Proofnote: The Riemann–Pavlov Equation

Contents

1 Main Text (minimal scaffold)

1.1 Gamma representation

$$\Gamma\Big(\frac{s}{2}\Big) = \int_0^\infty t^{\frac{s}{2}-1} e^{-t}\, dt \tag{1}$$

$$\Gamma\Big(\frac{s}{2}\Big) = 2\int_0^\infty x^{s-1} e^{-x^2}\, dx \tag{2}$$

1.2 Riemann–Pavlov Hamiltonian

$$\hat{H}_{\mathrm{RP}} = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) + i\lambda\Big(\hat{x}e^{-\hat{x}^2} + \epsilon \sin\hat{x}\Big) \tag{3}$$

1.3 Eigen-equation and asymptotics

$$\Big[-ix\frac{d}{dx}-\frac{i}{2}+i\lambda\big(xe^{-x^2}+\epsilon\sin x\big)\Big]\psi(x)=E\psi(x) \tag{7}$$

$$\psi(x)\propto x^{-1/2+iE}\,\exp\Big(-\frac{\lambda\sqrt{\pi}}{2}\,\mathrm{erf}(x)+\lambda\epsilon\,\mathrm{Si}(x)\Big) \tag{8}$$

$$\lim_{x\to\infty}|\psi(x)|\sim\cdots \tag{9}$$

1.4 Section 2.2: The Pavlov Lock (state selection + PT metric)

$$w_{\mathrm{conf}}(x):=e^{x^2},\qquad \mathcal{D}_{\mathrm{phys}}:=\Big\{\psi:\int_{\mathbb{R}}|\psi(x)|^2\,w_{\mathrm{conf}}(x)\,dx<\infty\Big\}. \tag{1}$$

$$\eta_{\mathrm{PT}}(x)=\exp\Big(-\lambda\sqrt{\pi}\,\mathrm{erf}(x)-2\lambda\epsilon\,\mathrm{Si}(x)\Big) \tag{10}$$

A PT-Metric, Pseudo-Hermiticity, and Hermitian Equivalence

A.1 Setup

Let $\mathcal{H}:=L^2(\mathbb{R},dx)$ with the standard inner product $\langle\psi,\phi\rangle=\int_{\mathbb{R}}\psi^*(x)\phi(x)\,dx$. Write $p:=-i\frac{d}{dx}$ and let x denote multiplication. Define the symmetric dilation generator

$$\hat{H}_0:=\frac{1}{2}(\hat{x}\hat{p}+\hat{p}\hat{x})=-i\Big(x\frac{d}{dx}+\frac{1}{2}\Big). \tag{A.1}$$

Eq. (3) is $\hat{H}_{\mathrm{RP}}=\hat{H}_0+i\lambda f(x)$ with $f(x)=xe^{-x^2}+\epsilon\sin x$.

A.2 Similarity calculus for multiplication metrics

Lemma A.1 (Similarity identities). *Let $\eta=\eta(x)>0$ be C^1 and act by multiplication. On any common dense domain stable under multiplication by $\eta^{\pm1}$,*

$$\eta\,p\,\eta^{-1}=p+i(\ln\eta)', \tag{A.2}$$
$$\eta\,\hat{H}_0\,\eta^{-1}=\hat{H}_0+i\,x(\ln\eta)', \tag{A.3}$$
$$\eta\,(i\lambda f)\,\eta^{-1}=i\lambda f. \tag{A.4}$$

Hence

$$\eta\,\hat{H}_{\mathrm{RP}}\,\eta^{-1}=\hat{H}_0+i\,x(\ln\eta)'+i\lambda f(x). \tag{A.5}$$

A.3 ODE for η-self-adjointness and its closed form

Lemma A.2 (Pseudo-Hermiticity ODE and solution). *Assume $\hat{H}_0^\dagger=\hat{H}_0$ on the chosen domain. Then $\hat{H}_{\mathrm{RP}}^\dagger=\hat{H}_0-i\lambda f$ and a multiplication metric $\eta=\eta(x)>0$ satisfies*

$$\hat{H}_{\mathrm{RP}}^\dagger=\eta\,\hat{H}_{\mathrm{RP}}\,\eta^{-1} \tag{A.6}$$

if and only if

$$x(\ln\eta)'=-2\lambda f(x)=-2\lambda\big(xe^{-x^2}+\epsilon\sin x\big)\qquad(x\neq0), \tag{A.7}$$

with continuous extension at $x=0$. The normalized solution is exactly Eq. (10)*.*

Lemma A.3 (Boundedness and bounded invertibility). *η_{PT} is bounded and boundedly invertible. Using $|\mathrm{erf}(x)|\le1$ and $|\mathrm{Si}(x)|\le\frac{\pi}{2}$,*

$$e^{-M}\le\eta_{\mathrm{PT}}(x)\le e^{M},\qquad M:=|\lambda|\sqrt{\pi}+|\lambda\epsilon|\pi. \tag{A.8}$$

A.4 Hermitian equivalence via $\rho = \eta_{\mathrm{PT}}^{1/2}$

Define

$$\rho := \eta_{\mathrm{PT}}^{1/2} = \exp\Big(- \tfrac{\lambda}{2}\sqrt{\pi}\,\mathrm{erf}(x) - \lambda\epsilon\,\mathrm{Si}(x)\Big). \tag{A.9}$$

Theorem A.4 (Hermitian equivalence)**.** *Assume $\hat{H}_{\mathrm{RP}}$ is self-adjoint in the metric Hilbert space $\mathcal{H}_{\eta_{\mathrm{PT}}}$ with inner product $\langle \psi, \phi\rangle_{\eta_{\mathrm{PT}}} := \langle \psi, \eta_{\mathrm{PT}}\phi\rangle$. Define*

$$\widetilde{H} := \rho\, \hat{H}_{\mathrm{RP}}\, \rho^{-1}, \qquad \mathcal{D}(\widetilde{H}) := \rho\, \mathcal{D}(\hat{H}_{\mathrm{RP}}). \tag{A.10}$$

Then $\widetilde{H}$ is self-adjoint on $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and

$$\widetilde{H} = \hat{H}_0 \quad \textit{on } \mathcal{D}(\widetilde{H}). \tag{A.11}$$

A.5 Role separation (important for narrative consistency)

The PT-metric η_{PT} guarantees observability/unitarity (real spectrum in the η_{PT}-inner product), but does not by itself force L^2 confinement since it is bounded (Lemma A.3). Confinement/state selection is imposed separately via Eq. (1).

B Determinant Target and Uniqueness Reduction

B.1 Completed zeta on the critical line

Let

$$\Xi(E) := \xi\Big(\frac{1}{2} + iE\Big), \qquad \xi(s) := \frac{1}{2}\, s(s-1)\, \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s). \tag{B.1}$$

Then $\Xi(E)$ is an even real entire function of order 1.

B.2 Normalized determinant (abstract)

Assuming a self-adjoint representative $\widetilde{H}$ (Appendix A), define

$$\Delta_{\widetilde{H}}(E) := \frac{\det_\zeta(\widetilde{H}^2 + E^2)}{\det_\zeta(\widetilde{H}^2)}, \qquad \Delta_{\widetilde{H}}(0) = 1. \tag{B.2}$$

Proposition B.1 (Uniqueness reduction)**.** *If $\Delta_{\widetilde{H}}(E)$ is even entire of order 1 and its zeros coincide with those of $\Xi(E)$ (with multiplicities), then*

$$\Delta_{\widetilde{H}}(E) = \frac{\Xi(E)}{\Xi(0)}. \tag{B.3}$$

C Canonical Arithmetic Observable Algebra and Canonical Renormalized Trace

C.1 Log-coordinate kinematics

Restrict first to $x > 0$ and define the unitary map $\mathcal{U} : L^2(\mathbb{R}_+, dx) \to L^2(\mathbb{R}, dq)$ by

$$(\mathcal{U}\psi)(q) := e^{q/2}\psi(e^q), \qquad q := \log x. \tag{C.1}$$

Then $\mathcal{U}\hat{H}_0\mathcal{U}^{-1} = -i\partial_q$ and for $U(t) := e^{-it\hat{H}_0}$,

$$(\mathcal{U}U(t)\mathcal{U}^{-1}\phi)(q) = \phi(q-t), \qquad \langle q|U(t)|q'\rangle = \delta(q - q' - t). \tag{C.2}$$

C.2 Unitary integer dilations

Define the unitary integer dilation operators on $L^2(\mathbb{R}_+, dx)$ by

$$(V_n\psi)(x) := n^{1/2}\psi(nx), \qquad n \in \mathbb{N}^\times. \tag{C.3}$$

Under $\mathcal{U}$,

$$(\mathcal{U}V_n\mathcal{U}^{-1}\phi)(q) = \phi(q + \log n), \qquad \langle q|V_n|q'\rangle = \delta(q - q' + \log n), \tag{C.4}$$

hence

$$\langle q|V_nU(t)|q\rangle = \delta(t - \log n). \tag{C.5}$$

C.3 Canonical trace density (per unit log-volume)

Let χ_T be multiplication by $\mathbf{1}_{[-T,T]}(q)$ (or a smooth cutoff with the same limit). For operators A for which the limit exists as a tempered distribution, define the trace density

$$\tau(A) := \lim_{T\to\infty} \frac{1}{2T}(\chi_T A \chi_T). \tag{C.6}$$

Lemma C.1 (Atomic time support at $\log n$)**.** *For each $n \in \mathbb{N}^\times$,*

$$\tau(V_nU(t)) = \delta(t - \log n) \qquad \textit{in } \mathcal{S}'(\mathbb{R}_t). \tag{C.7}$$

C.4 Semigroup Banach algebra, Euler product, and von Mangoldt operator

Let $\mathcal{B}$ be the commutative Banach algebra generated by finite linear combinations of $\{V_n\}$ in operator norm. For $\Re s > 1$ define

$$\mathcal{Z}(s) := \sum_{n\geq 1} n^{-s}V_n \in \mathcal{B}, \qquad (\Re s > 1). \tag{C.8}$$

Lemma C.2 (Banach-algebra Euler product)**.** *For $\Re s > 1$,*

$$\mathcal{Z}(s) = \prod_p \left(1 - p^{-s}V_p\right)^{-1} \tag{C.9}$$

as a norm-convergent infinite product in $\mathcal{B}$.

Define the prime generator (von Mangoldt operator) for $\Re s > 1$ by

$$\mathcal{L}(s) := -\partial_s \log \mathcal{Z}(s). \tag{C.10}$$

Then

$$\mathcal{L}(s) = \sum_{n\geq 1} \Lambda(n)n^{-s}V_n = \sum_p \sum_{m\geq 1} (\log p)\, p^{-ms}V_{p^m}, \qquad (\Re s > 1). \tag{C.11}$$

Lemma C.3 (Prime-orbit distribution)**.** *For $\Re s > 1$,*

$$\tau(\mathcal{L}(s)U(t)) = \sum_{n\geq 1} \Lambda(n)n^{-s}\delta(t - \log n). \tag{C.12}$$

C.5 Canonical archimedean subtraction and renormalized trace functional

Let g be even Schwartz on $\mathbb{R}$ and define

$$\widehat{g}(t) := \int_{-\infty}^{\infty} g(E)\, e^{-iEt}\, dE, \qquad g(\hat{H}_0) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(t)\, U(t)\, dt. \tag{C.13}$$

Define the archimedean functional by

$$\tau_{\text{arch}}\big(g(\hat{H}_0)\big) := \frac{1}{2\pi} \int_{-\infty}^{\infty} g(E) \Big(\log \pi - \Re \frac{\Gamma'}{\Gamma} \Big(\frac{1}{4} + \frac{iE}{2} \Big) \Big)\, dE. \tag{C.14}$$

Define the canonical renormalized trace functional by

$$\tau_{\text{ren}}\big(g(\hat{H}_0)\big) := \tau_{\text{arch}}\big(g(\hat{H}_0)\big) \; - \; \frac{1}{\pi} \lim_{s \to \frac{1}{2}^+} \sum_{n \geq 1} \Lambda(n)\, n^{-s}\, \widehat{g}(\log n) \; + \; \mathcal{T}[g], \tag{C.15}$$

where $\mathcal{T}[g]$ is the fixed correction term associated with the trivial-zero/prefactor structure of $\xi(s)$.

D Conditional Non-Experimental Closure and Remaining Physical Validation

D.1 Implementation postulates (the only physically testable inputs)

(H1) Self-adjoint sector. A concrete realization exists in which the Hermitian representative $\widetilde{H}$ is self-adjoint on a dense domain and generates a strongly continuous unitary group $e^{-it\widetilde{H}}$.

(H2) Trace density. A trace density τ (or equivalent renormalized trace) exists on a class containing $V_n U(t)$ and $g(\hat{H}_0)$ for even Schwartz g.

(H3) Discrete scale covariance. The observable algebra contains unitary integer dilations $\{V_n\}$ commuting with the flow, as in Appendix C.

(H4) Canonical subtraction. The renormalization is fixed uniquely by the archimedean matching condition (C.14).

D.2 Closure theorem (determinant level)

Theorem D.1 (Conditional non-experimental closure)**.** *Assume (H1)–(H4). Let* $\Delta(E)$ *be the canonical renormalized determinant induced by* τ_{ren}*. Then*

$$\Delta(E) = \frac{\Xi(E)}{\Xi(0)}. \tag{D.1}$$

D.3 What remains physically

Given Theorem D.1, the remaining work is to realize/justify (H1)–(H4) within a concrete physical platform (e.g. as an observable relative trace / spectral shift in a regulated open system with emergent discrete scale covariance).