$$H^{\mathrm{RS}} = N + \left(\frac{1}{c}\right)^2 H^{\mathrm{CM}} + \mathcal{O}\left(\left(\frac{1}{c}\right)^4\right).$$

Proof. This is a quick computation with Mathematica ($\rightarrow$ `RSHamiltonianExpansion.nb`). □

Definition 1.1.9. The function H^{CM} from equation (1.2) defines the Hamiltonians for the *classical Calogero-Moser models*.

The Hamiltonian (1.2) of the rational Calogero-Moser model evidently describes point particles repelling each other through centrifugal inverse-cube forces. [Rui87] hence argues that the rational Ruijsenaars-Schneider model describes the same phenomenon in the relativistic setting with finite speed of light c. We further remark that the trigonometric case might be thought of as having periodic coordinates after taking the position variables y_i to be purely imaginary, hence describing particles on a circle. Our focus will be on the rational Ruijsenaars-Schneider model and the trigonometric Calogero-Moser model.

Proposition 1.1.10. *Lax matrices ensuring Liouville integrability for the rational Ruijsenaars-Schneider and the trigonometric Calogero-Moser model are given by*

$$L_{ij}^{\mathrm{RS}} = \frac{\eta}{y_i - y_j + \eta}\left(\prod_{k\neq j}\frac{y_j - y_k - \eta}{y_j - y_k}\right)e^{-p_j} \tag{1.3}$$

and

$$L_{ij}^{\mathrm{CM}} = \delta_{ij}p_i + (1-\delta_{ij})\eta\theta_{ij}, \quad \theta_{ij} := \frac{e^{y_i}}{e^{y_i} - e^{y_j}}.$$

Proof. See [Aru20]. □

Remark. For the rational Ruijsenaars-Schneider model, we will make use of the alternative Hamiltonian

$$\operatorname{tr} L^{\mathrm{RS}} = \sum_i\left(\prod_{i\neq j}\frac{y_i - y_j - \eta}{y_i - y_j}\right)e^{-p_i}. \tag{1.4}$$

Choosing this Hamiltonian is mostly a matter of convention, as $\operatorname{tr} L^{\mathrm{RS}} + \operatorname{tr}(L^{\mathrm{RS}})^{-1}$ is equivalent to (1.1) up to a canonical transformation [Aru20]. Note that for the trigonometric Calogero-Moser model we already have

$$\tfrac{1}{2}\operatorname{tr}(L^{\mathrm{CM}})^2 = H^{\mathrm{CM}}$$

due to $\theta_{ij}\theta_{ji} = 1/(4\sinh^2((y_i - y_j)/2))$.

1.2 Quantum integrability

Integrability in the quantum context is generally considered to be a more diffuse concept than classical integrability. The basic definition generalizing Liouville integrability is the existence of a large set of mutually commuting quantum observables that include the Hamiltonian. Such can often be guaranteed by the existence of operators fulfilling the quantum Yang-Baxter equation or the applicability of various Bethe ansätze [Aru]. This usually comes hand-in-hand with the existence of large symmetries, especially *Hecke symmetries* and *Yangian symmetries*, which automatically generate sets of commuting operators via their commutative subalgebras.

In order to quantize the rational Ruijsenaars-Schneider and trigonometric Calogero-Moser model, we will start with a discussion of Hecke algebras and their representations, after which we move on to the Yangian of $\mathfrak{gl}_\ell$, whose representation theory gives rise to the Heisenberg model.

1.2.1 Weyl groups and Hecke algebras of type A

Definition 1.2.1. (i) The Weyl group of type A_{N-1} is the symmetric group S_N on N letters. We write cycles in standard form $(1\ 2\ 3)$, mapping 1 to 2, 2 to 3, and 3 to 1, and denote the simple transpositions $(i\ i+1)$ by s_i for $1 \le i < N$. Finally, we let $w := s_1 \cdots s_{N-1} = (1\ \cdots\ N)$ denote the Coxeter element.

(ii) Define the *Jucys-Murphy elements* $Y_i := \sum_{k=1}^{i-1} (k\ i) \in \mathbb{C}[S_N]$. They commute among each other and fulfill $s_i Y_j = Y_j s_i$ for $|i-j| > 1$ as well as $s_i Y_i = Y_{i+1} s_i - 1$.

(iii) Given a standard Young tableau t of shape λ partitioning N, define the *Specht polynomial*

$$f_t(y_1, ..., y_N) := \prod_{i<j \text{ in a column of } t} (y_j - y_i).$$

(iv) The *Specht module* $S(\lambda)$ is the $\mathbb{C}[S_N]$-module spanned by f_t, where t ranges over all standard Young tableaux of shape λ and S_N acts by permutation of variables.

Proposition 1.2.2. *(i) The Specht modules $S(\lambda)$, for λ ranging over Young diagrams partitioning N, exhaust all finite-dimensional irreducible representations of $\mathbb{C}[S_N]$.*

(ii) The Specht module $S(\lambda)$ has a basis $\{v_t\}$ labeled by standard Young tableaux t of shape λ such that

$$Y_i v_t = c_i(t) v_t,$$

where $c_i(t)$ is the content of i in t.

Proof. For (i), see (tba), for (ii), see [Mur81]. □

Definition 1.2.3. (i) The *affine Weyl group of type A_{N-1}* is the group $\dot{S}_N := S_N \ltimes \mathbb{Z}^N$, where S_N acts on $\mathbb{Z}^N$ by permutation of coordinates. The group algebra $\mathbb{C}[\dot{S}_N]$ is canonically generated by the group algebra $\mathbb{C}[S_N]$ of the symmetric group and an algebra of Laurent polynomials $\mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}]$. In fact, it is generated by the simple transpositions s_i and the element $\pi := X_1 w$.

(ii) The *degenerate double affine Hecke algebra (of type A_{N-1})* is the $\mathbb{C}[\eta,\hbar]$-algebra $\ddot{H}_N$ generated by the group algebra $\mathbb{C}[\dot{S}_N]$ and the polynomial algebra $\mathbb{C}[y_1,...,y_N]$ subject to the cross-relations

$$s_i y_j = y_j s_i, \quad s_i y_i = y_{i+1} s_i + \eta, \quad \pi y_i \pi^{-1} = y_{i+1}, \quad \pi y_N \pi^{-1} = y_1 + \mathrm{i}\hbar,$$

where $1 \leq i < N, 1 \leq j \leq N$ and $|i-j| > 1$. It can thus be seen to be generated by the simple transpositions s_i as well as π and y_1. Let us also remark that these relations admit an anti-automorphism

$$s_i \mapsto s_i, \quad X_i \mapsto X_i^{-1}, \quad y_i \mapsto y_i,$$

making left modules equivalent to right modules.

(iii) The $\mathbb{C}[\eta]$-subalgebra $\dot{H}_N \subset \ddot{H}_N$ generated only by $\mathbb{C}[S_N]$ and $\mathbb{C}[y_1,...,y_N]$ is the *degenerate affine Hecke algebra (of type A_{N-1})*. It is also generated by the s_i and y_1. There is an evaluation homomorphism

$$\mathrm{ev} : \dot{H}_N \to S_N[y], \quad s_i \mapsto s_i, y_i \mapsto y - \eta Y_i.$$

(iv) Let $e := \frac{1}{N!}\sum_{\sigma\in S_N} \sigma \in \ddot{H}_N$ be the symmetrizer. Then $S\ddot{H}_N := e\ddot{H}_N e$ is the *spherical degenerate double affine Hecke algebra (of type A_{N-1})*, or *spherical subalgebra* for short.

The spherical degenerate double affine Hecke algebra will be of great importance, since it produces families of commuting operators by the following proposition:

Proposition 1.2.4. *(i) $S\ddot{H}_N$ is finitely generated and commutative.*

(ii) $S\ddot{H}_N$ is generated as a commutative algebra by elementary symmetric polynomials in $X_1,...,X_N$ and $y_1,...,y_N$.

(iii) The degenerate double affine Hecke algebra $\ddot{H}_N$ is finite over $S\ddot{H}_N$.

(iv) Specializing to $\hbar = 0$, there is the Satake isomorphism $Z(\ddot{H}_N) \xrightarrow{\sim} S\ddot{H}_N, z \mapsto ze$, while one has $Z(\ddot{H}_N) = \mathbb{C}$ for $\hbar \neq 0$.

(v) The spectrum of $S\ddot{H}_N$ is the configuration space of the classical rational Ruijsenaars-Schneider and trigonometric Calogero-Moser model.

Proof. See [Obl03]. □

Much of the following is parallel to [LPS22], where the non-degenerate case is discussed. We construct a representation of $\dot{H}_N$ on polynomials $\mathbb{C}[y_1,...,y_N]$. This is supposed to become the wave function representation for the rational Ruijsenaars-Schneider model. Let us begin by considering the action where S_N acts by permutation of the variables. This allows us to introduce the following operators:

(i) The *divided difference operators* are defined as

$$\Delta_i := (y_i - y_{i+1})^{-1}(1 - s_i).$$

Note that the anti-symmetrization implies that $\Delta_i^2 = 0$ and that $\Delta_i f$ for any polynomial f will again be a polynomial despite its denominator. We further note that $\Delta_i y_i = y_{i+1}\Delta_i + 1$.

(ii) The *T-operators* are defined as

$$T_i := s_i + \eta\Delta_i = \frac{y_i - y_{i+1} - \eta}{y_i - y_{i+1}} s_i + \frac{\eta}{y_i - y_{i+1}}.$$

Lemma 1.2.5. *The mapping*

$$s_i \mapsto T_i, \quad y_i \mapsto y_i$$

gives rise to a representation of $\dot{H}_N$ on $\mathbb{C}[y_1, ..., y_N]$.

Proof. It is clear that T_i and T_j as well as T_i and y_j commute for $|i - j| > 1$. We also have

$$T_i^2 = 1 + s_i\eta\Delta_i + \eta\Delta_i s_i = 1 + \eta\Delta_i - \eta\Delta_i = 1.$$

as well as the braid relation $T_i T_{i-1} T_i = T_{i-1} T_i T_{i-1}$, which may be quickly checked in Mathematica ($\rightarrow$ `TOperatorRelations.nb`). The relation $T_i y_i = y_{i+1} T_i + \eta$ follows readily from $\Delta_i y_i = y_{i+1}\Delta_i + 1$. □

Proposition 1.2.6. *Let π act on $\mathbb{C}[y_1, ..., y_N]$ by*

$$(\pi f)(y_1, ..., y_N) := f(y_2, ..., y_N, y_1 + \mathrm{i}\hbar).$$

The mapping

$$s_i \mapsto T_i, \quad X_i \mapsto T_{i-1} \cdots T_1 \pi T_{N-1} \cdots T_i, \quad y_i \mapsto y_i$$

gives rise to a representation of $\ddot{H}_N$ on $\mathbb{C}[y_1, ..., y_N]$.

Proof. It is clear that the action of π fulfills the necessary relations between π and the y_i. Furthermore, we see that $X_1 = \pi w^{-1}$ acts as $\pi T_{N-1} \cdots T_1$, from which the general form for X_i follows. □

There is an obvious representation of $\dot{S}_N$ on Laurent polynomials $\mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}]$ where S_N permutes the variables and X_i acts simply by multiplication. This representation also extends to $\ddot{H}_N$ in the following way:

Proposition 1.2.7. *Let $\theta_{ij} := X_i/(X_i - X_j)$. The mapping*

$$s_i \mapsto s_i, \quad X_i \mapsto X_i, \quad y_i \mapsto -\mathrm{i}\hbar X_i \partial_i - i\eta + \eta \sum_{j<i} \theta_{ji}(1 - (i\ j)) - \eta \sum_{j>i} \theta_{ij}(1 - (i\ j))$$

gives rise to a representation of $\ddot{H}_N$ on $\mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}]$.

Proof. We refer to appendix B, lemma B.1. □

Remark. The operators X_i acting on $\mathbb{C}[y_1, ..., y_N]$ are called *Macdonald operators*, while the operators y_i acting on $\mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}]$ are called *Dunkl operators*.

1.2.2 Quantization of Ruijsenaars-Schneider and Calogero-Moser models

We will now show that the former polynomial representation above yields the rational Ruijsenaars-Schneider model, while the latter yields the trigonometric Calogero-Moser model. More precisely, we will see that certain symmetric polynomials in the X_i and y_i respectively yield the correct Hamiltonians. This manifests the bispectral duality between those two systems [Cha00] as a remnant of the S-duality exchanging the two sets of Laurent generators of the non-degenerate double affine Hecke algebra.

Definition 1.2.8. The *quantum rational Ruijsenaars-Schneider model* is essentially given by the former polynomial representation of the degenerate double affine Hecke algebra. Concretely, its Hilbert space of wave functions is nothing but the vector space $\mathbb{C}[y_1, ..., y_N]$, where the polynomial generators $y_1, ..., y_N$ provide position operators while the Laurent generators $X_1, ..., X_N$ provide operators involving momenta. Of particular interest is the spherical degenerate double affine Hecke algebra $S\ddot{H}_N$, from which we get commuting Hamiltonians

$$D_k := e_k(X_1, ..., X_N) \in S\ddot{H}_N,$$

with e_k the kth elementary symmetric polynomial.

Lemma 1.2.9. *Let*

$$x_{ji} := \frac{y_i - y_j - \eta}{y_i - y_j} + \frac{\eta}{y_i - y_j}(i\ j).$$

Then

$$X_i = x_{i,i-1} \cdots x_{i1} e^{\mathrm{i}\hbar\partial_i} x_{iN} \cdots x_{i,i+1}.$$

when acting on $\mathbb{C}[y_1, ..., y_N]$.

Proof. Observe that $T_i s_i = x_{i+1,i}$ and $s_i T_i = x_{i,i+1}$, hence

$$\begin{aligned} T_{i-1} \cdots T_1 \pi T_{N-1} \cdots T_i &= T_{i-1} \cdots T_1 e^{\mathrm{i}\hbar\partial_1} s_1 \cdots s_{N-1} T_{N-1} \cdots T_i \\ &= T_{i-1} \cdots T_1 s_1 \cdots s_{i-1} e^{\mathrm{i}\hbar\partial_i} s_i \cdots s_{N-1} T_{N-1} \cdots T_i \\ &= x_{i,i-1} \cdots x_{i1} e^{\mathrm{i}\hbar\partial_i} x_{iN} \cdots x_{i,i+1}. \end{aligned}$$

□

Proposition 1.2.10. *On symmetric polynomials,* i.e. *bosonic wave functions, D_1 reduces to the canonically quantized Hamiltonian (1.4) of the rational Ruijsenaars-Schneider model:*

$$D_1 = \sum_i \left(\prod_{i \neq j} \frac{y_i - y_j - \eta}{y_i - y_j} \right) e^{\mathrm{i}\hbar\partial_i}.$$

Proof. We proceed in analogy to [JKK+95]. We know that X_i acts as

$$x_{i,i-1} \cdots x_{i1} e^{\mathrm{i}\hbar\partial_i} x_{iN} \cdots x_{i,i+1}$$

by the previous lemma. It is clear that x_{ij} acts as the identity on symmetric polynomials. Note however that while D_1 does preserve the space of symmetric polynomials, $e^{\mathrm{i}\hbar\partial_i}$ on its own does not, so the x_{ij} to the left of $e^{\mathrm{i}\hbar\partial_i}$ act non-trivially. Pulling the permutations to the right yields

$$\left(\prod_{j<i}\frac{y_i - y_j - \eta}{y_i - y_j}\right)e^{\mathrm{i}\hbar\partial_i} + (\text{terms for } e^{\mathrm{i}\hbar\partial_j} \text{ with } j < i).$$

In particular, we see that only X_N contributes to the coefficient in front of $e^{\mathrm{i}\hbar\partial_N}$, which is exactly $\prod_{N\neq j}\frac{y_N - y_j - \eta}{y_N - y_j}$. The symmetry $(j\ \cdots\ N)D_1(N\ \cdots\ j) = D_1$ for $j < N$ shows that the coefficients in front of $e^{\mathrm{i}\hbar\partial_j}$ also have this form. □

Let us now move on to the trigonometric Calogero-Moser model:

Definition 1.2.11. The *quantum trigonometric Calogero-Moser model* is given by the latter polynomial representation of the degenerate double affine Hecke algebra, *i.e.* $\mathbb{C}[X_1^{\pm}, ..., X_N^{\pm}]$ is its Hilbert space of wave functions with position operators $X_1, ..., X_N$ and momentum-like operators $y_1, ..., y_N$. We again have a set of commutating Hamiltonians in the spherical degenerate double affine Hecke algebra given by

$$C'_k := \frac{1}{k}p_k(y_1, ..., y_N) \in S\ddot{H}_N,$$

with p_k the kth power sum symmetric polynomial. To compare with the classical case, [Eti09] considers the conjugates

$$C_k := \delta^{-1}C'_k\delta, \quad \delta := \prod_{i<j}(X_i - X_j)^{2\mathrm{i}\eta/\hbar}.$$

Proposition 1.2.12. *On* $\delta^{-1}\mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}]^{S_N}$*,* C_1 *reduces to the canonically quantized total momentum of the trigonometric Calogero-Moser model, while* C_2 *reduces to the Hamiltonian:*

$$C_1 = -\mathrm{i}\hbar\sum_i X_i\partial_i, \quad C_2 = -\frac{\hbar^2}{2}\sum_i (X_i\partial_i)^2 + \frac{\eta(\eta - \mathrm{i}\hbar)}{2}\sum_{i\neq j}\theta_{ij}\theta_{ji}.$$

Proof. We refer to appendix B, lemma B.2. □

Remark. The reparametrization $X_i = e^{x_i}$ yields

$$X_i\frac{\partial}{\partial X_i} = \frac{\partial}{\partial x_i}, \quad \theta_{ij}\theta_{ji} = \frac{1}{4\sinh^2((x_i - x_j)/2)},$$

so C_2 will indeed reduce to the classical Hamiltonian (1.2) upon sending $\hbar$ to zero.

1.2.3 The Yangian of $\mathfrak{gl}_\ell$

In order to define the Heisenberg model, we will make use of the representation theory of the *Yangian.* The Yangian $Y(\mathfrak{gl}_\ell)$ was first defined by Drinfeld in his seminal paper [Dri85] introducing quantum groups. An extensive textbook review can be found in [Mol07], which we follow closely.

Definition 1.2.13. The *Yangian* for the complex Lie algebra $\mathfrak{gl}_\ell$, written $Y(\mathfrak{gl}_\ell)$, is defined as the unital associative $\mathbb{C}[\eta]$-algebra with the following presentation: It has generators $t_{ij}^{(r)}$ for $1 \leq i, j \leq \ell$ and $r \geq 1$, which are subject to the relations

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = \eta(t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)}), \tag{1.5}$$

with $r, s \geq 0$ making use of $t_{ij}^{(0)} := \delta_{ij}/\eta$. Note that specializing to $\eta = 0$ yields the relations for the current algebra $\mathfrak{gl}_\ell[z^{-1}]$, hence the Yangian may be understood as a deformation thereof. Also note that the Yangian is a filtered algebra where we let $t_{ij}^{(r)}$ have degree $r-1$. Denote its filtered pieces by $F_i Y(\mathfrak{gl}_\ell)$.

Remark. It is often useful to introduce these generators as coefficients of a formal power series

$$t_{ij}(z) := \eta \sum_{r \geq 0} t_{ij}^{(r)} z^{-r} \in Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!].$$

The parameter z is called the *spectral parameter*. With this notation, equation (1.5) becomes

$$(z - w)[t_{ij}(z), t_{kl}(w)] = \eta(t_{kj}(z) t_{il}(w) - t_{kj}(w) t_{il}(z)) \in Y(\mathfrak{gl}_\ell)[\![z^{-1}, w^{-1}]\!], \tag{1.6}$$

which should be understood as an equality of the coefficients in each degree.

One might ask how $Y(\mathfrak{gl}_\ell)$ is related to $\mathfrak{gl}_\ell$. To this end, we let e_{ij} denote the basis of matrix units for $\mathfrak{gl}_\ell$ generating the universal enveloping algebra $U(\mathfrak{gl}_\ell)$. We observe from the defining relations for $r = 0$ and $s = 1$ that

$$[t_{ij}^{(1)}, t_{kl}^{(1)}] = \delta_{kj} t_{il}^{(1)} - t_{kj}^{(1)} \delta_{il},$$

which are the defining relations for the Lie algebra $\mathfrak{gl}_\ell$. This motivates the following:

Proposition 1.2.14. *(i) There is an injective homomorphism*

$$U(\mathfrak{gl}_\ell) \to Y(\mathfrak{gl}_\ell), \quad e_{ij} \mapsto t_{ij}^{(1)}.$$

(ii) There is a surjective homomorphism

$$\mathrm{ev}_0 : Y(\mathfrak{gl}_\ell) \to U(\mathfrak{gl}_\ell), \quad t_{ij}^{(1)} \mapsto e_{ij}, \quad t_{ij}^{(p)} \mapsto 0, \quad p > 1$$

called the evaluation homomorphism. *On the level of power series, it is given by*

$$Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!] \to U(\mathfrak{gl}_\ell)[\![z^{-1}]\!], \quad t_{ij}(z) \mapsto \delta_{ij} + \frac{\eta e_{ij}}{z}.$$

which should be understood coefficient-wise.

We saw that it can be notationally convenient to introduce the formal parameter z and work inside the algebra $Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!]$. In the same vein, it will also be convenient to work inside the algebra $\operatorname{End} V \otimes Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!]$, where $V := \mathbb{C}^\ell[\eta]$ and write the generators of $Y(\mathfrak{gl}_\ell)$ in matrix form:

$$T(z) := \sum_{ij} e_{ij} \otimes t_{ij}(z) \in \operatorname{End} V \otimes Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!].$$

The additional vector space V is usually called the *auxiliary space* to distinguish it from representation spaces, usually called *quantum spaces.* The expression might be thought of as a power series of matrices with coefficients in $Y(\mathfrak{gl}_\ell)$. With this in hand, define the following elements in $\operatorname{End} V \otimes Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!]^{\otimes N}$:

$$T_{[a]}(z) := \sum_{ij} \overset{0}{e_{ij}} \otimes \overset{1}{1} \otimes \cdots \otimes \overset{a}{t_{ij}(z)} \otimes \cdots \otimes \overset{N}{1},$$

where we have used 0 to denote the auxiliary space index and $1, ..., a, ..., N$ to denote quantum space indices. We can now define the map

$$\operatorname{End} V \otimes Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!] \to \operatorname{End} V \otimes Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!]^{\otimes 2}, \quad T(z) \mapsto T_{[1]}(z) T_{[2]}(z).$$

Using the identity $e_{ij} e_{kl} = \delta_{jk} e_{il}$, we see that on $Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!]$ this reduces to

$$Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!] \to Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!]^{\otimes 2}, \quad t_{ij}(z) \mapsto \sum_k t_{ik}(z) \otimes t_{kj}(z),$$

which degree-wise gives the map

$$\Delta : Y(\mathfrak{gl}_\ell) \to Y(\mathfrak{gl}_\ell) \otimes Y(\mathfrak{gl}_\ell), \quad t_{ij}^{(r)} \mapsto \sum_k \sum_{s=0}^{r} t_{ik}^{(s)} \otimes t_{kj}^{(r-s)}. \tag{1.7}$$

Proposition 1.2.15. *The map Δ from (1.7) is an algebra homomorphism. It equips $Y(\mathfrak{gl}_\ell)$ with the structure of a bialgebra with counit*

$$\epsilon : Y(\mathfrak{gl}_\ell) \to \mathbb{C}[\eta], \quad t_{ij}^{(r)} \mapsto 0.$$

Furthermore, $Y(\mathfrak{gl}_\ell)$ is a Hopf algebra. The antipode can be represented on the level of the algebra $\operatorname{End} V \otimes Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!]$ as

$$S : T(z) \mapsto T(z)^{-1}.$$

The inverse $T(z)^{-1}$ exists since the leading term in the series is the identity matrix.

Proof. This is theorem 1.5.1 of [Mol07]. □

Remark. Similar to the definition of the $T_{[a]}(z)$, we may define the following elements of the algebra $(\operatorname{End} V)^{\otimes k} \otimes Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!]$:

$$T_a(z) := \sum_{ij} \overset{1}{1} \otimes \cdots \otimes \overset{a}{e_{ij}} \otimes \cdots \otimes \overset{k}{1} \otimes \overset{0}{t_{ij}(z)},$$

where we have used $1, ..., a, ..., k$ to denote auxiliary space indices and 0 to denote the quantum space index. Letting $P := \sum_{ij} e_{ij} \otimes e_{ji} \in (\operatorname{End} V)^{\otimes 2}$ be the permutation operator and

$$R(z) := 1 - \frac{\eta P}{z}$$

be *Yang's rational R-matrix*, we obtain the following proposition:

Proposition 1.2.16 (RTT relation)**.** *The defining relations (1.6) of the Yangian may equivalently be written as*

$$R_{12}(z-w)T_1(z)T_2(w) = T_2(w)T_1(z)R_{12}(z-w), \tag{1.8}$$

where $R_{12}(z-w)$ is understood to act on the two auxiliary spaces labeled 1 *and* 2*.*

Proof. Consider the result of acting on $e_j \otimes e_l \in V \otimes V$. The left hand side gives

$$\sum_{ik} e_i \otimes e_k \otimes t_{ij}(z)t_{kl}(w) - \frac{\eta}{z-w}\sum_{ik} e_i \otimes e_k \otimes t_{kj}(z)t_{il}(w)$$

while the right hand side gives

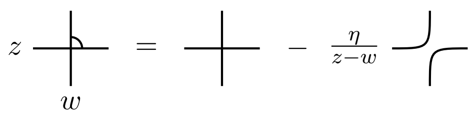
$$\sum_{ik} e_i \otimes e_k \otimes t_{kl}(w)t_{ij}(z) - \frac{\eta}{z-w}\sum_{ik} e_i \otimes e_k \otimes t_{kj}(z)t_{il}(w).$$

But this becomes exactly (1.6) after multiplying with $z-w$. $\square$

Remark. Note that the RTT relation is invariant under multiplying the R-matrix by a power-series in $z-w$. There are different conventions for the choice of normalization, which we briefly summarize:

Yang's convention	$R(z) = 1 - \frac{\eta}{z}P$
polynomial convention	$\hat{R}(z) = z - \eta P$
unitary convention	$\check{R}(z) = \frac{z}{z-\eta} - \frac{\eta}{z-\eta}P$

Diagrammatically, we write the R-matrix according to Yang's convention as

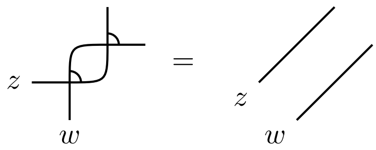


where we indicate the R-matrix at a crossing with an arc, signifying that it acts from south-west to north-east. Mathematically, these are to be read as string diagrams. Physically, the diagrams show two interacting particles: The diagram for the identity shows two particles passing through each other without interaction and the diagram for the permutation operator shows two particles repelling each other, weighted by the coupling constant and inverse distance.

We will mostly use Yang's convention, matching [Mol07] and many other sources. The polynomial convention matches much of [Aru]. We will later also make use of the unitary convention, which is arguably the most natural since it satisfies *unitarity*:

$$\check{R}(z-w)\check{R}(w-z) = 1,$$

illustrated with the diagram



which is essentially the second Reidemeister move. We can always switch from one convention to the other at the cost of normalization factors. The crucial difference between conventions lies in their analytic structure, which we will make use of extensively. More specifically:

$$\begin{aligned}
1 &= R(\infty) = \lim_{z\to\infty} \hat{R}(z)/z = \check{R}(\infty),\\
P &= -\operatorname{Res}_{z=0} R(z)/\eta = -\hat{R}(0)/\eta = \check{R}(0),\\
\Pi^+ &= R(-\eta)/2 = -\hat{R}(-\eta)/2\eta = \check{R}(-\eta),\\
\Pi^- &= R(\eta)/2 = \hat{R}(\eta)/2\eta = \operatorname{Res}_{z=\eta} \check{R}(z)/2\eta,
\end{aligned}$$

where $\Pi^\pm = (1 \pm P)/2$ are the symmetrizer and antisymmetrizer respectively. This is where the R-matrix becomes singular.

Definition 1.2.17. (i) There is a homomorphism of $\mathbb{C}[\eta]$-algebras

$$\exp(-y\partial) : Y(\mathfrak{gl}_\ell) \to Y(\mathfrak{gl}_\ell)[y], \quad t_{ij}(z) \mapsto t_{ij}(z-y).$$

that specializes to a *shift automorphism* of the Yangian for fixed values of y. This is possible because we can formally expand $(z-y)^{-r}$ as a power series in z^{-1} such that the coefficients are polynomials in y:

$$(z-y)^{-r} = \sum_{s=r}^{\infty} \binom{s-1}{r-1} y^{s-r} z^{-s}.$$

This also yields a general evaluation homomorphism

$$\mathrm{ev} : Y(\mathfrak{gl}_\ell) \to U(\mathfrak{gl}_\ell)[y], \quad t_{ij}(z) \mapsto \delta_{ij} + \frac{\eta e_{ij}}{z-y}.$$

(ii) There is the *transposition automorphism* $T(z) \mapsto T^t(-z)$, or $t_{ij}(z) \mapsto t_{ji}(-z)$.

(iii) Given any $f(z) \in \mathbb{C}[\eta][\![z^{-1}]\!]$ with leading term equal to one, there is an automorphism $T(z) \mapsto f(z)T(z)$.

(iv) Given any $g \in \mathrm{GL}_\ell$, there is an automorphism $T(z) \mapsto gT(z)g^{-1}$.

1.2.4 The dual Yangian of $\mathfrak{gl}_\ell$

In addition to the Yangian $Y(\mathfrak{gl}_\ell)$, we will later make use of the *dual Yangian* $Y^\vee(\mathfrak{gl}_\ell)$, which was studied in detail in the paper [Naz19]. Essentially, while the Yangian describes the disk at infinity with local coordinate z^{-1}, the dual Yangian describes the disk around zero with local coordinate z. We proceed parallel to the construction of the Yangian:

Definition 1.2.18. The *dual Yangian* for the complex Lie algebra $\mathfrak{gl}_\ell$, written $Y^\vee(\mathfrak{gl}_\ell)$, is defined as the unital associative $\mathbb{C}[\eta]$-algebra with the following presentation: It has generators $t_{ij}^{(-r)}$ for $1 \leq i, j \leq \ell$ and $r \geq 1$. To describe the defining relations, we again let $t_{ij}^{(0)} = \delta_{ij}/\eta$ and introduce

$$T^\vee(z) := \sum_{ij} t_{ij}^\vee(z) \otimes e_{ij} \in Y^\vee(\mathfrak{gl}_\ell)[\![z]\!] \otimes V, \quad t_{ij}^\vee(z) = \eta \sum_{r \geq 0} t_{ij}^{(-r)} z^r \in Y^\vee(\mathfrak{gl}_\ell)[\![z]\!],$$

as well as

$$T_a^\vee(z) = \sum_{ij} \overset{0}{t_{ij}^\vee(z)} \otimes \overset{1}{1} \otimes \cdots \otimes \overset{a}{e_{ij}} \otimes \cdots \otimes \overset{k}{1}.$$

The defining relation of the dual Yangian then is the *TTR relation*:

$$T_1^\vee(z) T_2^\vee(w) R_{12}(z-w) = R_{12}(z-w) T_2^\vee(w) T_1^\vee(z). \tag{1.9}$$

We may equip the dual Yangian with a descending filtration by giving $t_{ij}^{(-r)}$ the degree r. Denote the Nth filtered piece by $F_N Y^\vee(\mathfrak{gl}_\ell)$ and the completion with respect to this filtration by $\hat{Y}^\vee(\mathfrak{gl}_\ell)$.

Proposition 1.2.19. *The dual Yangian $Y^\vee(\mathfrak{gl}_\ell)$ is a bialgebra with comultiplication and counit*

$$\Delta : t_{ij}^\vee(z) \mapsto \sum_k t_{ik}^\vee(z) \otimes t_{kj}^\vee(z), \quad \epsilon : t_{ij}^\vee(z) \mapsto \delta_{ij}.$$

Proof. This appears in [Naz19]. □

Remark. The bialgebra structure of $Y^\vee(\mathfrak{gl}_\ell)$ extends to the completion $\hat{Y}^\vee(\mathfrak{gl}_\ell)$. We may additionally define an antipode on $\hat{Y}^\vee(\mathfrak{gl}_\ell)$, see [Naz19], making $\hat{Y}^\vee(\mathfrak{gl}_\ell)$ into a Hopf algebra.

Proposition 1.2.20. *There is a canonical non-degenerate pairing*

$$\langle \cdot, \cdot \rangle : Y(\mathfrak{gl}_\ell) \otimes Y^\vee(\mathfrak{gl}_\ell) \to \mathbb{C}[\eta],$$

such that multiplications and comultiplications become dual to each other in the sense that $\langle a, bc \rangle = \langle \Delta(a), b \otimes c \rangle$ and $\langle ab, c \rangle = \langle a \otimes b, \Delta(c) \rangle$, understanding that $\langle a \otimes b, c \otimes d \rangle := \langle a, c \rangle \langle b, d \rangle$.

Proof. This appears in [Naz19]. □

Lemma 1.2.21. *There is a basis $X_1, X_2, \ldots$ of $Y(\mathfrak{gl}_\ell)$ and a basis $X_1^\vee, X_2^\vee, \ldots \in Y^\vee(\mathfrak{gl}_\ell)$, both with increasing filtration degrees, such that $\langle X_i, X_j^\vee \rangle = \delta_{ij}$.*

Proof. This is a result of [Naz19]. □

Definition 1.2.22. The elements

$$\mathcal{R}_N := \sum_{\deg X_i \leq N} X_i^\vee \otimes X_i \in Y^\vee(\mathfrak{gl}_\ell)/F_{N+1} Y^\vee(\mathfrak{gl}_\ell) \otimes F_N Y(\mathfrak{gl}_\ell) \tag{1.10}$$

define an element $\mathcal{R} \in Y^\vee(\mathfrak{gl}_\ell) \hat{\otimes} Y(\mathfrak{gl}_\ell)$ called *the universal R-matrix.*

1.2.5 Representations of the Yangian

We briefly recall the representation theory of $\mathfrak{gl}_\ell$ before moving on to the Yangian. Consider a representation V of $U(\mathfrak{gl}_\ell)$. A non-zero element $\omega \in V$ is a highest weight vector with highest weight $\lambda = (\lambda_1, ..., \lambda_\ell)$ for $\lambda_i \in \mathbb{C}$ if the following relations hold:

$$e_{ij}\omega = 0, \quad 1 \leq i < j \leq \ell$$
$$e_{ii}\omega = \lambda_i\omega, \quad 1 \leq i \leq \ell.$$

If V is generated by ω, V is a highest weight representation with highest weight λ. Clearly, any highest weight representation is a quotient of the Verma module $M(\lambda)$, which is defined as $U(\mathfrak{gl}_\ell)$ quotiented by the left ideal generated by the coefficients of e_{ij} for $i < j$ as well as $e_{ii} - \lambda_i$. The Verma module has a unique maximal submodule, so that it has a unique simple quotient, which is denoted $L(\lambda)$. These are finite-dimensional if and only if $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$ for $i = 1, ..., \ell - 1$, which means that λ can be thought of as a complex number λ_1 together with a Young diagram with at most ℓ rows, see appendix A for more on Young diagrams. They exhaust all finite-dimensional irreducible polynomial representations of $U(\mathfrak{gl}_\ell)$.

We now proceed similarly with the Yangian:

Definition 1.2.23. Let V be a representation of $Y(\mathfrak{gl}_\ell)$. A non-zero element $\omega \in V$ is of *highest weight* $\lambda(z) = (\lambda_1(z), ..., \lambda_\ell(z))$ for $\lambda_i(z) \in \mathbb{C}[\eta][\![z^{-1}]\!]$ if the following relations hold:

$$t_{ij}(z)\omega = 0, \quad 1 \leq i < j \leq \ell$$
$$t_{ii}(z)\omega = \lambda_i(z)\omega, \quad 1 \leq i \leq \ell.$$

If V is generated by ω, we call V a *highest weight representation* with highest weight $\lambda(z)$. Again, any highest weight representation is a quotient of the *Verma module* $M(\lambda(z))$, which is just $Y(\mathfrak{gl}_\ell)$ quotiented by the left ideal generated by the coefficients of $t_{ij}(z)$ for $i < j$ as well as $t_{ii}(z) - \lambda_i(z)$. The Verma module has a unique maximal submodule, so that it has a unique simple quotient, which we denote by $L(\lambda(z))$.

Theorem 1.2.24. *Every finite-dimensional irreducible representation of $Y(\mathfrak{gl}_\ell)$ is a highest weight representation of the form $L(\lambda(z))$ for some highest weight $\lambda(z)$ and has a unique highest weight vector ω up to rescaling.*

Proof. This is theorem 3.2.7 of [Mol07]. □

Theorem 1.2.25. *The irreducible highest weight representation $L(\lambda(z))$ of $Y(\mathfrak{gl}_\ell)$ is finite-dimensional if and only if*

$$\frac{\lambda_i(z)}{\lambda_{i+1}(z)} = \frac{p_i(z+\eta)}{p_i(z)}$$

for $i = 1, ..., \ell - 1$ and unique monic polynomials $p_i(z) \in \mathbb{C}[\eta, z]$ called Drinfeld polynomials.

Proof. This is theorem 3.4.1 of [Mol07]. We consider the case $\ell = 2$. Essentially, one first finds a power series $f(z) \in \mathbb{C}[\eta][\![z^{-1}]\!]$ such that $f(z)\lambda_1(z)$ and $f(z)\lambda_2(z)$ are polynomials, so that we can say without loss of generality that

$$\lambda_1(z) = (1 + \alpha_1 z^{-1}) \cdots (1 + \alpha_k z^{-1}), \quad \lambda_2(z) = (1 + \beta_1 z^{-1}) \cdots (1 + \beta_k z^{-1}).$$

for certain complex numbers $\alpha_1, ..., \alpha_k, \beta_1, ..., \beta_k$. One shows that finite-dimensionality implies $(\alpha_i - \beta_i)/\eta \in \mathbb{Z}_+$ for all $i = 1, ..., k$ after some renumbering. Define the *string*

$$S(\alpha_i, \beta_i) := \{\beta_i, \beta_i + \eta, ..., \alpha_i - 2\eta, \alpha_i - \eta\}.$$

We now set

$$p(z) := \prod_{i=1}^{k} \prod_{\gamma \in S(\alpha_i, \beta_i)} (z + \gamma),$$

which fulfills $\lambda_1(z)/\lambda_2(z) = p(z + \eta)/p(z)$ as can be seen by their poles and zeros. $\square$

Corollary 1.2.26. *Finite-dimensional irreducible representations of $Y(\mathfrak{gl}_\ell)$ are parametrized by tuples $(f(z), p_1(z), ..., p_{\ell-1}(z))$ for $f(z)$ a power series in z^{-1} with constant term one and $p_1(z), ..., p_{\ell-1}(z)$ monic polynomials.*

Proof. The $p_i(z)$ correspond to the Drinfeld polynomials and $f(z)$ to $\lambda_\ell(z)$. $\square$

Definition 1.2.27. Given a weight $\lambda = (\lambda_1, ..., \lambda_\ell) \in \mathbb{C}^\ell$ for $\mathfrak{gl}_\ell$, we can pull the irreducible representation $L(\lambda)$ of $\mathfrak{gl}_\ell$ back along the evaluation homomorphism. Due to surjectivity of the evaluation homomorphism, the resulting representation of $Y(\mathfrak{gl}_\ell)$ will still be irreducible and it will be a highest weight representation with highest weight components $\lambda_i(z) = 1 + \eta\lambda_i z^{-1}$. For simplicity, we also write $L(\lambda)$ for this $Y(\mathfrak{gl}_\ell)$-module. Such modules are called *evaluation modules*. When they are finite-dimensional, they have Drinfeld polynomials

$$p_i(z) = (z + \eta\lambda_{i+1})(z + \eta\lambda_{i+1} + \eta) \cdots (z + \eta\lambda_i - 2\eta)(z + \eta\lambda_i - \eta),$$

which is makes sense due to $(\lambda_i - \lambda_{i+1})/\eta \in \mathbb{Z}_+$. We may twist evaluation modules by the shift automorphism for $y \in \mathbb{C}[\eta]$ or the transposition and respectively obtain modules we denote by $L(\lambda)_y$ and $L(\lambda)^t$ as well as $L(\lambda)^t_y$.

Remark. Let $y_1, ..., y_N \in \mathbb{C}[\eta]$ and consider a tensor product

$$L(\lambda^{(1)})^t_{y_1} \otimes \cdots \otimes L(\lambda^{(N)})^t_{y_N},$$

which inherits the structure of a $Y(\mathfrak{gl}_\ell)$-module via the coproduct. Let ω_j denote the highest weight vector of $L(\lambda^{(j)})^t_{y_j}$, respectively, and define $\omega := \omega_1 \otimes \cdots \otimes \omega_N$, which is a highest weight vector with highest weight components

$$\lambda_i(z) := \left(1 - \frac{\eta\lambda_i^{(1)}}{z - y_1}\right) \cdots \left(1 - \frac{\eta\lambda_i^{(N)}}{z - y_N}\right).$$

Note that these weight components are not linear polynomials anymore. This is because such representations are genuine representations of $Y(\mathfrak{gl}_\ell)$, not of $\mathfrak{gl}_\ell$. Physically, this reflects the fact that local sites usually transform in representations of $\mathfrak{gl}_\ell$ while the extended Yangian symmetry acts non-locally, *i.e.* on multiple tensorands. Hence, a large class of physically relevant representations of the Yangian are given by representations of this form:

Definition 1.2.28. Representations of the form

$$L(\lambda^{(1)})^t_{y_1} \otimes \cdots \otimes L(\lambda^{(N)})^t_{y_N},$$

as above are called *monodromy representations* with *inhomogeneities* $y_1, ..., y_N$. The highest weight vector ω is called the *pseudovacuum*.

We now take a closer look at how transfer matrices act on monodromy representations. Let us introduce the notation $V_y^- := L(\square)_y$ for the vector representation, and $V_y^+ := L(\square)^t_y$ for the covector representation. Our starting observation is the following:

Lemma 1.2.29. *The generator matrix $T(z)$ acts on V_y^+ via the R-matrix:*

$$R(z-y) = 1 - \frac{\eta P}{z-y}.$$

Proof. Inserting the definition of evaluation modules as well as the transposition and shift automorphisms, we see that $T(z)$ acts as

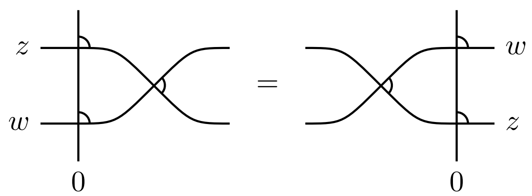
$$\sum_{ij} e_{ij} \otimes \left(\delta_{ij} - \frac{\eta e_{ji}}{z-y}\right) = \sum_i e_{ii} \otimes e_{ii} - \frac{\eta}{z-y} \sum_{ij} e_{ij} \otimes e_{ji} = 1 - \frac{\eta P}{z-y}.$$

$\square$

Remark. This shows that the RTT relation (1.8) in the fundamental representation becomes the *quantum Yang-Baxter equation*

$$R_{12}(z-w)R_{13}(z)R_{23}(w) = R_{23}(w)R_{13}(z)R_{12}(z-w) \in (\operatorname{End} V)^{\otimes 3},$$

where the subscript again denotes which spaces the R-matrix acts on. Diagrammatically, this reads



which is the third Reidemeister move.

Lemma 1.2.30. *Similarly, $T(z)$ acts on V_y^- via*

$$R^t(z-y) := 1 - \frac{\eta P^t}{z-y},$$

where the superscript t denotes transposition in the second space, i.e. $P^t = \sum_{ij} e_{ij} \otimes e_{ij}$.

Having $V_y^{\pm}$ under our belt, let us look at monodromy representations of the type

$$V_{y_1}^+ \otimes \cdots \otimes V_{y_N}^+,$$

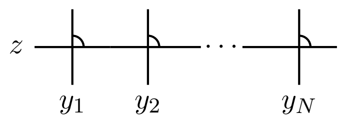
which we call *fundamental.*

Proposition 1.2.31. *The generator matrix $T(z)$ acts on fundamental monodromy representations via the* monodromy matrix

$$M(z) := R_{0N}(z - y_N) \cdots R_{01}(z - y_1),$$

where 0 *is the auxiliary space index and* $1, ..., N$ *are quantum space indices. In terms of diagrams, this reads*

$M(z) =$



Proof. This follows from the previous lemma and the formula for the coproduct (1.7), except the order is reversed since we used the transposition automorphism. □

We remark that V_0^- and V_0^+ are dual as $U(\mathfrak{gl}_\ell)$-modules, as they are both given by $\mathbb{C}^\ell$ with the standard and negative transposed action, respectively. This duality extends to the Yangian up to a shift of the spectral parameter:

Lemma 1.2.32. *The standard scalar product on $\mathbb{C}^\ell$ gives a $Y(\mathfrak{gl}_\ell)$-linear map*

$$V_y^+ \otimes V_{y+\ell\eta}^- \to \mathbb{C}[\eta], \quad e_p \otimes e_q \mapsto \delta_{pq}.$$

Proof. The Yangian generator $t_{ij}(z)$ acts on $V_{y_1}^+ \otimes V_{y_2}^-$ in the following way:

$$\begin{aligned}
&\sum_k \left(\delta_{ik} - \frac{\eta e_{ki}}{z - y_1}\right) e_p \otimes \left(\delta_{kj} + \frac{\eta e_{kj}}{z - y_2}\right) e_q \\
&= \sum_k \delta_{ik} e_p \otimes \delta_{kj} e_q + \frac{\eta}{z - y_2} \sum_k \delta_{ik} e_p \otimes e_{kj} e_q \\
&\qquad - \frac{\eta}{z - y_1} \sum_k e_{ki} e_p \otimes \delta_{kj} e_q - \frac{\eta^2}{(z - y_1)(z - y_2)} \sum_k e_{ki} e_p \otimes e_{kj} e_q \\
&= \delta_{ij} e_p \otimes e_q + \frac{\eta}{z - y_2} \delta_{jq} e_p \otimes e_i - \frac{\eta}{z - y_1} \delta_{ip} e_j \otimes e_q - \frac{\eta^2}{(z - y_1)(z - y_2)} \delta_{ip} \delta_{jq} \sum_k e_k \otimes e_k
\end{aligned}$$

which the standard scalar product maps to

$$\delta_{ij}\delta_{pq} \left(1 + \frac{\eta}{z - y_2} - \frac{\eta}{z - y_1} - \frac{\eta^2 \ell}{(z - y_1)(z - y_2)}\right).$$

This reduces to $\delta_{ij}\delta_{pq}$ when $y_1 = y_2 - \ell\eta$. □

Lemma 1.2.33. *The standard basis of $\mathbb{C}^\ell$ gives a $Y(\mathfrak{gl}_\ell)$-linear map*

$$\mathbb{C}[\eta] \to V^-_{y+\ell\eta} \otimes V^+_y, \quad 1 \mapsto \sum_p e_p \otimes e_p.$$

Proof. The Yangian generator $t_{ij}(z)$ acts in the following way:

$$\begin{aligned}
&\sum_p \sum_k \left(\delta_{ik} + \frac{\eta e_{ik}}{z-y_1}\right) e_p \otimes \left(\delta_{kj} - \frac{\eta e_{jk}}{z-y_2}\right) e_p \\
&= \sum_p \sum_k \delta_{ik} e_p \otimes \delta_{kj} e_p - \frac{\eta}{z-y_2} \sum_p \sum_k \delta_{ik} e_p \otimes e_{jk} e_p \\
&\quad + \frac{\eta}{z-y_1} \sum_p \sum_k e_{ik} e_p \otimes \delta_{kj} e_p - \frac{\eta^2}{(z-y_1)(z-y_2)} \sum_p \sum_k e_{ik} e_p \otimes e_{jk} e_p \\
&= \delta_{ij} \sum_p e_p \otimes e_p - \frac{\eta}{z-y_2} \sum_p \delta_{ip} e_p \otimes e_j + \frac{\eta}{z-y_1} \sum_p \delta_{jp} e_i \otimes e_p - \frac{\eta^2 \ell}{(z-y_1)(z-y_2)} e_i \otimes e_j \\
&= \delta_{ij} \sum_p e_p \otimes e_p + \left(-\frac{\eta}{z-y_2} + \frac{\eta}{z-y_1} - \frac{\eta^2 \ell}{(z-y_1)(z-y_2)}\right) e_i \otimes e_j
\end{aligned}$$

The second term becomes zero when $y_1 = y_2 + \ell\eta$. $\square$

Proposition 1.2.34. *The left and right duals of V^+_y are $V^-_{y+\ell\eta}$ and $V^-_{y-\ell\eta}$.*

Proof. The fact that $V^-_{y+\eta\ell}$ is the left dual for V^+_y follows from the previous lemmata. Twisting by the transposition automorphism of the Yangian turns $V^\pm_y$ into $V^\mp_y$, showing that $V^-_{y-\eta\ell}$ is the right dual by the same argument. $\square$

1.2.6 The Heisenberg model

Definition 1.2.35. The *twisted inhomogeneous Heisenberg $\mathfrak{gl}_\ell$-spin chain* of *length N* with invertible *twist matrix* $g = \operatorname{diag}(\gamma_1, ..., \gamma_\ell)$ and *inhomogeneities* $y_1, ..., y_N$, or *Heisenberg model* for short, has as Hilbert space the fundamental monodromy representations of the Yangian with inhomogeneities $y_1, ..., y_N$. Here η plays the role of Planck's constant. The generating function for the Hamiltonians is the *g-twisted transfer matrix*

$$\tau^g(z) := \operatorname{tr}_0 g_0 T(z),$$

where 0 denotes the auxiliary space index. In the fundamental monodromy representation, this takes the form

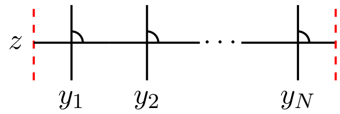
$$\tau^g(z) = \operatorname{tr}_0 g_0 R_{0N}(z-y_N) \cdots R_{01}(z-y_1),$$

or, using the the unitary convention,

$$\check{\tau}^g(z) = \operatorname{tr}_0 g_0 \check{R}_{0N}(z-y_N) \cdots \check{R}_{01}(z-y_1).$$

The corresponding diagram reads

$$\tau^g(z) =$$



where the dashed red lines indicate the twist matrix and are identified so that the whole diagram wraps around a cylinder, yielding the trace over the auxiliary space. One now defines commuting *non-local Hamiltonians* by

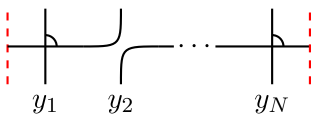
$$\begin{aligned}
H_i &:= -\operatorname{Res}_{z=y_i} \tau^g(z)/\eta \\
&= \operatorname{tr}_0 g_0 R_{0N}(y_i - y_N) \cdots R_{0,i+1}(y_i - y_{i+1}) P_{0i} R_{0,i-1}(y_i - y_{i-1}) \cdots R_{01}(y_i - y_1) \\
&= \operatorname{tr}_0 R_{0,i-1}(y_i - y_{i-1}) \cdots R_{01}(y_i - y_1) g_0 R_{0N}(y_i - y_N) \cdots R_{0,i+1}(y_i - y_{i+1}) P_{0i} \\
&= \operatorname{tr}_0 P_{0i} R_{i,i-1}(y_i - y_{i-1}) \cdots R_{i1}(y_i - y_1) g_i R_{iN}(y_i - y_N) \cdots R_{i,i+1}(y_i - y_{i+1}) \\
&= R_{i,i-1}(y_i - y_{i-1}) \cdots R_{i1}(y_i - y_1) g_i R_{iN}(y_i - y_N) \cdots R_{i,i+1}(y_i - y_{i+1}),
\end{aligned}$$

similarly

$$\begin{aligned}
\check{H}_i &:= \check{\tau}^g(y_i) = \left(\prod_{i \neq j} \frac{y_i - y_j}{y_i - y_j - \eta} \right) H_i \\
&= \check{R}_{i,i-1}(y_i - y_{i-1}) \cdots \check{R}_{i1}(y_i - y_1) g_i \check{R}_{iN}(y_i - y_N) \cdots \check{R}_{i,i+1}(y_i - y_{i+1}).
\end{aligned}$$

As an example, for $i = 2$, this yields the diagram

$$H_2 =$$



To get a Hamiltonian describing local interactions, we have to restrict to $y_1, ..., y_N = 0$. Define the operators

$$\begin{aligned}
\mathcal{P} &:= g_1 P_{12} \cdots P_{N-1,N} \text{ and} \\
\mathcal{H} &:= -\sum_{i=1}^{N-1} P_{i,i+1} - g_1 g_N^{-1} P_{1N}.
\end{aligned}$$

We call $\mathcal{P}$ the *total momentum* and $\mathcal{H}$ the *local Hamiltonian*.

Proposition 1.2.36. *Let $y_1, ..., y_N = 0$. Then $\tau^g(z)$ has the following expansion around $z = 0$:*

$$\tau^g(z) = \frac{\eta^N}{z^N} \mathcal{P} - \frac{\eta^{N-1}}{z^{N-1}} \mathcal{H}\mathcal{P} + \mathcal{O}(z^{-N+2})$$

Proof. Let us expand $z^N \tau^g(z)$ around $z = 0$. To zeroth order, we have

$$\begin{aligned}
z^N \tau^g(z)|_{z=0} &= \eta^N \operatorname{tr}_0 g_0 P_{0N} \cdots P_{01} \\
&= \eta^N \operatorname{tr}_0 P_{0N} \cdots P_{01} g_0 \\
&= \eta^N \operatorname{tr}_0 P_{12} \cdots P_{N-1,N} g_N P_{0N} \\
&= \eta^N \operatorname{tr}_0 g_1 P_{12} \cdots P_{N-1,N} P_{0N} \\
&= \eta^N g_1 P_{12} \cdots P_{N-1,N} = \eta^N \mathcal{P}.
\end{aligned}$$

To first order, we have

$$\begin{aligned}
\frac{\partial}{\partial z} z^N \tau^g(z)|_{z=0} &= \eta^{N-1} \sum_i \operatorname{tr}_0 g_0 P_{0N} \cdots P_{0,i+1} P_{0,i-1} \cdots P_{01} \\
&= \eta^{N-1} \sum_i \operatorname{tr}_0 P_{0N} \cdots P_{0,i+1} P_{0,i-1} \cdots P_{01} g_0 \\
&= \eta^{N-1} \sum_i \operatorname{tr}_0 P_{12} \cdots P_{i-1,i-2} P_{i-1,i+1} P_{i+1,i+2} \cdots P_{N-1,N} g_N P_{0N} \\
&= \eta^{N-1} \sum_i g_1 P_{12} \cdots P_{i-1,i-2} P_{i-1,i+1} P_{i+1,i+2} \cdots P_{N-1,N}.
\end{aligned}$$

Now observe for $1 \leq i < N$:

$$\begin{aligned}
& g_1 P_{12} \cdots P_{i-1,i-2} P_{i-1,i+1} P_{i+1,i+2} \cdots P_{N-1,N} \mathcal{P}^{-1} \\
&= g_1 P_{12} \cdots P_{i-2,i-1} P_{i-1,i+1} P_{i+1,i} P_{i,i-1} P_{i-1,i-2} \cdots P_{21} g_1^{-1} \\
&= P_{12} \cdots P_{i-2,i-1} P_{i-1,i+1} P_{i+1,i} P_{i,i-1} P_{i-1,i-2} \cdots P_{21} g_1 g_1^{-1} \\
&= P_{12} \cdots P_{i-2,i-1} P_{i-1,i+1} P_{i+1,i-1} P_{i+1,i} P_{i-1,i-2} \cdots P_{21} \\
&= P_{12} \cdots P_{i-2,i-1} P_{i+1,i} P_{i-1,i-2} \cdots P_{21} = P_{i,i+1},
\end{aligned}$$

while $i = N$ gives

$$\begin{aligned}
& g_1 P_{12} \cdots P_{N-2,N-1} \mathcal{P}^{-1} \\
&= g_1 P_{12} \cdots P_{N-2,N-1} P_{N,N-1} P_{N-1,N-2} \cdots P_{21} g_1^{-1} \\
&= g_1 P_{1N} g_1^{-1} = g_1 g_N^{-1} P_{1N}.
\end{aligned}$$

Hence $z^N \tau(z) = \eta^N \mathcal{P} - z\eta^{N-1} \mathcal{H}\mathcal{P} + \mathcal{O}(z^2)$. □

In the case $\ell = 2$, we can further rewrite this in a way that makes apparent how the Hamiltonian of the Heisenberg model describes twisted-periodic alignment of nearest neighbor spins, *i.e.* ferromagnetic materials:

Corollary 1.2.37. *Let $\ell = 2$ and $\sigma^x, \sigma^y, \sigma^z$ be the Pauli matrices. Then*

$$\begin{aligned}
\mathcal{H} = &-\frac{N-1}{2} - \frac{1}{2} \sum_{i=1}^{N-1} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z \right) \\
&- \frac{1}{2} g_1 g_N^{-1} - \frac{1}{2} \left(g_1 \sigma_1^x g_N^{-1} \sigma_N^x + g_1 \sigma_1^y g_N^{-1} \sigma_N^y + g_1 \sigma_1^z g_N^{-1} \sigma_N^z \right).
\end{aligned}$$

Proof. This follows from the well-known identity $1 + \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \sigma^z \otimes \sigma^z = 2P$. □

Chapter 2

Duality

2.1 Quantum-classical duality from functional relations

There is an enormous body of work on solving the Heisenberg model, *i.e.* diagonalizing its Hamiltonians, by various loosely related methods. Prime focus is given to a variety of approaches called *Bethe ansätze*, particularly the algebraic Bethe ansatz, see [Aru], which produces eigenvectors from the pseudovacuum using ladder operators supplemented by auxiliary equations called *Bethe equations*. In the coming sections, we will focus on an orthogonal approach employing functional relations between transfer matrices. This approach is originally due to [KNS94] and has been further developed in [Aru], where it is shown how the Lax matrix of the classical rational Ruijsenaars-Schneider model secretly controls the fusion relations, giving a first hint of quantum-classical duality.

2.1.1 Functional relations

Our key to solving the Heisenberg model is to enlargen our consideration of the g-twisted transfer matrix $\tau^g(z)$ to a big commutative subalgebra that contains it and make use of functional relations inside this subalgebra to arrive at a set of polynomial equations, the *spectral equation*, that the eigenvalues of the non-local Hamiltonians $H_1, ..., H_N$, *i.e.* the residues of $\tau^g(z)$, must satisfy. To this end, we consider the *Bethe subalgebra*:

Definition 2.1.1. The *g-twisted Bethe subalgebra* is the subalgebra $B^g(\mathfrak{gl}_\ell)$ of $Y(\mathfrak{gl}_\ell)$ generated by the coefficients of the higher g-twisted transfer matrices $\tau^g_\lambda(z)$, where λ ranges over Young diagrams.

Theorem 2.1.2. *The Bethe subalgebra $B^g(\mathfrak{gl}_\ell)$ is a maximal commutative subalgebra of $Y(\mathfrak{gl}_\ell)$ whenever g has simple spectrum.*

Proof. This appears in [NO96]. □

Now, we have left out what we mean by the higher g-twisted transfer matrices $\tau^g_\lambda(z)$. The basic idea is to take the definition of the usual transfer matrix and switch the auxiliary space from the fundamental representation to a higher representation labeled by a Young diagram λ. This defines the higher transfer matrices $\tau^g_\lambda(z)$. In particular of course, $\tau^g_\square(z)$ will coincide with the usual transfer matrix.

Definition 2.1.3. Let λ be a Young diagram. Let $g_\lambda, (e_{ij})_\lambda \in \operatorname{End} L(\lambda)$ denote the action of the twist matrix and the matrix units on the highest weight representation $L(\lambda)$. Define the *higher g-twisted transfer matrix of shape λ*

$$\tau^g_\lambda(z) := \operatorname{tr}_\lambda g_\lambda T_\lambda(z), \quad T_\lambda(z) = \sum_{ij} (e_{ij})_\lambda \otimes t_{ij}(z).$$

The functional relations between higher transfer matrices come from the so called *fusion relations*, which categorify to short exact sequences of representations of the Yangian. Of particular importance will be the following short exact sequence:

$$0 \to L([2, 1^{k-1}])^t_0 \to L([1^k])^t_0 \otimes L(\square)^t_\eta \to L([1^{k+1}])^t_\eta \to 0.$$

the rule for traces over short exact sequences will then give the functional relation

$$\tau^g_{[1^k]}(z)\tau^g_\square(z+\eta) = \tau^g_{[2,1^{k-1}]}(z) + \tau^g_{[1^{k+1}]}(z+\eta),$$

which will be our basis for deriving the spectral equation. The analogous short exact sequence for $\mathfrak{sl}_2$ was originally established in [CP90]. In terms of Young diagrams, the case $k = 3$ simply reads

$$\ydiagram{1,1} \otimes \ydiagram{1} = \ydiagram{2,1} \oplus \ydiagram{1,1,1},$$

which is the usual Littlewood-Richardson rule, except crucially we have an added dependence on the parameter z, which makes this short exact sequence non-split in general.

Let us now set out to show this, starting with a discussion of the *fusion procedure* [Mol08], originating in works such as [KRS81]. To this end, fix a standard Young tableau t_λ of shape λ with content vector $(c_1, ..., c_k)$ and let

$$R(z_1, ..., z_k) := \overrightarrow{\prod_{i<j}} R_{ij}(z_i - z_j),$$

where the arrow over the product indicates multiplication in lexicographical order. We note that successive application of the RTT relation (1.8) yields

$$R(z_1, ..., z_k)T_1(z_1)\cdots T_k(z_k) = T_k(z_k)\cdots T_1(z_1)R(z_1, ..., z_k). \tag{2.1}$$

Proposition 2.1.4. *Taking the following consecutive limits of $R(z_1, ..., z_k)$ yields a well-defined* fusion projector

$$\Pi_{t_\lambda} = \frac{d_\lambda}{k!} \lim_{z_k \to \eta c_k} \cdots \lim_{z_1 \to \eta c_1} R(z_1, ..., z_k),$$

with d_λ the dimension of the Specht module $S(\lambda)$ given by the hook formula. The fusion projectors form a complete set of primitive orthogonal idempotents decomposing the $\mathfrak{gl}_\ell$-module $(\mathbb{C}^\ell)^{\otimes k}$ into irreducible parts

$$L_{t_\lambda} := \Pi_{t_\lambda}(\mathbb{C}^\ell)^{\otimes k}$$

with $L_{t_\lambda} \cong L(\lambda)$ as $\mathfrak{gl}_\ell$-modules.

Proof. See [Mol08] or section 6.4 in [Mol07]. □

Proposition 2.1.5. *Consider $(\mathbb{C}^\ell)^{\otimes k}$ under the $Y(\mathfrak{gl}_\ell)$-action of the monodromy representation*

$$V^+_{-\eta c_1} \otimes \cdots \otimes V^+_{-\eta c_k}.$$

Then L_{t_λ} is a $Y(\mathfrak{gl}_\ell)$-submodule isomorphic to $L(\lambda)^t_0$.

Proof. This is proposition 6.5.1 in [Mol07]. □

This procedure makes it possible to reduce calculations in higher representations to tensor products of the covector representation. It is called the *fusion procedure*. In particular, we can express higher transfer matrices in a very concrete way using $(\mathbb{C}^\ell)^{\otimes k}$ as the auxiliary space. To see this, define

$$T_{t_\lambda}(z) := T_k(z+\eta c_k)\cdots T_1(z+\eta c_1), \quad g_{t_\lambda} := g_k \cdots g_1.$$

Then $L_{t_\lambda} \otimes Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!]$ is invariant under $g_{t_\lambda}T_{t_\lambda}(z)$ due to

$$\Pi_{t_\lambda}T_1(z+\eta c_1)\cdots T_k(z+\eta c_k) = T_k(z+\eta c_k)\cdots T_1(z+\eta c_1)\Pi_{t_\lambda},$$

which is derived from the higher RTT relation (2.1) by taking consecutive limits. It follows that

Proposition 2.1.6. *We have the identity*

$$\tau^g_\lambda(z) = \mathrm{tr}_{t_\lambda}\, g_{t_\lambda}T_{t_\lambda}(z)$$

where tr_{t_λ} denotes the partial trace over L_{t_λ}.

Example. Consider $\tau^g_{[1^2]}(z)$. With the previous proposition, we can write this as

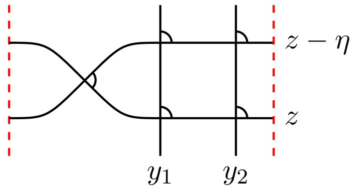
$$\tau^g_{[1^2]}(z) = \mathrm{tr}_{t_{[1^2]}}\, g_{t_{[1^2]}}T_{t_{[1^2]}}(z) = \mathrm{tr}_{12}\,\Pi_{t_{[1^2]}}g_2g_1T_2(z)T_1(z-\eta).$$

In particular, in the fundamental monodromy representation $L(\square)^t_{y_1} \otimes L(\square)^t_{y_2}$, this becomes

$$\tau^g_{[1^2]}(z) = \mathrm{tr}_{12}\,\Pi^-_{12}g_2g_1R_{23}(z-y_1)R_{24}(z-y_2)R_{13}(z-\eta-y_1)R_{14}(z-\eta-y_2).$$

In terms of diagrams, this reads

$$\tau^g_{[1^2]}(z) =$$



Note that the auxiliary space loops twice around the cylinder, exactly shifting by $\pm\eta$ when crossing the seam.

We can now also see that the highest transfer matrix $\tau^g_{[1^\ell]}(z)$ has a particularly nice description in terms of a Leibniz-type formula:

Corollary 2.1.7.

$$\tau^g_{[1^\ell]}(z) = \sum_\sigma \operatorname{sgn}\sigma \cdot \gamma_1\gamma_2 \cdots \gamma_\ell \cdot t_{\sigma(\ell)\ell}(z - \eta(\ell-1)) \cdots t_{\sigma(1)1}(z).$$

Proof. From the previous proposition, we have

$$\tau^g_{[1^\ell]}(z) = \operatorname{tr}_{t_{[1^\ell]}} g_{t_{[1^\ell]}} T_{t_{[1^\ell]}}(z) = \operatorname{tr}_{1,\dots,\ell} \Pi_{t_{[1^\ell]}} g_k \cdots g_1 T_\ell(z - \eta(\ell-1)) \cdots T_1(z).$$

Knowing that $\Pi_{t_{[1^\ell]}} = \sum_\sigma \operatorname{sgn}\sigma \cdot \sigma$, we apply the right hand side to $e_1 \otimes \cdots \otimes e_\ell$ and obtain

$$\sum_{i_1,\dots,i_\ell} \Pi_{t_{[1^\ell]}}(e_{i_1} \otimes \cdots \otimes e_{i_\ell}) \otimes t_{i_\ell,\ell}(z - \eta(\ell-1)) \cdots t_{i_1,1}(z).$$

But $\Pi_{t_{[1^\ell]}}(\gamma_{i_1} e_{i_1} \otimes \cdots \otimes \gamma_{i_\ell} e_{i_\ell})$ is only non-zero when the $i_1, \dots, i_\ell$ define a permutation σ, in which case it reduces to $\operatorname{sgn}\sigma \cdot \gamma_1 \cdots \gamma_\ell (e_1 \otimes \cdots \otimes e_\ell)$. ☐

Definition 2.1.8. Due to this fact, $\tau^g_{[1^\ell]}(z)$ deserves the name *g-twisted quantum determinant*. We hence also write

$$\operatorname{qdet}^g T(z) := \tau^g_{[1^\ell]}(z) \text{ and } \operatorname{qdet} T(z) := \tau^1_{[1^\ell]}(z).$$

Proposition 2.1.9. *The center of* $Y(\mathfrak{gl}_\ell)$ *is freely generated by the coefficients of* $\operatorname{qdet}^g T(z)$.

Proof. This is theorem 1.7.5 in [Mol07]. ☐

Corollary 2.1.10. *Let* V *be a highest weight representation of* $Y(\mathfrak{gl}_\ell)$ *with highest weight* $\lambda(z) = (\lambda_1(z), \dots, \lambda_\ell(z))$. *Then* $\operatorname{qdet}^g T(z)$ *acts as a scalar of the form*

$$\gamma_1\gamma_2 \cdots \gamma_\ell \lambda_1(z)\lambda_2(z-\eta) \cdots \lambda_\ell(z - \eta(\ell-1)).$$

Proof. This is proposition 3.2.5 of [Mol07]. Since $T(z)$ acts as a lower-triangular matrix on the highest weight vector, the only non-zero term in the Leibniz formula above is the term for $\sigma = \mathrm{id}$, which acts exactly as described. Since $\operatorname{qdet}^g T(z)$ lies in the center of $Y(\mathfrak{gl}_\ell)$, it will act on all vectors of V via this scalar. ☐

Definition 2.1.11. Let λ, μ be Young diagrams and t_λ, t_μ standard tableaux of shape λ, μ with content vectors $(c_1, ..., c_k), (d_1, ..., d_l)$. Define the *higher R-matrices*

$$R_{t_\lambda,t_\mu}(z) := \overleftarrow{\prod_i}\overrightarrow{\prod_j} R_{i,k+j}(z + \eta c_i - \eta d_j) \in (\operatorname{End}\mathbb{C}^\ell)^{\otimes k} \otimes (\operatorname{End}\mathbb{C}^\ell)^{\otimes l}[\![z^{-1}]\!].$$

Repeated application of the RTT relation (1.8) implies

$$R_{t_\lambda,t_\mu}(z-w)T_{t_\lambda}(z)T_{t_\mu}(w) = T_{t_\mu}(w)T_{t_\lambda}(z)R_{t_\lambda,t_\mu}(z-w).$$

Proposition 2.1.12. *Letting*

$$R^{(21)}_{t_\lambda,t_\mu}(z) := \overleftarrow{\prod_j}\overrightarrow{\prod_i} R_{k+j,i}(z + \eta d_j - \eta c_i)$$

we obtain

$$R_{t_\lambda,t_\mu}(z)R^{(21)}_{t_\mu,t_\lambda}(-z) = \prod_{ij}\left(1 - \frac{\eta^2}{(z+\eta c_i - \eta d_j)^2}\right)$$

In particular, higher R-matrices are invertible for all but a discrete set of values of z.

Proof. tba □

Corollary 2.1.13. *The transfer matrices $\tau^g_\lambda(z)$ commute among each other, making $B^g(\mathfrak{gl}_\ell)$ into a commutative algebra.*

Proof. Since $R_{t_\lambda,t_\mu}(z)$ is generically invertible by the previous propsition, we can give the following argument:

$$\begin{aligned}
\tau^g_\lambda(z)\tau^g_\mu(w) &= \operatorname{tr}_{t_\lambda,t_\mu} g_{t_\lambda}T_{t_\lambda}(z)g_{t_\mu}T_{t_\mu}(w)\\
&= \operatorname{tr}_{t_\lambda,t_\mu} g_{t_\lambda}g_{t_\mu}T_{t_\lambda}(z)T_{t_\mu}(w)\\
&= \operatorname{tr}_{t_\lambda,t_\mu} g_{t_\lambda}g_{t_\mu}R_{t_\lambda,t_\mu}(z-w)^{-1}T_{t_\mu}(w)T_{t_\lambda}(z)R_{t_\lambda,t_\mu}(z-w)\\
&= \operatorname{tr}_{t_\lambda,t_\mu} g_{t_\lambda}g_{t_\mu}R_{t_\lambda,t_\mu}(z-w)R_{t_\lambda,t_\mu}(z-w)^{-1}T_{t_\mu}(w)T_{t_\lambda}(z)\\
&= \operatorname{tr}_{t_\lambda,t_\mu} g_{t_\lambda}g_{t_\mu}T_{t_\mu}(w)T_{t_\lambda}(z)\\
&= \operatorname{tr}_{t_\lambda,t_\mu} g_{t_\mu}T_{t_\mu}(w)g_{t_\lambda}T_{t_\lambda}(z)\\
&= \tau^g_\mu(w)\tau^g_\lambda(z).
\end{aligned}$$

□

To finish off this section, let us finally prove the exactness of our sequence:

Proposition 2.1.14. *There is an exact sequence*

$$0 \to L([2,1^{k-1}])^t_\eta \to L([1^k])^t_\eta \otimes L(\square)^t_0 \to L([1^{k+1}])^t_0 \to 0.$$

Proof. Consider the fundamental monodromy representation

$$V_\eta^+ \otimes V_{2\eta}^+ \otimes \cdots \otimes V_{k\eta}^+ \otimes V_0^+.$$

By proposition 2.1.5, the fusion projectors $\Pi_{t_{[2,1^{k-1}]}}$ and $\Pi_{t_{[1^k]}} \otimes 1$ project onto the left and middle pieces $L([2,1^{k-1}])_\eta^t$ and $L([1^k])_\eta^t \otimes L(\Box)_0^t$. The Littlewood-Richardson rule gives

$$\Pi_{t_{[1^k]}} \otimes 1 = \Pi_{t_{[2,1^{k-1}]}} + \Pi_{t_{[1^{k+1}]}},$$

so that $\Pi_{t_{[2,1^{k-1}]}}$ actually factors through $\Pi_{t_{[1^k]}} \otimes 1$, which yields the inclusion from the left to the middle piece. Then $\Pi_{t_{[1^{k+1}]}}$ will be the projection onto the cokernel of this inclusion. The RTT relation supplemented by $\Pi_{t_{[1^{k+1}]}}\Pi_{t_{[1^k]}} = \Pi_{t_{[1^{k+1}]}}$ gives

$$\begin{aligned}
&\Pi_{t_{[1^{k+1}]}} R_{01}(z-\eta)R_{02}(z-2\eta)\cdots R_{0k}(z-k\eta)R_{0,k+1}(z)\\
&= \Pi_{t_{[1^{k+1}]}}\Pi_{t_{[1^k]}} R_{01}(z-\eta)R_{02}(z-2\eta)\cdots R_{0k}(z-k\eta)R_{0,k+1}(z)\\
&= \Pi_{t_{[1^{k+1}]}} R_{0k}(z-k\eta)\cdots R_{02}(z-2\eta)R_{01}(z-\eta)R_{0,k+1}(z)\Pi_{t_{[1^k]}}\\
&= R_{0,k+1}(z)R_{01}(z-\eta)R_{02}(z-2\eta)\cdots R_{0k}(z-k\eta)\Pi_{t_{[1^{k+1}]}}\Pi_{t_{[1^k]}}\\
&= R_{0,k+1}(z)R_{01}(z-\eta)R_{02}(z-2\eta)\cdots R_{0k}(z-k\eta)\Pi_{t_{[1^{k+1}]}},
\end{aligned}$$

so $\Pi_{t_{[1^{k+1}]}}$ is in fact $Y(\mathfrak{gl}_\ell)$-linear, as required. □

Corollary 2.1.15. *We obtain the functional relation*

$$\tau^g_{[1^k]}(z-\eta)\tau^g_{\Box}(z) = \tau^g_{[2,1^{k-1}]}(z-\eta) + \tau^g_{[1^{k+1}]}(z). \tag{2.2}$$

Proof. By the previous proposition, we have a linear map of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & L([2,1^{k-1}])_0 \otimes Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!] \arrow[r] \arrow[d, "g_{[2,1^{k-1}]}T_{[2,1^{k-1}]}(z-\eta)"] & L([1^k])_0 \otimes L(\Box)_\eta \otimes Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!] \arrow[r] \arrow[d, "g_{[1^k]}T_{[1^k]}(z-\eta)g_{\Box}T_{\Box}(z)"] & L([1^{k+1}])_\eta \otimes Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!] \arrow[r] \arrow[d, "g_{[1^{k+1}]}T_{[1^{k+1}]}(z)"] & 0\\
0 \arrow[r] & L([2,1^{k-1}])_0 \otimes Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!] \arrow[r] & L([1^k])_0 \otimes L(\Box)_\eta \otimes Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!] \arrow[r] & L([1^{k+1}])_\eta \otimes Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!] \arrow[r] & 0
\end{tikzcd}$$

Taking the trace over everything except $Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!]$ yields

$$0 = \tau^g_{[2,1^{k-1}]}(z-\eta) - \tau^g_{[1^k]}(z-\eta)\tau^g_{\Box}(z) + \tau^g_{[1^{k+1}]}(z)$$

as elements of $Y(\mathfrak{gl}_\ell)[\![z^{-1}]\!]$. □

2.1.2 Spectral equation

In this section, we will derive the *spectral equation* for the non-local Hamiltonians of the Heisenberg model from the functional relation (2.2). This is where we will see the Lax matrix of the rational Ruijsenaars-Schneider model appear seemingly out of nowhere. We will work along the lines of [Aru], but our approach is slightly different since we use Yang's convention for R-matrices instead of the polynomial convention and work in the g-twisted setting.

Lemma 2.1.16. *Let χ_λ be the character of the highest weight module $L(\lambda)$. Then*

$$\tau_\lambda^g(z) = \chi_\lambda(g) + \sum_j \frac{\operatorname{Res}_{z=y_j} \tau_\lambda^g(z)}{z - y_j}.$$

Proof. This is just the canonical partial fraction decomposition of $\tau_\lambda^g(z)$. It still remains to show that $\tau_\lambda^g(\infty) = \chi_\lambda(g)$. Let us take the column tableau t_λ, giving

$$\tau_\lambda^g(\infty) = \operatorname{tr}_{t_\lambda} g_{t_\lambda} T_{t_\lambda}(z).$$

But $T_{t_\lambda}(\infty)$ is just the identity matrix, yielding

$$\tau_{[1^k]}^g(\infty) = \operatorname{tr}_{t_\lambda} g_{t_\lambda},$$

which is exactly the character of the highest weight module $L(\lambda)$. □

Lemma 2.1.17. *We have*

$$\operatorname{Res}_{z=y_i} \tau_{[1^{k+1}]}^g(z) = \chi_{[1^k]}(g) H_i + \sum_j \frac{\eta H_i}{y_i - y_j - \eta} \operatorname{Res}_{z=y_j} \tau_{[1^k]}^g(z).$$

Proof. The functional relation (2.2) can be rewritten as

$$\tau_{[1^{k+1}]}^g(z) = \tau_{[1^k]}^g(z-\eta)\tau_{\square}^g(z) - \tau_{[2,1^{k-1}]}^g(z-\eta).$$

Let us take the residue at $z = y_i$. We remark that $\tau_{[1^k]}^g(z-\eta)$ does not have a pole at y_i, which can immediately be seen by expanding it in terms of R-matrices using the fusion procedure. On the other hand, taking the row tableau $t_{[2,1^{k-1}]}$, we find

$$\tau_{[2,1^{k-1}]}^g(z-\eta) = \operatorname{tr}_{1\ldots k} \Pi_{t_{[2,1^{k-1}]}} g_k \cdots g_1 M_k(z-\eta(k-1)) \cdots M_3(z-2\eta) M_2(z) M_1(z-\eta).$$

Introduce the partial monodromy matrix

$$M_a^{[i,j]}(z) := R_{ai}(z-y_i) \cdots R_{aj}(z-y_j)$$

and compute

$$\begin{aligned}
\operatorname{Res}_{z=y_i} M_2(z) M_1(y_i-\eta) &= M_2^{[N,i+1]}(y_i)\eta P_{1i} M_2^{[i-1,1]}(y_i) M_1^{[N,i+1]}(y_i-\eta) 2\Pi_{2i}^+ M_1^{[i-1,1]}(y_i-\eta) \\
&= M_2^{[N,i+1]}(y_i) M_1^{[N,i+1]}(y_i-\eta)\eta P_{1i} 2\Pi_{2i}^+ M_2^{[i-1,1]}(y_i) M_1^{[i-1,1]}(y_i-\eta) \\
&= M_2^{[N,i+1]}(y_i) M_1^{[N,i+1]}(y_i-\eta) 2\Pi_{21}^+ \eta P_{1i} M_2^{[i-1,1]}(y_i) M_1^{[i-1,1]}(y_i-\eta) \\
&= \Pi_{21}^+ M_1^{[N,i+1]}(y_i-\eta) M_2^{[N,i+1]}(y_i) 2\Pi_{21}^+ \eta P_{1i} M_2^{[i-1,1]}(y_i) M_1^{[i-1,1]}(y_i-\eta) \\
&= \Pi_{21}^+ M_2^{[N,i+1]}(y_i) M_1^{[N,i+1]}(y_i-\eta) 2\Pi_{21}^+ \eta P_{1i} M_2^{[i-1,1]}(y_i) M_1^{[i-1,1]}(y_i-\eta) \\
&= \Pi_{12}^+ \operatorname{Res}_{z=y_i} M_2(z) M_1(y_i-\eta).
\end{aligned}$$

But $\Pi_{t_{[2,1^{k-1}]}}$ antisymmetrizes over the indices 1 and 2, implying $\Pi_{t_{[2,1^{k-1}]}} \Pi_{12}^+ = 0$ and thus $\operatorname{Res}_{z=y_i} \tau_{[2,1^{k-1}]}^g(z-\eta) = 0$. With this, we finally arrive at

$$\operatorname{Res}_{z=y_i} \tau_{[1^{k+1}]}^g(z) = \tau_{[1^k]}^g(y_i-\eta) \operatorname{Res}_{z=y_i} \tau_{\square}^g(z),$$

such that lemma 2.1.16 yields the claim. □

In matrix notation, the previous lemma reads

$$\begin{pmatrix} \mathrm{Res}_{z=y_1} \tau^g_{[1^{k+1}]}(z) \\ \vdots \\ \mathrm{Res}_{z=y_N} \tau^g_{[1^{k+1}]}(z) \end{pmatrix} = \chi_{[1^k]}(g) \begin{pmatrix} H_1 \\ \vdots \\ H_N \end{pmatrix} - L^t \begin{pmatrix} \mathrm{Res}_{z=y_1} \tau^g_{[1^k]}(z) \\ \vdots \\ \mathrm{Res}_{z=y_N} \tau^g_{[1^k]}(z) \end{pmatrix}$$

with

$$L_{ij} := \frac{\eta H_j}{y_i - y_j + \eta} = \frac{\eta}{y_i - y_j + \eta} \left(\prod_{j \neq k} \frac{y_j - y_k - \eta}{y_j - y_k} \right) \check{H}_j.$$

This looks awfully close to the Lax matrix (1.3) of the rational Ruijsenaars-Schneider model when we substitute e^{-p_j} for $\check{H}_j$! Remember that X_j is the operator that under quantization corresponds to e^{-p_j}, which leads us to hypothesize that X_j and $\check{H}_j$ will in some sense turn out to be one and the same operator. This the main result of the next section.

Iterating the matrix equation and combining with the formula for the quantum determinant from corollary 2.1.10 finally gives the spectral equation, resembling a Cayley-Hamilton-type identity:

Theorem 2.1.18 (Spectral equation)**.** *The non-local Hamiltonians $H_1, ..., H_N$ of the spin chain fulfill the* spectral equation

$$\chi_{[1^\ell]}(g) \begin{pmatrix} \mathrm{Res}_{z=y_1} \operatorname{qdet}^1 T(z) \\ \vdots \\ \mathrm{Res}_{z=y_N} \operatorname{qdet}^1 T(z) \end{pmatrix} = \sum_{k=1}^{\ell} \chi_{[1^{\ell-k}]}(g)(-L^t)^{k-1} \begin{pmatrix} H_1 \\ \vdots \\ H_N \end{pmatrix}$$

Proof. tba □

Corollary 2.1.19. *The eigenvalues of $H_1, ..., H_N$ are exactly the complex roots of the spectral equation.*

Proof. ?? tba □

2.2 Generalized Schur-Weyl duality

Why should the Lax matrix of the rational Ruijsenaars-Schneider model appear in the spectral equation? A first clue is given by the fact that the Yangian allows for residual symmetries through its various automorphisms, the most peculiar of which is the shift automorphism. It has no analogue for $\mathfrak{gl}_\ell$ and is thus special to the Yangian. This non-rigidity gives additional degrees of freedom for monodromy representations: the inhomogeneities. They seem to play the role of the position variables of the rational Ruijsenaars-Schneider model. What is the mathematical reason for their appearance? Most simply, it is because the Schur-Weyl dual of the Yangian is the degenerate affine Hecke algebra, whose generators include polynomial generators that act as inhomogeneities. To introduce this, we will start with a discussion of classical Schur-Weyl duality between the symmetric groups S_N and the Lie algebras $\mathfrak{gl}_\ell$. A detailed account of various generalized Schur-Weyl dualities can be found in [Ant20].

2.2.1 Classical Schur-Weyl duality

Classical Schur-Weyl duality establishes a link between the representation theory of the symmetric group S_N and the representation theory of the Lie algebra $\mathfrak{gl}_\ell$. Let $\mathbb{C}^\ell$ be the vector representation of $\mathfrak{gl}_\ell$ and consider the N-fold tensor product representation $(\mathbb{C}^\ell)^{\otimes N}$. Clearly, $\mathfrak{gl}_\ell$ acts from the left via the coproduct of $U(\mathfrak{gl}_\ell)$. However, there is also a right action of S_N by permuting tensorands. Schur-Weyl duality now states the following:

Theorem 2.2.1. *(i) The actions of $U(\mathfrak{gl}_\ell)$ and $\mathbb{C}[S_N]$ on $(\mathbb{C}^\ell)^{\otimes N}$ are each others centralizer.*

(ii) We have the decomposition

$$(\mathbb{C}^\ell)^{\otimes N} = \bigoplus_\lambda L(\lambda) \otimes S(\lambda),$$

where λ ranges over all Young diagrams with N boxes and at most ℓ rows, $L(\lambda)$ is the corresponding irreducible highest weight representation of $\mathfrak{gl}_\ell$ and $S(\lambda)$ is the corresponding Specht module of S_N, compare proposition 2.1.4.

Corollary 2.2.2. *We have $L(\lambda) \cong (\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} S(\lambda)$.*

The last part can be nicely organized in categorical language, following [DM10]. Define a monoidal category S_* with objects $[N]$ for $N \in \mathbb{N}$ and monoidal product $[N] \otimes [M] := [N+M]$ as well as morphisms

$$\mathrm{Hom}_{S_*}([N],[M]) := \begin{cases} \mathbb{C}[S_N], & N = M \\ \varnothing, & N \neq M \end{cases}$$

with the monoidal product of morphisms given by the natural map $\mathbb{C}[S_N] \otimes \mathbb{C}[S_M] \to \mathbb{C}[S_{N+M}]$. We now take the following $\mathbb{C}$-linear closure

$$\mathsf{C}(S_*) := [S_*, \mathsf{Vect}] \simeq \bigoplus_N \mathbb{C}[S_N]\mathsf{Mod},$$

where we have a tensor product given by Day convolution:

$$U \otimes_{S_*} W := \mathbb{C}[S_{N+M}] \otimes_{\mathbb{C}[S_N] \otimes \mathbb{C}[S_M]} (U \otimes W).$$

which gives a monoidal fiber functor

$$F_\ell : \mathsf{C}(S_*) \to \mathsf{Vect}, \quad U \in \mathbb{C}[S_N]\mathsf{Mod} \mapsto (\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} U.$$

The fact that $U(\mathfrak{gl}_\ell)$ centralizes the action of the symmetric groups gives a homomorphism $U(\mathfrak{gl}_\ell) \to \mathrm{End}(F_\ell)$ and we obtain a commutative diagram

$$\begin{tikzcd} \mathsf{C}(S_*) \arrow[rr, "F_\ell"] \arrow[dr, "SW_\ell"'] & & \mathsf{Vect} \\ & U(\mathfrak{gl}_\ell)\mathsf{Mod} \arrow[ur] & \end{tikzcd}$$

Hence we may say that the algebras $U(\mathfrak{gl}_\ell)$ are Tannaka dual to the algebras $\mathbb{C}[S_N]$.

Theorem 2.2.3 (Classical Schur-Weyl duality)**.** *The functor*

$$SW_{\ell,N} : \mathbb{C}[S_N]\mathsf{Mod} \to U(\mathfrak{gl}_\ell)\mathsf{Mod}, \quad U \mapsto (\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} U$$

is full and also faithful when $\ell \geq N$. Its essential image are $U(\mathfrak{gl}_\ell)$-modules of weight N.

2.2.2 Schur-Weyl duality for the Yangian

The important observation now is that classical Schur-Weyl duality may be generalized from the Lie algebra $\mathfrak{gl}_\ell$ to the Yangian $Y(\mathfrak{gl}_\ell)$ by replacing the symmetric group with the degenerate affine Hecke algebra. The additional action by shifts of the spectral parameter z of the Yangian is defined using the action of the polynomial generators y_i of the degenerate affine Hecke algebra.

Proceeding as above, following the language of [DM10], we define a monoidal category $\dot{H}_*$ with objects $[N]$ for $N \in \mathbb{N}$, monoidal product $[N] \otimes [M] := [N+M]$ as well as morphisms

$$\mathrm{Hom}_{\dot{H}_*}([N],[M]) := \begin{cases} \dot{H}_N, & N = M \\ \varnothing, & N \neq M \end{cases}$$

with the monoidal product of morphisms given by the natural map $\dot{H}_N \otimes \dot{H}_M \to \dot{H}_{N+M}$. We again take the $\mathbb{C}$-linear closure

$$\mathsf{C}(\dot{H}_*) := [\dot{H}_*, \mathsf{Vect}] \simeq \bigoplus_N \dot{H}_N\mathsf{Mod},$$

equipped with the Day convolution tensor product

$$U \otimes_{\dot{H}_*} W := \dot{H}_{N+M} \otimes_{\dot{H}_N \otimes \dot{H}_M} (U \otimes W).$$

We may define a fiber functor $\mathsf{C}(\dot{H}_*) \to \mathsf{Vect}$ that factors through a functor $D_\ell : \mathsf{C}(\dot{H}_*) \to Y(\mathfrak{gl}_\ell)\mathsf{Mod}$. Its components are called *Drinfeld functors*.

Definition 2.2.4. We define the *Drinfeld functor*

$$D_{\ell,N} : \dot{H}_N\mathsf{Mod} \to Y(\mathfrak{gl}_\ell)\mathsf{Mod}, \quad U \mapsto (\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} U$$

by first considering the case $N = 1$ and introducing a $Y(\mathfrak{gl}_\ell)$-module structure on the tensor product $\mathbb{C}^\ell \otimes U$ via

$$t_{ij}^{(r)}(v \otimes u) := -\eta e_{ji} v \otimes y_1^{r-1} u,$$

which becomes

$$t_{ij}(z) \mapsto \delta_{ij} - \frac{\eta e_{ji}}{z - y_1} \quad \text{or} \quad T(z) \mapsto R(z - y_1)$$

in power series and matrix notation, respectively. The coproduct of the Yangian extends the definition to the remaining cases $N > 1$:

$$T(z) \mapsto R_{0N}(z - y_N) \cdots R_{01}(z - y_1).$$

Theorem 2.2.5 (Schur-Weyl duality for the Yangian)**.** *The functor D_ℓ is full and also faithful when $\ell > N$. Its essential image are $Y(\mathfrak{gl}_\ell)$-modules of weight N.*

Proof. This is the main theorem of [Dri86]. Drinfeld's original proof has never been published, but [CP95] contains a detailed and long-winded proof for the analogous case of affine quantum groups. $\square$

Proposition 2.2.6. *The functor D_ℓ is a monoidal functor, which implies that there exist natural isomorphisms*

$$D_{\ell,N+M}(U \otimes_{\dot{H}_*} W) \cong D_{\ell,N}(U) \otimes D_{\ell,M}(W).$$

Proof. This already appears in [Dri86]. $\square$

Proposition 2.2.7. *The following diagram commutes on the nose:*

$$\begin{array}{ccc} \mathbb{C}[S_N]\mathsf{Mod} & \xrightarrow{SW_{\ell,N}} & U(\mathfrak{gl}_\ell) \\ \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{ev}} \\ \dot{H}_N\mathsf{Mod} & \xrightarrow{D_{\ell,N}} & Y(\mathfrak{gl}_\ell) \end{array}$$

Proof. tba $\square$

We can already see how this is beginning to resemble the structure we are looking for: The Drinfeld functor takes representations of the degenerate affine Hecke algebra $\dot{H}_N$, such as the wave function representation of the quantum rational Ruijsenaars-Schneider model, and produces a representation of the Yangian clearly resembling fundamental monodromy representations with inhomogeneities given by the polynomial generators y_i of the degenerate affine Hecke algebra. More precisely, specializing the polynomial variables $y_1, ..., y_N$ to $\bar{y}_1, ..., \bar{y}_N \in \mathbb{C}[\eta]$, we have an isomorphism

$$D_{\ell,N}(\mathbb{C}[y_1, ..., y_N])|_{y_i=\bar{y}_i} \cong V^+_{\bar{y}_1} \otimes \cdots \otimes V^+_{\bar{y}_N}$$

of $Y(\mathfrak{gl}_\ell)$-modules.

However, we are still missing two important ingredients: Firstly, the twist matrix does not enter into the structure at any point, and secondly, we are disregarding the Laurent generators X_i of the degenerate double affine Hecke algebra $\ddot{H}_N$, which play an important role in defining the Hamiltonians of the quantum rational Ruijsenaars-Schneider model. These shortcomings will be remedied in the next section.

2.2.3 Twisted Schur-Weyl duality for the Yangian

It is clear that any $\ddot{H}_N$-module restricts to an $\dot{H}_N$-module to which we can apply the Drinfeld functor, giving a new functor

$$\ddot{H}_N\mathsf{Mod} \to Y(\mathfrak{gl}_\ell)\mathsf{Mod}, \quad U \mapsto (\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} U.$$

Crucially however, we still have the action of the Laurent generators X_i on $(\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} V$ left over. Since we tensor over the symmetric group, we are required to restrict to operators that are symmetric in the X_i. Such operators are provided by the spherical degenerate double affine Hecke algebra $S\ddot{H}_N$. Naively incorporating this action yields a functor

$$\ddot{H}_N\mathsf{Mod} \to S\ddot{H}_N \# Y(\mathfrak{gl}_\ell)\mathsf{Mod},$$

where "#" denotes the free product. Let us now twist this using the twist matrix g:

Definition 2.2.8. Define the *preaffine Drinfeld functor*

$$D^g_{\ell,N} : \ddot{H}_N\mathsf{Mod} \to S\ddot{H}_N \# Y(\mathfrak{gl}_\ell)\mathsf{Mod}, \quad U \mapsto (\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} U,$$

by letting X_i act on $(\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} U$ via

$$X_i(v \otimes u) := g_i v \otimes X_i u.$$

The y_i act untwisted.

With this definition, we are finally ready to show explicitly how the preaffine Drinfeld functor maps the quantum rational Ruijsenaars-Schneider model to the twisted inhomogeneous Heisenberg model in the limit $\hbar \to 0$. This is the aim of the next section.

2.3 Quantum-classical duality as Schur-Weyl duality

2.3.1 Fundamental spin chain

We are now ready to show how the Hamiltonian operators acting on the wave function representation $\mathbb{C}[y_1, ..., y_N]$ of the quantum rational Ruijsenaars-Schneider model produce the Hamiltonian operators on the fundamental spin chain, *i.e.*

$$V^+_{y_1} \otimes \cdots \otimes V^+_{y_N},$$

via the preaffine Drinfeld functor. This result rests on the following key observation:

Lemma 2.3.1. *On $(\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} \mathbb{C}[y_1, ..., y_N]$, we have*

$$g_i v \otimes X_i f = \check{R}_{i,i-1}(y_i - y_{i-1}) \cdots \check{R}_{i1}(y_i - y_1)(g_i \otimes e^{\mathrm{i}\hbar\partial_i})\check{R}_{iN}(y_i - y_N) \cdots \check{R}_{i,i+1}(y_i - y_{i+1})(v \otimes f).$$

where we have introduced

$$\check{R}_{ij}(y_i - y_j)(v \otimes f) := v \otimes \frac{y_i - y_j}{y_i - y_j - \eta} f - vs_i \otimes \frac{\eta}{y_i - y_j - \eta} f.$$

Proof. Since we are tensoring over S_N, we know that $vs_i \otimes f = v \otimes T_i f$, *i.e.*

$$\begin{aligned}
vs_i \otimes f &= v \otimes \left(\frac{y_i - y_{i+1} - \eta}{y_i - y_{i+1}} s_i + \frac{\eta}{y_i - y_{i+1}} \right) f \\
\Leftrightarrow vs_i \otimes f - v \otimes \frac{\eta}{y_i - y_{i+1}} f &= v \otimes \frac{y_i - y_{i+1} - \eta}{y_i - y_{i+1}} s_i f \\
\Leftrightarrow vs_i \otimes \frac{y_i - y_j}{y_i - y_j - \eta} f - v \otimes \frac{\eta}{y_i - y_j - \eta} f &= v \otimes s_i f,
\end{aligned}$$

or in short:

$$\check{R}_{i,i+1}(y_i - y_{i+1})(vs_i \otimes f) = v \otimes s_i f.$$

In combination, we obtain

$$v \otimes x_{i,i+1} f = v \otimes s_i T_i f = \check{R}_{i,i+1}(y_i - y_{i+1})(vs_i \otimes T_i f) = \check{R}_{i,i+1}(y_i - y_{i+1})(v \otimes f).$$

It follows that

$$\begin{aligned} g_i v \otimes X_i f &= g_i v \otimes x_{i,i-1} \cdots x_{i1} e^{\mathrm{i}\hbar\partial_i} x_{iN} \cdots x_{i,i+1} f \\ &= \check{R}_{i,i-1}(y_i - y_{i-1}) \cdots \check{R}_{i1}(y_i - y_1)(g_i v \otimes e^{\mathrm{i}\hbar\partial_i} x_{iN} \cdots x_{i,i+1} f) \\ &= \check{R}_{i,i-1}(y_i - y_{i-1}) \cdots \check{R}_{i1}(y_i - y_1)(g_i \otimes e^{\mathrm{i}\hbar\partial_i})(v \otimes x_{iN} \cdots x_{i,i+1} f) \\ &= \check{R}_{i,i-1}(y_i - y_{i-1}) \cdots \check{R}_{i1}(y_i - y_1)(g_i \otimes e^{\mathrm{i}\hbar\partial_i})\check{R}_{iN}(y_i - y_N) \cdots \check{R}_{i,i+1}(y_i - y_{i+1})(v \otimes f). \end{aligned}$$

□

We are now ready to state the main theorem of this chapter, showing that the Hamiltonians of the rational Ruijsenaars-Schneider model get mapped to the Hamiltonians of the twisted inhomogeneous Heisenberg model under the preaffine Drinfeld functor:

Theorem 2.3.2. *The operator*

$$\sum_i \frac{\eta}{z - y_i} \left(\prod_{k \neq i} \frac{y_i - y_k - \eta}{y_i - y_k} \right) X_i \in S\ddot{H}_N[\![z^{-1}]\!]$$

acts on $D^g_{\ell,N}(\mathbb{C}[y_1, ..., y_N])$ *as the transfer matrix* $\tau^g(z)$ *when* $\hbar = 0$.

Proof. We compare the residues. By lemma 2.3.1, we see that $\left(\prod_{k \neq i} \frac{y_i - y_k - \eta}{y_i - y_k} \right) X_i$ acts as

$$R_{i,i-1}(y_i - y_{i-1}) \cdots R_{i1}(y_i - y_1)(g_i \otimes e^{\mathrm{i}\hbar\partial_i}) R_{iN}(y_i - y_N) \cdots R_{i,i+1}(y_i - y_{i+1}),$$

which for $\hbar = 0$ coincides with $\mathrm{Res}_{z=y_i} \tau^g(z)$. □

Let us finish off this section by giving a Rosetta stone for quantum-classical duality:

twisted inhomogeneous Heisenberg model	rational Ruijsenaars-Schneider model
Yangian $Y(\mathfrak{gl}_\ell)$	degenerate affine Hecke algebra $\dot{H}_N$
Bethe subalgebra $B^g(\mathfrak{gl}_\ell)$	spherical subalgebra $S\ddot{H}_N$
fundamental monodromy representation	wave function representation
ith atom	ith particle
inhomogeneities y_i	positions y_i
non-local Hamiltonians $\check{H}_i$	Macdonald operators X_i at $\hbar = 0$
Planck constant η	coupling constant η

2.3.2 Higher spin chain

The way we have introduced the duality between the Heisenberg and Ruijsenaars-Schneider models easily lends itself to a generalization to spins in non-fundamental representations: Consider a Young diagram λ of N boxes with Specht module $S(\lambda)$ and pull back along the evaluation morphism. This yields the $\dot{H}_N$-module $S(\lambda)[y]$ with y_i acting as $y - \eta c_i$ on the basis vector v_{t_λ} labeled by a standard Young tableau t_λ of shape λ with content vector $(c_1, ..., c_N)$. We also know that $S(\lambda)[y]$ maps to $L(\lambda)[y]$ under the Drinfeld functor. On the other hand, we may equivalently try to apply the fusion procedure, taking the limit $y_i \to y - \eta c_i$ and projecting onto the subspace $L_{t_\lambda} \subset V^+_{y-\eta c_1} \otimes \cdots \otimes V^+_{y-\eta c_N}$ isomorphic to $L(\lambda)^t_y$.

We conclude that a single atom with spin representation labeled by λ is equivalently a *stack of N particles* that sit at relative positions determined by content vectors for standard Young tableaux of shape λ. A general quantum state is a superposition of such configurations and transforms under S_N in the representation given by the Specht module $S(\lambda)$.

twisted inhomogeneous Heisenberg model	rational Ruijsenaars-Schneider model
ith atom	ith stack of particles
spin in representation λ	stack satisfying λ-exchange statistics
inhomogeneities y_i of spins	positions y_i of stacks
Planck constant η	coupling constant η

What is the correspondence for Hamiltonians?

Example. Let us look at the two site spin chain

$$L(\square\square)^t_{y_1} \otimes L(\square\square)^t_{y_2} \subseteq V^+_{y_1} \otimes V^+_{y_1'} \otimes V^+_{y_2} \otimes V^+_{y_2'}.$$

for $y_i' = y_i - \eta$ with projector $\Pi_{\square\square} = (1 + (1\ 2))/2$ for $\ell = 2$. Solving this case explicitly using the spectral equation derived earlier proves difficult for the following reasons:

(i) We cannot take the limit $y_i' \to y_i - \eta$ in the spectral equation, since its coefficients have a pole at $y_i' = y_i - \eta$.

(ii) We might try to solve the spectral equation for generic inhomogeneities and *then* take the limit $y_i' \to y_i - \eta$ in the solution we obtained. However, the spectral equation for a spin chain with four sites and generic inhomogeneities contains polynomials in four variables and degree $\ell = 2$. Solving such an equation analytically is computationally intractable.

Instead, solving the Wronskian equation [Aru] for the $N = 4$ $\mathfrak{gl}_2$ spin chain with generic inhomogeneities and performing the fusion procedure as well as going to the homogeneous limit yields the solution

$$(z - \eta)^2(2z^2 + 4\eta z + 4\eta^2)$$

for the $\square\square\square\square$-multiplet, which is exactly correct up to the factor $(z - \eta)^2$, which results from using the polynomial convention.

2.4 S-duality

2.4.1 Schur-Weyl duality for the loop Yangian

Let us reexamine the preaffine Drinfeld functor

$$D^{g}_{\ell,N} : \ddot{H}_N\mathsf{Mod} \to S\ddot{H}_N\#Y(\mathfrak{gl}_\ell)\mathsf{Mod}.$$

Note that there is an asymmetry in its definition: It prioritizes the polynomial generators y_i by incorporating them in the action of the Yangian, while the S-dual Laurent generators X_i are artificially added on. It is known that there is a generalized Schur-Weyl duality between the affine symmetric group, which is the source of the Laurent generators, and the loop algebra $L(\mathfrak{gl}_\ell) := U(\mathfrak{gl}_\ell[t^{\pm 1}])$. Thus, we might hope to put both sides on a more equal footing by incorporating an action of the loop algebra on $(\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} U$ in addition to the action of the Yangian, making use of the Laurent generators X_i:

$$(xt^r)(v \otimes u) := \sum_i x_i v \otimes X_i^r u.$$

This yields an action of $L(\mathfrak{gl}_\ell)\#Y(\mathfrak{gl}_\ell)$ on $(\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} U$. This action in fact descends to the *loop Yangian* $LY(\mathfrak{gl}_\ell)$, which is a $\mathbb{C}[\eta, \hbar]$-algebra and a quotient of $L(\mathfrak{gl}_\ell)\#Y(\mathfrak{gl}_\ell)$:

Theorem 2.4.1. *The action of the Yangian $Y(\mathfrak{gl}_\ell)$ and the loop algebra $L(\mathfrak{gl}_\ell)$ glue together to an action of the loop Yangian $LY(\mathfrak{gl}_\ell)$.*

Proof. This is proved in [Gua05], also see [Kod16]. □

Definition 2.4.2. This defines the *affine Drinfeld functor*

$$\dot{D}_{\ell,N} : \ddot{H}_N\mathsf{Mod} \to LY(\mathfrak{gl}_\ell)\mathsf{Mod}.$$

We can now identify the Hamiltonians of the quantum rational Ruijsenaars-Schneider model and the quantum trigonometric Calogero-Moser model as elements of the loop Yangian, or more precisely of the centers of the loop algebra and the Yangian:

Proposition 2.4.3. *On $(\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} \mathbb{C}[y_1, ..., y_N]$, we have*

$$t^k(v \otimes f) = v \otimes p_k(X_1, ..., X_N)f,$$

In particular, t acts as $p_1(X_1, ..., X_N) = D_1$, which is the Hamiltonian of the quantum rational Ruijsenaars-Schneider model.

Proof. We immediately see this from the definition of the action of the loop algebra:

$$t^k(v \otimes f) = \sum_i v \otimes X_i^k f = v \otimes p_k(X_1, ..., X_N)f.$$

□

Proposition 2.4.4. *On $(\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} \mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}]$, the quantum determinant has the following expansion:*

$$\begin{aligned}\operatorname{qdet}^1 T(z) = 1 - \frac{\eta}{z} N - \frac{\eta}{z^2}\Big(- \mathrm{i}\hbar \sum_i X_i \partial_i + \eta \frac{N(N-1)}{2}\Big) \\ - \frac{\eta}{z^3}\Big(S_2 - \mathrm{i}\hbar\eta(N-1)\sum_i X_i\partial_i + \eta^2 \frac{N(N-1)(N-2)}{4}\Big)\end{aligned}$$

In particular, the second order coefficient is a conjugate of the total momentum and the third order coefficient is a conjugate of the Hamiltonian of the quantum trigonometric Calogero-Moser model.

Proof. tba, see [BGHP93].

$$R(z-y) = 1 - \frac{\eta P}{z-y} = 1 - \frac{\eta}{z} P - \frac{\eta}{z^2} y P - \frac{\eta}{z^3} y^2 P + \cdots$$

$$\begin{aligned}\operatorname{qdet}^1 T(z) &= \operatorname{tr}_{1\ldots\ell} \Pi^-_{1\ldots\ell} \prod_{ij} R_{ij}(z - y_j - i\eta) \\ &= \operatorname{tr}_{1\ldots\ell} \Pi^-_{1\ldots\ell}(1 - \frac{\eta}{z}\sum_{ij} P_{ij} - \frac{\eta}{z^2}\sum_{ij}\Big((y_j - i\eta)P_{ij} + \sum_{kl} P_{ij}P_{kl}\Big) \\ &+ \frac{\eta}{z^3}\sum_{ij}\Big((y_j - i\eta)^2 P_{ij} + \sum_{kl}(y_j - i\eta)P_{ij}P_{kl} + \sum_{klmn} P_{ij}P_{kl}P_{mn}\Big))\end{aligned}$$

□

2.4.2 The trigonometric Gaudin model

In order to define the trigonometric Gaudin model, let us introduce

$$G(t) = \sum_i \frac{\eta t P_{0i}}{t - X_i}.$$

We then have

$$\frac{1}{2}\operatorname{tr} G(t)^2 = \frac{1}{2}\sum_{ij}\frac{\eta^2 t^2 P_{ij}}{(t - X_i)(t - X_j)} = \frac{1}{2}\sum_i \frac{\eta^2 t^2}{(t - X_i)^2} + \sum_i \frac{\eta X_i G_i}{t - X_i},$$

where we have introduced the *Gaudin Hamiltonian*

$$G_i := \sum_{i \neq j} \frac{\eta X_i P_{ij}}{X_i - X_j}.$$

Theorem 2.4.5. *The elements*

$$\frac{1}{2}\sum_i \frac{\eta^2 t^2}{(t - X_i)^2} + \sum_i \frac{\delta^{-1} y_i \delta + \eta + \frac{\eta}{2}\sum_{i\neq j}(i\ j)}{t - X_i} \in \delta^{-1} S\ddot{H}_N \delta[\![t^{\pm 1}]\!]$$

act on $D_{\ell,N}(\mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}])$ in the same way as $\frac{1}{2}\operatorname{tr} G(t)^2$.

Proof. We check that the residues coincide.

$$\begin{aligned}\delta^{-1}y_i\delta+\eta+\frac{\eta}{2}\sum_{i\neq j}(i\ j)&=y_i+\eta+\eta\sum_{i\neq j}\theta_{ij}+\eta\sum_{j<i}P_{ij}\\&=-\hbar X_i\partial_i+\eta\sum_{i\neq j}\frac{X_iP_{ij}}{X_i-X_j}\end{aligned}$$

When $\hbar=0$, this becomes the Gaudin Hamiltonian G_i. □

Chapter 3

Geometry

3.1 The geometry behind quantum-classical duality

3.1.1 In terms of functorial quantum field theory

The aim of this section is to reformulate Schur-Weyl duality for the Yangian in terms of a higher functorial quantum field theory that will later turn out to represent four-dimensional Chern-Simons theory. The mathematical structures used so far have been heavily representation theoretical. Nonetheless, we have seen some hints that there is an underlying geometry at play: We have made use of braids on cylinders as well as complex coordinates that may be reinterpreted as $(1+N)$-pointed Riemann spheres.

These data are combined in the complex Teichmüller tower, whose genus zero part describes $(1+N)$-pointed Riemann spheres $(\mathbb{P}^1; \infty, y_1, ..., y_N; -\partial_{z^{-1}}, v_1, ..., v_N)$, where $y_1, ..., y_N$ are distinct complex numbers and $v_1, ..., v_N$ are non-zero tangent vectors at $y_1, ..., y_N$, respectively. The morphisms in genus zero are generated by braidings and twists, visualized in figure 2. In [BK01], it is proved that the complex Teichmüller groupoid is equivalent to the topological Teichmüller groupoid, coming from the following construction:

Definition 3.1.1. (i) An *extended surface* is a smooth oriented 2-manifold Σ with parametrized boundary circles.

(ii) The *topological Teichmüller groupoid* Teich is the groupoid of extended surfaces and orientation-preserving homeomorphisms up to isotopy.

Note that the parametrizations of the boundary components of an extended surface Σ may equivalently be thought of as an embedding of a disk D^2 into the surface that results from gluing the boundary component of Σ closed. It will thus be enough to look at the following category of embeddings:

Definition 3.1.2. Let $\mathsf{Mfld}^2_{\mathrm{fr}}$ be the $(\infty, 1)$-category whose objects are framed 2-manifolds and whose 1-morphisms are framed smooth embeddings, equipped with the compact-open topology.

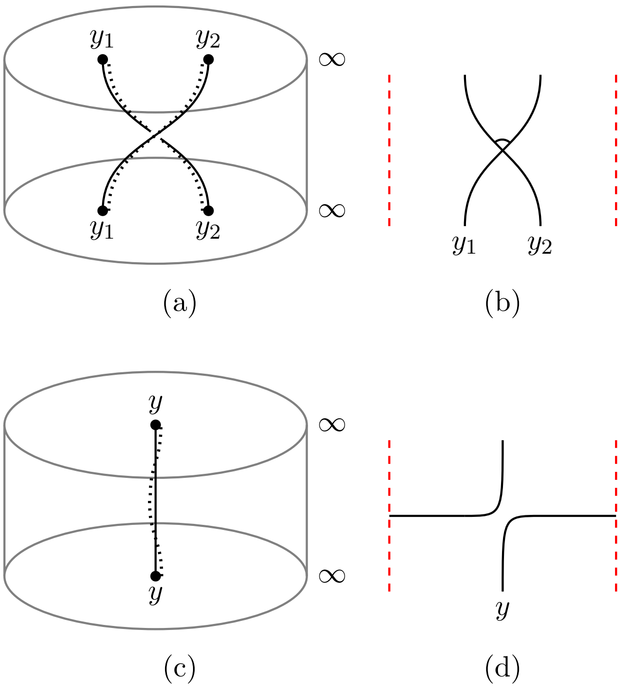


Figure 2: (a) A framed braiding in the $(1+2)$-pointed Riemann sphere. (b) The corresponding diagram on a cylinder as we would have drawn it until now. (c) A counter-clockwise 2π-twist of the framing around a single strand in the $(1+1)$-pointed Riemann sphere. (d) The corresponding diagram on a cylinder.

This means that 2-morphisms are isotopies between embeddings and higher morphisms are higher homotopies between these.

In [BBJ18a], a 2-functorial field theory on $\mathsf{Mfld}^2_{\mathrm{fr}}$ is constructed via factorization homology, taking an abelian braided rigid tensor category as the (framed) E_2-algebra of coefficients. In our case, such coefficients are readily provided by a certain category of $Y(\mathfrak{gl}_\ell)$-modules:

Definition 3.1.3. Consider η as a non-zero purely imaginary number. Let $Y(\mathfrak{gl}_\ell)\mathsf{Mod}^{\mathbb{R}}$ be the (ind-completion of the) subcategory of $Y(\mathfrak{gl}_\ell)\mathsf{Mod}$ generated by modules of the form

$$V^{\pm}_{y_1} \otimes \cdots \otimes V^{\pm}_{y_N},$$

under finite (co)limits. Here, the superscript $(-)^{\pm}$ indicates either $V^+_{y_i}$, in which case we take $y_i \in \mathbb{R} + 2\ell\eta\mathbb{Z}$, or $V^-_{y_i}$, in which case we take $y_i \in \mathbb{R} + \ell\eta + 2\ell\eta\mathbb{Z}$.

Proposition 3.1.4. *$Y(\mathfrak{gl}_\ell)\mathsf{Mod}^{\mathbb{R}}$ is an abelian braided rigid tensor category.*

Proof.
1. Abelian: The category is preabelian by closedness under finite (co)limits and preadditivity. Normality of mono- and epimorphisms is inherited from $Y(\mathfrak{gl}_\ell)\mathsf{Mod}$.
2. Braidings for generators: Let $\check{R}_+(z) := P\check{R}(z) = \frac{z}{z-\eta}P - \frac{\eta}{z-\eta}$ and $\check{R}_-(z) := \frac{z}{z-\eta}P - \frac{\eta}{z-\eta}P^t$. Define the braidings

$$\begin{aligned} c_{\pm\pm} &:= \check{R}_+(y_1 - y_2) : V^{\pm}_{y_1} \otimes V^{\pm}_{y_2} \to V^{\pm}_{y_2} \otimes V^{\pm}_{y_1} \\ c_{\pm\mp} &:= \check{R}_-(y_2 - y_1 - \ell\eta) : V^{\pm}_{y_1} \otimes V^{\mp}_{y_2} \to V^{\mp}_{y_2} \otimes V^{\pm}_{y_1}. \end{aligned}$$

   One may check that $c_{\pm\mp}$ and c_{--} come about by dualizing the legs of c_{++}. We have a problem however when $y_2 - y_1 = \ell\eta$ for $c_{\pm\mp}$, since $\check{R}_-(0) = P^t$ is singular.
3. Duals for generators: Note that the right dual of any generator module is again a generator module. It is obtained by reversing the order of the tensor product, flipping the signs, and shifting by $+\ell\eta$. The same holds for left duals, except we shift by $-\ell\eta$. Double right dualization induces a shift by $2\ell\eta$, which is an equivalence. In particular, right dualization is exact on generator modules, so we can extend right dualization to all of $Y(\mathfrak{gl}_\ell)\mathsf{Mod}^{\mathbb{R}}$.
4. Everything in sight commutes with finite (co)limits, so we may extend the braided rigid structure to all of $Y(\mathfrak{gl}_\ell)\mathsf{Mod}^{\mathbb{R}}$. ☐

Lemma 3.1.5. *The map*

$$\delta : V^+_{y_1} \otimes \cdots \otimes V^+_{y_N} \to Y^\vee(\mathfrak{gl}_\ell) \otimes V^+_{y_1} \otimes \cdots \otimes V^+_{y_N}, \quad v \mapsto T^\vee_N(y_N) \cdots T^\vee_1(y_1)(1 \otimes v)$$

equips $V^+_{y_1} \otimes \cdots \otimes V^+_{y_N}$ with the structure of a $Y^\vee(\mathfrak{gl}_\ell)$-comodule internal to $Y(\mathfrak{gl}_\ell)\mathsf{Mod}^{\mathbb{R}}$.

Proof. tba ☐

Lemma 3.1.6. *There exist finitely many tuples $(y_1, ..., y_N)$ such that $x \in F_N Y(\mathfrak{gl}_\ell)$ is zero if and only if x acts as zero on all $V^+_{y_1} \otimes \cdots \otimes V^+_{y_N}$.*

Proof. Let $\rho_{y_1,...,y_N} : Y(\mathfrak{gl}_\ell) \to \operatorname{End} V^+_{y_1} \otimes \cdots \otimes V^+_{y_N}$ denote the action. Given any $x \in F_N Y(\mathfrak{gl}_\ell)$, it is a result of [Naz19] that $\rho_{y_1,...,y_N}(x) = 0$ for all values of $y_1, ..., y_N$ implies $x = 0$. But $\rho_{y_1,...,y_N}(x)$ is polynomial in $y_1, ..., y_N$, so it is enough to know that it vanishes at finitely many points. $\square$

Proposition 3.1.7. *The dual Yangian $Y^\vee(\mathfrak{gl}_\ell)$ is the coend $\int^{M \in Y(\mathfrak{gl}_\ell)\mathsf{Mod}^\mathbb{R}} M \otimes M^\vee$.*

Proof. Let M_N be the full subcategory spanned by $V^+_{y_1} \otimes \cdots \otimes V^+_{y_N}$ for the finitely many tuples $(y_1, ..., y_N)$ of the previous lemma. Let

$$E_N := \bigoplus_{M \in \mathsf{M}_N} M \otimes M^\vee \in Y(\mathfrak{gl}_\ell)\mathsf{Mod}^\mathbb{R}$$

with the special element $1_N : \mathbb{C}[\eta] \to E_N$ representing the sum of identities. By the previous lemma, we can identify $F_N Y(\mathfrak{gl}_\ell)$ as a subspace of E_N via the injective map $Y(\mathfrak{gl}_\ell) \to E_N$ induced by the jointly faithful actions. Further let $L := \int^{M \in Y(\mathfrak{gl}_\ell)\mathsf{Mod}^\mathbb{R}} M \otimes M^\vee$. The coactions of the dual Yangian give a map $\psi : L \to Y^\vee(\mathfrak{gl}_\ell)$ that commutes with the coactions $E_N \to L \otimes E_N$ and $E_N \to Y^\vee(\mathfrak{gl}_\ell) \otimes E_N$. We may look at the image of 1_N

$$\widetilde{\mathcal{R}}_N : \mathbb{C}[\eta] \to E_N \to L \otimes F_N Y(\mathfrak{gl}_\ell) \subseteq L \otimes E_N,$$

which under $\psi \otimes \mathrm{id}$ gets mapped to $\mathcal{R}(1 \otimes 1_N)$, which is nothing but $\mathcal{R}_N \in Y^\vee(\mathfrak{gl}_\ell) \otimes F_N Y(\mathfrak{gl}_\ell)$, confer (1.10). In the limit, we obtain an element

$$\widetilde{\mathcal{R}} \in \varprojlim_N L \otimes F_N Y(\mathfrak{gl}_\ell)$$

mapping to the universal R-matrix $\mathcal{R}$. The element $\widetilde{\mathcal{R}}$ then witnesses that the composition $Y(\mathfrak{gl}_\ell) \otimes L \to Y(\mathfrak{gl}_\ell) \otimes Y^\vee(\mathfrak{gl}_\ell) \to \mathbb{C}[\eta]$ is a non-degenerate pairing, implying that ψ is an isomorphism (tba). $\square$

Theorem 3.1.8. *Factorization homology gives an $(\infty, 1)$-functor*

$$\int_{(-)} Y(\mathfrak{gl}_\ell)\mathsf{Mod}^\mathbb{R} : \mathsf{Mfld}^2_{\mathrm{fr}} \to \mathsf{Pr}_c$$

such that

(i) The disk D^2 (with blackboard framing) gets mapped to $Y(\mathfrak{gl}_\ell)\mathsf{Mod}^\mathbb{R}$.

(ii) The torus with one boundary component $T^2 \setminus D^2$ gets mapped to the category $Y^\vee(\mathfrak{gl}_\ell)^{\otimes 2}\text{-}\mathsf{mod}_{Y(\mathfrak{gl}_\ell)\mathsf{Mod}^\mathbb{R}}$ of $Y^\vee(\mathfrak{gl}_\ell)^{\otimes 2}$-modules internal to $Y(\mathfrak{gl}_\ell)\mathsf{Mod}^\mathbb{R}$.

(iii) The N-fold disk embeddings $(D^2)^{\sqcup N} \to D^2$ get mapped to the N-fold tensor product functor $(Y(\mathfrak{gl}_\ell)\mathsf{Mod}^\mathbb{R})^{\boxtimes N} \to Y(\mathfrak{gl}_\ell)\mathsf{Mod}^\mathbb{R}$. The action of the braid group on such embeddings gets mapped to an action on the N-fold tensor product functor via the braiding.

Proof. These are results of [BBJ18a]. $\square$

Corollary 3.1.9. *The induced action of the braid group on fundamental monodromy representations is given by the action of $S_N \subset \dot{H}_N$.*

This may be seen as a geometric reason for generalized Schur-Weyl duality between the Yangian $Y(\mathfrak{gl}_\ell)$ and the degenerate affine Hecke algebra $\dot{H}_N$: The Yangian provides the local observables of the functorial field theory while the degenerate affine Hecke algebra controls the global, topological braid group action commuting with the Yangian action via S_N, as well as the parametrization of the complex structure via $y_1, ..., y_N$ in the setting of genus zero.

However, we are again missing an action of the Laurent generators $X_1, ..., X_N$. Pictorially, we know that these should represent Dehn twists, which are only allowed in the case of *oriented* surfaces since they distort the blackboard framing. If we were working over oriented manifolds, we would be required to equip our category of coefficients with a balancing, or equivalently a pivotal structure. It is at this point that we face an obstruction: The Yangian is a well-known example of a Hopf algebra whose module category is *not* pivotal. We have already seen that this is the case, since the double (left) dual of V_y^+ is $V_{y+2\ell\eta}^+$, which is non-isomorphic to V_y^+. In short, we may say that our functorial field theory has a *framing anomaly*, which forces us to work in the category of framed surfaces.

One would expect as before that the action of the Laurent generators $X_1, ..., X_N$ finds its geometric interpretation once we incorporate the full loop Yangian $LY(\mathfrak{gl}_\ell)$. However, due its complex nature, more work needs to be done in order to establish some necessary properties of $LY(\mathfrak{gl}_\ell)$, such as the construction of a dual loop Yangian, so we leave this for future work.

We might also consider our functorial quantum field theory on the annulus A^2, which gives us the category of $Y^\vee(\mathfrak{gl}_\ell)$-modules internal to $Y(\mathfrak{gl}_\ell)\mathsf{Mod}^{\mathbb{R}}$. Again, we may look at embeddings $(D^2)^{\sqcup N} \to A^2$, which by [BBJ18a] all correspond to the functor

$$Y(\mathfrak{gl}_\ell)\mathsf{Mod}^{\boxtimes N} \to Y^\vee(\mathfrak{gl}_\ell)\text{-}\mathsf{mod}_{Y(\mathfrak{gl}_\ell)\mathsf{Mod}^{\mathbb{R}}}, \quad V_1 \boxtimes \cdots \boxtimes V_N \mapsto Y^\vee(\mathfrak{gl}_\ell) \otimes V_1 \otimes \cdots \otimes V_N.$$

This gives an action of the braid group of the annulus.

Definition 3.1.10. The *braid group $B_N(A^2)$ on N strands in A^2* is generated by elements $b_1, ..., b_{N-1}$ subject to the braid relations and Z_1. We may recursively define $Z_i := b_{i-1}Z_{i-1}b_{i-1}$.

The action action of $B_N(A^2)$ on $Y^\vee(\mathfrak{gl}_\ell) \otimes V_{y_1}^+ \otimes \cdots \otimes V_{y_N}^+$ is such that the simple braids act on the last tensorands via the braiding and the element Z_i should act as

$$R_{i,i-1}(y_i - y_{i-1}) \cdots R_{i,1}(y_i - y_1) T_i^\vee(y_i) R_{i,N}(y_i - y_N) R_{i,N-1}(y_i - y_{N-1}) \cdots R_{i,i+1}(y_i - y_{i+1}).$$

If we do a quantum Hamiltonian reduction $Y^\vee(\mathfrak{gl}_\ell) \to \underline{\mathrm{End}}(U)$ such that $T^\vee(y_i) = g$, we might recover the action we had before.

Remark. There is an unrelated appearance of the *trigonometric* Ruijsenaars-Schneider model of ℓ particles in [BBJ18b] when the category of coefficients is the module category of a quantum group $U_q(\mathfrak{gl}_\ell)$.

3.1.2 Meromorphic factorization homology

Let us start with the naive definition:

Definition 3.1.11. Let $\mathbb{C}\mathsf{Cob}'$ denote the monoidal $(2,1)$-category whose

(i) objects are natural numbers $[N]$, thought of as a disjoint union of N circles,

(ii) 1-morphisms $[N] \to [M]$ are Riemann surfaces marked with N ingoing and M outgoing ordered points, each equipped with a non-zero tangent vector. Composition is defined by gluing and identities are given by $\mathbb{P}^1$ marked with 0 and ∞ with non-zero tangent vectors ∂_z and $-\partial_{z^{-1}}$,

(iii) 2-morphisms are homotopy classes of paths in the moduli space of Riemann surfaces,

(iv) the tensor product is disjoint union.

Let $\mathbb{C}\mathsf{Cob}'_0$ denote the same $(2,1)$-category, but where we additionally require morphisms to have genus zero and have exactly one outgoing point per connected component. This last condition is needed to make the set of genus zero morphisms closed under composition.

We remark:

(i) The objects of $\mathbb{C}\mathsf{Cob}'_0$ are generated under the tensor product by the formal disks D. There are no relations.

(ii) The 1-morphisms of $\mathbb{C}\mathsf{Cob}'_0$ are generated under composition by the pairs-of-pants that are Riemann spheres $\mathbb{P}^1$, marked with the two ingoing points $y_1 \neq y_2$ in the complex plane with non-zero tangent vectors $X_1, X_2 \in \mathbb{C}^\times$ and outoing point ∞ with tangent vector $-\partial_{z^{-1}}$ modulo translations of y_1, y_2. Hence, such a pair-of-pants is labeled by three non-zero complex numbers $y_1 - y_2, X_1, X_2$. The only relation is associativity. (What about the twisting cylinder? It seems to give an isomorphism, modding out the variation in the tangent vectors)

(iii) The 2-morphisms of $\mathbb{C}\mathsf{Cob}'_0$ modulo higher morphisms are generated under composition by the Dehn twist on the identity cylinder and the braiding on the pair of pants. The relations are those of the framed braid group $B_N \ltimes \mathbb{Z}^N$.

With this, we would like to be able to define a monoidal $(2,1)$-functor

$$\mathcal{Z}'_0 : \mathbb{C}\mathsf{Cob}'_0 \to \mathsf{Rex}$$

by sending

(i) the formal disk D to $Y(\mathfrak{gl}_\ell)\mathsf{Mod}$,

(ii) the twisting cylinders to the shift functors,

(iii) the pair-of-pants labeled by y_1, y_2 to the tensor product functor

$$Y(\mathfrak{gl}_\ell)\mathsf{Mod}^{\boxtimes 2} \to Y(\mathfrak{gl}_\ell)\mathsf{Mod}, \quad U \boxtimes V \mapsto U(y_1) \otimes V(y_2).$$

(iv) the action of the framed braid group on 2-morphisms to the action of the framed braid group on the tensor product functor.

It is here that we run into problems: The braiding of the Yangian is not invertible for all values of the spectral parameter. For example, the braiding degenerates when we try to braid an object with its dual. Rather, the braiding is invertible *almost* everywhere. To get something well-defined, we hence should replace single complex curves by sheaves on the moduli space $M_{0,1+N}$.

Definition 3.1.12. Let $\mathbb{C}\mathsf{Cob}_0$ denote the 2-category whose

(i) objects are natural numbers $[N]$,

(ii) 1-morphisms $[N] \to [1]$ are quasi-coherent sheaves on $M_{0,N+1}$, or more generally for $[N] \to [M]$ on

$$M^{\mathrm{nc}}_{0,N+M} := \bigsqcup_{N_1+\cdots+N_M=N} M_{0,N_1+1} \times \cdots \times M_{0,N_M+1}.$$

(iii) 2-morphisms are morphisms of quasi-coherent sheaves,

Definition 3.1.13. A quasi-coherent sheaf U on $M_{0,N+1}$ is an $S_N \times \mathbb{C}$-equivariant $\mathbb{C}[y_1, ..., y_N]_\delta$-module. This means that U comes equipped with an action of the group $S_N \times \mathbb{C}$ such that

$$\sigma(fu) = \sigma(f)\sigma(u), \quad \sigma \in S_N \times \mathbb{C}, \quad f \in \mathbb{C}[y_1, ..., y_N]_\delta, \quad u \in U,$$

where $\sigma(f)$ is defined using the natural action of $S_N \times \mathbb{C}$ on $\mathbb{C}[y_1, ..., y_N]_\delta$ by permutations and shifts. A morphism $\alpha : U_1 \to U_2$ is a $\mathbb{C}[y_1, ..., y_N]_\delta$-linear and $S_N \times \mathbb{C}$-linear map.

Lemma 3.1.14. *A quasi-coherent sheaf on $M_{0,N+1}$ is equivalently a $\mathbb{C}$-equivariant module over $S_N \ltimes \mathbb{C}[y_1, ..., y_N]_\delta$.*

Remark. The operators

$$T_i := \frac{y_i - y_{i+1} - \eta}{y_i - y_{i+1}} s_i + \frac{\eta}{y_i - y_{i+1}}$$

together with $y_1, ..., y_N$ generate a subalgebra of $S_N \ltimes \mathbb{C}[y_1, ..., y_N]_\delta$ isomorphic to $\dot{H}_N$. Also note that $s_i = \frac{y_i - y_{i+1}}{y_i - y_{i+1} - \eta} T_i - \frac{\eta}{y_i - y_{i+1} - \eta}$, so that localizing at $y_i - y_{i+1}$ and $y_i - y_{i+1} - \eta$ makes both algebras isomorphic.

Theorem 3.1.15. *There is a 2-functor*

$$\mathcal{Z}_0 : \mathbb{C}\mathsf{Cob}_0 \to \mathsf{Rex}$$

that sends

(i) $[1]$ *to* $Y(\mathfrak{gl}_\ell)\mathsf{Mod}_f$*, the category of Yangian modules of finite level,*

(ii) 1-morphisms $U : [N] \to [1]$, which are modules over $S_N \ltimes \mathbb{C}[y_1, ..., y_N]_\delta$, to the right exact functor

$$Y(\mathfrak{gl}_\ell)\mathsf{Mod}_f^{\boxtimes N} \to Y(\mathfrak{gl}_\ell)\mathsf{Mod}_f, \quad V_1 \boxtimes \cdots \boxtimes V_N \mapsto (V_1 \otimes \cdots \otimes V_N) \otimes_{S_N} U,$$

(iii) 2-morphisms get mapped to the corresponding natural transformation.

Remark. The topologically accurate definition would have to make use of the full equivariant dg category of sheaves. We are just looking at zeroth cohomology.

To extend this definition to the case of non-zero genus, we need to use the dual Yangian and introduce an orientation on circles. We then have connected Riemann surfaces $+- \to \varnothing$ and $\varnothing \to -+$. The first is the usual Hom. The second usually needs a finiteness condition to exist, so we have to use a "completed tensor product". From this, we can glue a punctured torus: Compose the cup with the pair-of-pants. This gives something like the functor $\mathsf{Vect} \to Y(\mathfrak{gl}_\ell)\mathsf{Mod}, \mathbb{C} \mapsto Y^\vee(\mathfrak{gl}_\ell) \otimes_{S_2} U$.

To get quasi-coherent sheaves on $M_{1,N+1}$, start with $\mathbb{C}[\tau^\pm, y_1, ..., y_N][(y_i + n + \tau m)^{-1}, (y_i - y_j + n + \tau m)^{-1}]$. This has an action of $S_N \times GL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ by

$$\sigma \cdot (n, m) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(\tau, y_i) := f\left(\frac{a\tau + b}{c\tau + d}, y_{\sigma(i)} + n + \tau m\right).$$

Then take equivariant modules. We can define $y_0 := 0$. Then look at $\mathbb{C}[\tau^\pm, y_1, ..., y_N][\![(y_i - y_j + n + \tau m)^{-1}]\!]$, which contains $\wp(y_i)$ and $\wp(y_i - y_j)$ as well as the derivatives.

We identify the Hilbert space on the circle with the space of operators on the disk via $[0] \to [1]$. This corresponds to the canonical functor $\mathsf{Vect} \to Y(\mathfrak{gl}_\ell)\mathsf{Mod}$. This is not enough, we have to consider the twice-categorified picture.

In terms of double categories:

Definition 3.1.16. Let STCob_0 denote the double category whose

(i) objects are pairs of finite pointed sets $[N, M]$, representing a product of a meromorphic circle and a topological circle,

(ii) horizontal morphisms $[N, M] \to [1, M']$ are genus zero Riemann surfaces with $N+1$ marked points,

(iii) vertical morphisms $[N, M] \to [1, M']$ are topological cobordisms $(S^1)^{\sqcup M} \to (S^1)^{\sqcup M'}$,

(iv) squares homotopy classes of smooth families of Riemann surfaces and diffeomorphisms of topological surfaces.

Theorem 3.1.17. *There is a double functor*

$$\mathscr{Z}_0 : \mathsf{STCob}_0 \to \mathsf{Rex}$$

that sends

(i) *$[N,M]$ to* $Y^\vee(\mathfrak{gl}_\ell)^{\otimes M}\mathsf{mod}^{\boxtimes N}_{Y(\mathfrak{gl}_\ell)\mathsf{Mod}}$,

(ii) *a genus zero Riemann surface $[N,0] \to [1,0]$ to the tensor functor inside* $Y(\mathfrak{gl}_\ell)\mathsf{Mod}$,

(iii) *the pair-of-pants cobordism $[1,2] \to [1,1]$ goes to the tensor product inside* $Y^\vee(\mathfrak{gl}_\ell)\mathsf{mod}_{Y(\mathfrak{gl}_\ell)\mathsf{Mod}}$

(iv) *braids act on tensor functors. The topological direction also gives a Dehn twist.*

We may insert representations on the topological circles via quantum Hamiltonian reduction. The Dehn twist will then give the $X_1, ..., X_N$.

In the elliptic case, we want to look at $E_\tau \times S^1$, seen as a horizontal morphism $[0,1] \to [1,1]$. The structure sheaf on $M_{1,N+1}$ gives a functor

$$\mathsf{Vect} \to Y^\vee(\mathfrak{gl}_\ell)\mathsf{mod}_{Y(\mathfrak{gl}_\ell)\mathsf{Mod}}, \quad \mathbb{C} \mapsto Y^\vee(\mathfrak{gl}_\ell)$$

and we should interpret this as an algebra object in the ambient category $Y(\mathfrak{gl}_\ell)\mathsf{Mod}$. This algebra object is the $Y(\mathfrak{gl}_\ell)$-equivariant elliptic double Yangian $EY(\mathfrak{gl}_\ell) = Y^\vee(\mathfrak{gl}_\ell)^{\otimes 2}$!! See [BJ17]. Making insertions on the circle makes a quantum Hamiltonian reduction [BBJ18b] on one of the factors of $EY(\mathfrak{gl}_\ell)$. The resulting Dehn twist operators will give us the $X_1, ..., X_N$.

There are various problems with finiteness conditions. For example, $Y^\vee(\mathfrak{gl}_\ell)$ is not finite-dimesional, even though it arises as a (commutative) Frobenius algebra assigned to a circle in a TFT. This problem could be remedied by quantum Hamiltonion reducing $Y^\vee(\mathfrak{gl}_\ell)$ to a finite-dimensional endomorphism algebra. We also have to restrict to certain representations of $Y(\mathfrak{gl}_\ell)$ that have a symmetric meromorphic braiding *and* are closed under the tensor functors we defined.

How do we write this in the Morita picture? On the object level, we have internal algebra objects inside $Y(\mathfrak{gl}_\ell)\mathsf{Mod}$. On the horizonal morphism level, we have bimodules over the internal algebras. On the vertical morphism level, we have internal algebra morphisms. On the square level, we have internal equivariant bimodule maps?

Instead of punctured Riemann surfaces, we might want to look at Riemann surfaces equipped with a 1-form that only has double poles and no zeros.

We should put the topological direction into the tight morphisms, since they compose strictly.

The tensor product is disjoint union in the topological direction.

3.1.3 In terms of functorial quantum field theory

Definition 3.1.18. Let STCob_0 denote the partially monoidal double category whose

(i) objects are pairs $[N,M]$ of natural numbers,

(ii) tight morphisms $\Sigma : [N,M] \to [N,M']$ are oriented cobordisms $(S^1)^{\sqcup M} \to (S^1)^{\sqcup M'}$,

(iii) loose morphisms $U : [N,M] \to [1,M]$ are quasi-coherent sheaves on $\overline{M}_{0,N+1}$,

(iv) squares

$$\begin{array}{ccc} [N,M] & \xrightarrow{U} & [1,M] \\ \big\downarrow{\scriptstyle\Sigma} & & \big\downarrow{\scriptstyle\Sigma'} \\ [N,M'] & \xrightarrow{V} & [1,M'] \end{array}$$

are isotopy classes of diffeomorphisms $\psi : \Sigma \xrightarrow{\sim} \Sigma'$ equipped with a morphism $\alpha : U \to V$ of quasi-coherent sheaves,

(v) tight composition is given by gluing cobordisms,

(vi) loose composition of sheaves $U_1 : [N_1, M] \to [1, M]$ and $U_2 : [N_2, M] \to [1, M]$ with $W : [2, M] \to [1, M]$ is given by $\gamma_*(W \boxtimes U_1 \boxtimes U_2)$, where γ is the gluing map

$$\gamma : \overline{M}_{0,2+1} \times \overline{M}_{0,N_1+1} \times \overline{M}_{0,N_2+1} \to \overline{M}_{0,N_1+N_2+1},$$

(vii) composition of squares is composition of diffeomorphisms and sheaf morphisms,

(viii) the partial tensor product is given by $[N, M] \otimes [N, M'] := [N, M + M']$ as well as disjoint union on tight morphisms and direct sum on loose morphisms.

Definition 3.1.19. Let $Y(\mathfrak{gl}_\ell)\mathsf{Alg}$ denote the monoidal double category whose

(i) objects are algebras internal to $Y(\mathfrak{gl}_\ell)\mathsf{Mod}$,

(ii) tight morphisms are algebra maps internal to $Y(\mathfrak{gl}_\ell)\mathsf{Mod}$,

(iii) loose morphisms are bimodules internal to $Y(\mathfrak{gl}_\ell)\mathsf{Mod}$,

(iv) squares are equivariant bimodule maps internal to $Y(\mathfrak{gl}_\ell)\mathsf{Mod}$,

(v) tight composition is composition of algebra maps,

(vi) loose composition is tensor product of bimodules internal to $Y(\mathfrak{gl}_\ell)\mathsf{Mod}$,

(vii) composition of squares is composition of equivariant bimodule maps,

(viii) the tensor product is given by the tensor product in $Y(\mathfrak{gl}_\ell)\mathsf{Mod}$.

Theorem 3.1.20. *There is a monoidal double functor $\mathcal{Z}_0 : \mathsf{STCob}_0 \to Y(\mathfrak{gl}_\ell)\mathsf{Alg}$ that sends*

(i) $[N, M]$ to $(\mathbf{1}^{\oplus N}) \otimes Y^\vee(\mathfrak{gl}_\ell)^{\otimes M}$,

(ii) tight morphisms to the corresponding algebra map between copies of $Y^\vee(\mathfrak{gl}_\ell)$ using its Frobenius algebra structure,

(iii) loose morphisms $U : [N, 0] \to [1, 0]$ to $Y(\mathfrak{gl}_\ell)^{\otimes N} \otimes_{S_N} U$, (??)

(iv) squares to the corresponding action of the mapping class group as in [BBJ18a].

We may modify this by introducing marking on the topological circles or by making the cobordisms non-compact.

3.1.4 In terms of four-dimensional Chern-Simons theory

It is evident that the above description looks like the two-dimensional part of a fully extended four-dimensional functorial field theory. Hence, it is natural to ask which physical theory gives rise to it. Considering that we are looking at complex curves as 1-morphisms and braids in two dimensions as 2-morphisms, we are already given a hint that the theory should have two holomorphic directions and two topological directions. The functors should assign holonomies of some flat connection on $S^1 \times \mathbb{R}$ to the braids. Furthermore, the Yangian controls the local observables. These observations all point to four-dimensional Chern-Simons theory.

Four-dimensional Chern-Simons theory was first described in [Cos13], a recent review can be found in [Lac22]. It is easily constructed by promoting one coordinate of the usual three-dimensional Chern Simons theory to a complex coordinate z on a complex curve C and introducing an additional meromorphic 1-form $\omega = \varphi(z)dz$ on C to be able to integrate the Chern-Simons 3-form over a four-dimensional manifold $C \times \Sigma$ with Σ an oriented surface, yielding the following action:

$$S_\omega(A) = \frac{i}{4\pi} \int_{C\times\Sigma} \omega \wedge \mathrm{CS}(A), \quad \mathrm{CS}(A) := \langle A, dA + \tfrac{2}{3} A \wedge A \rangle.$$

The field A is a $\mathfrak{gl}_\ell$-valued 1-form on $C \times \Sigma$ that has no components in the $z, \bar{z}$ direction, meaning we can write it as $A = A_t dt + A_x dx$ up to some gauge transformation. We interpret A as a Lax connection on Σ with spectral parameter z, writing $A(z) \in \Omega^1(\Sigma, \mathfrak{gl}_\ell)$. It is known [LV21] that the poles of $A(z)$ lie at the zeros of ω, which are called *disorder defects*. In order to compare with the previous section, we set

$$\omega = \frac{(z - y_1 - \eta)\cdots(z - y_N - \eta)}{(z - y_1)\cdots(z - y_N)} dz.$$

The classical solutions for $A(z)$ determined in [LV21] give $A(y_i) = J_i$ and $A(z)$ has poles at $y_i + \eta$. This is congruent to the analytic structure of the R-matrix $\check{R}(z - y_i)$. We obtain the Cauchy matrix $C_{ij} = \frac{1}{y_i - y_j + \eta}$, which has inverse

$$(C^{-1})_{ij} = -(y_j - y_i + \eta) \prod_{k\neq i} \frac{y_j - y_k + \eta}{y_i - y_k} \prod_{k \neq j} \frac{y_k - y_i + \eta}{y_i - y_k}$$

The Cauchy matrix should be seen as a linear map $\mathfrak{gl}_\ell^{(\zeta)} \to \mathfrak{gl}_\ell^{(z)}$. We then have $\mathrm{diag}(X_1, ..., X_N)$, which we could let act on $\mathfrak{d}$ by the adjoint action, or just left-multiply in the enveloping algebra.

Essentially: Take τ in the pole representation $\tau(y_j)$, put it into the zero representation $\tau(y_i + \eta)$, then multiply by X_i, then go back to the pole representation. This should give the Lax matrix of the RS model, so its just a conjugation away! It also makes apparent why the elementary symmetric polynomials in the X_i are good invariants.

3.2 Extending to the elliptic case

3.2.1 The trigonometric case

Before we move onto the elliptic case, let us warm up with the trigonometric case. The rational RS model corresponds to the 4d geometry $\mathbb{R}^2 \times S^1 \times \mathbb{R}$. The trigonometric RS model corresponds to $\mathbb{R} \times (S^1)^2 \times \mathbb{R}$ and the elliptic RS model corresponds to $(S^1)^3 \times \mathbb{R}$. This could become a problem, since our TQFT only considers two fully extended dimensions. We can account for two of the S^1 by considering the torus, which gets mapped to $Y(\mathfrak{gl}_\ell)$-equivariant modules over $EY(\mathfrak{gl}_\ell)$. The third S^1 would have to come from some additional functors between these categories. However, the need for this third S^1 would be eliminated if we were to consider a balancing.

The TQFT assigns to $(S^1)^2 \times \mathbb{R}$ just the elliptic Yangian/the free module functor over the elliptic Yangian. On the other hand, to $(S^1)^2 \times S^1$, it will assign the composition of the dualizing maps, which is nothing but the coend over the category assigned to the torus. The question is: What happens when we do quantum Hamiltonian reduction. The boundary components give module structures over the elliptic Yangian. The math in Ben-Zvi et al. only accounts for line markings in 3d, but we would need point markings in 3d to get modules.

3.2.2 The elliptic case

In this section, we explore the representation theory set up by the functorial quantum field theory representing four-dimensional Chern-Simons theory on $C \times S^1 \times \mathbb{R}$ in the case where C is an elliptic curve. Topologically, this corresponds to the case of a torus $T^2 \setminus D^2$ with one boundary component. We already know that our functorial quantum field theory sends the torus with one (marked) boundary component to the category of $Y^\vee(\mathfrak{gl}_\ell)^{\otimes 2}$-modules internal to $Y(\mathfrak{gl}_\ell)\mathsf{Mod}^{\mathbb{R}}$. Again, we may look at embeddings $(D^2)^{\sqcup N} \to T^2 \setminus D^2$, which by [BBJ18a] all correspond to the functor

$$Y(\mathfrak{gl}_\ell)\mathsf{Mod}^{\boxtimes N} \to Y^\vee(\mathfrak{gl}_\ell)^{\otimes 2}\text{-}\mathsf{mod}_{Y(\mathfrak{gl}_\ell)\mathsf{Mod}^{\mathbb{R}}}, \quad V_1 \boxtimes \cdots \boxtimes V_N \mapsto Y^\vee(\mathfrak{gl}_\ell)^{\otimes 2} \otimes V_1 \otimes \cdots \otimes V_N.$$

We again obtain an action of the braid group of the torus, where the braids going through the handle add two additional sets of affine generators.

We are led to consider modules over the algebra $EY(\mathfrak{gl}_\ell) := Y^\vee(\mathfrak{gl}_\ell) \otimes Y^\vee(\mathfrak{gl}_\ell)$, which we call the *elliptic Yangian*. Let $A(z)$ and $B(w)$ denote the canonical elements $EY(\mathfrak{gl}_\ell) \otimes \mathrm{End}(V_z^+) \otimes \mathrm{End}(V_w^+)$. Then

$$R_{12}(z-w)A_1(z)R_{21}(w-z)B_2(w) = B_2(w)R_{12}(z-w)A_1(z)R_{21}(w-z).$$

This is exactly what we would expect from the relations for the torus/quantum Heisenberg double.

Similarly, we may consider the torus with one boundary component, to which the TQFT assigns $Y(\mathfrak{gl}_\ell)$-equivariant modules over $Y^\vee(\mathfrak{gl}_\ell) \otimes Y^\vee(\mathfrak{gl}_\ell)$. We have the $y_1, ..., y_N$ from the rational case as well as the $X_1, ..., X_N$ that arise from the transfer matrix. In addition, we also have $Z_1, ..., Z_N$ and $W_1, ..., W_N$ arising from the additional two tensorands, having their own equivalent of transfer matrices σ_A, σ_B. We may hopefully combine the Zs, Ws, and ys into elliptic functions after some reparametrization. We need functions of the form $\Phi(z, q_{ij} + \gamma)$ as well as $\Phi(\gamma, q_{kj})$ or $\sigma(q_{ij} + \eta)/\sigma(q_{ij})$. What are the relations among τ, σ_A, σ_B? They should come from the torus braid group, meaning σ_A and σ_B commute among each other and something else for τ and $\sigma_{A/B}$.

Structurally, we have $\Phi(z, x)\Phi(z, y) \sim \Phi(z, x + y)$, meaning $\Phi(z, q_{ij})$ could come about by some product $W_i W_j^{-1}$. The R-matrix quantization suggests that what previously was the transfer matrix arises as a product of the form $ZW^{-1}P$. This could be something like $\sigma_A \sigma_B^{-1} \tau$.

Since we should have a duality with the XYZ Heisenberg spin chain, we may compare with that literature. An important quantity is a transfer matrix of the form

$$\sigma(z) = \operatorname{tr} A(z)T(z)B(z)T(-z)^{-1},$$

where A and B satisfy the reflection equation. This also fits the data we have. However, the matrices A and B are used for open boundary conditions, apparently. This is probably what happens if we consider a disk with two marked points. But gluing the marked points together yields the torus, so the same formula should work. There should also be an additional spectral parameter?

I would expect the elliptic transfer matrix to act on a spin chain tensored with two reps of the dual Yangian. It will also satisfy fusion relations, but the expansion in terms of residues will be different, since we are doing it on an elliptic curve. Do we have something like

$$f(y_i + \eta; z) = \sigma(0; z) + \sum_i \Phi(z, y_i - y_j + \eta) \operatorname{Res}_{y_i} f(-; z) \frac{dx}{y}$$

If σ is doubly periodic, then it is amenable to such "elliptic partial fraction" expansions.

Previously, we had the parameter k in the Young diagram $[1^k]$ labeling the winding number for the higher transfer matrices. Now we should have two additional parameters m, n, labeling the winding number for the toroidal directions? These could label the additional representations we are inserting at the ends of the spin chain. We then get complicated fusion relations and $k = m = n = \ell$ gives a nice case from which we can determine the spectrum. But we would have three L-matrices, one for each direction. Their product might be the actual L-matrix we are after.

The elliptic Cauchy matrix is $\Phi(z, y_i - y_j + \eta)$.

Chapter 4

Conclusion

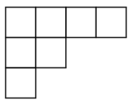
Starting from the appearance of the Lax matrix of the classical rational Ruijsenaars-Schneider model in the fusion relations of the Heisenberg model, we have teased out a beautifully symmetric structure: The four arms emerging from four-dimensional Chern-Simons theory are all related by clear dualities as summarized in figure 1. (tba)

Future work

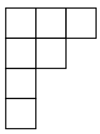
Appendix A

Young diagrams

Definition A.1. (i) A *partition* of a natural number N is a sequence $\lambda = [\lambda_1, ..., \lambda_\ell]$ of weakly decreasing natural numbers such that $\sum_i \lambda_i = N$. We adopt exponent notation in which $[2, 1^k]$ means that 1 appears k times, so $[2, 1^k]$ would be a partition of $k + 2$. Partitions can be visualized using *Young diagrams*, by drawing rows of boxes with decreasing row lengths, each corresponding to λ_i. For example, the partition $[4, 2, 1]$ of 7 would be drawn as:



Every partition λ has a *dual partition* λ' given by transposing the Young diagram. In the above case, this would be $[3, 2, 1^2]$, which we draw here:



(ii) Let λ be a partition of N. A *Young tableau* t *of shape* λ is a choice of bijectively labeling the boxes of the corresponding Young diagram by the numbers $1, ..., N$. Clearly, there are $N!$ such tableau. As an example, we take the following Young tableau of shape $[4, 2, 1]$:

3	5	1	6
7	2		
4			

Given a number $k \in \{1, ..., N\}$, we define its *content* to be $c_k(t) := i - j$, where i and j, respectively, are the row and column in which k appears on the Young tableau. Then $(c_1(t), ..., c_N(t))$ is called the *content vector* of t_λ. The tableau above has content vector $(-2, 0, 0, 2, -1, -3, 1)$. A Young tableau is *standard*, if the numbers are increasing from left to right and from top to bottom. The tableau above is *not* standard.

Appendix B

Dunkl operators

Lemma B.1 (See proposition 1.2.7). *Let* $\theta_{ij} := X_i/(X_i - X_j)$. *The mapping*

$$s_i \mapsto s_i, \quad X_i \mapsto X_i, \quad y_i \mapsto -\mathrm{i}\hbar X_i \partial_i - i\eta + \eta \sum_{j<i} \theta_{ji}(1-(i\ j)) - \eta \sum_{j>i} \theta_{ij}(1-(i\ j))$$

gives rise to a representation of $\ddot{H}_N$ *on* $\mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}]$.

Proof. It is clear that the necessary relations hold for the s_i and X_i. It is also clear that $s_i y_j = y_j s_i$ for $|i-j| > 1$. Let $1 \leq i < N$ and check the other relations:

(i) $y_i y_j = y_j y_i$: tba

(ii) Note that $s_i \theta_{i+1,i} = \theta_{i,i+1} s_i$ and $\theta_{i,i+1} = 1 - \theta_{i+1,i}$, leading to

$$\begin{aligned} s_i y_i = s_i \Big(& -\mathrm{i}\hbar X_i \partial_i - i\eta + \eta \sum_{j<i} \theta_{ji}(1-(i\ j)) \\ & - \eta \theta_{i,i+1}(1-(i\ i+1)) - \eta \sum_{j>i+1} \theta_{ij}(1-(i\ j)) \Big) \\ = \Big(& -\mathrm{i}\hbar X_{i+1} \partial_{i+1} - i\eta + \eta \sum_{j<i} \theta_{j,i+1}(1-(i+1\ j)) \\ & - \eta \theta_{i+1,i}(1-(i\ i+1)) - \eta \sum_{j>i+1} \theta_{i+1,j}(1-(i+1\ j)) \Big) s_i \\ = \Big(& -\mathrm{i}\hbar X_{i+1} \partial_{i+1} - i\eta + \eta \sum_{j<i} \theta_{j,i+1}(1-(i+1\ j)) \\ & - \eta(1-\theta_{i,i+1})(1-(i\ i+1)) - \eta \sum_{j>i+1} \theta_{i+1,j}(1-(i+1\ j)) \Big) s_i \\ = \Big(& -\mathrm{i}\hbar X_{i+1} \partial_{i+1} - (i+1)\eta + \eta \sum_{j<i+1} \theta_{j,i+1}(1-(i+1\ j)) \\ & + \eta s_i - \eta \sum_{j>i+1} \theta_{i+1,j}(1-(i+1\ j)) \Big) s_i \\ = {} & y_{i+1} s_i + \eta. \end{aligned}$$

(iii) We see that

$$
\begin{aligned}
&X_1 w\Big(-\mathrm{i}\hbar X_i\partial_i - i\eta + \eta\sum_{j<i}\theta_{ji}(1-(i\ j)) - \eta\sum_{j>i}\theta_{ij}(1-(i\ j))\Big)w^{-1}X_1^{-1}\\
&= X_1\Big(-\mathrm{i}\hbar X_{i+1}\partial_{i+1} - i\eta + \eta\sum_{j=1}^{i-1}\theta_{j+1,i+1}(1-(i+1\ j+1))\\
&\qquad - \eta\sum_{j=i+1}^{N}\theta_{i+1,j+1}(1-(i+1\ j+1))\Big)X_1^{-1}\\
&= X_1\Big(-\mathrm{i}\hbar X_{i+1}\partial_{i+1} - i\eta + \eta\sum_{j=2}^{i}\theta_{j,i+1}(1-(i+1\ j))\\
&\qquad - \eta\sum_{j=i+2}^{N}\theta_{i+1,j}(1-(i+1\ j)) - \eta\theta_{i+1,1}(1-(i+1\ 1))\Big)X_1^{-1}\\
&= X_1\Big(-\mathrm{i}\hbar X_{i+1}\partial_{i+1} - i\eta + \eta\sum_{j=2}^{i}\theta_{j,i+1}(1-(i+1\ j))\\
&\qquad - \eta\sum_{j=i+2}^{N}\theta_{i+1,j}(1-(i+1\ j)) - \eta(1-\theta_{1,i+1})(1-(i+1\ 1))\Big)X_1^{-1}\\
&= X_1\Big(-\mathrm{i}\hbar X_{i+1}\partial_{i+1} - (i+1)\eta + \eta\sum_{j=2}^{i}\theta_{j,i+1}(1-(i+1\ j))\\
&\qquad - \eta\sum_{j=i+2}^{N}\theta_{i+1,j}(1-(i+1\ j)) + \eta\theta_{1,i+1}(1-(i+1\ 1)) + \eta(i+1\ 1)\Big)X_1^{-1}
\end{aligned}
$$

but

$$
\begin{aligned}
X_1(\theta_{1,i+1}(i+1\ 1) - (i+1\ 1))X_1^{-1} &= X_1(\theta_{1,i+1}-1)(i+1\ i)X_1^{-1}\\
&= -X_1X_{i+1}^{-1}\theta_{i+1,1}(i+1\ i)\\
&= \theta_{1,i+1}(i+1\ i),
\end{aligned}
$$

which finally yields $\pi y_i\pi^{-1} = y_{i+1}$. A similar calculation for $i = N$ yields $\pi y_N\pi^{-1} = y_1 + \mathrm{i}\hbar$, remembering that $X_1(-\mathrm{i}\hbar X_1\partial_1)X_1^{-1} = -\mathrm{i}\hbar X_1\partial_1 + \mathrm{i}\hbar$. $\square$

Lemma B.2 (See proposition 1.2.12). *On $\delta^{-1}\mathbb{C}[X_1^{\pm1},...,X_N^{\pm1}]^{S_N}$, C_1 reduces to the canonically quantized total momentum of the trigonometric Calogero-Moser model, while C_2 reduces to the Hamiltonian:*

$$
C_1 = -\mathrm{i}\hbar\sum_i X_i\partial_i,\quad C_2 = -\frac{\hbar^2}{2}\sum_i(X_i\partial_i)^2 + \frac{\eta(\eta-\mathrm{i}\hbar)}{2}\sum_{i\neq j}\theta_{ij}\theta_{ji}.
$$

Proof. Observe

$$
\delta^{-1}(-\mathrm{i}\hbar X_i\partial_i)\delta = -\mathrm{i}\hbar X_i\partial_i - \delta^{-1}X_i\sum_{j<i}\frac{2\eta\delta}{X_j-X_i} + \delta^{-1}X_i\sum_{j>i}\frac{2\eta\delta}{X_i-X_j} = -\mathrm{i}\hbar X_i\partial_i + 2\eta\sum_{i\neq j}\theta_{ij}.
$$

Noting that $(1-(i\ j))$ acts as zero on symmetric polynomials and $2\sum_{i\neq j}\theta_{ij} = \binom{N}{2}$, we derive the identity

$$C_1 = \sum_i \delta^{-1} y_i \delta = -\mathrm{i}\hbar \sum_i X_i \partial_i + 2\eta \sum_{i\neq j} \theta_{ij} - \eta \binom{N}{2} = -\mathrm{i}\hbar \sum_i X_i \partial_i.$$

tba $\square$

Notation

$\mathbb{N}$	natural numbers with zero
$\mathbb{Z}$	ring of integers
$\mathbb{C}$	field of complex numbers
i	imaginary unit
$\hbar$	Planck's constant for particles
η	coupling constant for particles or Planck's constant for spins
N	rank of symmetric group and Hecke algebras
S_N	symmetric group
$\dot{S}_N$	affine symmetric group
$\dot{H}_N$	degenerate affine Hecke algebra
$\ddot{H}_N$	degenerate double affine Hecke algebra
$S\ddot{H}_N$	spherical degenerate double affine Hecke algebra
X_i	Laurent generators of $\dot{S}_N$ and $\ddot{H}_N$
y_i	polynomial generators of $\dot{H}_N$ and $\ddot{H}_N$
D_k	Hamiltonians of the quantum rational Ruijsenaars-Schneider model
C_k	Hamiltonians of the quantum trigonometric Calogero-Moser model
ℓ	rank of general linear Lie algebra
$\mathfrak{gl}_\ell$	general linear Lie algebra
V	η-extended vector space $\mathbb{C}^\ell[\eta]$
$Y(\mathfrak{gl}_\ell)$	Yangian of the general linear Lie algebra
$Y^\vee(\mathfrak{gl}_\ell)$	Dual Yangian of the general linear Lie algebra
$L(\mathfrak{gl}_\ell)$	loop algebra of the general linear Lie algebra
$LY(\mathfrak{gl}_\ell)$	loop Yangian of the general linear Lie algebra
z, w	spectral parameter
P	permutation operator
$R(z)$	Yang's R-matrix $1 - \eta P/z$
$\hat{R}(z)$	polynomial R-matrix $z - \eta P$
$\check{R}(z)$	unitary R-matrix $(z - \eta P)/(z - \eta)$
$\Pi^\pm$	(anti-)symmetrizer $(1 \pm P)/2$
λ	weight for $\mathfrak{gl}_\ell$ or Young diagram
t_λ	Young tableau of shape λ
Π_{t_λ}	fusion projector for t_λ

$L(\lambda)$	irreducible $\mathfrak{gl}_\ell$-module with highest weight λ
$S(\lambda)$	Specht module for λ
$L(\lambda)^t_y$	evaluation module of $Y(\mathfrak{gl}_\ell)$, transposed and shifted by $y \in \mathbb{C}$
$V^\pm_y$	$L(\Box)^t_y$ for "+" or $L(\Box)_y$ for "−"
g	twist matrix
γ_i	diagonal components of the twist matrix
$\tau^g(u)$	g-twisted (fundamental) transfer matrix
$\tau^g_\lambda(u)$	g-twisted transfer matrix of shape λ
C	complex curve
Σ	oriented surface
ω	meromorphic 1-form on C
A	$\mathfrak{gl}_\ell$-valued 1-form on Σ

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Erklärung

Hiermit versichere ich an Eides statt, dass ich die vorliegende Arbeit im Masterstudiengang Mathematik selbstständig verfasst und keine anderen als die angegebenen Hilfsmittel – insbesondere keine im Quellenverzeichnis nicht benannten Internet-Quellen – benutzt habe. Alle Stellen, die wörtlich oder sinngemäß aus Veröffentlichungen entnommen wurden, sind als solche kenntlich gemacht. Ich versichere weiterhin, dass ich die Arbeit vorher nicht in einem anderen Prüfungsverfahren eingereicht habe.

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