Quantum-classical duality

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State of the art

It was observed by Gorsky, Zabrodin and Zotov [GZZ14], that the spectrum of the twist matrix g of the quantum inhomogeneous Heisenberg model coincides with the spectrum of the Lax matrix L of the classical rational Ruijsenaars-Schneider model under the following substitutions:

ith particle	ith spin
position y_i	inhomogeneity y_i
velocity $\dot{y}_i$	non-local Hamiltonian H_i

The non-local Hamiltonians are given by

$$H_i := \mathrm{Res}_{z=y_i}\, \tau^g(z),$$

where $\tau^g(z)$ denotes the g-twisted transfer matrix. We call this fact *quantum-classical duality*.

Functional relations

Recently, Arutyunov [Aru] formulated a system of polynomial equations for the spectrum of $H_1, \ldots, H_N$ in terms of functional relations between higher transfer matrices $\tau_\lambda^g(z)$, where λ is a Young diagram.
The basic functional relation is

$$\tau_{[1^{k+1}]}(z) = \tau^g(z)\tau^g_{[1^k]}(z-\eta) - \tau_{[2,1^{k-1}]}(z-\eta),$$

from which we derive the recursion relation

$$\mathrm{Res}_{z=y_i}\, \tau_{[1^{k+1}]}(z) = e_k(g)H_i + \sum_j \underbrace{\frac{\eta H_i}{y_i - y_j - \eta}}_{\text{Lax matrix}} \mathrm{Res}_{z=y_j}\, \tau_{[1^k]}(z).$$

We see the Lax matrix of the rational RS model appearing!

Spectral equation

Iterating the recursion relation yields a Cayley-Hamilton-like identity consistent with [GZZ14]:

$$\sum_{k=0}^{\ell} e_k(g) L^k \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix} = 0,$$

where

$$b_i := \prod_{i \neq j} \frac{y_i - y_j - \eta}{y_i - y_j}.$$

This system of N equations of order ℓ form the *spectral equations*.

Representation theory

These facts spark the quest for a deeper reason behind the correspondence between both models.

To this end, let us look at the representation theoretic structure of the Heisenberg model: The observables of the Heisenberg model live inside the Yangian $Y(\mathfrak{gl}_\ell)$ and its Hilbert space is given by the representation

$$\mathbb{C}^\ell[y_1] \otimes \cdots \otimes \mathbb{C}^\ell[y_N]|_{y_i=u_i}.$$

This has a hidden symmetry given by the degenerate affine Hecke algebra

$$\dot{H}_N = \mathbb{C}[S_N] \otimes \mathbb{C}[y_1, ..., y_N]$$

via *Schur-Weyl duality*.

Schur-Weyl duality

Schur-Weyl duality means that there is a bimodule structure

$$Y(\mathfrak{gl}_\ell) \curvearrowright \mathbb{C}^\ell[y_1] \otimes \cdots \otimes \mathbb{C}^\ell[y_N] \curvearrowleft \dot{H}_N$$

that induces a functor

$$\begin{aligned} D_{\ell,N} : \dot{H}_N\mathrm{Mod} &\to Y(\mathfrak{gl}_\ell)\mathrm{Mod}, \\ U &\mapsto (\mathbb{C}^\ell[y_1] \otimes \cdots \otimes \mathbb{C}^\ell[y_N]) \otimes_{\dot{H}_N} U, \end{aligned}$$

called the *Drinfeld functor* [Dri86]. This restricts to the well-known correspondence between representations of S_N and $\mathfrak{gl}_\ell$.

Quantum rational (spin) RS model

There is a representation of $\dot{H}_N$ on polynomials $\mathbb{C}[y_1, .., y_N]$ that extends to a representation of the degenerate double affine Hecke algebra

$$\ddot{H}_N := \mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}] \otimes \mathbb{C}[S_N] \otimes \mathbb{C}[y_1, ..., y_N].$$

The elementary symmetric polynomials $e_k(X_1, ..., X_N)$ are commuting Hamiltonians for the quantum rational RS model.

We can then look at

$$D_{\ell,N}(\mathbb{C}[y_1, ..., y_N]) \cong (\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} \mathbb{C}[y_1, ..., y_N],$$

which contains both the RS and Heisenberg model. It may be thought of as the Hilbert space of the rational *spin* RS model, in analogy to [LPS22].

Extending the Drinfeld functor

We may extend and g-twist the Drinfeld functor, yielding

$$D^g_{\ell,N} : \ddot{H}_N \to \ddot{H}_N^{S_N} \# Y(\mathfrak{gl}_\ell)\mathrm{Mod}.$$

Theorem (Quantum-classical duality)

The element

$$\operatorname{tr} g + \sum_i \frac{\eta b_i X_i}{z - y_i} \in \ddot{H}_N^{S_N}[\![z^{-1}]\!]$$

acts on $D^g_{\ell,N}(\mathbb{C}[y_1, \dots, y_N])$ *in the same way as* $\tau^g(z)$ *when* $\hbar_{RS} = 0$.

Slogan

The g-twisted Drinfeld functor maps the Hamiltonians of the rational RS model to an $\hbar_{RS}$-deformation of the Hamiltonians of the Heisenberg model.

Loop Yangian

What happens when $\hbar_{RS} \neq 0$? Here we resort to a result of Guay [Gua05], constructing a Schur-Weyl functor

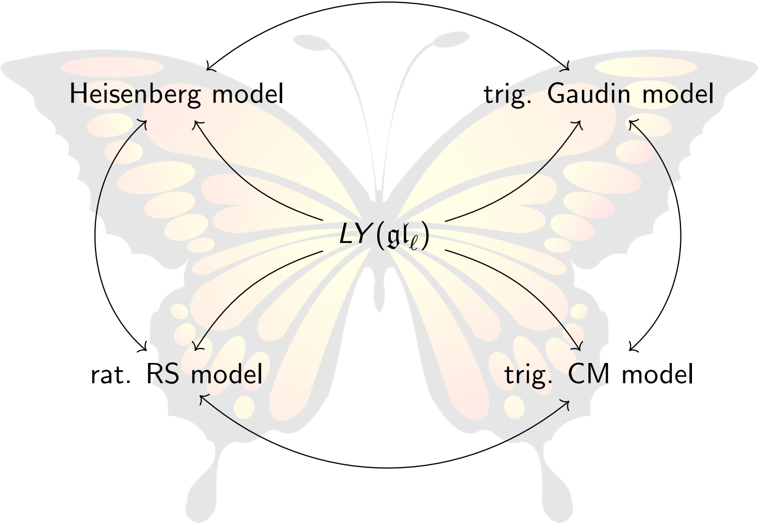
$$LD_{\ell,N} : \ddot{H}_N\mathrm{Mod} \to LY(\mathfrak{gl}_\ell)\mathrm{Mod},$$

where $LY(\mathfrak{gl}_\ell)$ is the *loop Yangian*, which is a quotient of $L(\mathfrak{gl}_\ell)\# Y(\mathfrak{gl}_\ell)$ as well as $Y(\widehat{\mathfrak{gl}_\ell})$.

Theorem

The loop variable t living in the center of $L(\mathfrak{gl}_\ell)$ acts on $LD_{\ell,N}(\mathbb{C}[y_1, ..., y_N])$ in the same way as the standard Hamiltonian of the rational RS model.

The full picture



Comparison with classical models

The loop Yangian $LY(\mathfrak{gl}_\ell)$ should be seen as the quantization of the Poisson structure found by Arutyunov and Frolov [AF98]:

They consider the rational spin RS and trigonometric spin CM models by Hamiltonian reduction of $T^*GL_N \times \mathfrak{gl}_N^*$ and derive two symmetries: The rational spin RS model has a loop symmetry and the trigonometric spin CM model has a Yangian symmetry.

4d Chern-Simons theory

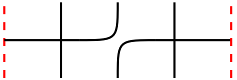
It is reasonable to ask for a geometric picture of the representation theoretic structures thus far. A natural guess where to look is 4d Chern-Simons theory, since it is known to assign the (delooping of the) category of representations of the Yangian to the formal punctured disc.

4d Chern-Simons theory is defined for a $\mathfrak{gl}_\ell$-gauge field A on a 4-manifold $C \times \Sigma$, where C is a complex curve equipped with a meromorphic 1-form ω and Σ is an oriented surface. In our case, we look at $C = \mathbb{P}^1$ with $\omega = dz$ and $\Sigma = S^1 \times [0, 1]$.

It was realized by Costello [Cos13] that Wilson lines along $S^1 \times [0, 1]$ with constant coordinate in $\mathbb{P}^1$ recover the Heisenberg model.

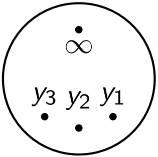
Double category of cobordisms

Such Wilson lines organize into a double category of cobordisms. Its objects are finite disjoint unions of $D^\times \times S^1_*$ and it has two types of morphisms: The first are purely topological cobordisms, such as



$(S^1 \times [1,0])^{\sqcup 3}$

the second are purely holomorphic cobordisms



$\mathbb{P}^1$

where we allow $y_1, ..., y_N \in \mathbb{P}^1 \setminus \{\infty\}$ with $y_i - y_j \neq 0, \eta$. Denote the space of such cobordisms by Y_N.

QC duality in terms of 4d CS theory

Proposition

Quasi-coherent sheaves on Y_N are equivalently modules over

$$\underline{\dot{H}}_N := \dot{H}_N[(y_i - y_j)^{-1}, (y_i - y_j - \eta)^{-1}].$$

With this identification, we can view the Drinfeld functor as a functor

$$\mathrm{Qcoh}(Y_N) \to Y(\mathfrak{gl}_\ell)\mathrm{Mod},$$

giving the part of 4d Chern-Simons theory that assigns Yangian modules to $\mathbb{P}^1 \times S^1$ with N distinguished points on which Wilson lines may end.

4d CS on $E \times S^1 \times [0,1]$

The above considerations motivate the study of 4d Chern-Simons theory on $E \times S^1 \times [0,1]$, where $E = \mathbb{C}/\Lambda$ is an elliptic curve. We similarly look at configurations of points Y_N^{ell} and let $\underline{\dot{H}}_N^{\text{ell}}$ denote the algebra whose modules model quasi-coherent sheaves on Y_N^{ell}. I expect that there is a Schur-Weyl bimodule

$$E(\mathfrak{gl}_\ell) \curvearrowright (\mathbb{C}^\ell)^{\otimes N} \otimes \mathcal{O}(Y_N^{\text{ell}}) \curvearrowleft \underline{\dot{H}}_N^{\text{ell}}.$$

where $E(\mathfrak{gl}_\ell)$ is Belavin's elliptic quantum group [CWY18, ES98]. Tensoring with $\mathcal{O}(Y_N^{\text{ell}})$ over $\underline{\dot{H}}_N^{\text{ell}}$ should give the Hilbert space for the elliptic spin RS model. The Hamiltonians are Ruijsenaars' elliptic difference operators acting on $\mathcal{O}(Y_N^{\text{ell}})$.

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