PhD

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Abstract

Acknowledgments

Contents

Chapter 1

Notes

1.1 Basics of elliptic structures

1.1.1 Elliptic functions and theta functions

Definition 1.1.1. An elliptic curve E (over $\mathbb{C}$) is a smooth projective curve or Riemann surface of genus 1. These are of the form $E \cong \mathbb{C}/\Lambda$ for $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ for $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$. This does not faithfully parametrize elliptic curves, but $M_{1,1} := \mathbb{H}/SL_2(\mathbb{Z})$ does, where $SL_2(\mathbb{Z})$ acts by Möbius transformations. Algebraically, every elliptic curve may be brought into the form

$$Y^2 Z = 4X(X - Z)(X - \lambda Z),$$

where λ is the λ-invariant of E, which is also not faithful up to an action of $SL_2(\mathbb{Z})/\Gamma(2) \cong S_3$. The invariant combination

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$$

is the j-invariant, yielding a bijection $j : M_{1,1} \to \mathbb{C}$. Stacky points are at $j = 0$ and $j = 12^3 = 1728$, where Λ becomes the lattice of the Eisenstein and Gauss integers, respectively, with automorphism groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ (plus the usual involution $Y \mapsto -Y$, giving $\mathbb{Z}_4$ and $\mathbb{Z}_6$). There is an isomorphism

$$\mathbb{C}/\Lambda \to E(\mathbb{C}), \quad z + \Lambda \mapsto \begin{cases} [\wp(z|\tau) : \wp'(z|\tau) : 1], & z \notin \Lambda \\ [0 : 1 : 0], & z \in \Lambda. \end{cases}$$

An elliptic function is a meromorphic function $f : \mathbb{C} \to \mathbb{C}$ that is Λ-periodic such that it descends to a meromorphic function on E. Theta functions are sections of certain line bundles over E, which may be represented as entire functions $\vartheta : \mathbb{C} \to \mathbb{C}$ satisfying

$$\vartheta(z + 1|\tau) = \vartheta(z|\tau), \quad \vartheta(z + \tau|\tau) = \exp(-\pi \mathrm{i}\tau - 2\pi \mathrm{i} z)\vartheta(z|\tau)$$

This line bundle has in fact only one section up to a prefactor, and this is the Jacobi theta function.

1.1.2 Belavin's elliptic R-matrix

[ES98]

Definition 1.1.2. Let $\xi = \exp(2\pi \mathrm{i}/\ell)$. Define a projectively flat rank ℓ vector bundle on $E = \mathbb{C}/\Lambda$ by the two monodromies

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \xi & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi^{\ell-1} \end{pmatrix}, \quad B = \exp(-\pi \mathrm{i}\tfrac{\ell-1}{\ell}) \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix},$$

satisfying $A^\ell, B^\ell = 1$ and $BA = \xi AB$ giving a flat/holomorphic PGL_ℓ-bundle $P \to E$. Belavin's elliptic R-matrix is the unique R-matrix satisfying the QYBE and unitarity as well as

(i) $R^B(z)$ has simple poles only at $\eta + \Lambda$,

(ii) $R^B(0) = P$,

(iii) $R^B(z+1) = A_1 R^B(z) A_1^{-1} = A_2^{-1} R^B(z) A_2$, $R^B(z+\tau) = B_1 R^B(z) B_1^{-1} = B_2^{-1} R^B(z) B_2$.

In particular, the R-matrix lives on the ℓ-fold cover $\bar{E} = \mathbb{C}/\ell\Lambda$, where P is trivialized. We may view the R-matrix as an element of

$$\operatorname{End} \mathbb{C}^\ell \otimes \Gamma(E, P \ltimes \operatorname{End} \mathbb{C}^\ell),$$

where $\Gamma(E, P \ltimes \operatorname{End} \mathbb{C}^\ell)$ are meromorphic sections of the associated bundle $P \ltimes \operatorname{End} \mathbb{C}^\ell$.

1.1.3 RTT representations

[ES98]

Definition 1.1.3. Define the category C_B whose objects are vector spaces V equipped with an invertible element

$$L(z) \in \mathrm{Mat}_\ell(\operatorname{End} V \otimes \mathcal{M}(\bar{E})),$$

thus having values in $\operatorname{End} \mathbb{C}^\ell \otimes \operatorname{End} V$, satisfying

$$R^B_{12}(z-w) L_1(z) L_2(w) = L_2(w) L_1(z) R^B_{12}(z-w).$$

Morphisms are linear maps $f : V \to V'$ satisfying $\varphi L(z) = L'(z)\varphi$. There is a tensor structure via

$$(V, L(z)) \otimes (V', L'(z)) := (V \otimes V', L_{12}(z) L'_{13}(z))$$

and finite-dimensional objects $(V, L(z))$ have duals $(V^*, L^*(z))$ with $L^*(z) = (L(z)^{-1})^{t_2}$. Belavin's R-matrix ensures the existence of a vector representation $(\mathbb{C}^\ell, R^B(z))$.

1.1.4 Elliptic Drinfeld functor

In order to define an elliptic Drinfeld functor, we first need an analog of the Yangian representation on $\mathbb{C}^\ell[y]$. For fixed z, we can view the coefficients of $R^B(z-y)$ as meromorphic $\operatorname{End}\mathbb{C}^\ell$-valued functions $f_{ij}(y)$ with at most a simple pole at $z-\eta+\Lambda$. They satisfy $f_{ij}(y+1) = \operatorname{Ad}_A^{-1} f_{ij}(y)$ and $f_{ij}(y+\tau) = \operatorname{Ad}_B^{-1} f_{ij}(y)$ and are thus meromorphic sections of the adjoint bundle $\operatorname{Ad} P$. These naturally act on sections of the bundle associated to $\mathbb{C}^\ell$. Let us abbreviate the space of such sections as

$$\Theta := \{f : \mathbb{C} \to \mathbb{C}^\ell \text{ meromorphic} \mid f(y+1) = A^{-1}f(y), f(y+\tau)B^{-1}f(y)\}.$$

We obtain an RTT representation on Θ. More generally, we may define

$$L_N(z) := R^B_{01}(z-y_1)\cdots R^B_{0N}(z-y_N)$$

whose coefficients act on the space

$$\Theta_N := \{f : \mathbb{C}^N \to (\mathbb{C}^\ell)^{\otimes N} \text{ meromorphic} \mid f(y_i+1) = A_i^{-1}f(y_i), f(y_i+\tau)B_i^{-1}f(y_i)\}.$$

These are the sections of a vector bundle $V_N \to E^N$, which can be pulled back to the (η-deformed) configuration space of points of E or even on $M_{1,1+N}$ such that the R-matrix allows us to put an RTT representation on its sections.

There is a commuting action of S_N from the right on Θ_N via R-matrices: $(i\ j) \mapsto R^B_{ij}(y_i-y_j)$, as well as sections of the structure sheaf of the configuration space of points on E. Together, these form a generalization of the degenerate affine Hecke algebra. If these sections of the structure sheaf are replaced by the theta functions of [Has95], we may let the elliptic difference operators act. This space is

$$Th^l = \{f : \mathbb{C}^N \to \mathbb{C} \text{ holomorphic} \mid f(y+e_i) = f(y), f(y+\tau e_i) = \exp(-\pi \mathrm{i} l\tau - 2\pi \mathrm{i} l y_i)f(y)\}.$$

Then the S_N-invariant subspace is spanned by the $\binom{N+l}{l}$ $\hat{\mathfrak{gl}}_N$-characters of level l. This defines a line bundle L on E^N.

All in all, we may twist the vector bundle V_N by L, obtaining $V_N \otimes L \to E^N$ and we have an action of S_N giving the descent data for a vector bundle W on the quotient stack $E^N \mathbin{/\mkern-6mu/} S_N$. Its space of meromorphic sections gives an object in C_B via $L_N(z)$. This defines a functor from quasi-coherent modules on $E^N \mathbin{/\mkern-6mu/} S_N$ to C_B.

Define the space

$$\Theta^l_{\ell,N} = \{f : \mathbb{C}^N \xrightarrow{\text{mer.}} (\mathbb{C}^\ell)^{\otimes N} \mid f(y_i+1) = A_i^{-1}f(y_i), f(y_i+\tau) = \exp(-\pi \mathrm{i} l\tau - 2\pi \mathrm{i} l y_i)B_i^{-1}f(y_i)\}.$$

These are sections of a vector bundle on E^N and we have actions

$$E(\mathfrak{gl}_\ell) \curvearrowright \Theta^\bullet_{\ell,N} \curvearrowleft S_N \ltimes Th^\bullet$$

via $L_N(z)$, permutations acting via R-matrices, and theta functions acting by scalar multiplication. This is graded by the level l. We now want to compute $\Theta^l_{\ell,N} \otimes_{S_N \ltimes Th^l} Th^l$. This is done by projecting out the action of S_N on $\Theta^\bullet_{\ell,N}$. This is the Hilbert space of the elliptic spin RS model, you might call it the space of non-abelian characters of $A^{(1)}_{N-1}$ graded by level. Then we let Ruijsenaars difference operators act after Hasegawa. The reason this also acts on the sections of the vector bundle is the existence of a connection.

1.1.5 Generalized Schur-Weyl duality

In general, we would like to construct a Schur-Weyl duality for any bundle of conformal blocks for any genus. Braid/Hecke generators are obtained as monodromies along the configuration space, which become R-matrices, the coordinates become one set of generators and tangent vectors give a second set of generators. For genus zero, this gives the Schur-Weyl duality between the loop Yangian and the degenerate double affine Hecke algebra, while for genus one, this gives a Schur-Weyl duality between a degenerate elliptic double affine Hecke algebra and a loop elliptic quantum group $LE(\mathfrak{gl}_\ell)$.

1.2 Loop Yangian

1.2.1 As quantization of rational spin RS Poisson algebra

[AF98]

The Hamiltonian reduction in the classical case is done as follows: Start with $(A, g, S) \in \mathfrak{gl}^*_N \times GL_N \times \mathfrak{gl}^*_N$ and factorize $S_{ij} = \sum_\alpha a^\alpha_i b^\alpha_j$, defining $S^{\alpha\beta}_{ij} := a^\alpha_i b^\beta_j$, where α, β can range in $1, ..., \ell$. Then

$$T^{\alpha\beta}(z) = \delta^{\alpha\beta} + \mathrm{tr}\,\frac{S^{\alpha\beta}}{z - A} = \delta^{\alpha\beta} + \sum_{n\geq 0} T^{\alpha\beta}_n z^{-n-1}, \quad T^{\alpha\beta}_n = \mathrm{tr}\, A^n S^{\alpha\beta}$$

generates the classical Yangian. Letting $J^{\alpha\beta}_n := \mathrm{tr}\, g^n S^{\alpha\beta}$, we define

$$J^{\alpha\beta}(z) = \sum_{n=-\infty}^{\infty} J^{\alpha\beta}_n z^{-n-1},$$

which generates the classical loop algebra. On the reduced phase space, they are also given by

$$J^{\alpha\beta}_n = \sum_{ij} (\mathbf{L}^{n-1})_{ij} \mathbf{a}^\alpha_j \mathbf{c}^\beta_i,$$

where $\mathbf{L}, \mathbf{a}, \mathbf{c}$ are the invariant versions of $L = TgT^{-1}, a$, and c (T begin the diagonalizer for A). This makes it clear how the Lax matrix corresponds to the monodromy around a loop. Thus, it is known that

$$J^{\alpha\beta}_1 = \sum_i S^{\beta\alpha}_i, \quad S^{\alpha\beta}_i = \mathbf{c}^\alpha_i \mathbf{a}^\beta_i, \quad \mathbf{c}^\alpha_i = \sum_\beta S^{\alpha\beta}_i, \quad \mathbf{a}^\alpha_i = \frac{S^{\beta\alpha}_i}{\sum_\gamma S^{\beta\gamma}_i}$$

Recall that the loop Yangian $LY(\mathfrak{gl}_\ell)$ is Schur-Weyl dual to the degenerate double affine Hecke algebra $\ddot{H}_N$ and that the center of the Yangian $Y(\mathfrak{gl}_\ell)$ generated by the quantum determinant gives Hamiltonians for the quantum trigonometric spin CM model, while the center of the loop algebra $L(\mathfrak{gl}_\ell)$ gives Hamiltonians for the quantum rational spin RS model. This gives natural quantizations to $T_n^{\alpha\beta}$ and $J_n^{\alpha\beta}$. The formula for $J_1^{\alpha\beta}$ suggests the quantization rule

$$S_i^{\alpha\beta} \to e_i^{\alpha\beta} \otimes X_i,$$

where $e_i^{\alpha\beta}$ is a matrix unit acting on the ith tensorand and X_i is the ith Laurent generator of $\ddot{H}_N$. The following Poisson rules remain to be checked:

$$\{S_i^{\alpha\beta}, S_j^{\mu\nu}\} = \frac{1}{y_i - y_j}(S_i^{\mu\beta} S_j^{\alpha\nu} + S_i^{\alpha\nu} S_j^{\mu\beta}) - \frac{\delta^{\beta\mu}}{y_i - y_j + \eta}(S_i S_j)^{\alpha\nu} + \frac{\delta^{\alpha\nu}}{y_j - y_i + \eta}(S_j S_i)^{\mu\beta}$$

and $\{y_i, S_j^{\alpha\beta}\} = S_j^{\alpha\beta}\delta_{ij}$. This second rule follows directly, the first is very non-trivial and should be checked using Mathematica. The first term seems unusual, but the last two terms look like the usual commutation relations in $\mathfrak{gl}_\ell$.

1.2.2 Fock space representation

[Kod16]

We construct the level one Fock space. Let $U = \mathbb{C}^\ell[X^{\pm 1}]$. Note that we have an isomorphism

$$(\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} \mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}] \cong \bigwedge^N U, \quad X_1^{m_1} \cdots X_N^{m_N} \otimes (e_{j_1} \otimes \cdots \otimes e_{j_N}) \mapsto e_{j_1} X^{m_1} \wedge \cdots \wedge e_{j_N} X^{m_j}.$$

This clearly has an action of the affine Yangian by Schur-Weyl duality. Note that we recover the trigonometric Calogero-Moser system. The Fock space is obtained by the inverse limit over N respecting a certain grading.

We clearly have an R-matrix $R(y_1 - y_2) \in \mathrm{End}(U^{\otimes 2})$ acting via matrix-differential operators. This should generalize to an R-matrix in $\mathrm{End}((\bigwedge^N U)^{\otimes 2})$. If the affine Yangian acts faithfully, we may obtain an RTT presentation this way.

1.2.3 Conjectural presentation from 4d CS

There is the Gauss decomposition $T(z) = F(z)H(z)E(z)$ constructed from quantum minors with diagonal $H(z)$. The coefficients are $f_{ij}(z), h_i(z), e_{ij}(z)$. Let $e_i(z) = e_{i,i+1}(z)$ and $f_i = f_{i+1,i}(z)$. Then we can present the Yangian in Drinfeld form using the coefficients of $f_i(z), h_i(z), e_i(z)$.

We know that the representations of the Yangian control Wilson lines. 't Hooft lines on the other hand modify the underlying principal bundle, for example by introducing a twist g (a conjugacy class of $\mathfrak{gl}_\ell$).

We should have that $S^1 \times *$ gets mapped to the 2-category $\mathcal{B}Y(\mathfrak{gl}_\ell)\mathsf{Mod}$, while $* \times S^1$ gets mapped to the 2-category $\mathcal{B}U(\dot{\mathfrak{gl}}_\ell)\mathsf{Mod}$ and $S^1 \times S^1$ gets mapped to $Y(\dot{\mathfrak{gl}}_\ell)\mathsf{Mod}$. Let A be the annulus with the outer boundary component marked. Then $A \times S^1$ should give the $Y(\dot{\mathfrak{gl}}_\ell)$-module $\mathbb{C}^\ell[y]$. On the other hand, $S^1 \times A$ should give the $Y(\dot{\mathfrak{gl}}_\ell)$-module $\mathbb{C}^\ell[X^{\pm 1}]$. From this, we have to be able to derive relations for the affine Yangian in the spirit of [CWY18].

Let us encode the lowest level bordisms (unions of $S^1 \times *$ and $* \times S^1$) as pairs $[N, M]$ of natural numbers. 1-morphisms $[N, M] \to [N', M]$ is combinatorially represented as (Γ, σ) with a composition $\Gamma : N \to N'$ with a permutation $\sigma \in S_M$ (a union of N' orderedly punctured discs times the permutation bordism). This however disregards closed 1-morphisms of the form $S^1 \times S^1$. Then 2-morphisms are mapping classes of surfaces, which includes permutation of punctures and Dehn twists. The field theory gives a map

$$\mathrm{End}((S^1 \times S^1)^{\sqcup N}) \to S\ddot{H}_N.$$

This tells us that the elements of the affine Yangian do not come from the plain field theory itself but from operators (natural transformations) on the field theory.

A Wilson line with incoming state e_i and outgoing state e_j on $A \times S^1 \times [0, 1]$ along $[0, 1]$ gives an operator $T_{ij}(z) = \sum_{r \in \mathbb{N}} T_{ij}^{(n)} z^{-r} : \mathbb{C}^\ell[y] \to \mathbb{C}^\ell[y]$ determined by the R-matrix. Similarly, a 't Hooft line on $[0, 1] \times S^1 \times A$ gives an operator $J_{ij}(q) = \sum_{n \in \mathbb{Z}} J_{ij}^{(n)} q^{-n-1} : \mathbb{C}^\ell[X^{\pm 1}] \to \mathbb{C}^\ell[X^{\pm 1}]$ determined by

$$J_{ij}^{(n)} : e_j \otimes X^k \mapsto e_i \otimes X^{k+n}.$$

These are the Fourier modes of the flux through the line. Both are operators on $A \times A$, so both act on $\mathbb{C}^\ell[y]$ and $\mathbb{C}^\ell[X^{\pm 1}]$ and satisfy non-trivial relations. These give the affine Yangian relations when $N \to \infty$. The mapping classes relate this to Dehn twists with the two deformation parameters mapped to η and $\hbar$. We can cross two T-matrices and get the universal R-matrix of the Yangian. We can also cross two J-matrices and get a trivial R-matrix. The interesting stuff happens when we cross a T-matrix and a J-matrix. In the vector representation $\mathbb{C}^\ell[y]$, this should have the effect of a twist g and a shift $e^{\hbar \partial}$.

All of this can be summarized in a graphical calculus on the cylinder $S^1 \times [0, 1]$ with two types of lines in the axial direction: Wilson and 't Hooft lines. Wilson lines are labeled by a complex coordinate y and 't Hooft lines are labeled by a mode number n. We can act on the diagrams via mapping classes. This gives a graphical way of determining the relations.

What does a crossing of a Wilson line and a 't Hooft line look like? It has the form

$$Q^{(n)}(y) : \mathbb{C}^\ell[y] \otimes \mathbb{C}^\ell[X^{\pm 1}] \to \mathbb{C}^\ell[y] \otimes \mathbb{C}^\ell[X^{\pm 1}]$$

and from the graphical calculus we can derive an RQQ relation, so this has to give a Yangian representation on $\mathbb{C}^\ell[X^{\pm 1}]$, on the other hand an affine algebra representation on $\mathbb{C}^\ell[y]$. One possibility would be

$$Q^{(n)}(y) = (1 - \frac{\eta P}{d - y})_n,$$

taking the degree n piece in the expansion, where d is the Dunkl operator. The expansion is well-defined since d acts nilpotently.

Where do η and $\hbar$ appear in the graphical calculus? We know that η appears as Planck's constant in the framing anomaly affecting Wilson lines, saying $(\mathbb{C}^\ell_y)^\vee = (\mathbb{C}^\ell)^*_{y+\ell\eta}$. What about the framing anomaly for 't Hooft lines? They should not have a framing anomaly, as the affine algebra is pivotal. I would expect $\hbar$ to be connected to the central charge, which appears when commuting two 't Hooft lines. If they are labeled by the identity matrix and mode numbers n, m, we get

$$[1 \otimes t^n, 1 \otimes t^m] = \ell n \delta_{n+m} c,$$

meaning we can extract the central charge as follows:

$$c = \frac{1}{\ell}[1 \otimes t, 1 \otimes t^{-1}]$$

This central charge is zero in the representation for the ratRS model.

This way, we can generate the affine Yangian by taking a triangle of lines and letting one of them take values in the algebra, the others in the vector representation.

Meeting 21.05.24: Try to work at the classical level to uncover the classical affine Yangian by taking the Poisson structure from [AF98] and commuting the current J with the T-matrix. Look at the r-matrix structure and quantize.

The affine algebra relation is

$$\begin{aligned}
[J^{\alpha\beta}(z), J^{\gamma\delta}(w)] &= \sum_{nm} [J^{\alpha\beta}_n, J^{\gamma\delta}_m] z^{-n-1} w^{-m-1} \\
&= \sum_{nm} (\delta_{\beta\gamma} J^{\alpha\delta}_{n+m} - \delta_{\delta\alpha} J^{\gamma\beta}_{n+m}) z^{-n-1} w^{-m-1} \\
&= \sum_{k} (\delta_{\beta\gamma} J^{\alpha\delta}_{k} - \delta_{\delta\alpha} J^{\gamma\beta}_{k}) w^{-k-1} z^{-1} \sum_n (z/w)^{-n} \\
&= \sum_{km} (\delta_{\beta\gamma} J^{\alpha\delta}_{k} - \delta_{\delta\alpha} J^{\gamma\beta}_{k}) z^{-k+m-1} w^{-m-1} \\
&= \sum_{k} (\delta_{\beta\gamma} J^{\alpha\delta}_{k} - \delta_{\delta\alpha} J^{\gamma\beta}_{k}) z^{-k-1} w^{-1} \sum_m (w/z)^{-m}
\end{aligned}$$

or

$$[J_1(z), J_2(w)] = \delta(z/w)[P, J_2(w)] = \delta(w/z)[P, J_1(z)], \quad \delta(z/w) = z^{-1} \sum_{n \in \mathbb{Z}} (z/w)^n.$$

Let us now introduce the crossing matrices

$$X(q) : \mathbb{C}^\ell[y] \otimes \mathbb{C}^\ell[X^{\pm 1}] \to \mathbb{C}^\ell[y] \otimes \mathbb{C}^\ell[X^{\pm 1}].$$

The two types of Wilson/'t Hooft-mixed YBEs tell us that

$$R_{12} X_{13}(q) X_{23}(q) = X_{23}(q) X_{13}(q) R_{12}, \quad [X_{31}(p), X_{32}(q)] = \delta(p/q)[P, X_{32}(q)]$$

Notation

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Erklärung