MATH GR 5320
Financial Risk Management and Regulation

Lecture 9: CVA

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Fall 2016

Compilation: November 1, 2016 at 13:14

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Outline

Review

Review

Major points:

- Credit risk is factored into OTC derivative valuations.
- Qualitative analysis of risk mitigants:
  - Netting agreements.
  - Collateralization.
  - Use of clearing houses.
  - Each risk mitigant introduces other risks.
- Qualitative analysis of counterparty credit risk:
  - Long only positions (like bonds) have a very different risk profile from derivatives that can change from assets to liabilities.
  - For assets that can change (like swaps), credit exposure is nonlinear (like a call option).

Review

CDS pricing, assuming zero correlation between numéraire and default, and constant known recovery rate.

$$\sum^{n} C\alpha(t_i)D(t_i)s(t_i) + \int C\alpha^*(u)D(u)p(u)du - \int (1-R)D(u)p(u)du$$

with:

C	CDS spread
$t_i, \ldots, t_n$	Payment times
$\alpha(t_i)$	Accrual factor for payment on t_i (roughly $1/2$ for semiannual payments)
$\alpha^*(t)$	Accrued interest factor, $\approx \frac{t-t_{i-1}}{t_i-t_{i-1}}\alpha(t_i)$
$s(t)$	Survival probability = probability that default occurs after t
$p(t)$	Default time PDF

Default probabilities are WRT pricing measure Q, not real world measure P.

Review

CDS bootstrap par spread:

$$\begin{aligned}
C(T_n) &= \frac{\int (1-R)D(u)p(u)du}{\sum^n \alpha(t_i)D(t_i)s(t_i) + \int \alpha^*(u)D(u)p(u)du} \\
s(t) &= e^{-\int_0^t \lambda(u)du} \\
p(t) &= -ds/dt = s(t)\lambda(t) \\
T_i &= \text{maturity dates of quoted CDS contracts} \\
\lambda(t) &= \lambda_i \text{ for } T_{i-1} < t \le T_i
\end{aligned}$$

I.e. stripped with hazard rate ($\lambda(t)$) being piecewise constant.

$\lambda(t)$ can be though of as the default arrival rate at time t.

CVA

Credit Valuation Adjustments

How does the counterparty exposure and the risk of default impact the value of the security?

- The Credit Valuation Adjustment (CVA) is the cost of the potential loss.
- Risk free price - CVA = price of risky security.

Counterparty risk — long only vs long/short

Counterparty risk calculations are far more complicated for instruments that are a combination of long/short positions than for long only instruments.

- In a long only instrument, (like a bond position), counterparty risk can be judged by using models that can incorporate a discount curve shift.
- In a long/short instrument, (like a swap position), the instrument can potentially be an asset or a liability. When it's an asset, default results in a loss. When it's a liability, default results in no change.

Valuation

Bond credit risk

If a bond with a coupon of C pays f times per year at times t_i, with maturity t_n, the value of the bond is:

$$\sum^{n} \frac{C}{f} D(t_i)s(t_i) + 100D(t_n)s(t_n) + \int^{t_n} 100RD(t)p(t)dt$$

- $p(t)$ — default probability density function.
- $s(t) = 1 - \int^{t} p(u)du$ — survival probability for time t (the probability of no default before time t).
- R — bond recovery rate.
- $D(t)$ — risk free discount factor for time t.

This is the above CDS pricing model applied to a fixed coupon bond. Note that it assumes independence of rates and default.

One of the original references on the subject is Altman and Kishore [AK96].

Bond model risk

What is the model risk in analyzing bonds by:

$$\sum^{n} \frac{C}{f} D(t_i)s(t_i) + 100D(t_n)s(t_n) + \int^{t_n} 100RD(t)p(t)dt$$

Assumptions:

- No mitigating terms in the bond prospectus.
- Interest rates and default are independent.
- Recovery rates are known.
- Default probabilities are known and reliable.
- Swap spreads are known and reliable.
- Curve stripping methods have no impact.
- We have the right coupon and frequency.
- We recover immediately at default time.

If using CDS spreads to calculate default probabilities, also:

- No idiosyncratic bond behavior (bond/cds basis = 0).
- CDS spreads are known and reliable.

Equivalent par curve

To get a feel for the impact of credit spreads on bond values, we can compute the par curve for the risky bond. For $t_i = i/f$, and for each n, solve for $C(t_n)$ such that

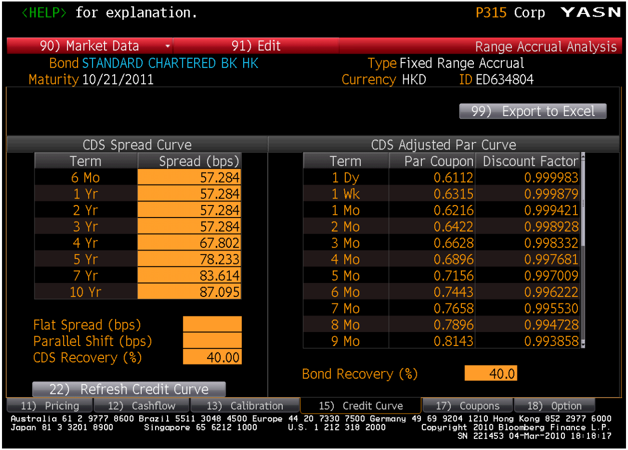
$$100 = \sum^{n} \frac{C(t_n)}{f} D(t_i)s(t_i) + 100D(t_n)s(t_n) + \int^{t_n} 100RD(t)p(t)dt,$$

Then $C(t_n)$ is the implied par curve — the coupons that the issuer with this CDS spread curve would theoretically use to issue debt at par.

This is essentially the common convention of discounting at a spread.

Equivalent par curve example

On the Bloomberg terminal, we do this calculation in YASN - the structured notes calculation screen.



Par curves vs default probabilities

Once the risky par curve is computed, one is tempted to strip it and use it for discounting.

Good points:

- This is simple, straight forward, and in line with common practices.
- This will properly price par bonds back to par.
- If the bond recovery rate is zero, this properly prices *all* bonds!

In the zero recovery rate case, this makes the risky discount factors

$$D(t_i)s(t_i),$$

so the risky spot rate curve $\bar{r}$ is given by

$$\bar{r}(t_i) = -\log(D(t_i)s(t_i))/t_i = r(t_i) - \log(s(t_i))/t_i,$$

where r is the risk free rate. So, the survival probabilities add a spread of $-\log(s(t_i))/t_i$ (the average hazard rate) to the risk free rate. This spread is roughly the CDS spread, adjusted by the CDS recovery rate.

Par curves vs default probabilities

Using the risky par curve directly for calculations also has drawbacks.

- For nonzero recovery rates, prices produced by this method on non-par bonds will differ from the default based method.

There is a difference between the two methods when the bond is a couple of hundred basis points away from par (roughly 5 to 15 basis points for a 100 bp CDS spread), but given other general uncertainties (such as the recovery rate, or the spread between bonds and CDS), this is a reasonable margin of error, and can be folded into OAS adjustments.

Par curves and OAS

If the spread curve is flat, it roughly amounts to a shift of the par curve, which is roughly adding an OAS.

Using the risky par curve in such a calculation is an improvement, in that it factors in the shape of the CDS spread curve.

Counterparty risk in swaps — Characteristics

Consider a five year swap in a flat interest rate environment.

There is volatility dependent risk:

- Zero volatility:
  - Swap is always nearly zero market value.
  - Minimal default risk.
- High volatility:
  - Swap value in future can be substantial.
  - Potentially substantial loss upon default.

There is curve dependent risk:

- In a steep interest rate environment, the swap is expected to be heavily off the money for the duration of its life.
- Far more counterparty risk.

Counterparty risk in swaps — Valuation

Let V_t be the value of the risk-free swap at time t, and let R be the recovery rate on the underlying swap.

If the counterparty defaults at time τ, the payoff for holding the swap is:

- If $V_\tau > 0$ we get $R \times V_\tau$.
- If $V_\tau < 0$ we still owe V_τ.

The above payoff is

$$\begin{aligned} &R \max(V_\tau, 0) + \min(V_\tau, 0) \\ &= V_\tau - (1 - R) \max(V_\tau, 0) \end{aligned}$$

The quantity $\max(V_\tau, 0)$ is the payoff of an option to enter into what's left of the swap at time τ — i.e. - a swaption maturing at time τ.

References: Alavian et al. [Ala+09], Brigo and Masetti [BM05], Cooper and Mello [CM91], Jarrow and Yu [JY01], Pykhtin and Zhu [PZ07], Stein and Lee [SL11], and Stein [Ste07].

Counterparty risk in swaps — Valuation

The loss at default time τ is:

$$(1-R)\max(V_\tau, 0)$$

The cost of this loss is:

$$\begin{aligned} CVA &= N_0 E^Q[(1-R)\max(V_\tau,0)/N_\tau 1_{\tau<T}] \\ &= (1-R)N_0 E^Q[\int_0^T \max(V_t,0)/N_t \delta(t-\tau)dt] \\ &= (1-R)N_0 \int_0^T E^Q[\max(V_t,0)/N_t \delta(t-\tau)]dt \end{aligned}$$

Counterparty risk in swaps — Valuation

If $\max(V_t, 0)/N_t$ (the value of the call) and $\delta(t-\tau)$ (the default event) are independent, then the expectation factors and the CVA is:

$$
\begin{aligned}
CVA &= (1-R)N_0 \int_0^T E^Q[\max(V_t,0)/N_t \delta(t-\tau)]dt \\
&= (1-R) \int_0^T N_0 E^Q[\max(V_t,0)/N_t] E^Q[\delta(t-\tau)]dt \\
&= (1-R) \int_0^T S(t)p(t)dt
\end{aligned}
$$

where $S(t)$ is the current value of the swaption to enter into the remainder of the swap at time t, and $p(t)$ is the default time probability density function.

Practical valuation

CVA valuation can be a little subtle. First of all, it's impractical to compute the above integral. One approach is discretization. Divide the time interval $[0, T]$ into periods $[t_i, t_{i+1}]$, and select $\bar{t}_i \in [t_i, t_{i+1}]$. Then

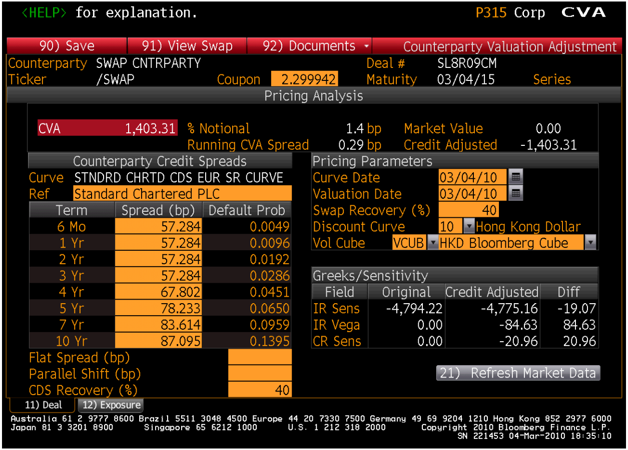
$$\begin{aligned} CVA &= (1-R)\int_0^T S(t)p(t)dt \\ &\approx (1-R)\sum S(\bar{t}_i)\bar{p}(t_i) \end{aligned}$$

where $\bar{p}(t_i) = \int_{t_i}^{t_{i+1}} p(t)dt = s(t_i) - s(t_{i+1})$ is the probability of defaulting in interval $[t_i, t_{i+1}]$.

The finer one subdivides the time period, the more accurate the calculation. It suffices to divide the time interval according to the cashflows of the swap as long as one values the call options at the midpoints of the intervals.

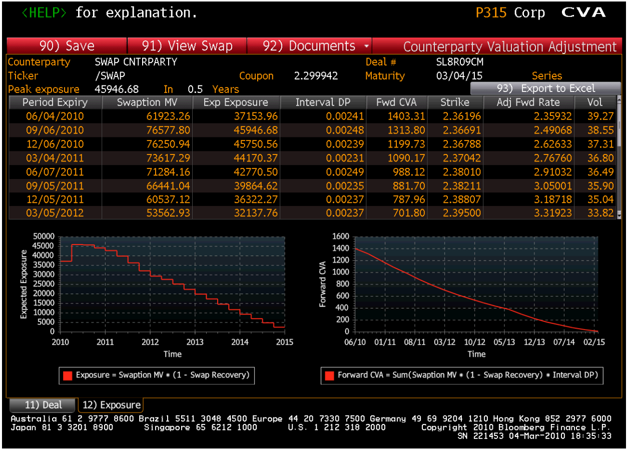
Counterparty risk in swaps — CVA<Go>

On the Bloomberg, the CVA function does this calculation.



Visualizing counterparty risk

The calculation and risks can be better visualized by exposure graphs over time:



CVA Model Risk

$$CVA = (1 - R)\int_0^T S(t)p(t)dt \approx (1 - R)\sum S(\bar{t}_i)\bar{p}(t_i)$$

What risks do we have?

- Interest rates and default are independent.
- Recovery rate is currently known.
- CDS and swap spreads are known and reliable.
- CDS and swap curve stripping methods have no impact.
- We have the right coupon and frequency.
- Option valuation is accurate — quoted vols, methodology, approximations, . . .
- We recover immediately at default time.
- Accurately modeling collateral in above.
- Can recover collateral.

Questions:

- Is this a hedgeable price?
- If cleared, does CVA = 0 (is there clearing house risk)?

Wrong way risk

If interest rates and default are correlated, the above breaks down.

Wrong way risk When values go up as default risk goes up.

Examples:

- Oil swap with Exxon:
  - You pay current oil price, Exxon pays fixed.
  - Oil prices tank:
    - Swap value goes up.
    - Exxon is making less money.
- Cross-currency swaps with Greece.

Mitigation:

- Bump up CVA, but how much?
- Separately managed.

Hedging

Hedging CVA

Hedging counterparty risk can be difficult. Since

$$CVA \approx (1 - R) \sum S(\bar{t}_i) \bar{p}(t_i)$$

It's natural to hedge with a portfolio of swaptions and CDSs:

- Take (1-R)$\bar{p}(t_i)$ positions in swaptions maturing at $\bar{t}_i$

and

- Take positions in CDS that neutralize exposure to credit moves

Hedging issues

By virtue of the swaption positions (and the low sensitivity of CDS to interest rates:

- Changes in CVA due to changes in interest rates and interest rate volatility are hedged.
- Changes in CVA due to changes in credit spreads are hedged.

However

- CDS position needs rebalancing when interest rates and vols change — Dynamic hedging with CDS is expensive.
- Swaption position needed changes when CDS spreads change — Dynamic hedging with swaptions is expensive.
- Cross gamma risk.
- Where can you get a risk free swap?
- What about the CVA of the CDS and the swaptions for that matter?
- How do you neutralize credit exposure and default exposure at the same time?

No CDS CDS

If CDS are not available for a given name, there are alternative ways to estimate default probabilities:

- Back out a CDS spread from a bond par curve.
- Use a generic spread (CDS indices).
- Use CDS spreads from a similar name.
- Use CDS spreads from a similar sector and credit rating.
- Use historically estimated default probabilities for a similar sector and credit rating.
- Estimate credit risks from low strike puts.

Each of these methods has its own problems.

Bond spread method

Backing out spreads from bonds is closest to yielding the price of a hedge. Problems include:

- Liquidity risk is priced into bond yields.
- Issued bonds might trade rarely or not trade at all.
- To actualize the hedge, one would have to short the bond.

Proxy CDS method

Using other CDS spreads includes:

- Use a generic spread (CDS indices).
- Use CDS spreads from a similar name.
- Use CDS spreads from a similar sector and credit rating.

On the plus side:

- Can give reasonable estimates of CDS spreads.
- Can be highly correlated with a particular name.

On the minus side:

- They all break down when that name deteriorates.
- They fail to capture idiosyncratic changes.
- They fail to capture the actual default event.
- In the latter case, a name will deteriorate well before its credit rating changes, leading to an underestimate of credit risk.

Historical default probabilities

Using historical default probabilities has its own problems.

- Same problems as the sector/credit rating approach.
- Cannot even hedge market moves.
- Does not respond to changing market conditions.
- Not a market price (historical, not implied).

Option estimation approach

Using low strike puts to estimate and hedge credit risk is an interesting approach.

- Options can be purchased to effect the hedge.
- Options capture the default event.
- Good correlation to credit risk.

On the other hand:

- Can be a disconnect between stock prices and credit (GM and Kerkorian in 2005).
- Adds substantial model risk — model must capture equity moves along with credit risk and default events.
- Too volatile — adds exposure to spot price moves.

General proxy issues

More generally, if a proxy is used which cannot effect a hedge, then

- The price is not an arbitrage free price.
- The credit risk cannot be hedged.

In this situation, one should consider managing the risk differently:

- Collateralize.
- Capitalize based on real world potential future exposures.

Portfolios

Portfolio counterparty risk

In the presence of netting agreements, the counterparty exposure must be computed on the portfolio of securities covered by the netting agreement.

- Loss given default is no longer the sum of the losses in the individual positions.
- Loss given default is $(1 - R)$ of the payoff of a call on the underlying *portfolio*.

Calculating portfolio counterparty risk

Three possible calculation approaches:

1. Feed call options on the portfolio to your defaultable interest rate derivatives valuation system.
2. Compute value of appropriate call options on the portfolio via your interest rate derivatives valuation system, and proceed as above.
3. Make some assumptions and get formulas.

Calculating portfolio counterparty risk

Brigo and Masetti [BM05] take the 3rd approach.

- Net all of the interest rate swaps covered by a given netting agreement.
- The floating "leg" of the portfolio is now some sort of amortizing floating leg with time dependent leverage.
- The fixed "leg" is now some sort of amortizing step coupon fixed leg.

Difficulties ensue because the leverage could be positive or negative (i.e. - at some times the aggregate can be a payer swap while at other times it can be a receiver swap).

Portfolio woes

Consider a portfolio consisting of an at the money 5 year payer swap with fixed rate F_1, and an at the money 3 year receiver swap with fixed rate F_2.

- Credit exposure 1 year out is to the difference between the value of the two tails being positive.
- Similar to the difference between the 4 year swap rate and the 2 year swap rate.

The portfolio behaves roughly like a spread between two rates, so the credit exposure is like a spread option, and thus, difficult to price.

Summary

Summary

CVA

- Credit Valuation Adjustment.
- CVA = risk free asset price - risky asset price.
- Option like exposure in general
  - Sensitive to shape of curve.
  - Sensitive to volatility.
- CVA approximation for long only positions – discount at a spread.

Summary

Bond price:

$$\sum^{n} \frac{C}{f} D(t_i)s(t_i) + 100D(t_n)s(t_n) + \int^{t_n} 100RD(t)p(t)dt$$

Swap CVA:

$$\begin{aligned} CVA &= (1-R)\int_0^T S(t)p(t)dt \\ &\approx (1-R)\sum S(\bar{t}_i)\bar{p}(t_i) \end{aligned}$$

Summary

Issues to keep in mind:

- Model assumptions.
- Wrong way risk.
- Hedging difficulties.
- Breaking assumptions with proxies.
- Portfolio level CVA inaccuracies.

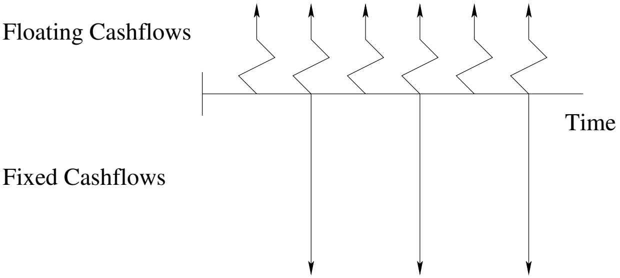
Appendix – CVA vs Swaptions

Appendix – CVA vs Swaptions

It would appear that swaption pricing would enable CVA calculations, but adjustments are needed.

The issue is in accurately valuing $S(\bar{t}_i)$. This is the value of the option to enter into the tail of the swap, which is slightly different from the swaption maturing at time $\bar{t}_i$.

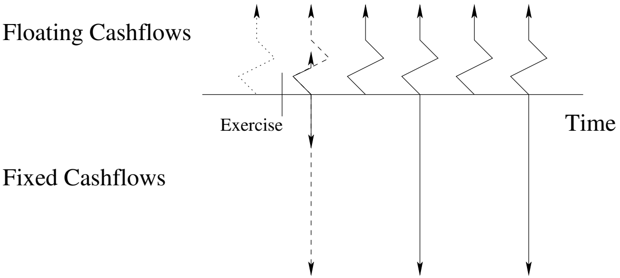
Swap cash flows:



CVA subtleties

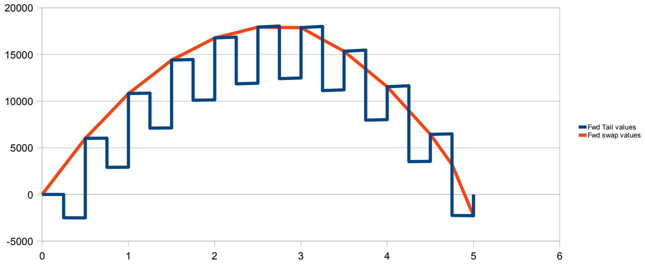
When exercising a swaption to enter into a swap with an odd first coupon, the first coupon is adjusted — the floating leg references a shortened index, and the fixed leg only accrues over the remainder of the period.

However, in the event of default at that time, the full cash flows are lost.



CVA subtleties

The difference between the value of the forward start swap and the corresponding tail of the underlying swap will typically be substantial, as is illustrated in the following graph of the two for a 5 year at the money payer swap, on a 1 million notional in a rising interest rate environment.



Forward swap rates

Consider the pay fixed swap with fixed rate F that the holder of a swaption would receive on exercise (at time t).

Let the underlying floating (fixed) leg pay at times t_i (t'_i). Then the time t value of the swap is[1]

$$S(t) = \sum L(t, t_i, t_{i+1}) Z(t, t_{i+1}) \alpha_i - \sum F \alpha_i^* Z(t, t'_i),$$

Since $t \leq t_1$,

$$L(t, t_i, t_{i+1}) = (1/\alpha_i^{**})(Z(t, t_i)/Z(t, t_{i+1}) - 1)$$

(with Libor accrual factor α_i^{**}), so if we assume $\alpha_i = \alpha_i^{**}$, then $S(t)$ can be written in terms of Z as:

$$S(t) = Z(t, t_1) - Z(t, t_n) - \sum F \alpha_i^* Z(t, t'_i),$$

This gives us the standard expression for the time t value of a forward start swap.

[1] $L(w, x, y)$ is the time w forward Libor rate setting at time x and paying at time y, $Z(x, y)$ be the time x price of a zero coupon bond paying at time y, and α_i (α_i^*) are the accrual fractions for floating (fixed) payment periods t_i to t_{i+1} (t'_i to t'_{i+1}), respectively.

Tail swap rate

The tail of the existing swap is slightly different. Its cash flows and accruals are the same as for the forward swap, except for the first period, where the reset date ($\bar{t}_1$) is earlier than the valuation time t, and the accrual factors correspond to the full period rather than the partial period. The time t value of the first floating cash flow is

$$\frac{Z(t,t_2)}{Z(\bar{t}_1,t_2)} - Z(t,t_2),$$

so the value of the tail is

$$\begin{aligned}
\bar{S}(t) &= \frac{Z(t,t_2)}{Z(\bar{t}_1,t_2)} - Z(t,t_2) + \sum_2 L(t,t_i,t_{i+1})Z(t,t_{i+1})\alpha_i \\
&\qquad - F\bar{\alpha}_1^* Z(t,t_1) - \sum_2 F\alpha_i^* Z(t,t_i') \\
&= \frac{Z(t,t_2)}{Z(\bar{t}_1,t_2)} - Z(t,t_n) - F\bar{\alpha}_1^* Z(t,t_1) - \sum_2 F\alpha_i^* Z(t,t_i').
\end{aligned}$$

Forward swaps and tail swaps compared

The difference between the values of the two is

$$\bar{S}(t) - S(t) = \frac{Z(t, t_2)}{Z(\bar{t}_1, t_2)} - Z(t, t_1) - F(\bar{\alpha}_1^* - \alpha_1^*)Z(t, t_1).$$

This is the adjustment that needs to be made to convert the swaption payoff to the payoff of the option on the tail swap.

References: Stein and Lee [SL11] and Stein [Ste07].

Forward swap vs tail

Under the equivalent martingale measure with respect to numeraire N, the time zero value of the forward start swaption is

$$N_0 E_0[\max(S(t), 0)/N_t]$$

Instead of this, we need the value of the option on the tail of the swap. Concentrating on the payoff, if

$$D = \bar{S}(t) - S(t) = \frac{Z(t, t_2)}{Z(\bar{t}_1, t_2)} - Z(t, t_1) - F(\bar{\alpha}_1^* - \alpha_1^*)Z(t, t_1)$$

we have

$$\max(\bar{S}(t), 0) = \max(S(t) + D, 0) = \max(S(t), -D) + D$$

So the option to enter into the tail of the swap is the same as the option to enter into the swaption with an associated fee of $S(t) - \bar{S}(t)$ with an additional cash component of $\bar{S}(t) - S(t)$.

One could apply a convexity adjustment technique to work this out, but it replacing $Z(t, t_1)$ and $Z(t, \bar{t}_1)$ with their corresponding time zero forward values is a reasonable approximation and it makes these adjustments fixed values and thus easily handled.

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