MATH GR 5320
Financial Risk Management and Regulation

Lecture 7: Credit risk modeling II

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Outline

Review

Review

Categories of credit modeling:

- Loan underwriting:
    - Analyze borrower, charge for risks.
- Investment decisions:
    - Relationship between credit spreads and other parts of the market.
- Credit risk:
    - Model default events.

Review

Loan underwriting:

- Balance sheet.
- FICO scores.
- Income and expenditures.
- Set rates based on riskiness.
- Position limits to limit losses.
- Done *before* loan is made.

Spread modeling

- Decompose spreads into default spread + "market price of risk" spread: $S = S_d + S_\lambda$.
- Relate spreads to survival probabilities and losses.
- Model S_λ as proportional to volatility.
- $S_d = -1/T \log(1 + \mathrm{LGD} \times \bar{s}(T))$.
- $S_\lambda = \lambda\sigma$.
- Relate spread components to CDS indices and use for investment or hedging.

Review

Credit risk:

- Explicitly model defaults for pricing risky instruments.
- Two kinds of models:
  - Structural models – model assets of firm.
    - Merton
    - Black-Cox
  - Reduced form – model statistics of time to default.
- Alternative view – information about default.
- Alternative view – everything is a reduced form model.

Review

Reduced form:

- Model the properties of default the time τ.
- Assume a deterministic hazard rate $\lambda(t)$.
- Survival time $s(t) = \text{Prob}(\tau > t) = e^{-\int_0^t \lambda(u)du}$.
- Default time pdf $p(t) = -ds/dt$.

Properties:

- Default time is unpredictable.
- Deterministic forward spreads.

Usage

- Piecewise constant λ used for pricing CDS.
- Used for CVA when pricing is independent of default time.
- Can't be used for market risk (no spread volatility).

Extend by making hazard rate stochastic.

Single default modeling

Single default modeling

Credit risk default modeling

Two kinds of models:

- Reduced form – model statistics of time to default.
- Structural models – model assets of firm:
  - Merton
  - Black-Cox

Last lecture – reduced form.

This lecture – structural.

Structural models

If we model the balance sheet of the firm, we get a structural model.

Notions:

- Firm asset value follows a process.
- Firm defaults if asset value gets too low.

Variants depend on exactly how the above two notions are modeled.

Merton

The first structural model is known as the Merton model. [Mer74].

Assumptions:

- Assets of firm follow GBM: $dV = \mu V dt + \sigma V dW$.
- Only debt of firm is one zero coupon bond maturing at time T with face value B.
- For simplicity use a constant risk free rate of r, and assume no dividends.
- Standard option pricing assumptions.
- Default occurs at time T if the assets are insufficient to pay off the debt.

Conclusions:

- At time T, creditors receive $\min(V_T, B) = B - (B - V_T)^+$, B minus the value of a put with strike B.
- At time T, equity owners retain $\max(V_T - B, 0)$, the payoff of a call with strike B.

What are debt and equity values before T?

Merton

Insert figure illustrating Merton here.

Merton bond and equity prices

By option pricing theory:

- Time $t < T$ value of equity (S_t) is value of call on assets of firm with strike B.
- Time $t < T$ value of debt (D_t) is $Be^{-r(T-t)}$ minus put on assets with strike B.

Conclude:

$$
\begin{aligned}
S_t &= \Phi(d_1)V_t - \Phi(d_2)Be^{-r(T-t)} \\
d_1 &= \frac{\log(V_t/B) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= \frac{\log(V_t/B) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\
D_t &= V_t - S_t \text{ (by put-call parity)}
\end{aligned}
$$

Merton default probabilities

Merton default probability at time t:

$$p_t(T) = Q(V_T \leq B) = E^Q[1_{V_T < B}]$$
$$V_t - B = \mathrm{VaR}^Q(V, T, 1 - p_t(T))$$

So we see that default probabilities in the Merton model are just one minus the percentiles for the VaR that brings the asset value down to the level of the debt at the debt horizon date.

So we get (from the derivation of GBM VaR):

$$\begin{aligned} p_t(T) &= Q(V_T \leq B) \\ &= \Phi\left(\frac{\log\left(\frac{B}{V_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}\right) \end{aligned}$$

Merton implications

Implications of Merton's model:

- No default before time T – CDSs maturing before T have zero spread.
- Equity prices do *not* follow GBM!
- Default is predictable. Informally:
  - Probability of default goes to 1 as $t \to T$ when $V_t < B - \epsilon$
  - Probability of default goes to 0 as $t \to T$ when $V_t > B + \epsilon$
- Formally
  - There exist stopping times $\tau_n < \tau$ that converge to τ (an "announcing sequence").
  - Announcing sequence: Let τ_n be the first $t > T - 1/n$ for which $V_t \leq B$.
- Recovery rate is stochastic – $R = V_T/B$ if the firm defaults.

Note: As in the reduced form model, default probabilities are risk-neutral.

Merton reduced

In terms of stochastic default processes, for $t \leq u \leq T$, we have:

$$\tau = T1_{V_T < B} + \infty 1_{V_T \geq B}$$

$$s_t(u) = 1 - \Phi(-d_2)1_{u=T}$$

$$p_t(u) = \Phi(-d_2)\delta(u - T)$$

$$\lambda_t(u) = \frac{\Phi(-d_2)\delta(u - T)}{1 - \Phi(-d_2)}$$

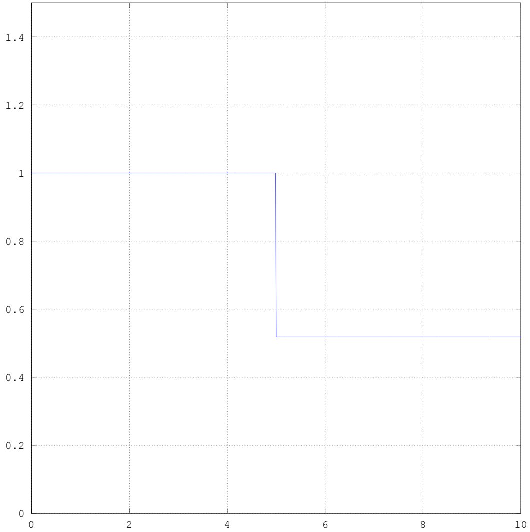
$$d_2 = \frac{\log(V_t/B) + (r - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

This phrases the Merton model as a reduced form model.

The default probability density, the survival probability and the hazard rate are now all stochastic processes and no longer deterministic.

Merton survival curve

Survival curve. Initial asset value = \$1,000,000, debt = \$900,000, $\sigma = 0.3$, $r = .03$, payment in 5 years:



Black-Cox

The Merton model is not commonly used because no default until time T is unrealistic.

The generalization that is commonly used is the Black-Cox model [BC76]

Merton:

- Default when $V_T \leq B$ for a fixed T.

Black-Cox:

- Default the first time $t < T$ that $V_t \leq B$, or at T if $V_T < K$.

Instead of passing a level at time T, we pass a level at any time – like a barrier option.

Other generalizations include:

- Stochastic interest rates [LS95].
- Time dependent barriers.
- Asset value following a jump diffusion.

Black-Cox specification

Black-Cox model:

- Asset value follows GBM. In equivalent martingale measure wrt money market numéraire ($B_t = e^{rt}$):
  - $dV = rVdt + \sigma VdW$
  - $V_t = V_0 e^{(r-\frac{\sigma^2}{2})t+\sigma W_t}$
- Default occurs when:
  - $V_T < K$, or
  - $V_t < B$.
- At default time τ:
  - Bond holders receive V_τ.
  - $V_\tau = B$ if $\tau < T$.

Black-Cox specification

Insert figure illustrating Black-Cox here.

Black-Cox formulas

To compute default probabilities under Black-Cox, we need to know when the asset value will hit the barrier.

References: Lecture 5 of Lalley [Lal01], or Chapter 5 of Steele [Ste01].

In Black-Cox, the asset value follows GBM:

$$V_t = V_0 e^{(r-\frac{\sigma^2}{2})t+\sigma W_t}$$

Define:

$$\tau \equiv \inf\{t | V_t = B\}$$

Then τ is the default stopping time for a default barrier of B in the Black-Cox model.

$$\begin{aligned} & V_t = B \\ \iff & (r-\sigma^2/2)t + \sigma W_t = \log(B/V_0) \\ \iff & W_t = \log(B/V_0)/\sigma + (\sigma/2 - r/\sigma)t \end{aligned}$$

So, the default stopping time is equivalently the first time that Brownian motion hits a tilted barrier.

Black-Cox formulas

Insert tilted barrier figure here.

Untilting

Take $V_0 = 1$ for simplicity. We need to know:

$$\tau = \inf\{t | W_t = \log(B)/\sigma + (\sigma/2 - r/\sigma)\, t\}$$

Define:

$$W_t^* = W_t - (\sigma/2 - r/\sigma)t$$

By applying the change of measure to make W^* a Brownian motion (say Q^*), we can untilt the barrier. In terms of W^*:

$$\tau = \inf\{t | W_t^* = \log(B)/\sigma\}$$

Then

$$\begin{aligned}\mathrm{Prob}(\tau < t) &= E^Q[1_{\tau<t}] \\ &= E^{Q^*}[1_{\tau<t} C_t^*]\end{aligned}$$

where $C_t^* = e^{-(\sigma/2 - r/\sigma)W_t + (\sigma/2 - r/\sigma)^2 t/2}$ is the change of measure factor.

Plan:

- Compute for W^* hitting a flat barrier under measure making it a BM.
- Use change of measure to compute for tilted barrier.

Black-Cox plan

Define

$$M_T \equiv \min_{0 \leq s \leq T}(W_s^*)$$

Then

$$\begin{aligned} & M_T \leq \log(B)/\sigma \\ \iff & \tau \leq T \end{aligned}$$

This gives us our survival probabilities:

$$\begin{aligned} s(T) &= \text{Prob}(\tau > T) \\ &= \text{Prob}(M_T > \log(B)/\sigma) \end{aligned}$$

So, we just need to know the distribution of the min of a Brownian motion, and then apply a measure change back to the original tilted barrier.

Reflection principle

The distribution of the min of a Brownian motion is computed using the reflection principle:

If $\tau < \infty$ a.e. is a stopping time for a Brownian motion W, then the process

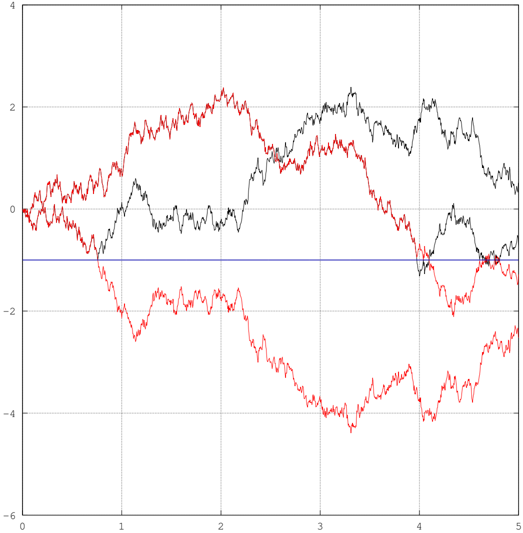
$$\overline{W}_t = \begin{cases} W_t & \text{for } t < \tau \\ W_\tau - (W_t - W_\tau) & \text{for } t \geq \tau \end{cases}$$

is also a Brownian motion. This is W_t reflected at τ.

In other words, when a path reaches time τ, we can reflect all the future values around the level at that time and the result is still a standard Brownian motion.

Reflection principle

Reflection principle for τ being the first time that W hits -1:



Hitting times and min of BM

We have[1]

$$
\begin{aligned}
M_T &\equiv \min_{0\le s\le T}(W_s)\\
\tau &= \inf\{t|W_t = \log(B)/\sigma\}\\
\overline{W}_t &= W_t \text{ reflected at } \tau\\
a &= \log(B)/\sigma\}
\end{aligned}
$$

Then

$$
\begin{aligned}
&\{M_T < a\}\\
\iff &\{W_T < a\} \text{ or } \{M_T < a \text{ and } W_T \ge a\}
\end{aligned}
$$

These two sets of events are disjoint, so

$$
\text{Prob}(M_T < a) = \text{Prob}(W_T < a) + \text{Prob}(M_T < a \text{ and } W_T \ge a)
$$

[1] We suppress the asterisk superscripts so as to avoid cluttering the formulas.

Brownian barrier hitting times

We have that:

$$\mathrm{Prob}(M_T < a) = \mathrm{Prob}(W_T < a) + \mathrm{Prob}(M_T < a \text{ and } W_T \geq a)$$

But

$$\begin{aligned}\mathrm{Prob}(M_T < a \text{ and } W_T \geq a) &= \mathrm{Prob}(M_T < a \text{ and } \overline{W}_T \leq a) \\ &= \mathrm{Prob}(\overline{W}_T \leq a) \\ &= \mathrm{Prob}(W_T \leq a)\end{aligned}$$

So

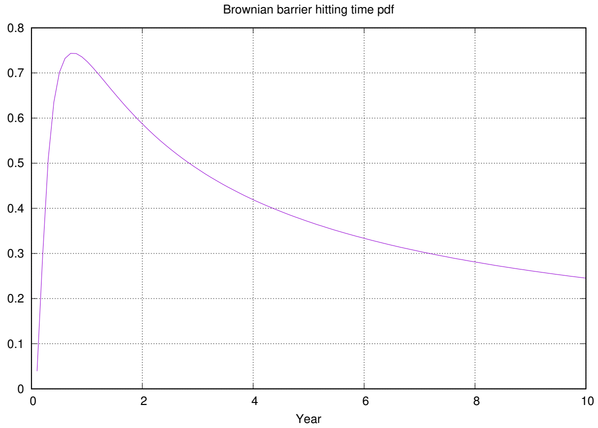
$$\begin{aligned}\mathrm{Prob}(\tau < T) &= \mathrm{Prob}(M_T < a) \\ &= 2\,\mathrm{Prob}(W_T < a) \\ &= 2\Phi\left(\frac{a}{\sqrt{T}}\right)\end{aligned}$$

$$\mathrm{PDF}(\tau) = -\frac{a}{T^{3/2}}\phi\left(\frac{a}{\sqrt{T}}\right)$$

($a < 0$ because this is a BM descending to a barrier.)

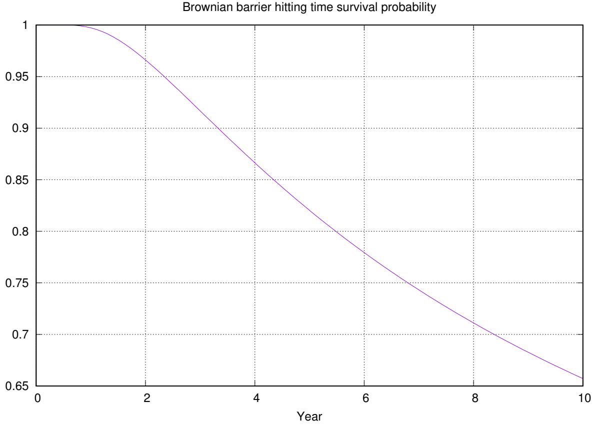
BM default time pdf

Default PDF for BM hitting a barrier of -3.



BM default pdf

Survival probabilities for BM hitting a barrier of -3.



Note that slope at $t = 0$ is always equal to zero.

Tilting the barrier

For convenience, let $a = \log(B)/\sigma$ and $b = \sigma/2 - r/\sigma$.

In terms of a and b,

$$\begin{aligned}
\tau &= \inf\{t | W_t = a + bt\} \\
W_t^* &= W_t - bt \\
C_t^* &= e^{-bW_t + b^2 t/2} \\
&= e^{-bW_t^* - b^2 t/2}
\end{aligned}$$

We need $\text{Prob}(\tau < t) = E^Q[1_{\tau<t}]$

$$\begin{aligned}
E^Q[1_{\tau<t}] &= E^{Q^*}[1_{\tau<t} C_t^*] \\
&= E^{Q^*}[1_{\tau<t} e^{-bW_t^* - b^2 t/2}]
\end{aligned}$$

If we could eliminate the exponential, we'd be done.

Elimination

We play some tricks...

$$
\begin{aligned}
E^Q[1_{\tau<t}] &= E^{Q^*}\left[1_{\tau<t}e^{-bW_t^*-\frac{b^2}{2}t}\right] \\
&= E^{Q^*}\left[1_{\tau<t}e^{-b(W_\tau^*+W_t^*-W_\tau^*)-\frac{b^2}{2}(\tau+t-\tau)}\right] \\
&= E^{Q^*}\left[1_{\tau<t}e^{-bW_\tau^*-\frac{b^2}{2}\tau}e^{b(W_t^*-W_\tau^*)-\frac{b^2}{2}(t-\tau)}\right]
\end{aligned}
$$

The third factor is the increment of W^* starting at τ. It's independent of the other two factors, so the expectation factors.

Moreover, the 3rd factor is a martingale, so its expectation is its initial value, which is 1. So it drops out.

Cleanup

Removing the 3rd term leaves us with a little algebra:

$$
\begin{aligned}
E^{Q}[1_{\tau<t}] &= E^{Q^*}\left[1_{\tau<t}e^{-bW^*_\tau-\frac{b^2}{2}\tau}e^{b(W^*_t-W^*_\tau)-\frac{b^2}{2}(t-\tau)}\right] \\
&= E^{Q^*}\left[1_{\tau<t}e^{-bW^*_\tau-\frac{b^2}{2}\tau}\right] \\
&= E^{Q^*}\left[1_{\tau<t}e^{-ab-\frac{b^2}{2}\tau}\right] \\
&= e^{-ab}E^{Q^*}\left[1_{\tau<t}e^{-\frac{b^2}{2}\tau}\right] \\
&= e^{-ab}\int_0^t e^{-\frac{b^2}{2}x}\,\mathrm{PDF}^{Q^*}(\tau)dx
\end{aligned}
$$

$$
\mathrm{PDF}(\tau) = -\frac{ae^{-ab-\frac{b^2}{2}T}}{T^{3/2}}\phi\left(\frac{a}{\sqrt{T}}\right) = -\frac{a}{T^{3/2}}\phi\left(\frac{a+bt}{\sqrt{T}}\right)
$$

The CDF is gotten by manipulating the integral to convert it to a cumulative normal.

Black-Cox default time CDF

The lazy way (via Mathematica):

$$
\begin{aligned}
In[1] &:= dist = NormalDistribution[0,1] \\
Out[1] &= NormalDistribution[0,1] \\
In[2] &:= p = PDF[dist, x] \\
Out[2] &= e^{-x^2/2}/\sqrt{2\pi} \\
In[3] &:= tstop = (p/.x- > (a+bx)/\sqrt{x})/x^{3/2} \\
Out[3] &= e^{(-(a+bx)^2/(2x))}/(\sqrt{2\pi}x^{3/2}) \\
In[4] &:= tstopcdf = Integrate[tstop, x]//FullSimplify \\
Out[4] &= (e^{-ab-\sqrt{a^2}\sqrt{b^2}} \\
&\qquad \times (-2 + Erfc[(\sqrt{a^2} - \sqrt{b^2}x)/(\sqrt{2x})] \\
&\qquad\qquad + e^{2\sqrt{a^2}\sqrt{b^2}} Erfc[(\sqrt{a^2} + \sqrt{b^2}x)/(\sqrt{2x})]))/(2\sqrt{a^2})
\end{aligned}
$$

Properties

Black-Cox properties:

- Spreads still tend to zero as $t \to 0$
- $\lambda_t(t) = 0$
- Default is predictable:
  - Let τ_n^1 be the hitting time for the barrier $B + 1/n$.
  - Let τ_n^2 be as in Merton.
  - Let $\tau_n = \min(\tau_n^1, \tau_n^2)$.
- Similar to Merton, we can rephrase as a reduced form model.

Information analysis

From a risk and valuation point of view, the difference between classical reduced form models and structural models is largely in the nature of the default event:

- Reduced form:
  - Unpredictable arrival of default.
  - No prior information about default event.
- Structural:
  - Predictable arrival of default.
  - Default time is known with greater and greater certainty as the default event is approached.

Informational analysis tries to unify and clarify these relationships by explicitly working with the information (i.e. the sigma algebra filtration).

References: Jarrow and Protter [JP04], Cetin et al. [Cet+04], Giesecke [Gie06]

Information analysis example

Forgetful Merton, as in Giesecke [Gie06] or Cetin et al. [Cet+04]:

- We know r, V_0, σ, B, and T.
- We don't know V_t for $t > 0$ – i.e. take the filtration of V out of what we know.

Analysis:

- Let $p = P(A_T < N | A_0)$.
- $s_0(u) = 0$ for $u < T$
- $s_0(u) = 1 - p$ for $u \geq T$.

As for $s_t(u)$ for $0 < t < T$:

- No new information.

Therefore:

$$s_t(u) = \begin{cases} 1 - p 1_{u \geq T} & \text{if } t < T \\ 1_{A_T > N} & \text{if } t \geq T \end{cases}$$

The model becomes like flipping a biased coin at time T to determine default.

Shortcomings

For risk analysis, all of these models have a significant shortcoming:

- Default probabilities are risk neutral.

For risk analysis, one needs real world default probabilities. Options:

- Incorporate the market price of risk to convert back to real world default rates
- Calibrate to history instead of to market prices

Both approaches are followed in practice.

Joint default

Joint default

Joint default modeling is important for:

- Modeling impact of default on a portfolio.
- Incorporating CVA into CDS.
- Pricing of exotic default options and correlation products (CDOs).

We look for:

- Realistic behavior.
- Default clustering.
- Ease of use.
- Calibration to markets or to historical data.
- Rating transition modeling.

Reduced joint I

A simple approach in reduced form modeling:

- Make hazard rates stochastic and correlated.
- $d\lambda_i = \mu_i \lambda dt + \sigma_i \lambda dW_i$.
- $dW_i dW_j = \rho_{ij} dt$.

Properties:

- Spreads are a function of hazard rates, so does a good job of correlating spreads.

Issues:

- Doesn't help much in correlating default times – two hazard rates going up doesn't mean default occurs.
- Calibration issue – where does correlation come from?
  - Pick ρ_{ij} so that spread correlation is correct.
  - Not so easy...

Reduced joint II

Alternative approach to correlating default in reduced form models:

- Model joint default.
- τ_1 Default time for just stock 1 defaulting.
- τ_2 Default time for just stock 2 defaulting.
- τ_{12} Default time for stock 1 and 2 defaulting together.

Properties:

- Easily incorporate appropriate joint default rates.

Issues:

- Joint default is simultaneous default.
- For n, get $O(n^2)$ extra default pairs.
- What about default triplets, ...?
- No rating transitions.

Structural joints

Structural joint default modeling:

- Can't use classic Merton unless bond maturities are at the same time.
- Use Black-Cox with correlated BMs.
- Correlate asset values by historical stock price correlations.
- Introduces significant correlation between default times.

Properties:

- Correlates default times and spreads.
- Use multiple barriers to simulate rating transitions.

Issues:

- Does not calibrate well to market.
- Not clear how to convert between real world and risk neutral.

Reduced redux

Third reduced form joint framework method – copula method:

- Default time distribution is marginal distribution of default time for each name.
- Create joint distribution by linking marginals with a copula.

Theorem (Sklar):
If the CDF of X_i is $F_i(x)$, and the joint CDF of the X_i is $F(x_1, \ldots, x_n)$, then there exists a unique function C such that

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

Then C is the copula associated with F.

Properties of copulas:

- Constructs a joint distribution directly from the marginals.
- Correlation structure separated from marginal behavior.
- Correlation introduction does not impact individual correlations.

Copula methodology

To use copulas:

- Pick a copula from a known joint distribution (Gaussian copula, etc).
- Apply it to the default time CDFs.
- Play tricks to reduce the dimensionality.

Typically model losses instead of default times:

- Loss distribution is default $+ (1 - R)$ times notional of defaulting name.
- Smooth in space and time and get a loss distribution for each time t.
- Integrate as appropriate.

Copula usage

Properties:

- Correlates default times.
- When applied to basic reduced form model, still no spread volatility.
- Losing processes.

Usage:

- Commonly used for CDO pricing.

Reference: Burtschell, Gregory, and Laurent [BGL05]

Bad rep

For five years, Li's formula, known as a Gaussian copula function, looked like an unambiguously positive breakthrough, a piece of financial technology that allowed hugely complex risks to be modeled with more ease and accuracy than ever before. With his brilliant spark of mathematical legerdemain, Li made it possible for traders to sell vast quantities of new securities, expanding financial markets to unimaginable levels.

His method was adopted by everybody from bond investors and Wall Street banks to ratings agencies and regulators. And it became so deeply entrenchedand was making people so much moneythat warnings about its limitations were largely ignored.

Then the model fell apart. Cracks started appearing early on, when financial markets began behaving in ways that users of Li's formula hadn't expected. The cracks became full-fledged canyons in 2008 when ruptures in the financial system's foundation swallowed up trillions of dollars and put the survival of the global banking system in serious peril.

Recipe for Disaster: The Formula that Killed Wall Street Salmon [Sal09]

Analysis

Is Salmon correct?

- Was model risk ignored?
- Was it really the usage of copulas?
- Is David Li to blame for the entire financial crisis?

Summary

Summary

Credit risk:

- Explicitly model defaults for pricing risky instruments.
- Two kinds of models:
  - Structural models – model assets of firm:
    - Merton
    - Black-Cox
  - Reduced form – model statistics of time to default.
- Alternative view – information about default.
- Alternative view – everything is a reduced form model.

Summary

Reduced form:

- Model the properties of default the time τ.
- Assume a deterministic hazard rate $\lambda(t)$.
- Survival time $s(t) = \text{Prob}(\tau > t) = e^{-\int_0^t \lambda(u)du}$.
- Default time pdf $p(t) = -ds/dt$.

Properties

- Default time is unpredictable.
- Deterministic forward spreads.

Usage:

- Piecewise constant λ used for pricing CDS.
- Used for CVA when pricing is independent of default time.
- Can't be used for market risk (no spread volatility).

Extend by making hazard rate stochastic.

Summary

Merton:

- Assets of firm follow GBM: $dV = \mu V dt + \sigma V dW$.
- Default at time T if $V_T < B$.

Properties:

- Equity is call option on assets with strike B and maturity T.
- Debt is ZCB paying B at T minus a put on the assets with strike B.
- Default time is predictable – announcing sequence given by τ_n being the first $t \geq T - 1/n$ for which $V_t \leq B$.
- Can be rephrased as a reduced form model:
  - $\tau = T 1_{V_T < B} + \infty 1_{V_T \geq B}$
  - $s_t(u) = 1 - \phi(-d_2) 1_{u=T}$
  - $p_t(u) = \Phi(-d_2)\delta(u - T)$
  - $\lambda_t(u) = \frac{\Phi(-d_2)\delta(u-T)}{1-\Phi(-d_2)}$

Summary

Black-Cox:

- Assets of firm follow GBM: $dV = \mu V dt + \sigma V dW$.
- Default if $V_t \leq B$ or if $V_T \leq K$.
- Formulas:
  - $a = \log(B/V_0)$.
  - $b = \sigma/2 - r/\sigma$.
  - PDF(τ): $-\frac{a}{T^{3/2}}\phi\left(\frac{a+bt}{\sqrt{T}}\right)$.
  - $CDF(\tau)$: $\frac{e^{-ab-|ab|}}{2|a|}(Erfc[\frac{|a|-|b|x}{\sqrt{2x}}] + e^{2|ab|}Erfc[\frac{|a|+|b|x}{\sqrt{2x}}] - 2)$

Properties:

- Spreads still always start out at 0.
- Default is predictable:
  - Let τ_n^1 be the hitting time for the barrier $B + 1/n$.
  - Let τ_n^2 be as in Merton.
  - Let $\tau_n = \min(\tau_n^1, \tau_n^2)$.
- Better match to real world.
- Still does not calibrate well.
- Formula derivations complex – involve reflection principle, usage of martingales, etc.

Summary

Joint default modeling:

- Needed for CDOs, portfolio losses, etc.
- Options:
  - Stochastic correlated hazard rates.
  - Joint default stopping time.
  - Correlated BMs in Black-Cox.
  - Copula method.

In general:

- Complicated.
- Issues in switching between risk neutral and real world.

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