CALCUL NUMERIC

TEMA 4

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Exercitiul 1

Fie matricea $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$**. Sa se afle manual descompunerea** QR **prin metoda Givens. Sa se rezolve sistemul** $Ax = b$**, pentru** $b = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$**.**

Vom efectuam transformari de rotatie asupra matricei A, in vederea obtinerii unei matrice superior triunghiulara (R).

Urmarim sa obtinem elemente cu valoarea 0 sub diagonala principala.

$$\begin{pmatrix} 1 & 1 & 0 \\ \boxed{1} & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Pentru a transforma valoarea elementului a_{21} in 0, vom calcula valorile de rotatie astfel:

$$s = \frac{a_{21}}{\sqrt{a_{11}^2 + a_{21}^2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$c = \frac{a_{11}}{\sqrt{a_{11}^2 + a_{21}^2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\boldsymbol{R_{(21)}} = \begin{pmatrix} \boldsymbol{\sqrt{2}/2} & \boldsymbol{\sqrt{2}/2} & \mathbf{0} \\ \boldsymbol{-\sqrt{2}/2} & \boldsymbol{\sqrt{2}/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

Aplicam transformarea asupra matricei A:

$$\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \boxed{1} & 1 \end{pmatrix}$$

Pentru a transforma valoarea elementului a_{32} in 0, vom calcula valorile de rotatie astfel:

$$s = \frac{a_{32}}{\sqrt{a_{22}^2 + a_{32}^2}} = \frac{1}{\sqrt{\frac{3}{2}}} = \frac{\sqrt{6}}{3}$$

$$c = \frac{a_{22}}{\sqrt{a_{22}^2 + a_{32}^2}} = -\frac{\frac{\sqrt{2}}{2}}{\sqrt{\frac{3}{2}}} = -\frac{\sqrt{3}}{3}$$

$$\boldsymbol{R_{(32)}} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{-\sqrt{3}/3} & \boldsymbol{\sqrt{6}/3} \\ \mathbf{0} & \boldsymbol{-\sqrt{6}/3} & \boldsymbol{-\sqrt{3}/3} \end{pmatrix}$$

Aplicam transformarea si obtinem matricea superior triunghiulara R:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{3}/3 & \sqrt{6}/3 \\ 0 & -\sqrt{6}/3 & -\sqrt{3}/3 \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sqrt{2}} & \boldsymbol{\sqrt{2}/2} & \boldsymbol{\sqrt{2}/2} \\ \mathbf{0} & \boldsymbol{\sqrt{6}/2} & \boldsymbol{\sqrt{6}/6} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{-2\sqrt{3}/3} \end{pmatrix} = \boldsymbol{R}$$

Matricea ortogonala Q o vom obtine astfel: $\boldsymbol{Q = (R_{(32)} \cdot R_{(21)})^T}$.

$$Q^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{3}/3 & \sqrt{6}/3 \\ 0 & -\sqrt{6}/3 & -\sqrt{3}/3 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ \sqrt{6}/6 & -\sqrt{6}/6 & \sqrt{6}/3 \\ \sqrt{3}/3 & -\sqrt{3}/3 & -\sqrt{3}/3 \end{pmatrix}$$

$$\boldsymbol{Q = \begin{pmatrix} \sqrt{2}/2 & \sqrt{6}/6 & \sqrt{3}/3 \\ \sqrt{2}/2 & -\sqrt{6}/6 & -\sqrt{3}/3 \\ 0 & \sqrt{6}/3 & -\sqrt{3}/3 \end{pmatrix}}$$

Am obtinut, astfel, factorizarea $\boldsymbol{A = Q \cdot R}$.

In continuare, vom rezolva sistemul dat, folosind descompunerea $A = QR$ calculata:

$$QRx = b$$

Cunoastem ca matricea Q este ortogonala, asadar vom folosi proprietatea $Q^T = Q^{-1}$:

$$\boldsymbol{Q^T| \;\; QRx = b \;\; \Rightarrow \;\; I_3Rx = Q^Tb \;\; \Rightarrow \;\; Rx = Q^Tb}$$

$$\begin{pmatrix} \sqrt{2} & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{6}/2 & \sqrt{6}/6 \\ 0 & 0 & -2\sqrt{3}/3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ \sqrt{6}/6 & -\sqrt{6}/6 & \sqrt{6}/3 \\ \sqrt{3}/3 & -\sqrt{3}/3 & -\sqrt{3}/3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{2} & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{6}/2 & \sqrt{6}/6 \\ 0 & 0 & -2\sqrt{3}/3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3\sqrt{2}/2 \\ 3\sqrt{6}/2 \\ -2\sqrt{3} \end{pmatrix}$$

$$\begin{aligned} -2\sqrt{3}/3 \cdot x_3 = -2\sqrt{3} \;&\Rightarrow\; \boldsymbol{x_3 = 3} \\ \sqrt{6}/2 \cdot x_2 + \sqrt{6}/6 \cdot x_3 = 3\sqrt{6}/2 \;&\Rightarrow\; \boldsymbol{x_2 = 2} \\ \sqrt{2} \cdot x_1 + \sqrt{2}/2 \cdot x_2 + \sqrt{2}/2 \cdot x_3 = 3\sqrt{2}/2 \;&\Rightarrow\; \boldsymbol{x_1 = -1} \end{aligned}$$

Pentru sistemul $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$, am obtinut solutia $\boldsymbol{x = \{-1,\; 2,\; 3\}}$.

Exercitiul 3

Fie matricea $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$**. Sa se calculeze, conform definitiei, valorile proprii ale matricei** A**.**

Conform definitiei, numim valoare proprie a matricei A, un scalar λ cu proprietatea $A \cdot v = \lambda \cdot v$, pentru un vector nenul v.

Vectorul v care indeplineste conditia se numeste vector propriu asociat matricei A.

$$\left.\begin{matrix} A \cdot v = \lambda \cdot v \\ v = I_3 \cdot v \end{matrix}\right\} \Rightarrow \left.\begin{matrix} (A - \lambda \cdot I_3) \quad v = 0_3 \\ v \neq 0_3 \end{matrix}\right\} \Rightarrow \text{coloanele matricei } (A - \lambda I_3) \text{ sunt liniar dependente}$$

$$\Downarrow$$

matricea $(A - \lambda I_3)$ nu este inversabila

$$\Downarrow$$

$$det(A - \lambda I_3) = 0$$

$$A - \lambda I_3 = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{pmatrix}$$

$$\begin{aligned} det(A - \lambda I_3) =& (3-\lambda)((3-\lambda)^2 - 1) - 1(3 - \lambda - 1) + 1(1 - (3-\lambda)) = \\ =& (3-\lambda)(9 - 6\lambda + \lambda^2) - 3 + \lambda + 1 + 1 - 3 + \lambda = \\ =& -\lambda^3 + 9\lambda^2 - 24\lambda + 20 \end{aligned}$$

$-\lambda^3 + 9\lambda^2 - 24\lambda + 20 = 0$

$-\lambda^3 + 2\lambda^2 + 7\lambda^2 - 14\lambda - 10\lambda + 20 = 0$

$-\lambda^2(\lambda - 2) + 7\lambda(\lambda - 2) - 10(\lambda - 2) = 0$

$(\lambda - 2)(-\lambda^2 + 7\lambda - 10) = 0$

$(\lambda - 2)(-\lambda^2 + 2\lambda + 5\lambda - 10) = 0$

$(\lambda - 2)(-\lambda(\lambda - 2) + 5(\lambda - 2)) = 0$

$(\lambda - 2)(\lambda - 2)(5 - \lambda) = 0$

$$\Rightarrow \begin{cases} \lambda - 2 = 0 \Rightarrow \underline{\boldsymbol{\lambda_1 = 2}} \\ \lambda - 2 = 0 \Rightarrow \underline{\boldsymbol{\lambda_2 = 2}} \\ 5 - \lambda = 0 \Rightarrow \underline{\boldsymbol{\lambda_3 = 5}} \end{cases}$$

$$\left.\begin{array}{l}(A-\lambda_1\cdot I_3)\quad v=0_3\\ \lambda_1=2\end{array}\right\}\Rightarrow \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=0 \Rightarrow v_1+v_2+v_3=0 \Rightarrow v_1=-v_2-v_3$$

$$\boldsymbol{v_{1_{(\lambda_1=2)}}} = \left\{ \begin{pmatrix}\boldsymbol{v_1}\\\boldsymbol{v_2}\\\boldsymbol{v_3}\end{pmatrix} = \boldsymbol{a}\begin{pmatrix}\boldsymbol{-1}\\\boldsymbol{1}\\\boldsymbol{0}\end{pmatrix}+\boldsymbol{b}\begin{pmatrix}\boldsymbol{-1}\\\boldsymbol{0}\\\boldsymbol{1}\end{pmatrix} |\boldsymbol{a},\ \boldsymbol{b}\in\mathbb{C} \right\}$$

$$\left.\begin{array}{l}(A-\lambda_2\cdot I_3)\quad v=0_3\\ \lambda_2=2\end{array}\right\}\Rightarrow \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=0 \Rightarrow v_1+v_2+v_3=0 \Rightarrow v_1=-v_2-v_3$$

$$\boldsymbol{v_{2_{(\lambda_2=2)}}} = \left\{ \begin{pmatrix}\boldsymbol{v_1}\\\boldsymbol{v_2}\\\boldsymbol{v_3}\end{pmatrix} = \boldsymbol{a}\begin{pmatrix}\boldsymbol{-1}\\\boldsymbol{1}\\\boldsymbol{0}\end{pmatrix}+\boldsymbol{b}\begin{pmatrix}\boldsymbol{-1}\\\boldsymbol{0}\\\boldsymbol{1}\end{pmatrix} |\boldsymbol{a},\ \boldsymbol{b}\in\mathbb{C} \right\}$$

$$\left.\begin{array}{l}(A-\lambda_3\cdot I_3)\quad v=0_3\\ \lambda_3=5\end{array}\right\}\Rightarrow \begin{pmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=0 \Rightarrow v_1={}^1\!/_2\cdot v_2+{}^1\!/_2\cdot v_3$$

$$\boldsymbol{v_{3_{(\lambda_3=5)}}} = \left\{ \begin{pmatrix}\boldsymbol{v_1}\\\boldsymbol{v_2}\\\boldsymbol{v_3}\end{pmatrix} = \boldsymbol{a}\begin{pmatrix}\boldsymbol{1/2}\\\boldsymbol{1}\\\boldsymbol{0}\end{pmatrix}+\boldsymbol{b}\begin{pmatrix}\boldsymbol{1/2}\\\boldsymbol{0}\\\boldsymbol{1}\end{pmatrix} |\boldsymbol{a},\ \boldsymbol{b}\in\mathbb{C} \right\}$$

Multimea valorilor proprii asociate matricei A poate fi descrisa astfel:

$$\boldsymbol{(\lambda, v)} = \left\{ \boldsymbol{(2,} \begin{pmatrix}\boldsymbol{-a-b}\\\boldsymbol{a}\\\boldsymbol{b}\end{pmatrix}\boldsymbol{)}\ ,\ \boldsymbol{(2,} \begin{pmatrix}\boldsymbol{-a-b}\\\boldsymbol{a}\\\boldsymbol{b}\end{pmatrix}\boldsymbol{)}\ ,\ \boldsymbol{(5,} \begin{pmatrix}\boldsymbol{1/2\cdot a+1/2\cdot b}\\\boldsymbol{a}\\\boldsymbol{b}\end{pmatrix}\boldsymbol{)}\ |\boldsymbol{a},\ \boldsymbol{b}\in\mathbb{C} \right\}$$

Exercitiul 6

Sa se demonstreze ca daca $A \in \mathcal{M}_n(\mathbb{R})$ este nesingulara, atunci matricea $A^T A$ este pozitiv definita.

Indicatie: Se va folosi definitia unei matrice pozitiv definite.

$$\left.\begin{array}{lcr} A\ nesingulara & \Rightarrow & det(A) \neq 0 \\ det(A^T) = det(A) & \Rightarrow & det(A^T) \neq 0 \end{array}\right\} \Rightarrow \begin{array}{l} A^T,\ A\ sunt\ inversabile \\ A^T,\ A\ au\ coloane\ liniar\ independente \end{array}$$

$A^T A$ **pozitiv definita**

$$\Downarrow$$

$$\vec{x}^T A^T A \vec{x} > 0$$

$$\left.\begin{array}{l} \left.\begin{array}{l} \vec{x}^T A^T A \vec{x} = 0 \\ A^T,\ A\ inversabile \end{array}\right\} \Rightarrow \boldsymbol{\vec{x} = \vec{0}} \\ \left.\begin{array}{l} \vec{x}^T A^T A \vec{x} > 0 \\ \vec{x}^T A^T = (A\vec{x})^T \end{array}\right\} \Rightarrow (A\vec{x})^T (A\vec{x}) > 0 \\ \left.\begin{array}{l} (A\vec{x})^T (A\vec{x}) = ||A\vec{x}||^2 \\ ||A\vec{x}||^2 > 0 \end{array}\right\} \Rightarrow \vec{x}^T A^T A \vec{x} > 0 \end{array}\right\} \Rightarrow \textbf{Matricea } A^T A \textbf{ este pozitiv definita.}$$

Exercitiul 7

Fie λ valoare proprie pentru $A \in \mathcal{M}_n(\mathbb{R})$ si $x \neq 0_n$ un vector propriu asociat valorii proprii λ.

a. Sa se arate ca λ este valoare proprie si pentru A^T.

$$\left.\begin{array}{l} \left.\begin{array}{l} A \cdot x = \lambda \cdot x \\ x = I_n \cdot x \end{array}\right\} \Rightarrow (A - \lambda \cdot I_n) \quad x = 0_n \\ \qquad\qquad\qquad\qquad\qquad\quad x \neq 0_3 \end{array}\right\} \Rightarrow det(A - \lambda I_n) = 0$$

Cunoastem ca $det(B) = det(B^T)$. Atunci:

$$\left.\begin{array}{l} det(A - \lambda I_n) = det(A - \lambda I_n)^T = det(A^T - \lambda I_n^T) \\ matricea\ identitate\ I_n\ este\ simetrica\ :\ I_n^T = I_n \end{array}\right\} \Rightarrow det(A - \lambda I_n) = det(A^T - \lambda I_n)$$

$$det(A^T - \lambda I_n) = 0 \ \Rightarrow \ (A^T - \lambda I_n)x = 0_n \ \Rightarrow \ \boldsymbol{A^T \cdot x = \lambda \cdot x}$$

Exercitiul 7

Fie λ valoare proprie pentru $A \in \mathcal{M}_n(\mathbb{R})$ si $x \neq 0_n$ un vector propriu asociat valorii proprii λ.

b. Sa se arate ca λ^k este valoare proprie a matricei A^k cu vectorul propriu x.

$$\underline{Ax = \lambda x}$$

$$A \mid Ax = \lambda x \;\Rightarrow\; AAx = \lambda Ax \;\Rightarrow\; A^2x = \lambda\lambda x \;\Rightarrow\; A^2x = \lambda^2 x$$
$$A \mid A^2x = \lambda^2 x \;\Rightarrow\; AA^2x = \lambda^2 Ax \;\Rightarrow\; A^3x = \lambda^2\lambda x \;\Rightarrow\; A^3x = \lambda^3 x$$
$$\vdots$$
$$A \mid A^{k-1}x = \lambda^{k-1}x \;\Rightarrow\; AA^{k-1}x = \lambda^{k-1}Ax \;\Rightarrow\; A^kx = \lambda^{k-1}\lambda x \;\Rightarrow\; \boldsymbol{A^k x = \lambda^k x}$$

Exercitiul 7

Fie λ valoare proprie pentru $A \in \mathcal{M}_n(\mathbb{R})$ si $x \neq 0_n$ un vector propriu asociat valorii proprii λ.

c. Sa se arate ca daca matricea A este nesingulara, atunci $\frac{1}{\lambda}$ este valoare proprie a matricei A^{-1} cu vectorul propriu x.

$A\ nesingulara \ \Rightarrow\ det(A) \neq 0 \ \Rightarrow\ A\ inversabila$

$Ax = \lambda x \ \Rightarrow\ A^{-1}Ax = \lambda A^{-1}x \ \Rightarrow\ I_n x = \lambda A^{-1}x \ \Rightarrow\ \boldsymbol{A^{-1}x = \frac{1}{\lambda}x}$