PhD

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Ort und Datum:	Hamburg im ? 2027

Abstract

Acknowledgments

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Chapter 1

Notes

1.1 Basics of elliptic structures

1.1.1 Elliptic functions and theta functions

Definition 1.1.1. An elliptic curve E (over $\mathbb{C}$) is a smooth projective curve or Riemann surface of genus 1. These are of the form $E \cong \mathbb{C}/\Lambda$ for $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ for $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$. This does not faithfully parametrize elliptic curves, but $M_{1,1} := \mathbb{H}/SL_2(\mathbb{Z})$ does, where $SL_2(\mathbb{Z})$ acts by Möbius transformations. Algebraically, every elliptic curve may be brought into the form

$$Y^2 Z = 4X(X-Z)(X-\lambda Z),$$

where λ is the λ-invariant of E, which is also not faithful up to an action of $SL_2(\mathbb{Z})/\Gamma(2) \cong S_3$. The invariant combination

$$j = 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$$

is the j-invariant, yielding a bijection $j : M_{1,1} \to \mathbb{C}$. Stacky points are at $j = 0$ and $j = 12^3 = 1728$, where Λ becomes the lattice of the Eisenstein and Gauss integers, respectively, with automorphism groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ (plus the usual involution $Y \mapsto -Y$, giving $\mathbb{Z}_4$ and $\mathbb{Z}_6$). There is an isomorphism

$$\mathbb{C}/\Lambda \to E(\mathbb{C}), \quad z + \Lambda \mapsto \begin{cases} [\wp(z|\tau) : \wp'(z|\tau) : 1], & z \notin \Lambda \\ [0:1:0], & z \in \Lambda. \end{cases}$$

An elliptic function is a meromorphic function $f : \mathbb{C} \to \mathbb{C}$ that is Λ-periodic such that it descends to a meromorphic function on E. Theta functions are sections of certain line bundles over E, which may be represented as entire functions $\vartheta : \mathbb{C} \to \mathbb{C}$ satisfying

$$\vartheta(z+1|\tau) = \vartheta(z|\tau), \quad \vartheta(z+\tau|\tau) = \exp(-\pi \mathrm{i}\tau - 2\pi\mathrm{i}z)\vartheta(z|\tau)$$

This line bundle has in fact only one section up to a prefactor, and this is the Jacobi theta function.

1.1.2 Belavin's elliptic R-matrix

[ES98]

Definition 1.1.2. Let $\xi = \exp(2\pi \mathrm{i}/\ell)$. Define a projectively flat rank ℓ vector bundle on $E = \mathbb{C}/\Lambda$ by the two monodromies

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \xi & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi^{\ell-1} \end{pmatrix}, \quad B = \exp(-\pi \mathrm{i}\tfrac{\ell-1}{\ell}) \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix},$$

satisfying $A^\ell, B^\ell = 1$ and $BA = \xi AB$ giving a flat/holomorphic PGL_ℓ-bundle $P \to E$. Belavin's elliptic R-matrix is the unique R-matrix satisfying the QYBE and unitarity as well as

(i) $R^B(z)$ has simple poles only at $\eta + \Lambda$,

(ii) $R^B(0) = P$,

(iii) $R^B(z+1) = A_1 R^B(z) A_1^{-1} = A_2^{-1} R^B(z) A_2$, $R^B(z+\tau) = B_1 R^B(z) B_1^{-1} = B_2^{-1} R^B(z) B_2$.

In particular, the R-matrix lives on the ℓ-fold cover $\bar{E} = \mathbb{C}/\ell\Lambda$, where P is trivialized. We may view the R-matrix as an element of

$$\operatorname{End}\mathbb{C}^\ell \otimes \Gamma(E, P \ltimes \operatorname{End}\mathbb{C}^\ell),$$

where $\Gamma(E, P \ltimes \operatorname{End}\mathbb{C}^\ell)$ are meromorphic sections of the associated bundle $P \ltimes \operatorname{End}\mathbb{C}^\ell$.

1.1.3 RTT representations

[ES98]

Definition 1.1.3. Define the category C_B whose objects are vector spaces V equipped with an invertible element

$$L(z) \in \operatorname{Mat}_\ell(\operatorname{End} V \otimes \mathcal{M}(\bar{E})),$$

thus having values in $\operatorname{End}\mathbb{C}^\ell \otimes \operatorname{End} V$, satisfying

$$R^B_{12}(z-w) L_1(z) L_2(w) = L_2(w) L_1(z) R^B_{12}(z-w).$$

Morphisms are linear maps $f : V \to V'$ satisfying $\varphi L(z) = L'(z)\varphi$. There is a tensor structure via

$$(V, L(z)) \otimes (V', L'(z)) := (V \otimes V', L_{12}(z) L'_{13}(z))$$

and finite-dimensional objects $(V, L(z))$ have duals $(V^*, L^*(z))$ with $L^*(z) = (L(z)^{-1})^{t_2}$. Belavin's R-matrix ensures the existence of a vector representation $(\mathbb{C}^\ell, R^B(z))$.

1.1.4 Elliptic Drinfeld functor

In order to define an elliptic Drinfeld functor, we first need an analog of the Yangian representation on $\mathbb{C}^\ell[y]$. For fixed z, we can view the coefficients of $R^B(z-y)$ as meromorphic $\operatorname{End}\mathbb{C}^\ell$-valued functions $f_{ij}(y)$ with at most a simple pole at $z-\eta+\Lambda$. They satisfy $f_{ij}(y+1) = \operatorname{Ad}_A^{-1} f_{ij}(y)$ and $f_{ij}(y+\tau) = \operatorname{Ad}_B^{-1} f_{ij}(y)$ and are thus meromorphic sections of the adjoint bundle $\operatorname{Ad} P$. These naturally act on sections of the bundle associated to $\mathbb{C}^\ell$. Let us abbreviate the space of such sections as

$$\Theta := \{f : \mathbb{C} \to \mathbb{C}^\ell \text{ meromorphic} \mid f(y+1) = A^{-1}f(y), f(y+\tau)B^{-1}f(y)\}.$$

We obtain an RTT representation on Θ. More generally, we may define

$$L_N(z) := R^B_{01}(z-y_1)\cdots R^B_{0N}(z-y_N)$$

whose coefficients act on the space

$$\Theta_N := \{f : \mathbb{C}^N \to (\mathbb{C}^\ell)^{\otimes N} \text{ meromorphic} \mid f(y_i+1) = A_i^{-1}f(y_i), f(y_i+\tau)B_i^{-1}f(y_i)\}.$$

These are the sections of a vector bundle $V_N \to E^N$, which can be pulled back to the (η-deformed) configuration space of points of E or even on $M_{1,1+N}$ such that the R-matrix allows us to put an RTT representation on its sections.

There is a commuting action of S_N from the right on Θ_N via R-matrices: $(i\ j) \mapsto R^B_{ij}(y_i - y_j)$, as well as sections of the structure sheaf of the configuration space of points on E. Together, these form a generalization of the degenerate affine Hecke algebra. If these sections of the structure sheaf are replaced by the theta functions of [Has95], we may let the elliptic difference operators act. This space is

$$Th^l = \{f : \mathbb{C}^N \to \mathbb{C} \text{ holomorphic} \mid f(y+e_i) = f(y), f(y+\tau e_i) = \exp(-\pi \mathrm{i} l\tau - 2\pi \mathrm{i} l y_i)f(y)\}.$$

Then the S_N-invariant subspace is spanned by the $\binom{N+l}{l}$ $\hat{\mathfrak{gl}}_N$-characters of level l. This defines a line bundle L on E^N.

All in all, we may twist the vector bundle V_N by L, obtaining $V_N \otimes L \to E^N$ and we have an action of S_N giving the descent data for a vector bundle W on the quotient stack $E^N /\!\!/ S_N$. Its space of meromorphic sections gives an object in C_B via $L_N(z)$. This defines a functor from quasi-coherent modules on $E^N /\!\!/ S_N$ to C_B.

Define the space

$$\Theta^l_{\ell,N} = \{f : \mathbb{C}^N \xrightarrow{\text{mer.}} (\mathbb{C}^\ell)^{\otimes N} \mid f(y_i+1) = A_i^{-1}f(y_i), f(y_i+\tau) = \exp(-\pi \mathrm{i} l\tau - 2\pi \mathrm{i} l y_i)B_i^{-1}f(y_i)\}.$$

These are sections of a vector bundle on E^N and we have actions

$$E(\mathfrak{gl}_\ell) \curvearrowright \Theta^\bullet_{\ell,N} \curvearrowleft S_N \ltimes Th^\bullet$$

via $L_N(z)$, permutations acting via R-matrices, and theta functions acting by scalar multiplication. This is graded by the level l. We now want to compute $\Theta^l_{\ell,N} \otimes_{S_N \ltimes Th^l} Th^l$. This is done by projecting out the action of S_N on $\Theta^\bullet_{\ell,N}$. This is the Hilbert space of the elliptic spin RS model, you might call it the space of non-abelian characters of $A^{(1)}_{N-1}$ graded by level. Then we let Ruijsenaars difference operators act after Hasegawa. The reason this also acts on the sections of the vector bundle is the existence of a connection.

1.1.5 Generalized Schur-Weyl duality

In general, we would like to construct a Schur-Weyl duality for any bundle of conformal blocks for any genus. Braid/Hecke generators are obtained as monodromies along the configuration space, which become R-matrices, the coordinates become one set of generators and tangent vectors give a second set of generators. For genus zero, this gives the Schur-Weyl duality between the loop Yangian and the degenerate double affine Hecke algebra, while for genus one, this gives a Schur-Weyl duality between a degenerate elliptic double affine Hecke algebra and a loop elliptic quantum group $LE(\mathfrak{gl}_\ell)$.

Instead of coordinates plus tangent vectors, the generators can also come from the product of two surfaces, going into 4d CS theory.

1.2 Loop Yangian

1.2.1 As quantization of rational spin RS Poisson algebra

[AF98]

The Hamiltonian reduction in the classical case is done as follows: Start with $(A, g, S) \in \mathfrak{gl}^*_N \times GL_N \times \mathfrak{gl}^*_N$ and factorize $S_{ij} = \sum_\alpha a^\alpha_i b^\alpha_j$, defining $S^{\alpha\beta}_{ij} := a^\alpha_i b^\beta_j$, where α, β can range in $1, ..., \ell$. Then

$$T^{\alpha\beta}(z) = \delta^{\alpha\beta} + \operatorname{tr} \frac{S^{\alpha\beta}}{z - A} = \delta^{\alpha\beta} + \sum_{n \geq 0} T^{\alpha\beta}_n z^{-n-1}, \quad T^{\alpha\beta}_n = \operatorname{tr} A^n S^{\alpha\beta}$$

generates the classical Yangian. Letting $J^{\alpha\beta}_n := \operatorname{tr} g^n S^{\alpha\beta}$, we define

$$J^{\alpha\beta}(z) = \sum_{n=-\infty}^{\infty} J^{\alpha\beta}_n z^{-n-1},$$

which generates the classical loop algebra. On the reduced phase space, they are also given by

$$J^{\alpha\beta}_n = \sum_{ij} (\mathbf{L}^{n-1})_{ij} \mathbf{a}^\alpha_j \mathbf{c}^\beta_i,$$

where $\mathbf{L}, \mathbf{a}, \mathbf{c}$ are the invariant versions of $L = TgT^{-1}, a$, and c (T begin the diagonalizer for A). This makes it clear how the Lax matrix corresponds to the monodromy around a loop. Thus, it is known that

$$J_1^{\alpha\beta} = \sum_i S_i^{\beta\alpha}, \quad S_i^{\alpha\beta} = \mathbf{c}_i^\alpha \mathbf{a}_i^\beta, \quad \mathbf{c}_i^\alpha = \sum_\beta S_i^{\alpha\beta}, \quad \mathbf{a}_i^\alpha = \frac{S_i^{\beta\alpha}}{\sum_\gamma S_i^{\beta\gamma}}$$

Recall that the loop Yangian $LY(\mathfrak{gl}_\ell)$ is Schur-Weyl dual to the degenerate double affine Hecke algebra $\ddot{H}_N$ and that the center of the Yangian $Y(\mathfrak{gl}_\ell)$ generated by the quantum determinant gives Hamiltonians for the quantum trigonometric spin CM model, while the center of the loop algebra $L(\mathfrak{gl}_\ell)$ gives Hamiltonians for the quantum rational spin RS model. This gives natural quantizations to $T_n^{\alpha\beta}$ and $J_n^{\alpha\beta}$. The formula for $J_1^{\alpha\beta}$ suggests the quantization rule

$$S_i^{\alpha\beta} \to e_i^{\alpha\beta} \otimes X_i,$$

where $e_i^{\alpha\beta}$ is a matrix unit acting on the ith tensorand and X_i is the ith Laurent generator of $\ddot{H}_N$. The following Poisson rules remain to be checked:

$$\{S_i^{\alpha\beta}, S_j^{\mu\nu}\} = \frac{1}{y_i - y_j}(S_i^{\mu\beta}S_j^{\alpha\nu} + S_i^{\alpha\nu}S_j^{\mu\beta}) - \frac{\delta^{\beta\mu}}{y_i - y_j + \eta}(S_i S_j)^{\alpha\nu} + \frac{\delta^{\alpha\nu}}{y_j - y_i + \eta}(S_j S_i)^{\mu\beta}$$

and $\{y_i, S_j^{\alpha\beta}\} = S_j^{\alpha\beta}\delta_{ij}$. This second rule follows directly, the first is very non-trivial and should be checked using Mathematica. The first term seems unusual, but the last two terms look like the usual commutation relations in $\mathfrak{gl}_\ell$.

1.2.2 Fock space representation

[Kod16]

We construct the level one Fock space. Let $U = \mathbb{C}^\ell[X^{\pm 1}]$. Note that we have an isomorphism

$$(\mathbb{C}^\ell)^{\otimes N} \otimes_{S_N} \mathbb{C}[X_1^{\pm1}, ..., X_N^{\pm1}] \cong \bigwedge^N U, \quad X_1^{m_1} \cdots X_N^{m_N} \otimes (e_{j_1} \otimes \cdots \otimes e_{j_N}) \mapsto e_{j_1}X^{m_1} \wedge \cdots \wedge e_{j_N} X^{m_j}.$$

This clearly has an action of the affine Yangian by Schur-Weyl duality. Note that we recover the trigonometric Calogero-Moser system. The Fock space is obtained by the inverse limit over N respecting a certain grading.

We clearly have an R-matrix $R(y_1 - y_2) \in \mathrm{End}(U^{\otimes 2})$ acting via matrix-differential operators. This should generalize to an R-matrix in $\mathrm{End}((\bigwedge^N U)^{\otimes 2})$. If the affine Yangian acts faithfully, we may obtain an RTT presentation this way.

1.2.3 Conjectural presentation from 4d CS

Consider 4d CS theory on $\mathbb{P}^1 \times S^1 \times [0, 1]$. Wilson lines at constant $z \in \mathbb{P}^1$ can be represented pictorially on a 2d surface with a seam, giving the cylinder after gluing. The Yangian gives the 2-Hilbert space for $D^\times \times *$ and the relations can be reconstructed from lines on a cylinder: The R-matrix is a crossing of two normal lines labeled by the vector representation of $\mathfrak{gl}_\ell$, the T-matrix is a crossing of a normal line and a wavy line. The normal line crossing the seam gives the twist matrix g parametrizing the background GL_ℓ-principal bundle (Maybe add $e^{\hbar\partial}$). A wavy line crossing the seam gives an additional element $A(z)$, and we can derive an analog of the RTT relation:

$$gT(z-y)A(z) = A(z)T(z-y)g.$$

Writing

$$T(z-y) = \sum_{m=0}^{\infty} T^{(m)}(z-y)^{-m-1} = \sum_{m=0}^{\infty}\sum_{k=m+1}^{\infty} \frac{(k-m)\cdots(k-1)}{m!} T^{(m)} y^{k-m-1} z^{-k}$$

and

$$A(z) = \sum_{n=-\infty}^{\infty} a^{(n)} z^{-n-1}$$

gives

$$\sum_{m=0}^{\infty}\sum_{k=m+1}^{\infty}\sum_{n=-\infty}^{\infty} \frac{(k-m)\cdots(k-1)}{m!} gT^{(m)}a^{(n)}y^{k-m-1}z^{-k-n-1} = \sum_{m=0}^{\infty}\sum_{k=m+1}^{\infty}\sum_{n=-\infty}^{\infty} \frac{(k-m)\cdots(k-1)}{m!} a^{(n)}T^{(m)}gy$$

which would imply $gT^{(m)}a^{(n)} = a^{(n)}T^{(m)}g$, so the coefficients satisfy

$$\gamma_i t_{ij}^{(m)} a^{(n)} = a^{(n)} t_{ij}^{(m)} \gamma_j.$$

Hence we might think of the twist as a kind of R-matrix commuting $A(z)$ and $T(z-y)$. We have the consistency condition $R(z)(g\otimes g) = (g\otimes g)R(z)$ plus invertibility. $A(z)$ also has to be invertible.

There is the Gauss decomposition $T(z) = F(z)H(z)E(z)$ constructed from quantum minors with diagonal $H(z)$. The coefficients are $f_{ij}(z), h_i(z), e_{ij}(z)$. Let $e_i(z) = e_{i,i+1}(z)$ and $f_i = f_{i+1,i}(z)$. Then we can present the Yangian in Drinfeld form using the coefficients of $f_i(z), h_i(z), e_i(z)$.

Notation

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Erklärung