SIMDI 226
Time series analysis : II

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Plan

Innovation process

Let $(X_t)_{t\in\mathbb{Z}}$ be a centered weakly stationary process. Its linear past is defined as

$$\mathcal{H}_t^X = \overline{\mathrm{Span}}\left(X_s, s \le t\right) \ .$$

Innovation process

The innovation process $(\epsilon_t)_{t\in\mathbb{Z}}$ of X is defined by

$$\epsilon_t = X_t - \mathrm{proj}\left(X_t | \mathcal{H}_{t-1}^X\right), \qquad t \in \mathbb{Z} \ .$$

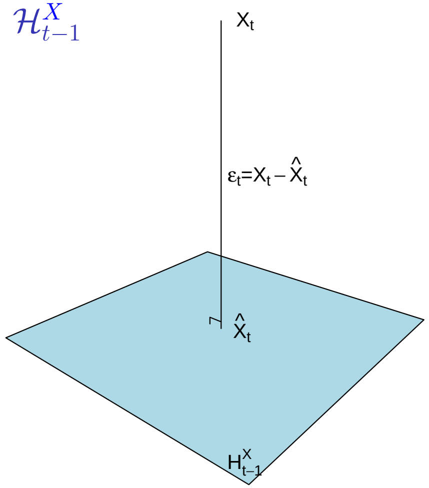
Here the projection is understood in the L^2 (Hilbert space) sense. It is characterized by

(i) $X_t - \epsilon_t \in \mathcal{H}_{t-1}^X$

(ii) $\epsilon_t \perp \mathcal{H}_{t-1}^X$

By definition, $(\epsilon_t)_{t\in\mathbb{Z}}$ is an orthogonal sequence.

Projection on $\mathcal{H}_{t-1}^X$



Innovation as a limit of finite order prediction

The linear past of order p is defined as

$$\mathcal{H}_{t,p}^X = \mathrm{Span}\left(X_s, t-p < s \le t\right) \ .$$

Then the prediction of order p is defined by

$$\mathrm{proj}\left(X_t | \mathcal{H}_{t-1,p}^X\right) = \sum_{k=1}^{p} \phi_{k,p} X_{t-k} \ ,$$

where $\phi_p = [\phi_{1,p}, \ldots, \phi_{p,p}]^T$ is independent of t.

Since $\mathcal{H}_t^X = \overline{\bigcup_{p\ge1} \mathcal{H}_{t,p}^X}$, we have

$$\lim_{p\to\infty} \mathrm{proj}\left(X_t | \mathcal{H}_{t-1,p}^X\right) = \mathrm{proj}\left(X_t | \mathcal{H}_{t-1}^X\right) \ .$$

The innovation process is a white noise

Consequence

The innovation process $(\epsilon_t)_{t\in\mathbb{Z}}$ of X is weakly stationary hence is a white noise. Its variance σ^2 is called the innovation variance of X.

Definition : regular/deterministic processes

A weakly stationary process is called regular if its innovation variance is strictly positive. Otherwise, we say that it is a deterministic process.

Examples

- If X is a white noise, $\epsilon = X$ (hence iff).
- A constant process $X_t = X_0$ is deterministic.
- Consider the harmonic process

$$X_t = \sum_{k=1}^{N} A_k \cos(\lambda_k t + \Phi_k) ,$$

where $(\lambda_k)_{1 \le k \le N} \in [-\pi, \pi]$ are N frequencies, $(\Phi_k)_{1 \le k \le N}$ are N i.i.d. random variables with a uniform distribution on $[-\pi, \pi]$, and independent of $(A_k)_{1 \le k \le N}$. Then X has covariance

$$\gamma(\tau) = \frac{1}{2} \sum_{k=1}^{N} \sigma_k^2 \cos(\lambda_k \tau) ,$$

where $\sigma_k^2 = \mathbb{E}\left[A_k^2\right]$, $k = 1, \ldots, N$. It follows that X is deterministic.

Purely non-deterministic processes

Let us define

$$\mathcal{H}_{-\infty}^X = \bigcap_{t \in \mathbb{Z}} \mathcal{H}_t^X .$$

- If X is deterministic then $X_t \in \mathcal{H}_{-\infty}^X$ for all $t \in \mathbb{Z}$.
- If $\mathcal{H}_{-\infty}^X = \{0\}$, we say that X is purely non-deterministic.

Example

If $X = \sum_{k \ge 0} \psi_k Z_{t-k}$ with $Z \sim \mathrm{WN}(0, \sigma^2)$ and $\psi \in \ell^2$ then X is purely non-deterministic.

Unfortunately, all regular processes are not purely non-deterministic : take the sum of a white noise with an uncorrelated constant process.

Projection on the innovation process

Let X be a centered regular weakly stationary process and let $(\epsilon_t)_{t \in \mathbb{Z}}$ be its innovation process and σ its innovation variance.

Define, for all $k = 0, 1, \ldots$

$$\psi_k = \frac{\langle X_t, \epsilon_{t-k} \rangle}{\sigma^2} \tag{1}$$

so that

$$U_t := \mathrm{proj}\left(X_t \middle| \mathcal{H}_t^\epsilon\right) = \sum_{k \ge 0} \psi_k \epsilon_{t-k} .$$

Note that $\psi_0 = 1$ and for all $s < t$,

$$\begin{aligned} \mathcal{H}_t^X &= \mathcal{H}_{t-1}^X \overset{\perp}{\oplus} \mathrm{Span}\left(\epsilon_t\right) \\ &= \mathcal{H}_s^X \overset{\perp}{\oplus} \mathrm{Span}\left(\epsilon_k,\ s < k \le t\right) . \end{aligned} \tag{2}$$

Then we get that

$$\mathcal{H}_{-\infty}^X \overset{\perp}{\oplus} \mathcal{H}_t^\epsilon = \mathcal{H}_t^X .$$

Wold decomposition

Define $V_t = \mathrm{proj}\left(X_t | \mathcal{H}^X_{-\infty}\right)$.

The decomposition $X_t = U_t + V_t$ is called the Wold decomposition.

The following facts follow.

- U and V are two uncorrelated processes.
- $(U_t)_{t\in\mathbb{Z}}$ is a regular purely non-deterministic process, $\mathcal{H}^U_t = \mathcal{H}^\epsilon_t$ and U has innovation ϵ.
- V is deterministic and $\mathcal{H}^V_{-\infty} = \mathcal{H}^X_{-\infty}$.

Convolution in ℓ^1

Denote

$$\ell^1 = \left\{ \psi \in \mathbb{C}^{\mathbb{Z}} \ : \ \sum_k |\psi_k| < \infty \right\} .$$

Define the linear filter with impulse response $\psi \in \ell^1$ by the convolution

$$\mathrm{F}_\psi : x = (x_t)_{t\in\mathbb{Z}} \mapsto y = \psi \star x \ , \quad y_t = \sum_{k\in\mathbb{Z}} \psi_k x_{t-k}, \quad t \in \mathbb{Z} \ .$$

Definition : types of filters

- If ψ is finitely supported, F_ψ is called a finite impulse response (FIR) filter.
- If $\psi_t = 0$ for all $t \leq 0$, F_ψ is said to be causal.
- If $\psi_t = 0$ for all $t > 1$, F_ψ is said to be anticausal.

Set of definition

FIR filter

When ψ is finitely supported, we may write

$$\mathrm{F}_\psi = \sum_{k \in \mathbb{Z}} \psi_k \, \mathrm{B}^k ,$$

where $\mathrm{B} = \mathrm{S}^{-1}$ is the Backshift operator.

If ψ is not finitely supported, it is well defined only on

$$\ell_\psi = \left\{ (x_t)_{t \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} \ : \ \text{for all } t \in \mathbb{Z}, \sum_{k \in \mathbb{Z}} |\psi_k \, x_{t-k}| < \infty \right\} .$$

Linear filtering of weakly stationary time series

Theorem

Let $\psi \in \ell^1$. Then, for all random process $X = (X_t)_{t \in \mathbb{Z}}$ such that

$$\sup_{t \in \mathbb{Z}} \mathbb{E}|X_t| < \infty ,$$

we have $X \in \ell_\psi$ a.s. If moreover

$$\sup_{t \in \mathbb{Z}} \mathbb{E}\left[|X_t|^2\right] < \infty ,$$

then the series

$$Y_t = \sum_{k \in \mathbb{Z}} \psi_k X_{t-k} ,$$

is absolutely convergent in L^2, and we have $(Y_t)_{t \in \mathbb{Z}} = \mathrm{F}_\psi(X)$ a.s.

Linear filtering of weakly stationary time series

Corollary

Let $\psi \in \ell^1$. Then, if $X = (X_t)_{t \in \mathbb{Z}}$ is weakly stationary then $Y = \mathrm{F}_\psi(X)$ is well defined and is an L^2 process.

2nd order properties

Moreover, Y is weakly stationary and, denoting by μ, γ and ν the mean, autocovariance function and spectral measure of X, those of Y are given by

- $\mu' = \mu \sum_k \psi_k$,
- $\gamma'(\tau) = \sum_{\ell,k} \psi_k \overline{\psi_\ell} \gamma(\tau + \ell - k)$,
- $\nu'(\mathrm{d}\lambda) = |\psi^*(\lambda)|^2 \ \nu(\mathrm{d}\lambda)$, with $\psi^*(\lambda) = \sum_k \psi_k \mathrm{e}^{-\mathrm{i}\lambda k}$.

All-pass filters

If $|\psi^*(\lambda)|^2 = 1$ for all λ, F_ψ is called an all-pass filter: it does not affect the spectral measure.

Examples

- Time shift : B^k
- Root inversion :$(1 - \alpha\,\mathrm{B})$ and $(1 - \overline{\alpha}^{-1}\,\mathrm{B}^{-1})$ have the same impact on the spectral measure but are different filters. Idea : can we define an all-pass filter as "$(1 - \alpha\,\mathrm{B})^{-1} \circ (1 - \overline{\alpha}^{-1}\,\mathrm{B}^{-1})$" ?

Composition

The convolution product $\star$ is commutative and associative in ℓ^1. So if $\psi, \phi \in \ell^1$, then for all $x \in \ell^1$,

$$\mathrm{F}_\psi \circ \mathrm{F}_\phi(x) = \psi \star (\phi \star x) = (\psi \star \phi) \star x = \mathrm{F}_{\psi\star\phi}(x) \; .$$

Theorem : composition

Let $\psi, \phi \in \ell^1$. Then, for all random process $X = (X_t)_{t\in\mathbb{Z}}$ such that

$$\sup_{t\in\mathbb{Z}} \mathbb{E}|X_t| < \infty \; ,$$

we have

$$\mathrm{F}_\psi \circ \mathrm{F}_\phi(X) = \mathrm{F}_\phi \circ \mathrm{F}_\psi(X) = \mathrm{F}_{\psi\star\phi}(X) \quad \text{a.s.}$$

Inversion

Definition : invertibility

Let $\psi \in \ell^1$ and $Y = \mathrm{F}_\psi(X)$. We will say that Y is invertible with respect to X if there exists $\phi \in \ell^1$, such that $X = \mathrm{F}_\phi(Y)$.

By the composition theorem, in the stationary case, it amounts to find ϕ such that

$$\psi \star \phi = \mathrm{e}_0 \Leftrightarrow \psi^* \times \phi^* = 1 \; ,$$

where e_0 is the impulse sequence, $\mathrm{e}_{0,k} = \mathbb{1}_{\{0\}}(k)$.

Inversion of a FIR filter

Causal FIR filters are of the form $P(\mathrm{B})$, where P is a polynomial, say $P(z) = \sum_{k=0}^{p} \mathrm{h}_k\, z^k$. Completing the sequence h by zeros, we have

$$P(\mathrm{B}) = \mathrm{F}_{\mathrm{h}} \quad \text{and} \quad \mathrm{h}^*(\lambda) = P(\mathrm{e}^{-\mathrm{i}\lambda}) \;.$$

Consequence

The problem of the inversion of a FIR filter is equivalent to find $\phi \in \ell^1$ such that

$$\frac{1}{P(z)} = \sum_{k\in\mathbb{Z}} \phi_k\, z^k$$

for all z on the unit circle Γ_1, which has a unique solution iff P does not vanish on Γ_1.

Inversion of a FIR filter: a special case

By the partial fraction decomposition, it suffices to consider the case

$$P(z) = 1 - \alpha z$$

- If $|\alpha| < 1$ we have, for all $z \in \Gamma_1$,

$$\frac{1}{1-\alpha z} = \sum_{k\geq 0} \alpha^k z^k \qquad \text{(Causal inverse filter)} \;.$$

- If $|\alpha| > 1$ we have, for all $z \in \Gamma_1$,

$$\frac{1}{1-\alpha z} = -\sum_{k\leq -1} \alpha^k z^k \qquad \text{(Anticausal inverse filter)} \;.$$

In all cases we obtain ϕ such that $\phi_k = O(\delta^k)$ as $k \to \pm\infty$ for $\delta = |\alpha| \wedge |\alpha|^{-1}$.

Applications

Rational filters

Let $\frac{P}{Q}$ be a rational function (with P and Q coprime polynomials). Suppose that Q does not vanish on Γ_1. Then there exists a unique $\phi \in \ell^1$ such that, for all $z \in \Gamma_1$,

$$\frac{P}{Q}(z) = \sum_{k} \phi_k z^k \;.$$

Moreover $\phi_k = O(\delta^k)$ as $k \to \pm\infty$ for some $\delta \in (0,1)$ and F_ϕ is causal iff Q does not vanish on the unit disk Δ_1.

Construction of an all-pass rational filter

Let $\alpha \notin \Gamma_1$ and define F_ψ by

$$\frac{1-\alpha z}{1-\overline{\alpha}^{-1} z^{-1}} = \sum_{k\in\mathbb{Z}} \psi_k z^k \;.$$

1 Wold decomposition

2 Convolution in ℓ^1

3 ARMA processes
- ARMA equations, stationary solutions
- Innovations of ARMA processes

MA(q) processes

Definition : MA(q) processes

Let $Z \sim \mathrm{WN}(0, \sigma^2)$ and Θ be a polynomial of degree q such that $\Theta(0) = 1$. The associated MA(q) equation is defined by

$$X = [\Theta(\mathrm{B})](Z) \Leftrightarrow X_t = Z_t + \sum_{k=1}^{q} \theta_k Z_{t-k} \text{ for all } t \in \mathbb{Z}.$$

($\Theta(z) = 1 + \sum_{k=1}^{q} \theta_k z^k$.) Then X is a centered weakly stationary process and its autocovariance function γ satisfies

$$\gamma(h) = \begin{cases} \sigma^2 \sum_{t=0}^{q-h} \theta_k \overline{\theta}_{k+h}, & \text{if } 0 \leq h \leq q \,, \\ \sigma^2 \sum_{t=0}^{q+h} \overline{\theta}_k \theta_{k-h}, & \text{if } -q \leq h \leq 0 \,, \\ 0, & \text{otherwise} \,, \end{cases}$$

It is called an MA(q) process.

AR(p) processes

Definition : AR(p) processes

Let $Z \sim \mathrm{WN}(0, \sigma^2)$ and Φ be a polynomial of degree p such that $\Phi(0) = 1$. The associated AR(p) equation is defined by

$$[\Phi(\mathrm{B})](X) = Z \Leftrightarrow X_t = \sum_{k=1}^{q} \phi_k X_{t-k} + Z_t \text{ for all } t \in \mathbb{Z}.$$

($\Phi(z) = 1 - \sum_{k=1}^{q} \phi_k z^k$.) If moreover X is weakly stationary, it is called an AR(p) process.

Theorem

The AR(p) equation admits a weakly stationary solution iff Φ does not vanish on Γ_1, in which case it is the unique one. Moreover, it is centered and admits a spectral density $f(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{|\Phi(\mathrm{e}^{-\mathrm{i}\lambda})|^2}$.

ARMA(p, q) processes

Definition : ARMA(p, q) processes

Let $Z \sim \mathrm{WN}(0, \sigma^2)$ and Θ, Φ be two coprime polynomials of degree q and p such that $\Theta(0) = \Phi(0) = 1$. The associated ARMA(p, q) equation is defined by

$$[\Phi(\mathrm{B})](X) = [\Theta(\mathrm{B})](Z) \Leftrightarrow X_t = \sum_{k=1}^{q} \phi_k X_{t-k} + Z_t + \sum_{k=1}^{q} \theta_k Z_{t-k} \text{ for all } t \in \mathbb{Z}.$$

If moreover X is weakly stationary, it is called an ARMA(p, q) process.

Theorem

The ARMA(p, q) equation admits a weakly stationary solution iff Φ does not vanish on Γ_1, in which case it is the unique one. Moreover, it is centered and admits a spectral density $f(\lambda) = \frac{\sigma^2}{2\pi} \frac{|\Theta(\mathrm{e}^{-\mathrm{i}\lambda})|^2}{|\Phi(\mathrm{e}^{-\mathrm{i}\lambda})|^2}$.

ARMA(p, q) representations

Consider an ARMA(p, q) process X solution to

$$[\Phi(\mathrm{B})](X) = [\Theta(\mathrm{B})](Z) .$$

Then X admits a linear representation $X = \mathrm{F}_\psi(Z)$ for a well chosen $\psi \in \ell^1$.

We say that the ARMA(p, q) representation is

- causal if F_ψ is causal. (iff Φ does not vanish on the unit closed disk Δ_1)
- invertible if $\mathrm{F}_\psi(Z)$ is invertible and the inverse filter is causal.(iff Θ does not vanish on the unit closed disk Δ_1)
- canonical if it is causal and invertible.

Existence of a canonical representation

Theorem

Consider an ARMA(p, q) process X solution to

$$[\Phi(\mathrm{B})](X) = [\Theta(\mathrm{B})](Z) .$$

Suppose that Φ does not vanish on the unit circle Γ_1. Then X admits a canonical representation

$$[\tilde{\Phi}(\mathrm{B})](X) = [\tilde{\Theta}(\mathrm{B})](\tilde{Z}) .$$

($\tilde{\Phi}$ and $\tilde{\Theta}$ do not vanish on Δ_1 and $\tilde{Z}$ is a white noise).

Idea of the proof

Consider $\Phi(z) = 1 - \alpha z$ with $|\alpha| > 1$. Observe that for all $z \in \Gamma_1$,

$$\Phi(z) = -\alpha\, z\, R(z)\tilde{\Phi}(z) ,$$

where we set $\tilde{\Phi}(z) = 1 - \overline{\alpha}^{-1} z$ and $R(z) = \frac{(1-\alpha^{-1} z^{-1})}{(1-\overline{\alpha}^{-1} z)}$.

Now $\tilde{\Phi}$ corresponds to a causally invertible filter and R^{-1} corresponds to an all-pass rational filter, say F_ϕ. Then

$$[\Phi(\mathrm{B})](X) = Z \Leftrightarrow [\tilde{\Phi}(\mathrm{B})](X) = \tilde{Z} ,$$

where $\tilde{Z} = -\alpha^{-1}\,\mathrm{F}_\phi \circ \mathrm{B}^{-1}(Z)$ is a white noise. We obtain a canonical representation.

Application : innovations of an ARMA process

Theorem

Let X be an ARMA(p, q) process with canonical representation

$$[\Phi(\mathrm{B})](X) = [\Theta(\mathrm{B})](Z) \ .$$

Then$Z$ is the innovation process of X.

Proof

The proof is in 3 steps

Step 1 Since Θ is causally invertible, $Z_t \in \mathcal{H}_t^X$ for all $t \in \mathbb{Z}$.

Step 2 Since Φ is causally invertible, $X_t \in \mathcal{H}_t^Z$ for all $t \in \mathbb{Z}$. Hence $Z_t \perp \mathcal{H}_{t-1}^X$.

Step 3 Hence, $\mathrm{proj}\left(X_t | \mathcal{H}_{t-1}^X\right) = \sum_{k=1}^p \phi_k X_{t-k} + \sum_{k=1}^q \theta_k Z_{t-k}$.