MATH GR 5320
Financial Risk Management and Regulation

Lecture 3: Risk Measurement

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Outline

Review

Market structure

To understand risks, we need to know how the markets operate

- Players – Who is participating.
- Pieces – What are they buying and selling.
- Moves – How the players operate.

Players

Each market participant has a different role. We need to understand the roles.

- Banks
  - Investment banks
  - Retail banks
  - Other banking institutions – credit unions, savings and loans, etc.
  - Meta-banks – bank holding companies
- Market facilitators
  - Exchanges
  - Clearing houses
  - Broker-dealers
  - Securities firms
- Investors
  - Individuals
  - Insurance companies
  - Hedge funds
  - Institutional investors
  - Pension funds
  - Corporate treasuries
- Regulators

Regulation

Post-crisis, regulation has become a much bigger part of the market.

- Deregulation prior to crisis
- Reregulation after the crisis

Major regulators:

- Federal Reserve Banks (FRB)
  - Bank holding companies, registered state banks
- US Treasury — Office of Comptroller of the Currency (OCC)
  - National banks
- Federal Deposit Insurance Corporation (FDIC) and The National Credit Union Administration (NCUA)
  - All depository institutions
- Securities and Exchanges Commission (SEC)
  - Trading – listings, security based swaps, hedge funds
- Commodity Futures Trading Commission (CFTC)
  - Trading – commodities, futures and options, other swaps
- Individual state regulatory agencies
  - State banks, community banks

Regulatory mechanisms

Banks are regulated by

- Monitoring activity and positions
- Analyzing potential losses of positions
- Requiring capital
- Reducing positions

Trading is regulated by

- Disclosure rules
- Trading rules (no insider trading)
- Investigations
- Capital requirements
- Margin requirements

The Pieces

What pieces do the players play with?

- Savings
- Loans
- Stocks
- Bonds
- Futures and options
- Structured products
- ...

The Moves

The activities are:

- Market making
- Transactional services
- Investing
- Hedging

Risk measurement

General risk measurement framework

To manage risk, we need to be able to measure risk.

In general, for a portfolio V, we:

- Pick a relevant horizon time T
- Compute some statistics of V_T, the portfolio value at time T.

Key issue:

- What is the distribution of V_T?

Picture

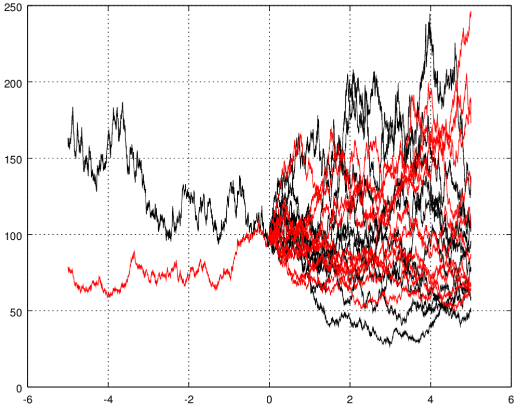


Figure: 10 sample paths for each of two stock processes, $dS_i = \mu_i S_i dt + \sigma_i S_i dW_i$, both start at 100, black with 30% vol and 7% return and red with 20% vol and 5% return.

Basic questions

Must answer the following questions in order to do risk calculations:

1. What factors affect the value of the portfolio?
2. What is the behavior of these factors?
3. What is the behavior of the P&L as a function of these factors?
4. How do I summarize the behavior of the P&L?

These are the four questions of risk analysis.

Each question is a step in the computation of a risk analytic.

Risk measurement steps

Detailed steps:

1. What factors affect the value of the portfolio?
   - Identify risk factors.
2. What is the behavior of these dependencies?
   - Model behavior of risk factors:
     - Develop a model.
     - Fit model to data.
     - Use model to yield future distribution of risk factors.
3. What is the behavior of the P&L as a function of these factors?
   - Use structure of portfolio and pricing functions to compute distribution of V_T from distribution of risk factors.
4. How do I summarize the behavior of the P&L?
   - Compute risk measures, which are statistics on V_T (such as mean, variance, percentiles, etc).

Numerical methods

Method used depends on form of risk measures and pricing models:

- Analytically.
- Semi-analytically.
- Use approximations.
- Make simplifying assumptions.
- Use Monte Carlo.

Example – scenario analysis

Scenario analysis:

- "How would X impact my portfolio?"
- "Can I survive the losses that would be incurred under each of the following scenarios?"
- Also known as stress testing (when scenarios are extreme or are chosen from a crisis).
- Quantitative component of the Central approach of the Comprehensive Capital Analysis and Review (CCAR) [Boa16]:
  - 3 scenarios – baseline, adverse and severely adverse.
  - Compare losses to capital to ensure bank can survive such scenarios.

Process:

- Identify risk factors.
- Risk factor model is a 1 point distribution – the change that a particular scenario would have on the risk factors.
- Behavior of P&L is computed by applying pricing model to new values of risk factors.
- Statistic describing behavior is loss under given scenario.

Stock example – scenario analysis

Consider a portfolio of stocks – $V = \sum a_i S_i$

Scenario analysis process:

- Risk factors can be the S_i themselves.
- Today's date is T_0.
- Pick 2 dates t_1 and t_2 from the 2008 crisis, or the Greece default, or the Asian crisis, or ...
- Or, make something up (like a 20% drop in the stock market).
- Determine individual risk factor changes:
  - Log return $R_i = \log(S_{i,t_2}/S_{i,t_1})$
- Determine market value after impact of chosen event:
  - $V_T = \sum a_i S_{i,T} = \sum a_i (S_{i,T_0} e^{R_i}) = \sum a_i (S_{i,T_0} S_{i,t_2}/S_{i,t_1})$
- Using loss under given scenario as our statistic;
  - Loss $= V_{T_0} - V_T = \sum a_i S_{i,T_0}(1 - S_{i,t_2}/S_{i,t_1})$

Sensitivity analysis

Sensitivity analysis:

- "Which risk factors might cause the biggest losses?"
- Direct analysis of the portfolio pricing function.
- Essentially just tabulating the deltas.

Scenario analysis process:

- Identify risk factors R_i.
- Model for distribution of risk factors is an infinitesimal perturbation.
- For each perturbation, compute losses.
- Risk factors whose perturbations cause the largest changes are the ones to be concerned about.

Sensitivity analysis example 1

Consider 100 shares of IBM, currently priced at \$155

- Current market price $= 100 \times 155 = 15,500$.
- Change in value with respect to change in stock price $= 100$.

Consider the option to buy 780 shares of IBM at \$155 in 1 year, priced at an implied volatility of 30% with an interest rate of 2%.

- Price of option to buy 1 share (Black-Scholes formula) $=$ \$19.87.
- Current value of position $= 780 \times 19.87 = 15,498.60$.
- Delta of option on 1 share $= 0.5858$.
- Change in value of position with respect to change in stock price $= 780 \times 0.5858 = 457$.

Conclusions:

- Two portfolios have the same market value.
- Option portfolio is riskier by a factor of 4.6.

Sensitivity analysis example 2

Start with \$15,500.

Write 355 call options on IBM struck at 155 and 502 put options on IBM struck at 155 (same market conditions).

- Price of option to buy 1 share = \$19.87.
- Price of option to sell 1 share = \$16.80.
- Amount received for options $= 355 \times 19.87 + 502 \times 16.80 =$ \$15,487.
- Total portfolio value
  - initial cash + cash received for writing options + value of options = 15,500 + 15,487 - 15,487 = \$15,500.
- Delta of call option on 1 share = 0.5858.
- Delta of put option on 1 share = -0.4142
- Change in value of position with respect to change in stock price $= -355 \times 0.5858 + 502 \times 0.4142 =$ \$-0.03.

Conclusions:

- Portfolios can have zero delta but still have substantial risk.

Stock GBM modeling

Consider a portfolio of stocks – $V = \sum a_i S_i$

- Risk factors can be the S_i themselves.
- Assume GBM:
  - $dS_i = \mu_i S_i dt + \sigma_i S_i dW_i$
  - $dW_i dW_j = \rho_{ij} dt$
- Fit μ_i, σ_i and ρ_{ij} to historical data.
- Given assumed processes, $V_T = \sum a_i S_{i,T}$.
- Can now compute statistics of V_T.

Stock factor model

The same stock portfolio can be modeled with a factor model:

- Select risk factors R_i, such as momentum, industry return, etc.
- Model stocks as $dS_i = \sum_j \beta_{ij} dR_j + d\epsilon_i$.
- Model risk factors as well — GBM or ABM.
- Fit model to historical data.
- Changes in V are now given by changes in R_j and in ϵ_i.
- Can now compute statistics of V_T.

Stock example – option portfolio

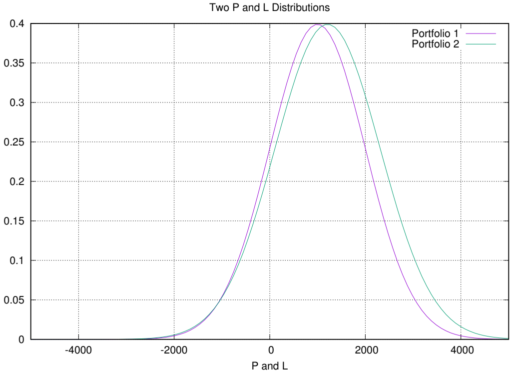
Consider a portfolio of call options with maturities T_i, strikes K_i on stocks S_i (if more than one option on the same stock, can have $S_i = S_j$).

- Pricing model:
  - $V_T = \sum a_i C(T, S_{i,T}, K_i, T_i, \sigma_i, r(T, T_i))$.
  - C is the Black-Scholes formula and σ_i is the implied vol at time T for options on S_i maturing at T_i with strike K_i.
  - r is the time T rate for discounting from T_i to T.
- Need to model the S_i.
- Should also model r and σ_i, not to mention correlations between all of them, lest volatility and interest rate risk is ignored.
- Calibrate the model to historical data.
- Start it off at the current market.
- The above pricing formula gives V_T as a function of the market factors.
- Now compute statistics.

Risk measures

Risk measures

Consider the P&L distribution of two different portfolios:



Which portfolio is riskier?

- Need a statistic of the P&L distribution that measures the riskiness.
- Such a statistic is called a risk measure.

Goals

Desirable characteristics of a risk measure:

- Easy to compute.
- Easy to validate (backtestable and elicitable).
- Actionable – value tells you what to do.
- Not deceptive (e.g. coherent).
- Robust – small measurement errors do not cause large changes in risk measure.

Coherence

One way for a risk measure to not be deceptive is if it is coherent.

We think of a risk measure as a function ρ of the P&L of a portfolio, where larger values mean greater risk.

A coherent risk measure is one that has the following desirable properties:

Normalized $\rho(0) = 0$

Monotonic $V \leq V' \implies \rho(V) \geq \rho(V')$

Positive homogeneity $\rho(\alpha V) = \alpha\rho(V)$ when $\alpha \geq 0$.

Translation invariance For a constant a, $\rho(V + a) = \rho(V) - a$

Subadditive $\rho(V + V') \leq \rho(V) + \rho(V')$

Some consider the larger class of convex risk measures. Instead of subadditivity and positive homogeneity, convexity:

Convexity $\rho(\lambda V + (1-\lambda)V') \leq \lambda\rho(V) + (1-\lambda)\rho(V')$

References: Artzner et al. [Art+99] and Föllmer and Schied [FS08]

Elicitability

We would like our risk measure to also be elicitable.

Risk measure ρ is elicitable if there exists a loss function L such that:

$$\rho(V) = \operatorname*{argmin}_X E[L(X, V)]$$

Think of $E[L(X, V)]$ as the magnitude of the error being made when using X as an estimate of $\rho(V)$.

Elicitability:

- Elicitable risk measures have good statistical tests for the accuracy of estimates.
- Risk measures that are not elicitable can still be backtested.

References: Gneiting [Gne11], Ziegel [Zie14], Emmer, Kratz, and Tasche [EKT13], Bellini and Bignozzi [BB13], and Acerbi and Szekely [AS14]

Robustness

Notions of robustness of a risk measure:

- Risk measure is continuous or uniformly continuous after making set of distributions into a metric space.
- Consider sensitivity of risk measure to changes in portfolio, changes in historical data, changes in parameters, etc.

But:

- New concept.
- Many functions are continuous but numerically unstable.
- More work needs to be done.

References: Cont, Deguest, and Scandolo [CDS10] and Emmer, Kratz, and Tasche [EKT13]

Common risk measures

Most common risk measurements:

- Variance at horizon T.
- p-th percentile VaR at horizon T.
- p-th percentile expected shortfall at horizon T.
- p-th expectile at horizon T.
- Sensitivity to risk factors.
- Scenario analyses.

Last two are commonly used, but don't exactly qualify as risk measures:

- Quantify potential size of losses.
- Fail to quantify probability of such losses.

But still useful.

Variance

Variance at horizon T:

$$
\begin{aligned}
E^P[(V_T - E^P[V_T])^2] &= E^P[V_T^2] - E^P[V_T]^2 \\
&= E^P[V_T^2] - \overline{V_T}^2
\end{aligned}
$$

- Measure of dispersion, and thus of risk.
- Excellent if V_T is normally distributed.
- Should easily pass back tests if calibrated to history.
- Relatively easy to compute.
- Robust by some measures.

Issues:

- Ignores asymmetry.
- Insufficient to estimate losses.
- Assuming normality can underestimate losses (e.g. 68% within 1 SD, 95.5% within 2 SD, 99.7% within 3 SD).

Variance and SD types

There are different types of standard deviation.

Standard deviation of a random variable:

- $\sqrt{E[(X-E[X])^2]}$
- $\sqrt{E[X^2]-E[X]^2}$

Population standard deviation:

- $\mu = \sum^n (X_i/n)$
- $\sqrt{\frac{1}{n}\sum^n (X_i-\mu)^2}$
- $\sqrt{\sum^n X_i^2/n-\mu^2}$

Sample standard deviation (corrected):

- $\sqrt{\frac{1}{n-1}\sum^n (X_i-\mu)^2}$

We will generally use the population standard deviation.

Note – Spreadsheets and mathematical packages often provide both. Pick the right one!

Variance example

Computing variance given possible future Apple prices:

ID	Apple	Square
1	116.52	13576.91
2	108.60	11793.96
3	101.21	10243.46
4	112.11	12568.65
5	111.25	12376.56
6	105.81	11195.76
7	109.67	12027.51
8	109.43	11974.92
9	105.08	11041.81
10	116.58	13590.90

- $E[V_T^2]$ = Average of squares = 12039.04
- $E[V_T]^2$ = Square of average = 12017.86
- Variance = 12039.04 - 12017.86 = 21.18
- SD = $\sqrt{\text{Variance}}$ = 4.6

VaR

The p level VaR is X if p of the time our losses are less than or equal to X:

$$\mathrm{VaR}(V, T, p) = G^{-1}(p), \text{ where}$$
$$G(X) = P[V_0 - V_T \leq X] = E^P[1_{V_0 - V_T \leq X}]$$

- Quantile of loss distribution.
- Precise percentile loss, so actionable.
- Essentially same as variance if distribution is normal.
- Elicitable.
- Robust.

Issues:

- Not coherent – fails subadditivity — diversification can increase risk.
- Is coherent for linear portfolios of elliptically distributed factors, but might as well use variance.
- Hard to back test accurately.
- Harder to compute than standard deviation.

VaR example

With the same Apple prices and starting price of 106.74.

Compute the VaR by sorting the losses:

ID	Apple	Loss
3	101.21	5.53
9	105.08	1.66
6	105.81	0.93
2	108.60	-1.86
8	109.43	-2.69
7	109.67	-2.93
5	111.25	-4.51
4	112.11	-5.37
1	116.52	-9.78
10	116.58	-9.84

10 samples, so:

- 90% VaR = worst loss = 5.53
- 80% VaR = 2nd worst = 1.66
- 70% VaR = 3rd worst = 0.93

VaR note

The p level VaR is X if p of the time our losses are less than or equal to X:

$$\begin{aligned}\mathrm{VaR}(V,T,p) &= G^{-1}(p), \text{ where}\\ G(X) &= P[V_0 - V_T \leq X] = E^P[1_{V_0 - V_T \leq X}]\end{aligned}$$

Often, given as:

$$\mathrm{VaR}(V,T,p) = \inf\{l | P[V_0 - V_T > l] \leq 1 - p\}$$

Our definition is a simplification – assumes the loss distribution does not have jumps.

VaR and jumps

But in the previous example, there are only jumps!

- 5.53 = Worst loss out of 10 samples.
- 1.66 = 2nd worst loss out of 10 samples.

Technically, we defined the VaR as $G^{-1}(0.9)$ so:

- VaR isn't a number in this case, it's an interval!
- VaR $= [1.66, 5.53)$
- More general definition resolves this by arbitrarily picking a point in this interval.
- Could consider these discrete samples of a continuous distribution which must then be estimated.

Why should the 90% VaR be 5.53, and not 1.66, or something in between?

- 90% of the time, our losses are 1.66 or less, but
- 10% of the time, our losses are 5.53 or greater.

For risk:

- Choose the most conservative estimate.
- Think of 90% VaR as 10% worst loss.

Expected shortfall

Expected shortfall

$$\mathrm{ES}(V, T, p) = -E^P[V_T - V_0 | V_T - V_0 < -\mathrm{VaR}(V_T, p)]$$

- Coherent measure of risk – diversification never increases risk.
- Mandated by latest Basel rules [Bas16].
- Robust by some measures.

Issues:

- Same as variance if distribution is normal.
- Harder to compute than VaR.
- Harder to back test (not elicitable, but still back-testable).
- Actionable?

ES example

With the same Apple prices and starting price of 106.74.

Sort and compute average of losses in tail:

ID	Apple	Loss	Sum	Count	ES
3	101.21	5.53	5.53	1	5.53
9	105.08	1.66	7.19	2	3.60
6	105.81	0.93	8.12	3	2.71
2	108.6	-1.86	6.26	4	1.57
8	109.43	-2.69	3.57	5	0.71
7	109.67	-2.93	0.64	6	0.11
5	111.25	-4.51	-3.87	7	-0.55
4	112.11	-5.37	-9.24	8	-1.16
1	116.52	-9.78	-19.02	9	-2.11
10	116.58	-9.84	-28.86	10	-2.89

10 samples, so:

- 90% ES = Worst loss = 5.53
- 80% ES = Average of 2 worst = 3.60
- 70% ES = Average of 3 worst = 2.71

Expectiles

Expectiles are like a weighted average of one sided variances

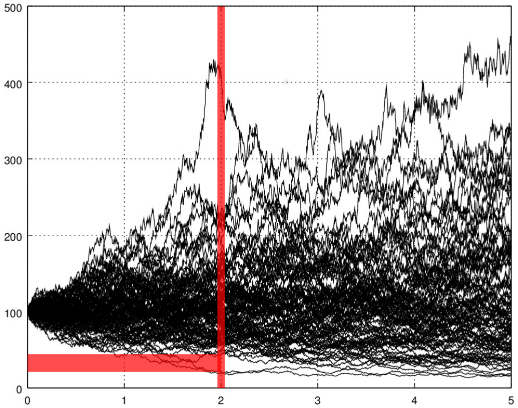
$$\operatorname*{argmin}_X E[|p - 1_{V<X}|(V - X)^2]$$

- Coherent.
- Elicitable.
- Robust by some measures.

Issues:

- Harder to compute than VaR – computationally similar to ES.
- Not clear exactly what it is.
- How is it actionable?

Picture



- 100 paths, 30% vol, 7% return
- 2 year 98% VaR = next to lowest path = high uncertainty
- 2 year 98% ES = average of these two paths
- 97% Var and ES – substantially lower

Formula derivations

Stock formulas

Portfolio:

- One share of stock S
- S follows GBM with known constant parameters

Formulas:

$$\begin{aligned} V_T &= S_T \\ dS_t &= \mu S_t dt + \sigma S_t dW_t \end{aligned}$$

We will compute:

$$\begin{aligned} \text{var}(V_T) \\ \text{VaR}(V, T, p) \\ \text{ES}(V, T, p) \end{aligned}$$

Basic facts

Brownian motion W_t:

- Almost all paths are continuous.
- Martingale: Expectation of future is current value:

$$W_t = E[W_T|\mathcal{F}_t], \text{ for } T \geq t$$

- $W_0 = 0$, so $E[W_t] = 0$ too.
- Independent increments: For times $t_i \leq t_{i+1}$, $W_{t_4} - W_{t_3}$ and $W_{t_2} - W_{t_1}$ are independent.
- Gaussian: $W_{t_2} - W_{t_1} \sim N(0, t_2 - t_1)$
- Covariance: $E[W_{t_2} W_{t_1}] = \min(t_2, t_1)$. (implies $\mathrm{var}[W_t] = t$)
- $dWdW = dt$

Ito's formula:

- If $S_t = f(t, V_t)$ then

$$df(t, V) = f_1 dt + f_2 dV + \frac{1}{2} f_{22} dVdV$$

Formula for GBM

Ito's formula shows:

$$S_T = S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma W_T}$$

Proof:

$$\begin{aligned} f(T, W) &= S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma W} \\ df(T, W) &= f_1 dT + f_2 dW + \frac{1}{2} f_{22} dW dW \\ &= (\mu - \frac{\sigma^2}{2}) S_T dt + \sigma S_T dW + \frac{\sigma^2}{2} S_T dt \\ &= \mu S_T dt + \sigma S_T dW \end{aligned}$$

Mean of GBM

GBM S is given by:

$$S_T = S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma W_T}$$

Know:

$$W_T \sim N(0, T), \text{ so}$$

$$\text{PDF}(W_T) = \frac{1}{\sqrt{2\pi T}} e^{\frac{-W^2}{2T}}$$

So, directly compute mean:

$$\begin{aligned} E[S_T] &= \int_\Omega S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma W_T} dP \\ &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma W} e^{\frac{-W^2}{2T}} dW \\ &= \frac{S_0 e^{(\mu - \frac{\sigma^2}{2})T}}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{\sigma W - \frac{W^2}{2T}} dW \end{aligned}$$

Mean of GBM

Have:

$$E[S_T] = \frac{S_0 e^{(\mu - \frac{\sigma^2}{2})T}}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{\sigma W - \frac{W^2}{2T}} dW$$

Complete the square:

$$\begin{aligned}
u &= \frac{W}{\sqrt{2T}} - \frac{\sqrt{2T}\sigma}{2} \\
u^2 &= \frac{W^2}{2T} - \sigma W + \frac{\sigma^2 T}{2} \\
du &= \frac{1}{\sqrt{2T}} dW \\
E[S_T] &= \frac{S_0 e^{(\mu - \frac{\sigma^2}{2})T}}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{-u^2 + \frac{\sigma^2 T}{2}} \sqrt{2T} du \\
&= \frac{S_0 e^{\mu T}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du \\
&= S_0 e^{\mu T}
\end{aligned}$$

EZ Mean of GBM

Alternatively, use martingales.

Define A_T by:

$$
\begin{aligned}
dA_T &= \sigma A_T dW_T \\
A_T &= e^{-\frac{\sigma^2}{2}T+\sigma W_T} \\
S_T &= S_0 e^{\mu t} A_T
\end{aligned}
$$

A_T is a martingale, so:

$$
\begin{aligned}
E[A_T] &= A_0 = 1 \\
E[S_T] &= E[S_0 e^{\mu T} A_T] \\
&= S_0 e^{\mu T} E[A_T] \\
&= S_0 e^{\mu T}
\end{aligned}
$$

This is why you should learn how to work with martingales!

EZ Variance of GBM

For the variance, we proceed analogously:

$$
\begin{aligned}
S_T &= S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma W_T} \\
S_T^2 &= S_0^2 e^{(2\mu - \sigma^2)T + 2\sigma W_T} \\
dB_T &= 2\sigma B_T dW_T \\
B_T &= e^{-2\sigma^2 T + 2\sigma W_T} \\
S_T^2 &= S_0^2 e^{(2\mu + \sigma^2)T} B_T
\end{aligned}
$$

B_T is a martingale, so:

$$
\begin{aligned}
E[S_T^2] &= E[S_0^2 e^{(2\mu + \sigma^2)T} B_T] \\
&= S_0^2 e^{(2\mu + \sigma^2)T} E[B_T] \\
&= S_0^2 e^{(2\mu + \sigma^2)T} \\
\mathrm{var}[S_T] &= E[S_T^2] - E[S_T]^2 \\
&= S_0^2 e^{(2\mu + \sigma^2)T} - S_0^2 e^{2\mu T} \\
&= S_0^2 (e^{\sigma^2 T} - 1) e^{2\mu T}
\end{aligned}
$$

EZer Variance of GBM

It would have been easier to use the mean formula:

$$
\begin{aligned}
S_T &= S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma W_T} \\
S_T^2 &= S_0^2 e^{(2\mu - \sigma^2)T + 2\sigma W_T} \\
&= S_0^2 e^{(2\mu + \sigma^2 - \frac{(2\sigma)^2}{2})T + 2\sigma W_T}
\end{aligned}
$$

Since $E[S_T] = S_0 e^{\mu T}$,

$$
E[S_T^2] = S_0^2 e^{(2\mu + \sigma^2)T}
$$

I.e. S_T^2 is also a GBM with a specific drift term, so the same formula applies.

Variances with numbers

For 5% growth rate and 30% volatility and an initial price of 100, we have the following variances:

Horizon (days)	Horizon (years)	Mean	$E[S^2]$	variance	SD	scaled 1 day
1	0.003968	100.02	10008	3.6	1.9	
5	0.019841	100.10	10038	17.9	4.2	4.23
10	0.039683	100.20	10076	35.9	6.0	5.98
20	0.079365	100.40	10152	72.3	8.5	8.45
252	1.000000	105.13	12092	1040.8	32.3	30.01

$$E[W_T] = 0$$
$$E[W_T^2] = T$$

So, for BM, we can scale by $\sqrt{T}$ to adjust for time

For GBM, it's close, but off by 7% at 1 year.

GBM VaR

To compute VaR, we need to know the probability of S_T being below a level X.

$$
\begin{aligned}
S_T &= S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma W_T} \\
P(S_T < X) &= P(\log(S_T) < \log(X)) \\
&= P(\log(S_0) + (\mu - \frac{\sigma^2}{2})T + \sigma W_T < \log(X)) \\
&= P\left(W_T < \frac{\log(\frac{X}{S_0}) - (\mu - \frac{\sigma^2}{2})T}{\sigma}\right)
\end{aligned}
$$

$W_T \sim N(0, T)$, so

$$
\begin{aligned}
P(W_T < a) &= \Phi(\frac{a}{\sqrt{T}}) \\
P(S_T < X) &= \Phi\left(\frac{\log(\frac{X}{S_0}) - (\mu - \frac{\sigma^2}{2})T}{\sqrt{T}\sigma}\right)
\end{aligned}
$$

GBM VaR

Continuing...

$$
\begin{aligned}
P(S_T < X) &= 1 - p \\
1 - p &= \Phi\left(\frac{\log(\frac{X}{S_0}) - (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \\
\Phi^{-1}(1-p) &= \frac{\log(\frac{X}{S_0}) - (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\
S_0 e^{\sigma\sqrt{T}\Phi^{-1}(1-p) + (\mu - \frac{\sigma^2}{2})T} &= X
\end{aligned}
$$

This yields:

$$
\mathrm{VaR}(S, T, p) = S_0 - S_0 e^{\sigma\sqrt{T}\Phi^{-1}(1-p) + (\mu - \frac{\sigma^2}{2})T}
$$

GBM VaR Values

Horizon (days)	Mean	SD	50% VaR	84% VaR	98% VaR	99.9% VaR	99.9999% VaR
1	100.02	1.9	0.00	1.86	3.80	5.67	9.36
5	100.10	4.2	-0.01	4.11	8.30	12.23	19.72
10	100.20	6.0	-0.02	5.75	11.53	16.85	26.69
20	100.40	8.5	-0.04	8.02	15.90	22.95	35.53
252	105.13	32.3	-0.50	25.42	45.73	60.23	78.88

Horizon (days)	16% VaR	2% VaR	0.1% VaR	0.00001% VaR
1	-1.90	-3.96	-6.02	-10.33
5	-4.30	-9.08	-13.96	-24.58
10	-6.14	-13.08	-20.31	-36.47
20	-8.81	-19.00	-29.90	-55.24
252	-35.44	-86.10	-153.98	-378.17

Expected shortfall

The formula for expected shortfall for $V_T = S_T$ is more complicated to derive

$$\begin{aligned}\mathrm{ES}(S,T,p) &= -E^P[S_T - S_0 | S_T - S_0 < -\mathrm{VaR}(S,T,p)] \\ &= S_0 - E^P[S_T | S_T < S_0 - \mathrm{VaR}(S,T,p)]\end{aligned}$$

$$E^P[S_T | S_T < X] = \frac{\int_{-\infty}^{X} S_T \, dP}{\int_{-\infty}^{X} dP}$$

$$\int_{-\infty}^{X} S_T \, dP = \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{Y} S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma W} e^{-\frac{W^2}{2T}} \, dW$$

where Y satisfies $S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma Y} = X$.

Now, complete the square, solve for Y, and plug in $X = S_0 - \mathrm{VaR}(S,T,p)$, and note that with such an X, the denominator above is just $1 - p$.

Summary

Summary

Risk measurement steps - answer the 4 questions:

1. What factors affect the value of the portfolio?
   - Identify risk factors.
2. What is the behavior of these dependencies?
   - Model behavior of risk factors:
     - Develop a model.
     - Fit model to data.
     - Use model to yield future distribution of risk factors.
3. What is the behavior of the P&L as a function of these factors?
   - Use structure of portfolio and pricing functions to compute distribution of V_T.
4. How do I summarize the behavior of the P&L?
   - Compute risk measures, which are statistics on V_T.

Summary

Method varies depending on form of risk measures, factor models, and pricing models:

- analytically
- semi-analytically
- using approximations
- making simplifying assumptions
- using Monte Carlo

Summary

Modeling examples:

- Direct modeling via fitted GBMs for stock portfolio
- Factor model for stock portfolio
- Option portfolio

Risk measures:

- Common measures
    - Variance
    - VaR
    - ES
    - Expectiles
    - Sensitivities (sort of)
    - Scenario analysis (sort of)
- Compare based on:
    - Coherence
    - Elicitability/backtestability
    - Robustness
    - Ease of use
- Need large numbers of samples for accuracy of statistical risk measures!

Summary

Computations:

$dS = \mu S dt + \sigma S dW$

- $E[S_T] = S_0 e^{\mu T}$
- $\mathrm{var}[S_T] = S_0^2 (e^{\sigma^2 T} - 1) e^{2\mu T}$
- $\mathrm{VaR}(S, T, p) = S_0 - S_0 e^{\sigma \sqrt{T} \Phi^{-1}(1-p) + (\mu - \frac{\sigma^2}{2}) T}$
- $\mathrm{ES}(S, T, p) = \text{homework}$
- Mostly easy to compute using martingale techniques

Still to come:

- Detailed VaR calculations
- More on VaR
- Credit risk
- Counterparty risk
- Regulation
- Case studies

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