MATH GR 5320
Financial Risk Management and Regulation

Lecture 6: Credit risk modeling I

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Outline

Review

Review

Model risk management:

- Fastest growing area in finance.
- 3 components:
  1. Effective governance.
  2. Robust model development, implementation and use.
  3. Sound model validation practices.

Governance:

- Firm-wide respect for model risk – "Safety first".
- Independent reporting lines.
- Auditing.

Review

Model development:

- Follow sound development practices:
  - Specification document.
  - Model design document.
  - Data analysis.
  - Software design document.
  - Code review and coding standards.
  - Testing.
  - System analysis and testing.
  - Change management.
- Followup on performance after deployment.

Review

Model validation:

- Separate team.
- Separate line of reporting.
- Critical review of models.
- Principal of “effective challenge”.

Review

Testing:

- Numerical code is more difficult to test than other code.
- Know the failure modes.
- Check special cases.
- Check overall behavior.
- Robustness testing.
- Backtesting.
- Data testing.

Credit modeling overview

Credit modeling overview

There are three types of credit analysis:

- Analyzing credit worthiness of borrowers – loan credit analysis and management.
- Analyzing impact of defaults and credit changes – trading book credit risk.
- Analyzing trading strategies – fixed income investment.

Each analysis is different and risk management in each area is different.

Loan credit analysis

Loan credit analysis – Important for the banking book.

Evaluate credit worthiness of borrower *before* making loans.

Analysis of commercial borrowers differs from that of individual borrowers:

- Commercial loans:
  - Balance sheet.
  - Cash flows.
  - Market performance.
- Retail loans:
  - FICO score.
  - Tax returns.
  - Outstanding loans.
  - Savings.
  - Job security.
  - Location.

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Loan credit risk management

Loans are in the banking book.

Managed differently from trading book.

- Analyze borrower before making a loan.
- Adjust interest rate to compensate for borrower risk.
- Avoid concentration risk.
- After making loans:
  - Maintain collateral for estimated losses.
  - Monitor payment timeliness.
  - Restructure if necessary.
  - Seize collateral, if any.

seasonal collateral

References: Office of the Comptroller of the Currency [Off98], Nordenfelt and Villasenor [NV08]

Trading book credit risk

Trading book credit risk – analysis of default risk.

- Default probabilities.
- Rating transitions.
- Recovery rates.
- Not spread volatility – that's in market risk.

Kinds of default models:

- Reduced form.
- Structural.

Fixed income investment

Fixed income investment - analysis of credit spreads.

- Volatility of spreads.
- Impact on prices.
- Relationships between spreads and market factors.
- Hedging relationships.

Types of modeling:

- Correlation of spreads to other factors. regressions

Default modeling overview

Default modeling

Default modeling – Explicitly model defaults to enable pricing of risky instruments or computing default risk

References:

- *Credit risk models*, Laurent, [Lau08]
- *A comparison of credit risk models*, Benito, Glavan, and Jacko, [BGJ05]
- *Credit risk modeling and valuation: An introduction*, Giesecke, [Gie04]
- *Statistical Methods in Credit Risk Modeling*, Zhang, [Zha09]
- *Credit risk modeling*, Bielecki, Jeanblanc-Picqué, and Rutkowski, [BJPR09]
- *Evaluating credit risk models*, Lopez and Saidenberg, [LS00]

Traditional default modeling

Traditional approaches:

- Reduced form models:
  - Artzner-Delbaen [AD95], Jarrow-Turnbull [JT95], ...
  - Directly model statistics of time to default, not the structure of the firm.
  - Exponentially distributed arrival time.
- Structural models:
  - Merton [Mer74], Black-Cox [BC76], Avellaneda-Zhu [AZ01], Longstaff-Schwartz [LS95], Duffie-Singleton [DS99], Hull-White [HW01], ...
  - Model assets and liabilities.
  - Assets follow a continuous process.
  - Default when assets cross a barrier (become too small relative to liabilities).
  - Directly model time to default as a function of above.

Contemporary default modeling

Today:

- Reduced form:
  - Any model where default time is directly modeled, even if it involves modeling assets and liabilities.
- Structural:
  - Any model where default occurs when a process crosses a barrier, even if not directly modeling assets and liabilities.

Modern classification

Classify by information. Two extremes:

- Reduced form:
  - Exponentially distributed arrival time.
  - Default time is unpredictable.
- Merton:
  - Continuous process hitting barrier.
  - Default time is predictable.

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Two ends of a continuum.

Adjust between them by modeling and adding uncertainty.

My classification

My classification:

- Reduced form
  - Hazard rate is deterministic.
- Merton
  - Hazard rate is stochastic and zero except for a spike in the future.

Two ends of a continuum.

All default models can be expressed and understood in terms of their hazard rate curve and survival curve processes. Adjust between them by modifying these processes.

Option pricing review

Option pricing review

Options are priced under the equivalent martingale measure with respect to a numéraire.

What is a numéraire?

- A numéraire is just an asset or self financing strategy that is always positive.
- We price derivatives by dividing by a numéraire and computing expectations.

What is the intuition?

- Dollars are not tradeable in the market, so quoting the prices of assets in terms of dollars is arbitrary.
- Multiplying all the prices by arbitrary functions of time would yield the same market.
- Dividing by the price of an asset in the market expresses prices in terms of shares of a tradeable asset, thus normalizing everything.

Numéraireology

How does dividing by the numéraire simplify calculations?

- The market is arbitrage free iff prices are given by integrating against measures in a consistent way [HK79; HP81; DS94].
- After dividing by a numéraire, the numéraire becomes an asset with a constant value of 1.
- These measures become probability measures and integrating against them becomes computing expectations.
- Each asset divided by the numéraire becomes a martingale:

$$S_t/N_t = E^Q[S_T/N_T|\mathcal{F}_t]$$

  This greatly simplifies calculations.
- But, remember $Q \neq P$!

Note — This can be done with *any* selection of numéraire, but different choices of numéraires will necessarily correspond to different measures.

Reduced form models

Reduced form models

Reduced form models directly model the statistics of the default (stopping) time τ.

Value of receiving X at T if no default first (else zero):

$$N_0 E^Q[X/N_T 1_{\tau>T}]$$

If τ and X/N_T are uncorrelated:

$$\begin{aligned} N_0 E^Q[X/N_T 1_{\tau>T}] &= N_0 E^Q[X/N_T] E^Q[1_{\tau>T}] \\ &= X_0 E^Q[1_{\tau>T}] \end{aligned}$$

Then Q (risk neutral) survival probability $s(T) = E^Q[1_{\tau>T}]$ is all we need to know to price X.

This is how CDSs are priced.

Characteristics

Default time distributions come up in survival analysis:

- τ Default time.
- $s(t)$ Survival probability.
- $p(t)$ Default time probability density.
- $\lambda(t)$ Hazard rate – instantaneous rate at which default events occur.

Relationships:

$$s(t) = \text{Prob}(\tau > t) = E[1_{\tau > t}] = 1 - \text{CDF}(\tau)$$

$$p(t) = -\frac{ds(t)}{dt}$$

$$s(t) = 1 - \int_0^t p(x)dx = \int_t^\infty p(x)dx = e^{-\int_0^t \lambda(x)dx}$$

$$\lambda(t) = \frac{p(t)}{s(t)} = -\frac{d\log(s(t))}{dt}$$

$$E[\tau] = \int_0^\infty tp(t)dt = \int_0^\infty s(t)dt$$

Example

Simple case – Constant hazard rate $\lambda(t) = \lambda$

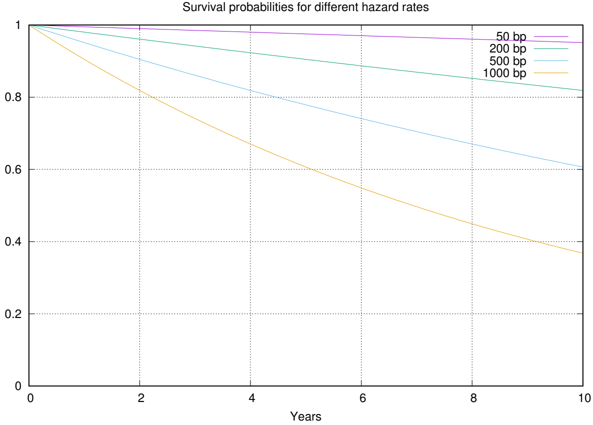
$$s(t) = e^{-\lambda t}$$
$$p(t) = \lambda e^{-\lambda t}$$
$$E[\tau] = 1/\lambda$$
$$\mathrm{var}[\tau] = 1/\lambda^2$$

Characteristics:

- Default time is exponentially distributed.
- Defining characteristic – distribution is memoryless:
  - Knowing that default hasn't occurred yet doesn't change chances of default.
  - Sort of stationary for survival analysis.
  - $\mathrm{Prob}(\tau > t | \tau > s) = \mathrm{Prob}(\tau > t - s)$.
  - Default time is unpredictable.
- Commonly used in modeling queues – Poisson arrival process has exponentially distributed times between arrivals – M/M/1 queues, etc.
- s is like a discount factor and λ is like a short rate.

Survival curves

Sample survival probability curves:



Note that in reduced form models, slope at zero is always < 0.

there is always prob to default

Pricing

Let X be a risky \$1 paid at time T, with recovery rate R (paid at T if default occurs).

Pricing is the same as when analyzing spread behavior if we assume independence:

$$
\begin{aligned}
\mathcal{V}(X) &= N_0 E^Q\left[\frac{X}{N_T}\right] \\
&= N_0 E^Q\left[\frac{1}{N_T} 1_{\tau>T} + \frac{R}{N_T} 1_{\tau\le T}\right] \\
&= N_0 E^Q\left[\frac{1}{N_T}\right] E^Q[1_{\tau>T}] + N_0 E^Q\left[\frac{R}{N_T}\right] E^Q[1_{\tau\le T}] \\
&= D(T)\left(s(T) + R(1 - s(T))\right) \\
&= D(T)\left(1 - \mathrm{LGD} \times (1 - s(T))\right) \\
&= D(T)\left(1 - \mathrm{LGD} \times (1 - e^{\int_0^T \lambda(u)du})\right)
\end{aligned}
$$

Conversions

Spread implied by a given hazard rate:

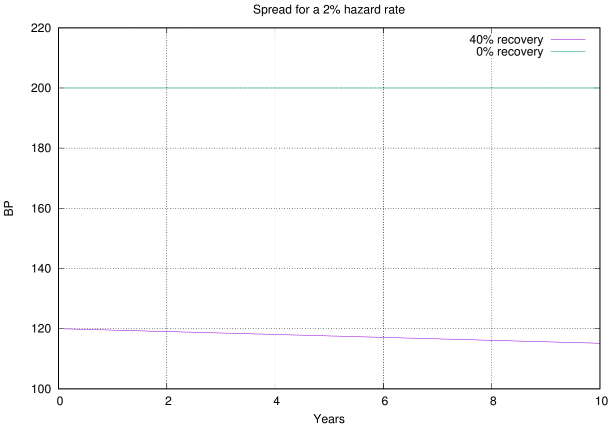
$$\begin{aligned} \mathcal{V}(X) &= e^{-(r+S)T} \\ &= e^{-rT}\left(1 - \mathrm{LGD} \times (1 - s(T))\right) \\ S &= -\frac{1}{T}\log\left(1 - \mathrm{LGD} \times (1 - e^{-\lambda T})\right) \end{aligned}$$

Hazard rate implied by a given spread:

$$\begin{aligned} e^{-ST} &= 1 - \mathrm{LGD} \times \left(1 - e^{-\lambda T}\right) \\ e^{-\lambda T} &= \frac{1 - e^{-ST}}{\mathrm{LGD}} - 1 \\ \lambda &= -\frac{1}{T}\log\left(1 - \frac{1 - e^{-ST}}{\mathrm{LGD}}\right) \end{aligned}$$

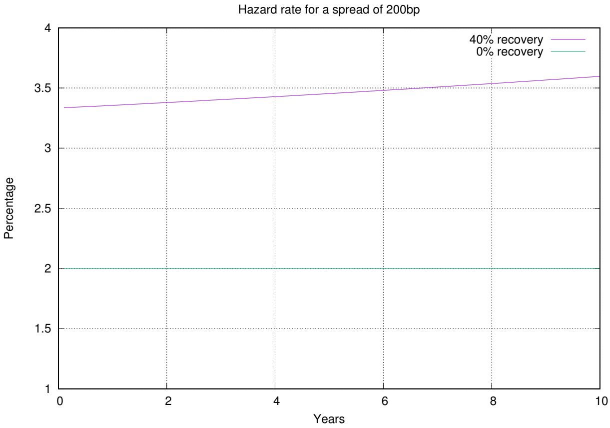
Spread for hazard

Spreads for a 2% hazard rate



Hazard for spread

Hazard rates for a 200 bp spread



General properties

General case, time dependent hazard rate:

- IID selection of waiting times – non-homogeneous Poisson process.
- Defining characteristic – $\text{Prob}(t_1 < \tau < t_2 | \mathcal{F}_t)$ is a constant for all $t < t_1$.
  default during t1 and t2
  - Process is no longer memoryless.
  - Knowing that default hasn't occurred yet changes chances of default because $\lambda(t)$ is changing.
  - Still essentially unpredictable.
- Piecewise constant hazard rate commonly used to calibrate to CDS spreads at discrete maturities.
- Spread as a function of time is deterministic.
- $s(t)$ monotonically decreasing, so $\lambda(t) \geq 0$.
- Any monotonically decreasing $s(t)$ can be put in this form (assuming conditional probabilities are constants).

Utility

Usage of default time distributions with time dependent hazard rates:

- CDS pricing, assuming some level of independence.
- Can extend to bond pricing.
- CVA calculations if portfolio is independent of default time.
- Pricing of anything that does not depend on forward probability volatility.
- Cannot be used for pricing options on CDS.
- Cannot be used for CVA when independence is violated (i.e. wrong way risk).
- Cannot be used for market risk (no spread volatility).

Extension

Can reduced form models be used more generally?

- Make hazard rate stochastic.

Behavior now depends on nature of hazard rate process.

Spread modeling

Spread modeling

Example of spread modeling from Benzschawel and Su [BS14].

S	Bond spread.
S_d	Spread due to real world default.
S_λ	Additional pricing spread.
R	Recovery rate (assume 40%).
LGD	Loss given default $\text{LGD} = 1 - R$, assume 60%.
$s(t)$	T forward measure survival probability for time t.
$\bar{s}(t)$	Real world survival probability for time t.

Convert risk neutral default probabilities to credit spreads.

Let $\mathcal{V}(X)$ be the value of a risky \$1 received at time T under the T forward measure Q (with ZCB price $Z_t(T)$):

$$
\begin{aligned}
X &= 1_{\tau > T} + R 1_{\tau \le T} \\
\mathcal{V}(X) &= Z_0(T) E^Q[(1_{\tau > T} + R 1_{\tau \le T})/Z_t(T)] \\
&= Z_0(T) \left(E^Q[1_{\tau > T}] + R E^Q[1_{\tau \le T}] \right) \\
&= Z_0(T) \left(s(T) + R(1 - s(T)) \right) \\
&= e^{-r_T T} \left(s(T) + (1 - \text{LGD})(1 - s(T)) \right) \\
&= e^{-r_T T} \left(1 - \text{LGD} \times (1 - s(T)) \right)
\end{aligned}
$$

Spread modeling

Express $\mathcal{V}(X)$ as a spread over the risky rate:

$$\begin{aligned}\mathcal{V}(X) &= e^{-(r_T+S)T} \\ &= e^{-r_T T}(1 - \mathrm{LGD} \times (1 - s(T))) \\ S &= -\frac{1}{T}\log(1 - \mathrm{LGD} \times (1 - s(T)))\end{aligned}$$

Use the same formula to convert real world default rates to a spread:

$$\begin{aligned}e^{-(r_T+S_d)T} &= e^{-r_T T}(1 - \mathrm{LGD} \times (1 - \bar{s}(T))) \\ S_d &= -\frac{1}{T}\log(1 - \mathrm{LGD} \times (1 - \bar{s}(T)))\end{aligned}$$

Spread modeling

Model the total spread as real default spread plus a “market price of risk” spread adjustment:

$$S = S_d + S_\lambda$$

Introduce:

- σ Spread volatility.
- λ Market price of risk.

Regressions (and option pricing theory) indicate:

- Residual spread S_λ should be proportional to volatility.

So:

$$S_\lambda = \lambda\sigma$$

Then λ is spread per unit of volatility.

Spread modeling

Usage:

- Model real world default rates.
- Model bonds as ZCB with principal paid at average bond duration.
- Regress $S - S_d$ against σ to get λ.
- Deduce various things about hedges.
- Potentially use to project spread changes for trading.

Summary

Summary

Categories of credit modeling:

- Loan underwriting:
  - Analyze borrower, charge for risks.
- Credit risk:
  - Model default events.
- Investment decisions:
  - Relationship between credit spreads and other parts of the market.

Summary

Loan underwriting:

- Balance sheet.
- FICO scores.
- Income and expenditures.
- Set rates based on riskiness.
- Position limits to limit losses.
- Done *before* loan is made.

Spread modeling:

- Decompose spreads into default spread + "market price of risk" spread: $S = S_d + S_\lambda$.
- Relate spreads to survival probabilities and losses.
- Model S_λ as proportional to volatility.
- $S_d = -1/T \log(1 + \mathrm{LGD} \times \bar{s}(T))$.
- $S_\lambda = \lambda\sigma$.
- Relate spread components to CDS indices and use for investment or hedging.

Summary

Credit risk:

- Explicitly model defaults for pricing risky instruments.
- Two kinds of models:
  - Structural models – model assets of firm.
    - Merton
    - Black-Cox
  - Reduced form – model statistics of time to default.
- Alternative view – information about default.
- Alternative view – everything is a reduced form model.

Summary

Reduced form:

- Model the properties of default the time τ.
- Assume a deterministic hazard rate $\lambda(t)$.
- Survival time $s(t) = \text{Prob}(\tau > t) = e^{-\int_0^t \lambda(u)du}$.
- Default time pdf $p(t) = -ds/dt$.

Properties:

- Default time is unpredictable.
- Deterministic forward spreads.

Usage:

- Piecewise constant λ used for pricing CDS.
- Used for CVA when pricing is independent of default time.
- Can't be used for market risk (no spread volatility).

Extend by making hazard rate stochastic.

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