CALCUL NUMERIC

TEMA 6

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Grupa 344

Exercitiul 1

Sa se adapteze teorema III.2. pentru cazul unidimensional, i.e. $n = 1$.

$$\left.\begin{aligned} & n = 1 \\ & ||v||_\infty = \max_{i=\overline{1,n}} |v_i| \end{aligned}\right\} \Rightarrow \quad ||G'(x)||_\infty = |G'(x)|$$

<u>**TEOREMA III.2**</u> (cazul unidimensional)

Fie $D \subset \mathbb{R}$ o multime convexa si $G : D \to \mathbb{R}$ o functie vectoriala astfel incat $G(x) \in D,\ \forall x \in D$. Mai mult, daca $|G'(x)| \leq 1,\ \forall x \in D$, atunci G admite un unic punct fix $x^* \in D$, iar sirul definit

$$\boldsymbol{x^{(k)} = G(x^{(k-1)})}, \text{ pentru } k \geq 1 \text{ si } x^{(0)} \in D \text{ arbitrar}$$

este convergent la x^* si are loc urmatoarea estimare:

$$|\boldsymbol{x^{(k)} - x^*}| \leq \boldsymbol{\frac{q^k}{1-q}}|\boldsymbol{x^{(1)} - x^{(0)}}|.$$

Exercitiul 3

Fie ecuatia $f(x) = 0$, unde $f(x) = x^4 + 2x^2 - x - 3$. Se considera functia $g(x) = (3 + x - 2x^2)^{\frac{1}{4}}$.

a. Aratati ca daca x^* este punct fix pentru g, atunci x^* este radacina functiei $f(x)$.

$$\underline{x^*\ punct\ fix\ al\ functiei\ g}$$

$$\Rightarrow\ x^* = g(x^*)$$

$$\Rightarrow\ x^* = \sqrt[4]{3 + x^* - 2{x^*}^2}$$

$$\Rightarrow\ {x^*}^4 = 3 + x^* - 2{x^*}^2$$

$$\Rightarrow\ {x^*}^4 + 2{x^*}^2 - 3 - x^* = 0 = f(x)$$

$$\Rightarrow\ \boldsymbol{x^*\ este\ radacina\ functiei\ f(x)}$$

Exercitiul 3

Fie ecuatia $f(x) = 0$, unde $f(x) = x^4 + 2x^2 - x - 3$. Se considera functia $g(x) = (3 + x - 2x^2)^{\frac{1}{4}}$.

b. Aflati domeniul de definitie $[a, b]$ al functiei g.

$$g : [a, b] \to \mathbb{R}, \quad \text{g(x)} = \sqrt[4]{3 + x - 2x^2}$$

$$\Rightarrow \sqrt[4]{3 + x - 2x^2} \in \mathbb{R}$$

$$\Rightarrow 3 + x - 2x^2 \geq 0$$

$$3 + x - 2x^2 = 0$$

$$\Rightarrow 2x^2 - 2x + 3x + 3 = 0$$

$$\Rightarrow (x + 1)(-2x + 3) = 0$$

$$\Rightarrow \begin{cases} x_1 = -1 \\ x_2 = 3/2 \end{cases}$$

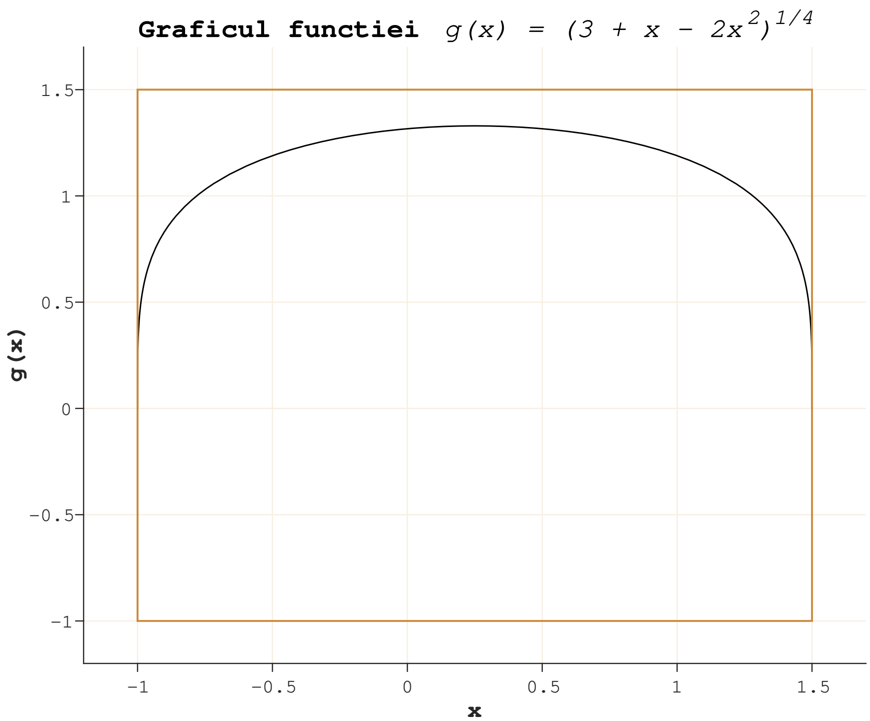
$$\Downarrow$$

$$\boldsymbol{g : [-1,\ 3/2] \to \mathbb{R}}$$

Exercitiul 3

Fie ecuatia $f(x) = 0$, unde $f(x) = x^4 + 2x^2 - x - 3$. Se considera functia $g(x) = (3 + x - 2x^2)^{\frac{1}{4}}$.

c. Construiti graficul functiei g pe intervalul $[a, b]$. Verificati daca $g(x) \in [a, b]$, $\forall x \in [a, b]$.

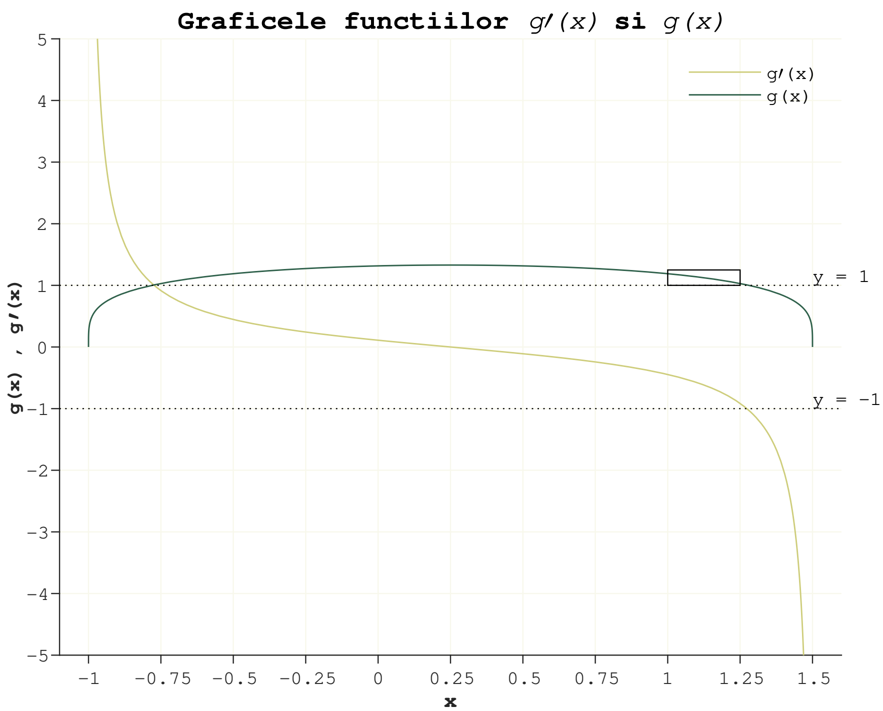


Observam ca $\forall x \in [-1, 3/2]$, obtinem $g(x) \in [-1, 3/2]$.

Exercitiul 3

Fie ecuatia $f(x) = 0$, unde $f(x) = x^4 + 2x^2 - x - 3$. Se considera functia $g(x) = (3 + x - 2x^2)^{\frac{1}{4}}$.

d. Construiti graficul functiei g' pe intervalul $[a, b]$ si doua segmente de dreapta situate pe dreptele $y = 1$ si $y = -1$. Alegeti conform graficului un interval $[a', b'] \subset [a, b]$, astfel incat $|g'(x)| < 1$. Verificati daca $g(x) \in [a', b']$, $\forall x \in [a', b']$.



Observam ca $\forall x \in [1,\ 1.25]$, obtinem $g(x) \in [1,\ 1.25]$ si $|g'(x)| < 1$.

Asadar, functia g admite un unic punct fix pe intervalul $[1,\ 1.25]$, iar sirul aproximarilor $x^{(k)}$ construit in baza functiei g, converge la solutia ecuatiei $f(x) = 0$.

Exercitiul 4.a

Fie ecuatia $f(x) = 0$, unde $f(x) = x^4 + 2x^2 - x - 3$. Se considera functia $g_1(x) = \left(\frac{x+3}{x^2+2}\right)^{\frac{1}{2}}$.

a. Aratati ca daca x^* este punct fix pentru g, atunci x^* este radacina functiei $f(x)$.

$$\underline{x^*\ punct\ fix\ al\ functiei\ g}$$

$$\Rightarrow\ x^* = g(x^*)$$

$$\Rightarrow\ x^* = \sqrt{\frac{x+3}{x^2+2}}$$

$$\Rightarrow\ x^{*^2} = \frac{x+3}{x^2+2}$$

$$\Rightarrow\ x^{*^2}(x^{*^2} + 2) - (x^* + 3) = 0$$

$$\Rightarrow\ x^{*^4} + 2x^{*^2} - 3 - x^* = 0 = f(x)$$

$$\Rightarrow\ \boldsymbol{x^*\ este\ radacina\ functiei\ f(x)}$$

Exercitiul 4.a

Fie ecuatia $f(x) = 0$, unde $f(x) = x^4 + 2x^2 - x - 3$. Se considera functia $g_1(x) = \left(\frac{x+3}{x^2+2}\right)^{\frac{1}{2}}$.

b. Aflati domeniul de definitie $[a, b]$ al functiei g.

$$g : [a, b] \to \mathbb{R}, \quad \mathrm{g(x)} = \sqrt{\frac{\mathrm{x}+3}{\mathrm{x}^2+2}}$$

$$\Rightarrow \sqrt{\frac{x+3}{x^2+2}} \in \mathbb{R}$$

$$\Rightarrow \frac{x+3}{x^2+2} \geq 0$$

$$x^2 + 2 > 0$$

$$\Rightarrow x + 3 \geq 0$$

$$\Rightarrow x \geq -3$$

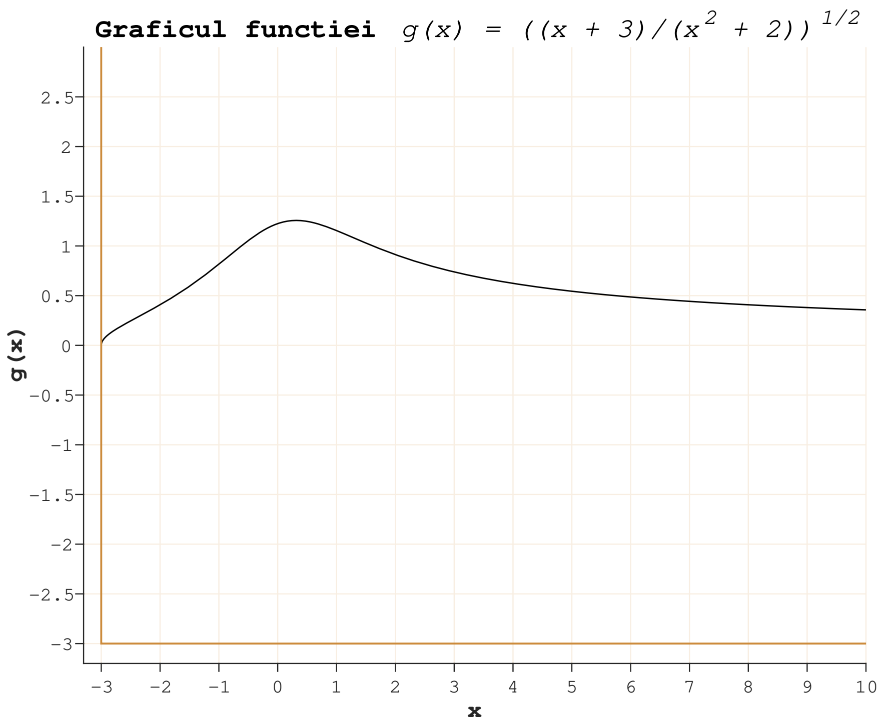
$$\Downarrow$$

$$\boldsymbol{g : [-3, \infty] \to \mathbb{R}}$$

Exercitiul 4.a

Fie ecuatia $f(x) = 0$, unde $f(x) = x^4 + 2x^2 - x - 3$. Se considera functia $g_1(x) = \left(\dfrac{x+3}{x^2+2}\right)^{\frac{1}{2}}$.

c. Construiti graficul functiei g pe intervalul $[a, b]$. Verificati daca $g(x) \in [a, b]$, $\forall x \in [a, b]$.

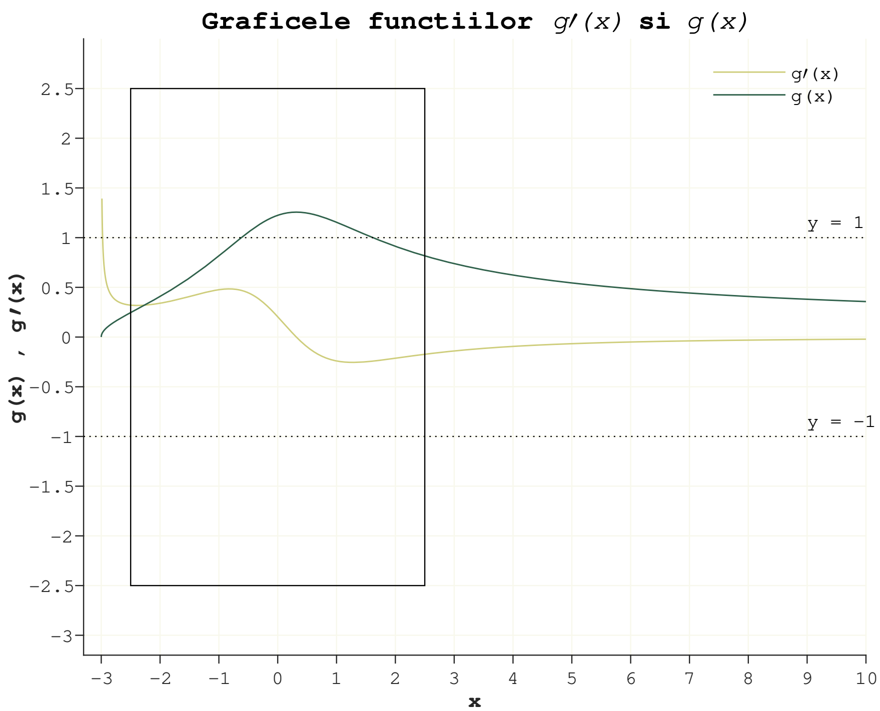


Observam ca $\forall x \in [-3, \infty]$, obtinem $g(x) \in [-3, \infty]$.

Exercitiul 4.a

Fie ecuatia $f(x) = 0$, unde $f(x) = x^4 + 2x^2 - x - 3$. Se considera functia $g_1(x) = \left(\dfrac{x+3}{x^2+2}\right)^{\frac{1}{2}}$.

d. Construiti graficul functiei g' pe intervalul $[a, b]$ si doua segmente de dreapta situate pe dreptele $y = 1$ si $y = -1$. Alegeti conform graficului un interval $[a', b'] \subset [a, b]$, astfel incat $|g'(x)| < 1$. Verificati daca $g(x) \in [a', b']$, $\forall x \in [a', b']$.



Observam ca $\forall x \in [-2.5,\ 2.5]$, obtinem $g(x) \in [-2.5,\ 2.5]$ si $|g'(x)| < 1$.

Asadar, functia g admite un unic punct fix pe intervalul $[-2.5,\ 2.5]$, iar sirul aproximarilor $x^{(k)}$ construit in baza functiei g, converge la solutia ecuatiei $f(x) = 0$.

Exercitiul 4.b

Fie ecuatia $f(x) = 0$, unde $f(x) = x^4 + 2x^2 - x - 3$. Se considera functia $g_2(x) = \left(\dfrac{x+3-x^4}{2}\right)^{\frac{1}{2}}$**.**

a. Aratati ca daca x^* este punct fix pentru g, atunci x^* este radacina functiei $f(x)$.

$$\underline{x^*\ punct\ fix\ al\ functiei\ g}$$

$$\Rightarrow\ x^* = g(x^*)$$

$$\Rightarrow\ x^* = \sqrt{\frac{x+3-x^4}{2}}$$

$$\Rightarrow\ {x^*}^2 = \frac{x+3-x^4}{2}$$

$$\Rightarrow\ 2{x^*}^2 - (x^* + 3 - x^4) = 0$$

$$\Rightarrow\ {x^*}^4 + 2{x^*}^2 - 3 - x^* = 0 = f(x)$$

$$\Rightarrow\ \boldsymbol{x^*\ este\ radacina\ functiei\ f(x)}$$

Exercitiul 4.b

Fie ecuatia $f(x) = 0$, unde $f(x) = x^4 + 2x^2 - x - 3$. Se considera functia $g_2(x) = \left(\frac{x+3-x^4}{2}\right)^{\frac{1}{2}}$.

b. Aflati domeniul de definitie $[a, b]$ al functiei g.

$$g : [a, b] \to \mathbb{R}, \quad \mathrm{g(x)} = \sqrt{\frac{\mathrm{x} + 3 - \mathrm{x}^4}{2}}$$

$$\Rightarrow \sqrt{\frac{x + 3 - x^4}{2}} \in \mathbb{R}$$

$$\Rightarrow \frac{x + 3 - x^4}{2} \geq 0$$

$$x + 3 - x^4 > 0$$

$$\Rightarrow \begin{cases} x_1 = -1.1640 \\ x_2 = 1.4526 \end{cases}$$

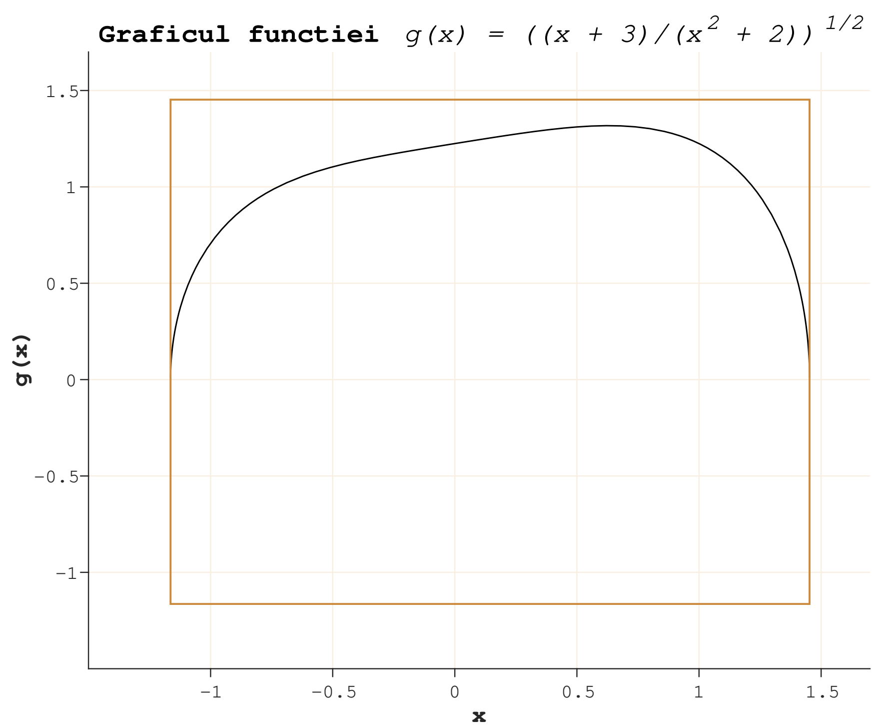
$$\Downarrow$$

$$\boldsymbol{g : [-1.1640, 1.4526] \to \mathbb{R}}$$

Exercitiul 4.b

Fie ecuatia $f(x) = 0$, unde $f(x) = x^4 + 2x^2 - x - 3$. Se considera functia $g_2(x) = \left(\frac{x + 3 - x^4}{2}\right)^{\frac{1}{2}}$.

c. Construiti graficul functiei g pe intervalul $[a, b]$. Verificati daca $g(x) \in [a, b]$, $\forall x \in [a, b]$.

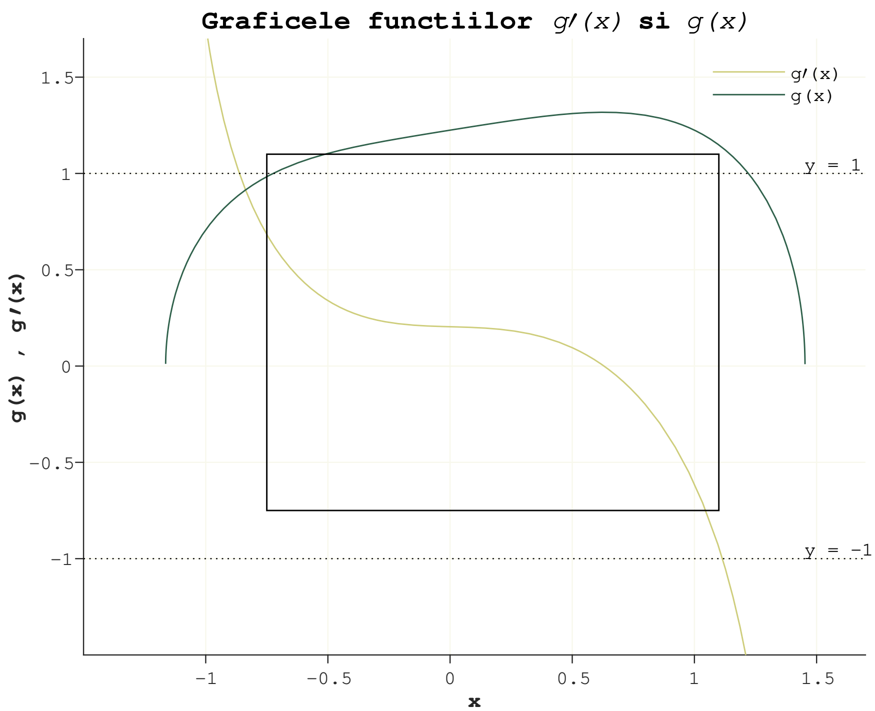


Observam ca $\forall x \in [-1.1640,\ 1.4526]$, obtinem $g(x) \in [-1.1640,\ 1.4526]$.

Exercitiul 4.b

Fie ecuatia $f(x) = 0$, unde $f(x) = x^4 + 2x^2 - x - 3$. Se considera functia $g_2(x) = \left(\dfrac{x+3-x^4}{2}\right)^{\frac{1}{2}}$.

d. Construiti graficul functiei g' pe intervalul $[a, b]$ si doua segmente de dreapta situate pe dreptele $y = 1$ si $y = -1$. Alegeti conform graficului un interval $[a', b'] \subset [a, b]$, astfel incat $|g'(x)| < 1$. Verificati daca $g(x) \in [a', b']$, $\forall x \in [a', b']$.



Observam ca $\forall x \in [-1.1640,\ 1.4526]$, obtinem $|g'(x)| < 1$, dar $g(x) \notin [-1.1640,\ 1.4526]$.

Asadar, sirul aproximarilor $x^{(k)}$ construit in baza functiei g prin metoda punctului fix, nu converge la solutia ecuatiei $f(x) = 0$.

Exercitiul 5

In baza unei functii $f \in C^2([a,b])$, derivata careia nu se anuleaza pe intervalul $[a,b]$, se construieste $g(x) = x - \frac{f(x)}{f'(x)}$.

a. Sa se arate ca x^* este radacina pentru f, daca si numai daca x^* este punct fix pentru g.

" $\Leftarrow$ " presupunem ca x^* este punct fix al functiei G

$$\Rightarrow\ x^* = g(x^*)$$
$$\Rightarrow\ x^* = x^* - \frac{f(x^*)}{f'(x^*)}$$
$$\Rightarrow\ x^* - x^* + \frac{f(x^*)}{f'(x^*)} = 0$$
$$\Rightarrow\ \frac{f(x^*)}{f'(x^*)} = 0$$
$$\Rightarrow\ f(x^*) = 0$$
$$\Rightarrow\ \boldsymbol{x^*\ este\ radacina\ a\ functiei\ f}$$

" $\Rightarrow$ " presupunem ca x^* este radacina a functiei F

$$\Rightarrow\ f(x^*) = 0 \mid /f'(x)$$
$$\Rightarrow\ \frac{f(x^*)}{f'(x^*)} = \frac{0}{f'(x^*)} \mid *(-1)$$
$$\Rightarrow\ -\frac{f(x^*)}{f'(x^*)} = 0 \mid +x^*$$
$$\Rightarrow\ x^* - \frac{f(x^*)}{f'(x^*)} = x^*$$
$$\Rightarrow\ g(x^*) = x^*$$
$$\Rightarrow\ \boldsymbol{x^*\ este\ punct\ fix\ al\ functiei\ g}$$

Exercitiul 6

Fie sistemul neliniar $\begin{cases} x_1^2 - 10x_1 + x_2^2 + 8 = 0 \\ x_1x_2^2 + x_1 - 10x_2 + 8 = 0 \end{cases}$ **.**

a. Sa se demonstreze ca x^* este solutie a sistemului neliniar daca si numai daca x^* este punct fix al functiei $G(x_1, x_2) = \left(\dfrac{x_1^2 + x_2^2 + 8}{10} \ , \ \dfrac{x_1x_2^2 + x_+8}{10} \right)$.

" $\Rightarrow$ " presupunem ca x^* este punct fix al functiei G

$$\Rightarrow \ x^* = G(x^*)$$

$$\Rightarrow \ (x_1^*, x_2^*) = \left(\frac{x_1^{*^2} + x_2^{*^2} + 8}{10} \ , \ \frac{x_1^* x_2^{*^2} + x_1^* + 8}{10} \right)$$

$$\Rightarrow \begin{cases} 10x_1^* - (x_1^{*^2} + x_2^{*^2} + 8) = 0 \\ 10x_2^* - (x_1^* x_2^{*^2} + x_1^* + 8) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1^{*^2} - 10x_1^* + x_2^{*^2} + 8 = 0 \\ x_1 x_2^{*^2} + x_1^* - 10x_2^* + 8 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} f_1(x_1^*, x_2^*) = 0 \\ f_2(x_1^*, x_2^*) = 0 \end{cases}$$

$$\Rightarrow \ \boldsymbol{x^* \ este \ radacina \ functiei \ F = (f_1(x_1, x_2), f_2(x_1, x_2))}$$

" $\Leftarrow$ " presupunem ca x^* este radacina a functiei F

$$\Rightarrow \begin{cases} x_1^2 - 10x_1 + x_2^2 + 8 = 0 \\ x_1x_2^2 + x_1 - 10x_2 + 8 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 10x_1^* - ({x_1^*}^2 + {x_2^*}^2 + 8) = 0 \\ 10x_2^* - (x_1^*{x_2^*}^2 + x_1^* + 8) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1^* = \frac{{x_1^*}^2 + {x_2^*}^2 + 8}{10} \\ x_2^* = \frac{x_1^*{x_2^*}^2 + x_1^* + 8}{10} \end{cases}$$

$$\Rightarrow (x_1^*, x_2^*) = \left(\frac{{x_1^*}^2 + {x_2^*}^2 + 8}{10} \; , \; \frac{x_1^*{x_2^*}^2 + x_1^* + 8}{10} \right)$$

$$\Rightarrow x^* = G(x^*)$$

$$\Rightarrow \boldsymbol{x^* \ este \ punct \ fix \ al \ functiei \ G}$$

Exercitiul 6

Fie sistemul neliniar $\begin{cases} x_1^2 - 10x_1 + x_2^2 + 8 = 0 \\ x_1x_2^2 + x_1 - 10x_2 + 8 = 0 \end{cases}$.

b. Sa se demonstreze conform teoremei III.2 ca functia G admite un unic punct fix pe domeniul $D = (x_1, x_2) \mid 0 \leq x_1, x_2 \leq 1.5$.

- Demonstram ca $G(x_1, x_2) \in [0, 1.5]$

$$g1(0,0) = \frac{8}{10} = \frac{4}{5} = 0.8 \ \in \mathbf{[0,\ 1.5]}$$
$$g1(3/2, 3/2) = \frac{2\frac{9}{4} + 8}{10} = \frac{50}{40} = 1.25 \ \in \mathbf{[0,\ 1.5]}$$
$$g2(0,0) = \frac{8}{10} = \frac{4}{5} = 0.8 \ \in \mathbf{[0,\ 1.5]}$$
$$g2(3/2, 3/2) = \frac{\frac{27}{8} + \frac{3}{2} + 8}{10} = \frac{103}{80} \approx 1.3 \ \in \mathbf{[0,\ 1.5]}$$

- Demonstram ca $||G'(x_1, x_2)|| < 1$

Vom evalua derivatele partiale pentru valorile maxime x_1 si x_2:

$$\frac{\delta g_1}{\delta x_1} = \frac{2x_1}{10} = \frac{3}{10} = 0.3$$
$$\frac{\delta g_1}{\delta x_2} = \frac{2x_2}{10} = \frac{3}{10} = 0.3$$
$$\frac{\delta g_2}{\delta x_1} = \frac{x_2^2 + 1}{10} = \frac{13}{40} = 0.325$$
$$\frac{\delta g_2}{\delta x_2} = \frac{2x_1x_2^2}{10} = \frac{x_1x_2}{5} = \frac{9}{20} = 0.45$$

$$||G'(x_1, x_2)||_\infty = max(0.3 + 0.3\ ,\ 0.325 + 0.45) = max(0.6\ ,\ 0.775) = 0.775 \ \mathbf{< 1}$$

$$\Downarrow$$

Functia G admite un unic punct fix pe domeniul D=[0, 1.5].