PhD

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Abstract

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Chapter 1

Notes

1.1 Basics of elliptic structures

1.1.1 Elliptic functions and theta functions

Definition 1.1.1. An elliptic curve E (over $\mathbb{C}$) is a smooth projective curve or Riemann surface of genus 1. These are of the form $E \cong \mathbb{C}/\Lambda$ for $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ for $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$. This does not faithfully parametrize elliptic curves, but $M_{1,1} := \mathbb{H}/SL_2(\mathbb{Z})$ does, where $SL_2(\mathbb{Z})$ acts by Möbius transformations. Algebraically, every elliptic curve may be brought into the form

$$Y^2 Z = 4X(X - Z)(X - \lambda Z),$$

where λ is the λ-invariant of E, which is also not faithful up to an action of $SL_2(\mathbb{Z})/\Gamma(2) \cong S_3$. The invariant combination

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$$

is the j-invariant, yielding a bijection $j : M_{1,1} \to \mathbb{C}$. Stacky points are at $j = 0$ and $j = 12^3 = 1728$, where Λ becomes the lattice of the Eisenstein and Gauss integers, respectively, with automorphism groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ (plus the usual involution $Y \mapsto -Y$, giving $\mathbb{Z}_4$ and $\mathbb{Z}_6$). There is an isomorphism

$$\mathbb{C}/\Lambda \to E(\mathbb{C}), \quad z + \Lambda \mapsto \begin{cases} [\wp(z|\tau) : \wp'(z|\tau) : 1], & z \notin \Lambda \\ [0 : 1 : 0], & z \in \Lambda. \end{cases}$$

An elliptic function is a meromorphic function $f : \mathbb{C} \to \mathbb{C}$ that is Λ-periodic such that it descends to a meromorphic function on E. Theta functions are sections of certain line bundles over E, which may be represented as entire functions $\vartheta : \mathbb{C} \to \mathbb{C}$ satisfying

$$\vartheta(z + 1|\tau) = \vartheta(z|\tau), \quad \vartheta(z + \tau|\tau) = \exp(-\pi\mathrm{i}\tau - 2\pi\mathrm{i}z)\vartheta(z|\tau)$$

This line bundle has in fact only one section up to a prefactor, and this is the Jacobi theta function.

1.1.2 Belavin's elliptic R-matrix

[ES98]

Definition 1.1.2. Let $\xi = \exp(2\pi \mathrm{i}/\ell)$. Define a projectively flat rank ℓ vector bundle on $E = \mathbb{C}/\Lambda$ by the two monodromies

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \xi & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi^{\ell-1} \end{pmatrix}, \quad B = \exp(-\pi \mathrm{i}\tfrac{\ell-1}{\ell}) \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix},$$

satisfying $A^\ell, B^\ell = 1$ and $BA = \xi AB$ giving a flat/holomorphic PGL_ℓ-bundle $P \to E$. Belavin's elliptic R-matrix is the unique R-matrix satisfying the QYBE and unitarity as well as

(i) $R^B(z)$ has simple poles only at $\eta + \Lambda$,

(ii) $R^B(0) = P$,

(iii) $R^B(z+1) = A_1 R^B(z) A_1^{-1} = A_2^{-1} R^B(z) A_2$, $R^B(z+\tau) = B_1 R^B(z) B_1^{-1} = B_2^{-1} R^B(z) B_2$.

In particular, the R-matrix lives on the ℓ-fold cover $\bar{E} = \mathbb{C}/\ell\Lambda$, where P is trivialized. We may view the R-matrix as an element of

$$\operatorname{End} \mathbb{C}^\ell \otimes \Gamma(E, P \ltimes \operatorname{End} \mathbb{C}^\ell),$$

where $\Gamma(E, P \ltimes \operatorname{End} \mathbb{C}^\ell)$ are meromorphic sections of the associated bundle $P \ltimes \operatorname{End} \mathbb{C}^\ell$.

1.1.3 RTT representations

[ES98]

Definition 1.1.3. Define the category C_B whose objects are vector spaces V equipped with an invertible element

$$L(z) \in \operatorname{Mat}_\ell(\operatorname{End} V \otimes \mathcal{M}(\bar{E})),$$

thus having values in $\operatorname{End} \mathbb{C}^\ell \otimes \operatorname{End} V$, satisfying

$$R^B_{12}(z-w) L_1(z) L_2(w) = L_2(w) L_1(z) R^B_{12}(z-w).$$

Morphisms are linear maps $f : V \to V'$ satisfying $\varphi L(z) = L'(z)\varphi$. There is a tensor structure via

$$(V, L(z)) \otimes (V', L'(z)) := (V \otimes V', L_{12}(z) L'_{13}(z))$$

and finite-dimensional objects $(V, L(z))$ have duals $(V^*, L^*(z))$ with $L^*(z) = (L(z)^{-1})^{t_2}$. Belavin's R-matrix ensures the existence of a vector representation $(\mathbb{C}^\ell, R^B(z))$.

1.1.4 Elliptic Drinfeld functor

In order to define an elliptic Drinfeld functor, we first need an analog of the Yangian representation on $\mathbb{C}^\ell[y]$. For fixed z, we can view the coefficients of $R^B(z-y)$ as meromorphic $\operatorname{End}\mathbb{C}^\ell$-valued functions $f_{ij}(y)$ with at most a simple pole at $z-\eta+\Lambda$. They satisfy $f_{ij}(y+1) = \operatorname{Ad}_A^{-1} f_{ij}(y)$ and $f_{ij}(y+\tau) = \operatorname{Ad}_B^{-1} f_{ij}(y)$ and are thus meromorphic sections of the adjoint bundle $\operatorname{Ad} P$. These naturally act on sections of the bundle associated to $\mathbb{C}^\ell$. Let us abbreviate the space of such sections as

$$\Theta := \{f : \mathbb{C} \to \mathbb{C}^\ell \text{ meromorphic} \mid f(y+1) = A^{-1}f(y), f(y+\tau)B^{-1}f(y)\}.$$

We obtain an RTT representation on Θ. More generally, we may define

$$L_N(z) := R^B_{01}(z-y_1)\cdots R^B_{0N}(z-y_N)$$

whose coefficients act on the space

$$\Theta_N := \{f : \mathbb{C}^N \to (\mathbb{C}^\ell)^{\otimes N} \text{ meromorphic} \mid f(y_i+1) = A_i^{-1}f(y_i), f(y_i+\tau)B_i^{-1}f(y_i)\}.$$

These are the sections of a vector bundle $V_N \to E^N$, which can be pulled back to the (η-deformed) configuration space of points of E or even on $M_{1,1+N}$ such that the R-matrix allows us to put an RTT representation on its sections.

There is a commuting action of S_N from the right on Θ_N via R-matrices: $(i\ j) \mapsto R^B_{ij}(y_i-y_j)$, as well as sections of the structure sheaf of the configuration space of points on E. Together, these form a generalization of the degenerate affine Hecke algebra. If these sections of the structure sheaf are replaced by the theta functions of [Has95], we may let the elliptic difference operators act. This space is

$$Th^l = \{f : \mathbb{C}^N \to \mathbb{C} \text{ holomorphic} \mid f(y+e_i) = f(y), f(y+\tau e_i) = \exp(-\pi \mathrm{i} l\tau - 2\pi \mathrm{i} l y_i)f(y)\}.$$

Then the S_N-invariant subspace is spanned by the $\binom{N+l}{l}$ $\hat{\mathfrak{gl}}_N$-characters of level l. This defines a line bundle L on E^N.

All in all, we may twist the vector bundle V_N by L, obtaining $V_N \otimes L \to E^N$ and we have an action of S_N giving the descent data for a vector bundle W on the quotient stack $E^N /\!\!/ S_N$. Its space of meromorphic sections gives an object in C_B via $L_N(z)$. This defines a functor from quasi-coherent modules on $E^N /\!\!/ S_N$ to C_B.

Define the space

$$\Theta^l_{\ell,N} = \{f : \mathbb{C}^N \xrightarrow{\text{mer.}} (\mathbb{C}^\ell)^{\otimes N} \mid f(y_i+1) = A_i^{-1}f(y_i), f(y_i+\tau) = \exp(-\pi \mathrm{i} l\tau - 2\pi \mathrm{i} l y_i)B_i^{-1}f(y_i)\}.$$

These are sections of a vector bundle on E^N and we have actions

$$E(\mathfrak{gl}_\ell) \curvearrowright \Theta^\bullet_{\ell,N} \curvearrowleft S_N \ltimes Th^\bullet$$

via $L_N(z)$, permutations acting via R-matrices, and theta functions acting by scalar multiplication. This is graded by the level l. We now want to compute $\Theta^l_{\ell,N} \otimes_{S_N \ltimes Th^l} Th^l$. This is done by projecting out the action of S_N on $\Theta^\bullet_{\ell,N}$. This is the Hilbert space of the elliptic spin RS model, you might call it the space of non-abelian characters of $A^{(1)}_{N-1}$ graded by level. Then we let Ruijsenaars difference operators act after Hasegawa. The reason this also acts on the sections of the vector bundle is the existence of a connection.

1.1.5 Generalized Schur-Weyl duality

In general, we would like to construct a Schur-Weyl duality for any bundle of conformal blocks for any genus. Braid/Hecke generators are obtained as monodromies along the configuration space, which become R-matrices, the coordinates become one set of generators and tangent vectors give a second set of generators. For genus zero, this gives the Schur-Weyl duality between the loop Yangian and the degenerate double affine Hecke algebra, while for genus one, this gives a Schur-Weyl duality between a degenerate elliptic double affine Hecke algebra and a loop elliptic quantum group $LE(\mathfrak{gl}_\ell)$.

1.2 Loop Yangian

1.2.1 As quantization of rational spin RS Poisson algebra

[AF98]

The Hamiltonian reduction in the classical case is done as follows: Start with $(A, g, S) \in \mathfrak{gl}_N^* \times GL_N \times \mathfrak{gl}_N^*$ and factorize $S_{ij} = \sum_\alpha a_i^\alpha b_j^\alpha$, defining $S_{ij}^{\alpha\beta} := a_i^\alpha b_j^\beta$, where α, β can range in $1, ..., \ell$. Then

$$T^{\alpha\beta}(z) = \delta^{\alpha\beta} + \mathrm{tr}\, \frac{S^{\alpha\beta}}{z - A} = \delta^{\alpha\beta} + \sum_{n \geq 0} T_n^{\alpha\beta} z^{-n-1}, \quad T_n^{\alpha\beta} = \mathrm{tr}\, A^n S^{\alpha\beta}$$

generates the classical Yangian. Letting $J_n^{\alpha\beta} := \mathrm{tr}\, g^n S^{\alpha\beta}$, we define

$$J^{\alpha\beta}(z) = \sum_{n=-\infty}^{\infty} J_n^{\alpha\beta} z^{-n-1},$$

which generates the classical loop algebra. On the reduced phase space, they are also given by

$$J_n^{\alpha\beta} = \sum_{ij} (\mathbf{L}^{n-1})_{ij} \mathbf{a}_j^\alpha \mathbf{c}_i^\beta,$$

where $\mathbf{L}, \mathbf{a}, \mathbf{c}$ are the invariant versions of $L = TgT^{-1}, a$, and c (T begin the diagonalizer for A). This makes it clear how the Lax matrix corresponds to the monodromy around a loop. Thus, it is known that

$$J_1^{\alpha\beta} = \sum_i S_i^{\beta\alpha}, \quad S_i^{\alpha\beta} = \mathbf{c}_i^\alpha \mathbf{a}_i^\beta, \quad \mathbf{c}_i^\alpha = \sum_\beta S_i^{\alpha\beta}, \quad \mathbf{a}_i^\alpha = \frac{S_i^{\beta\alpha}}{\sum_\gamma S_i^{\beta\gamma}}$$

Recall that the loop Yangian $LY(\mathfrak{gl}_\ell)$ is Schur-Weyl dual to the degenerate double affine Hecke algebra $\ddot{H}_N$ and that the center of the Yangian $Y(\mathfrak{gl}_\ell)$ generated by the quantum determinant gives Hamiltonians for the quantum trigonometric spin CM model, while the center of the loop algebra $L(\mathfrak{gl}_\ell)$ gives Hamiltonians for the quantum rational spin RS model. This gives natural quantizations to $T_n^{\alpha\beta}$ and $J_n^{\alpha\beta}$. The formula for $J_1^{\alpha\beta}$ suggests the quantization rule

$$S_i^{\alpha\beta} \to e_i^{\alpha\beta} \otimes X_i,$$

where $e_i^{\alpha\beta}$ is a matrix unit acting on the ith tensorand and X_i is the ith Laurent generator of $\ddot{H}_N$. The following Poisson rules remain to be checked:

$$\{S_i^{\alpha\beta}, S_j^{\mu\nu}\} = \frac{1}{y_i - y_j}(S_i^{\mu\beta} S_j^{\alpha\nu} + S_i^{\alpha\nu} S_j^{\mu\beta}) - \frac{\delta^{\beta\mu}}{y_i - y_j + \eta}(S_i S_j)^{\alpha\nu} + \frac{\delta^{\alpha\nu}}{y_j - y_i + \eta}(S_j S_i)^{\mu\beta}$$

and $\{y_i, S_j^{\alpha\beta}\} = S_j^{\alpha\beta}\delta_{ij}$. This second rule follows directly, the first is very non-trivial and should be checked using Mathematica. The first term seems unusual, but the last two terms look like the usual commutation relations in $\mathfrak{gl}_\ell$.

1.2.2 Fock space representation

[Kod16]

We construct the level one Fock space. Let $U = \mathbb{C}^\ell[X^{\pm 1}]$. Note that we have an isomorphism

$$(\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} \mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}] \cong \bigwedge^N U, \quad X_1^{m_1} \cdots X_N^{m_N} \otimes (e_{j_1} \otimes \cdots \otimes e_{j_N}) \mapsto e_{j_1} X^{m_1} \wedge \cdots \wedge e_{j_N} X^{m_j}.$$

This clearly has an action of the affine Yangian by Schur-Weyl duality. Note that we recover the trigonometric Calogero-Moser system. The Fock space is obtained by the inverse limit over N respecting a certain grading.

We clearly have an R-matrix $R(y_1 - y_2) \in \mathrm{End}(U^{\otimes 2})$ acting via matrix-differential operators. This should generalize to an R-matrix in $\mathrm{End}((\bigwedge^N U)^{\otimes 2})$. If the affine Yangian acts faithfully, we may obtain an RTT presentation this way.

1.2.3 Conjectural presentation from 4d CS

We know that the representations of the Yangian control Wilson lines. 't Hooft lines on the other hand modify the underlying principal bundle, for example by introducing a twist g (a conjugacy class of $\mathfrak{gl}_\ell$).

We should have that $S^1 \times *$ gets mapped to the 2-category $\mathcal{B}Y(\mathfrak{gl}_\ell)\mathsf{Mod}$, while $* \times S^1$ gets mapped to the 2-category $\mathcal{B}U(\dot{\mathfrak{gl}}_\ell)\mathsf{Mod}$ and $S^1 \times S^1$ gets mapped to $Y(\dot{\mathfrak{gl}}_\ell)\mathsf{Mod}$. Let A be the annulus with the outer boundary component marked. Then $A \times S^1$ should give the $Y(\dot{\mathfrak{gl}}_\ell)$-module $\mathbb{C}^\ell[y]$. On the other hand, $S^1 \times A$ should give the $Y(\dot{\mathfrak{gl}}_\ell)$-module $\mathbb{C}^\ell[X^{\pm 1}]$. From this, we have to be able to derive relations for the affine Yangian in the spirit of [CWY18].

Let us encode the lowest level bordisms (unions of $S^1 \times *$ and $* \times S^1$) as pairs $[N, M]$ of natural numbers. 1-morphisms $[N, M] \to [N', M]$ is combinatorially represented as (Γ, σ) with a composition $\Gamma : N \to N'$ with a permutation $\sigma \in S_M$ (a union of N' orderedly punctured discs times the permutation bordism). This however disregards closed 1-morphisms of the form $S^1 \times S^1$. Then 2-morphisms are mapping classes of surfaces, which includes permutation of punctures and Dehn twists. The field theory gives a map

$$\mathrm{End}((S^1 \times S^1)^{\sqcup N}) \to S\ddot{H}_N.$$

This tells us that the elements of the affine Yangian do not come from the plain field theory itself but from operators (natural transformations) on the field theory.

A Wilson line with incoming state e_i and outgoing state e_j on $A \times S^1 \times [0,1]$ along $[0,1]$ gives an operator $T_{ij}(z) = \sum_{r\in\mathbb{N}} T_{ij}^{(n)} z^{-r} : \mathbb{C}^\ell[y] \to \mathbb{C}^\ell[y]$ determined by the R-matrix. Similarly, a 't Hooft line on $[0,1] \times S^1 \times A$ gives an operator $J_{ij}(q) = \sum_{n\in\mathbb{Z}} J_{ij}^{(n)} q^{-n-1} : \mathbb{C}^\ell[X^{\pm 1}] \to \mathbb{C}^\ell[X^{\pm 1}]$ determined by

$$J_{ij}^{(n)} : e_j \otimes X^k \mapsto e_i \otimes X^{k+n}.$$

These are the Fourier modes of the flux through the line. Both are operators on $A \times A$, so both act on $\mathbb{C}^\ell[y]$ and $\mathbb{C}^\ell[X^{\pm 1}]$ and satisfy non-trivial relations. These give the affine Yangian relations when $N \to \infty$. The mapping classes relate this to Dehn twists with the two deformation parameters mapped to η and $\hbar$. We can cross two T-matrices and get the universal R-matrix of the Yangian. We can also cross two J-matrices and get a trivial R-matrix. The interesting stuff happens when we cross a T-matrix and a J-matrix. In the vector representation $\mathbb{C}^\ell[y]$, this should have the effect of a twist g and a shift $e^{\hbar\partial}$.

All of this can be summarized in a graphical calculus on the cylinder $S^1 \times [0,1]$ with two types of lines in the axial direction: Wilson and 't Hooft lines. Wilson lines are labeled by a complex coordinate y and 't Hooft lines are labeled by a mode number n. We can act on the diagrams via mapping classes. This way, we can generate the affine Yangian by taking a triangle of lines and letting one of them take values in the algebra, the others in the vector representation.

Meeting 21.05.24: Try to work at the classical level to uncover the classical affine Yangian by taking the Poisson structure from [AF98] and commuting the current J with the T-matrix. Look at the r-matrix structure and quantize.

The affine algebra relation is

$$\begin{aligned}
[J^{\alpha\beta}(z), J^{\gamma\delta}(w)] &= \sum_{nm} [J_n^{\alpha\beta}, J_m^{\gamma\delta}] z^{-n-1} w^{-m-1} \\
&= \sum_{nm} (\delta_{\beta\gamma} J_{n+m}^{\alpha\delta} - \delta_{\delta\alpha} J_{n+m}^{\gamma\beta}) z^{-n-1} w^{-m-1} \\
&= \sum_{k} (\delta_{\beta\gamma} J_{k}^{\alpha\delta} - \delta_{\delta\alpha} J_{k}^{\gamma\beta}) w^{-k-1} z^{-1} \sum_n (z/w)^{-n} \\
&= \sum_{km} (\delta_{\beta\gamma} J_{k}^{\alpha\delta} - \delta_{\delta\alpha} J_{k}^{\gamma\beta}) z^{-k+m-1} w^{-m-1} \\
&= \sum_{k} (\delta_{\beta\gamma} J_{k}^{\alpha\delta} - \delta_{\delta\alpha} J_{k}^{\gamma\beta}) z^{-k-1} w^{-1} \sum_m (w/z)^{-m}
\end{aligned}$$

or

$$[J_1(z), J_2(w)] = \delta(z/w)[P, J_2(w)] = \delta(w/z)[P, J_1(z)], \quad \delta(z/w) = z^{-1}\sum_{n\in\mathbb{Z}}(z/w)^n.$$

Let us now introduce the crossing matrices

$$Q(z,q) \in \mathrm{End}(\mathbb{C}^\ell[X^{\pm 1}] \otimes \mathbb{C}^\ell[y])$$

The two types of 3rd Reidemeister moves involving both Wilson and 't Hooft lines tell us that Q gives rise to representations of $Y(\mathfrak{gl}_\ell)$ and $\dot{\mathfrak{gl}}_\ell$ on $\mathbb{C}[X^{\pm 1}]$ and $\mathbb{C}[y]$, respectively:

$$R_{12}(z-w)Q_{13}(z,q)Q_{23}(w,q) = Q_{23}(w,q)Q_{13}(z,q)R_{12}(z-w), \quad [Q_{31}(z,p), Q_{32}(z,q)] = \delta(p/q)[P, Q_{32}(q)]$$

We already know that such representations may be constructed via Schur-Weyl duality with $T(z) \mapsto z - d - \eta P$ and $J(q) \mapsto \delta(q/D)P$, where d and D are the Dunkl and Macdonald operators. Hence I propose

$$Q(z,q) := \delta(q/D)(z - d - \eta P) \text{ or } Q(z,q) := \delta(q/X)(z - y - \eta P),$$

where D acts on $\mathbb{C}[y]$ and d acts on $\mathbb{C}[X^{\pm 1}]$. This is well-defined since $D^{\pm 1}$ acts nilpotently. This then generates the relations between T and J in a $T_1(z)Q_{12}(z,q)J_2(q) = J_2(q)Q_{12}(z,q)T_1(z)$ type relation.

Notation

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Erklärung