CALCUL NUMERIC

TEMA 12

Maciuca Gloria - Ruxandra

Grupa 344

Exercitiul 1

Sa se deduca formula cuadraturii Newton-Cotes inchisa $(n = 3)$**. Aceasta formula se mai numeste si formula de cuadratura Newton.**

Sa se deduca formula de cuadratura sumata Newton.

<u>Formula de cuadratura Newton</u> $(n = 3)$

$$\boldsymbol{I_3(f) = f(x_0)\int_{x_0}^{x_3} L_{3,0}(x)\ dx + f(x_1)\int_{x_0}^{x_3} L_{3,1}(x)\ dx + f(x_2)\int_{x_0}^{x_3} L_{3,2}(x)\ dx + f(x_3)\int_{x_0}^{x_3} L_{3,3}(x)\ dx}$$

$$\textit{pentru}\ \ \boldsymbol{[a,b] \sim [a = x_0 < x_1 < x_2 < x_3 = b]} \quad \textit{si} \quad \boldsymbol{x_i - x_{i-1} = h}$$

$$\boldsymbol{L_{3,0} = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}}$$

$$L_{3,0} = \frac{1}{(-h)(-2h)(-3h)}\ \Big(x^3 - x^2(x_1+x_2+x_3) + x(x_1x_2+x_1x_3+x_2x_3) - (x_1x_2x_3)\Big)$$

$$L_{3,0} = -\frac{1}{6h^3}\ \Big(x^3 - x^2(x_1+x_2+x_3) + x(x_1x_2+x_1x_3+x_2x_3) - (x_1x_2x_3)\Big)$$

$$\int_{x_0}^{x_3} L_{3,0} = -\frac{1}{6h^3}\left(\int_{x_0}^{x_3} x^3dx - (x_1+x_2+x_3)\int_{x_0}^{x_3} x^2dx + (x_1x_2+x_1x_3+x_2x_3)\int_{x_0}^{x_3} xdx - (x_1x_2x_3)\int_{x_0}^{x_3} dx\right)$$

$$\int_{x_0}^{x_3} x^3dx = \left.\frac{x^4}{4}\right|_{x_0}^{x_3} = \frac{x_3^4 - x_0^4}{4} = \frac{(x_0+3h)^4 - x_0^4}{4} = \frac{3h}{4}(4x_0^3 + 18x_0^2h + 36x_0h^2 + 27h^3)$$

$$\int_{x_0}^{x_3} x^2dx = \left.\frac{x^3}{3}\right|_{x_0}^{x_3} = \frac{x_3^3 - x_0^3}{3} = \frac{(x_0+3h)^3 - x_0^3}{3} = \frac{3h}{3}3(x_0^2 + 3x_0h + 3h^2)$$

$$\int_{x_0}^{x_3} xdx = \left.\frac{x^2}{2}\right|_{x_0}^{x_3} = \frac{x_3^2 - x_0^2}{2} = \frac{(x_0+3h)^2 - x_0^2}{2} = \frac{3h}{2}(2x_0 + 3h)$$

$$\int_{x_0}^{x_3} dx = x|_{x_0}^{x_3} = x_3 - x_0 = x_0 + 3h - x_0 = 3h$$

$$x_1 + x_2 + x_3 = x_0 + h + x_0 + 2h + x_0 + 3h = 3x_0 + 6h$$

$$x_1x_2 + x_1x_3 + x_2x_3 = (x_0+h)(x_0+2h) + (x_0+h)(x_0+3h) + (x_0+2h)(x_0+3h) = 3x_0^2 + 12x_0h + 11h^2$$

$$x_1x_2x_3 = (x_0+h)(x_0+2h)(x_0+3h) = x_0^3 + 6x_0^2h + 11x_0h^2 + 6h^3$$

$$-\frac{1}{6h^3}\cdot\frac{3h}{4}(4x_0^3+18x_0^2h+36x_0h^2+27h^3)=-\frac{1}{8h^2}(4x_0^3+18x_0^2h+36x_0h^2+27h^3)$$
$$\frac{1}{6h^3}\cdot\frac{3h}{3}\cdot 3(x_0^2+3x_0h+3h^2)(3x_0+6h)=\frac{1}{2h^2}(3x_0^3+15x_0^2h+27x_0h^2+18h^3)$$
$$-\frac{1}{6h^3}\cdot\frac{3h}{2}(2x_0+3h)(3x_0^2+12x_0h+11h^2)=-\frac{1}{4h^2}(6x_0^3+33x_0^2h+58x_0h^2+33h^3)$$
$$\frac{1}{6h^3}\cdot 3h(x_0^3+6x_0^2h+11x_0h^2+6h^3)=\frac{1}{2h^2}(x_0^3+6x_0^2h+11x_0h^2+6h^3)$$

$$\int_{x_0}^{x_3} L_{3,0}=\frac{1}{8h^2}\Big(-4x_0^3-18x_0^2h-36x_0h^2-27h^3+$$
$$+12x_0^3+60x_0^2h+108x_0h^2+72h^3-$$
$$-12x_0^3-66x_0^2h-116x_0h^2-66h^3+$$
$$+4x_0^3+24x_0^2h+44x_0h^2+24h^3\Big)$$
$$\int_{x_0}^{x_3} L_{3,0}=\frac{1}{8h^2}\cdot 3h^3$$
$$\Downarrow$$
$$\boldsymbol{\int_{x_0}^{x_3} L_{3,0}=\frac{3h}{8}}$$

$$\boldsymbol{L_{3,1}=\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}}$$
$$L_{3,1}=\frac{1}{h(-h)(-3h)}\ \Big(x^3-x^2(x_0+x_2+x_3)+x(x_0x_2+x_0x_3+x_2x_3)-(x_0x_2x_3)\Big)$$
$$L_{3,1}=\frac{1}{2h^3}\ \Big(x^3-x^2(x_0+x_2+x_3)+x(x_0x_2+x_0x_3+x_2x_3)-(x_0x_2x_3)\Big)$$
$$\int_{x_0}^{x_3} L_{3,1}=\frac{1}{2h^3}\left(\int_{x_0}^{x_3} x^3dx-(x_0+x_2+x_3)\int_{x_0}^{x_3} x^2dx+(x_0x_2+x_0x_3+x_2x_3)\int_{x_0}^{x_3} xdx-(x_0x_2x_3)\int_{x_0}^{x_3} dx\right)$$

$$\int_{x_0}^{x_3} x^3 dx = \left.\frac{x^4}{4}\right|_{x_0}^{x_3} = \frac{x_3^4 - x_0^4}{4} = \frac{(x_0+3h)^4 - x_0^4}{4} = \frac{3h}{4}(4x_0^3 + 18x_0^2 h + 36x_0h^2 + 27h^3)$$
$$\int_{x_0}^{x_3} x^2 dx = \left.\frac{x^3}{3}\right|_{x_0}^{x_3} = \frac{x_3^3 - x_0^3}{3} = \frac{(x_0+3h)^3 - x_0^3}{3} = \frac{3h}{3}3(x_0^2 + 3x_0h + 3h^2)$$
$$\int_{x_0}^{x_3} x dx = \left.\frac{x^2}{2}\right|_{x_0}^{x_3} = \frac{x_3^2 - x_0^2}{2} = \frac{(x_0+3h)^2 - x_0^2}{2} = \frac{3h}{2}(2x_0 + 3h)$$
$$\int_{x_0}^{x_3} dx = x|_{x_0}^{x_3} = x_3 - x_0 = x_0 + 3h - x_0 = 3h$$

$$x_0 + x_2 + x_3 = x_0 + x_0 + 2h + x_0 + 3h = 3x_0 + 5h$$

$$x_0x_2 + x_0x_3 + x_2x_3 = x_0(x_0+2h) + x_0(x_0+3h) + (x_0+2h)(x_0+3h) = 3x_0^2 + 10x_0h + 6h^2$$

$$x_0x_2x_3 = x_0(x_0+2h)(x_0+3h) = x_0^3 + 5x_0^2h + 6x_0h^2$$

$$\frac{1}{2h^3}\cdot\frac{3h}{4}(4x_0^3 + 18x_0^2h + 36x_0h^2 + 27h^3) = \frac{3}{8h^2}(4x_0^3 + 18x_0^2h + 36x_0h^2 + 27h^3)$$
$$-\frac{1}{2h^3}\cdot\frac{9h}{3}(x_0^2 + 3x_0h + 3h^2)(3x_0 + 5h) = -\frac{3}{2h^2}(3x_0^3 + 14x_0^2h + 24x_0h^2 + 15h^3)$$
$$\frac{1}{2h^3}\cdot\frac{3h}{2}(2x_0 + 3h)(3x_0^2 + 10x_0h + 6h^2) = \frac{3}{4h^2}(6x_0^3 + 29x_0^2h + 42x_0h^2 + 18h^3)$$
$$-\frac{1}{2h^3}\cdot 3h\cdot(x_0^3 + 5x_0^2h + 6x_0h^2) = -\frac{3}{2h^2}(x_0^3 + 5x_0^2h + 6x_0h^2)$$

$$\begin{aligned}\int_{x_0}^{x_3} L_{3,1} = \frac{3}{8h^2}\Big(&4x_0^3 + 18x_0^2h + 36x_0h^2 + 27h^3 - \\ &- 12x_0^3 - 56x_0^2h - 96x_0h^2 - 60h^3 + \\ &+ 12x_0^3 + 58x_0^2h + 84x_0h^2 + 36h^3 - \\ &- 4x_0^3 - 20x_0^2h - 24x_0h^2 - 0h^3\Big)\end{aligned}$$
$$\int_{x_0}^{x_3} L_{3,1} = \frac{3}{8h^2}\cdot 3h^3$$
$$\Downarrow$$
$$\boldsymbol{\int_{x_0}^{x_3} L_{3,1} = \frac{3h}{8}\cdot 3}$$

$$\boldsymbol{L_{3,2} = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}}$$

$$L_{3,2} = \frac{1}{2h \cdot h \cdot (-h)} \left(x^3 - x^2(x_0 + x_1 + x_3) + x(x_0x_1 + x_0x_3 + x_1x_3) - (x_0x_1x_3) \right)$$

$$L_{3,2} = -\frac{1}{2h^3} \left(x^3 - x^2(x_0 + x_1 + x_3) + x(x_0x_1 + x_0x_3 + x_1x_3) - (x_0x_1x_3) \right)$$

$$\int_{x_0}^{x_3} L_{3,2} = -\frac{1}{2h^3} \left(\int_{x_0}^{x_3} x^3 dx - (x_0 + x_1 + x_3) \int_{x_0}^{x_3} x^2 dx + (x_0x_1 + x_0x_3 + x_1x_3) \int_{x_0}^{x_3} x dx - (x_0x_1x_3) \int_{x_0}^{x_3} dx \right)$$

$$\int_{x_0}^{x_3} x^3 dx = \left.\frac{x^4}{4}\right|_{x_0}^{x_3} = \frac{x_3^4 - x_0^4}{4} = \frac{(x_0 + 3h)^4 - x_0^4}{4} = \frac{3h}{4}(4x_0^3 + 18x_0^2h + 36x_0h^2 + 27h^3)$$

$$\int_{x_0}^{x_3} x^2 dx = \left.\frac{x^3}{3}\right|_{x_0}^{x_3} = \frac{x_3^3 - x_0^3}{3} = \frac{(x_0 + 3h)^3 - x_0^3}{3} = \frac{3h}{3}3(x_0^2 + 3x_0h + 3h^2)$$

$$\int_{x_0}^{x_3} x dx = \left.\frac{x^2}{2}\right|_{x_0}^{x_3} = \frac{x_3^2 - x_0^2}{2} = \frac{(x_0 + 3h)^2 - x_0^2}{2} = \frac{3h}{2}(2x_0 + 3h)$$

$$\int_{x_0}^{x_3} dx = x|_{x_0}^{x_3} = x_3 - x_0 = x_0 + 3h - x_0 = 3h$$

$$x_0 + x_1 + x_3 = x_0 + x_0 + h + x_0 + 3h = 3x_0 + 4h$$

$$x_0x_1 + x_0x_3 + x_1x_3 = x_0(x_0 + h) + x_0(x_0 + 3h) + (x_0 + h)(x_0 + 3h) = 3x_0^2 + 8x_0h + 3h^2$$

$$x_0x_1x_3 = x_0(x_0 + h)(x_0 + 3h) = x_0^3 + 4x_0^2h + 3x_0h^2$$

$$-\frac{1}{2h^3} \cdot \frac{3h}{4}(4x_0^3 + 18x_0^2h + 36x_0h^2 + 27h^3) = -\frac{3}{8h^2}(4x_0^3 + 18x_0^2h + 36x_0h^2 + 27h^3)$$

$$\frac{1}{2h^3} \cdot \frac{9h}{3}(x_0^2 + 3x_0h + 3h^2)(3x_0 + 4h) = \frac{3}{2h^2}(3x_0^3 + 13x_0^2h + 21x_0h^2 + 12h^3)$$

$$-\frac{1}{2h^3} \cdot \frac{3h}{2}(2x_0 + 3h)(3x_0^2 + 8x_0h + 3h^2) = -\frac{3}{4h^2}(6x_0^3 + 25x_0^2h + 30x_0h^2 + 9h^3)$$

$$\frac{1}{2h^3} \cdot 3h \cdot (x_0^3 + 4x_0^2h + 3x_0h^2) = \frac{3}{2h^2}(x_0^3 + 4x_0^2h + 3x_0h^2)$$

$$\int_{x_0}^{x_3} L_{3,2} = \frac{3}{8h^2}\Big(-4x_0^3 - 18x_0^2h - 36x_0h^2 - 27h^3 +$$
$$+ 12x_0^3 + 52x_0^2h + 84x_0h^2 + 48h^3 -$$
$$- 12x_0^3 - 50x_0^2h - 60x_0h^2 - 18h^3 +$$
$$+ 4x_0^3 + 16x_0^2h + 12x_0h^2 + 0h^3\Big)$$

$$\int_{x_0}^{x_3} L_{3,2} = \frac{3}{8h^2} \cdot 3h^3$$

$$\Downarrow$$

$$\boldsymbol{\int_{x_0}^{x_3} L_{3,2} = \frac{3h}{8} \cdot 3}$$

$$\boldsymbol{L_{3,3} = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}}$$

$$L_{3,3} = \frac{1}{3h \cdot 2h \cdot h}\left(x^3 - x^2(x_0+x_1+x_2) + x(x_0x_1+x_0x_2+x_1x_2) - (x_0x_1x_2)\right)$$

$$L_{3,3} = \frac{1}{6h^3}\left(x^3 - x^2(x_0+x_1+x_2) + x(x_0x_1+x_0x_2+x_1x_2) - (x_0x_1x_2)\right)$$

$$\int_{x_0}^{x_3} L_{3,2} = \frac{1}{6h^3}\left(\int_{x_0}^{x_3} x^3dx - (x_0+x_1+x_2)\int_{x_0}^{x_3} x^2dx + (x_0x_1+x_0x_2+x_1x_2)\int_{x_0}^{x_3} xdx - (x_0x_1x_2)\int_{x_0}^{x_3} dx\right)$$

$$\int_{x_0}^{x_3} x^3dx = \frac{x^4}{4}\bigg|_{x_0}^{x_3} = \frac{x_3^4 - x_0^4}{4} = \frac{(x_0+3h)^4 - x_0^4}{4} = \frac{3h}{4}(4x_0^3 + 18x_0^2h + 36x_0h^2 + 27h^3)$$

$$\int_{x_0}^{x_3} x^2dx = \frac{x^3}{3}\bigg|_{x_0}^{x_3} = \frac{x_3^3 - x_0^3}{3} = \frac{(x_0+3h)^3 - x_0^3}{3} = \frac{3h}{3}3(x_0^2 + 3x_0h + 3h^2)$$

$$\int_{x_0}^{x_3} xdx = \frac{x^2}{2}\bigg|_{x_0}^{x_3} = \frac{x_3^2 - x_0^2}{2} = \frac{(x_0+3h)^2 - x_0^2}{2} = \frac{3h}{2}(2x_0 + 3h)$$

$$\int_{x_0}^{x_3} dx = x|_{x_0}^{x_3} = x_3 - x_0 = x_0 + 3h - x_0 = 3h$$

$$x_0 + x_1 + x_2 = x_0 + x_0 + h + x_0 + 2h = 3x_0 + 3h$$

$$x_0x_1 + x_0x_2 + x_1x_2 = x_0(x_0+h) + x_0(x_0+2h) + (x_0+h)(x_0+2h) = 3x_0^2 + 6x_0h + 2h^2$$

$$x_0x_1x_2 = x_0(x_0+h)(x_0+2h) = x_0^3 + 3x_0^2h + 2x_0h^2$$

$$\frac{1}{6h^3}\cdot\frac{3h}{4}(4x_0^3+18x_0^2h+36x_0h^2+27h^3)=\frac{1}{8h^2}(4x_0^3+18x_0^2h+36x_0h^2+27h^3)$$
$$-\frac{1}{6h^3}\cdot\frac{9h}{3}(x_0^2+3x_0h+3h^2)(3x_0+3h)=-\frac{1}{2h^2}(3x_0^3+12x_0^2h+18x_0h^2+9h^3)$$
$$\frac{1}{6h^3}\cdot\frac{3h}{2}(2x_0+3h)(3x_0^2+6x_0h+2h^2)=\frac{1}{4h^2}(6x_0^3+21x_0^2h+22x_0h^2+6h^3)$$
$$-\frac{1}{6h^3}\cdot 3h\cdot(x_0^3+3x_0^2h+2x_0h^2)=-\frac{1}{2h^2}(x_0^3+3x_0^2h+2x_0h^2)$$

$$\begin{aligned}\int_{x_0}^{x_3}L_{3,3}=\frac{1}{8h^2}\Big(&4x_0^3+18x_0^2h+36x_0h^2+27h^3-\\&-12x_0^3-48x_0^2h-72x_0h^2-36h^3+\\&+12x_0^3+42x_0^2h+44x_0h^2+12h^3-\\&-4x_0^3-12x_0^2h-8x_0h^2-0h^3\Big)\end{aligned}$$
$$\int_{x_0}^{x_3}L_{3,3}=\frac{1}{8h^2}\cdot 3h^3$$
$$\Downarrow$$
$$\boldsymbol{\int_{x_0}^{x_3}L_{3,3}=\frac{3h}{8}}$$

$$I_3(f)=f(x_0)\cdot\frac{3h}{8}+3f(x_1)\cdot\frac{3h}{8}+3f(x_2)\cdot\frac{3h}{8}+f(x_3)\cdot\frac{3h}{8}$$
$$\Downarrow$$
$$\boldsymbol{I_3(f)=\frac{3h}{8}\Big(f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\Big)}$$

Exercitiul 1

Sa se deduca formula cuadraturii Newton-Cotes inchisa ($n = 3$)**. Aceasta formula se mai numeste si formula de cuadratura Newton.**

Sa se deduca formula de cuadratura sumata Newton.

<u>Formula de cuadratura sumata Newton</u> ($n = 3$)

Cunoastem formula de calcul nesumata, pentru $[a, b] \sim [a = x_0 < x_1 < x_2 < x_3 = b]$:

$$I_3(f) = \frac{3h}{8}\Big(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\Big)$$

Observam ca avem nevoie de o discretizare a intervalului $[a, b]$ de $(3k + 3)$ noduri.

Asadar, pentru formula de calcul sumata, vom folosi urmatoarea discretizare:

$$\boldsymbol{[a, b] \sim [a = x_0 < x_1 < ... < x_{3k} = b]} \quad si \quad \boldsymbol{x_i - x_{i-1} = h}$$

Astfel, formula de cuadratura sumata Newton este:

$$I_3(f) = \frac{3h}{8}\Big(f(x_0) + 3f(x_{0+1}) + 3f(x_{0+2}) + f(x_{0+3})\Big) + ... + \frac{3h}{8}\Big(f(x_{3k}) + 3f(x_{3k+1}) + 3f(x_{3k+2}) + f(x_{3k+3})\Big)$$

Generalizand, obtinem:

$$\boldsymbol{I_3(f) = \frac{3h}{8}\sum_{i=0}^{k}\Big(f(x_{3k}) + 3f(x_{3k+1}) + 3f(x_{3k+2}) + f(x_{3k+3})\Big)}$$

Exercitiul 3

Fie ecuatia $x' = x \cdot cos(t) + x^2 \cdot cos(t)$**.**

a. Sa se demonstreze ca ecuatia admite proprietatea de E.U.L.

$$f(t,x) = x \cdot cos(t) + x^2 \cdot cos(t)$$
$$f(t,x) = cos(t) \cdot x \cdot (1+x)$$
$$\Downarrow$$

$f(t,x)$ este produs de functii elementare.

$f(t,x)$este continua pe $D = \mathbb{R}\text{x}\mathbb{R}$ **(1)**

$$\frac{\partial f}{\partial x}(t,x) = cos(t) + 2cos(t) \cdot x$$
$$\frac{\partial f}{\partial x}(t,x) = cos(t)(1+2x)$$
$$\Downarrow$$

$(t,x) \to \frac{\partial f}{\partial x}(t,x)$ este continua pe D

$f(t,x)$ este local Lipschitz in raport cu x **(2)**

(1) si **(2)**

$\Downarrow$ cf. Th. Cauchy-Lipschitz

ecuatia admite proprietatea E.U.L.

Exercitiul 3

Fie ecuatia $x' = x \cdot cos(t) + x^2 \cdot cos(t)$**.**

b. Sa se verifice ca $\varphi_K(t) = -\dfrac{e^{sin(t)}}{e^{sin(t)} + K}$**.**

$$\varphi'_K(t) = -\frac{e^{sin(t)} \cdot cos(t) \cdot (e^{sin(t)} + K)}{(e^{sin(t)} + K)^2} + \frac{e^{2 \cdot sin(t)} \cdot cos(t)}{(e^{sin(t)} + K)^2}$$

$$\varphi'_K(t) = -\frac{e^{sin(t)}}{e^{sin(t)} + K} \cdot cos(t) + \left(\frac{e^{sin(t)}}{e^{sin(t)} + K}\right)^2 \cdot cos(t)$$

$$\boldsymbol{\varphi'_K(t) = \varphi_K(t) \cdot cos(t) + \varphi_K^2(t) \cdot cos(t)}$$

Exercitiul 3

Fie ecuatia $x' = x \cdot cos(t) + x^2 \cdot cos(t)$**.**

c. Sa se afle solutia problemei Cauchy $(f, 0, 1)$**.**

$$\varphi_K(0) = 1$$

$$-\frac{e^{sin(0)}}{e^{sin(0)} + K} = 1$$

$$-\frac{1}{1 + K} = 1$$

$$1 + K = -1$$

$$\boldsymbol{K = -2}$$

$$\Downarrow$$

$$\varphi_0(t) = -\frac{e^{sin(t)}}{e^{sin(t)} - 2}$$

Exercitiul 3

Fie ecuatia $x' = x \cdot cost + x^2 \cdot cost$.

d. Sa se afle intervalul maximal $I(0,1) = (t^-(0,1), t^+(0,1))$.

$$conditie\ de\ existenta\ \varphi_0(t):$$

$$e^{sin(t)} - 2 \neq 0$$

$$e^{sin(t)} \neq 2$$

$$sin(t) \neq ln2$$

$$t \neq arcsin(ln2)$$

$$t \neq \pi - arcsin(ln2)$$

$$\Rightarrow t \in (-\infty\ ,\ arcsin(ln2))\ \cup\ (\pi - arcsin(ln2)\ ,\ \infty)$$

$$t_0 = 0 \in (-\infty\ ,\ arcsin(ln2))$$

$$\Downarrow$$

$$\boldsymbol{I(0,1) = (-\infty\ ,\ arcsin(ln2))}$$