1 Reminders, weak and strict stationarity

Exercise 1.1. Let X be a Gaussian vector, A_1 and A_2 two linear applications and set $X_1 = A_1X$ and $X_2 = A_2X$. Give the distribution of (X_1, X_2) and a necessary and sufficient condition for X_1 and X_2 to be independent.

Exercise 1.2. Let $(X_t)_{t\in\mathbb{Z}}$ be a second order stationary process with mean μ and autocovariance function γ. Let us denote

$$\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k \ .$$

1. Show that if $\gamma(t) \to 0$ as $t \to \infty$, then $\bar{X}_n \xrightarrow{P} \mu$ as $n \to \infty$.
2. Let us now assume that
$$\sum_{t\in\mathbb{Z}} |\gamma(t)| < \infty \ .$$
Show that $\mathrm{Var}(\bar{X}_n) = n^{-1}(c + o(1))$ and determine the constant c.
3. Deduce that for all $\epsilon > 0$,
$$\sum_{n\geq 1} \mathbb{P}(|\bar{X}_{n^2} - \mu| > \epsilon) < \infty \ .$$
4. Deduce that $\bar{X}_{n^2} \to \mu$ p.s. as $n \to \infty$.
5. Let us set $p_n = [\sqrt{n}]$. Show that $\mathrm{Var}\left(\bar{X}_n - \bar{X}_{p_n^2}\right) = O(n^{-3/2})$.
6. Conclude.

Exercise 1.3. Let X be a Gaussian random variable, with zero mean and unit variance, $X \sim \mathcal{N}(0, 1)$. Let $Y = X\mathbf{1}_{\{U=1\}} - X\mathbf{1}_{\{U=0\}}$ where U is a Bernoulli random variable with parameter $1/2$ independent of X. Show that $Y \sim \mathcal{N}(0, 1)$ and $\mathrm{Cov}(X, Y) = 0$ but also that X and Y are not independent. Use this to construct a weak white noise which is not a strong white noise, although the marginal distribution is invariant.

Exercise 1.4. Let $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ be two second order stationary processes that are uncorrelated in the sense that X_t and Y_s are uncorrelated for all t, s. Show that $Z_t = X_t + Y_t$ is a second order stationary process. Compute its autocovariance function, given the autocovariance functions of X and Y.

Exercise 1.5. Let $(\varepsilon_t)_{t\in\mathbb{Z}}$ be a strong white noise with $\mathbb{E}[\varepsilon_0] = 0$ and $\sigma^2 = \mathbb{E}[\varepsilon_0^2]$. Determine in each of the following cases, if the defined processes are second order stationary or strongly stationary. When appropriate, compute the mean and the autocovariance function.

1. $Y_t = a + b\varepsilon_t + c\varepsilon_{t-1}$ (a, b, c real numbers),
2. $Y_t = a + b\varepsilon_t + c\varepsilon_{t+1}$,
3. $Y_t = \sum_{j=0}^{+\infty} \rho^j \varepsilon_{t-j}$ for $|\rho| < 1$,
4. $Y_t = \varepsilon_t\varepsilon_{t-1}$, $Z_t = Y_t^2$ with $\mathbb{E}[\varepsilon_0^4] < \infty$.

2 Autocovariance function, Spectral measure, Herglotz theorem

Exercise 2.1. For $t \geq 2$, define

$$\Gamma_2 = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, ..., \Gamma_t = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \ddots & \vdots \\ \vdots & \ddots & 1 & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix}$$

1. For which values of ρ, is Γ_t guaranteed to be a covariance matrix for all values of t [Hint: write Γ_t as $\alpha I + A$ where A has a simple eigenvalue decomposition]?
2. Define a stationary process whose finite-dimensional covariance matrices coincide with Γ_t (for all $t \geq 1$).

Exercise 2.2. Show that the function γ defined by $\gamma(0) = 1$, $\gamma(1) = \gamma(-1) = \rho$ and $\gamma(k) = 0$ otherwise, is a covariance function if and only if $|\rho| \leq \frac{1}{2}$.

Exercise 2.3. Let (Y_t) be a weakly stationary process with spectral density f such that $0 \leq m \leq f(\lambda) \leq M < \infty$ for all $\lambda \in \mathbb{R}$. For $n \geq 1$, denote by Γ_n the covariance matrix of $[Y_1, \ldots, Y_n]'$. Show that the eigenvalues of Γ_n belong to the interval $[2\pi m, 2\pi M]$.

Exercise 2.4. Suppose that

$$Y_t = \beta t + S_t + X_t, \quad t \in \mathbb{Z} ,$$

where $\beta \in \mathbb{R}$, $(S_t)_{t\in\mathbb{Z}}$ is a 4-periodic weakly stationary process and $(X_t)_{t\in\mathbb{Z}}$ is a weakly stationary process such that (X_t) and (S_t) are uncorrelated.

1. Is (Y_t) weakly stationary ?
2. Which property is satisfied by the covariance function of (S_t) ? Define $(\bar{S}_t)$ as the process obtained by applying the operator $\sum_{i=0}^{3} B^i$ to (S_t), where B denotes the backshift operator. What can be said about $(\bar{S}_t)$?
3. Consider now (Z_t) obtained by successively applying $\sum_{i=0}^{3} B^i$ and $1 - B$ to (Y_t). Show that (Z_t) is weakly stationary and express its covariance function using that of (X_t).
4. Characterize the spectral measure ν of (S_t).
5. Compute the spectral measure of $(1 - B^4) \circ (Y_t)$ when (X_t) has spectral density f.

3 Linears models, the innovation process

Exercise 3.1. Let (X_t) be a centered weakly stationary regular process with autocovariance function γ; we denote by Γ_p the covariance matrix of $(X_1, X_2, \ldots, X_p)^T$.

1. By writing the orthogonality conditions, show that the predictor of order p

$$\mathrm{proj}\left(X_t \,\middle|\, \mathcal{H}^X_{t-1,p}\right) = \sum_{k=1}^{p} \phi_{k,p} X_{t-k}$$

   may be determined by solving a set of p equations known as the normal or Yule-Walker equations.

2. Express the variance of the prediction error σ_p^2 and show that it can also be written as

$$\sigma_p^2 = \frac{\det \Gamma_{p+1}}{\det \Gamma_p} \,.$$

Let us define $\epsilon_{1,0} = X_1$ and, for $t = 2, 3, \ldots$,

$$\epsilon_{t,t-1} = X_t - \sum_{k=1}^{t-1} \phi_{k,t-1} X_{t-k} \,.$$

3. Show that $(\epsilon_{t,t-1})_{t\geq 1}$ is an orthogonal sequence and compute their variances.

4. Express the covariance matrix of $\begin{bmatrix}\epsilon_{1,0} & \ldots & \epsilon_{p,p-1}\end{bmatrix}^T$ using Γ_p and deduce the Cholesky decomposition of Γ_p^{-1}. Compare with Question 2.

5. Show that the variance σ^2 of the innovation process satisfies

$$\log \sigma^2 = \lim_{p\to+\infty} \frac{1}{p} \log\det \Gamma_p \,.$$

Exercise 3.2. We consider in the following a MA(1) process:

$$X_t = Z_t + \theta Z_{t-1} \,,$$

where $(Z_t)_{t\in\mathbb{Z}}$ is a weak white noise of variance σ^2. We first assume that $|\theta| < 1$.

1. Show that Z_t may be written as $\sum_{k\geq 0} \psi_k X_{t-k}$ where $\psi \in \ell^1$.

2. Deduce from what precedes the form of the predictor $\mathrm{proj}\left(X_t \,\middle|\, \mathcal{H}^X_{t-1}\right)$ and that (Z_t) is the innovation process.

We now want to show that (Z_t) is also the innovation process when $\theta = -1$.

3. Check using the normal equations that

$$\mathrm{proj}\left(X_t \,\middle|\, \mathcal{H}^X_{t-1,p}\right) = -\sum_{j=1}^{p} \left(1 - j/(p+1)\right) X_{t-j}$$

4. Show that, in the L^2 sense, $\lim_{p\to+\infty} X_t - \mathrm{proj}\left(X_t \,\middle|\, \mathcal{H}^X_{t-1,p}\right) = Z_t$, and hence that (Z_t) still is the innovation process.

4 ARMA processes

Exercise 4.1 (Canonical ARMA representation)**.** Let $(X_t)_{t\in\mathbb{Z}}$ denote a second-order stationary process satisfying the following recurrence relation

$$X_t - 2X_{t-1} = Z_t + 4Z_{t-1}$$

where $(Z_t)_{t\in\mathbb{Z}}$ is a weak white noise with variance σ^2.

1. What is the power spectral density of (X_t)?
2. What is the canonical representation of (X_t)?
3. What is the variance of the innovation process associated to (X_t)?
4. How is it possible to write Z_t as a function of $(X_s)_{s\in\mathbb{Z}}$?

Exercise 4.2 (Sum of MA processes)**.** Let $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ denote two uncorrelated MA processes such that

$$\begin{aligned} X_t &= U_t + \theta_1 U_{t-1} + \ldots + \theta_q U_{t-q} \\ Y_t &= V_t + \rho_1 V_{t-1} + \ldots + \rho_p V_{t-p} \end{aligned}$$

where $(U_t)_{t\in\mathbb{Z}}$ and $(V_t)_{t\in\mathbb{Z}}$ are weak white noise processes with variances σ_U^2 and σ_V^2, respectively. Defining $Z_t = X_t + Y_t$,

1. Show that (Z_t) is an ARMA process.
2. Assuming that $q = p = 1$ and $0 < \theta_1, \rho_1 < 1$, compute the variance of the innovation process corresponding to (Z_t).

Exercise 4.3 (Sum of AR processes)**.** Let $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ denote two uncorrelated AR(1) processes :

$$\begin{aligned} X_t &= \phi X_{t-1} + U_t \\ Y_t &= \psi Y_{t-1} + V_t \end{aligned}$$

where $(U_t)_{t\in\mathbb{Z}}$ and $(V_t)_{t\in\mathbb{Z}}$ are two weak white noise processes with variances σ_U^2 and σ_V^2, respectively, $0 < \phi, \psi < 1$; define $Z_t = X_t + Y_t$.

1. Show that there exists a weak white noise $(W_t)_{t\in\mathbb{Z}}$ with variance σ^2 and θ with $|\theta| < 1$ such that
$$Z_t - (\phi + \psi)\, Z_{t-1} + \phi\psi Z_{t-2} = W_t - \theta W_{t-1}\,.$$
2. Check that $W_t = U_t + (\theta - \psi) \sum_{k=0}^{\infty} \theta^k U_{t-1-k} + V_t + (\theta - \phi) \sum_{k=0}^{\infty} \theta^k V_{t-1-k}$.
3. Determine the best linear predictor of Z_{t+1} in terms of $(X_s)_{s\le t}$ and $(Y_s)_{s\le t}$.
4. Determine the best linear predictor of Z_{t+1} based on the observation of $(Z_s)_{s\le t}$.
5. Among these two predictors, which one has the lowest error and why?

5 Linear forecasting

Exercise 5.1 (Levinson-Durbin algorithm)**.** For a centered weakly stationary process $(Y_t)_{t\in\mathbb{Z}}$, denote by $\text{proj}\left(Y_t \,\middle|\, \mathcal{H}^Y_{t-1,p}\right) = \sum_{j=1}^p \phi_{j,p} Y_{t-j}$ the linear projection of Y_t on $\text{span}\left(Y_{t-1}, \ldots, Y_{t-p}\right)$ and by $\sigma_p^2 = \mathbb{E}\left[Y_t - \text{proj}\left(Y_t \middle| \mathcal{H}^Y_{t-1,p}\right)\right]^2$ the variance of the corresponding projection error.

1. Show that $\phi_{1,1}$ coincides with the value of the autocorrelation function of (Y_t) at lag 1 and that $\sigma_1^2 = \gamma(0) - \phi_{1,1}\gamma(1)$, where γ is the autocovariance function of (Y_t).

2. Show that

$$\text{proj}\left(Y_t \,\middle|\, \mathcal{H}^Y_{t-1,p}\right) = \frac{\mathbb{E}\left[Y_t \epsilon^-_{t-p,p-1}\right]}{\mathbb{E}\left[\left(\epsilon^-_{t-p,p-1}\right)^2\right]} \epsilon^-_{t-p,p-1} + \text{proj}\left(Y_t \,\middle|\, \mathcal{H}^Y_{t-1,p-1}\right) ,$$

   where $\epsilon^-_{t-p,p-1} = Y_{t-p} - \text{proj}\left(Y_{t-p} \,\middle|\, \mathcal{H}^Y_{t-1,p-1}\right)$ denotes the *backward* prediction error at time $t-p$.

3. Find the expressions of $\epsilon^-_{t-p,p-1}$ and of its variance.

4. Deduce from the above that the prediction coefficients can be computed recursively, for $p = 1, 2, \ldots,$ according to

$$\begin{aligned} \phi_{p,p} &= \sigma_{p-1}^{-2}\left(\gamma(p) - \sum_{j=1}^{p-1} \phi_{j,p-1}\gamma(p-j)\right) \\ \phi_{j,p} &= \phi_{j,p-1} - \phi_{p,p}\phi_{p-j,p-1}\,, \text{for } j = 1, \ldots, p-1 \\ \sigma_p^2 &= \sigma_{p-1}^2\left(1 - \phi_{p,p}^2\right) \end{aligned}$$

   How does the number of operations required by this recursion scales with the order p of the predictor?

5. Why is the partial autocorrelation coefficient $\phi_{p,p}$ indeed a correlation?

Exercise 5.2 (Kalman filtering of an AR(1) process observed in noise)**.** Consider an AR(1) process (X_t) with canonical representation:

$$X_{t+1} = \phi X_t + W_{t+1} \tag{1}$$

where (W_t) is a centered white noise with known variance σ^2 and ϕ is also known. The process (X_t) is not directly observable and we have, for $t \geq 1$,

$$Y_t = X_t + V_t \,, \tag{2}$$

where (V_t) is a centered white noise with known variance ρ^2, that is uncorrelated with (W_t).

We denote by $\hat{X}_{t|t} = \mathrm{proj}\left(X_t \,\middle|\mathcal{H}^Y_{t,t}\right)$ the filtering estimate and by $\Sigma_{t|t} = \mathbb{E}(X_t - \hat{X}_{t|t})^2$ the corresponding projection error variance. Similarly, $\hat{X}_{t+1|t} = \mathrm{proj}\left(X_{t+1} \,\middle|\mathcal{H}^Y_{t,t}\right)$ is the best linear state predictor and $\Sigma_{t+1|t} = \mathbb{E}(X_{t+1} - \hat{X}_{t+1|t})^2$ the corresponding error variance.

1. Comment the differences between the model of Equations (1)–(2) and the general state-space representation.
2. Using the evolution equation (1), show that
$$\hat{X}_{t+1|t} = \phi \hat{X}_{t|t} \quad \text{and} \quad \Sigma_{t+1|t} = \phi^2 \Sigma_{t|t} + \sigma^2$$
3. Defining the innovation[1] by $I_{t+1} = Y_{t+1} - \mathrm{proj}\left(Y_{t+1} \,\middle|\mathcal{H}^Y_{t,t}\right)$. Using the observation equation (2), show that $I_{t+1} = Y_{t+1} - \hat{X}_{t+1|t}$.
4. Prove that $\mathbb{E}[I^2_{t+1}] = \Sigma_{t+1|t} + \rho^2$.
5. Why does the following expression holds:
$$\hat{X}_{t+1|t+1} = \hat{X}_{t+1|t} + k_{t+1} I_{t+1} \,,$$
where $k_{t+1} = \mathbb{E}[X_{t+1} I_{t+1}] / \mathbb{E}[I^2_{t+1}]$ (k_{t+1} is called *the Kalman gain*)?
6. Using the above expression of I_{t+1}, show that $\mathbb{E}[X_{t+1} I_{t+1}] = \Sigma_{t+1|t}$.
7. Show that $\Sigma_{t+1|t+1} = \Sigma_{t+1|t} - \mathbb{E}\left[(k_{t+1} I_{t+1})^2\right]$ and deduce from this that $\Sigma_{t+1|t+1} = (1 - k_{t+1}) \Sigma_{t+1|t}$.
8. Provide the complete set of equations that allows to compute $\hat{X}_{t|t}$ and $\Sigma_{t|t}$ iteratively for all $t \geq 1$.
9. Study the asymptotic behavior of $\Sigma_{t|t}$ as $t \to \infty$.

[1] Using the lecture notes' notation, this corresponds to $\epsilon^+_{t+1,t}$.

6 Statistical inference

Exercise 6.1. Let (X_t) be a weakly stationary process with mean μ and autocovariance function γ. We observe $\{X_1, \dots, X_n\}$.

1. Give the best linear *unbiased* estimator $\hat{\mu}_n$ that minimizes the mean squarred error

$$\mathrm{EQM} = \mathbb{E}[(\mu - \hat{\mu}_n)^2].$$

Exercise 6.2 (AR estimation using moments)**.** Let (X_t) be a centered weakly stationary process with covariance function γ. Denote

$$\Gamma_t = \mathrm{Cov}\left([X_1, \dots, X_t]^T\right) = (\gamma(i-j))_{1 \le i,j \le t} \quad \text{for all } t \ge 1.$$

In addition, for n successive observations $X_1, \dots, X_n$ of (X_t), define the *empirical covariance function* by $\hat{\gamma}$

$$\hat{\gamma}(k) = \begin{cases} n^{-1} \sum_{t=1}^{n-|k|} (X_t - \bar{X}_n)(X_{t+|k|} - \bar{X}_n) & \text{if } |k| \le n-1 \\ 0 & \text{otherwise} \end{cases}$$

where $\bar{X}_n = 1/n \sum_{t=1}^{n} X_t$.

1. We temporarily assume that there exists $k \ge 1$ such that Γ_k is invertible but Γ_{k+1} is not. Show that we can then write X_n as $\sum_{t=1}^{k} \alpha_t^{(n)} X_t$, where $\alpha^{(n)} \in \mathbb{R}^k$, for all $n \ge k+1$.
2. Show that the vectors $\alpha^{(n)}$ are bounded independently of n.
3. If we now assume $\gamma(0) > 0$ and $\gamma(t) \to 0$ as $t \to \infty$, can we find $t \ge 1$ such that Γ_t is singular ?
4. Show that $\hat{\gamma}$ is a covariance function. Deduce that the empirical covariance matrices $\hat{\Gamma}_k = (\hat{\gamma}(i-j))_{1 \le i,j \le k}$ are invertible for all $k \ge 1$ under a simple condition on $X_1, \dots, X_n$.

We now consider the problem of p-order linear prediction for the process (X_t). Denote by $\sum_{k=1}^{p} \phi_{k,p} X_{t-k}$ the best p-order linear predictor of X_t.

5. We assume that Γ_{p+1} is invertible. Show that the roots of $\Phi(z) = 1 - \sum_{k=1}^{p} \phi_{k,p} z^k$ have modulus in $(1, \infty)$.

Consider the AR(p) process

$$X_t = \sum_{k=1}^{p} \phi_k X_{t-k} + \varepsilon_t$$

where (ε_t) is a white noise with variance $\sigma^2 > 0$. To estimate $\phi_1, \dots, \phi_p$ and σ^2 we solve the Yule-Walker equations with the covariance function γ replaced by the empirical one, $\hat{\gamma}$.

6. Give a condition on $\phi_1, \dots, \phi_p$ for which this method appears to be appropriate.
7. Show that this approach does provide uniquely defined estimators $\hat{\phi}_1, \dots, \hat{\phi}_p$ and $\hat{\sigma}^2$.
8. Show that the filter $\hat{\Phi}(B) = 1 - \sum_{k=1}^{p} \hat{\phi}_k B^k$ is causally invertible.

7 Statistical inference (contd.)

Exercise 7.1 (Likelihood of Gaussian processes)**.** Consider n observations $X_1, \dots, X_n$ from a regular, centered, 2nd order stationary Gaussian process with autocovariance function γ_θ depending on an unknown parameter $\theta \in \Theta$. For a given value of θ, define the following innovation sequence

$$\begin{cases} I_{1,\theta} = X_1, \quad \upsilon_{1,\theta} = \gamma_\theta(0) \\ I_{t,\theta} = X_t - \hat{X}_{t,\theta}, \quad \upsilon_{t,\theta} = \mathrm{Var}_\theta(I_{t,\theta}) \quad \text{for } t = 2, \dots, n \end{cases}$$

where $\hat{X}_{t,\theta}$ denotes the L^2 projection of X_t onto $\mathrm{Span}(X_1, \dots, X_{t-1})$, under the parameter θ.

1. Show that the log-likelihood of θ can be written as

$$\log p_\theta(X_1, \dots, X_n) = -\frac{1}{2}\left[n \log(2\pi) + \sum_{t=1}^{n} \left\{ \log \upsilon_{t,\theta} + \frac{I_{t,\theta}^2}{\upsilon_{t,\theta}} \right\} \right]$$

2. Consider the AR(1) model $X_t = \phi X_{t-1} + \varepsilon_t$ where (ε_t) is a Gaussian white noise of variance σ^2 and define $\theta = (\phi, \sigma^2)$ and $\Theta = (-1, 1) \times (0, \infty)$. Show that the log-likelihood then satisfies

$$\log p_\theta(X_1, \dots, X_n) = -\frac{1}{2}\left[n \log(2\pi) + \log\left(\frac{\sigma^2}{1-\phi^2}\right) + \frac{X_1^2(1-\phi^2)}{\sigma^2} \right. \\ \left. + (n-1)\log\sigma^2 + \sum_{t=2}^{n} \frac{(X_t - \phi X_{t-1})^2}{\sigma^2} \right]$$

   Deduce from the above the expression of the "conditional" maximum likelihood estimator $\hat{\theta}_n = (\hat{\phi}_n, \hat{\sigma}_n^2)$, obtained by maximizing $\log p_\theta(X_2, \dots, X_n | X_1)$.

3. Show that the Fisher information matrix for θ is equivalent to nJ when $n \to \infty$, where J is a matrix to be determined.

4. The maximum likelihood estimator can be shown to be asymptotically efficient: $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}_2(0, J^{-1})$. Construct an asymptotic test for testing the null hypothesis $H_0 : \phi = 0$ against the alternative $H_1 : \phi \neq 0$; the decision threshold at level $\alpha \in (0, 1)$ of the (first type) risk will be expressed as a quantile of the $\mathcal{N}_1(0, 1)$ law.

Now consider the MA(1) model $X_t = \varepsilon_t + \rho\varepsilon_{t-1}$, where (ε_t) is a Gaussian white noise of variance σ^2 and $\theta = (\rho, \sigma^2) \in \Theta = (-1, 1) \times (0, \infty)$.

6. Show that the innovation sequence can be computed according to the following recursion:

$$\begin{cases} I_{1,\theta} = X_1, \quad \upsilon_{1,\theta} = (1+\rho^2)\sigma^2 \\ I_{t,\theta} = X_t - \rho\frac{\sigma^2}{\upsilon_{t-1,\theta}} I_{t-1,\theta}, \quad \upsilon_{t,\theta} = (1+\rho^2)\sigma^2 - \frac{\rho^2\sigma^4}{\upsilon_{t-1,\theta}} \quad \text{for } t = 2, \dots, n \end{cases}$$

7. Considering $\tilde{\upsilon}_{t,\theta} = \upsilon_{t,\theta}/\sigma^2$, obtain the expression of $\hat{\sigma}_n^2$ as a function of $\hat{\rho}_n$ and of the observations $X_1, \dots, X_n$.

8. Show that, for any $\epsilon > 0$, $\sup_{|\theta| \le 1-\epsilon} |\tilde{\upsilon}_{t,\theta} - 1| \to 0$.

8 ARCH and GARCH processes

A GARCH(q, p) process is defined as

$$X_t = \sigma_t Z_t \;, \tag{1}$$

$$\sigma_t^2 = a + \sum_{k=1}^{q} b_k X_{t-k}^2 + \sum_{k=1}^{p} c_k \sigma_{t-k}^2, \quad t \in \mathbb{Z} \;, \tag{2}$$

where (Z_t) is a centered process such that $\mathbb{E}[Z_t \mid \mathcal{F}_{t-1}] = 0$, $\mathbb{E}[Z_t^2 \mid \mathcal{F}_{t-1}] = 1$ and, for all t, σ_t is positive and $\mathcal{F}_{t-1}$-measurable, where $(X_t)_{t\in\mathbb{Z}}$ is adapted to $(\mathcal{F}_t)_{t\in\mathbb{Z}}$.

- How do (σ_t) and (Z_t) relate to (X_t)?
- The ARCH(q) process corresponds to the case where $p = 0$ and hence the second sum in equation (2) is absent. Show that in this case $(\mathcal{F}_t)_{t\in\mathbb{Z}}$ can be taken to be the natural filtration of $(X_t)_{t\in\mathbb{Z}}$.

Exercise 8.1 (Specific example of an ARCH(q) process)**.** Let $q \geq 1$, $p = 0$ and denote by $a, b_1, b_2, \ldots, b_q$ positive coefficients such that a and b_q are non-null. We consider the stationary solution of the following non-centered AR equation

$$Y_t = a + \sum_{i=1}^{q} b_i Y_{t-i} + U_t$$

where (U_t) is a centered strong white noise process of finite variance. In the following, assume that $P(1) > 0$, where $P(z) = 1 - \sum_{i=1}^{q} b_i z^i$.

1. Show that the filter associated to $P(B)$ is causally invertible.
2. Show that Y_t is a positive process under appropriate conditions on the support of the marginal distribution of U_t.
3. Let (V_t) denote an iid centered process such that $\mathbb{P}(V_t = \pm 1) = 1/2$ which is independent of the (U_t) process. Let $X_t = V_t\sqrt{Y_t}$. Show that (X_t) is then an ARCH(q) process.

Exercise 8.2 (Computation of the kurtosis of a conditionally Gaussian GARCH$(1, 1)$ process)**.** Assume that Equations (1) and (2) hold for all $t \in \mathbb{Z}$, where (Z_t) is iid with marginal distribution $\mathcal{N}(0, 1)$ and (σ_t^2) denotes the stationary solution that can be causally expressed as a function of (Z_{t-1}), that is, σ_t^2 is $\mathcal{G}_{t-1}$-measurable, where $\mathcal{G}_t$ denotes the σ-field generated by $(Z_s)_{s\leq t}$. Recall that for a $\mathcal{N}(0, 1)$-distributed random variable Z, we have $\mathbb{E}[Z^4] = 3$. Further assume that σ_t satisfies $\mathbb{E}[\sigma_t^4] < \infty$.

1. What is the conditional distribution of X_t given $\mathcal{G}_{t-1}$?
2. Denote by $\kappa = \frac{\mathbb{E}[X_t^4]}{(\mathbb{E}[X_t^2])^2}$ the *kurtosis* of X_t. Show that $\kappa = 3 + 3\frac{\mathrm{Var}(\mathbb{E}[X_t^2 \mid \mathcal{G}_{t-1}])}{(\mathbb{E}[X_t^2])^2}$.
3. For a GARCH(1,1) process, with $p = q = 1$ and $a, b = b_1, c = c_1 > 0$, show that

$$\kappa = 3 + \frac{6b^2}{1 - (c^2 + 2bc + 3b^2)} \;.$$

9 Multivariate processes

Exercise 9.1 (Vector AR)**.** Let $d \geq 1$. Consider the following vector AR(1) equation

$$X_t = \Phi X_{t-1} + Z_t \ , \tag{1}$$

where Φ is a $d \times d$ matrix, $(Z_t)_{t \in \mathbb{Z}}$ is a centered process taking its values in $\mathbb{R}^d$ such that $\mathbb{E}[Z_s Z_t^T]$ vanishes if $s \neq t$ and equals $\sigma^2 I$ if $s = t$. The operator norm of Φ is denoted by

$$\|\Phi\| = \sup_{x \in \mathbb{R}^d, \|x\| \leq 1} \|\Phi x\| \ .$$

and the spectral radius of Φ by $\rho(\Phi)$. We assume that $\rho(\Phi) < 1$.

1. Show that there exists a unique second order stationary vector process that satisfies (1).[**Indication**: use that $\|\Phi^k\| \leq C\rho^k$ for all $k \geq 1$, where $C > 0$ and $\rho(\Phi) < \rho < 1$ are some constants.]
2. Compute the autocovariance matrix function of (X_t).
3. Suppose that Φ is symmetric with null space reduced to $\{0\}$. Write (X_t) as a linear transform of d uncorrelated scalar AR(1) processes $(Y_t(1))$,..., $(Y_t(d))$.
4. Let (U_t) denote the sequence of the first entries of (X_t). Which ARMA representation does it satisfy ?
5. Compute the spectral density of (U_t).

Exercise 9.2 (Cointegration)**.** Consider a bivariate process X_t solution of the equation

$$\begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} = \underbrace{\begin{bmatrix} 1/2 & -1 \\ -1/4 & 1/2 \end{bmatrix}}_{\Phi} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix} , \tag{2}$$

where $(\epsilon_{1,t})$ and $(\epsilon_{2,t})$ are two uncorrelated scalar white noise sequences.

1. Show that Φ may be diagonalized as $P \Lambda P^{-1}$ where
$$P = \begin{pmatrix} 1 & 2 \\ -1/2 & 1 \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
is there a stationary solution (X_t) to Equation (2)?.
2. Define $Y_t = P^{-1} X_t$ and show that $(Y_{1,t})$ is an integrated process of order 1 while $(Y_{2,t})$ is a white noise process.
3. Show that there exists a unique linear combination of the form $\begin{bmatrix} 1 & \alpha \end{bmatrix} X_t$ which is stationary.