An Introduction to Equity Derivatives

An Introduction to Equity Derivatives

Theory and Practice

Second Edition

Sébastien Bossu
Philippe Henrotte

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Foreword

Today, equity derivatives are used by hundreds of thousands of people around the world – not only sophisticated investors such as hedge funds, institutional investors, or investment banks, but also private investors. Their popularity is due to the wide array of applications they offer: directional strategies, risk hedging, volatility trading, structured products, to name a few.

This book will be an ideal partner for anyone discovering equity derivatives or who wants to learn more about them. The text is remarkably well structured and accessible, starting with basic concepts and slowly increasing the level of complexity. The problems and accompanying solutions add a lot of insight, and the two new chapters on volatility trading and exotic options are a must-read.

I was very pleased to be asked to write the foreword of this new edition from Sébastien Bossu under the authority of my colleague Philippe Henrotte at HEC Paris. Once again, Sébastien demonstrates his ability to combine his sleek and sharp academic style together with his first-rate practical experience. Along with his constant interaction with many market practitioners, he continues to successfully leverage off his ongoing personal research on some of the most topical pricing and modeling challenges faced by our always-evolving industry.

Olivier Bossard
Senior Managing Director
Head of Derivatives Trading EMEA at Macquarie Bank

Before joining Macquarie, Olivier Bossard has developed from scratch and led over a decade the exceptional growth of Lehman's Structured Products business in Europe. He has twenty years of experience as an exotic option trader, and has also been teaching Financial Engineering at HEC Paris Business School since 1998.

Preface

For this new edition of our 2005 title *Finance and Derivatives* we have considerably redrafted our text and focused our attention on equity derivatives which is our core area of expertise. There are two new chapters, numerous chapter additions, several new problems with solutions, more figures and illustrations, and more examples. As before our aim is to suit the needs of both professionals and aspiring professionals discovering the field. No prior knowledge in finance is assumed, the only required background is an undergraduate level in mathematics.

The chapters form a sequence of gradual difficulty which we grouped within three parts:

- **Part I: Building Blocks (Chapters 1 to 4)** covers the fundamental concepts used in quantitative finance: interest rates, the time value of money, bonds and yields, portfolio valuation, risk and return, diversification.
- **Part II: First Steps in Equity Derivatives (Chapters 5 to 8)** covers forward contracts, options and option strategies, the binomial model, the lognormal model, Monte-Carlo simulations, and dynamic hedging. This part only relies on discrete time concepts in order to remain widely accessible.
- **Part III: Advanced Models and Techniques (Chapters 9 to 12)** goes one level higher into continuous time finance and covers models for asset prices, stochastic processes and calculus, the Black-Scholes model, volatility trading, exotic derivatives, and advanced models.

Our approach was to focus on the fundamentals while covering a fair amount of practical applications. We endeavored to keep our text as concise and straightforward as possible, leaving non-essential concepts and technical proofs to problems of higher difficulty which are identified with an asterisk (*).

The 2007–2008 financial crisis highlighted the fact that derivatives were often poorly understood. We do not think that the solution is to ban them altogether: when you are in the passenger seat and have just escaped a fatal car crash after speeding, you typically don't get rid of the car. Rather, we believe that more information and training is needed in the field (along with better drivers), and we hope that this new edition will prove useful and insightful to a large audience.

Disclaimer

This is a book about finance intended for professionals and future professionals. We are not trying to sell you any security, or give you any investment advice. The views expressed here are solely ours and do not necessarily reflect those of any entity directly or indirectly related to us. We took great care in proof-reading this book, but we disclaim any responsibility for any remaining errors and any use to which the contents of this book are put.

Addendum: A Path to Economic Renaissance

The following opinion piece only reflects the personal views of the author and does not engage any other contributor to this book.

This new publication provides me with the opportunity to comment on the current economic and cultural climate, which has changed markedly since the last edition. In particular, derivatives came into the spotlight and have been heavily criticized.

I want to emphasize that equity derivatives are not inherently harmful. When used competently, derivatives can reduce risk or, more precisely, they allow investors to select certain types of risks over others. While it is true that credit derivatives compounded losses early on in the recent economic crisis, they are not to be blamed for the culture of "real estate envy", cheap money and ostentation which then prevailed.

The crisis is far from being fully resolved. There is a distressing gap between the pessimism in mainstream political and management discourse, and the reality in large banks and corporations where profits are close to record highs and executive pay is on the rise.

To paraphrase Ronald Reagan, there is a rising sentiment that our leaders are the problem, not the solution, as expressed by many popular movements such as 'Occupy Wall Street'. Rather than reshuffling cards in favor of the next generation – a process known as 'creative destruction' in Schumpeter's theory – we just seem to be doubling down on the people who failed.

It is urgent, in my opinion, to take actions to increase the circulation of wealth in the economy in order to restore confidence in economic growth and progress. A few years ago, in a joint op-ed article published by a respected French economics newspaper, I proposed to cut on income taxes, which would give a much-needed break to the middle class, and introduce in its place a small annual tax on individual net worth (i.e. assets minus liabilities.) Unfortunately this piece was not published in equivalent newspapers or magazines in the US and the UK, perhaps because it was then perceived as too unorthodox.

Meanwhile, I have been distressed by the flurry of extravagant proposals dominating the media space: salary caps, bans on speculation, bans on derivatives, taxes on financial transactions, to name a few. All these proposals would result in costly bureaucratic rigidities at a time when we need to foster entrepreneurship, mobility and innovation.

The desire to protect consumers is of course legitimate, but the best protection is often provided through transparency. For example, I have suggested that financial retailers clearly break down the price of investments they offer between the present values of their fees and wholesale costs. This would help consumers understand how much of their money is effectually spent on financial assets, and promote competition between providers.

Every financial investment, from buying a house to purchasing Treasury bonds or options, is speculative in nature. Some people manage to get rich very quickly through talent, vision and hard work, and that’s admirable. Others manage to stay rich by promoting a culture of entitlement, status quo and cronyism, and we should resist against that.

I have no doubt we will get back on track as soon as the obvious choices are made. On the corporate side, expensive and redundant management layers must be cut in order to make room for new talent. On the political and economic side, we must promote a more equitable circulation of wealth. Above all, we must begin to select leaders not only because of their performance but also based on their ethics, bearing in mind the wisdom of ancient Greek philosophers who held that virtue cannot be taught: either you have it, or you don’t.

Sébastien Bossu, February 2012

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Part I

Building Blocks

1 Interest Rate

In this chapter we review the idea of interest rate and the closely related concepts of compounding and discounting.

1-1 Measuring Time

In finance the standard unit of time is the year. But can we safely assume that a year has 365 days? What about the 366 days of a leap year? What fraction of a year does the first six months represent: 0.5, or 181/365 (except, again, for leap years)?

Financial markets have regulations and conventions to answer these questions. The problem is that these conventions tend to vary by country. Worse still, within a given country different conventions may apply to different financial products.

We leave it to readers to become familiar with these day count conventions while in this book we will use the following rule, which professionals call **30/360** (Table 1-1 below). Note that the initial date starts at noon and the final date ends at noon; thus, there is only one whole day between 2 February 2012 and 3 February 2012.

Table 1-1 The 30/360 rule for measuring time

Rule	Result	Example: from 15 January 2012 to 13 March 2015
1. Count the number of whole years	Y	3 (from 15 January 2012 to 15 January 2015)
2. Count the number of remaining whole months and divide by 12	M/12	1/12 (from 15 January 2015 to 15 February 2015)
3. Count the number of remaining days (the last day of the month counting as the 30th unless it is the final date) and divide by 360	D/360	28/360 (under the 30/360 convention there are 16 days from 15 February 2015 to 1 March 2015 at noon and 12 days from 1 March 2015 to 13 March 2015)
TOTAL	Y + M/12 + D/360	3 + 1/12 + 28/360 = 3.161111...

From this rule we obtain the following simplified measures:

Semester (half year)	0.5 year
Quarter (three months)	0.25 year
Month	1/12 year
Week	7/360 year
Day	1/360 year

In practice. . .
The Excel function DAYS360(*Start_date*, *End_date*) counts the number of days on a 30/360 basis.

1-2 Interest Rate

In business life one can encounter two types of individuals whose *interests* are by definition opposed to each other:

- **Investors**, who have money and want to get richer while they remain idle;
- **Entrepreneurs**, who don't have money but want to become rich using the money of others.

Banks help to reconcile these two interests by acting as intermediaries, placing the money of the investor at the entrepreneur's disposal while taking the risk of bankruptcy (see Figure 1-1). In exchange, the bank demands that the entrepreneur pay *interest* at regular intervals, which serves to pay for the bank's service and the investor's capital.

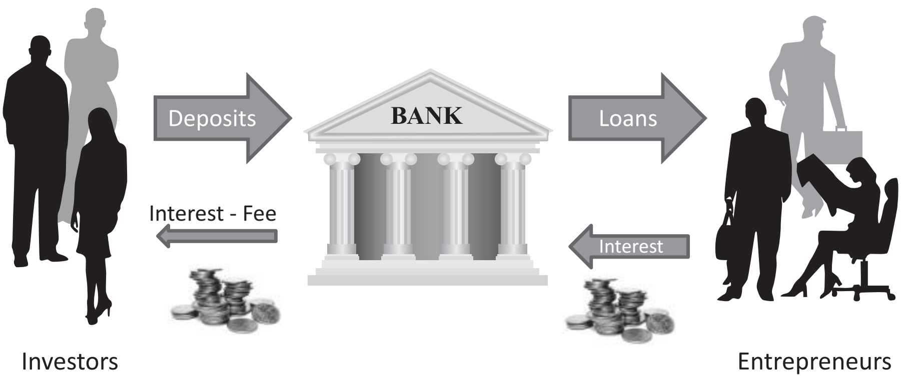

Figure 1-1 Banks are intermediaries between investors and entrepreneurs

1-2.1 Gross Interest Rate

Consider an investor who deposits \$100 and receives a total interest of \$12 over 2 years. His gross 2-year interest rate is then 12%. Generally, if I is the *total* interest paid on a capital K, the **gross interest rate** *over the period in consideration* is defined as:

$$r = \frac{I}{K}.$$

Examples
- €10 of interest paid over one year on a capital of €200 corresponds to a 5% annual gross interest rate.

- \$10 of interest paid *every year* for five years on a capital of \$200 corresponds to a 25% gross interest rate over five years, which is five times the above annual rate.

We must emphasize that **an interest rate is meaningless if no time period is specified**: a 5% gross interest rate every six months is far more lucrative than every year.

This rate is called 'gross' because it does not take into consideration the **compounding of interest**, which is explained next.

1-2.2 Compounding. Compound Interest Rate

When asked: "How much total interest does one collect after two years if the annual interest rate is 10%?," a distressing proportion of individuals reply in a single cry: "20%!" However, the correct answer is 21%, because **interest generates more interest**. In fact, a good capitalist, rather than foolishly spend the 10% interest paid by the bank after the first year, would immediately reinvest it the second year. Therefore, his total capital after one year is 110% of his initial investment on which he receives 10% interest the second year. His gross interest over the 2-year period is thus: 10% + 10% × 110% = 21%.

Generally, starting with initial capital K one may build a **compounding table** of capital at the end of each interest period (Table 1-2):

Table 1-2 Compounding table of capital K at interest rate r over n periods

Period	Capital	Example: $r = 10\%$
0	K	\$2,000
1	$K(1+r)$	2,000 × (1 + 10%) = \$2,200
2	$K(1+r)^2$	2,200 × (1 + 10%) = \$2,420
...	...	...
n	$K(1+r)^n$	$2{,}000 \times (1+10\%)^n$

From this table we derive a formula for the amount of accumulated interest after n periods:

$$I_n = K(1+r)^n - K.$$

We may now define the **compound interest rate** over n periods corresponding to the total accumulated interest:

$$r^{[n]} = \frac{I_n}{K} = (1+r)^n - 1.$$

(To avoid confusion we prefer the notation $r^{[n]}$ over r_n to indicate compounding over n periods, as r_n typically denotes a series of time-dependent variables.)

Example
The total accumulated interest over 3 years on an initial investment of \$2,000 at 5% semi-annual gross interest rate is $I_6 = 2{,}000 \times (1 + 0.05)^6 - 2{,}000 = \$680$. The compound interest rate over 3 years (6 semesters) is $r^{[6]} = 34\%$.

1-2.3 Conversion Formula

Two compound interest rates over periods τ_1 and τ_2 are said to be **equivalent** if they satisfy:

$$\left[1 + r^{[\tau_1]}\right]^{\frac{1}{\tau_1}} = \left[1 + r^{[\tau_2]}\right]^{\frac{1}{\tau_2}}. \tag{1-1}$$

Here τ_1 and τ_2 are measured in years (for instance $\tau_1 = 1.5$ represents a year and a half) and $r^{[\tau_1]}$ and $r^{[\tau_2]}$ are the equivalent interest rates over τ_1 and τ_2 years respectively.

Example
An investment at 5% semi-annual gross interest rate is equivalent to an investment at a 2-year compound rate of $r^{[2]} = (1 + 5\%)^{\frac{2}{0.5}} - 1 \approx 21.55\%$.

Equation (1-1) is very useful to convert a compound rate into a different period from the "physical" interest payment period. A good way to remember it is to think that for a given investment all expressions of the form $[1 + r^{[\text{period}]}]^{\text{frequency}}$ are equal, where frequency is the number of periods per year (the inverse of the period length).

1-2.4 Annualization

Annualization is the process of converting an interest rate into its annual equivalent. This allows one to quickly compare the profitability of investments whose interests are paid out over different periods.

In this book, unless mentioned otherwise, all interest rates are understood to be on an annual basis or annualized. With this convention, the compound interest rate over *T* years may always be written as:

$$\boldsymbol{r^{[T]} = (1 + r^{[\text{annual}]})^T - 1.}$$

Example
The annualized rate equivalent to a 5% semi-annual gross rate is:

$$r^{[1]} = (1 + 5\%)^{\frac{1}{0.5}} - 1 = 1.05^2 - 1 = 10.25\%,$$

from which we obtain the 2-year compound rate found in the previous example:

$$r^{[2]} = (1 + 10.25\%)^2 - 1 \approx 21.55\%.$$

1-3 Discounting

'*Time is money*.' In finance, this principle of the businessman has a very precise meaning: **a dollar today is worth more than a dollar tomorrow**. Two main reasons may be put forward:

- **Interest**: one dollar today produces interest between today and tomorrow;
- **Inflation**: the increase in consumer prices implies that one dollar buys more today than tomorrow.

With this principle in mind the next step is to determine **the value today of a dollar tomorrow** – or generally the **present value** of an amount received or paid out in the future.

1-3.1 Present Value

If \$100 invested today at 5% annual interest is worth $\$100 \times (1+5\%)^4 \approx \$121.55$ in four years, how much is \$100 received in four years worth today? The answer is given by a simple cross-division: $\$100/(1+5\%)^4 \approx \$100 \div 1.2155 \approx \$82.27$. In other words \$82.27 invested today at 5% interest rate grows to \$100 in four years. This amount is called the present value of \$100 received in four years.

Generally, the **present value** PV of an amount C paid or received in T years is the equivalent amount which, invested today at rate r, grows to C: $\mathrm{PV} \times (1+r)^T = C$, i.e.:

$$\mathrm{PV} = \frac{C}{(1+r)^T}.$$

Example
A European supermarket customarily pays its suppliers with a 3-month delay. With a 5% interest rate the present value of a delivery today of €1,000,000 worth of goods paid in 3 months is:

$$\frac{1{,}000{,}000}{(1+5\%)^{0.25}} \approx €987{,}877.$$

The 3-month payment delay is thus an implicit €12,123 discount, or 1.21%.

Discounting is the process of computing the present value of various future cash flows. It is a key concept in finance which brings all future amounts to their equivalent value as of today. Figure 1-2 below shows how discounting and compounding are reciprocal processes.

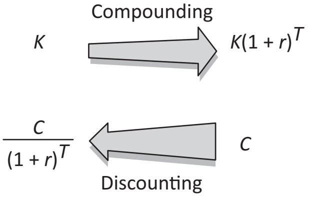

Figure 1-2 Compounding and discounting

1-3.2 Discount Rate and Required Return

In practice, the choice of the **discount rate** r is crucial when calculating a present value and depends on each investor's **required return**. The minimum required return is the interest rate offered by such "infallible" institutions as central banks or government treasury departments.

In the US, the generally accepted benchmark rate is the yield[1] of the 10-year Treasury Note. In Europe, the 10-year Gilt (UK), OAT (France), or Bund (Germany) are used, and in Japan the 10-year JGB.

However, an investor who is willing to take more risk should require a higher return and use a higher discount rate r. In investment banking it is not uncommon to use 10% to 20% discount rates when assessing the profitability of such risky investments as financing a film production or providing seed capital to a start-up company.

1-4 Problems

Problem 1: Measuring time
Calculate in years the time that passes between 30 November 2014 and 1 March 2016 on a 30/360 basis. What is the annualized interest rate of an investment at a gross rate of 10% over this period?

Problem 2: Savings account
On 1 January 2012 you deposit €1,000 in a savings account. On 1 January 2013 the bank sends a summary statement indicating that you received €40 of total interest in 2012.

(a) What is the gross annual interest rate of this savings account?
(b) How much interest will you get in 2013?
(c) Assuming interest is calculated and paid every month based on the account balance, how much interest would you have received if you had closed your account on 1 July 2012?

Problem 3: Ten years ago you invested £500 in a savings account. The last bank statement shows a balance of £1,030.52. What will your savings amount to in ten years if the interest rate stays the same?

Problem 4: From Greece with interest
You are a reputed financier and your personal credit allows you to borrow up to \$100,000 at a rate of 6.5% (with a little bit of imagination). On 22 September 2011 the interest offered on 1-year "deposits" with the Greek government is 135%, and the exchange rate of one euro is \$1.35. Your analysts believe that this exchange rate will remain stable during the coming year. Can you find a way to make money? Analyze your risks.

Problem 5: Sort the interest rates below from most lucrative to least lucrative:

(a) 6% per year;
(b) 0.5% per month;
(c) 30% every five years;
(d) 10% the first year then 4% the following two years.

[1] See Section 3-3 p.24 for the definition of yield.

Problem 6: Credit card
You receive a credit card offer with no minimum monthly payment and an annual interest rate of 17%, which is calculated and charged on your balance every month.

(a) What is the compound interest rate charged by the credit card company if you pay off your balance after one month? Eighteen months? Five years?
(b) Draw the curve of the compound interest rate as a function of time.
(c) When will the amount of interest charged exceed the initial balance?

Problem 7*: Continuous interest rate
(a) Consider the sequence $u_n = \left(1 + \frac{1}{n}\right)^n$ for $n \geq 1$. Show that (u_n) has limit e (Euler's constant: $e \approx 2.71828$.) *Hint:* $x^y = e^{y \ln x}$, $\ln(1+h) = h - \frac{h^2}{2} + \frac{h^3}{3} - \cdots$ *for small h.*
(b) If A_2 is a savings account with a 5% annual interest rate split into two payments of 2.5% every 6 months, what is the corresponding annualized interest rate r_2?
(c) Let A_n be a savings account with an annual interest rate of 5% split into n payments. Find the corresponding annualized interest rate r_n as a function of n.
(d) Find the limit r of r_n as n goes to infinity. How would you interpret the rate r?

Problem 8: Discounting. Using a discount rate of 4% per annum, calculate the present value of:

(a) €100,000 in one year?
(b) €1,000,000 in ten years?
(c) €100,000 ten years ago?

Problem 9: Expected return
After hesitating at length Mr Smith, an accomplished investment banker, eventually renounced an investment project offered at 30 million pounds sterling against a promised payoff of one billion pounds in twenty years' time. Can you estimate his required return?

Problem 10: Today's value of one dollar tomorrow
On 14 April 2005 the annualized interest rate on an overnight dollar deposit (i.e. between 14 April and 15 April 2005) was 2.77%. Calculate 'the value today of a dollar tomorrow,' that is the present value as of 14 April 2005 of one dollar collected on 15 April 2005.

Problem 11: Tuition planning
Mr and Mrs Jones are planning for their ten-year-old daughter Anna's future college education. According to College Board, a nonprofit membership association of more than 3,900 schools, colleges, and universities in the US, tuition at four-year private colleges and universities averaged $27,293 per year in 2010–11. The Jones have $20,000 in available savings and their bank offers a special deposit account with 5% guaranteed interest rate until Anna graduates from high school at the expected age of 18.

(a) Assuming no inflation in tuition costs and a $20,000 initial deposit, calculate how much the Joneses must save every year if they wish to cover for Anna's college education in full (room and board excepted)?

(b) In 2000–01 college tuition averaged approximately \$20,308. Calculate the corresponding annualized inflation rate, and re-answer the previous question taking tuition inflation into account.

Problem 12*: Rule of 72

In 1494, the Italian mathematician Luca Paciola wrote without proof: "In wanting to know for any percentage, in how many years the capital will be doubled, you bring to mind the rule of 72, which you always divide by the interest, and the result is in how many years it will be doubled. Example: When the interest is 6 percent per year, I say that one divides 72 by 6; obtaining 12, and in 12 years the capital will be doubled." Show that this is approximately correct when the annual interest rate r is small enough. *Hint:* $\ln(1 + h) \approx h$ *for small* h.

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2

Classical Investment Rules

In this chapter we review some of the classical rules used by financiers to assess the profitability of investment projects and decide whether or not to implement them.

2-1 Rate of Return. Time of Return

2-1.1 Gross Rate of Return (ROR)

The **gross rate of return** (**ROR**, also called return on investment or ROI) on a \$1,000 investment generating a total income of \$150 over 2 years is simply the ratio 150/1,000 = 15%. The ROR generalizes the concept of interest rate to any type of investment, such as buying a financial security[1] or upgrading a factory with new equipment. In general:

$$\text{ROR} = \frac{\text{Earnings}}{\text{Cost}} = \frac{\text{Income}}{\text{Price}}.$$

Note that in the case of a financial security such as a corporate stock, the change in price would be treated as earnings. If P_0 is the initial price, P_T the final price, and the security pays off a revenue R in-between, the gross rate of return over the period $[0, T]$ is:

$$\text{ROR} = \frac{P_T - P_0 + R}{P_0}.$$

Example
ABC Inc.'s stock trades at \$100 and analysts predict a \$4 dividend per share. If the price of ABC Inc. a year later is \$110, the 1-year ROR would be:

$$\text{ROR} = \frac{110 - 100 + 4}{100} = 14\%.$$

The ROR may be compared to the required return of an investor to decide whether to go ahead with the investment.

2-1.2 Time of Return (TOR)

Assume that our 2-year investment at 15% ROR delivers the same level of profitability over successive 2-year cycles (think of factory equipment which permanently increases productivity).

[1] See Section 3-1.1 p.19 for the definition of a financial security.

How long must we wait before we get our initial \$1,000 expense back? The answer is 1/15% ≈ 6.67 cycles of 2 years, i.e. about 13 years and 4 months, as illustrated in Figure 2-1 below.

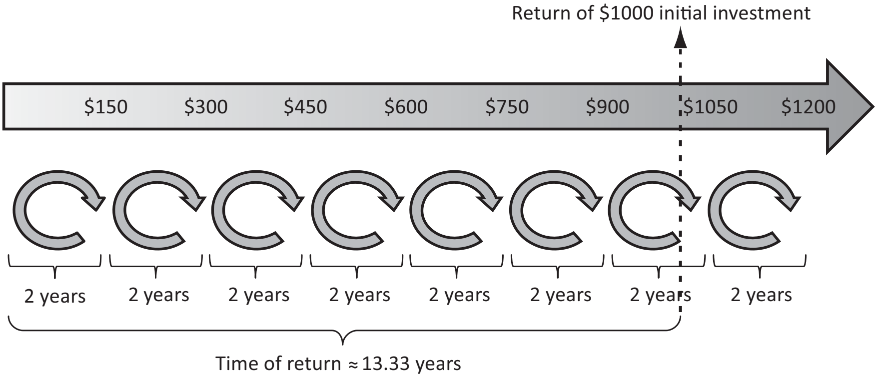

Figure 2-1 Time of return and investment cycles

This number is sometimes called the **time of return** or the **payback period** of an investment:

$$\text{TOR} = \frac{1}{\text{ROR}} = \frac{\text{Cost}}{\text{Earnings}} = \frac{\text{Price}}{\text{Income}}$$

The TOR measures the number of periods an investor has to wait until the initial cost is *returned* to him or her in earnings, assuming the latter remains constant and regular in time.

2-2 Net Present Value (NPV)

The gross rate of return is easy to calculate but only takes one period into consideration. When an investment generates several **cash flows** at different points in time, it is better to calculate its **net present value (NPV)**. By convention, a positive cash flow corresponds to an in-pocket gain for the investor, and a negative cash flow corresponds to an out-of-pocket expense, as illustrated below:

General form

Time	t_1	t_2	...	t_n
Cash flow	F_1	F_2	...	F_n

Example

Date (31st Dec)	2011	2012	2013	2014
Cash flow (mn€)	+100	−150	+200	+500

Cash flow tables are the bread and butter of financiers and accountants. There is, however, one major difference between the two professions: the accountant contemplates cash flows from the past while the financier fantasizes over *future* cash flows and entirely disregards *past* cash flows. This point is important to remember when historical data is provided, or when reasoning forward in time.

Given a cash flow table, the financier's first step is to discount the cash flows and calculate their aggregate **present value** (see Section 1-3.1 p.7):

General form	**Example with discount rate $r = 10\%$**
$\mathrm{PV} = \dfrac{F_1}{(1+r)^{t_1}} + \dfrac{F_2}{(1+r)^{t_2}} + \ldots + \dfrac{F_n}{(1+r)^{t_n}}$	$\mathrm{PV} = \dfrac{100}{1+10\%} + \dfrac{-150}{(1+10\%)^2} + \dfrac{200}{(1+10\%)^3} + \dfrac{500}{(1+10\%)^4} \approx €458.71\text{ mn}$

If the investment cost C_0 is already known, the **net present value** is defined as the aggregate present value *net of* the initial cost:

General form	**Example with initial cost $C_0 = €400$ mn**
$\mathrm{NPV} = -C_0 + \dfrac{F_1}{(1+r)^{t_1}} + \dfrac{F_2}{(1+r)^{t_2}} + \ldots + \dfrac{F_n}{(1+r)^{t_n}}$	$\mathrm{PV} = -400 + \dfrac{100}{1+10\%} + \dfrac{-150}{(1+10\%)^2} + \dfrac{200}{(1+10\%)^3} + \dfrac{500}{(1+10\%)^4}$ $\mathrm{PV} \approx €58.71\text{ mn} > 0$

There are then three cases:

- $\mathrm{NPV} > 0$: The investment is profitable and may be carried out (as shown in the example).
- $\mathrm{NPV} < 0$: The investment would be at a loss and should be rejected.
- $\mathrm{NPV} = 0$: The investment is neutral (theoretical case).

As always, the problem of selecting the appropriate discount rate is difficult and raises the issue of the investor's required return. Nevertheless the NPV is a better measure of an investment's profitability than the ROR because it takes interest compounding over multiple periods into account. It is the basic investment rule used by professionals.

2-3 Internal Rate of Return (IRR)

We can reverse the problem of selecting the discount rate and calculate instead the **internal rate of return** r which makes the NPV equal zero, in other words find the indifference point for the investor. In mathematical terms the IRR is the solution r^* to the equation $\mathrm{NPV}(r) = 0$, i.e.:

General form	**Example**
Find r such that $-C_0 + \dfrac{F_1}{(1+r)^{t_1}} + \dfrac{F_2}{(1+r)^{t_2}} + \ldots = 0$	$-400 + \dfrac{100}{1+r} + \dfrac{-150}{(1+r)^2} + \dfrac{200}{(1+r)^3} + \dfrac{500}{(1+r)^4} = 0$

Example
The IRR of the investment given in example is $r^* \approx 14.26\%$. Using a calculator we may verify that:

$$-400 + \frac{100}{1+14.26\%} + \frac{-150}{(1+14.26\%)^2} + \frac{200}{(1+14.26\%)^3} + \frac{500}{(1+14.26\%)^4} = 0.$$

The IRR rule is equivalent to the NPV and must be compared to the investor's required return to decide whether the investment should be accepted or rejected.

In practice. . .
The Excel functions XNPV and XIRR calculate the NPV and IRR of a cash flow table. You must install the 'Analysis ToolPack' add-in to access these functions. Note that these functions do not follow the 30/360 day count convention, which may result in small discrepancies.

	A	B	C	D	E	F
1	**Date**	31-Dec-10	31-Dec-11	31-Dec-12	31-Dec-13	31-Dec-14
2	**Cash flow**	-400	100	-150	200	500
3						
4	**NPV(10%)**	58.62		**IRR**	14.26%	
5	**=XNPV(10%,B2:F2,B1:F1)**			**=XIRR(B2:F2,B1:F1,10%)**		
6						

2-4 Other Investment Rules

Many accounting rules are also used in finance to assess the suitability of an investment, especially for corporate stocks. These rules are beyond the scope of this book and we only mention the most prevalent one: the **Price-to-Earnings Ratio** (**PER** or **P/E**), which is simply the TOR for a listed company. It is given as:

$$\text{PER} = \frac{\text{Price (per share)}}{\text{Earnings (per share and per annum)}}.$$

Example
On 31 December 2010, General Motors' stock price closed at \$36.86, while the 2010 earnings per share were \$2.88; its PER was thus 36.86/2.88 $\approx$ 12.8. This means that the stock price is worth nearly 13 years of profits. In other words, a new stockholder would have to wait for 13 years before General Motors' earnings would pay back his initial investment, assuming they remain constant and are entirely redistributed as dividends.

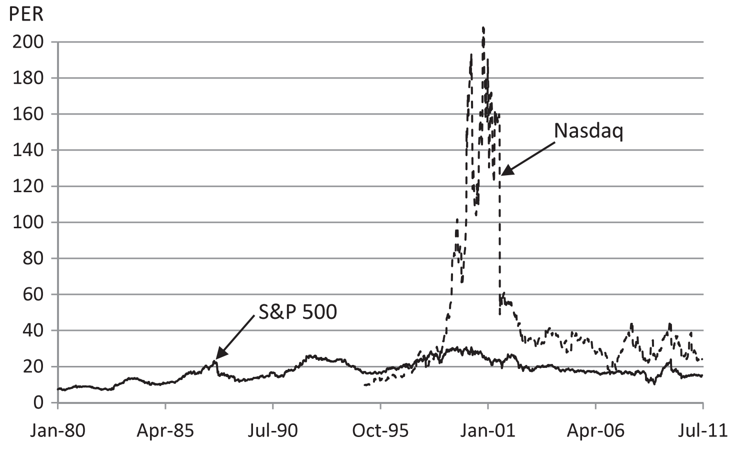

Figure 2-2 PER of S&P 500 and Nasdaq since 1980
(Source: Bloomberg)

Figure 2-2 above shows the history of the average PER for two stock indexes: the S&P 500, which includes the largest 500 US companies by market capitalization, and the Nasdaq Composite, which includes 3,000 technology and growth companies. In July 2011, the average PER was about 15 for S&P 500 companies and 24 for Nasdaq companies. Observe how the Nasdaq PER reached unprecedented levels during the Internet bubble in 2000–01.

Rapidly growing companies have a high PER: they reinvest their profits into their growth and do not redistribute much to stockholders. However, since their stock prices often increase at a faster pace, stockholders have the opportunity to be compensated through capital gains.

Conversely companies which have reached their maximum expansion have a low PER: they do not need to further invest in their activity and therefore redistribute most of their profits to stockholders. Dividends become regular and predictable and the stock price is close to the present value of the dividend flow.

2-5 Further Reading

- On ROR and IRR: Frank J. Fabozzi (2005) *The Handbook of Fixed Income Securities* 7th Edition, McGraw-Hill Trade: Chapter 6.
- On present value and investment decisions: Stephen A. Ross, Randolph W. Westerfield and Bradford D. Jordan (2008) *Fundamentals of Corporate Finance Standard Edition*: Chapters 6, 9, and 10.

2-6 Problems

Problem 1: ROR and TOR
What is the annualized rate of return and time of return of:

(a) A security purchased at $350 which analysts predict to be worth $400 in eighteen months;
(b) A winning lottery ticket purchased for €10 which will redeem €20 in one week;
(c) A security purchased for £1,000 which pays £100 in cash next year and whose price is thought to decrease by 5% after this payment;
(d) A 'payday loan' at the rate of $4 due in a month for every $3 borrowed?

Problem 2: NPV and IRR

You are offered an investment project with the following cash flows:

t (years)	2	4
Cash flow (% of initial investment)	−10	+150

(a) Calculate the NPV at 5% and 15% annual discount rates. Briefly comment on your results.
(b) Calculate the IRR. *You may either use a calculator or analytically solve the IRR equation.*
(c) What do you think of this investment?

Problem 3: Required return

Mr Smith offers an investment project with the following cash flows:

Date	Today	in 6 months	in 1 year	in 2 years and 3 months
Cash flow (mn$)	$-X$	−50	+20	+60

(a) What maximum price X are you willing to pay for this project if the relevant interest rate offered by the US Treasury is 4% per annum?
(b) Where does Mr Smith estimate your required return if $X = \$15$mn?

Problem 4*: Perpetuity, dividends, and stock value

(a) Show that for any real number $-1 < x < 1$: $1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}$.
(b) Using a 5% discount rate, determine the present value of a 'perpetuity' investment which pays an annual cash flow of \$5 forever. What is the IRR if the cost of the perpetuity is \$105?
(c) At the annual stockholders' meeting of Sky Inc., a dividend distribution of \$3 per share was voted. The CEO also announced that he was committed to increase the dividend by 1.5% every year. Calculate the theoretical value of Sky Inc.'s stock using a 4% annual discount rate.

Problem 5: Return and inflation

An apartment in Paris is listed at €425,000. You reckon that you could rent out the apartment at €1,500 per month for the next 20 years, and that its value in 20 years would be the same or higher.

(a) What would your IRR be if there is no real estate inflation? *Hint:* $1 + x + x^2 + \cdots + x^n = (1 - x^{n+1})/(1 - x)$
(b) What would your IRR be if there is a 2.5% annual inflation (monthly rent and resale price)?

Problem 6: Impact of corporate announcement

On 31 December 2011 the stock price of MetroTech SpA is €150 and analysts predict the following earnings per share (EPS) for the next 5 years:

2011	2012	2013	2014	2015	2016
20	40	47	55	15	12

Analysts also estimate that MetroTech SpA will make no profits after 2016.

(a) What is the current PER of MetroTech SpA?
(b) What is the value of MetroTech SpA to an investor who requires a 10% annual return?
(c) What is the market's expected return on MetroTech SpA?
(d) At a press conference, the CEO announces a project which will reduce the 2012 EPS by €5 and then increase future EPS by 10% from 2013 onwards. Can you estimate how the market will react?

An Introduction to Equity Derivative c uveory and ractice, Second dono ne
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3 Fixed Income

In this chapter we introduce financial securities, portfolios, and markets, and then focus on bonds and the yield curve.

3-1 Financial Markets

3-1.1 Securities and Portfolios

A financial security is a legal contract whereby two or more parties agree to exchange future cash flows.

Examples

- A company's stock is a financial security whose cash flows are the dividends.
- A fixed-interest rate loan is a financial security whose cash flows are the interest and principal repayments.

By extension certain assets[1] such as commodities and currencies are sometimes assimilated to securities even though there are no cash flows.

A financial portfolio is a collection of financial securities. By convention, a portfolio's cash flow at any given time is the sum of its securities' cash flows weighted by their quantities, as illustrated in Table 3-1 below.

Table 3-1 Portfolio made of 10,000 units of security A and 5,000 units of security B

		Cash flows ($)	
Security	**Quantity**	$t = 1$ **year**	$t = 2$ **years**
A	10,000	100	200
B	5,000	—	50
Portfolio P	1	1,000,000	2,250,000

3-1.2 Value and Price

Price is simply what you pay to own a security. **Value** is what it is worth to you. Price and value need not be equal.

[1] See Chapter 4, Footnote 1 for the definition of an asset.

Example
Mary is the manager of portfolio P shown in Table 3-1 above. Her target yield is 4% per annum. Using this figure as discount rate, she calculates that the portfolio's present *value* is $3,041,790. Pressed by her boss to sell the portfolio, she arranges an auction. The best buyer's quote is $3,000,000 and she trades at that *price*.

To present this in more detail:

- The *value* of a security is a positive or negative amount corresponding to the anticipated change in wealth of its owner. There may be several valuation methods producing different values for the same security. When there is no uncertainty on the cash flows, the standard valuation method is the present value (see Section 1-3.1 p.7 and Section 2-2 p.12).
- The *price* of a security is the amount of money agreed upon by two parties to trade that security. Typically the buyer pays the price to the seller but it may happen that the seller must pay the buyer in order to get rid of a security with negative value. Note that buyer and seller need not agree on "the" value of the security; and even if they do, nothing forces them to set the price at such value.

Price and value are often used interchangeably. Throughout this book, we have endeavored to maintain the distinction.

3-1.3 Financial Markets and Short-selling

Financial markets are physical or virtual marketplaces where one can buy and sell financial securities. They include organized exchanges (NYSE Euronext, Nasdaq, London Metal Exchange ...), as well as all **over-the-counter** (OTC) interactions taking place outside of exchanges (see Figure 3-1 overleaf).

Investors are normally allowed to **short-sell** securities they do not own.[2] This way, market participants may buy and sell securities at any time according to their views, and the law of supply and demand is continuously verified. Note that **a short-seller must pay all the security cash flows to the buyer**.

Example
After a corporate announcement, Sarah believes that the stock price of Bust Inc. will go down but she does not possess it in inventory. In the market, Bob is showing a bid price which is 5% lower than the last traded price. Sarah decides to short-sell the stock to Bob. Two months later Bust Inc. pays a dividend of 20¢ per share to its stockholders. Since Sarah is still short of the stock at this time she must pay this amount to Bob.

[2] In practice, a market participant who is short e.g. 1,000 stocks must immediately borrow them back from other participants on a short-term rolling basis, until he has repurchased the entire lot. This borrowing mechanism has a cost which depends on the quantity of available shares in circulation: the scarcer the security, the higher its borrow cost.

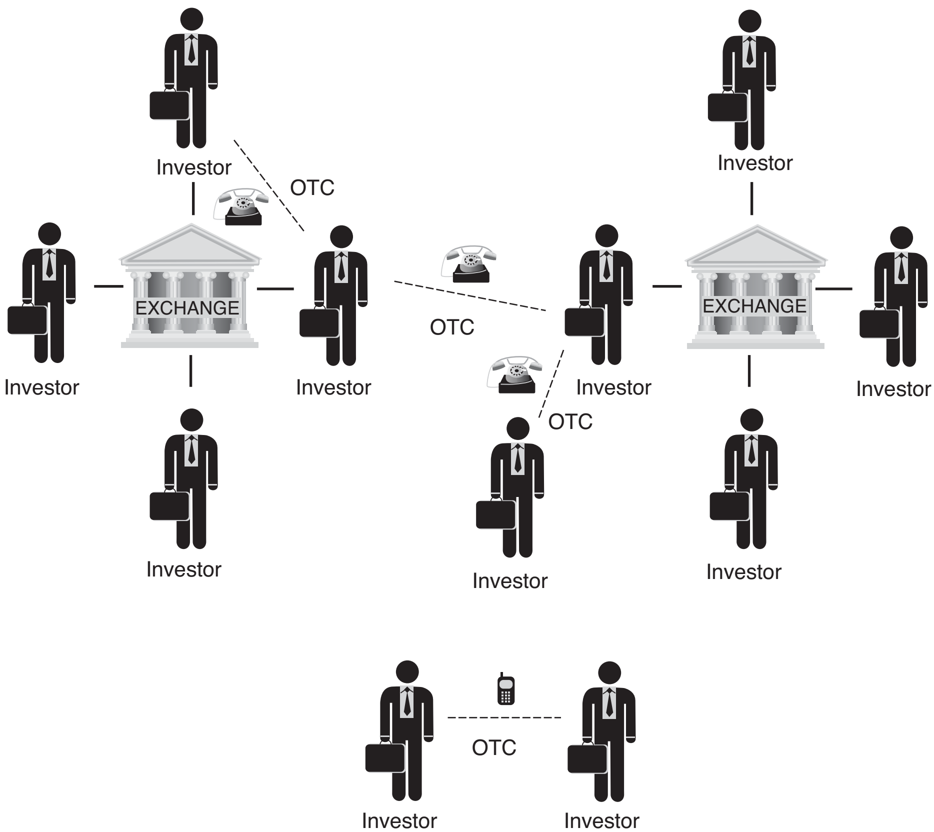


Figure 3-1 Financial markets include organized exchanges as well as OTC interactions

3-1.4 Arbitrage

Whenever an investor may trade securities for a positive profit (either today or at a future date) at strictly no cost and no risk, we say there is an **arbitrage opportunity**.

Example

Imagine there are two securities on the market both priced at €99: security STEEP, paying off €100 in 6 months, and security CHEAP, paying off €110 in 6 months. We could then make infinite amounts of money by repeatedly buying CHEAP and selling STEEP:

	Cash flows	
Transaction	**Today**	**6 months later**
Sell STEEP	+€99	−€100
Buy CHEAP	−€99	+€110
Total	**0 (no cost)**	**+€10 (positive profit)**

Sadly, if unsurprisingly, arbitrage opportunities are extremely rare. This is why financial theory always assumes the **absence of arbitrage opportunities**. At any rate, an arbitrage opportunity on the market could only exist for a very short period of time: in the above example all rational investors would buy CHEAP and sell STEEP, moving the prices of STEEP down and CHEAP up until the arbitrage disappears.

The no-arbitrage assumption, also called the '**no-free-lunch rule**' or the '**law of one price**,' allows us to find rules for the price of certain securities.

3-1.5 Price of a Portfolio

The distinction between value and price applies to portfolios of securities as well. However, under the assumptions of no arbitrage and infinite liquidity,[3] the **arbitrage price** of a portfolio of securities is simply the sum of each security's price multiplied by its respective quantity. Here 'arbitrage price' means that trading the portfolio at a different price would lead to an arbitrage opportunity, as illustrated in the example below:

Example

Consider a portfolio P made of 2 units of security A priced at €100 and 1 unit of security B priced at €50, with the following cash flows:

			Cash flows (€)	
Security	**Quantity**	**Unit price (€)**	$t = 1$ **year**	$t = 2$ **years**
A	2	**100**	10	110
B	1	**50**	30	30
Portfolio P	1	$\mathbf{2 \times 100 + 50 = 250}$	$2 \times 10 + 30 = 50$	$2 \times 110 + 30 = 250$

The arbitrage price of P is €250, as proved below:

- Suppose that P had a market price $X > 250$. In this case investors could carry out an arbitrage strategy by short-selling P at price X and buying 2 units of A and 1 unit of B for €250. Such a strategy would result in a profit of $X - 250 > 0$ today without any future cost:

Transaction	$t = 0$	$t = 1$	$t = 2$
Sell P	$+X$	−50	−250
Buy 2 A	−200	+20	+220
Buy B	−50	+30	+30
Total	$X - 250 > 0$	**0**	**0**

- Conversely, if we suppose that the market price of P is $X < 250$, investors could also make an arbitrage by buying P and short-selling 2 units of A and 1 unit of B.

Thus, in the absence of arbitrage opportunities, the market price of P must be €250, which is the weighted sum of the prices of securities A and B.

[3] A security is liquid if it can be purchased and sold in large quantities without affecting its price. In this book we always assume infinite liquidity; in other words we assume that we can buy and sell any given security in any desired quantity.

3-2 Bonds

Bonds are debt securities. The 2007–08 financial crisis abruptly reminded us that when lending money it is crucial to properly assess the borrower's capacity to repay. This is a difficult and widespread issue encompassing many practical areas of finance, from credit cards to mortgages to the funding of old and new businesses.

To keep matters simple we only consider bonds issued by the government and assume that they are **default**-free. This is already an oversimplification: some governments did default on their bonds in the past (e.g. Argentina in 2002) or are perceived to be more likely to default than others in the future (e.g. Greece vs. Germany in 2010–11).

3-2.1 Treasury Bonds

A treasury bond is a government-issued security with the following characteristics:

- A **face value** N (also called **par amount**, **principal amount**, or sometimes **notional amount**): the amount borrowed;
- A **maturity date** T: the date when the principal must be repaid;
- A series of **coupons** $C_{t_1}, C_{t_2}, \ldots, C_T$: interest amounts paid at dates $t_1, t_2, \ldots, T$.

Typically all coupon amounts are equal and expressed as a percentage of the face value, which is conventionally set at 100 if unspecified. Below is what a treasury bond's cash flows look like:

Cash flows of a bond

Time	t_1	t_2	$\ldots$	T
Cash flow	C_{t_1}	C_{t_2}	$\ldots$	$N + C_T$

Example: German Bund 4.25% 2014, issued 11 April 2009

Date	11 Apr 2010	11 Apr 2011	11 Apr 2012	11 Apr 2013	11 Apr 2014
Cash flow (€)	+4.25	+4.25	+4.25	+4.25	+104.25

3-2.2 Zero-Coupon Bonds

Zero-coupon bonds are bonds which pay no coupon: only the principal is repaid at maturity. Short-term bonds whose maturity is less than one year at issuance are usually zero-coupon bonds.

Zero-coupon bonds play an important role in financial theory (see Section 3-4 below).

3-2.3 Bond Markets

Bonds are issued by a government's treasury department through auctions on the **primary market** at a price close to the par amount N. Once issued they are traded on the **secondary market** at a fluctuating price. At maturity, bondholders surrender their securities to the issuer who repays the par amount N. At each coupon date (typically at every anniversary of the issue date) coupons are 'detached' and bondholders receive the coupon amount from the issuer.

3-3 Yield

Bond analysis essentially deals with two problems:

- Relative value analysis: Compare two bonds whose prices are known;
- Fundamental value analysis: Find the value of a bond whose price is unknown.

The classical approach, still commonly used in practice, relies on the concept of **yield to maturity**.

3-3.1 Yield to Maturity

Given the price P of a bond one may calculate its internal rate of return (see Section 2-3 p.13) which is called **yield to maturity** or simply **yield**. A bond's yield y is thus the solution to the equation:

$$P = \frac{C_{t_1}}{(1+y)^{t_1}} + \frac{C_{t_2}}{(1+y)^{t_2}} + \cdots + \frac{N + C_T}{(1+y)^T}.$$

Note that when the yield y increases the price P must decrease: '**when rates go up, prices go down.**'

Example
On the European market a 5-year bond with an annual coupon of 5% is priced at €99 while a 3-year bond with an annual coupon of 3% is priced at €95. Even though the 5-year bond is more expensive, its 5.23% yield is higher than the 4.83% yield on the 3-year bond. We may indeed verify that:

$$\frac{5}{1.0523} + \frac{5}{1.0523^2} + \frac{5}{1.0523^3} + \frac{5}{1.0523^4} + \frac{105}{1.0523^5} = €99$$

and:

$$\frac{3}{1.0483} + \frac{3}{1.0483^2} + \frac{103}{1.0483^3} = €95.$$

3-3.2 Yield Curve

Investors tend to prefer short maturity bonds for the following two reasons:

- **Lower interest rate risk**: The yield of a bond reflects the investor's actual wealth accrual only if
  (a) the investor is able to reinvest each detached coupon at the same yield, and
  (b) the investor holds the bond to maturity.

This could happen if the central bank did not change the interest rate, in which case bond yields would be stable over time. Since the central bank *does* change the interest rate, the price and yield of bonds fluctuate and an investor would normally prefer short-term bonds to avoid being locked up for too long with, e.g. a 4% yield when the interest rate is at, e.g. 8%.

- **Lower default risk**: Even though governments typically have a low default risk, the probability of default is nevertheless higher for long maturities (e.g. 30 years) than short maturities.

Issuers, on the other hand, tend to prefer long maturities in order to spread their debt over time. This divergence between demand (investors) and supply (issuers) normally results in long-maturity yields being higher than short-term ones. In other words the **yield curve** (also called the '**interest rate term structure**,' that is the structure of rates as a function of maturity or 'term'), usually slopes upward as illustrated in Figure 3-2 below.

Maturity	Rate (%)
9-Jun-11	0.008
11-Aug-11	0.023
10-Nov-11	0.074
3-May-12	0.165
30-Apr-13	0.522
15-May-14	0.924
30-Apr-15	1.334
30-Apr-16	1.802
30-Apr-17	2.155
30-Apr-18	2.481
15-May-19	2.729
15-May-20	2.971
15-May-21	3.148
15-Feb-26	3.712
15-Feb-31	3.981
15-Feb-36	4.174
15-May-41	4.273

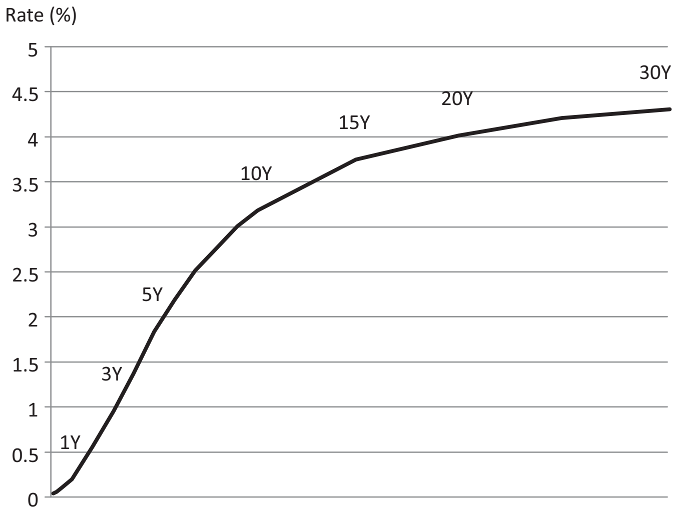


Figure 3-2 Yield curve of US government bonds as of 16 May 2011
(Source: Bloomberg)

In practice there are three typical shapes of the yield curve:

- **Upward sloping**: This is the most common case: the longer the maturity the higher the interest rate risk and default risk, which translate into a higher yield.
- **Downward sloping**: This happens when the market expects rates to decrease in the future.
- **Flat**: $y \equiv r$ (where r is constant). This is a theoretical case in which all rates are assumed to be the same regardless of maturity.

3-3.3 Approximate Valuation

Using the yield curve one may compute the approximate value $\tilde{V}$ of a bond whose price is unknown. To do this:

(1) Linearly interpolate the yields of the two bonds whose maturities T_1, T_2 tightly bound the target maturity T:

$$\tilde{y} = y_1 + \frac{y_2 - y_1}{T_2 - T_1}(T - T_1).$$

(2) Compute the present value of the bond using the interpolated yield $\tilde{y}$:

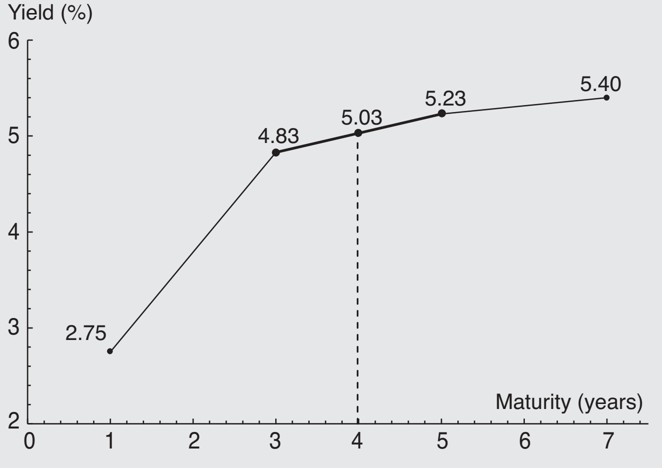
$$\tilde{V} = \frac{C_{t_1}}{(1+\tilde{y})^{t_1}} + \frac{C_{t_2}}{(1+\tilde{y})^{t_2}} + \cdots + \frac{N + C_T}{(1+\tilde{y})^T}. \tag{3-1}$$

Example
On the European bond market a 5-year bond with 5% annual coupon has a 5.23% yield, and a 3-year bond with 3% annual coupon has a 4.83% yield. The interpolated 4-year yield is thus:

$$\tilde{y} = \frac{4.83\% + 5.23\%}{2} = 5.03\%$$

and the approximate value of a 4-year bond with 7% annual coupon is:

$$\tilde{V} = \frac{7}{1.0503} + \frac{7}{(1.0503)^2} + \frac{7}{(1.0503)^3} + \frac{107}{(1.0503)^4} \approx €106.98$$



3-4 Zero-Coupon Yield Curve. Arbitrage Price

The concept of yield to maturity has only one merit: it makes gullible investors think that bond analysis is easy. We show the limits of this concept with an example.

Consider two bonds A and B both maturing in 2 years and having a \$1,000 face value. Bond A pays a \$100 coupon every year and bond B pays only one coupon of \$1,000 after one year, as shown below:

Maturity	**1**	**2**
Bond A	100	1,100
Bond B	1,000	1,000

Suppose that the bond prices are \$1,000 for A and \$1,735 for B. Which bond should we recommend buying?

The classical approach tells us to compare yields. But here the two bonds have the same yield: $y_A = y_B = 10\%$. Does this mean that one should be indifferent to buying A or B?

To help answer this question, suppose that a better-informed investor also looks at a zero-coupon bond C maturing in 1 year and priced at \$90.91 for \$100 face value. While gullible investors ponder over the pros and cons of investing in A or B, the better-informed investor can make infinite amounts of money by repeating the following arbitrage strategy:

	Cash flows		
Transaction	$t = 0$	$t = 1$	$t = 2$
Buy B	−1,735	+1,000	+1,000
Sell 10/11th A	+909.09	−90.91	−1,000
Sell 9 C	+828	−900	–
Total	+2.09	+9.09	–

This example shows why there is more to bond analysis than computing a yield to maturity. In fact, arbitrage-free bond analysis relies on the concept of **zero-coupon yield**.

3-4.1 Zero-Coupon Rate Curve

The **zero-coupon rate curve** is the arbitrage-free version of the yield curve. In mature markets such as US or French government bonds, the zero-coupon rate curve is directly observable on ‘strips’ which are government-issued zero-coupon bonds. In other markets it can be inferred from standard treasury bonds using the ‘bootstrapping’ method (see Section 3-4.3 below).

Specifically, given a zero-coupon bond with face value N, maturity T, and price P, the **zero-coupon rate** for maturity T is given as:

$$z(T) = \left(\frac{N}{P}\right)^{\frac{1}{T}} - 1.$$

Examples

- The 4-year zero-coupon rate corresponding to a 4-year zero-coupon bond priced at \$90 is:

$$z(4) = \left(\frac{100}{90}\right)^{1/4} - 1 = 2.67\%.$$

- On the US bond market the zero-rate curve is directly observable:

Maturity	Rate (%)
15-Aug-11	0.036
15-Nov-11	0.094
15-May-12	0.1555
15-May-13	0.4815
15-May-14	0.955
15-May-15	1.4045
15-May-16	1.894
15-May-17	2.29
15-May-18	2.611
15-May-19	2.9175
15-May-20	3.216
15-May-21	3.452
15-May-26	4.192
15-May-31	4.452
15-May-41	4.662

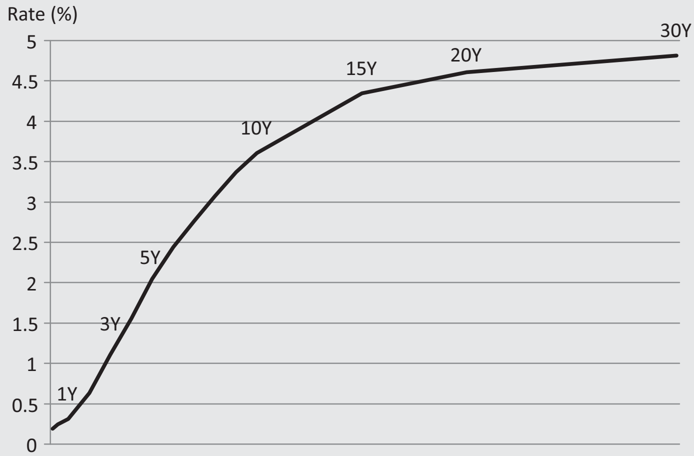


Figure 3-3 Zero-coupon rate curve of US government bonds as of 16 May 2011
(Source: Bloomberg)

3-4.2 Arbitrage Price of a Bond

Given a zero-coupon rate curve $z(t)$ we can find the **arbitrage price** P of any bond with face value N, maturity T, and coupons $C_{t_1}, C_{t_2}, \ldots, C_T$ paid at dates $t_1, t_2, \ldots, T$:

$$P = \frac{C_{t_1}}{(1 + z(t_1))^{t_1}} + \frac{C_{t_2}}{(1 + z(t_2))^{t_2}} + \cdots + \frac{N + C_T}{(1 + z(T))^T}. \tag{3-2}$$

Example

Using the zero-coupon rate curve shown in Figure 3-3 above, the arbitrage price of a 5-year bond with 5% coupon is:

$$P = \frac{5}{1.001555} + \frac{5}{1.004815^2} + \frac{5}{1.00955^3} + \frac{5}{1.014045^4} + \frac{105}{1.01894^5} \approx \$115.13.$$

Equation (3-2) extends the present value formula from Section 2-2 p.12 and generalizes to any **fixed income security**[4] paying a series of n cash flows $F_{t_1}, F_{t_2}, \cdots, F_{t_n}$ at future dates t_1, $t_2, \cdots, t_n$:

$$P = \frac{F_{t_1}}{(1+z(t_1))^{t_1}} + \frac{F_{t_2}}{(1+z(t_2))^{t_2}} + \cdots + \frac{F_{t_n}}{(1+z(t_n))^{t_n}}.$$

The formal proof of this result is based on a decomposition of the security's cash flows into a portfolio of zero-coupon bonds (see Problem 12).

3-4.3 Zero-Coupon Rate Calculation by Inference: the 'Bootstrapping' Method

As mentioned in Section 3-4.1 above, we may infer the zero-coupon rates from the prices of standard bonds if we assume that there is no arbitrage opportunity on the bond market. This is done by solving a system of linear equations, provided that all cash flow dates coincide.[5] We illustrate this approach known as '**bootstrapping**' with an example.

Consider three bonds A, B, C with the following cash flows:

Bond	**Price at $t = 0$**	**Cash flow at $t = 1$**	**Cash flow at $t = 2$**	**Cash flow at $t = 3$**
A	111.41	10	110	—
B	102.82	5	5	105
C	88.90	—	—	100

Our aim is to find zero-coupon rates $z(1)$, $z(2)$, $z(3)$ which satisfy:

$$\begin{cases} \dfrac{10}{1+z(1)} + \dfrac{110}{(1+z(2))^2} = 111.41 \\ \dfrac{5}{1+z(1)} + \dfrac{5}{(1+z(2))^2} + \dfrac{105}{(1+z(3))^3} = 102.82 \\ \dfrac{100}{(1+z(3))^3} = 88.90 \end{cases}$$

Writing $x_i = \frac{1}{(1+z(i))^i}$ we obtain the more familiar system of linear equations:

$$\begin{cases} 10x_1 + 110x_2 = 111.41 \\ 5x_1 + 5x_2 + 105x_3 = 102.82 \\ 100x_3 = 88.90 \end{cases}$$

[4] A fixed income security is any financial security whose cash flows are *fixed*, i.e. known in advance. A standard bond with predetermined coupons is a fixed income security. A stock is *not* a fixed income security because it pays variable dividends tied to the company's profits.

[5] In practice a variety of fixed income securities are used to determine the zero-coupon curve and their dates rarely match. A linear interpolation between the closest dates is commonly used to circumvent this problem.

whose solution is:

$$\begin{cases} x_1 = 0.9704 \\ x_2 = 0.9246 \\ x_3 = 0.8890 \end{cases}$$

Note that the '**discount factors**' x_1, x_2, x_3 are the respective prices of 1-year, 2-year, and 3-year zero-coupons with a face value of 1. Reverting to z through the definition of x we obtain the zero-coupon rates:

$$\begin{cases} z(1) = \dfrac{1}{x_1} - 1 = 3.05\% \\ z(2) = \left(\dfrac{1}{x_2}\right)^{1/2} - 1 = 4\% \\ z(3) = \left(\dfrac{1}{x_3}\right)^{1/3} - 1 = 4\% \end{cases}$$

3-5 Further Reading

- On bonds and the yield curve: Frank J. Fabozzi (2005) *The Handbook of Fixed Income Securities* 7th Edition, McGraw-Hill Trade: Chapters 1, 3, 4, and 5.
- On markets, arbitrage and short selling: John C. Hull (2009) *Options, Futures and Other Derivatives* 7th Edition, Prentice Hall: Chapters 1 and 5.
- On bonds: Stephen A. Ross, Randolph W. Westerfield, and Bradford D. Jordan (2008) *Fundamentals of Corporate Finance Standard Edition*: Chapter 7.

3-6 Problems

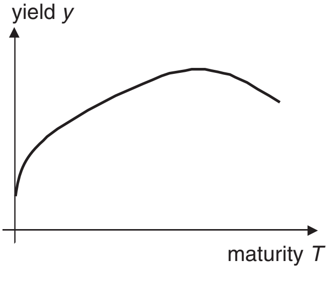
Problem 1: Yield

Compute the annual yield of the following bonds:

(a) Bond A – maturity: 30 years, annual coupon: 5%, price: €100.
(b) Bond B – maturity: 2 years, annual coupon: 6%, price: £106.
(c) Bond C – maturity: 1 year, zero coupon, price: $95.
(d) Bond D – maturity: 13 years, face value: $1,000, semi-annual coupon: $50, price: $1,000.

Problem 2: Yield curve

How would you interpret the shape of the following yield curve in terms of investors' anticipations of future interest rates?



Problem 3: True or False?

"On 1 January 2012 the 3-year yield is 4%. I can choose between bond A with 3% coupon and maturity date 15 November 2014, and bond B with 4.5% coupon and maturity date 15 February 2015. I should buy B because its coupon rate is higher than the 4% market yield."

Problem 4: Maximum return on a bond

(a) You just bought a 1-year bond with 4% annual coupon for \$98. What is your annual return if you hold the bond to maturity?
(b) One month later the yield curve collapses to zero. What is your gross return over 1 month? Is it better than the return found in question (a)?
(c) In general, what is the maximum gross return an investor can get on a bond with price P, annual coupon C, and a maturity of T whole years?

Problem 5*: Liquidity and arbitrage price of a portfolio

Explain why the assumption of infinite liquidity is needed to derive the arbitrage price of a portfolio in Section 3-1.5 p.22.

Problem 6: Approximate valuation

Using the yield curve of Figure 3-2 p.25, estimate the value of a bond with 5% annual coupon and maturity 15 May 2022.

Problem 7: Bond arbitrage

On the UK bond market investors can buy and sell:

- Bond A – maturity: 1 year, zero coupon, face value: £100, price: £90
- Bond B – maturity: 2 years, face value: £1,000, annual coupon: £50, price: £945
- Bond C – maturity: 2 years, face value: £1,000, annual coupon: 10%, price: £990

Can you find an arbitrage strategy which has no cost today and will make a profit in 2 years?

Probem 8: Dividend announcement

At the annual stockholders' meeting of Tankai Corp., a high-tech company headquartered in Tokyo, a dividend distribution of ¥400 per share was voted. The market reacted positively to the news and the stock price is about to close at ¥10,000. The dividend will be paid overnight to every owner of the stock, and the stock price will open the following day at level S.

(a) Suppose you know in advance that $S >$ ¥9,600. Find an arbitrage strategy.
(b) Suppose you know in advance that $S <$ ¥9,600. Find an arbitrage strategy.
(c) What can you conclude?

Problem 9: Arbitrage price

Using the zero-coupon rate curve in Figure 3-3 p.28, calculate the arbitrage price of a 10-year bond with \$500 face value and 6% annual coupon.

Problem 10: Zero-coupon bond portfolio

Using the following zero-coupon rate curve, build a bond portfolio which costs nothing and makes \$10,000 if all rates go up 25 basis points (i.e. a +0.25% parallel shift). *There are several possible answers to this problem.*

Maturity	3 months	6 months	1 year	2 years	5 years
Zero-coupon rate	3.72%	4.10%	4.44%	4.09%	3.89%

Problem 11: Zero-coupon rate curve

On the German bond market investors can buy and sell:

- Bond A – maturity: 1 year, zero coupon, price: €97
- Bond B – maturity: 2 years, fixed annual coupon, yield: 4%, price: €100
- Bond C – maturity: 3 years, annual coupon: 4%, price: €95

Find the 1-, 2-, and 3-year zero-coupon rates.

Problem 12*: Arbitrage price formula

Let A be a financial security that pays 3 annual cash flows F_1, F_2, F_3, and let X, Y, Z be the 1-, 2-, 3-year zero-coupon bonds with face value 1 and prices P_X, P_Y, P_Z respectively.

(a) Show that the arbitrage price of A is: $P = P_X F_1 + P_Y F_2 + P_Z F_3$.
(b) Compare P with the arbitrage price formula (Equation (3-2) p.28).

Problem 13: Price sensitivity and convexity. *It is recommended to use a spreadsheet to solve this problem.*

Suppose the US zero-coupon rate curve is given as:

Maturity (years)	1	2	3	4	5	6	7
Zero-coupon rate	7.5%	8%	8.25%	8.25%	8%	8%	7.75%

Consider the following three bonds with face value \$100:

Bond	Maturity	Annual Coupon
X	4 years	8%
Y	7 years	9%
Z	5 years	–

(a) Calculate the arbitrage price of each bond.
(b) The price sensitivity of a bond (also known as the 'dollar value of one basis point' or DV01) is defined as the change in price when all rates go up 1 basis point (i.e. a +0.01%

parallel shift of the entire zero-coupon rate curve). Compute the price sensitivity of each bond.

(c) Calculate the price of each bond in the following scenarios:
  (i) 10 basis point rate increase (+0.10%);
  (ii) 1 point rate increase (+1%).

(d) Compare your respective answers to question (c) with:
  (i) 10 times the price sensitivity;
  (ii) 100 times the price sensitivity.

(e) Based on this comparison, do you think that price sensitivity is a good indicator of the interest rate risk which bonds are exposed to?

(f) (*) Suppose that the zero-coupon rate curve is flat at rate r. Using a second-order Taylor expansion in r, identify a secondary indicator for the interest rate risk of a bond.

Problem 14: Zero-coupon rate curve and expectations.
The short-term zero-coupon rate curve of the euro zone is given as:

Maturity	1 day (1/360 yr)	1 week (1/52 yr)	2 weeks (1/24 yr)	1 month (1/12 yr)	2 months (1/6 yr)
Zero-coupon rate	2.78%	2.77%	2.75%	2.92%	3.08%

The current refinancing rate of the European Central Bank (ECB) is at 2.75%. This is the rate at which banks can borrow from the ECB for 2 weeks. The ECB Board of Governors will meet in 2 weeks and potentially decide on a new refinancing rate R.

(a) Without making any calculation can you guess if the market expects the ECB to:
  (i) Lower its rate by 25 bps (i.e. $R = 2.5\%$)?
  (ii) Leave its rate unchanged (i.e. $R = 2.75\%$)?
  (iii) Raise its rate by 25 bps (i.e. $R = 3\%$)?
  (iv) Raise its rate by 50 bps (i.e. $R = 3.25\%$)?
  (v) Any other scenario?

(b) The treasury department at Lezard Brothers, a reputed investment bank, must find €100mn in cash for the coming month. Find two ways to achieve this objective. What are the corresponding borrowing costs?

(c) Does your answer to question (b) support or invalidate your answer to question (a)?

(d) Fadeberg News, a financial news agency, recently published the following survey of the predictions from leading financial economists at top investment banks:

Rate R	**Number of predictions**
25 basis point drop	0
Unchanged	3
25 basis point hike	21
50 basis point hike	6

Based on this survey, Bernard Bull, a trader at Lezard Brothers, thinks that the 1-month zero-coupon rate of 2.92% is overvalued and comes to you, the head of trading at Lezard Brothers, with the following strategy:

- Invest €100mn at 2.92% over one month;
- Borrow €100mn at 2.75% for 2 weeks;
- In 2 weeks, roll over and borrow €100mn at rate R for 2 weeks.

  (i) Bernard says his strategy is a "fantastic arbitrage opportunity." Do you agree?
  (ii) Calculate the profit or loss of this strategy in each of the scenarios in the survey.
  (iii) Do you give a thumbs up to Bernard? *There is no unique answer to this question.*

An Introduction to Equity Derivatives: Theory and Practice, Second Edition
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4 Portfolio Theory

In this chapter we examine the relationship between the return on an asset[1] and its risk, and how the latter may be reduced through portfolio diversification.

4-1 Risk and Return of an Asset

4-1.1 Average Return and Volatility

When looking at a fixed stream of cash flows, the rate of return can be determined in advance. For example the yield-to-maturity of a bond with 5% coupon is simply 5% at inception.

But when an asset is traded on a market, the return typically fluctuates over time: one year after issuance the same bond might only yield e.g. 3.6% until maturity. Investors may therefore be at **risk** of earning a lower return than they thought.

Table 4-1 and Figure 4-1 below show the monthly returns on three assets in 2008:

- The stock of Kroger Co., a large supermarket chain in the United States;
- A US Treasury bond[2] with 4.75% coupon maturing in December 2008;
- A share in the Coast Value LP hedge fund.

We can see that the monthly returns of the Treasury bond were significantly less dispersed than those of Kroger Co. and Coast Value LP.

Table 4-1 Monthly returns of three assets in 2008

	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
Kroger	4.73%	−0.72%	0.35%	0.51%	2.39%	2.40%	−4.26%	−1.41%	−6.46%	−4.53%	4.95%	5.30%
T-Bond	1.35%	0.49%	0.36%	−0.05%	−0.06%	0.15%	0.31%	0.11%	0.39%	0.13%	0.08%	0.02%
Coast Value	3.18%	−0.60%	−3.88%	6.24%	3.80%	−1.45%	2.10%	0.76%	12.32%	−8.79%	7.11%	2.69%

(Source: Bloomberg, WRDS)

[1] An asset is any entity that has financial value, which includes securities as well as as immaterial entities, such as brands. However the term 'asset' is commonly used in finance as a synonym for 'security' and we will use these terms interchangeably in this book.

[2] Technically, this particular US government bond was known as a 'US Treasury Note' because its original maturity was more than 12 months and less than 10 years. The distinction between US Treasury Bills, Notes, and Bonds is purely formal and may be ignored for all practical purposes.

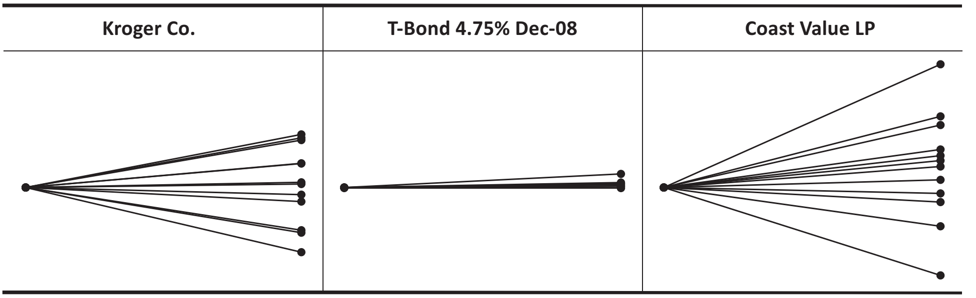

Figure 4-1 Monthly returns of three assets in 2008

The level of dispersion of returns of an asset is called **volatility** or **risk** and is commonly measured as the annualized standard deviation of returns:[3]

- Standard deviation: $\sigma_{\text{periodic}} = \sqrt{\frac{1}{N-1}\sum_{t=1}^{N}(r_t - \bar{r})^2}$;
- Volatility / Risk / Annualized standard deviation: $\sigma_{\text{annual}} = \sigma_{\text{periodic}} \times \sqrt{\text{Nb periods per year}}$

where $\bar{r} = \frac{1}{N}\sum_{t=1}^{N} r_t$ is the average periodic return and N is the number of observations. Table 4-2 below presents these statistics for Kroger Co., the Treasury bond and Coast Value LP.

Table 4-2 Return statistics

Statistic	Symbol	Kroger Co.	T-Bond	Coast Value LP
Average monthly return	$\bar{r}$	0.3%	0.3%	2.0%
Monthly volatility	σ_{monthly}	3.9%	0.4%	5.5%
Risk (annual volatility)	$\sigma_{\text{annual}} = \sigma_{\text{monthly}} \times \sqrt{12}$	13.6%	1.3%	18.9%

To understand what these numbers mean, consider the column for Kroger Co. A 0.3% average monthly return and 3.9% monthly volatility indicate that a typical stock price move over a month was 0.3%±3.9%, i.e. either +4.2% or −3.6%. In other words, we may think of Kroger Co.'s 2008 returns as equivalent to a sequence of 12 coin flips where each "heads" is worth +4.2% and each "tails" is worth −3.6%.

Similarly, a 13.6% *annual* volatility means that a typical stock price move over a year is ±13.9% on top of the average annual return. Note how **volatility grows with the *square root* of the time period**: ±3.9% per month is equivalent to ±13.6% per year or $3.9\% \times \sqrt{12}$, *not* ±47% or 3.9%×12! This is because month after month some fluctuations are expected to cancel out so that the most likely deviation after two months will *not* be twice the monthly deviation, and so on.

[3] Readers not familiar with the theory of statistical estimation may be surprised by the $N-1$ denominator in our definition of standard deviation. This is because using N as denominator would create a small bias.

In practice...
The Excel function STDEV(*Number1, Number2, ...*) calculates the standard deviation of a sample.

4-1.2 Risk-free Asset. Sharpe Ratio

When looking at several assets, the asset with zero volatility is called the **risk-free asset** and its return is called the **risk-free rate r_f**. In our example, the Treasury bond is a suitable proxy for the risk-free asset because its 1.3% volatility is close to zero; accordingly, the risk-free rate is about 3.3% p.a.

The risk-free rate is the default benchmark for the return performance delivered by other risky assets such as Kroger Co. or Coast Value LP. The difference $r_A - r_f$ between the return of a risky asset and the risk-free rate is called the **risk premium**.

Let us go back in time at the beginning of 2008: why would anyone want to invest in a blue-chip stock or a hedge fund, rather than the Treasury bond? The most common rationale would be to get a better return than 3.3% one year later. This implies that the ex-ante *expected* risk premium of any risky asset is typically positive.

Similarly, when comparing two risky assets, investors typically expect a higher premium from the riskier asset. As such, the return performance of an asset must be compared to the risk incurred. This is exactly what the **Sharpe ratio** does:

$$\text{Sharpe Ratio} = \frac{\text{Premium}}{\text{Risk}} = \frac{r_A - r_f}{\sigma_A}.$$

The Sharpe ratio is the premium per unit of risk incurred. The higher the risk premium and the lower the risk, the higher the ratio. A Sharpe ratio of 1.0 is usually considered very good and above 1.5 regarded as excellent.

Table 4-3 below shows what the ex-ante risk-return profiles of Kroger Co., the Treasury bond and Coast Value LP might have looked like at the beginning of 2008 in the eyes of two hypothetical investors: aggressive and conservative. In Section 4-4 p.43 we introduce a model which approaches the expected return in a more rational way rather than making empirical projections based on individual preferences.

Table 4-3 Examples of risk-return projections: aggressive and conservative

	Aggressive Investor			Conservative Investor		
Ex-ante projections as of 1 Jan. 2008	**T-Bond**	**Kroger Co.**	**Coast Value LP**	**T-Bond**	**Kroger Co.**	**Coast Value LP**
Expected annual return	3.3%	8%	15%	3.3%	7%	8%
Expected annual volatility	“0%”	15%	20%	“0%”	15%	25%
Expected risk premium	0	4.7%	11.7%	0	3.7%	4.7%
Expected Sharpe ratio	n/a	0.31	0.59	n/a	0.25	0.19

If we fast-forward to the end of 2008, we may compare these projections with ex-post *realized* risk-return profiles (Table 4-4 below). We can see that the risk premium and Sharpe ratio of Kroger Co. turned out to be slightly negative: it was retrospectively a bad investment which did worse than the risk-free asset. On the other hand, Coast Value LP delivered a very strong risk premium of nearly 21% and a Sharpe ratio above 1: it was retrospectively an excellent investment.

As always, when looking at past returns, every winning trade looks smart and every losing trade looks foolish. The tough part is to get the ex-ante expected returns right. In early 2008, before the financial crisis had reached its climax, very few investors did; and in fact Kroger Co. did pretty well compared to other blue-chip stocks.

Table 4-4 Realized risk-return profiles

Ex-post observations as of 31 Dec. 2008	T-Bond	Kroger Co.	Coast Value LP
Realized annual return	3.3%	2.4%	24.2%
Realized annual volatility	"0%" (1.3%)	13.6%	18.9%
Realized risk premium	0	−0.9%	20.9%
Realized Sharpe ratio	n/a	−0.07	1.10

4-2 Risk and Return of a Portfolio

4-2.1 Portfolio Valuation

In Section 3-1.2 p.19 we distinguished between the price and value of an asset, and in Section 3-1.5 p.22 we showed that under the assumptions of no arbitrage and infinite liquidity the arbitrage price of a portfolio is the sum of asset prices weighted by their quantities:

General Form

Asset	Price	Quantity
#1	p_1	q_1
#2	p_2	q_2
...		...
#n	p_n	q_n
Portfolio	$p_1 \times q_1 + p_2 \times q_2 + \cdots + p_N \times q_N$	

Example

Asset	Price	Quantity
Kroger Co.	$25	12,000
T-Bond	$100	-5,000
Portfolio	$25 \times 12{,}000 - 100 \times 5{,}000 = -\$200{,}000$	

This valuation method for portfolios is known as **mark-to-market**. A negative quantity is called a **short position**, and a positive quantity a **long position**. The example portfolio above is thus long 12,000 shares of Kroger Co. and short 5,000 T-Bonds.

The negative portfolio *value* of −$200,000 is *not* a latent loss. Rather, it is the amount of cash that the owner of the portfolio must pay to get rid of it. To do so, the owner must either find a buyer for the whole portfolio or **liquidate** each position individually, i.e. sell 12,000 shares of Kroger Co. and buy 5,000 T-Bonds back for a net $200,000 cash output.

4-2.2 Return of a Portfolio

Let us go back to the beginning of 2008 and consider a portfolio long $1,000,000 worth of Kroger Co.'s stock, $500,000 worth of Treasury bonds, and $500,000 worth of shares in Coast

Value LP. Table 4-5 overleaf shows the monthly mark-to-market price (MTM) and return of this portfolio based on the data from Table 4-1 p.35. We can see that at the end of 2008 the portfolio was worth about \$2.162mn and thus delivered a solid 0.162 / 2 = 8.1% annual return, for an annual volatility of 8.7%.

There is a shortcut to calculate a portfolio's return: simply take the average of all asset returns weighted by the asset proportions in the portfolio. In our example we find:

$$\frac{1{,}000{,}000}{2{,}000{,}000} \times 2.4\% + \frac{500{,}000}{2{,}000{,}000} \times 3.3\% + \frac{500{,}000}{2{,}000{,}000} \times 24.2\% = 8.1\%,$$

which is consistent with Table 4-5 overleaf. Generally, for a portfolio of n assets in *proportions* $w_1, w_2 \ldots, w_n$ (possibly negative), the **portfolio return** is given as:

$$R_{\mathrm{P}} = w_1 R_1 + w_2 R_2 + \cdots + w_n R_n$$

where $R_1, R_2, \ldots, R_n$ are the asset returns and the weights must sum to 1 (i.e. $w_1 + w_2 + \cdots + w_n = 1$).

4-2.3 Volatility of a Portfolio

The formula for **portfolio volatility** is more involved: if we were to compute the weighted average of annualized volatilities we would get 11.8% instead of the actual figure of 8.7% reported in Table 4-5 overleaf. This is because asset returns are imperfectly **correlated** and provide so-called '**gains of diversification**' as explained in more detail in Section 4-3 below.

- For a portfolio of two assets, volatility is given as:

$$\sigma_{\mathrm{P}} = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1 w_2 \sigma_1 \sigma_2 \rho}$$

where σ_1 and σ_2 are the asset volatilities, and ρ is the correlation coefficient between the two assets. This is merely a restatement of the formula for the variance of a sum of random variables (see Section A-5.5 p.210):

$$\sigma_{\mathrm{P}} = \sqrt{\mathbb{V}(w_1 R_1 + w_2 R_2)} = \sqrt{w_1^2\mathbb{V}(R_1) + w_2^2\mathbb{V}(R_2) + 2w_1 w_2 \operatorname{Cov}(R_{1,} R_2)}$$

Example
We know from Table 4-5 overleaf that the risks of Kroger Co. and Coast Value LP are 13.6% and 18.9% respectively, and their correlation can be computed to be about 0.07. Thus, the risk of a portfolio made of 70% Kroger and 30% Coast Value is about 11.4%, as shown below:

$$\sigma_{\mathrm{P}} = \sqrt{0.7^2 \times (13.6\%)^2 + 0.3^2 \times (18.9\%)^2 + 2 \times 0.7 \times 0.3 \times (13.6\%) \times (18.9\%) \times 0.07}$$
$$\approx 11.4\%.$$

Table 4-5 Monthly mark-to-market price and return of a portfolio

	Initial mark-to-market	Jan-08	Feb-08	Mar-08	Apr-08	May-08	Jun-08
Kroger Co.	$1,000,000	$1,047,331	$1,039,758	$1,043,355	$1,048,670	$1,073,729	$1,099,547
T-Bond	$500,000	$506,768	$509,246	$511,081	$510,849	$510,540	$511,329
Coast Value LP	$500,000	$515,900	$512,805	$492,908	$523,665	$543,565	$535,682
Portfolio MTM	**$2,000,000**	**$2,069,998**	**$2,061,808**	**$2,047,344**	**$2,083,185**	**$2,127,833**	**$2,146,558**
Portfolio Return		3.5%	−0.4%	−0.7%	1.8%	2.1%	0.9%

	Jul-08	Aug-08	Sep-08	Oct-08	Nov-08	Dec-08	Annual return	Annual risk
Kroger Co.	$1,052,701	$1,037,848	$970,816	$926,862	$972,727	$1,024,325	2.4%	13.6%
T-Bond	$512,892	$513,456	$515,479	$516,175	$516,594	$516,692	3.3%	1.3%
Coast Value LP	$546,932	$551,089	$618,982	$564,575	$604,716	$620,981	24.2%	18.9%
Portfolio MTM	**$2,112,525**	**$2,102,392**	**$2,105,278**	**$2,007,612**	**$2,094,038**	**$2,161,999**	**Portfolio return**	**Portfolio risk**
Portfolio Return	−1.6%	−0.5%	0.1%	−4.6%	4.3%	3.2%	8.1%	8.7%

Note how portfolio volatility decreases with correlation,[4] the most favorable case being that of negatively correlated assets: whenever one asset goes down, the other tends to go up and the portfolio fluctuates less overall.

- For a portfolio of three assets, the formula is:

$$\sigma_{\mathrm{P}} = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_1\sigma_2\rho_{1,2} + 2w_1w_3\sigma_1\sigma_3\rho_{1,3} + 2w_2w_3\sigma_2\sigma_3\rho_{2,3}} \tag{4-1}$$

  where $\rho_{1,2}$, $\rho_{1,3}$, $\rho_{2,3}$ are the pairwise correlation coefficients. Problem 5 verifies this formula for the portfolio in Table 4-5 and investigates the impact of correlation on the Sharpe ratio.
- Generally, for n assets, we have:

$$\begin{cases} \sigma_{\mathrm{P}} = \sqrt{\mathbb{V}(R_{\mathrm{P}})} \\ \mathbb{V}(R_{\mathrm{P}}) = \mathbb{V}\left(\sum_{i=1}^{n} w_i R_i\right) = \sum_{i=1}^{n} w_i^2 \underbrace{\sigma_i^2}_{\mathbb{V}(R_i)} + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} w_i w_j \underbrace{\sigma_i\sigma_j\rho_{i,j}}_{\mathrm{Cov}(R_i,R_j)} \end{cases} \tag{4-2}$$

  This formula may look a little daunting at first, yet it really is nothing else but the weighted sum of the n individual variances and all $n(n-1)$ covariance pairs as shown in the matrix in Figure 4-2 below. Because of symmetry across the diagonal, instead of summing all off-diagonal terms it is enough to sum e.g. the upper triangular terms with ad-hoc weights and multiply the result by 2.

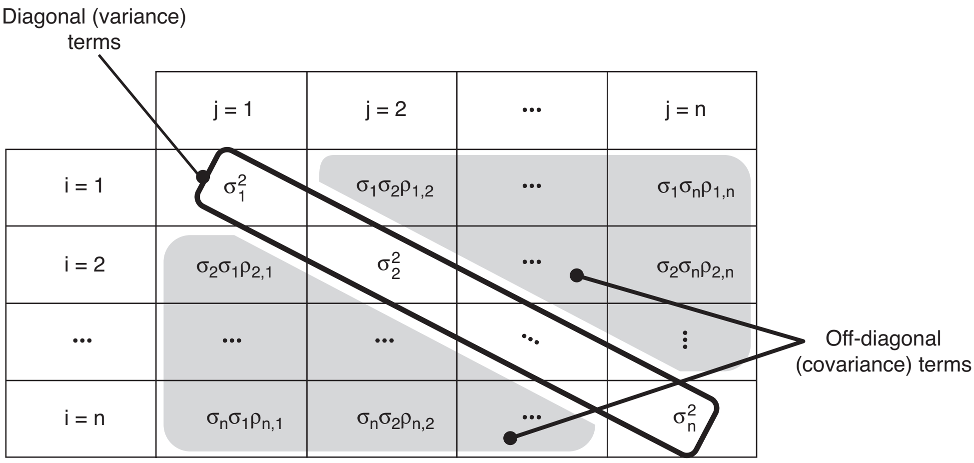

Figure 4-2 Covariance matrix

4-3 Gains of Diversification. Portfolio Optimization

Diversification is the scholarly term for not putting all of one's eggs in one basket. Figure 4-3 overleaf shows the volatility of weekly returns in 2000–10 for a portfolio made of a growing

[4] Provided weights are positive. When one weight is negative, the relationship is reversed.

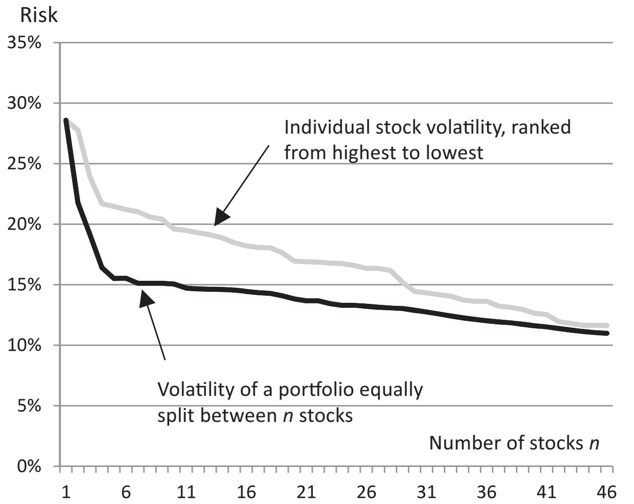

Figure 4-3 Volatility of a portfolio made of a growing number of stocks (2000–10)

number of stocks from the Dow Jones EuroStoxx 50 index. Observe how portfolio volatility dramatically decreases and then stabilizes somewhat as the number of stocks increases.

To understand the mechanics of diversification let us consider again the example of Kroger Co., the T-Bond, and Coast Value LP. Figure 4-4 below shows their risk-return profiles.

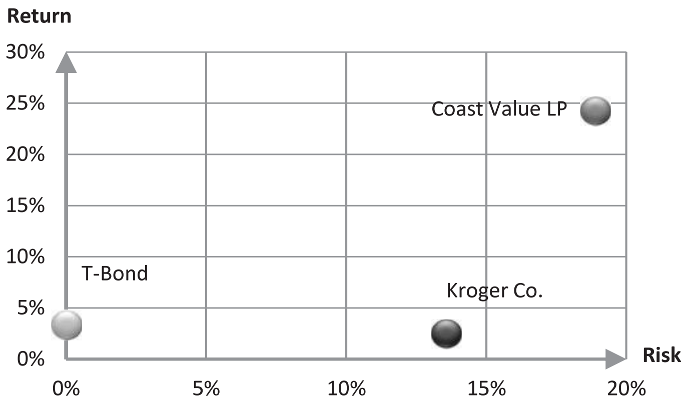

Figure 4-4 Risk-return plot of three assets

The correlation between Kroger Co. and Coast Value LP is 0.07, which is low: the two assets are almost uncorrelated. This implies large gains of diversification, as shown in Figure 4-5 overleaf where we can observe the path followed by a portfolio which is initially fully invested in Coast Value LP and gradually moves towards a full investment in Kroger Co.

The portfolio made of 70% Kroger Co. and 30% Coast Value LP has the lowest risk among all listed portfolios. Compared to a 100% investment in Kroger Co., this portfolio achieves a

Weight Kroger Co.	Risk	Return
0%	18.9%	24.2%
10%	17.1%	22.0%
20%	15.5%	19.8%
30%	14.1%	17.7%
40%	12.9%	15.5%
50%	12.0%	13.3%
60%	11.5%	11.1%
70%	11.4%	9.0%
80%	11.7%	6.8%
90%	12.5%	4.6%
100%	13.6%	2.4%

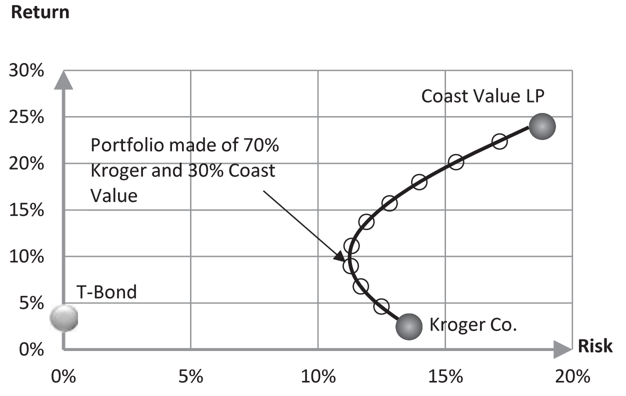

Figure 4-5 Risk-return evolution of a portfolio made of Kroger Co. and Coast Value LP

higher return (9% instead of 2.4%) together with a lower risk (11.4% instead of 13.6%), i.e. it has a better Sharpe ratio (0.5 instead of −0.07).

Generally, there is an **optimal portfolio** of Kroger Co. and Coast Value LP which minimizes the risk without sacrificing the return. In Problem 7, we show that this portfolio is made of about 67% Kroger Co. and 33% Coast Value LP.

Portfolio optimization may be performed on any number of assets, at the cost of more complex computations. Problem 6 looks at a three-asset case, while Problem 8 and Problem 9 look at n-asset cases.

4-4 Capital Asset Pricing Model

The two key principles of portfolio theory may be stated as follows:

(1) More risk means more expected return;
(2) More diversification means less risk.

However, these two principles are not always consistent, as shown in the following paradox. Consider two assets A and B with the same 10% expected return. The return of any portfolio made of A and B must thus be 10% as well. From Principle (2) we know that there is an optimal portfolio P with lower risk than A or B. As such, Principle (1) appears to be violated: A and B are both riskier than P, yet their returns are the same as P.

To resolve this paradox, the **CAPM (Capital Asset Pricing Model)** proposes to distinguish between two types of risk:

- **Market risk**, or 'systematic risk': Common to all risky assets, and reflecting general market trends. This type of risk cannot be eliminated by diversification and must be rewarded with a higher return.
- **Specific risk**, or 'idiosyncratic risk': Specific to each asset, and corresponding to price fluctuations stemming from the asset's own characteristics. This type of risk can be eliminated by diversification and therefore should not be rewarded with a higher return.

Example
The terrorist attack on the World Trade Center in New York on 11 September 2001 caused stock prices to collapse (inevitable market risk), particularly affecting the airline industry (specific risk which could have been avoided by investing in each one of the companies in the S&P 500 index, for instance).

Under certain assumptions which are beyond the scope of this book, the conclusion of the CAPM is that the expected return r_A of any asset A is the function of only three parameters: the risk-free rate r_f, the expected market risk premium $r_M - r_f$ and the asset's sensitivity to market movements β_{A}. Specifically:

$$r_A = r_f + \beta_A(r_M - r_f).$$

Example
The risk-free rate is 5%, the expected market return is 8%, and stock A is twice as sensitive to market movements as the S&P 500 index, i.e. when the index is up 1% the price of A is up 2%. According to the CAPM equation, the expected return on A is:

$$r_A = 5\% + 2 \times (8\% - 5\%) = 11\%.$$

4-5 Further Reading

- Frank J. Fabozzi and Harry M. Markowitz (2011) *The Theory and Practice of Investment Management* 2nd edition, John Wiley & Sons: Chapters 1, 2, 3, and 4.
- On the capital market line: Stephen A. Ross, Randolph W. Westerfield, and Bradford D. Jordan (2008) *Fundamentals of Corporate Finance Standard Edition*: Chapter 13.

4-6 Problems

Problem 1: True or False? *The three questions are independent.*

(a) "The average monthly return of Kroger Co. in 2009-10 was 0.28% (including dividends). Therefore, its annual return was $(1 + 0.28\%)^{12} - 1 \approx 3.4\%$."
(b) "To calculate the annual volatility of a series of monthly returns I may either compute their standard deviation and multiply it by $\sqrt{12}$, or equivalently I may annualize each monthly return and then compute the corresponding standard deviation."
(c) "The return of my portfolio is 15% per year and its risk is 25% per year. The stock of MeToo.Com has a 15% return and 30% risk. Hence, adding MeToo.Com to my portfolio would increase its risk but not its return."

Problem 2: Risk-free rate and Sharpe ratio

Using the data for the Treasury bond in Table 4-1 p.35, determine the theoretical risk-free rate r_f so that the Sharpe ratio of the T-Bond be equal to 1.

Problem 3: Risk and return of Richky Corp.

The table below gives the stock price of Richky Corp. at the end of each month over the past year. The risk-free rate was constant at 5%.

Jan.	Feb.	March	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
\$144	\$123	\$128	\$137	\$147	\$130	\$139	\$147	\$175	\$162	\$154	\$158

(a) Given a \$134 initial stock price at the end of the previous year and a \$13 dividend per share distributed on 30 June, calculate the monthly returns of Richky Corp. Assume that the dividend is reinvested in the stock.
(b) What is the realized risk-return profile of Richky Corp.?
(c) (*) You are the Chief Financial Officer of Richky Corp. At a business meeting, Mr David Haffmann, the Production Manager, proposes a project which would reduce annual production costs by \$1.2 per share on average with a standard deviation of \$0.6 per share. Do you approve this project? *There are several possible answers to this question.*

Problem 4: Risk premium and CAPM

Following the CAPM, the risk-free rate is 3% and the expected market return is 7%. Calculate the risk premium and Sharpe ratio of the following assets:

(a) Pschitzer Pharmaceuticals (stock): 15% volatility and 1.5 beta;
(b) T-Bond: 3% volatility and 0.2 beta;
(c) Goldy (mutual fund): 12% volatility and –0.5 beta. Is it worth investing in this asset?

Problem 5: Volatility and Sharpe ratio

(a) Based on Table 4-1 p.35, calculate the pairwise correlation coefficients $\rho_{1,2}$, $\rho_{1,3}$ and $\rho_{2,3}$ for Kroger Co., the T-Bond and Coast Value LP.
(b) Verify that Equation (4-1) p.41 matches the level of portfolio risk given in Table 4-5 p.40.
(c) Calculate the portfolio's Sharpe ratio and compare it to the Sharpe ratio of each asset.

Problem 6: Currency portfolio. *It is recommended that you solve this problem using a spreadsheet.*

You are a euro-zone investor with 1 billion euros to be invested in dollars (USD), yen (JPY), or pounds sterling (GBP). You are given the following market data and forecasts:

Currency	1-year interest rate	Exchange rate (per euro)	Exchange rate forecast in 1 year	Volatility forecast
USD	2.5%	\$1.30	\$1.40	10%
JPY	0.25%	¥130	¥120	8%
GBP	4.5%	£0.65	£0.65	6%

Correlation forecast	USD	JPY	GBP
USD	1	0.30	0.25
JPY	0.30	1	0.50
GBP	0.25	0.50	1

(a) Plot the three currencies on a risk-return chart, taking the interest produced by each currency into account.
(b) Draw the risk-return evolution of a portfolio which gradually switches from dollars to yen (i.e. 100% in dollars initially, then 90% in dollars and 10% in yen, etc.). Repeat this question for a portfolio which gradually switches from yen to pounds, and then from pounds to dollars.
(c) Plot the risk-return profiles of all possible portfolios made of the three currencies, considering only long investment positions in multiples of 5%.
(d) Which portfolio would you choose to obtain an expected return around 5.25%? Is this choice optimal?

Problem 7*: General portfolio optimization on 2 assets

Consider two assets A and B with returns R_A and R_B, volatilities σ_A and σ_B, and correlation ρ. Let P be a portfolio of A and B with weights w and $1 - w$ respectively.

(a) Can the sign of w be negative? Explain why or why not.
(b) Express the return R_P of the portfolio as a function of w, R_A and R_B.
(c) Express the risk σ_P of the portfolio as a function of w, σ_A, σ_B and ρ.
(d) Suppose $\rho = 1$. What is the shape of σ_P as a function of w? Is there an optimal value of w which minimizes σ_P? What about $\rho = -1$?
(e) Suppose $-1 < \rho < 1$.
  (i) What is the shape of σ_P^2 (the portfolio's *variance*) as a function of w?
  (ii) Show that the optimal value of w which minimizes σ_P is: $w^* = \dfrac{\sigma_B(\sigma_B - \sigma_A\rho)}{\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho}$.
  Find the value of w^* when $\sigma_A = \sigma_B$.
  (iii) Calculate the value of w^* in the case of Kroger Co. and Coast Value LP (see Section 4-3 p.41).
(f) Suppose A is the risk-free asset. Can you simplify the expression for σ_P from question (c)? Find the optimal portfolio of A and B which minimizes σ_P.

Problem 8*: Portfolio optimization on *n* correlated assets

Consider n assets all having the same volatility of 100% and pairwise correlation ρ. Let P be a portfolio with weights $w_1, w_2, \ldots, w_n$.

(a) Show that the variance of the portfolio can be rewritten as: $\sigma_P^2 = \rho + (1-\rho)\sum_{i=1}^{n} w_i^2$.
(b) Recall that $w_n = 1 - \sum_{j=1}^{n-1} w_j$. Using differentiation techniques, show that the minimum variance portfolio is equally weighted.
(c) Express the optimal portfolio variance as a function of ρ and n only, and find its limit as the number of assets goes to infinity.

Problem 9*: Portfolio optimization on *n* uncorrelated assets

Consider n uncorrelated assets with volatilities $\sigma_1, \sigma_2, \ldots, \sigma_n$. Let P be a portfolio with weights $w_1, w_2, \ldots, w_n$. Show that the minimum variance of the portfolio is $1/n^{\text{th}}$ of the harmonic average of individual variances, i.e.: $\sigma_P^{*2} = 1/\sum_{i=1}^{n} \frac{1}{\sigma_i^2}$. Is the optimal portfolio unique?

Assuming all volatilities are bounded from above by a constant K, what is the limit of σ_P^* as n goes to infinity?

An Introduction to Equity Derivative c uveory and ractice, Second domone
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Part II

FIRST STEPS IN EQUITY DERIVATIVES

5 Equity Derivatives

In this chapter we introduce equity derivative securities, which is the main subject matter of this book. These sophisticated securities, whose cash flows are tied to the prices of stocks or stock indexes, have become increasingly popular with investors since the 1970s. In 2010 their market size was over six trillion dollars according to the Bank for International Settlements.

5-1 Introduction

An **equity derivative security** is a financial security whose cash flows depend on the prices of one or more stocks or stock indexes called **underlying assets**. These cash flows are collectively known as the derivative's **payoff**.

Examples

1. Barack pledges to buy 1,000 shares of Kroger Co. from Michelle in a month's time at a pre-agreed price of \$25 per share (and Michelle also promises to sell the shares to Barack). If the underlying stock price rises from its current level of \$24 to e.g. \$27 after a month, Barack makes a good deal: he buys at \$25 something worth \$27 on the market. On the other hand, if the price of Kroger Co. remains below \$25 Barack is losing to Michelle. In Section 5-2.2 below we will see that, for this transaction to be fair, Michelle should give about \$958.80 in cash to Barack when entering the pledge.
2. Konstantin buys a certificate from his bank to receive the observed annualized volatility of the S&P 500 index over the next 3 months, converted at the rate of \$100 per percentage point. If volatility is 22%, the bank will pay Konstantin \$2,200.
3. Bill strikes a deal with Larry to have the option to swap Microsoft's annual return against Google's, converted at the rate of \$1mn per percentage point. If the return on Google after one year is 10%, while the return on Microsoft is 7%, Bill would exercise his option and receive a net cash flow of \$3mn from Larry. If instead Google is at 6% while Microsoft is at 12%, Bill would stay put.

For ease of analysis we focus on derivatives on a single underlying stock or index (Examples 1 and 2). Derivatives on multiple underlying assets (Example 3) are discussed in Chapter 12.

Formally, if D_t is the value of the derivative security at time t and S_t is the price of its underlying asset, we may often posit that there is a function f of time and the stock price such that:

$$D_t = f(t, S_t).$$

Example
Denoting $T = 1/12 \approx 0.0833$ the delivery date in Example 1, we have:

$$D_T = (S_T - 25) \times 1{,}000.$$

Thus if Kroger Co. trades at \$27 at $t = T$, the value of the contract at maturity is $D_T = \$2{,}000$. This is the profit that Barack would make by immediately reselling 1,000 shares of Kroger Co. on the market for \$27,000.

Finding an expression for $D_t = f(t, S_t)$ at an arbitrary point in time t is often difficult. Economic assumptions must be made, chief of which the absence of arbitrage opportunities. Fortunately the theory of derivatives valuation has been a very active and successful research area for the past forty years and its conclusions are widely accepted and implemented in practice. The aim of this book is to introduce its key results.

Equity derivatives are commonly split into two categories:

(1) **Forward and futures contracts**,[1] where two parties promise to exchange the underlying asset at a pre-agreed price and date. The contract on Kroger Co. (Example 1) falls in this category.
(2) **Options**, where one party has the right (but not the obligation) to exchange the underlying asset at a pre-agreed price and date with another party. This right is usually bought for a premium paid upfront to the option seller (comparable to a one-off insurance fee).

There are also more complex derivative securities called **exotic derivatives**, such as described in Examples 2 and 3, which we encounter in several problems and in Chapter 12.

While it is theoretically conceivable to create perpetual derivatives, in practice all derivatives have a final date called the **maturity date** past which they cease to exist.

5-2 Forward Contracts

The financial characteristics of a forward contract are:

- The **underlying asset** S: the stock or stock index to be exchanged in the future;
- A **maturity** or **delivery date** T: the date when the underlying is exchanged (or ‘delivered’);
- A **delivery price** or **strike**[2] K: the price to pay in exchange for the underlying asset.

In this book we denote ϕ_t the value at time t of a forward contract on one unit of underlying asset *from the buyer’s viewpoint*. The seller then lives in a symmetric world where the contract value is $-\phi_t$.

[1] Futures are traded on exchanges while forward contracts trade ‘over-the-counter’ (OTC). The margin call system of exchanges can result in a more or less significant price difference between these two types of derivatives, especially if long maturities are involved. However we will ignore these differences and use the terms ‘forward’ and ‘future’ interchangeably in this book.

[2] Historically ‘delivery price’ only applied to forward contracts while ‘strike’ only applied to options. The tendency is now to use ‘strike’ for either category.

5-2.1 Payoff

As illustrated in Figures 5-1 and 5-2 below, the **payoff** of a forward contract at maturity is:

- For the buyer: $\phi_T = S_T - K$;
- For the seller: $-\phi_T = K - S_T$.

This formula corresponds to the profit or loss at maturity T when the buyer receives one unit of underlying worth S_T and pays K to the seller. Note that this profit or loss is *latent* in the sense that to actually realize his gains or losses the buyer would need to immediately resell the underlying on the market.

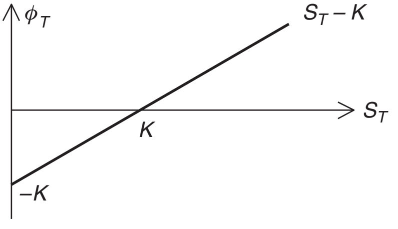


Figure 5-1 Payoff for the forward buyer at $t = T$

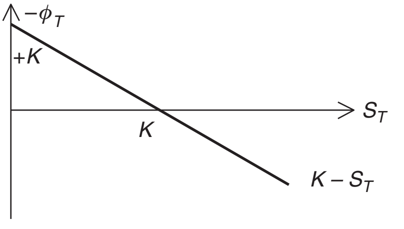


Figure 5-2 Payoff for the forward seller at $t = T$

5-2.2 Arbitrage Price

We now focus on the value of the forward contract at an arbitrary point in time $t \leq T$ and particularly on today's value ϕ_0. As mentioned earlier, equity derivatives valuation is often difficult and requires modeling the behavior of the underlying asset. However, forward contracts are a nice category of derivatives which require very few assumptions, namely:

- The underlying asset does not pay any dividend (this assumption is relaxed in Section 5-2.4 below);
- Investors may borrow and lend money over T years at interest rate r;
- There are no arbitrage opportunities, securities are infinitely liquid, and short-selling is feasible.

The arbitrage price at $t = 0$ of a forward contract is then:

$$\phi_0 = S_0 - \frac{K}{(1+r)^T}. \tag{5-1}$$

This formula is easily generalized to an arbitrary point in time $t \leq T$ (see Problem 5).

Example
The current price of Kroger Co. is \$24, the 1-month zero-coupon rate is 2%, and no dividend is scheduled within the next month. The arbitrage price of a forward contract to buy one share of Kroger Co. at \$25 in one month is:

$$\phi_0 = 24 - \frac{25}{(1+2\%)^{1/12}} \approx -\$0.9588.$$

This amount corresponds to the loss per share that Barack makes to Michelle when entering the pledge in Example 1. For 1,000 shares, Michelle should pay \$958.80 in cash to Barack for their transaction to be neutral.

The arbitrage argument establishing Equation (5-1) is as follows:

- Suppose $\phi_0 > S_0 - \frac{K}{(1+r)^T}$, i.e.: $\phi_0 + \frac{K}{(1+r)^T} > S_0$. One may then implement the arbitrage strategy below known as 'cash-and-carry':

Transaction	**Cash flow at $t = 0$**	**Cash flow at $t = T$**
Buy the underlying today on the 'cash market'[3] and carry it in inventory until the maturity date	$-S_0$	
Sell the forward contract today	$+\phi_0$	$-\phi_T = K - S_T$
Borrow the amount $\dfrac{K}{(1+r)^T}$ today at rate r until maturity	$+\dfrac{K}{(1+r)^T}$	$-\dfrac{K}{(1+r)^T} \times (1+r)^T = -K$
Sell the underlying at maturity		$+S_T$
Total	$\phi_0 + \dfrac{K}{(1+r)^T} - S_0 > 0$	0

Note that this strategy is fully covered because the arbitrageur possesses the underlying asset in inventory at time of delivery.

- Suppose $\phi_0 < S_0 - \frac{K}{(1+r)^T}$, i.e.: $\phi_0 + \frac{K}{(1+r)^T} < S_0$. Again one may implement an arbitrage strategy known as 'reverse cash-and-carry':

Transaction	**Cash flow at $t = 0$**	**Cash flow at $t = T$**
Sell the underlying today on the 'cash market'	$+S_0$	
Buy the forward contract today	$-\phi_0$	$\phi_T = S_T - K$
Lend the amount $\dfrac{K}{(1+r)^T}$ today at rate r until maturity	$-\dfrac{K}{(1+r)^T}$	$+\dfrac{K}{(1+r)^T} \times (1+r)^T = +K$
Buy the underlying at maturity		$-S_T$
Total	$S_0 - \phi_0 - \dfrac{K}{(1+r)^T} > 0$	0

Note that this strategy may only be implemented if either the arbitrageur already possesses the underlying asset in inventory at $t = 0$ (e.g. a shareholder), or short-selling is feasible.

[3] The cash market is the market where assets such as stocks are traded for immediate settlement against cash.

5-2.3 Forward Price

In practice, when entering a forward contract, the delivery price is set so that the contract has zero initial value, i.e.: $S_0 - \frac{K}{(1+r)^T} = 0$. Solving for K yields the so-called '**forward price**' of the underlying asset for maturity T:

$$F_0 = S_0(1+r)^T. \tag{5-2}$$

This quantity is also commonly denoted $F(0,T)$. It must not be confused with the current price S_0 of the underlying (also called the '**spot price**') nor the final spot price S_T of the underlying which is unknown until $t = T$.

Figure 5-3 below shows how the three prices relate through time. Observe how the forward price converges to the spot price on the delivery date.

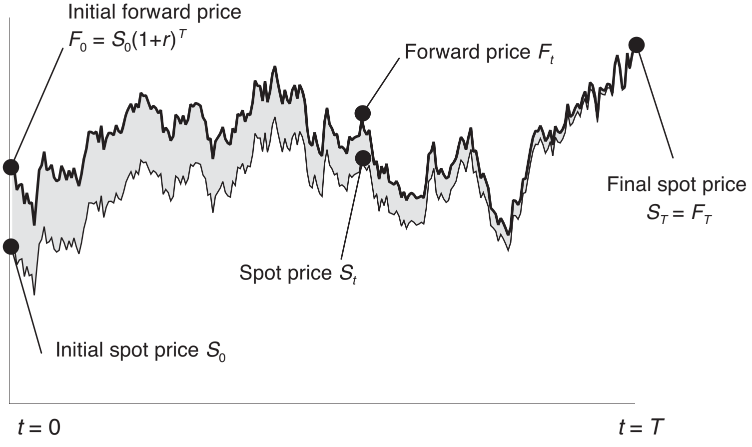


Figure 5-3 The forward price converges to the spot price on the delivery date

Example
Kroger Co. currently trades at \$24, the 1-month zero-coupon rate is 2%, and no dividend is scheduled within the next month. The 1-month forward price of Kroger Co. is:

$$F_0 = 24 \times (1 + 2\%)^{1/12} \approx \$24.0396.$$

In our Example 1, this price corresponds to Barack's implicit purchase price of Kroger Co. at maturity after being compensated with a \$958.80 lump sum from Michelle.

5-2.4 Impact of Dividends

When dividends are scheduled before the maturity date, the formulas given earlier (Equations (5-1) and (5-2)) are no longer valid. This is because dividends are a benefit to anyone who keeps the underlying asset in inventory. Assuming no arbitrage, this benefit must be reflected with a lower forward price.

For ease of exposure, we look at the impact of a single dividend payment in one of the following two forms:

- A fixed cash dividend D paid at time $0 \leq t_D \leq T$;
- A proportional dividend at rate d paid at time $0 \leq t_d \leq T$, corresponding to a variable cash amount of $d \times S_{t_d}$.

Note that the value of D or d is assumed to be known in advance, which is not entirely realistic, especially for long maturities. The study of multiple dividends is left to Problem 8.

5-2.4.1 Single Cash Dividend

A cash dividend D paid at time t_D being a fixed cash flow, we may calculate its present value and subtract it from the right-hand side of Equation (5-1):

$$\phi_0 = S_0 - \frac{D}{[1+z(t_D)]^{t_D}} - \frac{K}{[1+z(T)]^T} \tag{5-3}$$

where $z(t_D)$, $z(T)$ are the zero-coupon rates for the dividend and maturity dates, respectively.

Solving $\phi_0 = 0$ for K, we obtain the new forward price:

$$F_0 = \left(S_0 - \frac{D}{[1+z(t_D)]^{t_D}}\right)[1+z(T)]^T,$$

which is lower than the forward price without dividends (Equation (5-2)).

Example
The stock of ABC Inc. currently trades at \$100 and the annual cash dividend is \$4. The next dividend payment occurs in 3 months and the zero-coupon rate curve is flat at 2%. The price of a 1-year forward contract struck at \$100 on ABC Inc. is: $\phi_0 = 100 - \frac{4}{[1+2\%]^{0.25}} - \frac{100}{1+2\%} \approx -\$2.02$, and the forward price is: $F_0 = \left(100 - \frac{4}{[1+2\%]^{0.25}}\right) \times 1.02 \approx \$97.94$.

To properly establish Equation (5-3), we show how the 'cash-and-carry' argument is modified by the cash dividend (we leave the modification of the 'reverse cash-and-carry' argument to the reader). Suppose $\phi_0 > S_0 - \frac{D}{[1+z(t_D)]^{t_D}} - \frac{K}{[1+z(T)]^T}$, i.e.: $\phi_0 + \frac{D}{[1+z(t_D)]^{t_D}} + \frac{K}{[1+z(T)]^T} > S_0$. The arbitrage strategy is now:

Transaction	**Cash flow at $t = 0$**	**Cash flow at $t = t_D$**	**Cash flow at $t = T$**
Buy the underlying today on the 'cash market' and carry it in inventory until the maturity date, collecting the cash dividend D at time t_D.	$-S_0$	$+D$	
Sell the forward contract today.	$+\phi_0$		$-\phi_T = K - S_T$

Transaction	Cash flow at $t = 0$	Cash flow at $t = t_D$	Cash flow at $t = T$
Borrow the amount $\frac{K}{(1+z(T))^T}$ today at rate $z(T)$ until maturity.	$+\frac{K}{(1+z)(T)^T}$		$-K$
Borrow the amount $\frac{D}{(1+z(t_D))^{t_D}}$ today at rate $z(t_D)$ until time t_D.	$+\frac{D}{(1+z)(t_D)^{T_D}}$	$-D$	
Sell the underlying at maturity			$+S_T$
Total	$>$**0**	**0**	**0**

5-2.4.2 Single Proportional Dividend

A proportional dividend $d \times S_{t_d}$ paid at time t_d being a variable cash flow, the present value argument of Section 5-2.4.1 above no longer works. Instead, we may immediately reinvest the dividend into the underlying stock to purchase a quantity d. Therefore, it is enough to buy a smaller quantity $\frac{1}{1+d}$ of stock initially which will grow to exactly one share after the dividend reinvestment. The formula for the arbitrage price of the forward contract thus becomes:

$$\phi_0 = \frac{S_0}{1+d} - \frac{K}{(1+r)^T},$$

where r is the discount rate for maturity T. Solving $\phi_0 = 0$ for K yields the new forward price:

$$F_0 = \frac{S_0}{1+d}(1+r)^T, \tag{5-4}$$

which is lower than the forward price without dividends (Equation (5-2)).

Example
The stock of ABC Inc. currently trades at \$100 and the annual proportional dividend is 4%. The next dividend payment occurs in 3 months and the zero-coupon rate curve is flat at 2%. The price of a 1-year forward contract struck at \$100 on ABC Inc. is: $\phi_0 = \frac{100}{1+4\%} - \frac{100}{1+2\%} \approx -\$1.89$, and the forward price is: $F_0 = \frac{100}{1+4\%} \times (1+2\%) \approx \$98.08$.

5-3 'Plain Vanilla' Options

'Plain vanilla' options are derivative securities which confer the right but not the obligation to buy or sell the underlying asset S at a pre-agreed **strike price** K on or until a certain **maturity date** T. The typology is summarized below:

	Call: right to buy	**Put: right to sell**
European: at time T	Right to buy S at time T	Right to sell S at time T
American: until time T	Right to buy S until time T	Right to sell S until time T

These options are called 'plain vanilla' in contrast to **'exotic' options** which are more complex (see problems and Chapter 12). Sometimes the strike price is called the **exercise price**, and the maturity date is called the **exercise date or** the **expiry date**.

In this book, we denote c_t, C_t^{US} the value at time t of a European and American call, respectively; and p_t, P_t^{US} the value at time t of a European and American put, respectively. As with forward contracts, option values are expressed for one unit of underlying asset and from the option buyer's viewpoint.

5-3.1 Payoff

Options may be viewed as one-sided forward contracts in the sense that "the option owner can only win." Consider for example a 1-month European call on Kroger Co. struck at \$25. Let S_T be the spot price of Kroger Co. in one month. At this date the owner of the call must decide whether to exercise his right to buy the asset:

- If he exercises the option, the P&L is $S_T - 25$ (>0 for a profit, <0 for a loss);
- If he does not exercise, the P&L is zero.

Clearly a rational individual will only exercise the option if it is profitable to do so, i.e. if Kroger Co. trades above \$25 at maturity. The option's payoff is thus:

$$c_T = \max(0, S_T - 25).$$

Figures 5-4 and 5-5 below represent the general payoff of European calls and puts:

- For the call: $\max(0, S_T - K)$;
- For the put: $\max(0, K - S_T)$.

Note that the payoffs of corresponding American options are the same, with the enhancement that the owner may also exercise at any time τ before maturity T. In general, the exercise date τ cannot be known in advance and its optimal determination depends on market conditions.[4]

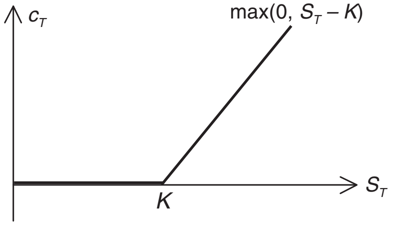


Figure 5-4 Payoff of the European call

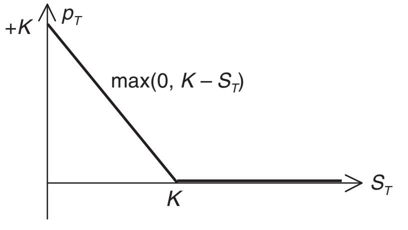


Figure 5-5 Payoff of the European put

[4] However, it can be shown that it is never optimal to exercise an American call on a non-dividend-paying stock before the maturity date.

5-3.2 Option Value

Option valuation is considerably more complex than the pricing of forward contracts; we discuss it in the following chapters. Assuming no arbitrage, infinite liquidity, and the ability to short-sell, we may however state some elementary properties (see Problem 6):

- All option values must be nonnegative: $c_t, C_t^{US}, p_t, P_t^{US} \geq 0$;
- An American option is worth at least as much as its European counterpart: $C_t^{US} \geq c_t$, $P_t^{US} \geq p_t$;
- The value of an American call cannot exceed the price of the underlying asset: $C_t^{US} \leq S_t$.

In practice, it is common to split an option value into two parts:

- **Intrinsic value**: how much the option would pay off if it were exercised immediately;
- **Time value**: how much the option is worth on top of its intrinsic value.

Figure 5-6 below shows the evolution of the time value. We can see that, despite some ups and downs, time value progressively vanishes as the maturity date approaches, at which point the option value converges to the intrinsic value.

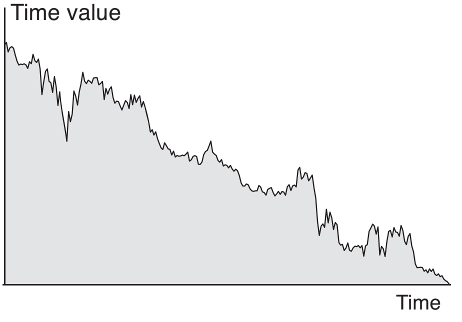


Figure 5-6 Time value decays through time

Additionally, practitioners have adopted the following terminology to characterize equity options:

- **'In-the-money' (ITM)**: the option has positive intrinsic value (i.e. $S_t > K$ for a call, $S_t < K$ for a put);
- **'At-the-money' (ATM)**: the stock price is at the strike price (i.e. $S_t = K$);
- **'Out-of-the-money' (OTM)**: the option is neither in- nor at-the-money (i.e. $S_t < K$ for a call, $S_t > K$ for a put).

Note that in other asset classes such as currencies, different conventions may exist.

5-3.3 Put-Call Parity

An option portfolio which is long one call and short one put with identical characteristics (underlying S, strike K, maturity T) has exactly the same payoff as a forward contract, as shown in Figure 5-7 overleaf:

$$c_T - p_T = \phi_T = S_T - K.$$

In other words: "call minus put equals forward." This means that we may switch between call and put positions by buying or selling the forward contract ("call = put + forward," "put = call – forward").

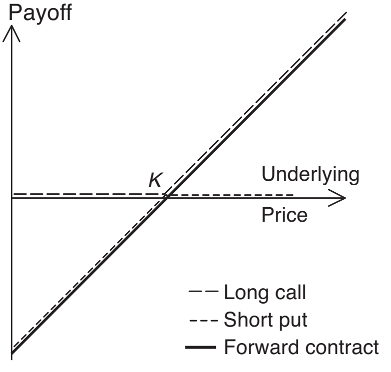


Figure 5-7 Put-call parity: long call and short put is the same as long forward contract

Assuming no arbitrage, infinite liquidity and the ability to short-sell, we obtain the **put-call parity** equation:

$$c_0 - p_0 = \phi_0.$$

This results from the law of one price (Section 3-1.4 p.21) applied to the option portfolio and the forward contract.

If there are no dividends we may substitute Equation (5-1) p.51 and write:

$$c_0 - p_0 = S_0 - \frac{K}{(1+r)^T}. \tag{5-5}$$

Note that **put-call parity does not hold for American options**.

5-3.4 Option Strategies

Equity options have become popular securities to make directional bets, hedge risks, and boost returns. They may also be combined to make non-directional bets. We now review some common uses and strategies.

5-3.4.1 Leverage

Suppose Kroger Co. currently trades at \$24, and that 1-month European calls on Kroger struck at \$25 trade at \$0.50. Instead of allocating \$100,000 of capital to buy 4,166.67 shares of Kroger, a bullish investor could spend only \$10,000 to buy 20,000 calls. Figure 5-8 overleaf compares the Net P&L of both strategies. We can see that the calls outperform the stocks whenever the stock price falls below ≈\$21 or rises above ≈\$26.

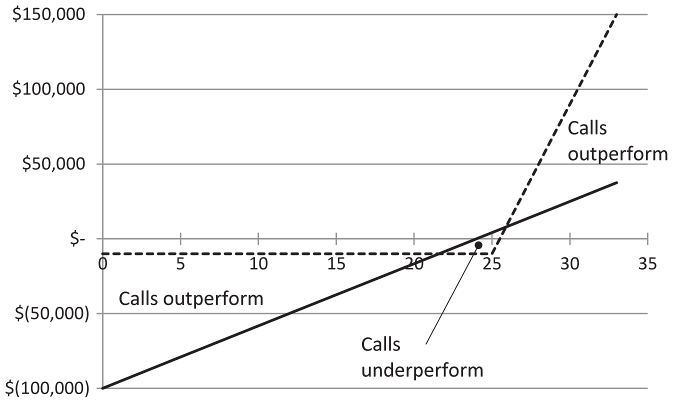


Figure 5-8 Net P&L of buying stocks vs. calls

5-3.4.2 Covered Call

A covered call is a short out-of-the-money call position covered by a long stock position. By put-call parity this strategy is equivalent to a short put combined with a zero-coupon bond. Figure 5-9 below shows the payoff.

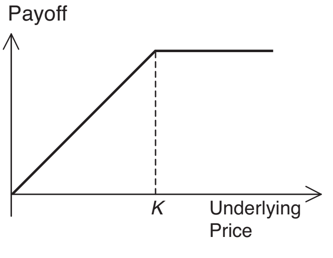


Figure 5-9 Covered call payoff

For example, a moderately bullish investor in Kroger Co.'s stock who does not believe that the price will go above $25 in a month can pick up an extra $0.50 per share by selling a call struck at $25. Figure 5-10 below compares the Net P&L of this covered call strategy vs. being long the stock.

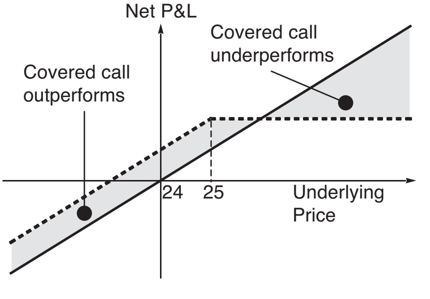


Figure 5-10 Net P&L of long stock vs. covered call strategy (not drawn to scale)

Covered call strategies are often executed on a recurring basis in an attempt to outperform a static investment in the stock in the long run. Figure 5-11 below compares the 2000–11 evolution of the "S&P 500 Buy-Write Strategy Index" (BXM) published by CBOE vs. the S&P 500 index itself (dividends included). We can see that the covered call strategy outperformed the underlying index during this period.

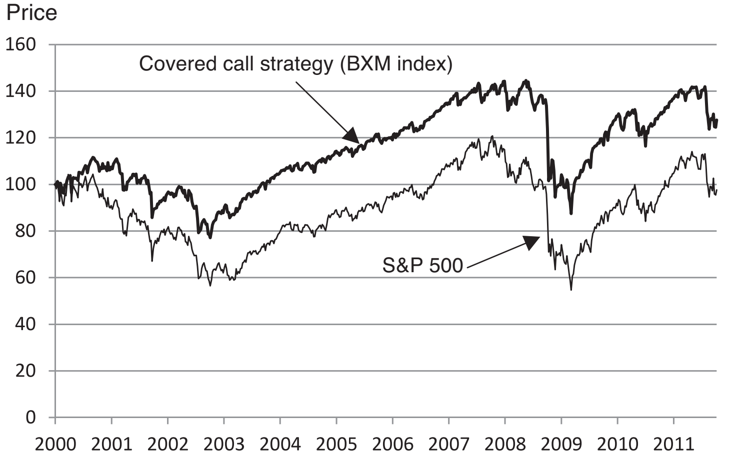


Figure 5-11 Covered call strategy on the S&P 500 versus the index starting at base 100 (2000–11) (Source: Bloomberg)

5-3.4.3 Straddle

A straddle is a call and a put with same strike and maturity. An investor who does not have a directional view on a stock price but thinks that it will move away from its current level may want to buy an at-the-money straddle. If instead he thinks the stock price will stay around its current level he may want to sell an at-the-money straddle. Figures 5-12 and 5-13 below show the Net P&L of both strategies.

Straddles are simple bets on the volatility of a stock: the more volatile a stock turns out to be, the more likely its spot price will move away from its initial level and the more profitable the straddle.

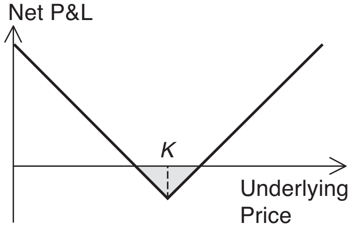


Figure 5-12 Net P&L of a long straddle

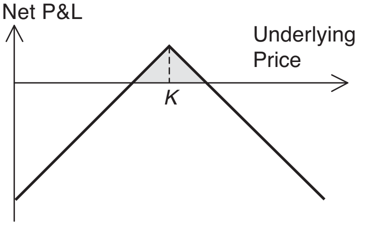


Figure 5-13 Net P&L of a short straddle

5-3.4.4 Butterfly

A butterfly is a "straddle with wings," i.e. long a call struck at K_1, short 2 calls struck at K_2 and long a call struck at K_3. Typically the strikes are chosen to obtain symmetric "wings," as illustrated in Figure 5-14 below. An investor who has the view that the underlying stock price will end up within an interval may want to buy a butterfly.

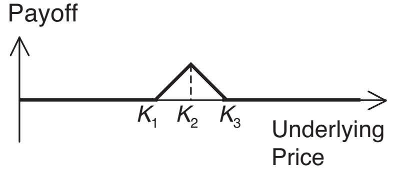


Figure 5-14 Payoff of butterfly strategy

5-4 Further Reading

John C. Hull (2009) *Options, Futures and Other Derivatives* 7th Edition, Prentice Hall: Chapters 5, 9, and 10.

5-5 Problems

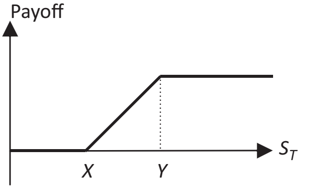
Problem 1: True or False? *The three questions are independent.*
(a) "The value of a derivative security is always positive."
(b) "With options you can never lose any money."
(c) "Assuming no dividends, the time value of a European in-the-money call exceeds the time value of the corresponding European out-of-the-money put with same strike and maturity."

Problem 2: Forward price and price of the forward contract
Microsona Corp.'s stock currently trades at ¥5,200 in Tokyo. The 1-year risk-free interest rate is 1% p.a. Calculate the 1-year forward price of Microsona Corp. and the price of a 6-month forward contract on Microsona Corp. struck at ¥5,000 in each of the following three scenarios:

(a) Microsona Corp. does not pay any dividend;
(b) Microsona Corp. will pay a cash dividend of ¥104 in 9 months, and the 9-month interest rate is 0.8% p.a.;
(c) Microsona Corp. will pay a proportional dividend of 2% in 3 months.

Problem 3: Option payoffs. *The three questions are independent.*
(a) In each of the three examples in Section 5-1 p.49 identify the underlying assets, the maturity date, and the payoff formula.

(b) Find a portfolio of European options on an underlying asset S with maturity T whose payoff matches the figure below:



(c) What is the payoff at maturity of a portfolio long a zero-coupon bond with face value \$90 and an in-the-money European call option struck at \$90 on an underlying stock S currently trading at \$100? Assuming no dividends, no arbitrage, infinite liquidity, and the ability to short-sell, show that this portfolio must be worth more than \$100.

Problem 4: Option strategies. *The two questions are independent.*

(a) Consider a 3-month at-the-money straddle on Kroger Co.'s stock which currently trades at \$24. The cost of the straddle is \$1.92 and the 3-month interest rate is 3%. At what stock price levels does the Net P&L break even? Can you estimate the corresponding annualized volatility?

(b) You are long an exotic option which will cause a loss of ¥100mn if the Nikkei 225 index trades anywhere below the 6,750 level in one year. The current spot price of the Nikkei is 8,750 and the table below gives some 1-year European option prices on the Nikkei. Find the cheapest option strategy to cover your potential loss.

Strike	Put price (¥)	Call price (¥)
6500	251	2,444
6750	261	2,205
7000	270	1,965

Problem 5: Forward price and price of the forward contract at an arbitrary time *t*

Generalize the formulas for the forward price and the arbitrage price of the forward contract (Equations (5-1) p.51 and (5-2) p.53) at an arbitrary point in time $0 \leq t \leq T$.

Problem 6: Arbitrage arguments

Establish the following relationships at an arbitrary point in time $t \leq T$. Assume no arbitrage, infinite liquidity, and the ability to short-sell. *All option characteristics are the same (underlying S, maturity T, strike K).*

(a) $0 \leq c_t \leq C_t^{\mathrm{US}}$;
(b) $\max(0, S_t - K) \leq C_t^{\mathrm{US}} \leq S_t$;
(c) $c_t \geq \phi_t$.

Problem 7: The S&P 500 index is worth \$1,200, the price of a 1-year at-the-money American call on the S&P 500 is \$144, the interest rate curve is flat at 1% per annum, and the annual dividend rate is 2%. What is the maximum price you would pay for a 1-year at-the-money European put on the S&P 500?

Problem 8: Successive dividends
Find formulas for the price ϕ_0 of the forward contract when:

(a) Two cash dividends D_1, D_2 are paid out at times $0 < t_1 < t_2 \leq T$;
(b) Two proportional dividends at rates d_1, d_2 are paid out at times $0 < t_1 < t_2 \leq T$;
(c) A cash dividend D and then a proportional dividend at rate d are successively paid out at times $0 < t_1 < t_2 \leq T$.

Problem 9: Barrier option
A 'knock-out barrier option' is a call or put option which may only be exercised at maturity if the price of the underlying never hits a pre-agreed barrier price H throughout the life of the option. Symmetrically, a 'knock-in barrier option' may only be exercised at maturity if the price of the underlying hits the barrier price H.

(a) Do you think barrier options should be:
  (i) more expensive than plain vanilla options of same characteristics?
  (ii) less expensive than plain vanilla options of same characteristics?
  (iii) more expensive in some cases and less expensive in other cases? (Please specify.)
(b) In this question we consider calls with strike 100 and 1-month maturity. The underlying spot price is 90.
  (i) What is the value of a knock-out call with barrier $H = 95$? What about a knock-in call with barrier $H = 95$?
  (ii) What qualitative difference do you see between a knock-out call with barrier $H = 80$ and a knock-out call with barrier $H = 110$?
  (iii) Suppose that the value of the plain vanilla call is 2 and the value of a knock-out call with barrier $H = 80$ is 1. Can you find the value of another barrier option?

Problem 10: Forward exchange rate. *This problem is about forward contracts in foreign exchange and goes beyond the scope of equity derivatives.*

The spot exchange rate of the euro is S dollars, i.e. to buy one euro one must pay S dollars. The euro zone yield curve is flat at r_{EU} while the American yield curve is flat at r_{US}. A forward contract on the euro-dollar is an agreement to receive euros and pay dollars at a pre-agreed date T and exchange rate F.

(a) Starting with €1, find two ways to have dollars in a year's time by investing or borrowing in either currency and exchanging between currencies through the spot and forward markets. Using an arbitrage argument, establish that the 1-year forward exchange rate of the euro must be:
$$F = S\frac{1 + r_{US}}{1 + r_{EU}}.$$
Can you make a parallel with a relevant forward price formula for a stock or equity index?
(b) Assume $r_{EU} = 2\%$, $r_{US} = 4\%$, $S = \$1.30$ for €1. What is the 1-year forward exchange rate? Can you also find the 2-year forward exchange rate?
(c) Interpret your results in terms of appreciation or depreciation of the *dollar*. Find at least one supporting argument for your interpretation and one opposing argument. Assuming both arguments weigh equally on the evolution of the euro-dollar exchange rate through time, what would be your best guess of the exchange rate in a year's time? Can you find a way to make money if this forecast proves to be accurate? Is this an arbitrage?

Problem 11: Forward interest rate. *This problem is about forward interest rates and goes beyond the scope of equity derivatives.*

Consider a market with no arbitrage and infinite liquidity where investors can lend and borrow money for any maturity T at the zero-coupon rate $z(T)$. Let $z(T, \tau)$ denote the *forward rate* agreed today on a loan beginning at time $t = T$ and ending at time $t = T + \tau$. For instance, the '6-month × 1-year' forward rate $z(0.5, 1)$ is the rate agreed today on a loan starting in six months and ending in eighteen months.

(a) In this question the zero-coupon rate curve is given as: $z(T) = 5\% + T \times 0.5\%$.
  (i) Draw $z(T)$ as a function of maturity T.
  (ii) Suppose that $z(1, 2) = z(3)$. Determine two ways to invest or borrow money over 3 years and show that there is an arbitrage opportunity.
  (iii) Calculate the rate $z(1, 2)$ which eliminates the arbitrage and generally $z(1, \tau)$ for any borrowing or lending period τ. Draw $z(1, \tau)$ as a function of τ.

(b) In this question the zero-coupon rate curve $z(T) \geq 0$ is arbitrary.
  (i) Show that the forward rate $z(T, \tau)$ must satisfy the equation:

$$[1 + z(T, \tau)]^{\tau} = \frac{[1 + z(T + \tau)]^{T+\tau}}{[1 + z(T)]^{T}}.$$

  (ii) Verify your results in question (a)(iii).
  (iii) Find an upper bound for $z(T)$ as a function of T, τ and $z(T + \tau)$. *Hint: Recall that a negative interest rate leads to an arbitrage.*

An Introduction to Equity Derivative cuveory and ractice, Second dcoone
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6 The Binomial Model

As we mentioned in Chapter 5 derivatives valuation is usually difficult and depends on several economic assumptions. While forwards and futures may be priced with few of these, options need to model the behavior of the underlying asset price.

In this chapter, we introduce the most straightforward and didactic approach to valuing an option: the binomial model proposed by Cox, Ross, and Rubinstein in 1979.

Notwithstanding its apparent simplicity, this model leads to a fairly robust valuation algorithm which was very popular on option trading floors in the 1980s and 1990s. Although it has progressively been replaced by more sophisticated approaches it is worth studying as a beginner's version of the Black-Scholes model presented in Chapter 10.

6-1 One-Step Binomial Model

In the binomial model, the future underlying stock price can only take two possible values: an 'up' value and a 'down' value. We begin our exposition with an example.

6-1.1 An Example

Suppose that Kroger Co.'s stock trades at \$24, that no dividend is scheduled, and that analysts expect it to either go up \$2 or down \$1 in a month's time, as shown in Figure 6-1 below where $T = 1/12$.

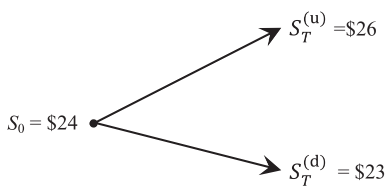


Figure 6-1 One-step binomial tree for Kroger Co.'s stock price

Now consider a 1-month European call option on Kroger Co. struck at \$25. Following the binomial model, the option payoff can only take two values:

- In the 'up' scenario, the call pays off: $c_T^{(u)} = 26 - 25 = \$1$;
- In the 'down' scenario, the call is worth zero: $c_T^{(d)} = 0$.

At time $t = 0$, the owner of the call option may either immediately sell the call on the market and cash in its value c_0, or take the risk to wait until maturity $t = T$ and cash in a random profit. This is summarized in Figure 6-2 overleaf.

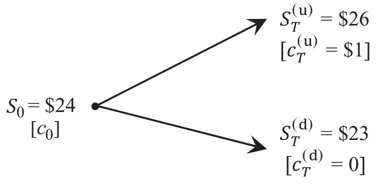


Figure 6-2 Stock price and call value at each node of the binomial tree

It turns out that there is a strategy to eliminate the risk of waiting until maturity: If the owner of the call sells a quantity Δ of stock today, the value of his portfolio (call and underlying) at each node of the tree is given in Figure 6-3 below. By choosing Δ such that the two portfolio values at maturity T are the same, the owner of the call eliminates the risk.

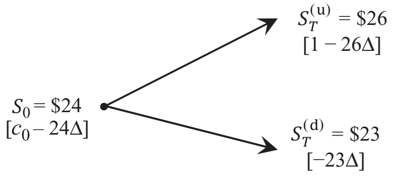


Figure 6-3 Value of a portfolio long one call and short Δ shares of Kroger Co.

Here we must choose Δ such that $1 - 26\Delta = -23\Delta$, i.e. $\Delta = 1/3$. The "long call, short 1/3 stock" portfolio is now always worth \$–7.67 at maturity[1] in both 'up' and 'down' scenarios.

Because the portfolio is now risk-free, its return between $t = 0$ and $t = T$ must equal the risk-free rate r, under penalty of arbitrage. Equivalently, we may treat the portfolio value at maturity as a fixed cash flow and calculate its present value:

$$\mathrm{PV} = -\frac{7.67}{(1+r)^T}.$$

But the price of the portfolio at time $t = 0$ is also:

$$c_0 - 24\Delta = c_0 - 24/3 = c_0 - 8.$$

Equating the two results yields **the value of the call at time $t = 0$**:

$$c_0 = 8 - \frac{7.67}{(1+r)^T}.$$

With a 1% risk-free rate and $T = 1/12$ we obtain: $c_0 \approx \$0.34$.

[1] Recall that a negative portfolio value does not correspond to a loss, but rather to the amount to pay to get rid of the portfolio (see Section 4-2.1 p.38).

6-1.2 General Formulas

Generally, let D_t be the value of a derivative security at time t on an underlying asset with spot price S_t. In the one-step binomial model over the time period $[0, T]$ the initial price S_0 of the underlying is known and the final price S_T may only take one of the two values $S_T^{(u)}$ and $S_T^{(d)}$. Figure 6-4 below shows the underlying price and the derivative's value at each node of the tree.

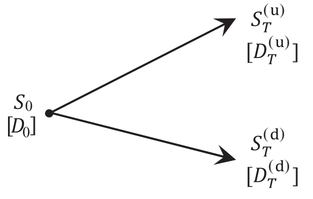


Figure 6-4 Spot price and derivative's value at each node of the binomial tree

If we build a portfolio which is long the derivative and short a quantity Δ of the underlying, its value at any point in time t is $P_t = D_t - \Delta S_t$. Choosing Δ such that $P_T^{(u)} = P_T^{(d)}$ we get a risk-free portfolio whose value at maturity T does not depend[2] on S_T. Thus, P_T is a fixed cash flow with present value $P_0 = \frac{P_T}{(1+r)^T}$, where r is the risk-free rate for maturity T. Substituting $P_0 = D_0 - \Delta S_0$ and rearranging terms yields the current value D_0 of the derivative:

$$\begin{cases} D_0 = \Delta \times S_0 + \dfrac{D_T^{(\bullet)} - \Delta \times S_T^{(\bullet)}}{(1+r)^T} \\ \Delta = \dfrac{D_T^{(u)} - D_T^{(d)}}{S_T^{(u)} - S_T^{(d)}} \end{cases} \tag{6-1}$$

where $S_T^{(\bullet)}$ and $D_T^{(\bullet)}$ may simultaneously be chosen in either the 'up' or 'down' scenario.

Note that Δ is the ratio of amplitude between the two possible final values for the derivative and the two possible final prices for the underlying asset: $\Delta = \frac{\delta D_T}{\delta S_T}$.

6-2 Multi-Step Binomial Trees

The binomial model is easily generalized to multiple steps. We illustrate how this is done along our example on Kroger Co. The current spot price is still at \$24 and the tree in Figure 6-5 overleaf shows the analysts' expectations of the stock price evolution over the next two months. Additionally, analysts believe that the risk-free rate will remain at 1% p.a. over each period.

With this information in hand, we may calculate the value p_0 of a 2-month European put on Kroger Co. struck at \$26. To do so we simply iterate the binomial model going backwards:

[2] Precisely, the value at maturity of the portfolio does not depend on the realization of either outcome $\{S_T = S_T^{(u)}\}$ or $\{S_T = S_T^{(d)}\}$.

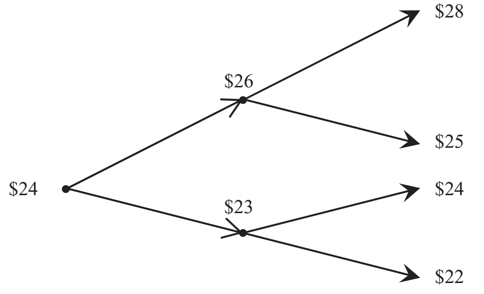


Figure 6-5 Two-step binomial tree for Kroger Co.'s stock price

- **Step t = 2 months**: The put value is the payoff $\max(0, K - S_T) = \max(0, 26 - S_{2/12})$. Figure 6-6 below shows this value at each terminal node.

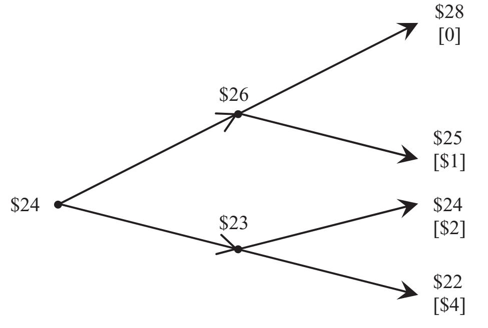


Figure 6-6 Put value at step $t = 2$ months

- **Step t = 1 month**: We apply the one-step model twice to find the put value at each of the middle 'up' and 'down' nodes. The results are shown in Figure 6-7 below.

  'Up' node: $\Delta^{(u)} = \dfrac{0-1}{28-25} = -\dfrac{1}{3}$ and $p_1^{(u)} = -\dfrac{1}{3} \times 26 + \dfrac{0-(-\frac{1}{3}) \times 28}{(1+1\%)^{1/12}} = \$0.659;$

  'Down' node : $\Delta^{(d)} = \dfrac{2-4}{24-22} = -1$ and $p_1^{(d)} = -23 + \dfrac{2-(-1) \times 24}{(1+1\%)^{1/12}} = \$2.978.$

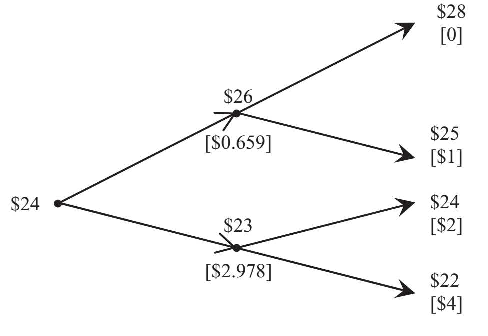


Figure 6-7 Put value at steps $t = 1$ month and $t = 2$ months

- **Step $t = 0$**: We use the one-step model one more time to get:

$$\Delta = \frac{0.659 - 2.978}{26 - 24} = -0.773$$
$$p_0 = -0.773 \times 24 + \frac{0.659 - (-0.773) \times 26}{(1 + 1\%)^{1/12}} = \$2.188.$$

Thus, the value of a 2-month European put on Kroger Co. struck at \$26 is \$2.188. Using put-call parity (Equation (5-5) p.58) we deduce the value of the European call with identical characteristics (underlying, strike, maturity):

$$c_0 = p_0 + \phi_0 = p_0 + S_0 - \frac{K}{(1+r)^T} = 2.188 + 24 - \frac{26}{(1 + 1\%)^{2/12}} = \$0.232.$$

6-3 Binomial Valuation Algorithm

While the one-step binomial model is clearly not realistic, we may split a given time period $[0, T]$ into a large number of smaller steps to create a tree which spans a more plausible range of stock prices over time. In general, if n is the number of steps, then the number of terminal nodes is 2^n which grows exponentially fast. Figure 6-8 below shows what a 5-step tree for the evolution of Kroger Co. over about two weeks looks like using up-moves of $+\$0.50$ and down-moves of $-\$0.25$ every three days.

The output option value depends on the magnitude of up- and down-moves, which raises the practical issue of how to pick these numbers in the first place. Additionally, there is no a priori requirement that up- and down-moves should be the same at every step, which leaves yet more freedom for parameterization.

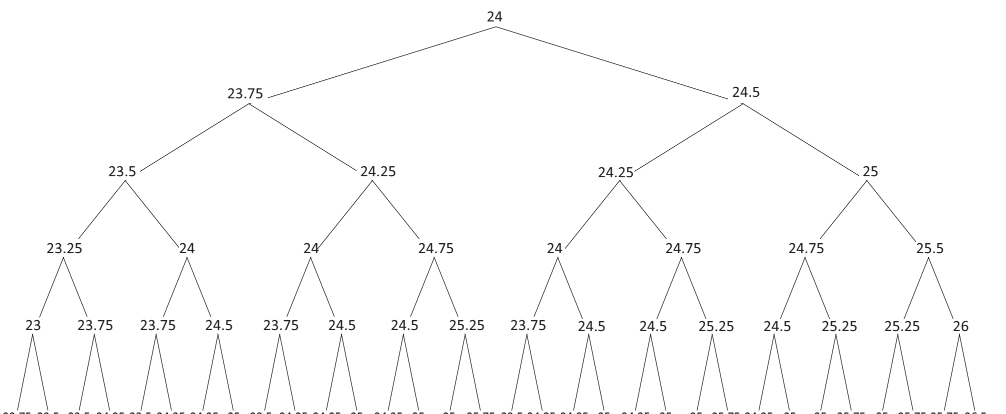


Figure 6-8 Five-step binomial tree for Kroger Co.'s stock price

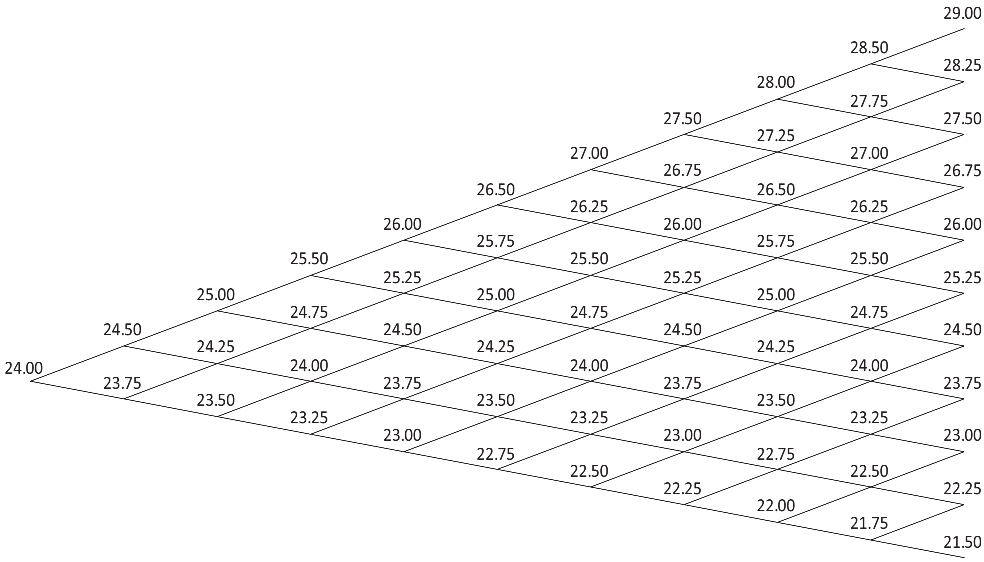


Figure 6-9 Ten-step recombining binomial tree for Kroger Co.'s stock price

We will not venture into these topics, mostly because more efficient algorithms are now in use. However, as mentioned in this chapter's introduction, the binomial model remains of didactic interest because of its simplicity and ease of implementation.

It is worth noting that a common trick to speed computations up is to choose up- and down-moves so as to obtain a **recombining tree** at every node, as illustrated in Figure 6-9 above. In this fashion, the number of terminal nodes is $n + 1$ instead of 2^n. This trick works perfectly well on vanilla options and many other derivatives, but breaks down on more complex options such as barrier options[3] with path-dependent payoffs (see Problem 7).

6-4 Further Reading

- John C. Hull (2009) *Options, Futures and Other Derivatives* 7th Edition, Prentice Hall: Chapter 11.
- Steven S. Shreve (2005) *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model*, Springer: Chapter 1.
- John C. Cox, Stephen A. Ross and Mark Rubinstein (1979) Option pricing: A simplified approach, *Journal of Financial Economics*, 7, 229–263.

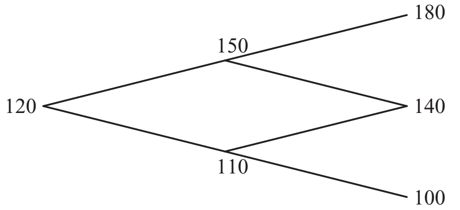
6-5 Problems

Problem 1: Two-step binomial tree

The stock of Schultz AG currently trades at €120 in Frankfurt. Below is a tree of your analysts' expectations for the evolution of the stock price over the next two years. Your analysts also

[3] A barrier option is a type of exotic option which additionally requires the underlying asset to hit, or never hit, a pre-agreed barrier level. For example, a 1-month barrier call option on Kroger Co. struck at \$25 with a \$20 'knock-out' barrier will pay off max(0, S_T – 25) at maturity only if the spot price S_t trades above \$20 at all times $0 \leq t \leq T$ until maturity.

predict that no dividend will be paid out, and that the risk-free rate will remain stable at 5% p.a.



(a) Calculate the value of a 2-year European call on Schultz AG struck at €130.
(b) Calculate the value of a 2-year European put on Schultz AG struck at €130.

Problem 2: 'At-the-money-forward' options

The stock of ABC Inc. currently trades at \$100 and does not pay any dividend. Analysts predict that the price may rise or fall by 10% every 6 months and that the risk-free rate will remain at 5% per annum for all maturities.

(a) Sketch a binomial tree for the evolution of the stock price over the next year.
(b) Calculate the 1-year forward price F of ABC Inc.'s stock.
(c) Calculate the value of a 1-year European call on ABC Inc. with strike F. What about a European put with identical characteristics (underlying, strike, maturity)?

Problem 3*: Binomial model and forward contracts

Consider the one-step binomial model.

(a) Show that the binomial model is consistent with the arbitrage price formula for forward contracts. In other words, apply the model to a forward contract with strike K and maturity T on a non-dividend-paying underlying stock and verify that your result agrees with Equation (5-1) p.51.
(b) Show that if $K > S_T^{(u)} > S_T^{(d)}$, the value of a put is equal to the arbitrage price of a short forward contract with same characteristics (underlying, strike, maturity). What happens if $K < S_T^{(d)} < S_T^{(u)}$?

Problem 4*: Binomial model and dividends

Consider a one-step binomial model on an underlying stock S which pays a cash dividend D on the maturity date T. Assume that the values $S_T^{(u)}$, $S_T^{(d)}$ already take into account any predictable stock price reduction caused by the dividend.

(a) What is the new final value at time T of a portfolio long one unit of derivative and short a quantity Δ of S?
(b) Find the quantity Δ such that the portfolio's final value is constant.
(c) Show that the current value of the derivative must be:

$$D_0 = \Delta \times S_0 + \frac{D_T^{(\bullet)} - \Delta \times (S_T^{(\bullet)} + D)}{(1+r)^T}.$$

(d) Calculate D_0 for the 1-month European call on Kroger Co. discussed in Section 6-1.1 p.65 when $D = \$0.20$.

Problem 5: American option

Compute the value of a 2-year *American* put with strike \$50 using a two-step binomial tree. The underlying stock pays no dividends, its current \$50 price may rise or fall by 20% each year, and the risk-free rate is constant at 5% in each annual period. *Hint: American puts can be exercised at each node of the tree with instant payoff $K-S_t$. A rational investor will thus choose to exercise early whenever the unexercised call is worth less.*

Problem 6: Binomial pricing algorithm

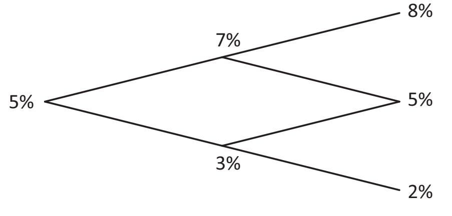
In a spreadsheet, build a 12-step binomial tree to calculate the value of a European call with a strike price of 100 and a maturity of one year. Assume that the 1-month interest rate is constant at 10% p.a., that the underlying stock pays no dividend and has a spot price of 100 which may go up or down by 2% every month. *Hint: Use a recombining tree.*

Problem 7: Barrier option. *This is a continuation of Chapter 5, Problem 9 p.63.*

With a 3-step binomial tree calculate the value of a knock-out call with barrier $H = 80$. Assume that the price of the underlying may increase or decrease by 15 at each step, that there are no dividends, and that the interest rate is zero.

Problem 8: Call on a bond. *This problem is about the valuation of a call on a bond and goes beyond the scope of equity derivatives.*

Consider a world where, at any given point in time t, the interest rate curve is flat at rate r_t which may vary. Currently, $r_0 = 5\%$ for all maturities. Below is a tree of various scenarios made by analysts for the evolution of r_t over the next two years.



(a) Calculate the price of a 5-year zero-coupon bond at each node of the tree. *Keep in mind that the maturity of the zero-coupon shortens at each step.*
(b) Using a two-step binomial model, calculate the value of a 2-year European call struck at 90 on this bond.

Problem 9*: Risk-neutral probability

Consider the one-step binomial model over the period $[0, T]$. Let $\omega^{(u)}$ denote the 'up' scenario and $\omega^{(d)}$ the 'down' scenario with respective probabilities $p^{(u)}$ and $p^{(d)} = 1-p^{(u)}$. The underlying asset S is worth S_t at any point in time t and does not pay any dividend. Let D_t be the value of a derivative on S at time t, r the annual risk-free rate and denote $r^{[T]} = (1+r)^T - 1$ the compound interest rate over the period $[0, T]$.

Assume that the final price of the underlying is:

- $S_T^{(u)} = S_0(1+u)$ in the 'up' scenario;
- $S_T^{(d)} = S_0(1+d)$ in the 'down' scenario,

where u and d are parameters satisfying: $u > r^{[T]} > d > -1$.

(a) In this question $S_0 = 100$, $T = 1$, $r = 5\%$, $u = 6\%$, $d = -4\%$. Calculate the value of a European call struck at 100. Does your result depend on the probabilities $p^{(u)}$ and $p^{(d)}$?

(b) In general, show that the value of the derivative at time $t = 0$ can be written:

$$D_0 = \frac{1}{(1+r)^T}\left[pD_T^{(u)} + (1-p)D_T^{(d)}\right],$$

where p is a function of $r^{[T]}$, u and d.

(c) Verify that $0 < p < 1$.

(d) Let $p^{(u)} = p$.
   (i) Verify that D_0 is equal to the expected present value of the payoff D_T.
   (ii) Find the expected gross rate of return on S over $[0, T]$. Why do you think p is called the 'risk-neutral probability' of S going up?

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7 The Lognormal Model

The binomial model presented in Chapter 6 envisages only two outcomes for the future price of the underlying asset – an unrealistic limitation. To make the model more convincing we must iterate it over a large number of steps.

In this chapter we develop a different approach to derivatives valuation based on probabilities. This approach allows for an infinite number of future underlying prices alongside a lognormal distribution. The resulting "lognormal model" aggregates several ideas borrowed from actuarial science or financial economics over the years, and as such it cannot be linked to a single author or group of authors.

7-1 Fair Value

Consider a derivative security on an underlying asset S which may or may not pay dividends. Assume no arbitrage, infinite liquidity, the ability to short-sell, and a payoff amount $D_T = f(T, S_T)$ on the maturity date T, where S_T is the price of S at time T.

Additionally, suppose that the final underlying price S_T follows a certain probability distribution (uniform, normal, lognormal, etc.). The current **fair value** D_0 of the derivative may now be defined as the expected discounted payoff:

$$D_0 = \mathbb{E}\left(\frac{f(T, S_T)}{(1+r)^T}\right),$$

where r is the annual discount rate, which might a priori differ from the zero-coupon rate $z(T)$.

We now discuss the two key assumptions:

- The probability distribution of the final underlying price S_T;
- The discount rate r.

7-1.1 Probability Distribution of S_T

The probability distribution of S_T must be chosen to span a plausible range of final underlying prices. In equities, the following three properties are typically sought:

1. The stock price must always be positive: $S_T > 0$.
2. Forward contracts must be correctly priced: $(D_T = S_T - K) \Rightarrow (D_0 = \phi_0)$. This property is also known as **forward-neutrality**.[1]
3. The probability of the stock price doubling must be the same as halving, and generally for any x-fold factor: $\mathbb{P}\{\frac{S_T}{S^*} \geq x\} = \mathbb{P}\{\frac{S_T}{S^*} \leq \frac{1}{x}\}$, where S^* is the median.

[1] Forward-neutrality is closely related to risk-neutrality (see Chapter 6, Problem 9 p.72).

These three requirements are satisfied by the lognormal distribution with mean $\mu = \ln F_0 - \frac{1}{2}\sigma^2 T$ and standard deviation $\sigma\sqrt{T}$, where F_0 is the forward price of the underlying for maturity T and σ is a free volatility parameter. Section A-3.4 p.209 gives the definition of the lognormal distribution and Problem 10 verifies the three properties.

Figures 7-1 and 7-2 below show the difference between the normal and lognormal distributions under two volatility assumptions: high (50%) and low (20%). While both distributions exhibit a bell-curve shape, we can see that they differ more significantly when volatility is high.

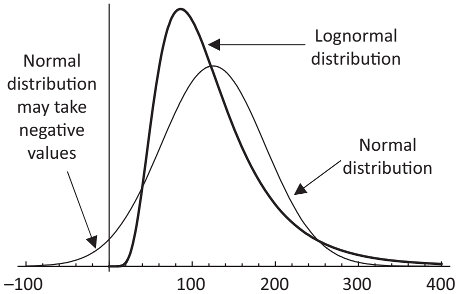

Figure 7-1 Normal and lognormal distribution $S_0 = 100$, $F_0 = 125$, $\sigma = 50\%$, $T = 1$

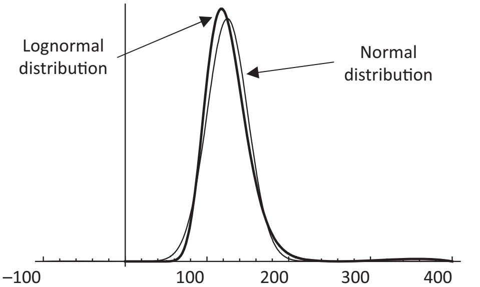

Figure 7-2 Normal and lognormal distribution $S_0 = 100$, $F_0 = 125$, $\sigma = 20\%$, $T = 1$

7-1.2 Discount Rate

Choosing an appropriate discount rate r is not as trivial as it might seem. For a fixed cash flow or a forward contract, arbitrage considerations demand the use of the risk-free zero-coupon rate $z(T)$ for maturity T. But for a variable, risky payoff, portfolio theory (see Section 4-4 p.43) suggests the choice of a higher discount rate.

It turns out that *even* for a risky payoff the risk-free rate $z(T)$ is appropriate. This is because the payoff risk may in theory be entirely eliminated by means of a dynamic trading strategy known as ‘delta-hedging’ (see Section 8-1.1 p.84).

As a corollary the forward-neutrality property (2) simplifies to:

$$(D_T = S_T) \Rightarrow (D_0 = S_0),$$

provided there is no dividend.

7-2 Closed-Form Formulas for European Options

The lognormal model yields the following closed-form formulas for the value of European calls and puts (see Problem 14 for derivations):

$$c_0 = \frac{1}{(1+r)^T}\left[F_0 N(d_1) - K N(d_2)\right]$$
$$p_0 = \frac{1}{(1+r)^T}\left[K N(-d_2) - F_0 N(-d_1)\right] \tag{7-1}$$

where F_0 is the forward price, K is the strike price, T is the maturity, r is the discount rate, σ is the volatility, $N(\cdot)$ is the standard normal cumulative distribution, and d_1, d_2 are the coefficients:

$$d_1 = \frac{\ln \frac{F_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln \frac{F_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \qquad \text{(7-2)}$$

Example

Standard & Logs Ltd. shares trade at 290 pence in London, pay no dividends and have a 40% volatility. The 3-month interest rate is 4%. Based on the closed-form formula, the value of a 3-month European call struck at 300 pence is about 20 pence:

- 3-month forward price of Standard & Logs Ltd.: $F_0 = 290 \times (1 + 4\%)^{0.25} \approx 292.86$.
- Coefficients d_1 and d_2:

$$d_1 \approx \frac{\ln \frac{292.86}{300} + \frac{1}{2} \times (0.4^2) \times 0.25}{0.4 \times \sqrt{0.25}} \approx -0.0205,\ d_2 = d_1 - 0.4 \times \sqrt{0.25} \approx -0.2205$$

- Normal cumulative distribution levels: $N(d_1) \approx 0.4918$ and $N(d_2) \approx 0.4127$
- Value of the call: $c_0 \approx \frac{1}{(1+4\%)^{0.25}}[292.86 \times 0.4918 - 300 \times 0.4127] \approx 20.01$ pence.

Using put-call parity (Equation (5-5) p.58) we obtain the value of the corresponding European put:

$$p_0 = c_0 - S_0 + \frac{K}{(1+r)^T} \approx 20.01 - 290 + \frac{300}{(1+4\%)^{0.25}} \approx 27.08 \text{ pence.}$$

Figures 7-3 and 7-4 below compare the value and payoff of a call and a put struck at K as a function of the spot price. The gray area (the difference between the option value and its payoff) is the time value of each option which decays as we approach maturity (see Section 5-3.2 p.57).

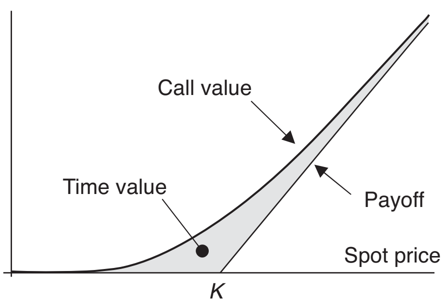

Figure 7-3 Value and payoff of a call

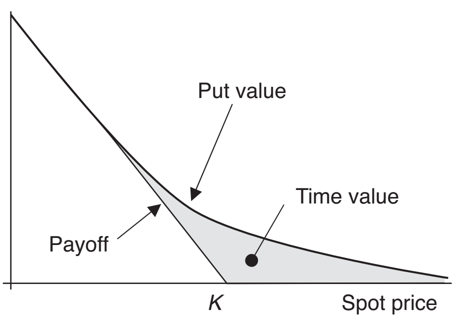

Figure 7-4 Value and payoff of a put

7-3 Monte-Carlo Method

Finding closed-form formulas for $D_0 = \mathbb{E}[\frac{D_T}{(1+r)^T}]$ in the lognormal model is rare. In most cases we may only perform a numerical evaluation. The **Monte-Carlo method** is one of the most popular techniques to do so.

The method consists in simulating a large number of values of S_T according to its probability distribution. The value D_0 is then approximately equal to the average discounted payoff $D_T = f(T, S_T)$:

$$D_0 \approx \frac{1}{(1+r)^T} \times \frac{1}{N} \sum_{i=1}^{N} f(T, s_i),$$

where N is the number of simulations and $s_1, s_2, \ldots, s_N$ are the simulated values of S_T.

In practice...
Monte-Carlo simulations are very easy to implement in a spreadsheet. In Excel, the following functions and formulas are useful:

- RAND() generates a uniformly distributed random number between 0 and 1.
- NORMSINV(RAND()) generates a normally distributed random number with zero mean and unit standard deviation (see Problem 13).
- F * EXP($-$0.5 * *sigma*^2 * T + *sigma* * SQRT(T) * NORMSINV(RAND())) simulates a lognormally distributed value for S_T, where F, *sigma*, T are the values of the forward price, volatility and maturity respectively.

Figure 7-5 overleaf shows 10,000 payoff simulations of an at-the-money call under the lognormal model, together with the empirical density of S_T. We can see that the density peaks near the forward price, suggesting that the method works well for payoffs which are not tied to rare events.

The central limit theorem (see Section A-5.7 p.211) tells us that for an infinite number of simulations the Monte-Carlo method converges to the fair value of the option. The main disadvantage of this method is that the convergence is relatively slow (the error is of order $1/\sqrt{N}$), but thanks to the ever-increasing speed of microprocessors this issue has become manageable.

7-4 Further Reading

On Monte-Carlo methods:

- Peter Jaeckel (2002) *Monte Carlo Methods in Finance*, John Wiley & Sons.
- Paul Glasserman (2010) *Monte Carlo Methods in Financial Engineering*, Springer.

7-5 Problems

Problem 1: True or False?
"The lognormal model builds on the idea that the log of the final underlying price S_T has a normal distribution centered on the log of the forward price F_0."

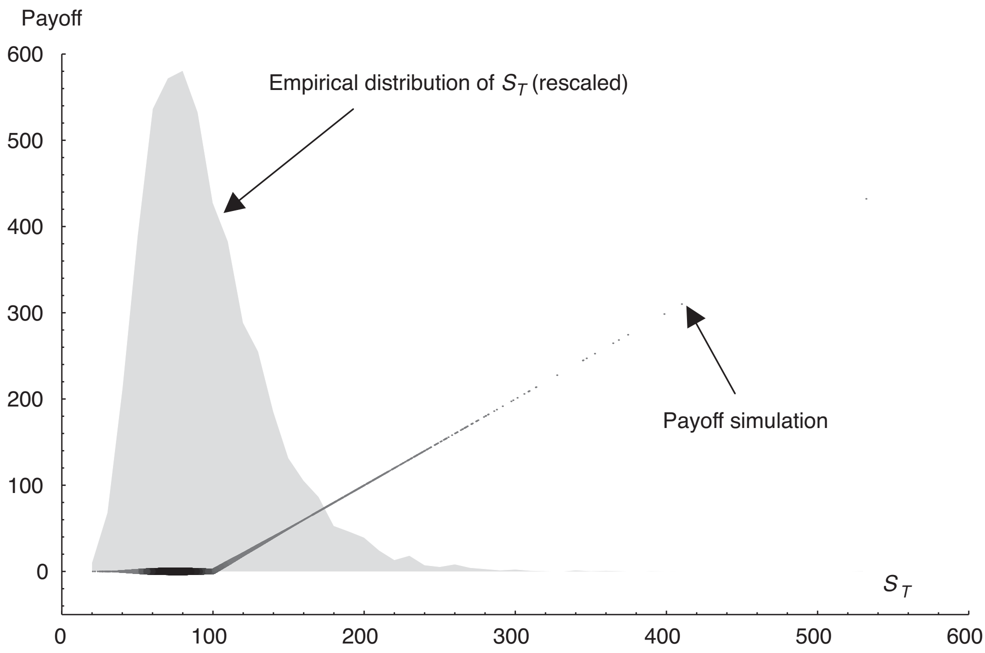

Figure 7-5 10,000 simulated call payoffs

Problem 2: Closed-form formulas. *The three questions are independent.*

(a) Based on the lognormal model, what is the value of a 1-month European call on Kroger Co. struck at \$25? Kroger Co. currently trades at \$24, no dividend is scheduled, the risk-free rate is 1% p.a., and volatility is 20%.

(b) Based on the lognormal model, what is the value today of a European call and a European put struck at \$50 and maturing in 6 months? The underlying asset is a stock with spot price \$50, no dividend, and 30% volatility. The risk-free rate is 10% per annum.

(c) Verify that, when substituting the volatility parameter $\sigma > 0$ with $-\sigma < 0$, the closed-form formula for c_0 yields $-p_0$.

Problem 3: Other distributions for S_T

Discuss the pros and cons of the following distribution assumptions to model the final underlying price S_T of a stock:

(a) Uniform: $S_T \sim \mathscr{U}[a, 2F_0 - a]$ with $0 \leq a < F_0$;

(b) Normal: $S_T \sim \mathscr{N}(F_0, b)$ where $b > 0$;

(c) (*) $S_T = \frac{2F_0}{\pi} X$ where X has cumulative distribution function $F_X(x) = 1 - \frac{1}{1+x^2}$ for $x > 0$, $F_X(x) = 0$ for $x \leq 0$. *Hint:* $\text{Arctan}' x = \frac{1}{1+x^2}$.

Problem 4: Plain vanilla option pricer

In a spreadsheet, build a pricer which computes the value of European calls and puts using the closed-form formulas (Equations (7-1) and (7-2) pp.76–77). The user must be able to input the following parameters: forward price F, strike K, maturity T, risk-free rate r, and volatility σ. Then, produce the following graphs:

(a) Value of a 1-year call struck at 100 as a function of F, with $r = 5\%$ and $\sigma = 20\%$. Also draw the discounted "forward-intrinsic" value on the same graph, i.e. $\frac{\max(0, F-K)}{(1+r)^T}$.
(b) Value of 1-year and 2-year puts struck at 100 as a function of $0 < \sigma \leq 100\%$, with $F = 100$ and $r = 10\%$.
(c) Value of a 2-year call as time passes for strikes $K = 90$, 100 and 110, with $F = 100$, $r = 5\%$, $\sigma = 40\%$.

Problem 5: Monte-Carlo pricer
In a spreadsheet, build a pricer which computes the value of the following derivative payoffs using the Monte-Carlo method with 5,000 simulations. The user must be able to input the following parameters: forward price F, strike K, maturity T, risk-free rate r, and volatility σ. Then, with a sensible choice of parameters, produce the graph of the value and discounted "forward-intrinsic" value of the derivative as a function of the forward price F. *Hint: use the same random sample while varying the forward price F.*

(a) 'Ballena call': $D_T = 100 \times \max(0, \frac{S_T - 100}{S_T})$;
(b) 'Kick-out put': $D_T = \max(0, 100 - S_T)$ if $S_T > 50$, 0 otherwise;
(c) 'Call spread': long call struck at 100, short call struck at 130.

Problem 6: Negative time value and arbitrage
(a) Based on the lognormal model, what is the value of a 3-year, European in-the-money put struck at €100 on a stock trading at €60 with a 10% volatility? The price of the 3-year zero-coupon is €80 and the stock does not pay any dividend.
(b) Compare your result with the intrinsic value of the put. Is there an arbitrage opportunity?

Problem 7: Butterflies and probabilities
Dull Inc.'s stock currently trades at \$100 in New York, and the interest rate curve is flat at zero. The fair values of three 1-year calls on Dull Inc. are shown below:

Strike	\$90	\$100	\$110
Fair value	\$20	\$15	\$12

(a) Draw the payoff function of a butterfly strategy on Dull Inc. which is long the \$90- and \$110-strike calls and short 2 at-the-money calls. What is the cost and maximum payoff of this strategy?
(b) Without making any specific assumption on the distribution of Dull Inc.'s stock price in one year, show that its probability to lie within the \$90–\$110 range is at least 20%. *Hint: Superimpose the payoff of a derivative which pays off \$10 whenever the final stock price is in the range, and zero outside.*
(c) (*) Let $f(x)$ be the distribution density of Dull Inc.'s stock price in one year, which may or may not be lognormal. Assume that you know all the fair values $c_0(K)$ of 1-year calls on Dull Inc. with strikes $90 \leq K \leq 110$. Using a limit argument, establish that:

$$f(100) = \frac{d^2 c_0}{dK^2}(100).$$

Problem 8: Rule of thumb for the value of an 'at-the-money-forward' call
Consider an 'at-the-money-forward' European call, i.e. a plain vanilla call whose strike K is equal to the underlying forward price F_0.

(a) What does the closed-form formula in the lognormal model become in this case?
(b) Using a first-order Taylor expansion of the standard normal cumulative distribution $N(\cdot)$, show that the value of the call is approximately:

$$c_0 \approx 40\% \times \frac{F_0}{(1+r)^T} \times \sigma\sqrt{T}.$$

(c) Compare this rule of thumb to the value of a 1-year at-the-money-forward call on an underlying stock with spot price 100 and volatility 40%, assuming no dividends and a 5% interest rate.

Problem 9: Digital option
The payoff of a digital option with strike K and maturity T is:

$$D_T = \begin{cases} 1 \text{ if } S_T > K \\ 0 \text{ otherwise} \end{cases}$$

(a) Draw the payoff function of a digital option with strike 100.
(b) Show that the fair value of the digital option in the lognormal model is $D_0 = \frac{N(d_2)}{(1+r)^T}$, where $d_2 = \frac{\ln(F/K)-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$.
(c) Compute D_0 for $F = 110$, $K = 100$, $T = 1$, $r = 5\%$, $\sigma = 30\%$.

Problem 10*: Lognormal distribution
Let X be a normally distributed random variable with mean μ and standard deviation σ, and let $Y = e^X$.

(a) Show that Y follows a lognormal distribution.
(b) Show that the expected value of Y is $\mathbb{E}(Y) = \exp(\mu + \frac{1}{2}\sigma^2)$.
(c) What is the expected value of Y^2? Find the variance of Y.
(d) Show that the lognormal distribution for the final underlying price S_T (Section 7-1.1 p.75) satisfies the three properties.
(e) Show that in the lognormal model $\mathbb{E}((\Delta S)^2) \sim S_0^2\sigma^2 T$ as $T \to 0$, where $\Delta S = S_T - S_0$.

Problem 11: Quadratic option
The payoff of a quadratic option with strike K and maturity T is: $D_T = (S_T - K)^2$.

(a) Draw the payoff function of a quadratic option with strike 100.
(b) Find a closed-form formula for the fair value of a quadratic option in the lognormal model. *Hint: consult Problem 10.*
(c) Compute the value of a 1-year quadratic option with strike £100 on a share trading at £105 that does not pay dividends. Volatility is 25% per year and the risk-free rate is 10% per annum.

Problem 12: Log-contract

The payoff of the log-contract on an underlying asset S is $D_T = -\ln(S_T / F_0)$, where F_0 is the forward price of S for maturity T.

(a) Draw the payoff function of the log-contract.
(b) Find the fair value D_0 of the log-contract in the lognormal model as a function of r, σ and T.
(c) (*) Imagine that, for a given maturity T, we know the prices of an infinite number of European calls and puts struck along a continuum between 0 and infinity. Consider a portfolio which is:
    (i) long a quantity dK / K^2 of puts struck at $K < F_0$;
    (ii) long a quantity dK / K^2 of calls struck at $K > F_0$.

In other words, the mark-to-market value of this portfolio is: $P_0 = \int_0^{F_0} \frac{1}{K^2} p_0(K)dK + \int_{F_0}^{\infty} \frac{1}{K^2} c_0(K)dK$, where $p_0(K)$ and $c_0(K)$ are the respective market prices of a put and a call struck at K. Show that P_0 must be equal to the fair value of the log-contract found in question (b).

Problem 13*: Simulating a normal distribution

Let X be a random variable with uniform distribution over the interval [0, 1], and let $Y = N^{-1}(X)$, where $N^{-1}(\cdot)$ is the inverse function of the standard normal cumulative distribution. Show that Y follows a standard normal distribution.

Problem 14*: Derivation of the closed-form formulas for European options

Let $D_T = \max(0, S_T - K)$ be the payoff of a European call with strike K and maturity T. The following questions provide a step-by-step derivation of the closed-form formulas for c_0 and p_0 (Equations (7-1) and (7-2) pp.76–77) in the lognormal model.

(a) Show that $\mathbb{E}(D_T) = \mathbb{E}(S_T I) - KN(d_2)$, where $I = 1$ if $S_T > K$ and $I = 0$ otherwise.
(b) Show that S_T has the same distribution as $Y = \exp(\mu + \sigma X \sqrt{T})$ where $\mu = \ln F_0 - \frac{1}{2}\sigma^2 T$ and $X \sim \mathcal{N}(0, 1)$.
(c) Show that $Y > K$ if and only if $X > d_2$.
(d) Using the change of variable $y = \sigma\sqrt{T} - x$, show that $\mathbb{E}(S_T I) = F_0 N(d_1)$.
(e) Establish the closed-form formulas for c_0 and p_0 (Equations (7-1) p.76 and (7-2) p.77).

Problem 15*: Breeden-Litzenberger formula

Imagine that, for a given underlying stock S and maturity T, we may trade an infinite number of calls whose strikes $K > 0$ form a continuum. Assume that $S_T > 0$ has a distribution density $f(x)$, which may or may not be lognormal.

(a) Show that, for any $K > 0$: $f(K) = (1 + r)^T \frac{d^2 c_0}{dK^2}(K)$, where $c_0(K)$ is the price of the call struck at K. *Hint*: $(1 + r)^T c_0(K) = \mathbb{E}[\max(0, S_T - K)] = \int_K^{\infty} (x - K) f(x)dx$.
(b) Let $D_T = g(S_T)$ be the payoff of a derivative on S, where $g(x)$ is a bounded, smooth (twice continuously differentiable) function with $g(0) = 0$. Using two successive integrations by parts, show that $D_0 = \int_0^{+\infty} g''(K)c_0(K)dK$.

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8 Dynamic Hedging

This chapter covers several practical aspects of option[1] trading under the lognormal model. Option traders typically monitor the risks embedded in options very closely and often endeavor to offset or ‘hedge’ them. In principle, such dynamic hedging allows option payoffs to be perfectly replicated at no risk, but in reality there is a mismatch which is related to the volatility of the underlying asset.

8-1 Hedging Option Risks

Calls and puts became popular with investors as a means to bet on the direction of a stock or index while limiting losses to the option premium. Figures 8-1 and 8-2 below show this asymmetry in **Net P&L** (payoff minus premium). Here, K is the strike price (the underlying price level to reach in order to get a positive payoff) and K' is the **break-even** price (the price level to reach to make a net profit). The difference between K and K' is roughly equal to the option premium.[2]

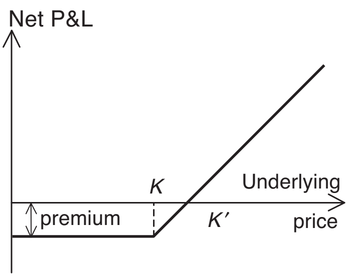

Figure 8-1 Net P&L of a call

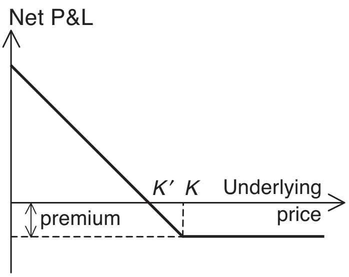

Figure 8-2 Net P&L of a put

On the other hand option sellers seem to face a much higher risk. Figures 8-3 and 8-4 overleaf show the Net P&L of short call and put positions, which are just flipped versions of Figures 8-1 and 8-2. Observe how much larger losses could be versus profits: why would anyone want to take this side of the bet?

It turns out that option sellers may virtually eliminate all directional risk by implementing a dynamic trading strategy known as **delta-hedging**. Most option trading desks use a version of

[1] Most concepts introduced in this chapter also apply to derivatives in general, not just vanilla options. We use the term ‘option’ for didactic reasons.

[2] Precisely, the difference between K and K' is equal to the option premium compounded at the zero-coupon rate r between $t = 0$ and $t = T$ (recall that the option premium is paid at $t = 0$ and not at maturity T).

this strategy, which also lies beneath the Black-Scholes option valuation model introduced in Chapter 10.

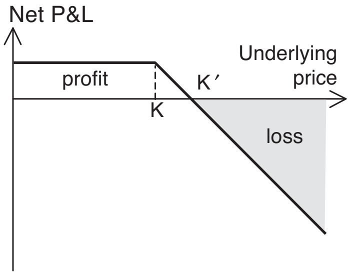

Figure 8-3 Net P&L of a short call

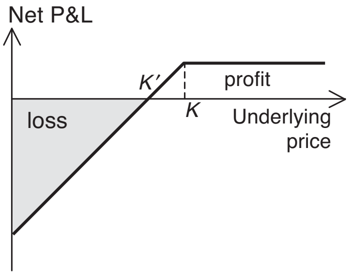

Figure 8-4 Net P&L of a short put

8-1.1 Delta-hedging

The **delta** is the change in option value caused by a change in stock price. For example, if Kroger Co. currently trades at \$24 and a European call on Kroger Co. is worth \$2 with a 0.5 delta, then a \$1 increase in Kroger Co.'s spot price from \$24 to \$25 roughly results in a \$0.5 increase in call value from \$2 to \$2.5.

Mathematically, the delta is the first-order derivative of the option value f with respect to the underlying spot price S:

$$\delta = \frac{\partial f}{\partial S}.$$

Hedging the delta of an option position means taking an opposite position in quantity δ of underlying. The resulting portfolio made of the option and the stock position has a delta of zero and is called '**delta-neutral**': it is immune to small changes in underlying price.

Example
To delta-hedge a short position of 1,000 calls on Kroger Co. with 0.5 delta, one must take a long position of $0.5 \times 1000 = +500$ stocks, i.e. *buy* 500 stocks. We may verify that the resulting portfolio is delta-neutral: if Kroger Co.'s price goes up \$1 the change in portfolio value is approximately:

$$-1000 \times (2.5 - 2) + 500 \times (25 - 24) = 0.$$

An option's delta varies over time as the underlying spot price moves and as maturity approaches. In order to maintain a delta-neutral portfolio at all times, the delta-hedge must be continuously adjusted by buying or selling the underlying.[3] The resulting trading strategy is called '**delta-hedging**' and, in theory, it replicates the option payoff at no risk.

[3] In practice continuous rebalancing of the delta-neutral portfolio is impossible because it would incur infinite transaction costs. See Problem 4 p.90 for an example of how delta-hedging would be carried out on a monthly basis.

Delta-hedging is an example of a **dynamic trading strategy**, in contrast with static strategies which are executed only once. For example, a static strategy to cover a short position of 1,000 calls on Kroger Co. struck at $25 could be to immediately buy 1,000 stocks on the market for $24,000 and wait until maturity. If the owner of the call exercises the option, the option seller can deliver the 1,000 stocks against $25,000, thus making a $1,000 profit. If instead the owner does not exercise the call, the option seller remains long the 1,000 stocks, which is risky because the price of Kroger Co. could fall well below $24.

8-1.2 Other Risk Parameters: the 'Greeks'

Option investors and traders are exposed to many more risks than delta. Table 8-1 below gives the other classical risk parameters called 'sensitivities' or '**Greeks**' – namely: convexity (Γ: gamma), time decay (Θ: theta), volatility risk ($\mathscr{V}$: vega[4]), interest rate risk (ρ: rho).

Table 8-1 The 'Greeks' of an option

δ or Δ (delta)	**Γ (gamma)**	**Θ (theta)**	**$\mathscr{V}$ (vega)**	**ρ (rho)**
$\frac{\partial f}{\partial S}$	$\frac{\partial^2 f}{\partial S^2} = \frac{\partial \Delta}{\partial S}$	$\frac{\partial f}{\partial t}$	$\frac{\partial f}{\partial \sigma}$	$\frac{\partial f}{\partial r}$
Change in option value when the underlying price S goes up $1	Change in delta when the underlying price S goes up $1	Change in option value due to the passage of time (generally converted into 1 day)	Change in option value when the volatility σ goes up 1 point (+1%)	Change in option value when the interest rate r goes up 100 basis points (i.e. +1%)
Example: at-the-money call on Kroger Co., zero dividend, $S_0 = \$24$, $c_0 = \$2$, strike $K = \$24$, maturity $T = 1$ year, volatility $\sigma = 20\%$, interest rate $r = 1\%$				
$\delta = 0.56$	$\Gamma = 0.082$	$\theta = –0.3$¢ per day	$\mathscr{V} = 9$¢ per volatility point	$\rho = 11$¢ per interest rate point (0.11¢ per basis point)
'If the stock price goes up $1 the option value will increase from $2 to $2.56'	'If the stock price goes up $1 the option's delta will increase from 0.56 to 0.64'	'By the end of the day the option's value will be down 0.3¢'	'If volatility goes up from 20% to 21%, the option value will be up 9¢'	'If interest rates go up 50 basis points the option value will be up 5.5¢'

The Greeks of European vanilla options have closed-form formulas listed in Section C-7 p.220. Note that there are other, less common Greek letters such as 'volga' ($\partial^2 f/\partial\sigma^2$, see Problem 10), 'vanna' ($\partial^2 f/\partial S\partial\sigma = \partial^2 f/\partial\sigma\partial S$), and more.

We must emphasize that each Greek letter is a first- or second-order, *ceteris paribus* approximation. For example, a daily theta of –0.3¢ means that the option loses approximately 0.3¢ in value after one day provided all other parameters (S, σ, r...) are unchanged.

[4] Hellenist readers may be puzzled at this unknown letter in the Greek alphabet. It was indeed invented by market practitioners. Rumor has it that it comes from the TV serial *Star Trek*.

Note that **Greeks may be generalized to portfolios of options on the same underlying S**: the delta of a portfolio long 1,000 calls with delta $\delta_1 = 0.5$ and short 500 calls with delta $\delta_2 = 0.4$ is $\delta_P = 1000\delta_1 - 500\delta_2 = 1000 \times 0.5 - 500 \times 0.4 = 300$. In other words if the underlying price goes up \$1 this portfolio gains \$300 in value.

8-1.3 Hedging the Greeks

In order to hedge Greeks other than delta, an option trader must buy or sell other options. This is where difficulties begin: every option comes with a bundle of Greeks attached to it which, when added to a portfolio, modify all the portfolio's Greeks. This makes the perfect hedging of risk parameters an impossible task in practice.

Since perfect hedging is impossible, an option trader is always left with some risk, and will strive to select those levels of risk he believes benefit him (e.g. being long vega if he thinks volatility is on the rise), or at least hurt him only a tolerable amount.

8-2 The P&L of Delta-hedged Options

Once delta-hedged, the P&L profile of an option differs from Figures 8-1 to 8-4 pp.83–84. It turns out that we may approximate it quite accurately with only two Greeks: gamma and theta.

8-2.1 Gamma

As a second-order derivative, the interpretation of gamma may be less straightforward than for other Greeks. The first interpretation is purely analytical and based on a second-order Taylor expansion:

$$f(S') - f(S) \approx \left(\frac{\partial f}{\partial S}\right)(S' - S) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial S^2}\right)(S' - S)^2,$$

$$\text{or: } \Delta f \approx \delta \times (\Delta S) + \frac{1}{2}\Gamma \times (\Delta S)^2.$$

In short, the change in option value is driven by the change in spot price multiplied by the delta and the square of the change in spot price multiplied by the half-gamma.

Example
For $\delta = 0.56$ and $\Gamma = 0.082$, if the price of Kroger Co. drops by $\Delta S = -\$2$ the call value goes down by $\Delta f \approx -0.56 \times 2 + {}^1/_2 \times 0.082 \times 2^2 \approx -\$0.96$.

When ΔS is small, the second term in $(\Delta S)^2$ is negligible and we have $\Delta f \approx \delta \times (\Delta S)$. For large price moves, however, the second term can be significant as illustrated in Figures 8-5 and 8-6 overleaf, especially if the Γ coefficient is high. As such, **the gamma measures the delta-hedging error**: when gamma is high, the delta-hedge should be rebalanced more frequently to maintain a delta-neutral portfolio.

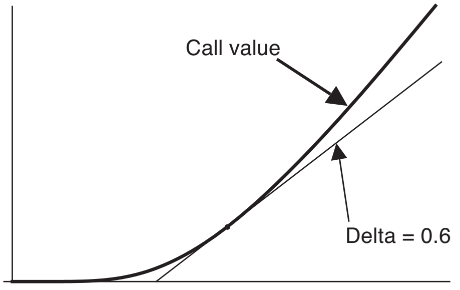

Figure 8-5 Value of a call vs. δ approximation

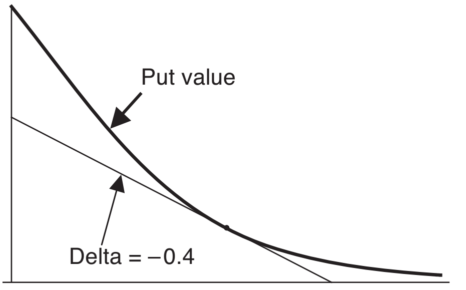

Figure 8-6 Value of a put vs. δ approximation

The second interpretation is of more fundamental nature and shows why option traders tend to be more comfortable with a 'long gamma' position, meaning that their option portfolio has positive gamma.

To understand this point, consider Figures 8-7 and 8-8 below showing the P&L of a delta-hedged call and a delta-hedged put as functions of the underlying price, i.e. the change in value of a portfolio long the option and short δ shares of stock or index units. We can make three observations:

- The two graphs are identical: **once delta-hedged, calls and puts have the same P&L profile**.
- The P&L is always positive: **a delta-hedged long call or put will always generate profits as the underlying price moves away from its initial price**. Unsurprisingly, there is a downside to this otherwise engaging statement which we investigate in Section 8-2.2 below.
- The Gamma-P&L prediction is very accurate around the initial spot price: the gamma is a good measure of the P&L of a delta-hedged option position resulting from changes in the underlying price. In particular, a positive gamma means profits and a negative gamma means losses. The larger the gamma and the change, the larger the profit or loss.

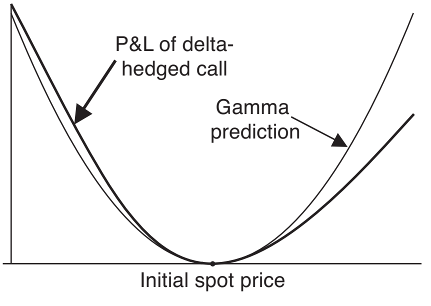

Figure 8-7 P&L and Gamma-P&L of a delta-hedged call

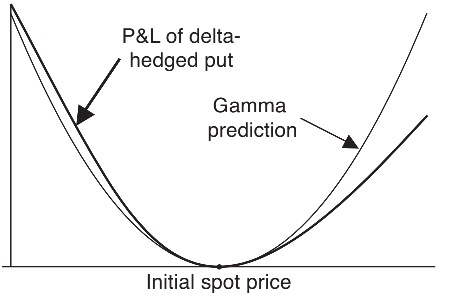

Figure 8-8 P&L and Gamma-P&L of a delta-hedged put

8-2.2 Theta

The daily theta of an option gives the P&L after one day if all other parameters – in particular the underlying price S – remain unchanged. For plain vanilla options, theta is negative: as time passes and maturity approaches, the option loses time value.

Figure 8-9 below shows how, for a delta-hedged option position, the P&L on theta counter-balances the P&L on gamma. When an option trader is long gamma, his profits as the spot price moves are gradually hurt by his losses as time passes, and conversely for a short gamma position. As such, the trader wants to be long gamma when the spot market is shaky and short gamma when it is quiet.

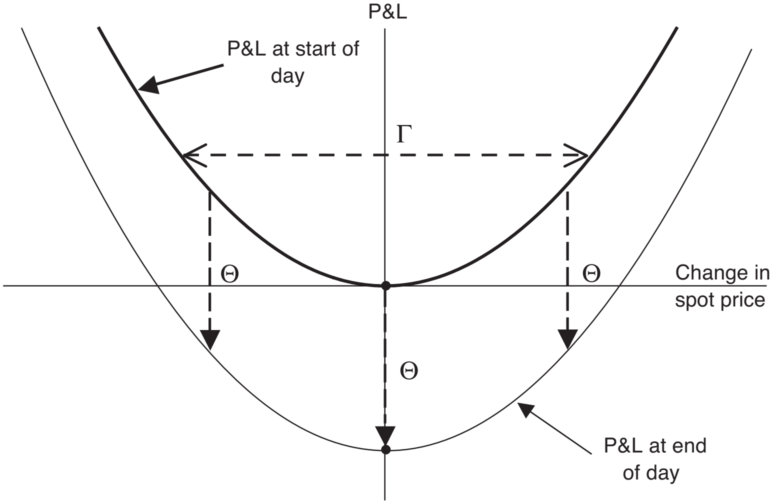

Figure 8-9 Effect of theta on the P&L of a delta-hedged call or put

In fact, there is a **reverse relationship between theta and gamma**:

$$\Theta \approx -\frac{1}{2}\Gamma\sigma^2 S^2, \tag{8-1}$$

where S is the underlying spot price and σ is the volatility. This is because, after a time period Δt, the loss on theta is $\Theta \times \Delta t$, while the profit on gamma is $^1/_2\ \Gamma \times (\Delta S)^2$. Assuming that, in a "fair game," the expected P&L should be zero, we may write $\Theta \times \Delta t + \frac{1}{2}\Gamma \times \mathbb{E}\left[(\Delta S)^2\right] = 0$. But for small Δt we have $\mathbb{E}\left[(\Delta S)^2\right] \approx \sigma^2 S^2 \Delta t$ (see Chapter 7, Problem 10 p.81 question (e)), which yields Equation (8-1) after dividing by Δt and rearranging terms. An exact version of the relationship is derived in Chapter 10, Problem 7 p.116.

8-2.3 Delta-hedging P&L Proxy

Generally, the trading P&L of an option position over a time period Δt may be broken down along the option Greeks:

$$\text{P\&L}_{\Delta t} = \delta \times (\Delta S) + \frac{1}{2}\Gamma \times (\Delta S)^2 + \Theta \times (\Delta t) + \mathscr{V} \times (\Delta\sigma) + \rho \times (\Delta r) + \cdots$$

where ΔS is the change in spot price, $\Delta\sigma$ is the change in volatility, Δr is the change in interest rate, and "..." are higher-order terms which complete the Taylor expansion.

If the option is delta-hedged and we assume that volatility and interest rates are constant, and that the higher-order terms are negligible, we may write:

$$\text{P\&L}_{\Delta t} \approx \frac{1}{2}\Gamma \times (\Delta S)^2 + \Theta \times (\Delta t).$$

Substituting $\Theta \approx -\frac{1}{2}\Gamma\sigma^2 S^2$ (Equation (8-1)) and factoring by S^2, we obtain the delta-hedging P&L proxy formula:

$$\text{P\&L}_{\Delta t} \approx \frac{1}{2}\Gamma S^2 \left[\left(\frac{\Delta S}{S}\right)^2 - \sigma^2 \Delta t\right]. \tag{8-2}$$

Example
For $\Gamma = 0.082$, $\sigma = 20\%$ and $\Delta t = 1/252$ (one trading day), if the price of Kroger Co. drops from \$24 to \$22 over one day the P&L on the delta-hedged at-the-money call is roughly $^1/_2 \times 0.082 \times 24^2 \times [(2 / 24)^2 - 0.2^2 / 252] \approx \$0.16$.

Equation (8-2) tells us that the P&L of a delta-hedged option over a period Δt is driven by the difference between the squared return on the underlying and the option's squared volatility, multiplied by a scaling factor of $^1/_2\Gamma S^2$ known as '**dollar gamma**'. In particular, the break-even points are $S(1 \pm \sigma\sqrt{\Delta t})$.

8-3 Further Reading

On delta-hedging and the Greek letters: John C. Hull (2009) *Options, Futures and Other Derivatives* 7th Edition, Prentice Hall: Chapter 17.

8-4 Problems

Problem 1: True or False?
"Delta-hedging eliminates the volatility risk of the underlying by creating a portfolio which is insensitive to all movements in the underlying price."

Problem 2: Greeks
You are an option trader in charge of a portfolio of options on Swinger Corp.'s stock whose Greeks are given in the table below. The current price of Swinger Corp. in New York is \$80, its annual volatility is 40% and the company does not pay any dividend. The risk-free rate is 3%.

Δ (per \$)	**Γ (per \$ per \$)**	**Θ (per day)**	**$\mathcal{V}$ (per %)**	**ρ (per %)**
1,500,000	−10,000	14,222	−500,000	300,000

(a) What is your P&L if the stock price of Swinger Corp. drops down by 1%? 10%?
(b) What is your P&L after one day if nothing happens on the markets?

(c) There are market rumors that the Federal Reserve is about to raise its target rate by 25 basis points to 3.25%. Estimate your gain or loss in this scenario.
(d) After a press release, the trading volume of Swinger Corp.'s stock significantly increases and you feel that the stock price becomes shaky, yet it stays around \$80. Do you think this situation is favorable, unfavorable or indifferent to you?

Problem 3: Delta-hedge, break-even levels
You just sold 1 million puts on Plumet SA's stock which currently trades at €20 in Paris. The put's delta is –0.3, its daily theta is €−0.002301 and its gamma is 0.0671831.

(a) You want to delta-hedge your position. What actions do you take?
(b) Once delta-hedged, what would your P&L be if the price of Plumet SA dropped by €0.50 after one day? By €5?
(c) Find the stock price level(s) for which your P&L breaks even.

Problem 4: Delta-hedging P&L
The table below gives the history of the share price of Range Ltd. over the past 12 months, together with the value and delta of a call maturing at the end of the 12-month period. At the beginning of the period you purchase 10,000 calls which you delta-hedge immediately and then on a monthly basis. Compute your monthly and cumulative P&L, including your purchase cost and option payoff. Assume zero interest rates.

Month	Range Ltd Price (£)	Call Value (£)	Delta (per £)
0	100	11.84	0.61
1	90	6.04	0.43
2	105	13.89	0.68
3	90	5.02	0.41
4	85	2.8	0.29
5	95	6.06	0.48
6	100	8.01	0.58
7	110	14.07	0.78
8	115	17.46	0.87
9	125	26.16	0.97
10	120	20.8	0.97
11	115	15.4	0.98
12	110	10	1

Problem 5: Greeks of the underlying asset and the forward contract
Assuming no dividends, find the Greeks of:

(a) the underlying asset S;
(b) a forward contract on S with delivery price K and maturity T.

How do your answers change if there is a single cash or proportional dividend?

Problem 6: Hedging multiple Greeks
The table below gives the value, delta, and gamma of two options on the same underlying asset S currently trading at $100. You are short one Option 1.

Security	Value ($)	Delta (per $)	Gamma (per $ per $)
Option 1	12.36	0.57	0.013076
Option 2	5.71	0.33	0.012153

(a) Calculate the quantities of S and Option 2 to buy or sell in order to make the portfolio both delta- and gamma-neutral.
(b) Show that your portfolio is approximately theta-neutral.
(c) Would your portfolio still be delta- and gamma-neutral if the underlying price went up $10?

Problem 7*: Delta of a European call or put
Consider the closed-form formulas for European call and put values in the lognormal model (Equations (7-1) and (7-2) pp.76–77).

(a) Verify that: $F_0 N'(d_1) = K N'(d_2)$ where $N'(\cdot)$ is the standard normal density.
(b) Assuming no dividends, show that the delta of a call is $\delta_{call} = \frac{\partial c}{\partial S} = N\,(d_1)$.
(c) What is the delta of a put with same characteristics (underlying, strike, maturity)?

Problem 8: Call and put have same gamma and vega
Using put-call parity, show that a European call must have the same gamma and vega as a European put with same characteristics (underlying, strike, maturity).

Problem 9: Log-contract. *This is a continuation of Chapter 7, Problem 12 p.81.*
The payoff of the log-contract is: $D_T = -\ln(S_T \,/\, F_0)$ where F_0 is the forward price of the underlying asset S at time $t = 0$. The zero-coupon rate curve is flat at rate r and there are no dividends.

(a) Show that the fair value D_t at time $t < T$ of the log-contract is:

$$D_t = -\frac{\ln(S_t/S_0) - \frac{1}{2}\sigma^2(T-t) - t\ln(1+r)}{(1+r)^{T-t}}.$$

(b) What is the dollar gamma of the log-contract?
(c) You are long 2 log-contracts maturing in 1 year, which you delta-hedge every trading day of length $\Delta t = 1/252$. Assuming $r = 0$, show that after one year your cumulative trading P&L is roughly:

$$\text{Cumulative P\&L} \approx \sigma^2_{\text{Realized}} - \sigma^2,$$

where σ_{Realized} is the historical volatility of daily returns on the underlying asset during the trading year (Section 4-1.1 p.35).

Problem 10: Volga

The 'vol-gamma' or 'volga' of an option is the second-order sensitivity of the option value with respect to the volatility parameter σ: Volga $= \partial^2 f / \partial \sigma^2$. You are a trader in charge of an option portfolio with zero vega and $-10{,}000$ volga (per % per %).

(a) What is your P&L if volatility goes down 1 point (i.e. -1%)? Up 5 points? Would you say your volga position is favorable or unfavorable?

(b) The current volatility level is 30%. You ask your quantitative analysts to provide a 95% confidence interval for the coming month. Their answer is: 27% to 33%. What is the maximum loss you could make over the next month at the 95% confidence level? Could you lose more?

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Part III

Advanced Models and Techniques

9 Models for Asset Prices in Continuous Time

A great deal of financial theory, including option valuation, may be carried out in discrete time. However, advanced theory often requires a continuous time framework which is an idealization of how markets and people operate.

In this chapter we have our first shot at continuous-time finance and introduce some key ideas, from continuous interest rates to Brownian motions and the Ito-Doeblin theorem. Sections 9-3 and 9-4 are more technical and abstract and may be skipped on first reading.

9-1 Continuously Compounded Interest Rate

9-1.1 Fractional Interest Rate

While interest is typically paid out annually, it may occasionally be split into several equal payments. For example, a US Treasury bond with \$100 face value and 5% nominal coupon[1] pays \$2.5 interest every six months; or a savings account with 6% p.a. nominal interest rate may pay interest every month on the account balance at the monthly rate of 0.5%.

Generally, we call "fractional" an interest rate $r_{[m]}$ which is split into m equal payments over the year, as shown in Table 9-1 below:

Table 9-1 Interest payments at fractional interest rate $r_{[m]}$

***t* (years)**	0	$\frac{1}{m}$	$\frac{2}{m}$	$\ldots$	$\frac{m}{m}=1$	$\ldots$
Cash flow	—	$\frac{r_{[m]}}{m} \times K_0$	$\frac{r_{[m]}}{m} \times K_{\frac{1}{m}}$	$\ldots$	$\frac{r_{[m]}}{m} \times K_{\frac{m-1}{m}}$	$\ldots$
Capital	K	$K_{\frac{1}{m}} = K\left(1+\frac{r_{[m]}}{m}\right)$	$K_{\frac{2}{m}} = K\left(1+\frac{r_{[m]}}{m}\right)^2$	$\ldots$	$K_1 = K\left(1+\frac{r_{[m]}}{m}\right)^m$	$\ldots$

The equivalent annual compound rate is thus $r = \left(1+\frac{r_{[m]}}{m}\right)^m - 1$. Note that $r_{[m]} < r$: a fractional rate underestimates its equivalent compound rate.

Examples

- The equivalent compound rate on a US Treasury bond with 5% nominal coupon is $r = \left(1+\frac{5\%}{2}\right)^2 - 1 \approx 5.06\%$ (assuming interest is reinvested into the bond).

[1] The nominal coupon, also called face coupon, is the coupon rate which is advertised at issuance and used to determine the actual coupon payments according to market conventions. In practice, most government bonds detach an annual coupon equal to the face coupon, a major exception being US government bonds which detach a semi-annual coupon equal to half the face coupon.

- The equivalent compound rate on a savings account with monthly interest payments at 6% p.a. nominal rate is $r = \left(1 + \frac{6\%}{12}\right)^{12} - 1 \approx 6.17\%$.

9-1.2 Continuous Interest Rate

In the limit as m goes to infinity, we obtain a so-called **continuous** or **continuously compounded interest rate** $\widehat{r}$ which is split an infinite number of times. The equivalent compound rate is $r = \lim_{m \to +\infty} \left(1 + \frac{\widehat{r}}{m}\right)^m - 1 = e^{\widehat{r}} - 1$ (see Chapter 1, Problem 7 p.9). For example, a savings account paying a continuous interest rate of 6% p.a. has an equivalent compound rate of $e^{6\%} - 1 \approx 6.18\%$.

Note the following expressions in relation to the continuous rate $\widehat{r}$:

- Capitalization over T years starting with initial capital K: $K e^{\widehat{r}T}$;
- Discounting over T years of a cash flow C: $C e^{-\widehat{r}T}$.

While continuous interest rates are almost never encountered in practice, they are very common in financial theory.

9-2 Introduction to Models for the Behavior of Asset Prices in Continuous Time

Historical market prices have been analyzed for centuries, often in the hope of making winning predictions. The first continuous-time model for the behavior of asset prices was proposed by Louis Bachelier in his 1900 thesis. Bachelier's approach was a visionary breakthrough, but his thesis was ill-received and he was blackballed from a first-rate career in academia as a result of the short-sightedness of his jury.

We now introduce some key ideas based on empirical observations. Consider Figure 9-1 overleaf which shows the daily prices of the S&P 500 index and their moving average over a 50-day rolling window.[2] We may empirically distinguish two components:

- A **general upward or downward trend** (in markets' jargon: a 'bull' or 'bear' trend), whose cycles are best observed on the moving average curve;
- **Random variations** about the trend, which positively or negatively add up to the latter.

Following this empirical analysis and denoting X_t the index level at time t, m_t the market trend, and Z_t the random variation, we may break down the daily change in S&P 500 as:

$$X_{t+1\,\text{day}} - X_t = (m_{t+1\,\text{day}} - m_t) + (Z_{t+1\,\text{day}} - Z_t).$$

Generally, over an infinitesimal time period dt:

$$X_{t+dt} - X_t = (m_{t+dt} - m_t) + (Z_{t+dt} - Z_t),$$

[2] Each point of the moving average curve corresponds to the average price of the S&P 500 over the preceding 50 days.

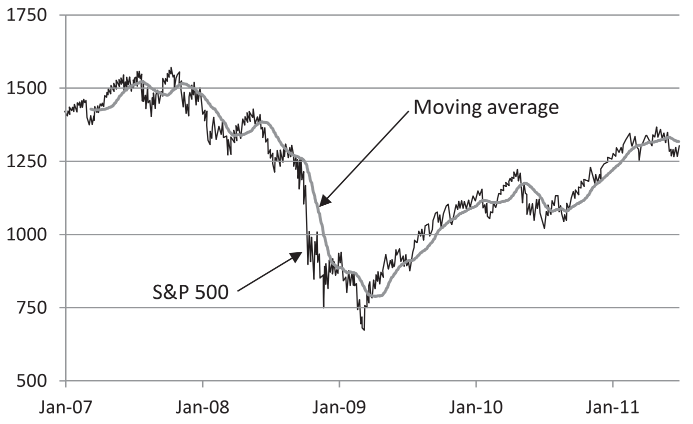


Figure 9-1 Daily historical prices of the S&P 500 index (January 2007 to June 2011)

or, with differential notations:

$$dX_t = dm_t + dZ_t. \tag{9-1}$$

With this basic decomposition in hand, a plausible model for the future behavior of X_t comes to mind:

- Assuming the general trend $m_t = m(t)$ is a reasonably smooth, differentiable function of time with first-order derivative $\mu_t = \frac{dm_t}{dt}$, we can write: $dm_t = \mu_t dt$. Equation (9-1) then becomes:

$$dX_t = \mu_t dt + dZ_t.$$

- In contrast, the random variations $Z_t = Z(t)$ make a fairly irregular, non-differentiable function of time. As such, we cannot express dZ_t with a derivative, and we must instead model it as a random variable. Because of its erratic nature, dZ_t is classically assumed to be normally distributed.

To verify that this approach is sensible we simulated a random series on a computer and visually compared the results to real data. Figure 9-2 overleaf shows one such simulation with $dt = 1/252$, $\mu_t = 40$, $dZ_t \sim \mathcal{N}(0, 19)$: the difference is virtually unnoticeable.

In the following sections we introduce in greater detail the classical random processes, also called **stochastic processes**, which are commonly used in finance to model the behavior of asset prices in continuous time.

9-3 Introduction to Stochastic Processes

The comprehensive study of stochastic processes is an entire field of mathematics and statistics which is beyond our scope. Our aim here is practical: to give the necessary concepts and vocabulary for an adequate understanding of the Black-Scholes model presented in Chapter 10. Readers who need a refresher in probability and statistics may consult Appendix A.

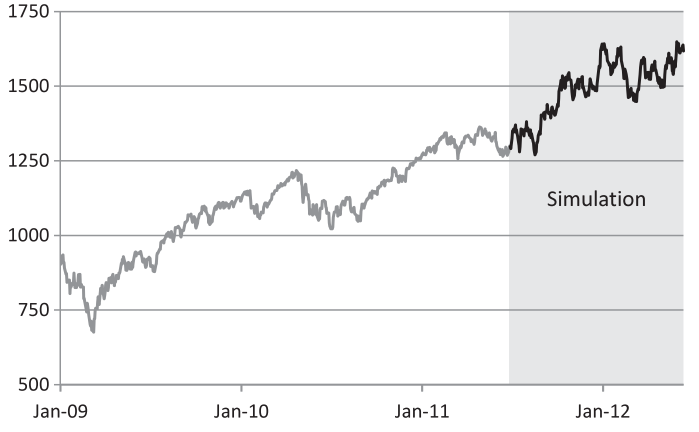


Figure 9-2 Simulated prices of S&P 500 using the random model $dX_t = \mu_t dt + dZ_t$

We begin with the usual representation of uncertainty as a universe Ω of all possible states of nature ω which may happen in the future at some probability.

A continuous-time **stochastic process** is a sequence $(X_t)_{t \geq 0}$ of random variables indexed by time t. In a given state of nature ω, the **path** followed by the process $(X_t)_{t \geq 0}$ is the function $t \mapsto X_t(\omega)$. The process $(X_t)_{t \geq 0}$ is called **continuous** whenever every path $t \mapsto X_t(\omega)$ is a continuous function of time for all states of nature ω in Ω.

In the rest of this section we review a selection of classical stochastic processes.

9-3.1 Standard Brownian Motion

A **standard Brownian motion**, also called **standard Wiener process**, is a *continuous* stochastic process $(W_t)_{t \geq 0}$ which satisfies:

(1) $W_0 = 0$;
(2) For all $0 \leq t < t'$, the increment variable $D = W_t' - W_t$ follows a normal distribution with zero mean and standard deviation $\sqrt{t' - t}$;
(3) For all $0 \leq t_1 < t_2 \leq t_3 < t_4$, the increment variables $D = W_{t_2} - W_{t_1}$ and $\Delta = W_{t_4} - W_{t_3}$ are independent, and generally so for any number of non-overlapping increments.[3]

Using properties (2) and (3) we may represent an infinitesimal increment $dW_t = W_{t+dt} - W_t$ as a random variable following a normal distribution with zero mean and standard deviation $\sqrt{dt}$. This is sometimes written: $dW_t \equiv \tilde{\varepsilon}_t \sqrt{dt}$, where $(\tilde{\varepsilon}_t)_{t \geq 0}$ is a continuous sequence of independent standard normals.

Figures 9-3 and 9-4 overleaf show the results of two computer simulations using steps $dt = 0.05$ and 0.01.

[3] In other words, any finite sequence of n increments $(W_{t_2} - W_{t_1}), (W_{t_4} - W_{t_3}), \ldots, (W_{t_{2n}} - W_{t_{2n-1}})$ with $0 \leq t_1 < t_2 \leq t_3 < t_4 \leq \cdots t_{2n-1} < t_{2n}$ is independent.

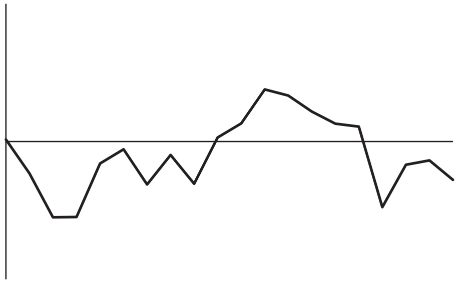


Figure 9-3 Brownian path simulated with step $dt = 0.05$

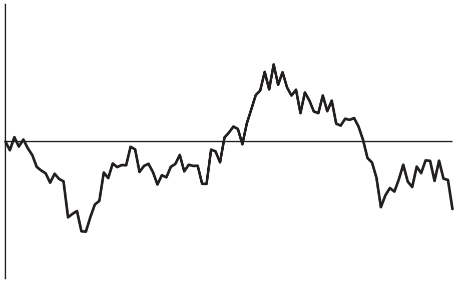


Figure 9-4 Brownian path simulated with step $dt = 0.01$

Brownian motions have some remarkable properties:

- Brownian paths tend to exhibit alternate cycles above and below the time axis, and it can be shown that the expected number of crosses of the axis is infinite. This is because Brownian increments have zero mean and the cumulative process has no general upward or downward trend.
- Brownian paths are continuous at every point in time but nowhere differentiable (see Problem 9 for a heuristic proof).

9-3.2 Generalized Brownian Motion

A **generalized Brownian motion** or generalized Wiener process $(X_t)_{t \geq 0}$ is a stochastic process with infinitesimal increments:

$$dX_t = a\,dt + b\,dW_t, \tag{9-2}$$

where a and b are constant parameters, and $(W_t)_{t \geq 0}$ is a standard Brownian motion.

Equation (9-2) may be difficult to grasp at first. It often helps to consider the two terms $a\,dt$ and $b\,dW_t$ separately, as illustrated in Figure 9-5 overleaf:

- $a\,dt$ is a deterministic upward or downward trend called **drift**. Omitting the $b\,dW_t$ term, we have $dX_t = adt$, i.e. $\frac{dX_t}{dt} = a$, and by integration we obtain the line $X_t = at + X_0$.
- $b\,dW_t$ is a random variation about the trend whose amplitude is controlled by b and the square root of the time period dt (recall that $dW_t \equiv \tilde{\varepsilon}_t\sqrt{dt}$ is normally distributed with zero mean and standard deviation $\sqrt{dt}$).

Integrating[4] both sides of the relationship $dX_t = a\,dt + b\,dW_t$, we obtain a useful analytical expression for the generalized Brownian motion:

$$X_t = X_0 + at + bW_t. \tag{9-3}$$

[4] Here we must emphasize that stochastic integration follows different rules than classical integration. However, in this case, we simply wrote that the sum of infinitesimal increments dX_t is equal to the cumulative process X_t.

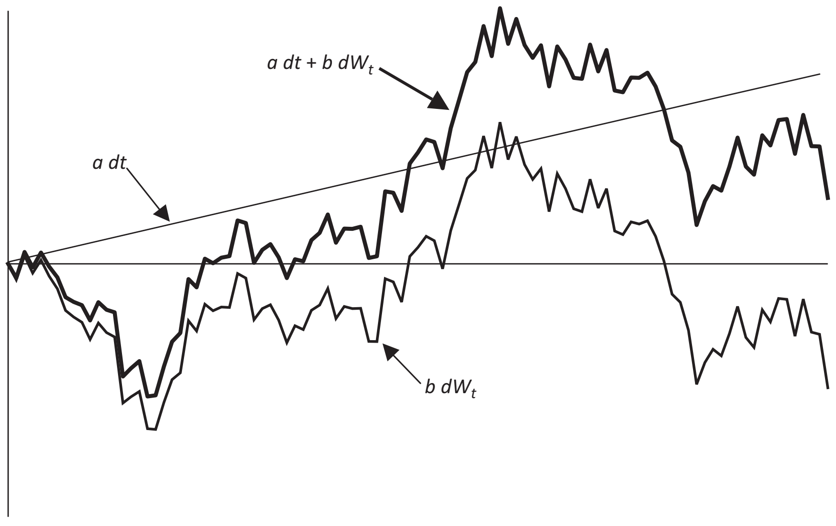


Figure 9-5 Path of a generalized Brownian motion simulated with $a = 15$, $b = 20$ and $dt = 0.01$

9-3.3 Geometric Brownian Motion

A **geometric Brownian motion** $(X_t)_{t \geq 0}$ is a stochastic process with infinitesimal increments:

$$dX_t = (aX_t)dt + (bX_t)dW_t,$$

where a and b are constant parameters, and $(W_t)_{t \geq 0}$ is a standard Brownian motion.

The financial interpretation of this definition becomes apparent when we divide both sides by X_t and write:

$$\frac{X_{t+dt} - X_t}{X_t} = adt + bdW_t.$$

When X_t is the price of a stock or index, the ratio $\frac{X_{t+dt}-X_t}{X_t}$ is the rate of return over an infinitesimal time interval dt. **A geometric Brownian motion is thus particularly appropriate to model the behavior of stock prices through their returns.** This is the preferred approach in finance.

In the example of Section 9-4.2 below we derive an analytical expression for the geometric Brownian motion:

$$X_t = X_0 e^{(a - \frac{1}{2}b^2)t + bW_t}, \tag{9-4}$$

and it is easy to verify that X_t is lognormally distributed.

9-4 Introduction to Stochastic Calculus

In ordinary calculus (see Section B-2.2 p.214), an infinitesimal change in the function $f(x, y)$ may be broken down along its first-order derivatives:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

It turns out that, when one of the variables is a stochastic process, this principle is no longer true. For example, if (X_t) is a geometric Brownian motion with parameters a and b, the infinitesimal change in $f(t, X_t)$ is:

$$df = \underbrace{\frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX_t}_{\text{Ordinary calculus terms}} + \underbrace{\frac{1}{2}\frac{\partial^2 f}{\partial X^2}b^2 X_t^2 dt}_{\text{Additional term}}.$$

This result is very relevant to derivatives valuation: if we assume that $D_t = f(t, S_t)$ is the value of a derivative security on an underlying S, and that (S_t) is a geometric Brownian motion, the equation above tells us how to calculate the change $dD_t = df$ in the derivative's value.

We now explain this result in more detail.

9-4.1 Ito Process

An **Ito process** $(X_t)_{t\geq 0}$ is a yet more general form of stochastic processes seen thus far, with infinitesimal increments:

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \tag{9-5}$$

where a and b are now two "sufficiently smooth"[5] *functions* of two variables. Note that if a and b are constant $(X_t)_{t\geq 0}$ is a generalized Brownian motion, and if $a(t, X) = \mu X$ and $b(t, X) = \sigma X$ then $(X_t)_{t\geq 0}$ is a geometric Brownian motion with parameters μ and σ.

Example
$dX_t = 0.05X_t dt + 0.5X_t^{1/(1+t)}dW_t$ is an Ito process with drift function $a(t, X) = 0.05X$ and volatility function $b(t, X) = 0.5X^{1/(1+t)}$. When t is small it behaves approximately as a geometric Brownian motion.

9-4.2 The Ito-Doeblin Theorem[6]

Consider an Ito process $(X_t)_{t\geq 0}$ and the attached process $Y_t = f(t, X_t)$ where f is a "sufficiently smooth" function of time and X. Then $(Y_t)_{t\geq 0}$ is also an Ito process, with

[5] By "sufficiently smooth" we mean that each function is continuously differentiable with respect to the time variable t and twice continuously differentiable with respect to the state variable X.

[6] Until recently academic literature referred to this theorem as 'Ito's lemma.' In a remarkable development of the story of probabilities, it turned out that the mathematician Wolfgang Doeblin had done similar work decades earlier but his notes remained sealed at the Académie des Sciences in Paris until May 2000 as a result of his tragic death during World War II.

infinitesimal increment:

$$dY_t = df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX_t + \frac{1}{2}b^2\frac{\partial^2 f}{\partial X^2}dt, \tag{9-6}$$

or, in longhand notation after collecting terms in dt:

$$dY_t = df(t, X_t) = \left\{\frac{\partial f}{\partial t}(t, X_t) + \frac{1}{2}[b(t, X_t)]^2\frac{\partial^2 f}{\partial X^2}(t, X_t)\right\}dt + \frac{\partial f}{\partial X}(t, X_t)dX_t.$$

Substituting Equation (9-5) into Equation (9-6) and collecting dt terms we equivalently have the expanded expression:

$$dY_t = df = \left(\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial X} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial X^2}\right)dt + \left(b\frac{\partial f}{\partial X}\right)dW_t. \tag{9-7}$$

Example

If (X_t) is a geometric Brownian motion with parameters μ and σ (i.e. $a = \mu X$ and $b = \sigma X$), and $Y_t = \ln X_t$, we may apply the Ito-Doeblin theorem on $f(t, X) = \ln X$ to get:

$$\begin{aligned} dY_t = df &= \left(\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial X} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial X^2}\right)dt + \left(b\frac{\partial f}{\partial X}\right)dW_t \\ &= \left(0 + \mu X_t\frac{1}{X_t} - \frac{1}{2}\sigma^2 X_t^2\frac{1}{X_t^2}\right)dt + \left(\sigma X_t\frac{1}{X_t}\right)dW_t \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t \end{aligned}$$

In other words (Y_t) is a generalized Brownian motion with parameters $(\mu - \frac{1}{2}\sigma^2)$ and σ. Substituting these values into Equation (9-3), we get:

$$Y_t = Y_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t.$$

Observing that $Y_t = \ln X_t$ and taking the exponential of both sides of this equation yields Equation (9-4).

The Ito-Doeblin theorem is the cornerstone of stochastic calculus. As unpalatable as it may look, it is a crucial technical step in the derivation of the Black-Scholes partial differential equation in Chapter 10.

9-4.3 Heuristic Proof of the Ito-Doeblin Theorem

A formal proof of the theorem requires advanced concepts in probability theory which are beyond the scope of this book. Instead, we sketch a derivation based on a Taylor expansion and the property that $(dW_t)^2 \equiv dt$ (see Problem 11).

A Taylor expansion of $f(t,x)$ is $df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2 + \cdots$. When $x = X_t$ is stochastic, we have $dx = dX_t = a(t, X_t)dt + b(t, X_t)dW_t$, which yields after substitution:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}(adt + bdW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}\left[a^2dt^2 + 2ab\,dt\,dW_t + b^2(dW_t)^2\right] + \cdots$$

In the limit as dt goes to 0, the dt^2 and $dt\,dW_t$ terms vanish but the $(dW_t)^2$ term "tends" to dt. Making appropriate substitutions and collecting terms, we obtain as required:

$$df = \left(\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial X} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial X^2}\right)dt + b\frac{\partial f}{\partial X}dW_t.$$

9-5 Further Reading

On Brownian motions and the Ito-Doeblin theorem:

- John C. Hull (2009) *Options, Futures and Other Derivatives* 7th Edition, Prentice Hall: Chapter 12.
- Steven Shreve (2004) *Stochastic Calculus Models for Finance II: Continuous-Time Models*, Springer: Chapters 3 and 4.

On Wolfgang Doeblin's story:

- Robert Jarrow and Philip Protter (2004) A short history of stochastic integration and mathematical finance the early years, 1880–1970, in *The Herman Rubin Festschrift*, IMS Lecture Notes 45; 75–91.

9-6 Problems

Problem 1: Fractional and continuous interest rates
Fill the gaps in the table below of equivalent annual interest rates.

From \ To	Annual gross rate	Fractional every 6 months	Fractional every month	Continuously compounded
Annual gross rate	10%			
Fractional every 6 months		10%		
Fractional every month			10%	
Continuously compounded				10%

Problem 2: Continuous compounding
Starting with an initial capital K, find the level K_t of capitalization after t years at the risk-free interest rate R when:

(a) interest is paid annually;
(b) interest is split into m equal payments over the year;

(c) interest is paid continuously. Using a financial argument, show that K_t must satisfy the ordinary differential equation: $dK_t = R\,K_t\,dt$, and verify your result. *Hint: the differential equation* $f' = af$ *has solutions* $f(x) = f(0)e^{ax} + c$*, where a and c are constants.*

Problem 3: Some econometrics

Look up the daily historical prices of a stock or equity index of your choice over a 1-year period and use a spreadsheet to answer the following questions:

(a) Calculate the mean and standard deviation of your time series.
(b) Produce a distribution histogram with 9 buckets and compare it to a normal distribution.
(c) Compute the series of daily returns and re-answer the two questions above.

Problem 4: Simulating Brownian motions

In a spreadsheet, simulate 100 normally distributed variables (see Section 7-3 p.78) and answer the following questions:

(a) Draw the path of a standard Brownian motion with time step $dt = 0.1$, then $dt = 0.01$.
(b) Draw the path of a generalized Brownian motion with initial value $X_0 = 100$, time step $dt = 0.01$, and parameters $a = 30$, $b = 5$.
(c) Draw the path of a geometric Brownian motion with initial value $X_0 = 100$, time step $dt = 0.01$, and parameters $a = 10\%$, $b = 40\%$.

Problem 5: Simulating market prices and option values

In a spreadsheet, simulate the daily prices S_t of an asset over a year on a 252 trading days basis using the model: $dS_t = \mu S_t dt + \sigma S_t dW_t$, with $S_0 = 100$, $\mu = 10\%$ and $\sigma = 40\%$ (*Hint: consult Problem 4*). Then use the closed-form formulas given in Section C-7 p.220 with an annual risk-free rate $r = 5\%$ and compute on each day:

- The value of a European call with strike 100 and initial maturity 1 year;
- The delta of the call;
- The gamma of the call;
- The theta of the call.

Draw the graph of the evolution of each quantity over time for two simulations and comment on your results.

Problem 6: Delta-hedging simulation

Using your spreadsheet from Problem 5, implement a daily delta-hedging strategy on a short 10,000 calls position with strike 100 and 1-year maturity. Compute on each day:

- Your portfolio value at the beginning of the day;
- The quantity of underlying you need to buy or sell;
- Your portfolio value at the end of the day;
- Your daily P&L (change in portfolio value and overnight financing charge or gain at 5% interest rate p.a. on the portfolio value);
- Your daily P&L proxy using Equation (8-2) p.89;
- Your cumulative P&L.

Comment on your results.

Problem 7: Applying the Ito-Doeblin theorem
Let (X_t) be a geometric Brownian motion with parameters μ and σ, and initial value $X_0 = 1$.

(a) What is the stochastic process followed by $Z_t = X_t{}^n$ (where n is a positive integer)?
(b) What is the stochastic process followed by $H_t = \frac{1}{X_t}$?

Problem 8: Standard Brownian motion. *The three questions are independent.*
Let $(W_t)_{t \geq 0}$ be a standard Brownian motion.

(a) Show that for any time $T > 0$, W_T is normally distributed. What is the expectation and standard deviation?
(b) For fixed $0 < t < t'$, are the random variables W_t and $W_{t'}$ independent? *Hint: calculate the covariance.*
(c) For $\sigma > 0$, define $H_t = \exp(\sigma W_t)$, $h_t = \mathbb{E}(H_t)$ and $G_t = H_t \,/\, h_t$. Find h_t as a function of σ and t. Using the Ito-Doeblin theorem, show that G_t is a geometric Brownian motion and find its parameters as functions of σ.

Problem 9: Continuity and non-differentiability of Brownian motion
Let $(W_t)_{t \geq 0}$ be a standard Brownian motion, $D_t = W_{t+h} - W_t$ the increment between t and $t + h$, and $\delta_t = D_t \,/\, h$ the corresponding slope, where $t > 0$ is fixed and $h > 0$.

(a) What is the distribution of D_t and δ_t? Give the expectation and variance as functions of t and h.
(b) What is the limit of D_t when h goes to 0? Interpret your result.
(c) Re-answer question (b) for δ_t.

Problem 10*: Positivity of geometric Brownian motion
Let $(X_t)_{t \geq 0}$ be a geometric Brownian motion with parameters (a, b) and initial value $X_0 = 1$. Let $t > 0$ be fixed, and $R_n = \frac{X_{t+\frac{1}{n}} - X_t}{X_t}$ for a positive integer n. Assume that X_t is known rather than random.

(a) Explain why, for large n, the distribution of R_n may be approximated by a normal distribution. Find the mean and standard deviation of this distribution.
(b) Let p_n be the probability that $R_n < -1$. Find the limit of p_n as n goes to infinity and interpret your result.

Problem 11*: Quadratic variation. *To solve this problem you must be familiar with the gamma distribution and its properties.*

Let (W_t) be a standard Brownian motion and $Q_n(t) = \sum_{k=0}^{n-1} \left(W_{\frac{k+1}{n}t} - W_{\frac{k}{n}t} \right)^2$ the sum of squared Brownian increments over $[0, t]$ at n regular subintervals.

(a) What is the expectation of $Q_n(t)$?
(b) Show that the distribution of $Q_n(t)$ is gamma and find its parameters.
(c) What is the limit distribution as n goes to infinity?
(d) Explain the rationale for writing: $(dW_t)^2 \equiv dt$.

Problem 12*: Stochastic integration
Let (W_t) be a standard Brownian motion and $X = \frac{1}{T}\int_0^T (T-t)dW_t$.

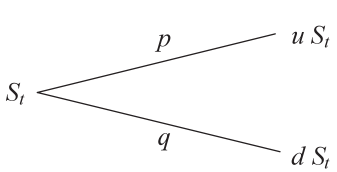
(a) For $n > 0$ write $X_n = \sum_{k=0}^{n-1}\left(1-\frac{k}{n}\right)\left(W_{\frac{k+1}{n}T} - W_{\frac{k}{n}T}\right)$. Show that X_n is normally distributed and then infer the distribution of X with its expectation and variance.
(b) Let $f(t,x) = (T-t)x$. Using the Ito-Doeblin theorem, show that $\frac{1}{T}\int_0^T W_t dt = X$.

Problem 13: Stock price modeling
In this problem, we consider two continuous-time models for the stock price (X_t) of Winner AG which currently trades at $X_0 = €100$ in Frankfurt. Based on historical data, the expected annual return of Winner AG is 12%, i.e. $\mathbb{E}(R) = 0.12$ with $R = \frac{X_1 - X_0}{X_0}$, and its volatility is 40%, i.e. $\mathbb{V}(R) = 0.4^2$.

(a) We model Winner AG's stock price process as a generalized Brownian motion with parameters m and s.
  (i) Find the expectation and variance of X_t as a function of m, s and t. What values would you propose for parameters m and s ?
  (ii) What is the probability of the stock price exceeding €120 in 1 year? 2 years?
  (iii) Can Winner AG's stock price become negative in this model?
(b) Re-answer question (a) if we model the stock price process as a geometric Brownian motion with parameters μ and σ.

Problem 14*: Binomial tree and Brownian motion. In this problem we show that the Brownian motion is the limit-case of a binomial tree as the number of subdivisions goes to infinity.

Consider a stock S whose future price S_t over the period $[0, T]$ is modeled as a *recombining* binomial tree with n steps of length $\tau = T / n$. The tree repeats the pattern shown below, where $u = \exp(\sigma\sqrt{\tau})$, $d = \exp(-\sigma\sqrt{\tau})$ for a given parameter $\sigma > 0$, and the probability to go up or down is $p = q = ½$.



(a) How many terminal nodes are there in the tree?
(b) How many paths are there between the initial price S_0 and a given final price $S_T = u^i d^j S_0$ (where i, j are non-negative integers)?
(c) What is the distribution of S_T?
(d) Let I_T be the (random) number of 'up' branches followed by the stock price over $[0, T]$.
  (i) Express I_T as a function of $\ln(S_T/S_0)$, σ, τ and n.
  (ii) Show that I_T has a binomial distribution and find the distribution parameters.

(e) Using the central limit theorem, show that $\ln(S_T)$ converges to a normal distribution as n goes to infinity and give its parameters.

(f) Generalizing your results, show that if $(\tilde{S}_t)_{0\leq t\leq T}$ is the limit-process as n goes to infinity, then $W_t = \frac{1}{\sigma}\ln\frac{\tilde{S}_t}{\tilde{S}_0}$ satisfies the three properties of a standard Brownian motion (Section 9-3.1 p.98). What is the distribution of $\tilde{S}_T$? Assuming zero interest rates, is this distribution consistent with the lognormal model (Section 7-1.1 p.75)?

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10 The Black-Scholes Model

In Chapters 6 and 7 we presented two different approaches to option valuation: the binomial model, based on an arbitrage argument recursively applied to a tree; and the lognormal model, based on a lognormal distribution of stock prices. The Black-Scholes-Merton model may be seen as a combination of these two approaches.

The model appeared in 1973 at a time of economic turmoil and new financial risks. Options were becoming increasingly popular, but no good or consistent valuation model existed. Black-Scholes came as a breakthrough and is widely viewed as having enabled the dramatic expansion of options markets. It remains the standard model for option valuation some three decades later and was distinguished in 1995 by a “Nobel prize” in economics.

10-1 The Black-Scholes Partial Differential Equation

Combining the features of the binomial and lognormal models,[1] Black-Scholes assumes that the price of the underlying asset between t and $t + dt$ is subject to a random change following a lognormal distribution, i.e. $S_{t+dt} = S_t(1 + X)$ where X is the normally distributed instantaneous return (see Figure 10-1 below).

In addition to the familiar assumptions of no arbitrage, infinite liquidity and the ability to short-sell, the Black-Scholes model also assumes that:

- The price (S_t) of the underlying stock or index S follows a geometric Brownian motion with parameters μ and σ (see Section 9-3.3 p.100):

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t;$$

- The yield curve is flat and constant throughout time at the *continuous* interest rate r;
- There are no dividends;

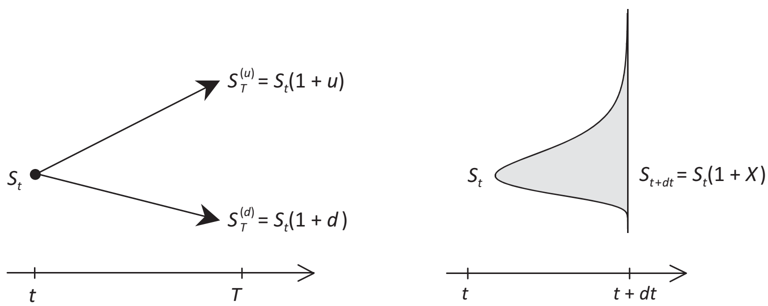

Figure 10-1 The Binomial and Black-Scholes models of asset prices

[1] Note that the binomial model appeared after Black-Scholes.

- The value of the derivative on S is a "sufficiently smooth"[2] function of time and the underlying asset price:

$$D_t = f(t,\ S_t) \text{ for all } t \geq 0.$$

Based on these assumptions, Black, Scholes, and Merton derived a partial differential equation for f which leads to the derivative's value. We now review their derivation.

10-1.1 Ito-Doeblin Theorem for the Derivative's Value

By the Ito-Doeblin theorem (Equation (9-7) p.102) the instant change in the derivative's value may be written as:

$$dD_t = df = \left(\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2}\right) dt + \left(\sigma S_t \frac{\partial f}{\partial S}\right) dW_t.$$

From a financial standpoint, this equation means that between times t and $t + dt$, the value of the derivative is exposed to a **drift** $\left(\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2}\right) dt$ (mean growth over time) and to a **volatility risk** $\left(\sigma S_t \frac{\partial f}{\partial S}\right) dW_t$. This risk is *proportional* to the volatility risk $\sigma S_t dW_t$ of the underlying, with a coefficient of proportionality $\frac{\partial f}{\partial S}$. Figure 10-2 below illustrates the one-to-one correspondence between the underlying price and the derivative's value.

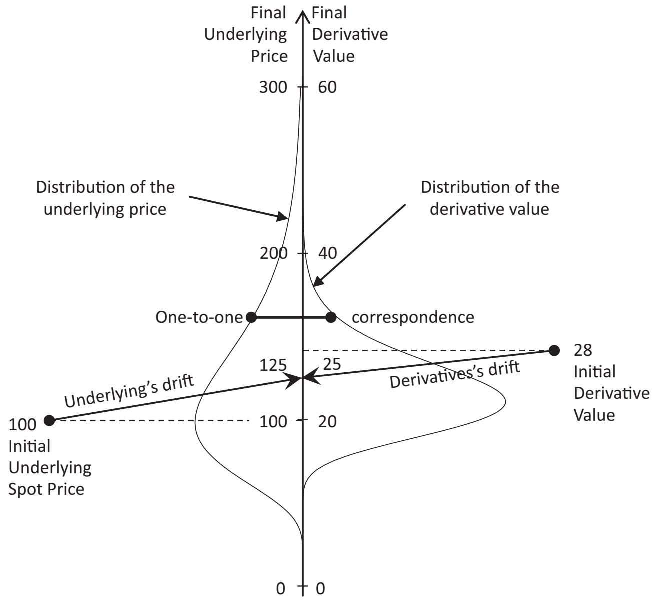

Figure 10-2 Drift and distribution of the underlying price and the derivative value

[2] Continuously differentiable with respect to t and twice continuously differentiable with respect to S.

10-1.2 Riskless Hedged portfolio

Similar to the binomial model the owner of one unit of derivative may, at any time t, eliminate all volatility risk at horizon $t + dt$ by selling δ shares of underlying stock. The value P_t of the resulting portfolio is then:

- $P_t = f(t,\ S_t) - \delta S_t$ at time t, and:
- $P_{t+dt} = f(t + dt,\ S_{t+dt}) - \delta S_{t+dt}$ at time $t + dt$.

Taking the difference between these two expressions we may write:

$$\begin{aligned} dP_t &= P_{t+dt} - P_t \\ &= df - \delta dS_t \\ &= \left[\left(\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2}\right) - \delta \mu S_t\right] dt + \left[\sigma S_t \frac{\partial f}{\partial S} - \delta \sigma S_t\right] dW_t \quad (10\text{-}1) \end{aligned}$$

For the portfolio to be riskless the change in value dP_t must have no random component, i.e. the second bracket must be zero:

$$\sigma S_t \frac{\partial f}{\partial S} - \delta \sigma S_t = 0.$$

Solving for δ yields $\delta = \frac{\partial f}{\partial S}$. This result is in line with our prior observation that the volatility risk of the derivative is exactly proportional to that of the underlying, with coefficient (or **hedge ratio**) $\frac{\partial f}{\partial S}$.

Substituting δ into Equation (10-1) and cancelling terms yields:

$$dP_t = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2}\right) dt. \quad (10\text{-}2)$$

10-1.3 Arbitrage Argument

Because the portfolio is riskless, its value must grow at the risk-free continuous interest rate r:

$$dP_t = r P_t dt.$$

Substituting $P_t = f(t, S_t) - \delta S_t$, we obtain another expression for dP_t:

$$dP_t = r\,(f - \delta S_t)\,dt = \left(rf - r\frac{\partial f}{\partial S} S_t\right) dt. \quad (10\text{-}3)$$

10-1.4 Partial Differential Equation

Connecting Equations (10-2) and (10-3) and rearranging terms, we get the **Black-Scholes partial differential equation**:

$$rf = \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}, \quad (10\text{-}4)$$

where, for ease of notation, we dropped the time subscript in the spot price S_t.

This partial differential equation has an infinite number of solutions which define the admissible (or tradable) derivatives on the underlying asset S. To find an individual solution, additional constraints must be specified, such as the payoff value at maturity. In the case of European vanilla options:

- For the vanilla call : $f(T, S_T) = \max(0, S_T - K)$;
- For the vanilla put : $f(T, S_T) = \max(0, K - S_T)$.

10-1.5 Continuous Delta-hedging

It is worth emphasizing that the hedge ratio $\delta = \frac{\partial f}{\partial S}$ continuously changes over time: the Black-Scholes model builds on a continuous delta-hedging trading strategy (see Section 8.1.1 p.84). By following this strategy an option trader may in theory replicate any option payoff at no risk.

10-2 The Black-Scholes Formulas for European Vanilla Options

For European calls and puts with strike K and maturity T, the Black-Scholes partial differential equation has the analytical solutions:

$$\begin{aligned} c_0 &= S_0 N(d_1) - K e^{-rT} N(d_2) \\ p_0 &= K e^{-rT} N(-d_2) - S_0 N(-d_1) \end{aligned} \tag{10-5}$$

where $N(\cdot)$ is the standard normal cumulative distribution, and d_1, d_2 are the coefficients:

$$\begin{aligned} d_1 &= \frac{\ln \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} \\ d_2 &= \frac{\ln \frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T} \end{aligned} \tag{10-6}$$

The step-by-step derivation of these formulas is quite technical and we refer interested readers to Wilmott (2007). Interestingly enough, they are very similar to the closed-form formulas obtained in the lognormal model (Equations (7-1) and (7-2) pp.76–77). A closer look reveals they are equivalent: e^{-rT} is the discount factor for maturity T, and substituting $F_0 = S_0 e^{rT}$ into the closed-form formulas yields the Black-Scholes formulas.

One may then question the benefit of Black-Scholes over the lognormal model. The differences are subtle yet significant. For one, Black-Scholes gives a "recipe" to replicate option payoffs by dynamically trading the underlying asset. Furthermore, such dynamic replication is theoretically perfect and riskless, which means that the appropriate discount rate in the lognormal model is the risk-free rate rather than a higher rate. Before Black-Scholes, nobody knew either of these two crucial facts.

10-3 Volatility

Remarkably enough, the drift parameter μ neither appears in the Black-Scholes partial differential equation (Equation (10-4)) nor in the closed-form formulas (Equations (10-5) and (10-6)): the value of an option simply does not seem to depend on the expected return[3] of its underlying asset.

This may be somewhat counter-intuitive, but it must be recalled that stock prices already reflect expectations of future growth. In other words, when a stock's expected return μ increases the spot price S_t usually increases as well, which in turn changes the option value. If the option value also depended on μ there would be double accounting.

On the other hand, the volatility parameter σ plays a crucial role: option values tend to be very sensitive to this parameter. For example, consider a 1-year European call struck at €110 on a stock currently trading at €100. The higher the volatility, the more likely it is that the stock price will end up well above €110 and the option pay off a large amount. Thus, the call value is also higher.

We now review two standard approaches to determine the volatility parameter σ: historical volatility and implied volatility.

10-3.1 Historical Volatility

Volatility may be estimated using historical prices. Given a sample $s_0, s_1, s_2, \ldots, s_n$ observed at regular time periods of length τ, we may compute the mean and variance estimates:[4]

$$\bar{u} = \frac{1}{n}\sum_{i=1}^{n} u_i, \quad \hat{s}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2,$$

where $u_i = \ln \frac{s_i}{s_{i-1}}$ is the log-return for observation i. An annualized estimate of volatility σ is then:

$$\hat{\sigma} = \frac{\hat{s}}{\sqrt{\tau}}.$$

For example the historical volatility of daily log-returns of the S&P 500 in July 2011 was 15.5% on the basis of 252 trading days per year (i.e. $\tau = 1/252$).

Log-returns are preferred to the more familiar arithmetic returns used in Section 4-1.1 p.35. This is for consistency with the geometric Brownian motion assumption in Black-Scholes.

The historical approach to estimating volatility has a number of issues. How far back in time should one go? Should prices be observed every second, hour, day, or month? Is past volatility a good predictor of future volatility? Because these questions do not have definite answers, historical volatility is only used as a very rough estimate to find the Black-Scholes value of an option.

[3] $\frac{dS_t}{S_1} = \frac{S_{t+dt} - S_t}{S_t}$ is the instantaneous rate of return on S. Since dW_t has zero mean we have: $\mathbb{E}\left(\frac{dS_t}{S_t}\right) = \mu dt$.

[4] Recall that $\frac{1}{n}\sum(u_i - \overline{u})^2$ is a *biased* estimation of the variance of a sample (u_i) of length n.

10-3.2 Implied Volatility

Suppose Kroger Co. currently trades at \$24, and that 1-year and 2-year at-the-money European calls on Kroger Co. trade at \$2 and \$2.50 respectively. How can we use this information to find a sensible value for an 18-month European at-the-money call on Kroger Co.?

The wrong way to go about this is to take the average of the two call prices and value the 18-month call at \$2.25. Similar to bonds, raw option prices do not mean much and cannot be interpolated. However, just as we extract yields from bond prices, we may reverse-engineer option prices in terms of volatility, as illustrated in Figures 10-3 and 10-4 below.

Assuming a flat interest rate curve at 1%, this approach gives a 20% volatility for the 1-year call and a 17% volatility for the 2-year call. We may thus estimate the volatility parameter of the 18-month call to be around 18.5% with a corresponding call value of \$2.33.

Generally, the level of parameter σ which matches an option's Black-Scholes value with its market price is called **implied volatility**, and it is usually calculated using a numerical equation solver. Note that implied volatility may not be the same for all options on a given underlying asset – in fact, it is typically observed that every vanilla option has a different implied volatility.

The implied approach to estimating volatility is more accurate than the historical approach, but it is not always available.

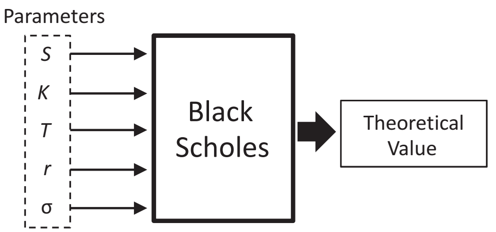

Figure 10-3 Black-Scholes valuation

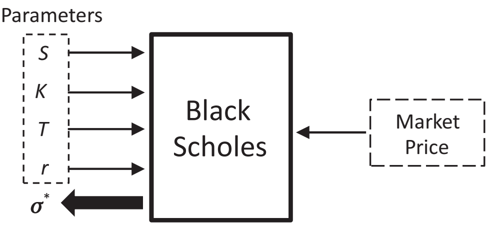

Figure 10-4 Implied volatility

10-4 Further Reading

- On the Black-Scholes partial differential equation: Paul Wilmott (2007) *Paul Wilmott Introduces Quantitative Finance* 2nd edition, John Wiley & Sons: Chapters 7 and 8.
- On the Black-Scholes model: John C. Hull (2009) *Options, Futures and Other Derivatives* 7th Edition, Prentice Hall: Chapter 13.
- Fischer Black and Myron Scholes (1973) The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, 81(3), pp. 637–654.
- Robert C. Merton (1973) Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science* (The RAND Corporation) 4(1), pp. 141–183.

10-5 Problems

Problem 1: True or False?
"In the Black-Scholes model, the volatility risk of the derivative is hedged by selling $\delta = \frac{\partial f}{\partial S}$ units of the underlying because the two risks are proportional. Since the drift coefficient μ of the underlying does not play any role, one could equivalently hedge the volatility risk of

the derivative by selling the same quantity of any other asset following a geometric Brownian motion with identical volatility parameter σ."

Problem 2: Limit-cases of the Black-Scholes formulas
Find the limit of the Black-Scholes formulas (Equations (10-5) and (10-6)) as $T \to 0$ or $\sigma \to 0$ and comment on your results.

Problem 3: Implied volatility 'smile'
The DAX is a total return index of 30 selected German blue chip stocks traded on the Frankfurt Stock Exchange. (A total return index assumes that dividends are reinvested.) On 7 July 2011 the spot price of the DAX was €7484.5 and the prices of European call options expiring on 21 December 2012 are shown in the table below. Assuming a 1.4% p.a. continuous interest rate, calculate the implied volatility of each option and produce the corresponding 'smile' curve.

Strike (€)	Call price (€)	Strike (€)	Call price (€)
6600	1405.4	7500	841.6
6700	1333.5	7600	758.1
6800	1263.1	7700	703.2
6900	1194.0	7800	650.5
7000	1126.6	7900	625.5
7100	1060.9	8000	551.3
7200	996.8	8100	505.0
7300	934.2	8200	461.1
7400	873.8		

Problem 4: Black-Scholes and forward contracts

(a) Under the assumptions of the Black-Scholes model given in Section 10-1 p.109, show that the arbitrage price at time t of a forward contract with strike K and maturity T is $\phi_t = S_t - Ke^{-r(T-t)}$, then verify that ϕ_t satisfies the Black-Scholes partial differential equation (Equation (10-4) p.111).
(b) Show that the Black-Scholes formulas (Equations (10-5) and (10-6) p.112) are consistent with put-call parity (Equation (5-5) p.58).

Problem 5: Exponential asset
In the Black-Scholes model, may a derivative security on an underlying asset S be worth $D_t = \exp(S_t)$ at all times $t \leq T$?

Problem 6: Stock-options
The employee compensation scheme of MeToo.Com ("MTC") includes stock-options, i.e. out-of-the-money call options on the company's stock. The stock-options are issued annually and allow stocks to be purchased two years later at $1.2 \times S$, where S is the stock price at time of issuance. MeToo.Com does not pay any dividend and the continuous interest rate curve is flat at 5% p.a.

(a) As an employee of MeToo.Com you just received 1,000 stock-options and you have gathered the following data. Can you estimate the value of your award?

Dow Jones Industrial Index (DJI)	10,000
MeToo.Com (MTC) Closing Price	\$10.00
DJI 1-year historical volatility	20%
MTC 1-year historical volatility	35%
MTC 1-year \$10 call implied volatility	25%
MTC 2-year \$10 call implied volatility	32%
MTC 2-year \$12 call implied volatility	30%

(b) As the director of Human Resources at MeToo.Com you just distributed 1,000,000 stock-options to the company's employees. You plan a 10% increase for the next distribution.
  (i) Estimate the cost of this increase, stating your assumptions and risks incurred.
  (ii) Silverman, Sacks & Co., a reputed investment bank, offers 100,000 forward start calls on MeToo.Com for \$1,360,000. The calls will strike at 120% of the stock price in 1 year and become exercisable 2 years later. Is this a good offer?

Problem 7: Exact relationship between gamma and theta
Consider an option on an underlying S which you delta-hedge at time t. Show that, in the Black-Scholes model, the option's gamma and theta must satisfy the equation:

$$\Theta + \frac{1}{2}\sigma^2 S_t^2 \Gamma = r P_t,$$

where P_t is the value of your delta-hedged portfolio. Compare this result with Equation (8-1) p.88.

Problem 8*: Black-Scholes with continuous dividends
Consider a stock S which pays a continuous dividend at the nominal annual rate q. All the assumptions of the Black-Scholes model otherwise hold.

(a) Show that the forward price of S is $F_0 = S_0 e^{(r-q)T}$, where S_0 is the spot price. *Hint: consult Chapter 5, Problem 8 p.63 and take the limit in the fashion of Chapter 1, Problem 7 p.9.*
(b) What are the corresponding closed-form formulas for the European call and put? *Hint: use Equations (7-1) and (7-2) pp.76–77.*
(c) Let P_t be the value at time t of a portfolio which is short one unit of derivative worth $D_t = f(t, S_t)$ and long $\delta = \frac{\partial f}{\partial S}(t, S_t)$ units of underlying.
  (i) Show that: $dP_t = -dD_t + \frac{\partial f}{\partial S} dS_t + q \frac{\partial f}{\partial S} S_t dt$.
  (ii) What does the Black-Scholes partial differential equation become?

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11 Volatility Trading

With the development of option markets the prices of many options became available in real time, and this additional information led to more sophisticated models and securities. Somewhere down the road it also became clear that option trading was all about volatility, specifically the gap between realized volatility, or how much a stock or equity index actually moves, and implied volatility, or how much it is expected to move by the market.

In this chapter we review the definitions of implied and realized volatilities and show how one may trade volatility using options.

11-1 Implied and Realized Volatilities

11-1.1 Realized Volatility

Realized volatility is another word for historical volatility: the magnitude of 'realized', i.e. observed, up- and down-moves. As discussed in previous chapters, it is calculated as the annualized standard deviation of daily, weekly or monthly returns:

$$\sigma_{\text{Realized}} = \sqrt{\frac{M}{N-1}\sum_{t=1}^{N}(r_t - \bar{r})^2},$$

where N is the number of observations, M is the number of periods per year (e.g. 252 for daily), $\bar{r} = \frac{1}{N}\sum_{t=1}^{N} r_t$ is the mean return, and r_t is either the arithmetic or log-return for observation t. On occasion, $\bar{r}$ is assumed to be zero, in which case the correct denominator is N instead of $N-1$.

Realized volatility may occasionally refer to the volatility parameter in the *assumed* behavior of real stock prices, distinct from the theoretical behavior posited by a model such as Black-Scholes. For example, one might assume that in the "real" world stock prices follow a geometric Brownian motion with realized volatility σ (i.e. $dS_t/S_t = \mu dt + \sigma dW_t$), distinct from the geometric Brownian motion in the Black-Scholes world with volatility σ^* (i.e. $dS_t/S_t = \mu^* dt + \sigma^* dW_t^*$). Of course these two assumptions contradict each other, and there is ample empirical evidence that actual stock prices do not follow a geometric Brownian motion in the first place, but this fiction can facilitate calculations and lead to useful findings.

11-1.2 Implied Volatility

Implied volatility is the value σ^* of the volatility parameter that makes the market price of a given call or put match the corresponding Black-Scholes value:

- For a European call trading at price A, σ^* is the solution to $S_0 N(d_1(\sigma)) - Ke^{-rT}N(d_2(\sigma)) = A$;

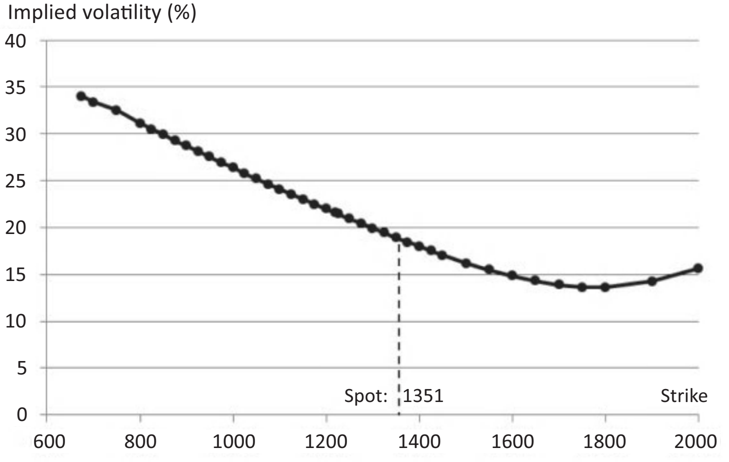

Figure 11-1 Implied volatility smile of options on S&P 500 index expiring 22-Dec-2011 as of 7-Jul-2011

(Source: Bloomberg)

- For a European put trading at price B, σ^* is the solution to $Ke^{-rT}N(-d_2(\sigma)) - S_0 N(-d_1(\sigma)) = B$;

where notations are those of Section 10-2 p.112 and the dependence on σ was made explicit. Because of put-call parity, European calls and puts of identical characteristics (underlying, strike, maturity) must have the same implied volatility.

On most equity option markets it is observed that every option has a different implied volatility. In particular, for a given maturity T, a so-called '**implied volatility smile**' is observed as shown in Figure 11-1 above for the S&P 500 index. We can see that low-strike options have a higher implied volatility than at-the-money and high-strike options.

As a phenomenon, the smile exposes some limitations of the Black-Scholes model. For example, it is "wrong" to calculate the value of a call spread using Black-Scholes, as shown in Problem 2. There are other, more sophisticated models which produce option values that are consistent with the smile, but they have their own limitations too. In a sense, all models are "wrong," but it does not mean they are useless.

Implied volatility may be thought of as the market's guess at future realized volatility in the scenario suggested by the strike. For example, low-strike puts protect investors against a market crash, in which case realized volatility spikes up: it makes sense for the market to see their implied volatility at a higher level.

11-2 Volatility Trading Using Options

In Section 8-2.3 p.88 we already saw that the P&L on a delta-hedged option position is the product of the option's dollar gamma and the difference between the squared realized return and implied variance:

$$\text{P\&L}_{\Delta t} \approx \frac{1}{2}\Gamma_t S_t^2 \left[\left(\frac{\Delta S_t}{S_t} \right)^2 - \sigma^2 \Delta t \right], \tag{11-1}$$

where Δt is the delta-hedging period, Γ_t is the option's gamma, S_t is the spot price, ΔS_t is the change in spot price, and σ is implied volatility. For example, if the dollar gamma is \$10,000 (per % per %), the spot price moves by 1% and implied volatility is 32%, we obtain a one-day P&L of $10{,}000 \times (1^2 - 32^2/252) \approx -\$30{,}000$.

The squared realized return in Equation (11-1) may be thought of as the instantaneous realized variance of the underlying. If we delta-hedge an option at N regular intervals of length Δt until maturity, we then capture the gap between realized and implied variances, weighted by dollar gammas:

$$\text{Cumulative P\&L} \approx \sum_{t=1}^{N} \Gamma_{t-1}^{\$} \left[r_t^2 - \sigma^2 \Delta t \right], \tag{11-2}$$

where r_t is the arithmetic return between $t - \Delta t$ and t and $\Gamma_{t-1}^{\$}$ is the dollar gamma at $t - \Delta t$.

Equation (11-2) shows that delta-hedging is essentially a bet on realized volatility versus implied. However, it is not a pure bet because dollar gammas generate a fair amount of **P&L path-dependency**: depending on the particular path followed by the spot price, the delta-hedging strategy may generate very different results.

Problem 7 looks into the continuous time version of Equation (11-2) which was initially derived by Carr and Madan (1998).

11-3 Volatility Trading Using Variance Swaps

As an alternative to delta-hedging, banks and exchanges created new derivative securities allowing investors to trade volatility directly. These new securities include:

- Volatility swaps, where two parties agree to exchange future realized volatility (converted into dollars) at a pre-agreed price and date;
- Variance swaps, where two parties agree to exchange future realized variance (converted into dollars) at a pre-agreed price and date;
- CBOE VIX Futures,[1] which allow investors to bet on the level of short-term implied volatility at a future date;
- Options on realized volatility, options on the VIX, VIX trackers...

Of all the securities above, variance swaps are by far the most robust because they can be approximately replicated with a static portfolio of vanilla options.

[1] The VIX or "Volatility Index" is calculated and maintained by the Chicago Board Options Exchange to reflect the level of implied volatility of options on the S&P 500 index which are near expiry.

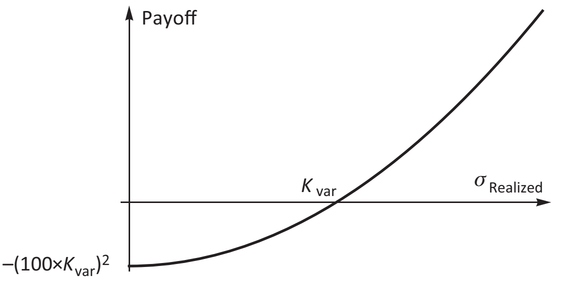

Figure 11-2 Variance swap payoff as a function of realized volatility

11-3.1 Variance Swap Payoff

From the buyer's viewpoint, a variance swap on an underlying S with strike K_{var} and maturity T pays off:

$$\text{Variance Swap Payoff} = 100^2 \times \left(\sigma^2_{\text{Realized}} - K^2_{\text{var}}\right),$$

where σ_{Realized} is the realized volatility of daily log-returns between $t = 0$ and $t = T$ with zero mean assumption. For example, a 1-year variance swap on the S&P 500 struck at 25% pays off $100^2 \times (0.26^2 - 0.25^2) = \$51$ if realized volatility in the forthcoming year is 26%.

Figure 11-2 above shows the variance swap payoff as a function of realized volatility. We can see that the shape is convex quadratic, which implies that profits are boosted and losses discounted as realized volatility goes away from the strike.

11-3.2 Variance Swap Market

Variance swaps trade mostly over-the-counter. Before the 2007–08 financial crisis they were very popular and liquid; after the crisis liquidity dried up and the market went into limbo for some time. Even though it would be technically easy to revive the market by moving to an exchange, regulators and traders did not catch up on this idea yet.

Table 11-1 overleaf shows some bid-offer market quotes for various underlyings and maturities.

11-3.3 Variance Swap Hedging and Pricing

On a mature option market with many liquid options of different strikes and maturities, a trader can approximately replicate the payoff of a variance swap with maturity T by setting up a portfolio of European calls and puts of the same maturity, and then delta-hedge it daily.

To see this, recall the delta-hedging cumulative P&L proxy (Equation (11-2)):

$$\text{Cumulative P\&L} \approx \sum_{t=1}^{N} \Gamma^{\$}_{t-1}\left[r_t^2 - \sigma^2 \Delta t\right].$$

Table 11-1 Variance swap market on three equity indexes as of 19-Sep-2011

Maturity	S&P 500 (Spot: 1182)	EuroStoxx 50 (Spot: 2096)	Nikkei 225 (Spot: 8860)
Oct11 (1 month)	33.70 / 34.70	43.95 / 45.45	32.00 / 34.50
Nov11 (2 months)	34.40 / 35.40	44.10 / 45.60	32.00 / 34.00
Dec11 (3 months)	34.60 / 35.60	43.50 / 45.00	32.00 / 34.00
Mar12	34.30 / 35.30		31.50 / 33.00
Jun12	34.00 / 35.00	40.50 / 42.00	31.00 / 32.50
Dec12	32.95 / 33.95	38.10 / 39.60	30.50 / 32.00
Dec13	32.20 / 33.20		30.00 / 31.50
Dec15	32.05 / 33.05		
Dec17	32.05 / 33.05		
Dec19	32.20 / 33.20		

(Source: Large US investment bank)

As noted earlier this equation is usually path-dependent. However, if we could find a portfolio of vanilla options with constant dollar gamma c, such path-dependency would disappear and the resulting P&L equation would closely resemble the payoff of a variance swap:

$$\text{Cumulative P\&L} \approx c\left[\sum_{t=1}^{N} r_t^2 - \sigma^2 T\right].$$

Figure 11-3 below shows the dollar gamma profiles of options on an underlying S trading at \$100 with strikes from \$25 to \$200 spaced \$25 apart. We can see that each profile peaks around the strike, and that the peaks lie on a diagonal.

A natural idea here is to bring all peaks on a horizontal line using option quantities inversely proportional to the strike, and hope that the resulting portfolio would have a nearly constant dollar gamma profile.

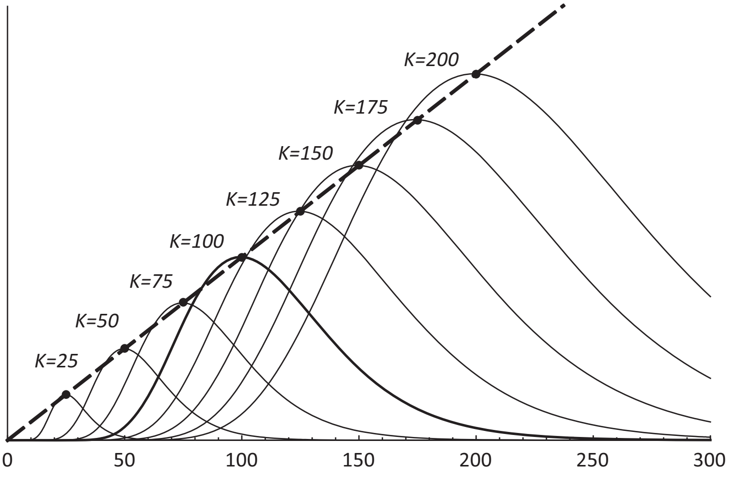

Figure 11-3 The dollar gammas of options with strikes spaced 25 apart

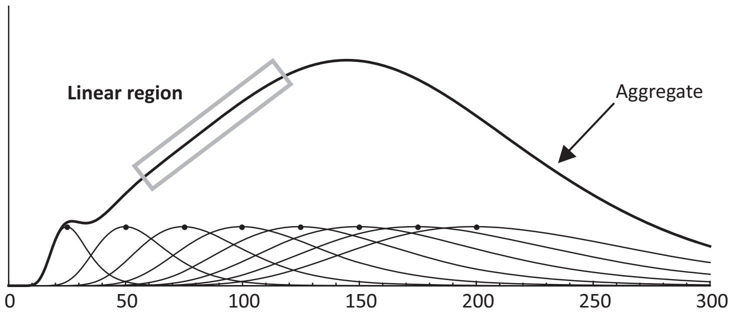

Figure 11-4 Dollar gamma of a portfolio of options with weights inversely proportional to the strike

This is shown in Figure 11-4 above, where we can see that the portfolio's dollar gamma is *not* constant as hoped. However, we can notice that there is a linear region in the 60–125 strike range. This suggests that using quantities inversely proportional to the *square* of strike, we would obtain a constant dollar gamma region, as confirmed in Figure 11-5 below.

A perfect hedge with constant dollar gamma for all underlying levels would take infinitely many options struck along a continuum between 0 and infinity and weighted in quantities inversely proportional to the square of strike. Note that this is a strong result, as the static hedge is both space (underlying level) and time independent.

The price of the perfect portfolio gives us the fair value of the variance swap. In practice, however, only a finite number of strikes are available and we have the approximation:

$$K_{\text{var}}^2 \approx \frac{2(1+r)^T}{T}\left[\sum_{i=1}^{n}\frac{p_0(K_i)}{K_i^2}\Delta K_i + \sum_{i=n+1}^{n+m}\frac{c_0(K_i)}{K_i^2}\Delta K_i\right],$$

where r is the interest rate for maturity T, $K_1 < \cdots < K_n \le F_0 \le K_{n+1} < \cdots < K_{n+m}$ are the successive strikes of n puts worth $p_0(K_i)$ and m calls worth $c_0(K_i)$, and $\Delta K_i = K_i - K_{i\text{-}1}$ is the strike step.

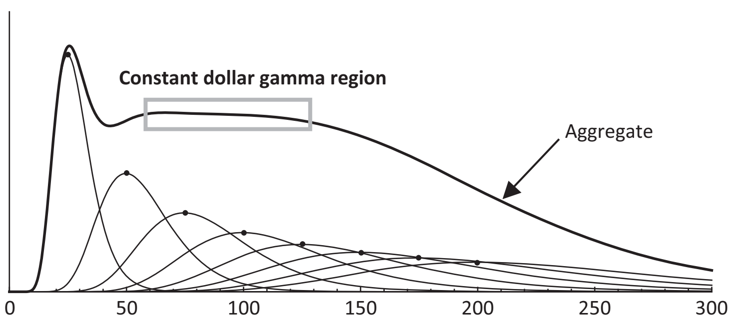

Figure 11-5 Dollar gamma of a portfolio of options with weights inversely proportional to the square of strike

An illustration of this calculation in the presence of the implied volatility smile is given in Problem 5.

11-4 Further Reading

- On variance swaps: Kresimir Demeterfi, Emanuel Derman, Michael Kamal and Joseph Zou (1999) A Guide to Volatility and Variance Swaps, *The Journal of Derivatives*, 6(4) Summer 1999, pp. 9–32.
- On variance swaps and volatility trading: Sébastien Bossu, Eva Strasser and Régis Guichard (2005) Just what you need to know about variance swaps, JPMorgan Equity Derivatives Report.
- Peter Carr and Dilip Madan (1998) Towards a theory of volatility trading, in R. Jarrow (editor), *Volatility*. Risk Publications, pp. 417–427.

11-5 Problems

Problem 1: Realized volatility. *The two questions are independent.*

(a) In early 2010, the Dow Jones Industrial Average index typically moved by roughly $\pm 0.8\%$ every day. On May 6th at 2:45pm a "flash crash" occurred and the index plunged about 9%. What would have been the impact on the 6-month realized volatility if the index had not recovered by the end of the day?
(b) A stock has 9-month and 1-year realized volatilities of 20% and 30% respectively. What happened during the 3-month window between 9 and 12 months?

Problem 2: Call spread and Black-Scholes
Bulls Inc.'s stock trades at \$100 in New York and does not pay any dividend. On the option market, 1-year at-the-money calls on Bulls Inc. trade at 35% implied volatility, and 1-year calls struck at \$125 trade at 25% implied volatility. The interest rate curve is flat at 0%.

(a) What is the price of a \$100–\$125 call spread on Bulls Inc.?
(b) What is the theoretical value of the call spread in the Black-Scholes model with a volatility parameter $\sigma = 35\%$? What about $\sigma = 25\%$?
(c) Using a solver can you find the level of σ which makes the Black-Scholes theoretical value match the market price?

Problem 3: Volatility trading with options
You are long a call which you delta-hedge daily until maturity. Your cumulative P&L proxy is given by Equation (11-2) p.119.

(a) Imagine the "real" stock price process is a geometric Brownian motion with zero drift and realized volatility $\sigma_R > \sigma$. Are you guaranteed to make money? If not, what can you say about your expected P&L?

(b) Imagine the "real" stock price process has increments $dS_t = \sigma_t S_t dW_t$ where the instantaneous realized volatility (σ_t) is a stochastic process independent from (S_t) which satisfies $\mathbb{E}(\sigma_t) > \sigma$. Are you guaranteed to make money? If not, what can you say about your expected P&L?

Problem 4: Forward variance swap

On the variance swap market of the Dow Jones EuroStoxx 50 index, 1-year variance trades at 40% and 2-year variance trades at 36%. A client comes to you asking for a quote on a 'forward variance swap' which pays off the *forward* realized variance $\sigma_{\mathrm{FR}}^2 = \sum_{t=253}^{504} r_t^2$ observed between 1 and 2 years, minus the squared strike $K_{\mathrm{fvar}}{}^2$.

(a) Assuming zero interest rates, find a replicating portfolio for the forward variance swap and calculate its fair strike K_{fvar}.

(b) Does your answer change if you assume a 5% flat interest rate curve?

Problem 5: Variance swap pricing. *This is a continuation of Chapter 10, Problem 3 p.115.*

Assume the implied volatility smile of options on the DAX is given by $\sigma^*(K) = 20.6\% - 0.2464 \times \ln \frac{K}{7640}$. Calculate out-of-the-money call and put values for strikes between 5,000 and 10,000 spaced 100 apart, and then estimate the fair strike of a variance swap.

Problem 6: Volatility swap

A volatility swap pays off $\$100 \times (\sigma_{\mathrm{Realized}} - K_{\mathrm{vol}})$. Suppose that both a volatility swap and a variance swap on the same underlying S and maturity T trade on the market. Show that if there are no arbitrages, liquidity is infinite, and short-selling is feasible, then $K_{\mathrm{vol}} < K_{\mathrm{var}}$.

Problem 7*: Continuous delta-hedging P&L equation

You are a trader who is long an option on a non-dividend paying stock S which you continuously delta-hedge until maturity T. At any given time t, you calculate the value $f(t, S_t)$ of the option using the Black-Scholes model with continuous interest rate r and volatility parameter σ. Your treasury department requires you to borrow or lend enough cash at all times to bring the total mark-to-market value of your positions (cash, stock, and option) to zero.

(a) Suppose that the value of your delta-hedged portfolio P_t is positive. How much cash do you need to lend or borrow? What is your lending profit or borrowing cost between times t and $t + dt$? What if $P_t < 0$?

(b) Show that your total P&L (net of treasury operations) between times t and $t + dt$ is:

$$\pi_t = \frac{1}{2}\Gamma_t S_t^2 \left[\left(\frac{dS_t}{S_t} \right)^2 - \sigma^2 dt \right],$$

where $\Gamma_t = \frac{\partial^2 f}{\partial S^2}(t, S_t)$ is the option's gamma. *Hint: use the Ito-Doeblin theorem to find the P&L on your option position, then consult Chapter 10, Problem 7 p.116 to make a substitution and cancel terms as appropriate in your total P&L expression.*

(c) Suppose that the real market dynamics of the stock is not the geometric Brownian motion posited by Black-Scholes, but instead a stochastic process with variable drift and volatility:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t.$$

Show that your cumulative P&L at maturity T is then:

$$\Pi_T = \frac{1}{2}\int_0^T e^{r(T-t)}\Gamma_t S_t^2(\sigma_t^2 - \sigma^2)dt.$$

Hint: consult Chapter 9, Problem 11 p.105 and argue that $(dS_t\,/\,S_t)^2 \equiv \sigma_t^2 dt$.

An Introduction to Equity Derivative c uveory and ractice, Second dcmone
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12 Exotic Derivatives

We encountered several exotic options in previous chapters. With the progress made in computing speed, the possibilities to design, value, and hedge complex payoffs have become virtually limitless. In this chapter we review the payoff and properties of the most widespread equity derivatives exotics, and then introduce several theoretical extensions of the Black-Scholes model.

12-1 Single-Asset Exotics

12-1.1 Digital Options

A **European digital** or **binary option** pays off \$1 if the underlying asset price is above the strike K at maturity T, and 0 otherwise:

$$\text{Digital Payoff} = \begin{cases} 1 & \text{if } S_T > K \\ 0 & \text{otherwise} \end{cases}$$

In its American version, which is more uncommon, the option pays off \$1 *as soon as* the strike level is hit.

Digital options are not easy to dynamically hedge because their delta can become very large near maturity (see Problem 1). Exotic traders tend to 'overhedge' them with a tight call spread.

Digital options have closed-form Black-Scholes formulas which only give a very rough proxy when there is an implied volatility smile. This is because of their connection with call spreads.

12-1.2 Asian Options

In an **Asian call** or **put**, the final underlying asset price is replaced by an average:

$$\text{Asian Call Payoff} = \max(0, A_T - K);$$
$$\text{Asian Put Payoff} = \max(0, K - A_T),$$

where $A_T = \frac{1}{n}\sum_{i=1}^{n} S_{t_i}$ for a set of pre-agreed 'fixing dates' $t_1 < t_2 < \ldots < t_n \leq T$. For example, a 1-year at-the-money Asian call on the S&P 500 index with quarterly fixings pays off $\max(0, \frac{S_{0.25}+S_{0.5}+S_{0.75}+S_1}{4} - S_0)$, where S_0 is the current spot price and $S_{0.25}, \ldots, S_1$ are the future spot prices observed every 3 months.

On occasion, the strike may also be replaced by an average, typically over a short initial observation period.

Fixed-strike Asian options are always cheaper than their European counterparts, because A_T is less volatile than S_T.

There are no closed-form Black-Scholes formulas for "arithmetic" Asian options; however, there are closed-form formulas for "geometric" Asian options, providing an upper or lower bound. Based on these formulas, we have the rule of thumb:

$$\text{Asian option value} \approx \text{European option value}/\sqrt{3},$$

provided the fixing dates are regularly spaced apart and the strike is nearly at-the-money-forward.

12-1.3 Barrier Options

In a **barrier call** or **put**, the underlying asset price must hit, or never hit, a certain barrier level H before maturity:

- For a 'knock-in' option the underlying must hit the barrier, or else the option pays nothing;
- For a 'knock-out' option the underlying must *never* hit the barrier, or else the option pays nothing.

Figures 12-1 and 12-2 below illustrate how a knock-out call and a knock-in put behave on two sample paths.

Barrier options are always cheaper than their European counterparts, because their payoff is subject to an additional constraint. On occasion, a fixed cash 'rebate' is paid out if the barrier condition is not met.

Similar to digital options, barrier options are not easy to dynamically hedge: their delta can become very large near the barrier level. Exotic traders tend to 'overhedge' them by shifting the barrier a little in their valuation model.

Barrier options have closed-form Black-Scholes formulas. When there is an implied volatility smile, however, these formulas are "wrong" and only provide a very rough proxy.

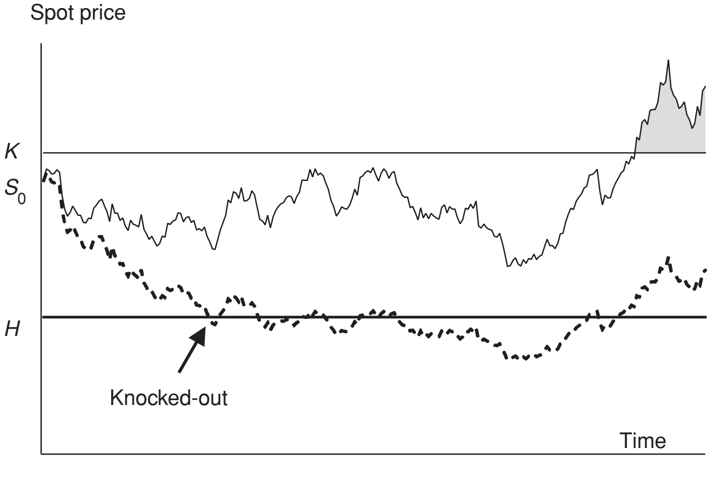

Figure 12-1 Stock price evolution and behavior of a knock-out call

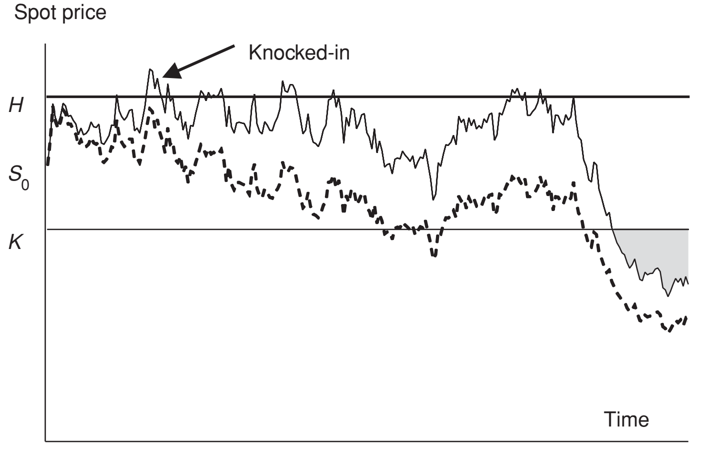

Figure 12-2 Stock price evolution and behavior of a knock-in put

In practice the barrier is often monitored on a set of pre-agreed 'fixing dates' $t_1 < t_2 < \ldots < t_n \leq T$ rather than continuously over the interval $[0, T]$. Monte-Carlo simulations are then commonly used for valuation.

Broadie-Glasserman-Kou (1997) derived a nice result to switch between continuous and discrete barrier monitoring by shifting the barrier level H by a factor $\exp(\pm\beta\sigma\sqrt{\Delta t})$ where $\beta \approx 0.5826$, σ is the underlying volatility, and Δt is the time between two fixing dates (see Problem 4 for an application).

12-1.4 Lookback Options

A **lookback call** or **put** is an option on the maximum or minimum price reached by the underlying asset until maturity:

$$\text{Lookback call payoff} = \max\left(0, \max_{0\leq t\leq T} S_t - K\right)$$

$$\text{Lookback put payoff} = \max\left(0, K - \min_{0\leq t\leq T} S_t\right)$$

Lookback options are always more expensive than their European counterparts: about twice as much when the strike is nearly at-the-money. They have closed-form Black-Scholes formulas, which only give a rough proxy when there is an implied volatility smile.

In practice the maximum or minimum is often monitored on a set of pre-agreed 'fixing dates' rather than continuously. Monte-Carlo simulations are then commonly used for valuation.

12-1.5 Forward Start Options

In a **forward start option** the strike is determined as a percentage k of the spot price on a future start date $t_0 > 0$:

$$\text{Forward start call payoff} = \max(0, S_T - kS_{t_0})$$

$$\text{Forward start put payoff} = \max(0, kS_{t_0} - S_T)$$

At $t = t_0$ a forward start option becomes a regular option. Note that the forward start feature is not specific to vanilla options and can be added to any exotic option which has a strike.

Exotic option payoffs are often based on the underlying gross rate of return over the period $[t_0, T]$ instead of the final spot price. In the case of a forward start call the payoff would read:

$$\text{Forward start ROR call payoff} = \max\left(0, \frac{S_T}{S_{t_0}} - k\right).$$

The value of a forward start ROR option then mostly depends on implied volatility and its delta is small. This can be seen on the Black-Scholes proxy value of an at-the-money-forward ROR call (see Chapter 7, Problem 8 pp.80–81):

$$\text{ATMF forward start ROR call or put value} \approx 0.4e^{-rt_0}\sigma^*\sqrt{T - t_0},$$

where σ^* is the implied volatility of an option with a strike equal to the underlying forward price $F(t_0, T)$ and time to maturity $T - t_0$. Note that σ^* constantly changes with time as option market conditions change.

The valuation and hedging of forward start options can be difficult and typically require a stochastic volatility model (see Section 12-3.2.1 p.135).

12-1.6 Cliquet Options

A **cliquet** or **ratchet option** consists in a series of consecutive forward start options, for example:

$$\text{Monthly cliquet option payoff} = \max\left[0, \sum_{i=1}^{12} \min\left(5\%, \frac{S_{i/12}}{S_{(i-1)/12}} - 1\right)\right],$$

where 5% is the 'local cap' amount. In other words, this particular cliquet option pays off the greater of zero and the sum of monthly returns, each capped at 5%.

Cliquet options can be very difficult to value and especially hedge.

12-1.7 Structured Products

Structured products combine several securities together, especially exotic options. They are typically sold as equity-linked notes (ELN) or mutual funds to small investors as well as large institutions. These notes and funds are sometimes traded on exchanges.

Examples

Capital Guaranteed Performance Note	**'Reverse Convertible' Note**
Issuer: ABC Bank Co.	Issuer: ABC Bank Co.
Notional amount: \$10,000,000 Issue date: [Today]	Notional amount: €2,000,000 Issue date: [Today]

Capital Guaranteed Performance Note	**'Reverse Convertible' Note**
Maturity date: [Today + 5 years] Underlying index: S&P 500 (SPX) Payoff: $\text{Notional} \times \left[100\% + \text{Participation} \times \max\left(0, \frac{SPX_{\text{final}}}{SPX_{\text{initial}}} - 1\right)\right]$	Maturity date: [Today + 3 years] Underlying stock: Kroger Co. (KR) Payoff: (a) If, between the start and maturity dates, Kroger Co. always trades above the Barrier level, Issuer will pay: $\text{Notional} \times \max\left(115\%, \frac{S_{\text{final}}}{S_{\text{initial}}}\right)$. (b) Otherwise, Issuer will pay: $\text{Notional} \times \frac{S_{\text{final}}}{S_{\text{initial}}}$.
Participation: 50%	Barrier level: 70%

In the Capital Guaranteed Performance Note, investors are guaranteed[1] to get their \$10 mn capital back after 5 years. This is much safer than a direct \$10 mn investment in the S&P 500 index which could result in a loss. In exchange for this protection, investors receive a smaller share in the S&P 500 performance: 50% instead of 100%.

In the Reverse Convertible Note, investors may lose on their €2 mn capital if Kroger Co. ever trades below the 70% barrier, but never more than a direct investment in the stock (ignoring dividends). Otherwise, investors receive at least €2.3 mn after 3 years, and never less than a direct investment in the stock (again, ignoring dividends).

In some cases it is possible to break down a structured product into a portfolio of securities whose prices are known and find its value. In all other cases the payoff is typically programmed on a Monte-Carlo simulation engine.

12-2 Multi-Asset Exotics

12-2.1 Spread Options

The payoff of a **spread option** is based on the difference in gross return between two underlying assets:

$$\text{Spread option payoff} = \max\left(0, \frac{S_T^{(1)}}{S_0^{(1)}} - \frac{S_T^{(2)}}{S_0^{(2)}} - k\right),$$

where k is the residual strike level (in %). For example, a spread option on Apple Inc. vs Google Inc. with 5% strike pays off the outperformance of Apple over Google in excess of 5%: if Apple's return is 13% and Google's is 4%, the option pays off $13\% - 4\% - 5\% = 4\%$.

The value of a spread option is very sensitive to the level of correlation between the two assets (see Problem 6). Specifically the option value increases as correlation decreases: the lower the correlation, the wider the two assets are expected to spread apart.

[1] Provided the issuer does not go bankrupt.

12-2.2 Basket Options

A **basket call** or **put** is an option on the gross return of a portfolio of n underlying assets:

$$\text{Basket call payoff} = \max\left(0, \sum_{i=1}^{n} w_i \frac{S_T^{(i)}}{S_0^{(i)}} - k\right)$$
$$\text{Basket put payoff} = \max\left(0, k - \sum_{i=1}^{n} w_i \frac{S_T^{(i)}}{S_0^{(i)}}\right)$$

where the weights $w_1, \ldots, w_n$ sum to 100% and the strike k is expressed as a percentage (e.g. 100% for at-the-money).

Example

Equally-weighted Stock Basket Call

Option seller: ABC Bank Co.

Notional amount: $20,000,000
Issue date: [Today]
Maturity date: [Today + 3 years]
Underlying stocks: IBM (IBM), Microsoft (MSFT), Google (GOOG)
Payoff:

$$\text{Notional} \times \max\left(0, \frac{1}{3}\left(\frac{IBM_{\text{final}}}{IBM_{\text{initial}}} + \frac{MSFT_{\text{final}}}{MSFT_{\text{initial}}} + \frac{GOOG_{\text{final}}}{GOOG_{\text{initial}}}\right) - 1\right)$$

Option price: 17.4%

The value of basket options is sensitive to the level of pairwise correlations between the assets. The lower the correlation, the less volatile the portfolio and the cheaper the basket option.

12-2.3 Worst-of and Best-of Options

A **worst-of call** or **put** is an option on the lowest gross return between n underlying assets:

$$\text{Worst-of call payoff} = \max\left(0, \min_{1 \le i \le n} \frac{S_T^{(i)}}{S_0^{(i)}} - k\right)$$
$$\text{Worst-of put payoff} = \max\left(0, k - \min_{1 \le i \le n} \frac{S_T^{(i)}}{S_0^{(i)}}\right)$$

where the strike k is expressed as a percentage (e.g. 100% for at-the-money). For example a worst-of at-the-money call on Apple, Google, and Microsoft pays off the worst stock return between the three companies, if positive.

Similarly, a **best-of call** or **put** is an option on the highest gross return between n underlying assets.

Worst-of calls and best-of puts are always cheaper than any of their single-asset European counterparts, while best-of calls and worst-of puts are always more expensive.

12-2.4 Quanto Options

The payoff of a **quanto option** is paid out in a different currency than the underlying asset's, at a guaranteed exchange rate. For example a call on the S&P 500 index quanto euro pays off $\max(0, S_T - K)$ in euros instead of dollars, thereby guaranteeing an exchange rate of 1 euro per dollar.

The actual exchange rate between the asset currency and the quanto currency is in fact an implicit additional underlying asset. The value of quanto options is very sensitive to the correlation between the primary asset and the implicit exchange rate.

Quanto options are an example of 'hybrid' exotic options involving different asset classes – here equity and foreign exchange.

12-2.5 Structured Products

Multi-asset structured products significantly expand the payoff possibilities of exotic options. They allow investors to bet on correlation and express complex investment views.

Example

Worst-of 'Reverse Convertible' Note Quanto CHF

Issuer: ABC Bank Co.

Notional amount: CHF 5,000,000
Issue date: [Today]
Maturity date: [Today + 3 years]
Underlying indexes: S&P 500 (SPX), EuroStoxx-50 (SX5E), Nikkei 225 (NKY)
Payoff: (a) If, between the start and maturity dates, all underlying indexes always trade above the Barrier level, Issuer will pay:

$$\text{Notional} \times \max\left(120\%, \min\left(\frac{SPX_{\text{final}}}{SPX_{\text{initial}}}, \frac{SX5E_{\text{final}}}{SX5E_{\text{initial}}}, \frac{NKY_{\text{final}}}{NKY_{\text{initial}}}\right)\right).$$

(b) Otherwise, Issuer will pay:

$$\text{Notional} \times \min\left(\frac{SPX_{\text{final}}}{SPX_{\text{initial}}}, \frac{SX5E_{\text{final}}}{SX5E_{\text{initial}}}, \frac{NKY_{\text{final}}}{NKY_{\text{initial}}}\right).$$

Barrier level: 50% of Initial Price

Multi-asset structured product valuation is almost always done using Monte-Carlo simulations. Hedging correlation risk is often difficult or expensive, and exotic trading desks tend to accumulate large exposures which can cause significant losses during a market crash.

12-2.6 Dispersion and Correlation Trading

A **dispersion trade** involves the spread between a multi-asset option and the corresponding portfolio of single-asset options:

$$\text{Vanilla dispersion payoff} = \underbrace{\max\left(0, \sum_{i=1}^{n} w_i \frac{S_T^{(i)}}{S_0^{(i)}} - k\right)}_{\text{Basket call}} - \sum_{i=1}^{n} w_i \underbrace{\max\left(0, \frac{S_T^{(i)}}{S_0^{(i)}} - k\right)}_{\text{Single-stock calls}}$$

$$\text{Variance dispersion payoff} = \left(\sigma_{\text{Basket}}^{\text{Realized}}\right)^2 - \sum_{i=1}^{n} w_i \left(\sigma_i^{\text{Realized}}\right)^2$$

Dispersion trades are popular with hedge funds and sophisticated investors who want to indirectly bet on correlation. A direct bet on realized correlation may also be made using a correlation swap whose payoff is:

$$\text{Correlation swap payoff} = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} \rho_{i,j}^{\text{Realized}} - K_{\text{correl}},$$

where $\rho_{i,j}^{\text{Realized}}$ is the observed correlation coefficient between the daily log-returns of stocks i and j over the period $[0, T]$, and K_{correl} is a number between 0 and 1 corresponding to the pre-agreed price of correlation.

12-3 Beyond Black-Scholes

Since its publication in 1973, many extensions of the Black-Scholes model have been put forward. A rather straightforward but key extension was to introduce multiple assets. Another key development was to relax the assumption of constant volatility σ and attempt to reproduce the implied volatility smile. In recent years, practitioners have been looking at volatility derivatives and correlation modeling, while many academics are still on the hunt for a new model which would supersede Black-Scholes in the hope of winning the next Nobel prize...

12-3.1 Black-Scholes on Multiple Assets

Extending Black-Scholes to a basket of n underlying assets may look daunting at first but it turns out to be fairly straightforward, except perhaps notation-wise.

Assume that the prices of the underlying stocks $S^{(1)}, S^{(2)}, \ldots, S^{(n)}$ follow n correlated geometric Brownian motions:

$$\begin{aligned}
dS_t^{(1)} &= \mu_1 S_t^{(1)} dt + \sigma_1 S_t^{(1)} dW_t^{(1)} \\
dS_t^{(2)} &= \mu_2 S_t^{(2)} dt + \sigma_2 S_t^{(2)} dW_t^{(2)} \\
&\vdots \\
dS_t^{(n)} &= \mu_n S_t^{(n)} dt + \sigma_n S_t^{(n)} dW_t^{(n)}
\end{aligned}$$

where $dW_t^{(i)} dW_t^{(j)} = \rho_{i,j} dt$. If the derivative value only depends on time and the n spot prices, we have $D_t = f(t, S_t^{(1)}, \cdots, S_t^{(n)})$ and we may apply the multidimensional version of the

Ito-Doeblin theorem to get:

$$dD_t = df = \frac{\partial f}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial f}{\partial S^{(i)}} dS_t^{(i)} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial S^{(i)}\partial S^{(j)}}\sigma_i\sigma_j S_t^{(i)} S_t^{(j)} \rho_{i,j}dt.$$

A portfolio long one unit of derivative and short $\delta_i = \frac{\partial f}{\partial S^{(i)}}$ units of each asset $S^{(i)}$ is then riskless, and by the same reasoning as in the single-asset case we obtain a multidimensional partial differential equation for f whose only parameters are the interest rate r, the volatilities $(\sigma_1, \ldots, \sigma_n)$ and the pairwise correlation coefficients $(\rho_{i,j})$.

In practice, the preferred valuation method for multi-asset exotics is Monte-Carlo simulation, which is more efficient than the binomial tree or a finite difference lattice for large baskets.

12-3.2 Fitting the Smile

When there is an implied volatility smile Black-Scholes does not correctly value call spreads nor most exotic option payoffs. Several extensions have been proposed which all generate a smile or fit the existing market smile, mainly: stochastic volatility, jumps, and 'local volatility.'

12-3.2.1 Stochastic Volatility

In a stochastic volatility model, the underlying asset price has increments:

$$dS_t = \mu(\cdots)S_t dt + \sigma_t S_t dW_t,$$

where (σ_t) is now itself a stochastic process. A popular specification is that of the Heston (1993) model, whereby the instant variance $v_t = \sigma_t^2$ is assumed to have increments:

$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}dZ_t,$$

where (Z_t) is another standard Brownian motion with $dW_t\, dZ_t = \rho dt$, θ is the long-run variance level, κ is the rate at which v_t reverts to θ, and ξ is the volatility of variance.

Using further arbitrage arguments involving another option, and an economic assumption on the risk premium for volatility, we obtain a new partial differential equation governing option values.

Figure 12-3 overleaf shows the resulting implied volatility surface (the smile curve along the expiry axis) for a particular choice of parameters chosen to fit the market surface of the S&P 500. We can see that the Heston model produces a plausible shape but is too flat for short expiries.

12-3.2.2 Jumps

Another popular extension of Black-Scholes is to add jumps, i.e. sudden price discontinuities:

$$dS_t = \mu S_t dt + \sigma S_t dW_t + dJ_t,$$

where J_t is the sum of all jumps up to time t. A popular specification for (J_t) is a compound Poisson process independent of the underlying price where jumps occur at frequency λ and each jump size is random with a common distribution.

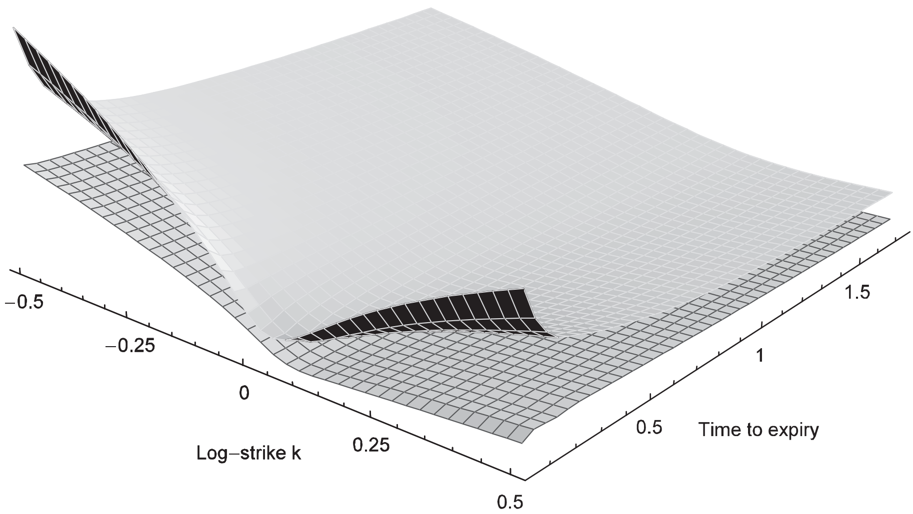

Figure 12-3 Comparison of the S&P 500 implied volatility surface (top) with its Heston fit (bottom) as of 15 September 2005

(Source: Jim Gatheral (2005), *The Volatility Surface*, John Wiley & Sons. Reproduced with permission)

One important theoretical issue with jump models is that it is no longer possible to set up a risk-free portfolio. The traditional arbitrage argument thus breaks down. A common trick used by mathematicians here is to argue that the *expected* return on the (now risky) delta-hedged portfolio should still be the risk-free rate, but it is a very debatable argument which goes against the teachings of portfolio theory.

Jump models also produce a plausible implied volatility smile, and as such appear more realistic than Black-Scholes.

12-3.2.3 Local Volatility

Derman and Kani (1994) and Dupire (1994) independently proposed to let the volatility coefficient depend on time and the spot price:

$$dS_t = \mu S_t dt + \sigma_{\text{loc}}(t, S_t) S_t dW_t.$$

This idea is best visualized on a tree, as shown in Figure 12-4 overleaf. Here, the so-called 'local volatility' function $\sigma_{\text{loc}}(t, S)$ is known in advance, but the actual volatility coefficients $\sigma_{\text{loc}}(t, S_t)$ depend on the particular path taken by the spot price.

In a hypothetical market situation where we know the prices of all European calls and puts for a continuum of strikes and maturities, the local volatility function is uniquely determined. In practice only a finite number of strikes and maturities are available, but using sensible interpolation and extrapolation techniques we may map out an entire local volatility function.

Because the local volatility model perfectly fits the entire implied volatility smile, it gives a correct value for call spreads, put spreads, and a wide range of exotic derivatives.

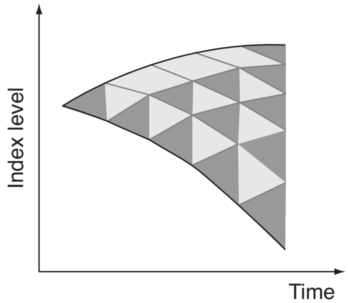

Figure 12-4 A binomial tree with local volatilities
(Source: Emanuel Derman (2004), *My Life As A Quant*, John Wiley & Sons. Reproduced with permission)

12-3.3 Discrete Hedging and Transaction Costs

Black-Scholes assumes continuous hedging and no transaction costs. In practice this is not possible. In the approaches discussed below, the delta-hedged portfolio is no longer risk-free and the argument that its expected return should nevertheless be the risk-free rate is debatable.

12-3.3.1 Discrete Hedging

Wilmott (1994) shows how the drift μ reappears in the partial differential equation for an option hedged at discrete time steps Δt:

$$\underbrace{rf = \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}}_{\text{Black}-\text{Scholes PDE}} + \frac{1}{2}\Delta t(\mu - r)(r - \mu - \sigma^2)S^2 \frac{\partial^2 f}{\partial S^2}.$$

Note that in this model we must hedge with a modified "better delta" to minimize the variance of the delta-hedged portfolio.

This approach is equivalent to Black-Scholes with a modified volatility parameter. As a result, whenever $\mu > r$ or $\mu < r - \sigma^2$, calls and puts are worth less than in the continuous hedging case. This may be a surprise because discrete hedging is intuitively riskier. However, in the model's framework, being exposed to a *known* drift μ between two hedging times is a source of profit reflected in discounted option values. Here, a crucial objection is that in the real world the drift is uncertain, making the "better hedge" very difficult to implement.

12-3.3.2 Transaction Costs

Leland (1985) shows that when every buy or sell order costs a small fraction κ of the value of the shares traded, hedging takes place at discrete time steps Δt, and the option's gamma is always positive, then we should value the option with an adjusted volatility parameter:

$$\sigma^- = \sigma\sqrt{1 - \frac{\kappa}{\sigma}\sqrt{\left(\frac{8}{\pi\Delta t}\right)}}.$$

Thus long calls or puts are less valuable than in Black-Scholes because every delta rebalancing incurs a cost. Symmetrically, short calls and puts are worth more and should be valued with the adjusted volatility parameter:

$$\sigma^+ = \sigma \sqrt{1 + \frac{\kappa}{\sigma}\sqrt{\left(\frac{8}{\pi \Delta t}\right)}}.$$

Note that these adjusments only work for small enough κ and long enough Δt. As Δt goes to zero, transaction costs unsurprisingly go to infinity.

12-3.4 Correlation Modeling

The study of equity correlations and how to model their behavior is a new and promising research area. Just as the Black-Scholes constant volatility assumption is "wrong," it turns out that assuming constant correlation in multi-asset models is also "wrong."

This is best observed by comparing the implied volatility smile on a stock index such as the Dow Jones EuroStoxx 50 against the "average" smile on its 50 constituent stocks (Figure 12-5 below). We can see that the slope of the index smile is a little steeper, a phenomenon which may only be reproduced by inputting a different correlation level for each strike.

On the fundamental side, correlation is embedded in index volatility. Assimilating an equity index to a portfolio of stocks we have the proxy:

$$\text{Average correlation} \approx \left(\frac{\text{Index volatility}}{\text{Average stock volatility}}\right)^2.$$

This relationship, which is derived in Problem 7, may be exploited for the valuation and hedging of correlation swaps as well as multi-asset exotics.

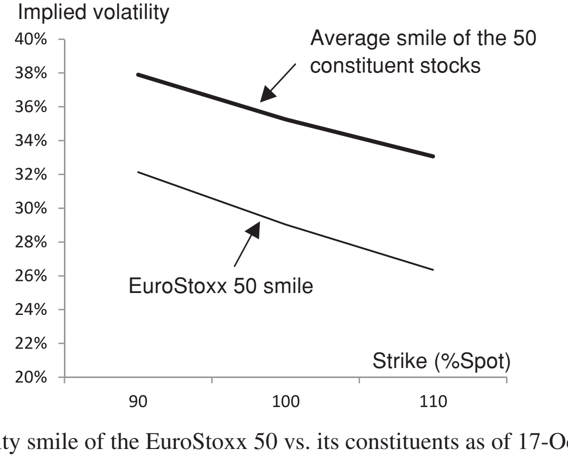

Figure 12-5 Volatility smile of the EuroStoxx 50 vs. its constituents as of 17-Oct-2011

(Source: Large US investment bank)

12-4 Further Reading

- On exotic options and models: John C. Hull (2009) *Options, Futures and Other Derivatives* 7th Edition, Prentice Hall: Chapters 24 and 26.
- On advanced volatility models: Jim Gatheral (2005) *The Volatility Surface*, John Wiley & Sons.
- On the emergence of “quants” in finance: Emanuel Derman (2004) *My Life As A Quant*, John Wiley & Sons.
- On discrete hedging and transaction costs: Paul Wilmott (2006) *Paul Wilmott on Quantitative Finance* 2nd edition, John Wiley & Sons: Chapters 47 and 48.

Research articles:

- Mark Broadie, Paul Glasserman, and Steve Kou (1997) A continuity correction for discrete barrier options, *Mathematical Finance*, Vol. 7, no. 4 (October 1997), pp. 325–348.
- Emanuel Derman and Iraj Kani (1994) Riding on a Smile, *Risk*, 7(2), February, pp. 139–145.
- Bruno Dupire (1994) Pricing with a smile, *Risk*, 7(1), January, pp. 18–20.
- H.E. Leland (1985) Option pricing and replication with transaction costs, *Journal of Finance*, 40, pp. 1283–1301.
- Steven L. Heston (1993) A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *The Review of Financial Studies*, 6(2), pp. 327–343.
- Paul Wilmott (1994) Discrete charm, *Risk*, 7(3), pp. 48–51.

12-5 Problems

Problem 1: Digital options. *This is a continuation of Chapter 7, Problem 9 p.81.*

(a) What is the delta of the digital call option with strike K and maturity T in the Black-Scholes model?

(b) Find the limit of the delta as $T \to 0$ and interpret your result in terms of delta-hedging near maturity.

(c) (*) Suppose that there is an implied volatility smile $\sigma^*(k)$ for maturity T. Using a limit argument show that the fair value of the digital call is $D_0 = e^{-rT} N[d_2(K)] - Ke^{-rT} N'[d_2(K)] \frac{\partial \sigma^*}{\partial k}(K)$, where $d_2(k) = \frac{\ln(F_0/k)}{\sigma^*(k)\sqrt{T}} - \frac{1}{2}\sigma^*(k)\sqrt{T}$. *Hint: consult Chapter 8, Problem 7 p.91.*

Problem 2: Asian options

(a) Assuming zero interest rates, show that the value of a 1-year Asian call on an underlying asset S with strike K and monthly fixing dates is at most $\frac{c_0(K,1/12)+c_0(K,2/12)+\cdots+c_0(K,1)}{12}$ where $c_0(K, T)$ is the value of a European call on S with strike K and maturity T.

(b) Consider a continuous Asian call on an underlying asset S with payoff $\max(0, \frac{1}{T}\int_0^T S_t dt - S_0)$. Assuming no dividends and zero interest rates, show that an approximate upper bound for the Asian call in the Black-Scholes model is two thirds of the value of the at-the-money European call. *Hint: consult Chapter 7, Problem 8 p.80.*

Problem 3: Single-asset structured products

Suppose that the implied volatility of the S&P 500 index (SPX) is 30% for all maturities and all strikes, the continuous dividend rate is 2.5% p.a., and the continuous interest rate curve is flat at 2.5% p.a.

(a) What is the participation level on a 5-year equity-linked note with payoff:

$$\text{Notional} \times \left[100\% + \text{Participation} \times \max\left(0, \frac{SPX_{\text{final}}}{SPX_{\text{initial}}} - 1\right)\right]?$$

(b) What is the floor level on a 10-year equity-linked note with payoff:

$$\text{Notional} \times \max\left[\text{Floor}, \frac{SPX_{\text{final}}}{SPX_{\text{initial}}}\right]?$$

(c) Given that a 3-year at-the-money knock-out put with barrier 60% $\times$ SPX_{initial} is worth 3.25%, and that a 1% increase in strike costs 0.2%, can you estimate the coupon level on a 3-year equity-linked note with payoff:
- Notional $\times$ max[1 + Coupon, $\frac{SPX_{\text{final}}}{SPX_{\text{initial}}}$] if SPX always trades above 60% of its initial level;
- Notional $\times$ $\frac{SPX_{\text{final}}}{SPX_{\text{initial}}}$ otherwise?

Problem 4: Barrier monitoring

MeToo.Com's stock currently trades at \$50 and has a flat 40% implied volatility for all strikes and maturities. A 2-year knock-in barrier call on MeToo.Com with strike \$50 and barrier \$40 is worth \$3.56, and the price sensitivity to a +\$1 barrier shift is \$0.56. Estimate the value of the same barrier call when the barrier is monitored every month rather than continuously.

Problem 5: Correlated Brownian motions

Consider two uncorrelated standard Brownian motions (V_t) and (W_t) and let $Z_t = \rho V_t + \sqrt{1-\rho^2}W_t$ where $-1 \leq \rho \leq 1$. Show that $(dV_t)(dZ_t) \equiv \rho dt$, then simulate the joint paths of (V_t) and (Z_t) on a computer with $dt = 0.01$ and $\rho = 0.7$.

Problem 6: Spread option

Consider a spread option between two non-dividend-paying stocks: MeToo.Com ($S^{(1)}$) and Giggle Inc. ($S^{(2)}$), with payoff $D_T = \max(0, \frac{S_T^{(1)}}{S_0^{(1)}} - \frac{S_T^{(2)}}{S_0^{(2)}})$. The volatility of $S^{(1)}$ is 40%, the volatility of $S^{(2)}$ is 35%, and their correlation is 65%. The interest rate curve is flat at 0%.

(a) Find the fair value of the spread option using the Monte-Carlo method with 5,000 simulations. *Hint: consult problems Chapter 7, Problem 5 p.80 and Problem 5 above.*

(b) What is the impact on the option value of a 1% increase in correlation?

Problem 7*: Correlation proxy

Consider n stocks $S^{(1)}, S^{(2)}, \ldots, S^{(n)}$ with realized volatilities $\sigma_1, \sigma_2, \ldots, \sigma_n$ and correlations $(\rho_{i,j})$ such that $0 < a \leq \sigma_i \leq b$ for all stocks and $\rho_{i,j} \geq c > 0$ for all stock pairs. Define the "volatility-weighted" average realized correlation as $\bar{\rho} = \dfrac{\sum_{i<j} \sigma_i \sigma_j \rho_{i,j}}{\sum_{i<j} \sigma_i \sigma_j}$.

(a) Verify that $0 < \bar{\rho} \leq 1$. What does $\bar{\rho}$ reduce to when all correlation coefficients are equal to c?

(b) Show that $\bar{\rho} = \dfrac{\sigma_{\mathrm{B}}^2 - s^2/n}{\bar{\sigma}^2 - s^2/n}$ where σ_{B} is the realized volatility of an equally weighted basket of the stocks, $\bar{\sigma} = \frac{1}{n}\sum\limits_{i=1}^{n} \sigma_i$ and $s^2 = \frac{1}{n}\sum\limits_{i=1}^{n} \sigma_i^2$.

(c) Show that $\bar{\rho} \sim (\dfrac{\sigma_{\mathrm{B}}}{\bar{\sigma}})^2$ as n goes to infinity. *Hint: show that* $\frac{1}{n}s^2 = o(\sigma_{\mathrm{B}}^2)$.

Problem 8: Zero-cost variance dispersion. *The notations are those of Problem 7.*

Denote v_i the fair strike of a variance swap on $\mathrm{S}^{(\mathrm{i})}$ and v_{B} the fair strike of a variance swap on the equally weighted basket. Assume zero interest rates.

(a) Find the cost of a standard variance dispersion trade as a function of v_i's, v_{B}, and n.

(b) A generalized variance dispersion trade with equal stock weights has payoff $\sigma_{\mathrm{B}}^2 - \beta s^2$. Find the ratio β for which the trade has zero cost as a function of v_i's, v_{B} and n.

(c) Show that the payoff of the zero-cost variance dispersion trade may be rewritten as $s^2 \times (\hat{\rho} - k)$ where $\hat{\rho} = \frac{\sigma_{\mathrm{B}}^2}{s^2}$ and $k = \dfrac{v_{\mathrm{B}}^2}{\frac{1}{n}\sum_{i=1}^{n} v_i^2}$.

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Solutions

Problem Solutions

Chapter 1

Problem 1: Measuring time

Rule	Result	From 30 November 2014 until 1 March 2016
1. Count the number of whole years	Y	1 (from 30 November 2014 until 30 November 2016)
2. Count the number of remaining months and divide by 12	M/12	3/12 (from 30 November 2016 until 29 February 2016)
3. Count the number of remaining days (the last day of the month counting as the 30th unless it is the final date) and divide by 360	D/360	1/360 (there is 1 day between 29 February 2016 and 1 March 2016)
TOTAL	Y + M/12 + D/360	1 + 3/12 + 1/360 = 1.2527 . . .

Using the conversion formula (Equation (1-1) p.6): $r^{[1]} = (1 + 10\%)^{1/1.2527} - 1 \approx 7.90\%$ per year.

Problem 2: Savings account

(a) The gross annual interest rate is $r = \frac{I}{K} = \frac{40}{1000} = 4\%$.
(b) The interest received in 2013 will be €41.60, as shown in the compounding table below:

Date	Balance	Interest
1 January 2012	€1,000	—
1 January 2013	$1{,}000 \times (1 + 4\%) =$ €1,040	€40
1 January 2014	$1{,}040 \times (1 + 4\%) =$ €1,081.60	€41.60

(c) Using the conversion formula (Equation (1-1) p.6) the monthly rate $r^{[1/12]}$ is

$$r^{[1/12]} = (1 + 4\%)^{1/12} - 1 \approx 0.327\%.$$

Compounded over six months, we get $r^{[1/2]} = (1 + 0.327\%)^6 - 1 \approx 1.98\%$.

The interest received after 6 months is thus €19.80, which is slightly less than €40/2 = €20 because of compounding. We could also have calculated the semi-annual interest rate directly: $r^{[1/2]} = (1 + 4\%)^{1/2} - 1 \approx 1.98\%$.

Problem 3: The total amount of interest accumulated over the past ten years was £530.52. The gross interest rate over ten years is thus $r = \frac{530.52}{500} \approx 106.11\%$. Based on the same interest rate, the savings in ten years will amount to: $1030.52 \times (1 + 106.11\%) \approx$ £2,124.

Problem 4: From Greece with interest

Below is a possible strategy to take advantage of the attractive 135% Greek interest rate:

(a) Borrow \$100,000 at 6.5% interest for one year.
(b) Convert this capital into $100{,}000/1.35 \approx$ €74,074.
(c) Deposit the euros at 135% interest for one year.
(d) After one year, collect $74{,}074 \times (1+135\%) \approx$ €174,074.
(e) Exchange the euros for $174{,}074 \times 1.35 =$ \$235,000.
(f) Repay the \$100,000 loan with interest, totaling \$106,500.

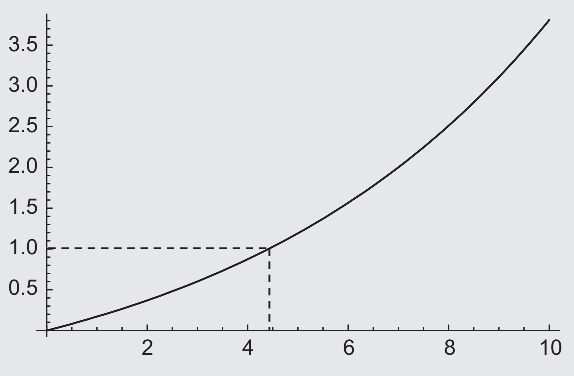
Bottom line: you made \$128,500.

The most important risk to this strategy is the **default risk** (bankruptcy) of the Greek government: it would be rather naïve to believe that an entity which offers a 135% interest rate can stay afloat for very long. Another risk incurred is the **exchange rate risk**: the euro could be worth less than \$1.35 after one year, reducing the strategy's profitability. However, if the Greek government does not default, the exchange rate would need to go lower than $106{,}500/174{,}074 \approx 0.61$ to prevent you from paying off your \$100,000 loan – an improbable scenario.

Problem 5: Using the conversion formula (Equation (1-1) p.6) we can annualize all interest rates and find that: **b** > **a** > **d** > **c**. (Note: for **d** the compounding over three years is given as: $(1 + 10\%) \times (1 + 4\%)^2$).

Problem 6: Credit card

(a) Using the conversion formula (Equation (1-1) p.6) the compound interest rate after τ years is given as $r^{[\tau]} = (1 + 17\%)^\tau - 1$. Thus: $r^{[1/12]} \approx 1.3\%$; $r^{[0.5]} \approx 8.2\%$; $r^{[1.5]} \approx 26.6\%$; $r^{[5]} \approx 119.2\%$.
(b) The curve is exponential:



(c) The interest charged will exceed the initial balance when the interest rate exceeds 100%. This will happen when $(1+17\%)^{\tau} = 1+100\% = 2$, i.e. when $\tau = \frac{\ln 2}{\ln 1.17} \approx 4.41$ years.

Problem 7*: Continuous interest rate

(a) Using the exponential form, we have for all $n \geq 1$: $u_n = e^{n\ln\left(1+\frac{1}{n}\right)}$. When n goes to infinity, $\frac{1}{n}$ goes to 0 and thus: $\ln\left(1+\frac{1}{n}\right) \approx \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots$ Multiplying both sides by n yields: $n\ln\left(1+\frac{1}{n}\right) \approx 1 - \frac{1}{2n} + \frac{1}{3n^2} - \ldots$. Therefore $n\ln\left(1+\frac{1}{n}\right)$ goes to 1 as n goes to infinity, from which we obtain: $\lim_{n\to+\infty} u_n = e^1 = e$.

(b) Using the conversion formula (Equation (1-1) p.6): $r_2 = (1+2.5\%)^2 - 1 \approx 5.06\%$.

(c) Similarly: $r_n = \left(1+\frac{5\%}{n}\right)^n - 1$.

(d) Following the same reasoning as in question (a) we find $r = \lim_{n\to+\infty} r_n = e^{5\%} - 1 \approx 5.13\%$. This is the annualized interest rate of an imaginary savings account whose interest is paid out continuously at each fraction of a second (see continuous compounding in Section 9-1.2 p.96).

Problem 8: Discounting. Based on a 4% discount rate the present values are:

(a) $\text{PV} = \frac{100{,}000}{1+4\%} \approx €96{,}154$;

(b) $\text{PV} = \frac{1{,}000{,}000}{(1+4\%)^{10}} \approx €675{,}564$;

(c) $\text{PV} = \frac{100{,}000}{(1+4\%)^{-10}} = 100{,}000 \times (1+4\%)^{10} \approx €148{,}024$ (this is actually compounding).

Problem 9: Expected return. The fact that Mr Smith hesitated at length indicates that the project's return was only slightly inferior to the required return r of the investment banker. Therefore, for Mr Smith, the value of one billion pounds in twenty years discounted at rate r is inferior but close to 30 million pounds today, i.e.: $30{,}000{,}000 \approx \frac{1{,}000{,}000{,}000}{(1+r)^{20}}$. Solving for r we find a required return of about 20%.

Problem 10: Today's value of one dollar tomorrow

The present value of a \$1 cash flow tomorrow is $\frac{1}{(1+2.77\%)^{\frac{1}{360}}} \approx \$0.999924105$.

Problem 11: Tuition planning

(a) Assuming no inflation, Anna's four-year college education will cost $4 \times 27{,}293 = \$109{,}172$. Starting with a \$20,000 deposit at 5% and adding an amount X every year, the Jones's total savings after 8 years will be:

$$20{,}000 \times 1.05^8 + X \times 1.05^7 + X \times 1.05^6 + \cdots + X \times 1.05 = 29{,}549.10 + X\frac{1.05 - 1.05^8}{1 - 1.05}.$$

Matching the right-hand side with \$109,172 and solving for X we find $X \approx \$9{,}313.70$.

(b) Tuition inflation was $\left(\frac{27{,}293}{20{,}308}\right)^{1/10} - 1 \approx 3\%$ p.a. Assuming the same rate, Anna's education will cost $27{,}293 \times \left(1.03^8 + 1.03^9 + 1.03^{10} + 1.03^{11}\right) \approx \$144{,}645$. To match this amount, the Jones must now save $X = (144{,}645 - 29{,}549.10)\frac{0.05}{1.05^8 - 1.05} \approx \$13{,}462.9$ every year.

Problem 12*: Rule of 72

Starting with capital K, an investment at rate r over t years yields an amount $K(1+r)^t$. The capital will double when $K(1+r)^t = 2K$, i.e. $t = \frac{\ln 2}{\ln(1+r)} \approx \frac{0.693}{r}$ which is not very far from the ratio of 72 to the percentage rate.

Chapter 2

Problem 1: ROR and TOR

(a) ROR: $r^{[1.5]} = \frac{400-350}{350} \approx 14.29\%$, thus $r^{[1]} = (1+r^{[1.5]})^{\frac{1}{1.5}} - 1 \approx 9.31\%$ p.a. TOR $=$ 1/14.29% $\approx$ 7 cycles of 18 months, i.e. about 10.5 years.

(b) ROR: $r^{[7/360]} = \frac{20-10}{10} = 100\%$, thus $r^{[1]} = (1+r^{[7/360]})^{\frac{360}{7}} - 1 \approx 3 \times 10^{17}\%$. TOR $=$ 1 week.

(c) ROR: $r^{[1]} = \frac{100+950-1000}{1000} = 5\%$. TOR $=$ 20 years.

(d) ROR: $r^{[1/12]} = \frac{4-3}{3} = 33.33\%$, thus $r^{[1]} = \left(1+r^{[1/12]}\right)^{12} - 1 = \left(\frac{4}{3}\right)^{12} - 1 \approx 3057\%$. TOR $=$ 3 months.

Problem 2: NPV and IRR

(a) NPV at 5% and 15% discount rates:

$$NPV(5\%) = -C_0 + \frac{-0.1 \times C_0}{1.05^2} + \frac{1.5 \times C_0}{1.05^4} \approx 14.33\% \times C_0 > 0$$
$$NPV(15\%) = -C_0 + \frac{-0.1 \times C_0}{1.15^2} + \frac{1.5 \times C_0}{1.15^4} \approx -21.80\% \times C_0 < 0$$

At 5% discount rate the investment project is acceptable, but at 15% it should be rejected. The indifference point which makes the NPV zero (alias IRR) must thus be between 5% and 15%.

(b) The IRR is the discount rate r^* such that NPV(r) $= 0$, i.e. $-C_0 + \frac{-0.1\times C_0}{(1+r)^2} + \frac{1.5\times C_0}{(1+r)^4} = 0$. Multiplying both sides by $-(1+r)^4 \ / \ C_0$ we get $(1+r)^4 + 0.1 \times (1+r)^2 - 1.5 = 0$. Writing $x = (1+r)^2$ we obtain a familiar quadratic equation whose positive solution is: $x^* = \frac{-0.1+\sqrt{0.1^2+4\times 1.5}}{2} \approx 1.1758$. The IRR is thus $r^* = \sqrt{x^*} - 1 \approx 8.43\%$.

(c) An 8.43% IRR is quite attractive. However, achieving this return entirely relies on the last cash flow of 150% actually being paid after 4 years: the investor is thus making a risky bet.

Problem 3: Required return

(a) Using a 4% discount rate, the present value of the project is:

$$PV(4\%) = -\frac{50}{1.04^{0.5}} + \frac{20}{1.04} + \frac{60}{1.04^{2.25}} \approx 25.13.$$

A rational investor would never pay more than \$25,130,000 for this project.

(b) Assuming Mr Smith is not a philanthropist, he must estimate our required return to be around the project's IRR:

$$-15 - \frac{50}{(1+r)^{0.5}} + \frac{20}{1+r} + \frac{60}{(1+r)^{2.25}} = 0.$$

Using a financial calculator or a spreadsheet, we find: $r^* \approx 14.48\%$.

Problem 4*: Perpetuity, dividends, and stock value

(a) For all x we have: $(1-x)\left(1+x+x^2+\cdots+x^n\right) = 1 - x + x - x^2 + \cdots - x^{n+1} = 1 - x^{n+1}$. For $-1 < x < 1$, x^{n+1} goes to 0 as n goes to infinity which proves the result.

(b) The present value of the perpetuity is $PV = \frac{5}{1.05} + \frac{5}{1.05^2} + \cdots + \frac{5}{1.05^n} + \cdots$. Factoring by 5/1.05 and applying the result from question (a) with $x = 1/1.05$, we get $PV = \frac{5}{1.05} \times \frac{1}{1-1/1.05} = \$100$. If the perpetuity costs \$105 the IRR solves $-105 + \frac{5}{1+r} + \frac{5}{(1+r)^2} + \cdots + \frac{5}{(1+r)^n} + \cdots = 0$, i.e. $\frac{5}{1+r} \times \frac{1}{1-1/(1+r)} = 105$, giving $r = 5/105 \approx 4.76\%$.

(c) Stocks are securities which pay dividends every year. If the current dividend is \$3 and future dividends increase by 1.5% per year, the table of dividend flow is:

***t* (years)**	0	1	2	...	n	...
Cash flow (\$)	+3	+3.045	+3.091	...	$3 \times (1 + 1.5\%)^n$	...

Using a 4% discount rate the theoretical value of Sky Inc. is:

$$\begin{aligned} PV &= 3 + \frac{3 \times 1.015}{1.04} + \frac{3 \times 1.015^2}{1.04^2} + \cdots + \frac{3 \times 1.015^n}{1.04^n} + \cdots \\ &= 3 \times \left[1 + \frac{1.015}{1.04} + \left(\frac{1.015}{1.04}\right)^2 + \cdots + \left(\frac{1.015}{1.04}\right)^n + \cdots\right] \\ &= 3 \times \frac{1}{1 - \dfrac{1.015}{1.04}} \approx \$124.80 \end{aligned}$$

Problem 5: Return and inflation

(a) Without inflation, the cash flow table for this real estate investment is:

***t* (years)**	0	1/12	...	239/12	240/12 = 20
Cash flow (€)	−423,500	1,500	...	1,500	425,000

The IRR solves the equation:

$$-425{,}000 + \sum_{i=0}^{239} \frac{1{,}500}{(1+r)^{\frac{i}{12}}} + \frac{425{,}000}{(1+r)^{20}} = 0.$$

Substituting $x = \left(\frac{1}{1+r}\right)^{\frac{1}{12}}$ and $\sum_{i=0}^{239} x^i = \frac{1-x^{240}}{1-x}$, we can write:

$$-425{,}000\left(1 - x^{240}\right) + 1{,}500\frac{1 - x^{240}}{1 - x} = 0$$

Dividing both sides by $1 - x^{240}$ and rearranging terms:

$$\frac{1}{1-x} = \frac{425{,}000}{1,500} \rightarrow x = \frac{423{,}500}{425{,}000} \approx 0.9965 \rightarrow r^* \approx 4.32\%$$

(b) With 2.5% annual inflation, the IRR equation becomes:

$$-425{,}000 + 1,500\sum_{i=0}^{239}\left(\frac{1+2.5\%}{1+r}\right)^{\frac{i}{12}} + 425{,}000\left(\frac{1+2.5\%}{1+r}\right)^{20} = 0$$

Substituting $R = \frac{1+r}{1+2.5\%} - 1$, we find again the same equation as in question (a), whose solution is $R^* \approx 4.32\%$. Thus, the IRR is $r^* = (1 + 4.32\%) \times (1 + 2.5\%) - 1 \approx 6.93\%$, which is approximately 2.5% higher than the ex-inflation IRR.

Problem 6: Impact of corporate announcement

(a) PER for 2011: $PER_{2011} = \frac{150}{20} = 7.5$. In other words, the stock price of MetroTech SpA is 7.5 times the 2011 EPS.

(b) Stock value to an investor with a 10% expected return:

$$\text{PV} = \frac{40}{1.1} + \frac{47}{1.1^2} + \frac{55}{1.1^3} + \frac{15}{1.1^4} + \frac{12}{1.1^5} = €134.22.$$

This value is lower than the stock price of €150: our investor would not invest in MetroTech SpA.

(c) The market's expected return on MetroTech SpA is the IRR which solves:

$$-150 + \frac{40}{1+r} + \frac{47}{(1+r)^2} + \frac{55}{(1+r)^3} + \frac{15}{(1+r)^4} + \frac{12}{(1+r)^5} = 0.$$

Using a financial calculator or a spreadsheet we find: $r^* = 5\%$.

(d) Using the market's expected return of 5%, we can estimate the post-announcement value of MetroTech SpA to be:

$$\text{PV} = \frac{35}{1.05} + \frac{47}{1.05^2} + \frac{60.5}{1.05^3} + \frac{16.5}{1.05^4} + \frac{13.2}{1.05^5} = €152.14$$

The stock price of MetroTech SpA should thus increase by €2.14.

Chapter 3

Problem 1: Yield

(a) 5% (The yield of a bond with a price of 100 is always equal to the coupon.)
(b) 2.87% (solution to the quadratic equation: $-106 + \frac{6}{1+y} + \frac{106}{(1+y)^2} = 0$).
(c) $z(1) = \frac{100}{95} - 1 \approx 5.26\%$.
(d) 10.25% (the yield of a bond with price 100 is always equal to the coupon which is 5% per semester; the annualized yield is thus $(1 + 5\%)^2 - 1 = 10.25\%$).

Problem 2: Yield curve. This curve is a hybrid of the upward and downward sloping cases listed in Section 3-3.2 p.23. This kind of curve can be observed when the market expects rates to decrease in the medium term, but not immediately.

Problem 3: True or False? False: this argument confuses yield to maturity with coupon rate. The coupon rate is fixed when the bond is issued. The bond yield, on the other hand, fluctuates until maturity; it corresponds to the coupon rate of a newly issued bond with same maturity. Only the yield can help decide whether a bond is 'cheap' or 'rich' as it takes into consideration both the price and the coupon rate.

In this problem the 3-year yield is 4%. The two bonds (15 November 2014 and 15 February 2015) both have maturities close to 3 years so their yield should be close to 4% regardless of the coupon rate. However, we cannot conclude anything without knowing the exact yield of the bonds.

Problem 4: Maximum return on a bond

(a) The ROR is $\frac{104-98}{98} \approx 6.12\%$.
(b) If the yield curve is flat at zero the bond price must be \$104. The gross return is then 6.12% over a month, which is better than 6.12% over a year.
(c) For a bond investor, the ideal scenario is when the yield curve immediately collapses to zero after buying the bond. In this case the new bond price is $P' = 100 + T \times C$ and the gross return is $\frac{100+T\times C-P}{P}$. Assuming $P \approx 100$ we have the rule of thumb: max return $\approx T \times C\%$.

Problem 5*: Liquidity and arbitrage price of a portfolio. Without the infinite liquidity assumption, every buy or sell order would impact the price of a security. For example, an investor who owns 1,000,000 shares of a stock trading at \$30 may not be able to sell all his shares for \$30,000,000. Instead, he might sell 10,000 shares at \$30 per share, then another 10,000 shares at \$29.95 per share, etc., so that his total proceeds might only be e.g. \$27,000,000 at the end of the selling process. The argument for the arbitrage price of a portfolio with large quantities of illiquid securities thus breaks down.

Problem 6: Approximate valuation. Based on the yield curve in Figure 3-1 p.21 the bonds maturing on 15 May 2021 and 15 February 2026 yield 3.148% and 3.712% respectively. Since there are 1710 days between the two maturities on a 30/360 basis, the interpolated yield for 15 May 2022 is:

$$\tilde{y} = 3.148\% + \frac{360}{1710}(3.712\% - 3.148\%) \approx 3.267\%.$$

Applying Equation (3-1) p.26 an estimate as of 16 May 2011 for the value of the 11-year bond is:

$$\tilde{V} = \frac{5}{1.03267} + \frac{5}{1.03267^2} + \cdots + \frac{105}{1.03267^{11}} \approx 115.80.$$

Problem 7: Bond arbitrage. Example of an arbitrage strategy which has no cost today and makes a profit in 2 years:

	Cash flows		
Bond	$t = 0$	$t = 1$	$t = 2$
A	90	100	—
B	945	50	1050
C	990	100	1100
Buy 2 C, sell 2 B and 1 A	0	0	$100 > 0$

Problem 8: Dividend announcement

(a) Arbitrage strategy if $S > 9600$:

Transaction	Cash flow
Buy one stock just before close of market	−¥10,000
Collect the dividend overnight	+¥400
Sell the stock when the market opens the following day	$+S$
Total	$S - 9{,}600 > 0$

(b) If $S <$ ¥9,600 the arbitrage strategy is the same as above after reversing transactions and signs.

(c) Although questions (a) and (b) have the flavor of an arbitrage proof that $S = 9{,}600$, we actually cannot make any conclusion here because we may never predict with certainty whether $S > 9{,}600$ or $S < 9{,}600$. However, in practice it is observed that the stock price approximately drops by the dividend amount after it is paid; and if S was the overnight forward price the arbitrage proof would hold.

Problem 9: Arbitrage price. Applying Equation (3-2) p.28 the arbitrage price of the bond is:

$$\begin{aligned} P &= \frac{30}{1 + z(1)} + \frac{30}{(1 + z(2))^2} + \cdots + \frac{530}{(1 + z(10))^{10}} \\ &= \frac{30}{1.0036} + \frac{30}{1.0094^2} + \cdots + \frac{530}{1.03452^{10}} \approx \$594.57. \end{aligned}$$

Problem 10: Zero-coupon bond portfolio. When rates go up bond prices go down. To make a profit in this situation one must thus sell bonds before prices drop. For instance, we may sell a 5-year zero-coupon with a million dollar face value, collecting $\frac{1{,}000{,}000}{(1+3.89\%)^5} = \$826{,}288$ in cash. To net out this cash flow we may for instance buy a 1-year zero-coupon with face value $826{,}288 \times (1 + 4.44\%) = \$862{,}975$. By construction the price of our portfolio is zero:

Asset	Position	Unit price	Mark-to-market
5-year zero-coupon	short 1,000,000	\$0.8262877	−\$826,288
1-year zero-coupon	long 862,975	\$0.9574875	\$826,288
Portfolio	**long 1**	–	**0**

If all rates go up 25 basis points the price of the portfolio will be:

Asset	Position	Unit price	Mark-to-market
5-year zero-coupon	short 1,000,000	\$0.8164172	−\$816,417
1-year zero-coupon	long 862,975	\$0.9552011	\$824,314
Portfolio	**long 1**	**–**	+\$**7,897**

Finally, to make exactly a \$10,000 profit we should **leverage** (i.e. multiply all quantities) approximately 1.27 times (10,000/7,897).

Problem 11: Zero-coupon rate curve. Following the bootstrapping method in Section 3-4.3 p.29, we want to find $z(1)$, $z(2)$, $z(3)$ such that:

$$\begin{cases} \dfrac{100}{1+z(1)} = 97 \\ \dfrac{4}{1+z(1)} + \dfrac{104}{(1+z(2))^2} = 100 \\ \dfrac{4}{1+z(1)} + \dfrac{4}{(1+z(2))^2} + \dfrac{104}{(1+z(3))^3} = 95 \end{cases}$$

The classic approach is to substitute with $x_i = \frac{1}{(1+z(i))^i}$ in order to get a familiar linear system. Here, however, it is possible to solve these equations one after the other:

$$\begin{cases} \dfrac{100}{1+z(1)} = 97 \rightarrow z(1) = 3.09\% \\ \dfrac{4}{1+3.09\%} + \dfrac{104}{(1+z(2))^2} = 100 \rightarrow z(2) = 4.02\% \\ \dfrac{4}{1+3.09\%} + \dfrac{4}{(1+4.02\%)^2} + \dfrac{104}{(1+z(3))^3} = 95 \rightarrow z(3) = 5.96\% \end{cases}$$

Problem 12*: Arbitrage price formula

(a) The cash flows of security A can be decomposed into a portfolio of X, Y, Z in quantities F_1, F_2, F_3 respectively. Under the assumptions of absence of arbitrage opportunities and infinite liquidity, the arbitrage price of this portfolio is: $P = P_X F_1 + P_Y F_2 + P_Z F_3$.

(b) Denoting $z(1)$, $z(2)$, $z(3)$ the 1-, 2-, 3-year zero-coupon rates respectively, the arbitrage price of the portfolio can be rewritten as:

$$P = \frac{F_1}{(1+z(1))} + \frac{F_2}{(1+z(2))^2} + \frac{F_3}{(1+z(3))^3}.$$

This is Equation (3-2) p.28 when there are only 3 annual cash flows.

Problem 13: Price sensitivity and convexity

(a) Arbitrage prices of the bonds:

$$P_X = \frac{8}{1.075} + \frac{8}{1.08^2} + \frac{8}{1.0825^3} + \frac{108}{1.0825^4} = 99.26$$
$$P_Y = \frac{9}{1.075} + \frac{9}{1.08^2} + \frac{9}{1.0825^3} + \frac{9}{1.0825^4} + \frac{9}{1.08^5} + \frac{9}{1.08^6} + \frac{109}{1.0775^7} = 106.18$$
$$P_Z = \frac{100}{1.08^5} = 68.06$$

(b) Price sensitivities:

$$\text{DV01}_X = \frac{8}{1.0751} + \frac{8}{1.0801^2} + \frac{8}{1.0826^3} + \frac{108}{1.0826^4} - 99.26 = -0.033$$
$$\text{DV01}_Y = \frac{9}{1.0751} + \frac{9}{1.0801^2} + \cdots + \frac{109}{1.0776^7} - 106.18 = -0.055$$
$$\text{DV01}_Z = \frac{100}{1.0801^5} - 68.06 = -0.033$$

(c) Bond prices in each scenario:
   (i) P_X goes down 33 cents from \$99.26 to \$98.93; P_Y goes down 54 cents from \$106.17 to \$105.63; P_Z goes down 32 cents from \$68.06 to \$67.74.
   (ii) P_X drops \$3.01 from \$99.26 to \$96.05; P_Y drops \$5.27 from \$106.17 to \$100.90; P_Z drops \$3.07 from \$68.06 to \$64.99.

(d) First-order comparison:
   (i) to 10-fold price sensitivity:

Bond	**DV01 × 10**	**Price change**	**difference (DV01 – variation)**
X	−0.33	−0.33	0
Y	−0.55	−0.54	−0.01
Z	−0.33	−0.32	+0.01

   (ii) to 100-fold price sensitivity:

Bond	**DV01 × 100**	**Price change**	**difference (DV01 – variation)**
X	−3.3	−3.01	−0.29
Y	−5.5	−5.27	−0.23
Z	−3.3	−3.07	−0.23

(e) Multiplying the price sensitivity by the basis point rate move gives a good approximation of the change in bond price for small moves. For large moves the approximation becomes inaccurate.

(f) (*) When the zero-coupon rate curve is flat at r, the bond price is a function $P(r)$ of only one variable. A second-order Taylor expansion of P around r is then:

$$P(r+\delta) \underset{\delta\to 0}{\overset{(2)}{\approx}} P(r) + \delta\frac{\partial P}{\partial r}(r) + \frac{1}{2}\delta^2\frac{\partial^2 P}{\partial r^2}(r).$$

For $\delta = 0.01\%$ we have:

$$\text{DV01} = P(r + 0.01\%) - P(r) \approx 0.01\% \times \frac{\partial P}{\partial r}(r) + \frac{1}{2} 0.01\%^2 \frac{\partial^2 P}{\partial r^2}(r).$$

Since 0.01% is very close to 0, the quadratic term is negligible and $DV01 \approx 0.01\% \times \frac{\partial P}{\partial r}(r)$, i.e.: $\frac{\partial P}{\partial r}(r) \approx \text{DV01} \times 10{,}000$. For a given parallel shift δ of the rate curve, the change in the bond price may thus be written:

$$P(r + \delta) - P(r) \underset{\delta \to 0}{\overset{(2)}{\approx}} 10{,}000 \times \delta \times \text{DV01} + \frac{1}{2}\delta^2 \frac{\partial^2 P}{\partial r^2}(r).$$

For example if $\delta = 1\%$:

$$P(r + 1\%) - P(r) \approx 100 \times \text{DV01} + \frac{1}{20{,}000} \frac{\partial^2 P}{\partial r^2}(r).$$

This equation helps us better understand the empirical finding in question (d) whereby DV01 is only an accurate indicator of the change in bond price for small rate moves. For large moves the second degree term has an impact. This term known as **bond convexity** is therefore a natural complement to DV01 when measuring the sensitivity of a given bond to interest rate risk.

Problem 14: Yield curve and expectations

(a) This question tricks many into the following "common sense" argument: "Since the 2-week rate is 2.75% and the 1-month rate is 2.92% the market expects an increase of $2.92\% - 2.75\% = 0.17\%$. The majority of market participants must thus expect the ECB to raise its refinancing rate by 25 basis points."
However, this argument ignores that the 2.92% rate also applies to the forthcoming 2 weeks. The correct answer is that the ECB is expected to raise its refinancing rate by 25 to 50 basis points, as demonstrated in question (b).

(b) *1st method*: Lezard Brothers borrows €100 mn for 1 month at 2.92%. Interest paid on this transaction is $100{,}000{,}000 \times [(1 + 2.92\%)^{1/12} - 1] =$ €240,136.21. *2nd method*: Lezard Brothers borrows €100 mn for 2 weeks at 2.75%, and rolls over the loan and interest in 2 weeks' time at rate R for the remaining 2 weeks. The total cost of this method is: $100{,}000{,}000 \times [(1 + 2.92\%)^{1/24} \times (1 + R)^{1/24} - 1]$.

(c) Equating the costs of both methods and solving for R we find:

$$R^* = \left(\frac{100{,}240{,}136.21}{100{,}113{,}100.02}\right)^{24} - 1 \approx 3.09\%.$$

The rate of 3.09% can be interpreted as the market's expectation of the ECB's decision in 2 weeks. In other words, the market expects an increase between 25 and 50 basis points which is different from the naive answer to question (a).

(d) (i) Bernard is not accurate: in 2 weeks the amount he needs to borrow is $100{,}000{,}000 \times (1 + 2.75\%)^{1/24} =$ €100,113,100.02 rather than €100,000,000. The exact cash flow table of Bernard's strategy is thus:

Transaction	today	in 2 weeks	in 1 month
Lend €100 mn at 2.92% over one month	−100 mn		+100,240,136.21
Borrow €100 mn at 2.75% for 2 weeks	+100 mn	−100,113,100.02	
In 2 weeks, borrow €100,113,100.02 at rate R for another 2 weeks		+100,113,100.02	$-100{,}113{,}100.02 \times (1+R)^{1/24}$
TOTAL	0	0	$100{,}240{,}136.21 - 100{,}113{,}100.02 \times (1+R)^{1/24}$

For Bernard's strategy to be a pure arbitrage the final cash flow should always be positive. Clearly this cannot always be true since the last cash flow depends on R. We must thus analyze the risks of Bernard's strategy along various scenarios for R.

(ii) Profit or loss of Bernard's strategy in various scenarios for R:

Scenario	a) −25 bps	b) no change	c) +25 bps	d) +50 bps
Profit / Loss	€+23,980.93	€+13,808.25	€+3,659.26	€−6,466.16

(iii) Bernard's strategy would make a small profit of €3,659.26 in the consensus scenario (25 basis point increase) but it would also cause a larger loss of €6,466.16 in the case of a 50 basis point increase. To see whether the risk is worth the effort we may, for example, calculate the expected profit or loss under the probabilities implied by the survey:

Scenario	a) −25 bps	b) no change	c) +25 bps	d) +50 bps
Profit / Loss	€+23,980.93	€+13,808.25	€+3,659.26	€−6,466.16
Probability	0%	10%	70%	20%

In this analysis the expected profit is €+2,649.07 which seems attractive. However, it should be pointed out that the order of magnitude of this profit (in the thousands) is very different from the principal amounts employed (€100 mn). To be truly profitable the strategy should be leveraged, which may pose some risk management problem.

Chapter 4

Problem 1: True or False?

(a) False: 3.4% is an annualized return which has no reason to match the actual annual return. For example consider the series of 12 alternating monthly returns (+10%, −8%, +10%, −8%, ..., +10%, −8%): the average monthly return is 1% equivalent to 12.68% annualized, yet the annual return is $1.1^6 \times 0.92^6 - 1 \approx +7.42\%$.

(b) False: the only correct method is to compute the standard deviation of monthly returns and multiply by $\sqrt{12}$. In the above example this gives ≈32.6%, whereas computing the standard deviation of annualized returns would give ≈145%.

(c) False: if MeToo.Com has a low or negative correlation with the portfolio, adding it could lower the risk below 25%.

Problem 2: Risk-free rate and Sharpe ratio. Risk-free rate so that the Sharpe ratio equals 1:

$$r_f = R_{\text{T-Bond}} - \text{Sharpe} \times \sigma_{\text{T-Bond}} = 3.3\% - 1 \times 1.3\% = 2\%.$$

Problem 3: Risk and return

(a) The \$13 dividend per share paid on 30 June allows one to buy $\frac{13}{130} = 0.1$ extra share. Taking this into account we get the following returns:

Month	Price	Quantity	Mark-to-market	Return
Dec. N - 1	134	1	134.00	
Jan.	144	1	144.00	7.46%
Feb.	123	1	123.00	−14.6%
March	128	1	128.00	4.1%
April	137	1	137.00	7.0%
May	147	1	147.00	7.3%
June	130	1.1	143.00	−2.7%
July	139	1.1	152.90	6.9%
Aug.	147	1.1	161.70	5.8%
Sept.	175	1.1	192.50	19.0%
Oct.	162	1.1	178.20	−7.4%
Nov.	154	1.1	169.40	−4.9%
Dec.	158	1.1	173.80	2.6%

(b) Using the data from the table above, the annual return of Richky Corp. is $\frac{173.80-134.00}{134.00} = 29.70\%$ with a monthly standard deviation of 8.76%, i.e. $8.76\% \times \sqrt{12} \approx 30.36\%$ per year. With a risk-free rate of 5%, the Sharpe ratio is $\frac{29.70\%-5\%}{30.36\%} = 0.81$, which is very good.

(c) At the current stock price of \$158 a \$1.2 annual saving means an increase in annual return by 0.76%, and a \$0.6 standard deviation means an extra risk of 0.38% per year. If risks added up we would get a standard deviation of 9.14% per month (32.56% per year) for an annual return of 30.46%.

However, when correlation is less than 1, risks do not add up: for example, the total risk when correlation is zero would be $\sqrt{8.76\%^2 \times 12 + 0.38\%^2 \times 12} = 30.39\%$ in which case the Sharpe ratio would improve to 0.84.

A more detailed analysis shows that the Sharpe ratio is in fact a function of the correlation between the project and stock returns:

$$Sharpe(\rho) = \frac{32.56\% - 5\%}{\sqrt{12 \times \left(8.76\%^2 + 0.38\%^2 + 2\rho \times 8.76\% \times 0.38\%\right)}}.$$

From this relationship we may deduce that the correlation must be smaller than 0.69 in order for the project to improve the risk-return profile of the company.

Problem 4: Risk premium and CAPM

(a) Pschitzer Pharmaceuticals (stock): Risk premium: $r_A - r_f = \beta_A(r_M - r_f) = 1.5 \times (7\% - 3\%) = 6\%$. Sharpe ratio:

$$Sharpe_A = \frac{r_A - r_f}{\sigma_A} = \frac{6\%}{15\%} = 0.4.$$

(b) T-Bond: Risk premium: 0.8%. Sharpe ratio: 0.27.
(c) Goldy (mutual fund): Risk premium: −2%. Sharpe ratio: −0.17. At first sight investing in Goldy would be irrational since the risk premium is negative: the 1% expected return is less than the risk-free rate. However, the negative beta pinpoints to a benefit: when the market is down 1% the fund is up 0.5%. Thus, investing in Goldy helps reduce losses in a bear market and it makes sense for investors to pay a negative risk premium for this protection.

Problem 5: Volatility and Sharpe ratio

(a) Pairwise correlation coefficients: Kroger and T-Bond: 0.083, Kroger and Coast Value: 0.069, T-Bond and Coast Value: 0.019.
(b) Portfolio risk:

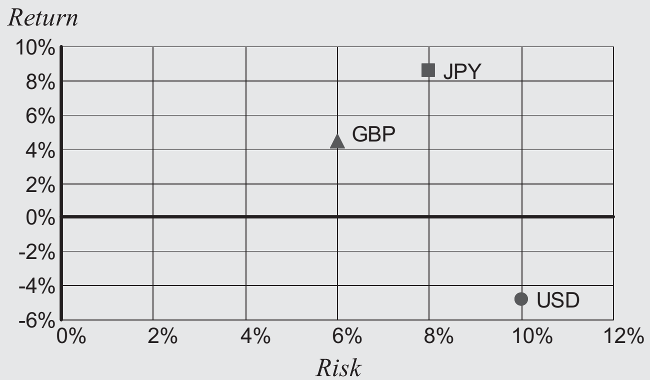
$$\begin{aligned}\sigma_{\mathrm{P}}^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_1\sigma_2\rho_{1,2} + 2w_1w_3\sigma_1\sigma_3\rho_{1,3} + 2w_2w_3\sigma_2\sigma_3\rho_{2,3}\\ &= 0.5^2 \times 13.6\%^2 + 0.25^2 \times 1.3\%^2 + 0.25^2 \times 18.9\%^2\\ &\quad +2 \times 0.5 \times 0.25 \times 13.6\% \times 1.3\% \times 0.083\\ &\quad +2 \times 0.5 \times 0.25 \times 13.6\% \times 18.9\% \times 0.069\\ &\quad +2 \times 0.25^2 \times 1.3\% \times 18.9\% \times 0.019\\ &= 0.007353 \Rightarrow \sigma_P = 8.6\%\end{aligned}$$

This matches the 8.7% figure found in Table 4.5 p.40 (give or take rounding errors).
(c) The portfolio's Sharpe ratio is (8.1% − 3.3%)/8.7% ≈ 0.55, which is roughly half-way between Kroger Co. and Coast Value LP.

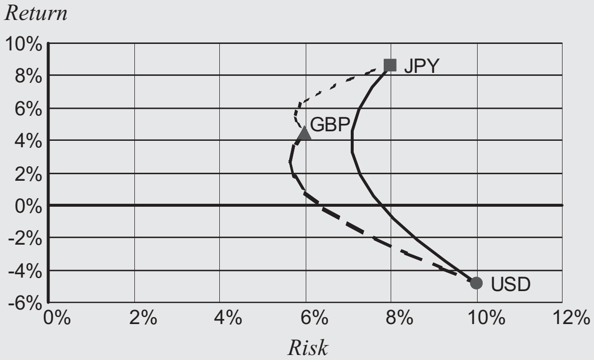
Problem 6: Currency portfolio

(a) We first need to calculate the return on each currency. For example, consider the yen: to invest all the capital in yen, we must first exchange 1 billion euros for 130 billion yen. At 0.25% interest, this amount will grow to ¥130,325,000,000 after 1 year which will then be worth €1,086,041,670 according to the exchange rate forecast of ¥120 per euro. The return on the yen is thus 8.60%.

The same argument gives a return of −4.86% on the dollar and 4.50% on the pound. We may now plot the risk-return profiles of the three currencies:



(b) Evolution of the risk-return profile of a portfolio gradually switching from dollars to yens to pounds to dollars:



(c) In this question we must determine the risk and return of all possible portfolios made of the three currencies. Let w_{USD}, w_{JPY} and w_{GBP} denote the weights of the currencies. Since we only consider long positions we have the following constraints:

$$\begin{cases} 0 \leq w_{USD} \leq 1,\ 0 \leq w_{JPY} \leq 1,\ 0 \leq w_{GBP} \leq 1 \\ w_{USD} + w_{JPY} + w_{GBP} = 1 \end{cases}$$

Under these constraints, the risk-return profile of a portfolio P = (w_{USD}, w_{JPY}, w_{GBP}) is given as:

$$R_P = -4.82\% \times w_{USD} + 8.60\% \times w_{JPY} + 4.50\% \times w_{GBP}$$

$$\sigma_P = \sqrt{\sum_{(i,j)\in\{USD,JPY,GBP\}^2} w_i w_j \sigma_i \sigma_j \rho_{i,j}}$$

where σ_i is the risk of currency i and $\rho_{i,j}$ is the correlation coefficient between currencies i and j ($\rho_{i,j} = 1$ when $i = j$).

Following the steps below we can determine the risk and return of a finite number of portfolios in a spreadsheet:

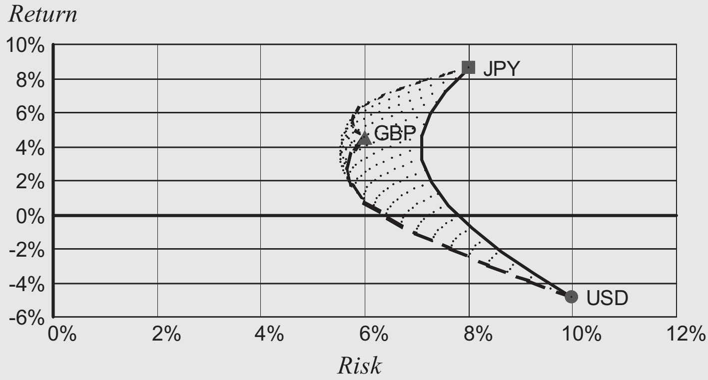
(i) Build a table of weights as below:

↓ w_{JPY} w_{USD} →	0%	5%	10%	...
0%	$w_{GBP} = 1 - w_{USD} - w_{JPY} = 100\%$	95%	90%	...
5% ...	95%	90%	...	...

(ii) Build a table of portfolio returns using the table of weights above and the returns of each currency.

(iii) Build a table of portfolio volatilities using the table of weights above and the volatility and correlations forecasts for the three currencies.

This methodology produces the graph below. It is interesting to note that some risk-return coordinates are outside of the frontiers of the previous graph. This means that there exists an optimal portfolio of the three currencies which has e.g. the same return as the pound with a lower risk.



(d) A mix of 25% in dollars and 75% in yen would give a 5.25% annual return and a risk of 7.16%. However, we know from question (c) that there exists a better solution. Indeed, we find that a mix of 5% in dollars, 30% in yen, and 65% in pounds gives a 5.27% annual return for a lower annual risk of 5.68%.

Problem 7*: General portfolio optimization on 2 assets

(a) Weights can be negative in case of short positions.

(b) Portfolio return: $R_P = wR_A + (1-w)R_B = R_B + w(R_A - R_B)$

(c) Portfolio risk:

$$\begin{aligned}\sigma_{\mathrm{P}} &= \sqrt{\mathbb{V}\,(wR_A + (1-w)R_B)} \\ &= \sqrt{w^2\sigma_A^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\sigma_A\sigma_B\rho} \\ &= \sqrt{(\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho)w^2 + 2\sigma_B(\sigma_A\rho - \sigma_B)w + \sigma_B^2}\end{aligned}$$

(d) If $\rho = 1$ we have $\sigma_P = |\, w\sigma_A + (1\text{-}w)\sigma_B \,|$ after recognizing a perfect square. The graph of σ_P consists of two line segments whose minimum is reached at $w^* = -\frac{\sigma_B}{\sigma_A - \sigma_B}$, with $\sigma_P(w^*) = 0$. If, for example, the risk of A is higher than B we would take a short position on A and a long position on B, in proportions such that each position has the same variance. Since the assets are perfectly correlated, any profit on A would be exactly offset by a loss on B and vice versa. By construction this portfolio is riskless and must earn the risk-free rate r_f.

The case where $\rho = -1$ is similar with the benefit of dealing only with long positions.

(e) Suppose $-1 < \rho < 1$.

(i) If $|\,\rho\,| < 1$, σ_P^2 is a quadratic function $aw^2 + bw + c$ whose coefficients a, b, c can be found in question (c). The shape is parabolic.

(ii) The parabola reaches its minimum at $w^* = -\frac{b}{2a} = \frac{\sigma_B(\sigma_B - \sigma_A\rho)}{\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho}$. Note that the minimum value σ_P^* can be shown to be strictly positive: if A and B are not perfectly correlated, risk cannot be entirely eliminated. When $\sigma_{\mathrm{A}} = \sigma_{\mathrm{B}}$ we have $w^* = 0.5$.

(iii) In the case of Kroger Co. and Coast Value LP, the optimal portfolio has weight:

$$w^* = \frac{18.9\% \times (18.9\% - 13.6\% \times 0.07)}{13.6\%^2 + 18.9\%^2 - 2 \times 13.6\% \times 18.9\% \times 0.07} \approx 67\%.$$

(f) If A is risk-free we have $\sigma_A = 0$. $R_A = r_f$ is then a constant random variable and its correlation with R_B is zero. In this case $\sigma_P = \sqrt{(1-w)^2\sigma_B^2} = |1-w|\,\sigma_B$: the risk of the portfolio is the risk of B multiplied by its weight. The minimum is unsurprisingly reached at $w = 1$ when the portfolio is fully invested in the risk-free asset.

Problem 8*: Portfolio optimization on *n* correlated assets

(a) Using Equation (4-2) p.41 with $\sigma_i = 1$ we get $\sigma_P^2 = \sum_{i=1}^n w_i^2\sigma_i^2 + 2\sum_{i<j} w_i w_j \sigma_i \sigma_j \rho_{i,j} = \sum_{i=1}^n w_i^2 + 2\rho\sum_{i<j} w_i w_j$. Using $\left(\sum_{i=1}^n w_i\right)^2 = \sum_{i=1}^n w_i^2 + 2\sum_{i<j} w_i w_j$ we can write $\sigma_P^2 = \rho\left(\sum_{i=1}^n w_i\right)^2 + (1-\rho)\sum_{i=1}^n w_i^2 = \rho + (1-\rho)\sum_{i=1}^n w_i^2$ because $\Sigma w_i = 1$.

(b) Substituting $w_n = 1 - \sum_{i=1}^{n-1} w_i$ we have $\sigma_P^2 = \rho + (1-\rho)\sum_{i=1}^{n-1} w_i^2 + (1-\rho)\left(1 - \sum_{i=1}^{n-1} w_i\right)^2$. Taking first-order derivatives with respect to $w_1, \ldots, w_{n\text{-}1}$, we find that the optimal weights must satisfy:

$$2(1-\rho)\left[w_i - \left(1 - \sum_{j=1}^{n-1} w_j\right)\right] = 0 \Rightarrow w_i = w_n.$$

The optimal portfolio is thus equally weighted: $w_i = 1/n$.

(c) The optimal portfolio variance is $\sigma_P^{*2} = \rho + (1-\rho)\frac{1}{n} \underset{n\to\infty}{\longrightarrow} \rho$.

Problem 9*: Portfolio optimization on *n* uncorrelated assets

When correlation is zero, the portfolio variance is $\sigma_P^2 = \sum_{i=1}^n w_i^2\sigma_i^2 = \sum_{i=1}^{n-1} w_i^2\sigma_i^2 + \sigma_n^2\left(1 - \sum_{i=1}^{n-1} w_i\right)^2$. First-order conditions yield for all i: $2\sigma_i^2 w_i - 2\sigma_n^2\left(1 - \sum_{j=1}^{n-1} w_j\right) = 0 \Rightarrow w_i = \frac{\sigma_n^2}{\sigma_i^2} w_n$. Since $\sum_{i=1}^n w_i = 1$ we get $w_n = 1/\sum_{i=1}^n \frac{\sigma_n^2}{\sigma_i^2}$. Thus $w_i = 1/\sum_{j=1}^n \frac{\sigma_i^2}{\sigma_j^2}$ and the minimum portfolio variance is $\sigma_P^{*2} = \sum_{i=1}^n \sigma_i^2 / \left(\sum_{j=1}^n \frac{\sigma_i^2}{\sigma_j^2}\right)^2 = 1/\sum_{i=1}^n \frac{1}{\sigma_i^2}$. This minimum is unique, and if $\sigma_i \le K$ then $\sigma_P^* \to 0$ as $n \to \infty$.

Chapter 5

Problem 1: True or False?

(a) False: Forward contracts are derivative securities, yet they may have a negative value. However, plain vanilla options always have a positive value. Generally, the value of a derivative security whose payoff is positive is always positive.

(b) False: While the payoff of a plain vanilla option is always positive, the Net P&L (payoff minus premium) may be negative.

(c) True: Using put-call parity (Equation (5-5) p.58) we may write:

$$\begin{aligned}\text{Time value of ITM call} &= c_0 - (S_0 - K)\\ &= p_0 + \phi_0 - S_0 + K\\ &= p_0 + K\left(1 - \frac{1}{(1+r)^T}\right) > p_0 = \text{Time value of OTM put.}\end{aligned}$$

Problem 2: Forward price and price of the forward contract

(a) No dividends: 1-year forward price:

$$F_0 = S_0(1+r)^T = 5{,}200 \times (1+1\%)^1 = 5{,}252.$$

Price of the 6-month forward contract:

$$\phi_0 = S_0 - \frac{K}{(1+r)^T} = 5{,}200 - \frac{5{,}000}{(1+1\%)^{0.5}} = ¥224.81.$$

(b) Cash dividend of ¥104 in 9 months: 1-year forward price:

$$\begin{aligned} F_0 &= \left(S_0 - \frac{D}{(1+z(t_D))^{t_D}}\right)(1+r)^T \\ &= \left(5{,}200 - \frac{104}{(1+0.8\%)^{0.75}}\right)(1+1\%)^1 \approx ¥5{,}147.58. \end{aligned}$$

Price of the 6-month forward contract: ¥224.81 (the dividend is paid after the contract expires).

(c) Proportional dividend of 2% in 3 months: 1-year forward price:

$$F_0 = \frac{S_0}{1+d}(1+r)^T = \frac{5{,}200}{1+2\%}(1+1\%)^1 \approx ¥5{,}149.02.$$

Price of the 6-month forward contract:

$$\phi_0 = \frac{S_0}{1+d} - \frac{K}{(1+r)^T} = \frac{5{,}200}{1+2\%} - \frac{5{,}000}{(1+1\%)^{0.5}} = ¥147.54.$$

Problem 3: Option payoffs

(a) *Example 1*: Underlying asset: Kroger Co. Maturity: 1 month. Payoff: $S_T - 25$ where $T = 1/12$. *Example 2*: Underlying asset: S&P 500 index. Maturity: 3 months. Payoff: $10{,}000 \times \sqrt{\frac{252}{63}\sum_{t=1}^{63} r_t^2}$ where $r_t = \frac{S_t - S_{t-1}}{S_{t-1}}$ is the arithmetic return on day t out of 63 trading days. *Example 3*: Underlying assets: Google Inc. and Microsoft Inc. Maturity: 1 year. Payoff: $100{,}000{,}000 \times \max\left(0, \frac{G_T}{G_0} - \frac{M_T}{M_0}\right)$ where $T = 1$, G_t is Google's price and M_t is Microsoft's price.

(b) Option portfolio: long call struck at X, short call struck at Y (this is known as a 'call spread').

(c) Portfolio payoff: $\max(90, S_T) \geq S_T$. Since there are no dividends this implies that the portfolio value must exceed $S_0 = \$100$ under penalty of arbitrage (the inequality is strict because there are scenarios in which the payoff is greater than the stock price).

Problem 4: Option strategies

(a) The Net P&L breaks even whenever the stock price hits $\$24 \pm 1.92 \times (1+3\%)^{0.25} \approx \$22.06$ or \$25.94 (recall that the \$1.92 premium is paid immediately rather than on the maturity date). A ± 1.92 move intuitively corresponds to an $1.92/24 = 8\%$ volatility of quarterly returns. We may thus estimate the annualized volatility to be around $8\% \times \sqrt{4} = 16\%$.

(b) The cheapest strategy is to buy a 6,750–7,000 put spread (long the 7,000 put and short the 6,750 put) at $270 - 261 = ¥9$ per spread. Each spread will pay off $7{,}000 - 6{,}750 =$

¥250 whenever the Nikkei trades at 6,750 or below, hence we need to buy 100 mn/250 = 400,000 spreads for a total cost of ¥3.6 mn.

Problem 5: Forward price and price of the forward contract at an arbitrary time *t*
Equation (5-1) p.51 and Equation (5-2) p.53 are valid for $t = 0$. It is easy to see that the distance between an arbitrary point in time t and the maturity T is the same as the distance between times 0 and $T - t$. This observation instantly gives the generalized formulas:

$$\phi_t = S_t - \frac{K}{(1+r)^{T-t}} \quad \text{(same proof as in Section 5-2.2 p.51)}$$
$$F_t = S_t(1+r)^{T-t} \quad \text{(obtained by solving } \phi_t = 0 \text{ for } K\text{)}.$$

Problem 6: Arbitrage arguments

(a) $c_t \geq 0$ because the call payoff is always nonnegative. $C_t^{\text{US}} \geq c_t$ because the owner of an American call can always choose to exercise on the maturity date.

(b) $\max(0, S_t - K) \leq C_t^{\text{US}}$ because exercising the American call at time t is worth the intrinsic value $\max(0, S_t - K)$. If $C_t^{\text{US}} > S_t$ we would have the following arbitrage strategy: buy S and sell the American call for an immediate profit; if the owner of the call ever exercises deliver S against $K > 0$; otherwise, at maturity, sell S on the market against $S_T > 0$.

(c) If $c_0 < S_0 - \frac{K}{(1+r)^T}$ we would have an arbitrage: buy the call and sell a forward contract on S with strike K for an immediate profit; at maturity collect $\max(0, S_T - K) - (S_T - K) \geq 0$.

Problem 7: From put-call parity: $p_t = c_t - \phi_t$, and from Problem 6 question (a) we have $C_t^{\text{US}} \geq c_t$. Thus $p_t \leq 144 - [1{,}200/(1+2\%)^1 - 1{,}200/(1+1\%)^1] \approx \$155.65$.

Problem 8: Successive dividends

(a) Two cash dividends: $\phi_0 = S_0 - \dfrac{D_1}{[1+z(t_1)]^{t_1}} - \dfrac{D_2}{[1+z(t_2)]^{t_2}} - \dfrac{K}{[1+z(T)]^T}$.

(b) Two proportional dividends: $\phi_0 = \dfrac{S_0}{(1+d_1)(1+d_2)} - \dfrac{K}{[1+z(T)]^T}$;

(c) Cash dividend then proportional dividend: $\phi_0 = \dfrac{S_0 - D/[1+z(t_1)]^{t_1}}{1+d} - \dfrac{K}{[1+z(T)]^T}$.

Problem 9: Barrier option

(a) Answer (ii): Compared to plain vanilla options, barrier options must satisfy additional conditions in order to be exercised. The chances of receiving the vanilla option payoff are thus lower when there are barriers, which implies that the value of barrier options must be lower than that of plain vanilla ones.

(b) (i) Figures S-1 and S-2 below show the payoff of each barrier call. The knock-out call has no value: starting from 90, if the underlying price exceeds the strike of 100 at maturity it must go through the 95 barrier during the life of the option, in which case it becomes worthless. Symmetrically, the knock-in call has the same value as a plain vanilla call: for the underlying price to exceed the strike it must go through the barrier during the life of the option which activates the option.

(ii) The knock-out call with barrier $H = 80$ will lose its value if the underlying price goes down from 90 to 80 or less. However, since the strike is 100, the option has no intrinsic value when the barrier is hit: only the time value is lost. On the other hand, the knock-out call with barrier $H = 110$ has intrinsic value of 10 when the barrier is hit; this type of barrier option is sometimes referred to as a '**kick-out call**' because it can cause more financial "pain."

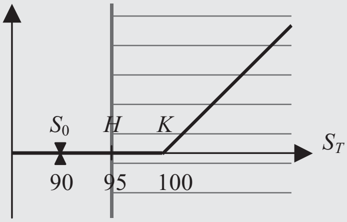


Figure S-1 Payoff of a knock-out call with strike $K = 100$ and barrier $H = 95$

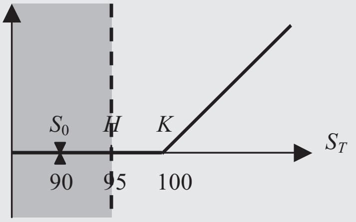


Figure S-2 Payoff of a knock-in call with strike $K = 100$ and barrier $H = 95$

(iii) A portfolio which is long a knock-in call and a knock-out call with the same characteristics (underlying, strike, maturity, barrier H) has the same payoff as a plain vanilla call. Thus, if the plain vanilla call is worth 2 and the knock-out call is worth 1 the knock-in call must also be worth 1.

Problem 10: Forward exchange rate

(a) *1st method*: exchange €1 for \$$S$ today and invest the dollars at rate r_{US} for a year; the final dollar amount in a year's time is \$$S(1 + r_{US})$. *2nd method*: invest €1 at rate r_{EU} for a year and then exchange the resulting capital of €$(1 + r_{EU})$ into dollars at the forward rate F agreed today; the final dollar amount in a year's time is \$$F(1 + r_{EU})$. (See Figure S-3 below.)

Both methods are equivalent ways of exchanging euros into dollars in a year's time. Thus they must yield the same dollar amount per euro, under penalty of arbitrage. As a result: $S(1 + r_{US}) = F(1 + r_{EU})$, which proves the required formula.

This formula is the same as Equation (5-4) p.55 for the forward price of a stock with dividend rate $d = r_{EU}$ and interest rate $r = r_{US}$.

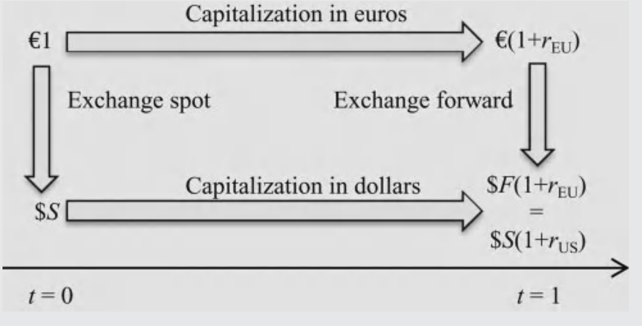


Figure S-3 Two methods to convert euros into dollars in a year's time

(b) Applying the formula derived in question (a): $F = 1.30 \times \frac{1+4\%}{1+2\%} \approx \$1.3255$ for € 1. For the 2-year forward exchange rate, we may either repeat the argument in question 1 with an investment horizon of 2 years, or iterate the 1-year forward exchange rate formula a second time:

$$F(0, 2) = 1.30 \times \frac{(1 + 4\%)^2}{(1 + 2\%)^2} = 1.3255 \times \frac{1 + 4\%}{1 + 2\%} \approx 1.3515.$$

(c) For both the 1-year and 2-year forward exchange rates, the number of dollars required to buy one euro is higher than the spot rate. The forward markets thus imply a depreciation of the dollar through time.

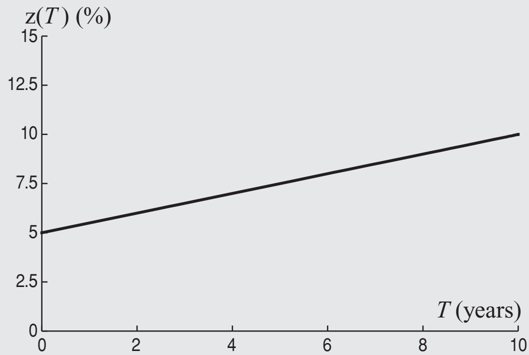
One possible explanation is that the higher interest rate in the US reflects a higher inflation in consumer prices than in the euro zone: the demand for dollars should thus decrease. However, the higher interest rate in the US should attract foreign investments from the euro zone to the dollar zone: the demand for dollars should thus increase.

Assuming both factors weigh equally on the exchange rate, the best forecast of the euro-dollar exchange rate in one year would be the current spot rate $S = 1.30$ rather than the forward rate $F = 1.3255$.

An investor with euros today could thus enter a strategy to invest his euros at a 2% rate and simultaneously buy dollars forward at 1.3255. In a year's time, if the exchange rate is still at 1.30, his profit would be $1.3255 \times 1.02 - 1.30 = \$0.052$ for every euro invested. This strategy is clearly not an arbitrage: if the euro-dollar exchange rate in one year is above $\$1.3255 \times 1.02 = 1.3520$, the strategy loses money.

Problem 11: Forward interest rate

(a) (i) Curve of $z(T) = 5\% + T \times 0.5\%$:



(ii) *1st method*: Invest (borrow) €1 over 3 years at the zero-rate $z(3) = 6.5\%$; after interest compounding we get $1.065^3 = €1.208$. *2nd method*: Invest (borrow) €1 over 1 year at the zero-rate $z(1) = 5.5\%$ and get €1.055; then invest (borrow) €1.055 over the following 2 years at the forward rate $z(1, 2) = 6.5\%$; after interest compounding we get $1.055 \times 1.065^2 = €1.197$.

These two methods are equivalent ways of investing or borrowing euros over 3 years but yield different results. Hence there is an arbitrage opportunity: invest €1 through the first method and borrow €1 through the second method to collect $1.208 - 1.197 = €0.011$.

(iii) The following equation must be satisfied to ensure no arbitrage opportunities:

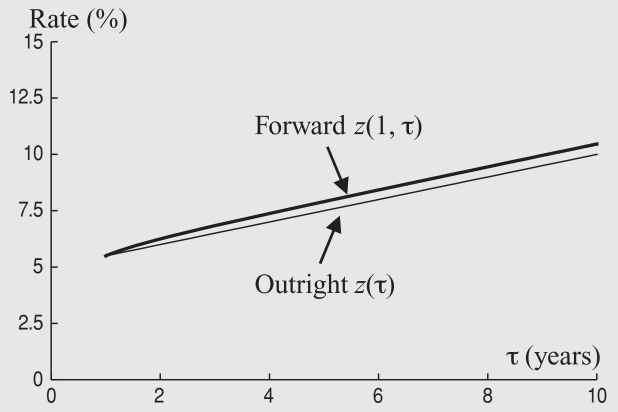
$$1.065^3 = 1.055 \times (1 + z(1, 2))^2.$$

Solving for $z(1,2)$ we get 7%. Generally:

$$(1 + z(\tau + 1))^{\tau+1} = 1.055 \times (1 + z(1, \tau))^\tau.$$

Solving for the forward rate $z(1, \tau)$ we get:

$$z(1,\tau) = \left[\frac{(1+z(\tau+1))^{\tau+1}}{1.055}\right]^{\frac{1}{\tau}} - 1 = \frac{(1.05+0.005\tau)^{1+\frac{1}{\tau}}}{1.055^{\frac{1}{\tau}}} - 1.$$



(b) (i) Suppose that $(1+z(T,\tau))^{\tau} > \frac{(1+z(T+\tau))^{T+\tau}}{(1+z(T))^{T}}$, i.e. $(1+z(T+\tau))^{T+\tau} < (1+z(T))^{T}(1+z(T,\tau))^{\tau}$.

In other words direct capitalization over $T+\tau$ years yields less than indirect capitalization over T and then τ years. Hence, there is an arbitrage opportunity:

Transaction	$t = 0$	$t = T$	$t = T + \tau$
Borrow €1 over $T+\tau$ years at rate $z(T+\tau)$	$+1$		$-(1+z(T+\tau))^{T+\tau}$
Invest €1 over T years at rate $z(T)$	-1	$+(1+z(T))^{T}$	
Invest €$(1+z(T))^{T}$ forward between T and $T+\tau$ at rate $z(T,\tau)$		$-(1+z(T))^{T}$	$+(1+z(T))^{T}(1+z(T,\tau))^{\tau}$
Total	**0**	**0**	$>$**0**

Conversely, if $(1+z(T,\tau))^{\tau} < \frac{(1+z(T+\tau))^{T+\tau}}{(1+z(T))^{T}}$, we again have an arbitrage by reversing transactions and signs.

(ii) Plugging numerical values in, we get: $z(1,\tau) = \left[\frac{(1+z(\tau+1))^{\tau+1}}{(1+z(1))^{1}}\right]^{\frac{1}{\tau}} - 1 = \frac{(1.05+0.005\tau)^{1+\frac{1}{\tau}}}{1.055^{\frac{1}{\tau}}} - 1.$

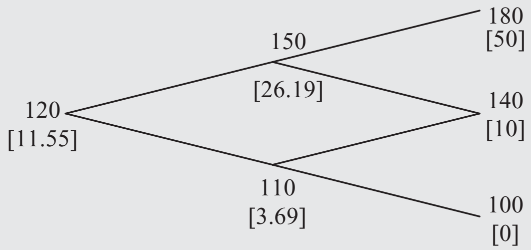
(iii) Since $z(T) \geq 0$ we get $z(T,\tau) = \left[\frac{(1+z(\tau+T))^{\tau+T}}{(1+z(T))^{T}}\right]^{\frac{1}{\tau}} - 1 \leq (1+z(\tau+T))^{1+\frac{T}{\tau}} - 1.$

Note that for $z(\tau+T) \approx 0$ we have the proxy $(1+z(\tau+T))^{1+\frac{T}{\tau}} - 1 \approx \left(1+\frac{T}{\tau}\right) z(\tau+T)$.

Chapter 6

Problem 1: Two-step binomial tree

(a) Value of a 2-year European call on Schultz AG struck at €130:

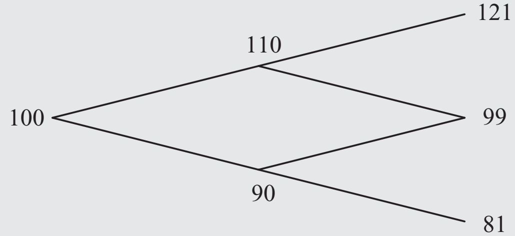


- Step $t = 1$: 'Up' scenario: $\Delta_1^{(u)} = \frac{50-10}{180-140} = 1,\ c_1^{(u)} = 150 - \frac{130}{1+5\%} = €26.19$. 'Down' scenario: $\Delta_1^{(d)} = \frac{10-0}{140-100} = 0.25,\ c_1^{(d)} = 0.25 \times 110 + \frac{0-0.25\times100}{1.04} = €3.69$.
- Step $t = 0$: $\Delta_0 = \frac{26.19-3.69}{150-110} = 0.5625,\ c_0 = 0.5625 \times 120 + \frac{26.19-0.5625\times150}{1.04} = €11.55$.

(b) Value of a 2-year European put on Schultz AG struck at €130: $p_0 = €9.47$ (using the binomial model or put-call parity).

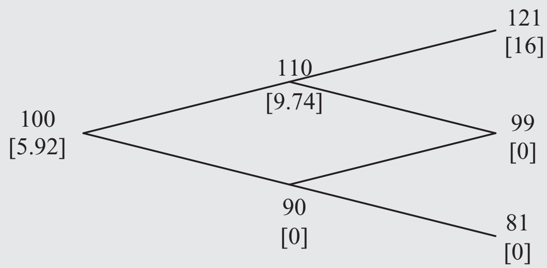
Problem 2: 'At-the-money-forward' options

(a) Binomial tree for the evolution of the stock price:



(b) 1-year forward price: $F = 100 \times (1 + 5\%) = \$105$.

(c) Value of a 1-year European call on ABC Inc. with strike F:



- Step $t = 0.5$: $\Delta_{0.5}^{(u)} = \frac{16-0}{121-99} = 0.7272$; $c_{0.5}^{(u)} = 0.7272 \times 110 + \frac{0-0.7272\times99}{(1+5\%)^{0.5}} = \$9.74$.
- Step $t = 0$: $\Delta_0 = \frac{9.74-0}{110-90} = 0.4868$; $c_0 = 0.4868 \times 100 + \frac{0-0.4868\times90}{(1+5\%)^{0.5}} = \$5.92$.

  According to put-call parity: $c_0 - p_0 = \phi_0$. Since the strike price is equal to the forward price F we have $\phi_0 = 0$; hence the European put is also worth \$5.92.

Problem 3*: Binomial model and forward contracts

(a) The payoff of a forward contract is $\phi_T = S_T - K$. Therefore:

$$\Delta = \frac{\left(S_T^{(u)} - K\right) - \left(S_T^{(d)} - K\right)}{\left(S_T^{(u)} - S_T^{(d)}\right)} = 1,$$

and $\phi_0 = \Delta S_0 - \frac{\left(S_T^{(d)} - K\right) - \Delta S_T^{(d)}}{(1+r)^T} = S_0 - \frac{K}{(1+r)^T}$, which agrees with Equation (5-1) p.51.

(b) If $K > S_T^{(u)} > S_T^{(d)}$ then $p_T^{(u)} = \max(K - S_T^{(u)}, 0) = K - S_T^{(u)}$, $p_T^{(d)} = \max(K - S_T^{(d)}, 0) = K - S_T^{(d)}$. In other words the payoff of the put is that of a short forward contract in both scenarios ('up' and 'down'). Thus, we know from question (a) that $p_0 = -\phi_0$. If $K < S_T^{(u)} < S_T^{(d)}$ then $p_T^{(u)} = p_T^{(d)} = 0$ and thus $p_0 = 0$.

Similar results apply to call options and often simplify calculations in multi-step binomial trees.

Problem 4*: Binomial model and dividends

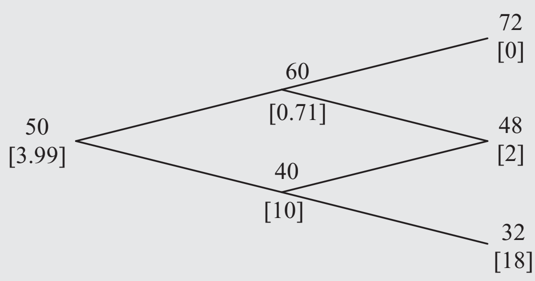
(a) The new final value of the portfolio is $P_T = D_T - \Delta \times (S_T + D)$.

(b) We want Δ such that $D_T^{(\mathrm{u})} - \Delta \times (S_T^{(\mathrm{u})} + D) = D_T^{(\mathrm{d})} - \Delta \times (S_T^{(\mathrm{d})} + D)$. Since the $\Delta \times D$ terms cancel we obtain the same formula: $\Delta = \frac{D_T^{(\mathrm{u})} - D_T^{(\mathrm{d})}}{S_T^{(\mathrm{u})} - S_T^{(\mathrm{d})}}$.

(c) Following the same reasoning as in Section 6-1.2 p.67 we get $D_0 - \Delta \times S_0 = \frac{D_T^{(\cdot)} - \Delta \times (S_T^{(\cdot)} + D)}{(1+r)^T}$ which yields the desired result.

(d) For the call on Kroger Co.: $D_0 = \frac{1}{3} \times 24 + \frac{1 - \frac{1}{3} \times (26 + 0.2)}{(1+1\%)^{1/12}} \approx \$0.27$.

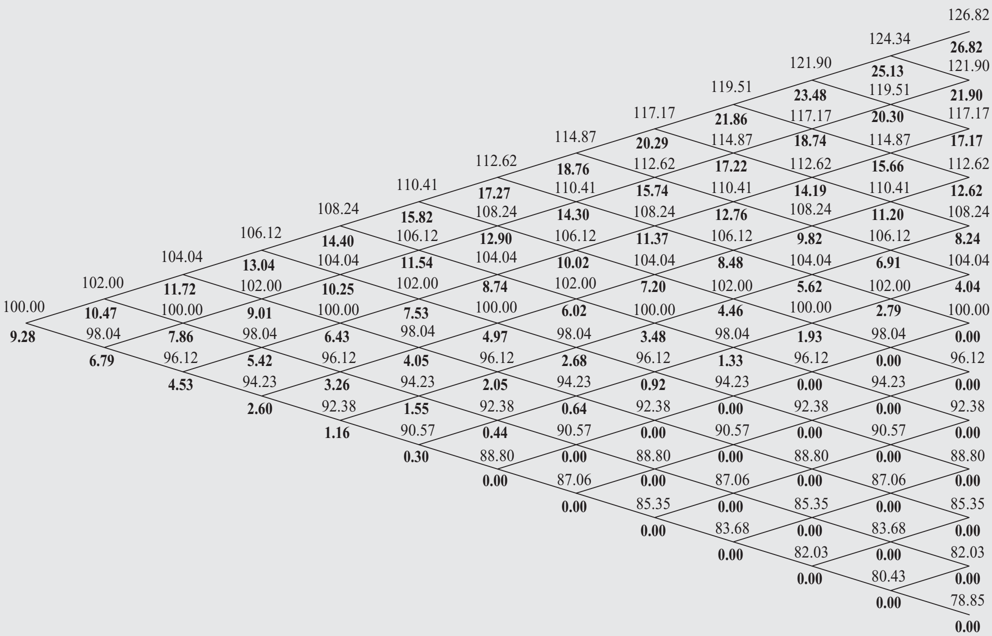
Problem 5: American option



- Step $t = 1$: 'Up' scenario: $\Delta_1^{(u)} = \frac{0-2}{72-48} = -0.0833$, $p_1^{(u)} = -0.0833 \times 60 + \frac{0 + 0.0833 \times 72}{1+5\%} = \$0.71$; with an underlying price at \$60 the option has zero intrinsic value and it would be crazy to exercise early, thus $P_1^{US(u)} = \$0.71$. 'Down' scenario: $\Delta_1^{(d)} = \frac{2-18}{48-32} = -1$, $p_1^{(d)} = -40 + \frac{50}{1+5\%} = \$7.62$; with an underlying price at \$40 an early exercise allows one to cash in \$10 which is better, thus: $P_1^{US(d)} = \$10$.
- Step $t = 0$: $\Delta_0 = \frac{0.71-10}{60-40} = 0.4645$, $p_0 = -0.4645 \times 50 + \frac{10 + 0.4645 \times 40}{1+5\%} = \$3.99$; exercising early would destroy the option value, thus $P_0^{US} = \$3.99$.

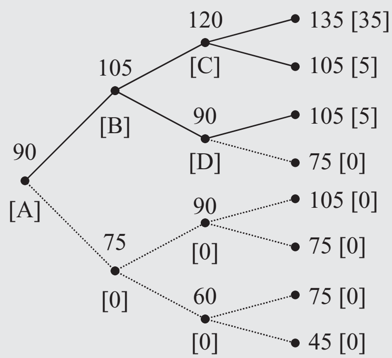
Problem 6: Binomial pricing algorithm

The tree below displays the results for this spreadsheet-based exercise. There are 12 periods and 13 columns. The monthly interest rate to be used in the calculations is $1.10^{1/12} - 1 \approx 0.8\%$.



Problem 7: Barrier option

The value of a barrier option is path-dependent: at each node of the tree we must determine whether the underlying previously breached the barrier or not. Hence we must use a non-recombining binomial tree, eliminating branches whenever the underlying price goes to 80 or below:



- Step $t = 2$, nodes C and D:

$$\begin{aligned}\Delta_D &= 1\\ D &= 120 - 100 = 20\\ \Delta_C &= \frac{5-0}{105-75} = 0.1667\\ C &= 0.1667 \times 90 + (0 - 0.1667 \times 75) = 2.5\end{aligned}$$

- Step $t = 1$, node B:

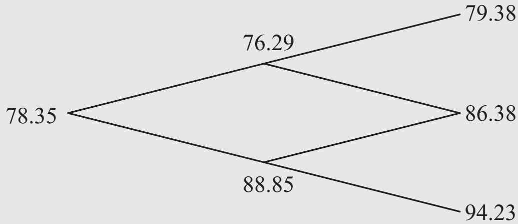
$$\begin{aligned}\Delta_B &= \frac{20-2.5}{120-90} = 0.5833\\ B &= 0.5833 \times 105 + (20 - 0.5833 \times 120) = 11.25\end{aligned}$$

- Step $t = 0$, node A:

$$\begin{aligned}\Delta_A &= \frac{11.25-0}{105-75} = 0.375\\ A &= 0.375 \times 90 + (0 - 0.375 \times 75) = 5.625\end{aligned}$$

Problem 8: Call on a bond

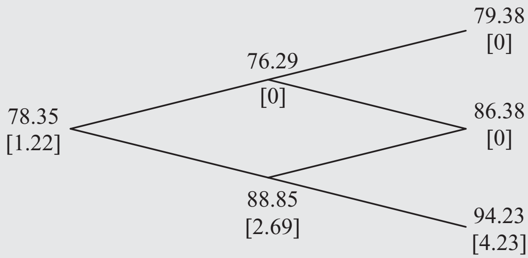
(a) Price of a T-year zero-coupon bond: $P = \frac{100}{(1+z(T))^T}$. Applying this formula at each node of the tree using the appropriate rate we obtain the following zero-coupon prices:



For example, the middle terminal node is calculated as follows:

$$P_2^{(ud)} = \frac{100}{(1+5\%)^3} = 86.38.$$

(b) When iterating Equation (6-1) p.67 going backwards we must be careful to use the appropriate discount rate which varies at each node:



- Step $t = 1$:

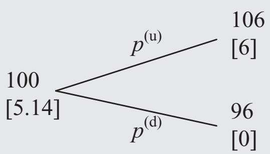
$$\Delta_1^{(d)} = \frac{0 - 4.23}{86.38 - 94.23} = 0.5389;\ c_1^{(d)} = 0.5389 \times 88.85 + \frac{0 - 0.5389 \times 86.38}{1 + 3\%} = 2.69.$$

- Step $t = 0$:

$$\Delta_0 = \frac{0 - 2.69}{76.29 - 88.85} = 0.2139;\ c_0 = 0.2139 \times 78.35 + \frac{0 - 0.2139 \times 76.29}{1 + 5\%} = 1.22.$$

Problem 9*: Risk-neutral probability

(a) Call value: $\Delta = \frac{6-0}{106-96} = 0.6$; $c_0 = 0.6 \times 100 + \frac{0-0.6\times 96}{1+5\%} = 5.14$. This value clearly does not depend on probabilities $p^{(\mathrm{u})}$ and $p^{(\mathrm{d})}$.



(b) We want to find p such that: $D_0 = \frac{1}{(1+r)^T}\left[pD_T^{(\mathrm{u})} + (1-p)D_T^{(\mathrm{d})}\right]$, i.e.:

$$\Delta S_0 + \frac{D_T^{(\cdot)} - \Delta S_T^{(\cdot)}}{(1+r)^T} = \frac{1}{(1+r)^T}\left[pD_T^{(\mathrm{u})} + (1-p)D_T^{(\mathrm{d})}\right].$$

Choosing the 'down' scenario for the numerator $D_T^{(\cdot)} - \Delta S_T^{(\cdot)}$ on the left-hand side, then solving for p, we get:

$$p = \frac{\Delta\left(S_0(1+r)^T - S_T^{(\mathrm{d})}\right)}{D_T^{(\mathrm{u})} - D_T^{(\mathrm{d})}} = \frac{S_0(1+r)^T - S_T^{(\mathrm{d})}}{S_T^{(\mathrm{u})} - S_T^{(\mathrm{d})}} = \frac{r^{[T]} - d}{u - d}.$$

(c) Since $u > r^{[T]} > d > -1$, we have $u - d > r^{[T]} - d > 0$ which implies that $0 < p < 1$.

(d) (i) Substituting $p^{(\mathrm{u})} = p$ and $p^{(\mathrm{d})} = 1 - p$ into the expression for D_0 derived in question (b) we obtain:

$$D_0 = \frac{1}{(1+r)^T}\left[p^{(\mathrm{u})}D_T^{(\mathrm{u})} + p^{(\mathrm{d})}D_T^{(\mathrm{d})}\right] = \frac{1}{(1+r)^T}\mathbb{E}(D_T) = \mathbb{E}\left(\frac{D_T}{(1+r)^T}\right)$$

(ii) Expected return on S over $[0, T]$:

$$\mathbb{E}\left(\frac{S_T - S_0}{S_0}\right) = p\frac{S_0(1+u) - S_0}{S_0} + (1-p)\frac{S_0(1+d) - S_0}{S_0}$$
$$= pu + (1-p)d = r^{[T]}.$$

In other words, although S is a risky asset, its expected return equals the risk-free rate, which contradicts portfolio theory (see Section 4-4 p.43). This suggests that the probabilities $p^{(\mathrm{u})} = p$ and $p^{(\mathrm{d})} = 1 - p$ have a "special" meaning and do not coincide with the objective probabilities used in portfolio theory.

In fact this choice of probabilities allows one to express the value of a derivative security as the discounted expectation of its payoff. A remarkable property is that they make the expected return on the underlying asset S equal the risk-free rate, as though we were neutral to the risk of S, whence the name 'risk-neutral probabilities.'

Chapter 7

Problem 1: True or False?

False: By definition, a random variable X is lognormally distributed with parameters (μ, σ) if the variable $Y = \ln(X)$ is normally distributed with parameters (μ, σ). In the lognormal model, the final price of the underlying S_T is lognormally distributed with parameters $(\ln F_0 - \frac{1}{2}\sigma^2 T$, $\sigma\sqrt{T})$, which means that $\ln(S_T)$ is normally distributed with mean $\ln F_0 - \frac{1}{2}\sigma^2 T$ rather than $\ln F_0$ as incorrectly stated. The corrective term $-\frac{1}{2}\sigma^2 T$ is required so that the expectation of S_T matches the forward price F_0 (see Problem 10).

Problem 2: Closed-form formulas

(a) Applying Equations (7-1) p.76 and (7-2) p.77 with a forward price $F_0 = 24(1 + 1\%)^{1/12} \approx \$24.02$:

$$d_1 = \frac{\ln(24.02/25) + \frac{1}{2}0.2^2/12}{0.2/\sqrt{12}} \approx -0.6638; \;\; d_2 = -0.6638 - 0.2/\sqrt{12} \approx -0.7215;$$
$$c_0 = 24N(-0.6638) - \frac{25}{(1+1\%)^{1/12}}N(-0.7215) \approx \$0.2042.$$

(b) Call value today: $F_0 = 50(1+10\%)^{0.5} \approx \$52.44$, $d_1 = \dfrac{\ln(52.44/50) + \frac{1}{2}0.3^2/2}{0.3/\sqrt{2}} \approx 0.3307$, $d_2 \approx 0.1185$, $c_0 = 50N(0.3307) - \frac{50}{(1+10\%)^{0.5}}N(0.1185) \approx \$5.39$. Put value: $p_0 = c_0 - \phi_0 = 5.39 - 50 + \frac{50}{(1+10\%)^{0.5}} \approx \$3.06$.

(c) Substituting $\sigma > 0$ with $-\sigma < 0$ we get:

$$d_1' = -\frac{\ln\frac{F_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = -d_1; \;\; d_2' = -d_1 + \sigma\sqrt{T} = -d_2;$$
$$c_0' = \frac{1}{(1+r)^T}\left[F_0 N(-d_1) - KN(-d_2)\right] = -p_0$$

Problem 3: Other distributions for S_T

(a) The uniform distribution $[a, 2F_0 - a]$ with $0 \le a < F_0$ satisfies properties (1) and (2) of Section 7-1.1 p.75 but not property (3) (as a counter-example take $a = 0$ and $x = 1.5$).

(b) The normal distribution $[F_0, b]$ where $b > 0$ satisfies property (2) but not (1) and (3).

(c) (*) $S_T = \frac{2F_0}{\pi} X$ satisfies all properties:
  1. $S_T \geq 0$ because $X \geq 0$.
  2. To prove $\mathbb{E}(S_T) = F_0$ we need to show that $\mathbb{E}(X) = \int_0^{+\infty} x f_X(x)dx = \frac{\pi}{2}$, where $f_X(x) = F_X{}'(x) = \frac{2x}{(1+x^2)^2}$. Hence $\mathbb{E}(X) = \int_0^{+\infty} x \frac{2x}{(1+x^2)^2} dx$. Integrating by parts we get:

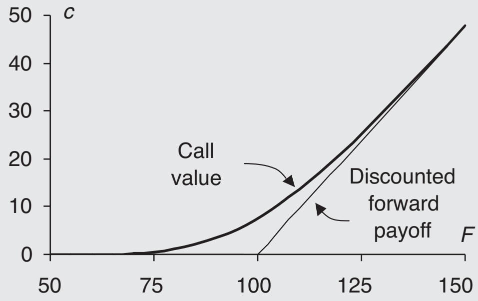
$$\mathbb{E}(X) = \left[\frac{x}{1+x^2}\right]_0^{+\infty} - \int_0^{+\infty} \frac{1}{1+x^2} dx = 0 - [\text{Arctan}(x)]_0^{+\infty} = \frac{\pi}{2}.$$

  3. The median of X is the value x such that $F_X(x) = \frac{1}{2}$; solving for x we find $x = 1$. Hence the median of S_T is $S^* = \frac{2F_0}{\pi}$. Furthermore $\mathbb{P}\left\{\frac{S_T}{S^*} \geq x\right\} = \mathbb{P}\{X \geq x\} = \frac{1}{1+x^2}$, which matches $\mathbb{P}\left\{\frac{S_T}{S^*} \leq \frac{1}{x}\right\} = \mathbb{P}\left\{X \leq \frac{1}{x}\right\} = 1 - \frac{1}{1+(1/x)^2} = \frac{1}{1+x^2}$.

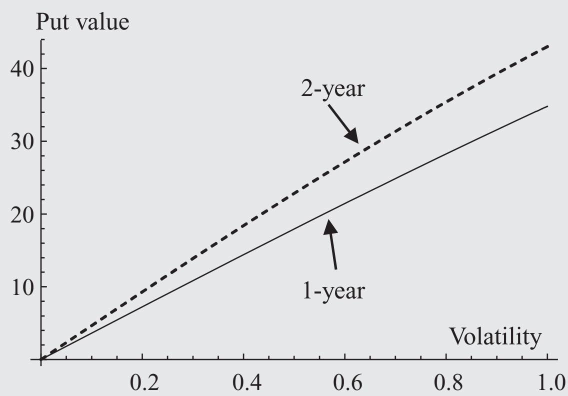
This distribution is thus plausible (the graph of f_X is even bell-shaped). However it does not have any volatility parameter and the median is far from the mean F_0, which is counter-intuitive.

Problem 4: Plain vanilla option pricer

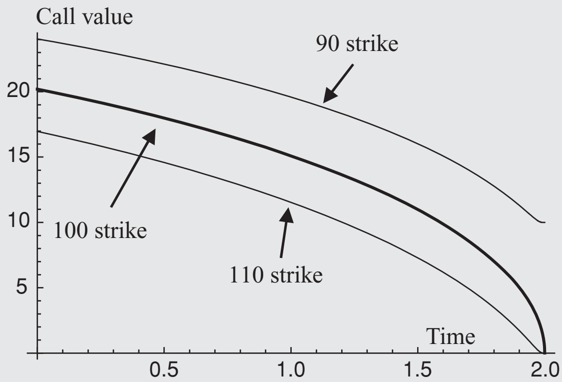
(a) Value of a 1-year call struck at 100 as a function of F:



(b) Value of 1-year and 2-year puts struck at 100 as a function of volatility:



(c) Value of 2-year calls as time passes:



Problem 5: Monte-Carlo pricer

The screen capture below shows our solution for the ‘Ballena Call’. Note that we used the TABLE function in Excel to automatically calculate the price and discounted forward payoff for various forward prices.

	A	B	C	D	E	F	G	H	I
1		**Forward**	110						
2		**Maturity**	2	years					
3		**Risk-free rate**	6.00%						
4		**Volatility**	30%						
5									
6		**Price**	12.15						
7		**Disc. Fwd Payoff**	8.09						
8									
9	**Simulation**	**Uniform**	S_T	**Payoff**		**Forward**	**Price**	**Disc. Fwd Payoff**	
10	1	0.885956609	167.64	40.35		**110**	12.15	8.09	
11	2	96	40	00		**50**	0.39	0.00	
12	3	17	65	39		**60**	1.16	0.00	
13	4	19	84	81		**70**	2.48	0.00	
14	5	91	39	25.59		**80**	4.30	0.00	
15	6	49	59	0.00		**90**	6.60	0.00	
16	7	84	60	0.00		**100**	9.27	0.00	
17	8	73	93	46.50		**110**	12.15	8.09	
18	9	92	74	0.00		**120**	15.13	14.83	
19	10	0.156409396	65.51	0.00		**130**	18.14	20.54	
20	11	0.308008368	81.26	0.00		**140**	21.14	25.43	
21	12	0.161508195	66.10	0.00		**150**	24.06	29.67	
22	13	0.383307603	88.64	0.00		**160**	26.90	33.37	
23	14	0.675298416	121.92	17.98		**170**	29.63	36.65	
24	15	0.317074791	82.15	0.00		**180**	32.23	39.56	
25	16	0.202852292	70.65	0.00		**190**	34.69	42.16	
26	17	0.423689874	92.65	0.00		**200**	37.00	44.50	
27	18	0.269583157	77.47	0.00					
28	19	0.787513608	141.03	29.09					
29	20	0.382469661	88.56	0.00					
5002	4993	0.643433402	117.50	14.90					
5003	4994	0.370767142	87.40	0.00					
5004	4995	0.515425176	102.20	2.15					
5005	4996	0.20197664	70.56	0.00					
5006	4997	0.82391307	149.19	32.97					
5007	4998	0.861587123	159.47	37.29					
5008	4999	0.858459816	158.52	36.92					
5009	5000	0.145181732	64.19	0.00					
5010									

=AVERAGE(D10:D5009)/(1+C3)^C2

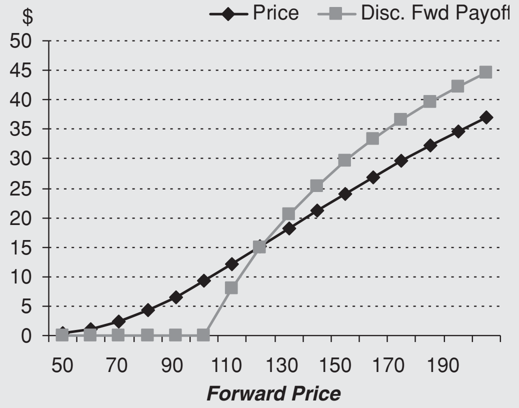
Cells generated with RAND() then **Copy / Paste Special... Values**

=$C$1*EXP($C$4*SQRT($C$2)*NORMSINV(B10)-0.5*$C$4^2*$C$2)

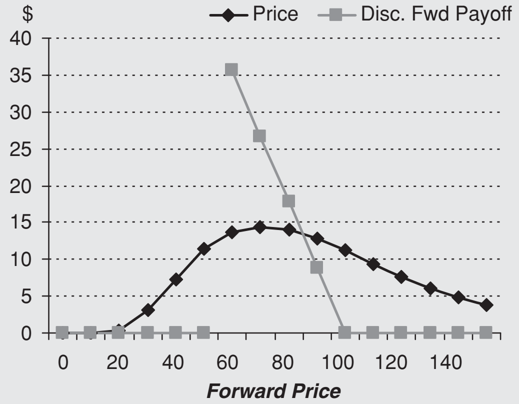
=100*MAX(1-100/C10,0)

Using this spreadsheet we obtained the following graphs:

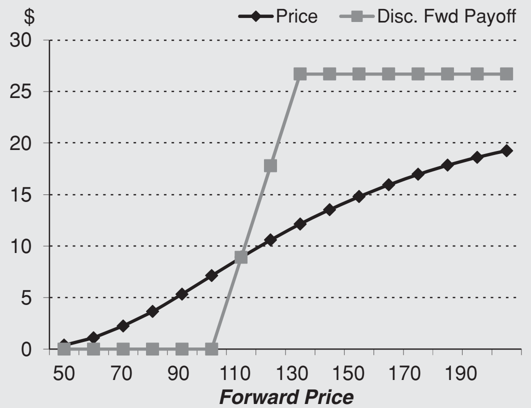
(a) 'Ballena Call':



(b) 'Kick-out put':



(c) 'Call spread':



Problem 6: Negative time value and arbitrage

(a) Value of the 3-year put: The 3-year zero-coupon rate is $z(3) = \left(\frac{100}{80}\right)^{\frac{1}{3}} - 1 = 7.7217\%$. Forward price: $F_0 = 60 \times (1 + 7.7217\%)^3 = \frac{60}{0.8} =$ €75. Coefficients d_1 and d_2:

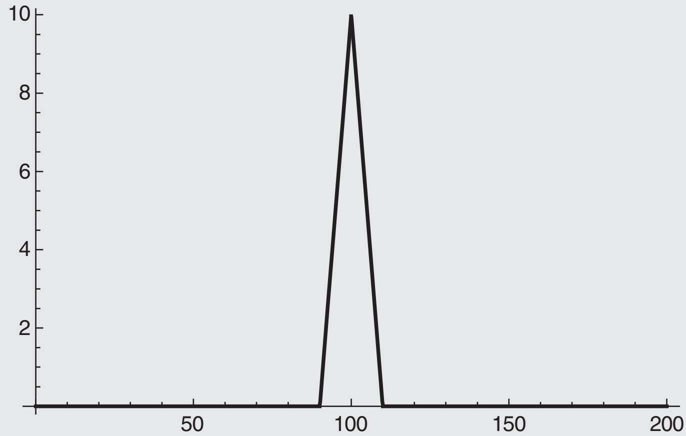
$$d_1 = \frac{\ln \frac{75}{100} + 0.5 \times (0.1^2) \times 3}{0.1 \times \sqrt{3}} \approx -1.5743; \;\; d_2 = -1.5743 - 0.1 \times \sqrt{3} \approx -1.7475.$$

Put value: $p_0 = 0.8 \times [100 \times N(1.7475) - 75 \times N(1.5743)] =$ €20.24.

(b) Intrinsic value of the put: $100 - 60 =$ €40. The put is thus worth about half its intrinsic value, which might seem surprising. However, this is not an arbitrage opportunity: the put is European and cannot be exercised before maturity.

Problem 7: Butterflies and probabilities

(a) Butterfly payoff:



Strategy cost $= 20 + 12 - 2 \times 15 = \$2$. Maximum payoff $= \$10$.

(b) The payoff of a derivative paying \$10 whenever the stock price is in the range and 0 otherwise is always greater than or equal to the butterfly payoff. Its value must thus be at least \$2 under penalty of arbitrage. Furthermore, the fair value can be written $\mathbb{E}\left(10 \times I_{\{90 \leq S_1 \leq 110\}}\right) = 10\mathbb{P}\{90 \leq S_1 \leq 110\}$ where $I_{\{...\}}$ is the indicator variable taking the value 1 when the stock price is in the range and 0 otherwise. Therefore $\mathbb{P}\{90 \leq S_1 \leq 110\} \geq 2/10 = 20\%$.

(c) (*) The payoff of a quantity $1/\varepsilon$ of butterfly with strikes 100-ε, 100 and 100+ε provides a lower bound for the indicator variable that the stock price lies in the range [100-ε, $100 + \varepsilon$]. In the limit as $\varepsilon \to 0$ the strategy's payoff matches exactly the indicator variable that $S_1 = 100$. However, the corresponding probability of $S_1 = 100$ vanishes and we must divide it by ε beforehand to get the density:

$$\begin{aligned} f(100) &= \lim_{\varepsilon \to 0} \frac{\mathbb{P}\{100 - \varepsilon \leq S_1 \leq 100 + \varepsilon\}}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{c_0(100 - \varepsilon) - 2c_0(100) + c_0(100 + \varepsilon)}{\varepsilon^2}. \end{aligned}$$

Recognizing the limit of a second-order finite difference, we conclude: $f(100) = \frac{d^2 c_0}{dK^2}(100)$.

Problem 8: Rule of thumb for the value of an 'at-the-money-forward' call

(a) The coefficients d_1 and d_2 simplify to:

$$d_1 = \frac{\ln \frac{F_0}{F_0} + \frac{\sigma^2}{2} T}{\sigma\sqrt{T}} = \frac{\sigma\sqrt{T}}{2}$$
$$d_2 = d_1 - \sigma\sqrt{T} = -\frac{\sigma\sqrt{T}}{2} = -d_1.$$

The value of the call is then:

$$\begin{aligned} c_0 &= \frac{1}{(1+r)^T}\left[F_0 N(d_1) - F_0 N(d_2)\right] \\ &= \frac{F_0}{(1+r)^T}\left[N(d_1) - N(-d_1)\right] \\ &= \frac{F_0}{(1+r)^T}(2N(d_1) - 1) \end{aligned}$$

because $N(-x) = 1 - N(x)$. Note that if the underlying does not pay any dividend then $\frac{F_0}{(1+r)^T} = S_0$.

(b) A first-order Taylor expansion of $N(x)$ at 0 is:

$$N(x) \underset{x \to 0}{\overset{(1)}{\approx}} N(0) + N'(0)(x-0), \text{ i.e.: } N(x) \underset{x \to 0}{\overset{(1)}{\approx}} \frac{1}{2} + \frac{x}{\sqrt{2\pi}}.$$

Assuming that the coefficient d_1 is close to 0 we may write:

$$\begin{aligned} c_0 &= \frac{F_0}{(1+r)^T}\left[2N(d_1) - 1\right] \\ &\approx \frac{F_0}{(1+r)^T}\left[2\left(\frac{1}{2} + \frac{d_1}{\sqrt{2\pi}}\right) - 1\right] \\ &\approx \frac{2}{\sqrt{2\pi}} \times \frac{F_0 \times \sigma\sqrt{T}}{(1+r)^T} \end{aligned}$$

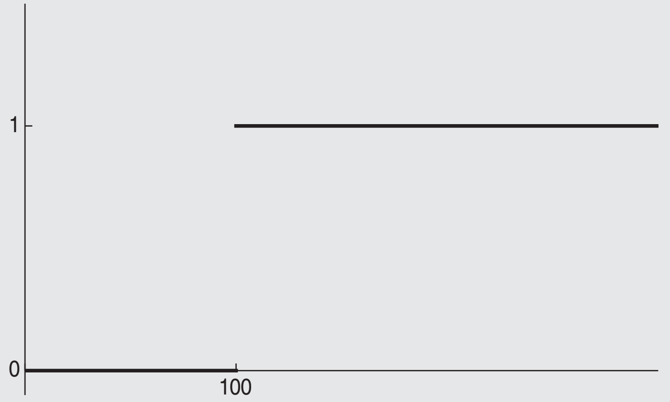
Noting that $\frac{2}{\sqrt{2\pi}} \approx 0.4$ we obtain the required formula. Note that if the underlying does not pay dividends this expression simplifies to $c_0 \approx 40\% \times S_0 \times \sigma\sqrt{T}$.

(c) Rule of thumb: $c_0 \approx 0.4 \times 100 \times 0.4\sqrt{1} = 16$. True value:

$$c_0 = 100 \times \left(2N\left(\frac{1}{2}0.4\sqrt{1}\right) - 1\right) \approx 15.85.$$

Problem 9: Digital option

(a) Digital payoff:



(b) Fair value of the digital option:

$$D_0 = \mathbb{E}\left(\frac{D_T}{(1+r)^T}\right) = \frac{\mathbb{P}\{S_T > K\}}{(1+r)^T} = \frac{1}{(1+r)^T}\mathbb{P}\left\{\frac{\ln(S_T) - \ln(F) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} > -d_2\right\}$$
$$= \frac{1 - N(-d_2)}{(1+r)^T} = \frac{N(d_2)}{(1+r)^T},$$

because S_T is lognormally distributed with parameters ($\ln F - \frac{1}{2}\sigma^2 T$, $\sigma\sqrt{T}$), and $N(-x) = 1 - N(x)$.

(c) $D_0 = \dfrac{N\left(\frac{\ln(110/100) - \frac{1}{2}0.3^2\times 1}{0.3\sqrt{1}}\right)}{(1+5\%)^1} \approx \$0.54.$

Problem 10*: Lognormal distribution

(a) Y is lognormally distributed because $\ln(Y) = X$ is normally distributed.

(b) Using the transfer formula (see Section A-5.6 p.211): $\mathbb{E}(Y) = \mathbb{E}(e^X) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^x e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$. Writing $e^x e^{-\frac{(x-\mu)^2}{2\sigma^2}} = e^{-\frac{(x-\mu)^2 - 2\sigma^2 x}{2\sigma^2}}$ and noting that: $(x-\mu)^2 - 2\sigma^2 x = x^2 - 2(\mu+\sigma^2)x + \mu^2 = (x-\mu-\sigma^2)^2 + \sigma^2(2\mu+\sigma^2)$, we obtain: $\mathbb{E}(Y) = e^{\mu+\frac{1}{2}\sigma^2}\int_{-\infty}^{+\infty}\frac{e^{-\frac{(x-\mu-\sigma^2)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}dx$. Recognizing the integral of the density of a normal distribution with mean $\mu+\sigma^2$ and standard deviation σ, whose value is 1 by definition, we obtain the desired result.

(c) From the definition of Y: $\mathbb{E}(Y^2) = \mathbb{E}(e^{2X})$. Since the variable $X' = 2X$ is normally distributed with mean $\mu' = 2\mu$ and standard deviation $\sigma' = 2\sigma$, we have $\mathbb{E}(Y^2) = e^{2\mu+2\sigma^2}$ and thus:

$$\mathbb{V}(Y) = \mathbb{E}(Y^2) - [\mathbb{E}(Y)]^2 = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right).$$

(d) Let X be a normally distributed random variable with mean $\ln F_0 - \frac{1}{2}\sigma^2 T$ and standard deviation $\sigma\sqrt{T}$. Choosing $S_T = e^X$ we verify the three properties:

1. $S_T = e^X > 0$.
2. From question (b) we get $\mathbb{E}(S_T) = \exp\left(\ln F_0 - \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T\right) = F_0$.

3. It is easy to verify that $S^* = F_0 e^{-\frac{1}{2}\sigma^2 T}$ is the median of S_T. Thus:

$$\mathbb{P}\left\{\frac{S_T}{S^*} \geq x\right\} = \mathbb{P}\left\{\ln\left(\frac{S_T}{S^*}\right) \geq \ln x\right\}$$
$$= \mathbb{P}\left\{X - \ln(F_0) + \frac{1}{2}\sigma^2 T \geq \ln x\right\}$$
$$= \mathbb{P}\left\{X^* \geq \frac{\ln x}{\sigma\sqrt{T}}\right\}$$

where $X^* = \frac{X - \ln(F_0) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$ has a standard normal distribution. Finally:

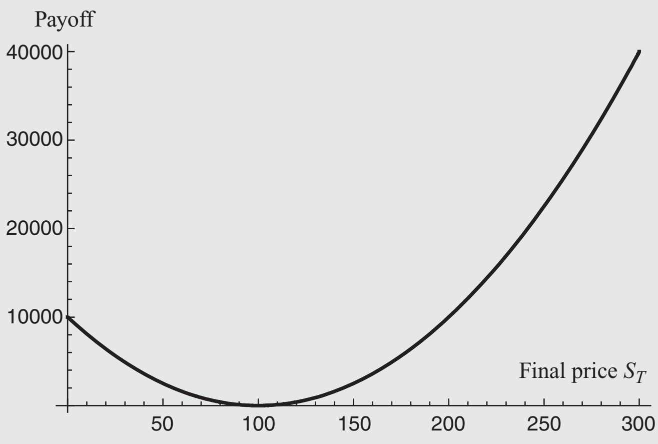
$$\mathbb{P}\left\{\frac{S_T}{S^*} \geq x\right\} = \mathbb{P}\left\{X^* \geq \frac{\ln x}{\sigma\sqrt{T}}\right\} = \mathbb{P}\left\{X^* \leq -\frac{\ln x}{\sigma\sqrt{T}}\right\} = \mathbb{P}\left\{X^* \leq \frac{\ln \frac{1}{x}}{\sigma\sqrt{T}}\right\}$$
$$= \mathbb{P}\left(\left\{\frac{S_T}{S^*} \leq \frac{1}{x}\right\}\right),$$

because $1 - N(x) = N(-x)$.

(e) Write $(\Delta S)^2 = (S_T - F_0 + F_0 - S_0)^2 = (S_T - F_0)^2 + (F_0 - S_0)^2 + 2(S_T - F_0)(F_0 - S_0)$. Thus $\mathbb{E}\left((\Delta S)^2\right) = \mathbb{V}(S_T) + (F_0 - S_0)^2$ since $\mathbb{E}\,(S_T) = F_0$. From question (c) we have $\mathbb{V}(S_T) = \exp\left(2\ln F_0 - \sigma^2 T + \sigma^2 T\right)\left(\exp(\sigma^2 T) - 1\right) = F_0^2\left(e^{\sigma^2 T} - 1\right)$. But $F_0 \to S_0$ and $e^{\sigma^2 T} - 1 \sim \sigma^2 T$ as $T \to 0$, which proves the desired result.

Problem 11: Quadratic option

(a) Quadratic option payoff:



(b) Expanding the square: $D_T = S_T^2 - 2KS_T + K^2$. Discounting both sides of this equation and then taking expectations we find that the fair value of the quadratic option at $t = 0$ may be written:

$$D_0 = \frac{\mathbb{E}(D_T)}{(1+r)^T} = \frac{1}{(1+r)^T}\left[\mathbb{E}(S_T^2) + 2K\,\mathbb{E}(S_T) + K^2\right].$$

From property (2) in the lognormal model we get $\mathbb{E}\left(S_T\right) = F_0$. From Problem 10 we get:

$$\mathbb{E}(S_T^2) = \exp\left(2\ln F_0 - \sigma^2 T + 2\sigma^2 T\right) = F_0^2 e^{\sigma^2 T}.$$

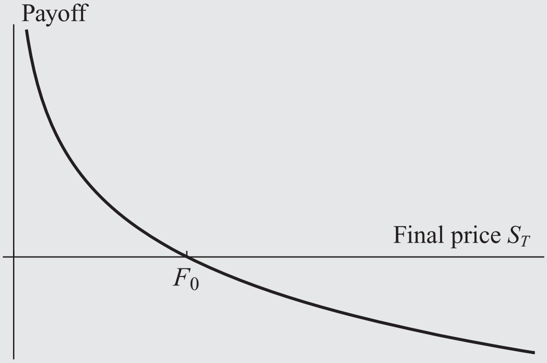
It follows that $D_0 = \frac{1}{(1+r)^T}\left[F_0^2 e^{\sigma^2 T} - 2K F_0 + K^2\right]$.

(c) Value of the 1-year quadratic option: Forward price: $F_0 = 105 \times (1+10\%) = £115.50$. Option value:

$$D_0 = \frac{1}{1+10\%}\left[115.5^2 \times e^{0.25^2} - 2 \times 100 \times 115.5 + 100^2\right] \approx £1000.$$

Problem 12: Log-contract

(a) Log-contract payoff:



(b) In the lognormal model $\ln S_T \sim \mathcal{N}(\ln F_0 - \frac{1}{2}\sigma^2 T, \sigma\sqrt{T})$. The fair value of the log-contract is thus:

$$D_0 = \frac{1}{(1+r)^T}\mathbb{E}\left[-\ln\frac{S_T}{F_0}\right] = \frac{1}{(1+r)^T} \times \frac{1}{2}\sigma^2 T.$$

(c) (*) To show that the portfolio value P_0 must be equal to the fair value D_0 of the log-contract we first look at the portfolio payoff:

$$P_T = \int_0^{F_0} \frac{1}{K^2} p_T(K)dK + \int_{F_0}^{\infty} \frac{1}{K^2} c_T(K)dK = \int_0^{F_0} \frac{1}{K^2}\max(0, K - S_T)\,dK$$
$$+ \int_{F_0}^{\infty} \frac{1}{K^2}\max(0, S_T - K)\,dK.$$

If $S_T < F_0$ then the second integral vanishes and integrating by parts leads to:

$$\int_{S_T}^{F_0} \frac{1}{K^2}(K - S_T)dK = [\ln K]_{S_T}^{F_0} + S_T\left[\frac{1}{K}\right]_{S_T}^{F_0} = -\ln\frac{S_T}{F_0} + \left(\frac{S_T}{F_0} - 1\right).$$

Similarly if $S_T > F_0$ then the first integral vanishes and the expression reduces to:

$$\int_{F_0}^{S_T} \frac{1}{K^2}(S_T - K)dK = -S_T\left[\frac{1}{K}\right]_{F_0}^{S_T} - [\ln K]_{F_0}^{S_T} = -\ln\frac{S_T}{F_0} + \left(\frac{S_T}{F_0} - 1\right).$$

In either case $P_T = D_T + \left(\frac{S_T}{F_0} - 1\right)$. Therefore, the portfolio of puts and calls has fair value $P_0 = \frac{\mathbb{E}(D_T)}{(1+r)^T} + \frac{1}{(1+r)^T}\left(\frac{\mathbb{E}(S_T)}{F_0} - 1\right)$ whose first term is D_0 and second term vanishes because of forward-neutrality (property (2) in Section 7-1.1 p.75).

Problem 13*: Simulating a normal distribution

We want to show that the cumulative distribution function of Y is the same as that of the standard normal distribution. Let $U(z) = z$ denote the cumulative distribution function of the uniform distribution over the interval [0, 1]. For any real number x we have:

$$\begin{aligned}\mathbb{P}\{Y \le x\} &= \mathbb{P}\left\{N^{-1}(X) \le x\right\} = \mathbb{P}\left\{N\left(N^{-1}(X)\right) \le N(x)\right\} = \mathbb{P}\{X \le N(x)\}\\ &= U\left(N(x)\right) = N(x).\end{aligned}$$

Problem 14*: Derivation of the closed-form formulas for European options

(a) Because $D_T = (S_T - K)I$ and I is the payoff of a digital option (see Problem 9) we have $\mathbb{E}(D_T) = \mathbb{E}(S_T I) - K\mathbb{E}(I) = \mathbb{E}(S_T I) - KN(d_2)$.

(b) Taking the log of both sides of $Y = \exp\left(\mu + \sigma X\sqrt{T}\right)$ we get a normal distribution with mean μ and standard deviation $\sigma\sqrt{T}$. Hence Y has the same lognormal distribution as S_T.

(c) Taking the log of both sides of $Y > K$ and rearranging terms we get the equivalent inequality:

$$\sigma X\sqrt{T} > \ln K - \mu = \ln(K/F_0) + \frac{1}{2}\sigma^2 T.$$

Dividing both sides by $\sigma\sqrt{T}$ we equivalently get $X > -d_2$.

(d) Using the results of questions (b) and (c) we may write:

$$\mathbb{E}(S_T I) = \mathbb{E}\left(\exp(\mu + \sigma X\sqrt{T})J\right) = \int_{-d_2}^{+\infty} \exp\left(\mu + \sigma x\sqrt{T}\right) n(x)dx,$$

where J is the indicator variable that $X > -d_2$ and $n(\cdot)$ is the density of the standard normal distribution. But $\exp\left(\mu + \sigma x\sqrt{T}\right) n(x) = \exp\left(\mu - x^2/2 + \sigma x\sqrt{T}\right)/\sqrt{2\pi}$, and $-x^2/2 + \sigma x\sqrt{T} = -\frac{1}{2}\left(x - \sigma\sqrt{T}\right)^2 + \frac{1}{2}\sigma^2 T$; thus: $\mathbb{E}(S_T I) = \frac{e^{\mu+\sigma^2 T/2}}{\sqrt{2\pi}} \int_{-d_2}^{+\infty} \exp\left(-\frac{1}{2}\left(x - \sigma\sqrt{T}\right)^2\right)dx$. Recognizing that $e^{\mu+\sigma^2 T/2} = F_0$ and applying the affine change of variable $y = \sigma\sqrt{T} - x$ we obtain:

$$\mathbb{E}(S_T I) = \frac{F_0}{\sqrt{2\pi}} \int_{-\infty}^{d_2+\sigma\sqrt{T}} \exp\left(-\frac{1}{2}y^2\right)dy = F_0 N(d_1)$$

(e) Combining the results of questions (a) and (d) we get $\mathbb{E}(D_T) = F_0 N(d_1) - KN(d_2)$; dividing both sides by $(1+r)^T$ yields the closed-form formula for c_0. The closed-form formula for p_0 is a straightforward consequence of put-call parity: $p_0 = c_0 + (F_0 - K)/(1+r)^T$.

Problem 15*: Breeden-Litzenberger formula

(a) Noting that $(1+r)^T c_0(K) = \mathbb{E}\left[\max(0, S_T - K)\right] = \int_K^{\infty}(x-K)f(x)dx = \int_K^{+\infty} xf(x)dx - K\int_K^{+\infty} f(x)dx$ and differentiating with respect to K we get: $(1+r)^T \frac{dc_0}{dK}(K) = -Kf(K) - \int_K^{+\infty} f(x)dx + Kf(K) = -\int_K^{+\infty} f(x)dx$. Differentiating once more we obtain the desired result.

(b) Writing $D_0 = \frac{\mathbb{E}(g(S_T))}{(1+r)^T} = \int_0^{+\infty} g(x)\frac{f(x)}{(1+r)^T}dx$ and substituting the result of question (a), we get $D_0 = \int_0^{+\infty} g(x)\frac{d^2c_0}{dK^2}(x)dx$. Integrating by parts yields $D_0 = \left[g(x)\frac{dc_0}{dK}(x)\right]_0^{+\infty} - \int_0^{+\infty} g'(x)\frac{dc_0}{dK}(x)dx$, and the bracket vanishes because infinite-strike calls are worthless and $g(0) = 0$. Integrating by parts once more we get the desired result. Intuitively this result means that one may perfectly replicate any option payoff g with a continuum of calls struck at K in quantities $g''(K)dK$.

Chapter 8

Problem 1: True or False?

False: Delta-hedging only hedges against small underlying price moves. In case of large moves the delta-hedger is still exposed to some risk measured by the gamma. In practice, the hedger is also exposed to changes in other parameters such as the volatility σ or the short-term interest rate r.

Problem 2: Greeks

(a) At \$80 stock price, a 1% decrease is worth \$0.80. Since the delta is 1,500,000, the portfolio's loss will be roughly $\delta \times \Delta S = 1{,}500{,}000 \times (-0.8) = -\$1{,}200{,}000$. In case of a 10% drop worth \$8 the first-order approximation in delta is inaccurate and we must take gamma into account:

$$\begin{aligned}\text{P\&L} &\approx \delta \times \Delta S + \frac{1}{2}\Gamma \times (\Delta S)^2\\ &\approx 1{,}500{,}000 \times (-8) - \frac{1}{2} \times 10{,}000 \times (-8)^2\\ &\approx -\$12{,}320{,}000\end{aligned}$$

(b) If nothing happens on the market, the P&L will be equal to the daily theta i.e. +\$14,222.

(c) Interest rate risk is measured by the rho of \$300,000 per point (100 basis points). Therefore, if the interest rate increases by a quarter point the P&L is +\$75,000.

(d) A high trading volume may indicate that there is a transfer of risk on the market. Usually, an increase in trading volume is associated with large price movements but in this example the price remains stable. However, a likely outcome is that the volatility parameter will increase, which is unfavorable since the vega is negative (−\$500,000 per volatility point).

Problem 3: Delta-hedge, break-even levels

(a) A delta of −0.3 means that the value of the put goes down €0.3 when the stock price of Plumet SA goes up €1. Here we are *short* 1 million puts, which means that we *gain* €300,000 when the stock price goes up €1. Hence, to hedge the delta we need to sell 300,000 stocks of Plumet SA.

(b) A theta of €−0.002301 per day on a short position of 1 million puts means that we gain €2,301 per day if nothing happens (in particular the underlying stock price must remain constant). Having delta-hedged the position, only the gamma will have an impact on our P&L as the stock price of Plumet SA drops by €0.50:

$$\begin{aligned}\text{Gamma P\&L} &= \frac{1}{2}\Gamma(\Delta S)^2 \\ &= -1{,}000{,}000 \times \frac{1}{2} \times 0.0671831 \times (0.50)^2 \\ &\approx -€8{,}398\end{aligned}$$

Hence, our total P&L at the end of the day will be a loss of (8,398 − 2,301) = €6,097. In case the price drops by €5 (10 times more), the loss in gamma will be 100 times higher due to the quadratic nature of the gamma P&L, and our total P&L will be (839,800 − 2,301) = €837,500.

(c) The P&L breaks even when the loss in gamma is exactly offset by the profit in theta. This will happen when the change in underlying stock price satisfies $\frac{1}{2}\,\Gamma \times (\Delta S)^2 = -\Theta_{\text{daily}}$, i.e. $\frac{1}{2} \times 0.0671831 \times (\Delta S)^2 = 0.002301$. This equation has two solutions with opposite signs:

$$\Delta S = \pm\sqrt{\frac{2 \times 0.002301}{0.0671831}} = \pm 0.2617€.$$

In other words the P&L breaks even when the stock price of Plumet SA either goes down to (20 − 0.2617) = €19.7383 or up to (20 + 0.2617) = €20.2617 after one day.

Problem 4: Delta-hedging P&L

Month	Range Ltd Price (£)	Call Value (£)	Delta (per £)	"Dollar" Gamma	Stock position*	Monthly P&L** (£)	Cumulative P&L (£)
0	100	11.84	0.61	76.61	−6,100	−118,400†	−118,400
1	90	6.04	0.43	73.96	−4,300	3,000	−115,400
2	105	13.89	0.68	82.02	−6,800	14,000	−101,400
3	90	5.02	0.41	80.57	−4,100	13,300	−88,100
4	85	2.8	0.29	70.87	−2,900	−1,700	−89,800
5	95	6.06	0.48	99.12	−4,800	3,600	−86,200
6	100	8.01	0.58	110.57	−5,800	−4,500	−90,700
7	110	14.07	0.78	100.73	−7,800	2,600	−88,100
8	115	17.46	0.87	83.66	−8,700	−5,100	−93,200
9	125	26.16	0.97	31.12	−9,700	0	−93,200
10	120	20.8	0.97	38.37	−9,700	−5,100	−98,300
11	115	15.4	0.98	41.42	−9,800	−5,500	−103,800
12	110	10	1	0		95,000‡	−8,800

*Stock position: Number of shares at the end of the month.
**Monthly P&L: 10,000 times the change in call value, plus the previous number of shares times the change in stock price.
†Purchase cost.
‡Includes the call payoff of £100,000.

Problem 5: Greeks of the underlying asset and the forward contract

The Greeks of the underlying asset and the forward contract are obtained by calculating the first or second-order derivative of their price with respect to the appropriate variable:

Asset price	**Δ**	**Θ**	**Γ**	$\mathcal{V}$	ρ
a) $f(t, S_t) = S_t$	1	0	0	0	0
b) $f(t, S_t) = S_t - \dfrac{K}{(1+r)^{T-t}}$	1	$-\dfrac{K\ln(1+r)}{(1+r)^{T-t}}$	0	0	$\dfrac{K(T-t)}{(1+r)^{T-t+1}}$

When there is a single cash dividend D at time t_D the price of the forward contract becomes $f(t, S_t) = S_t - \frac{D}{(1+r)^{t_D-t}} - \frac{K}{(1+r)^{T-t}}$ (see Section 5-2.4.1 p.54); thus $\Theta = -\frac{D\ln(1+r)}{(1+r)^{t_D-t}} - \frac{K\ln(1+r)}{(1+r)^{T-t}}$ and $\rho = \frac{D(t_D-t)}{(1+r)^{t_D-t+1}} + \frac{K(T-t)}{(1+r)^{T-t+1}}$. When there is a single proportional dividend at rate d the price of the forward contract becomes $f(t, S_t) = \frac{S_t}{1+d} - \frac{K}{(1+r)^{T-t}}$ and the delta becomes $1/(1+d)$ while the other Greeks are unchanged.

Problem 6: Hedging multiple Greeks

(a) Let x be the quantity of Option 2 and y the quantity of S. The delta and gamma of the portfolio are then:

$$\begin{cases} \delta = -0.57 + 0.33x + y \\ \Gamma = -0.013076 + 0.012153\,x \end{cases}$$

The condition $\Gamma = 0$ yields $x = \frac{0.013076}{0.012153} \approx 1.076$, and the condition $\delta = 0$ yields $y = 0.57 - 0.33 \times 1.076 \approx 0.215$. Hence, to construct a delta and gamma-neutral portfolio, we need to sell 1 unit of Option 1, buy 1.076 units of Option 2 and 0.215 units of the underlying asset.

(b) Using Equation (8-1) p.88 we can estimate the portfolio theta to be:

$$\Theta \approx -\frac{1}{2} \times 100^2 \times \sigma^2 \times (-0.013076 + x \times 0.012153) = 0.$$

(c) The portfolio would not be delta-neutral if the underlying price went up \$10. Because gamma is the derivative of delta with respect to the underlying spot price, the new option deltas are approximately $0.57 + 0.013076 \times 10 = 0.70076$ and $0.33 + 0.013153 \times 10 = 0.46153$ respectively, and the new portfolio delta is $-0.70076 + x \times 0.46153 + 0.215 \approx 0.011$. Here, we cannot calculate the new portfolio gamma but with more data we would find a similar result.

Problem 7*: Delta of a European call or put

(a) $N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is the density of the standard normal distribution. Hence:

$$\begin{aligned} F_0 N'(d_1) &= \frac{F_0}{\sqrt{2\pi}} \exp\left[-\frac{\left(\ln(F_0/K) + \frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}\right] \\ &= \frac{F_0}{\sqrt{2\pi}} \exp\left[-\frac{\left(\ln(F_0/K) - \frac{1}{2}\sigma^2 T\right)^2 + 2\ln(F_0/K)\sigma^2 T}{2\sigma^2 T}\right] \end{aligned}$$

$$= \frac{F_0}{\sqrt{2\pi}} \frac{K}{F_0} \exp\left[-\frac{\left(\ln(F_0/K) - \frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}\right]$$
$$= KN'(d_2)$$

(b) Assuming no dividends, the forward price is $F = S(1+r)^T$ where S is the current spot price, and Equation (7-1) p.76 becomes $c = SN(d_1) - \frac{K}{(1+r)^T} N(d_2)$. Differentiating with respect to S yields:

$$\delta_{call} = N(d_1) + S\frac{\partial d_1}{\partial S} N'(d_1) - \frac{K}{(1+r)^T} \frac{\partial d_2}{\partial S} N'(d_2).$$

Since $d_2 = d_1 - \sigma\sqrt{T}$ we have $\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$ and the above equation simplifies to:

$$\delta_{call} = N(d_1) + \frac{\partial d_1}{\partial S}\left[SN'(d_1) - \frac{K}{(1+r)^T} N'(d_2)\right] = N(d_1),$$

because the expression between brackets vanishes as a consequence of question (a).

(c) In the absence of dividends put-call parity ensures that $\delta_{call} - \delta_{put} = 1$, and from $N(x) = 1 - N(-x)$ we obtain $\delta_{put} = -N(-d_1)$.

Problem 8: Call and put have same gamma and vega

When there are no dividends, taking the second-order derivative of each side of the put-call parity equation (Equation (5-5) p.58) with respect to the spot price S yields that European puts and calls must have the same gamma. Assuming a proportional or cash dividend gives the same result. Similarly puts and calls must have the same vega.

Problem 9: Log-contract

(a) The fair value of the log-contract is $D_t = -\frac{\mathbb{E}[\ln(S_T/F_0)]}{(1+r)^{T-t}}$. Since S_T is lognormally distributed with parameters $\left(\ln F_t - \frac{1}{2}\sigma^2(T-t),\ \sigma\sqrt{T-t}\right)$ we have $D_t = -\frac{\ln(F_t/F_0) - \frac{1}{2}\sigma^2(T-t)}{(1+r)^{T-t}}$. In the absence of dividends we have $F_0 = S_0(1+r)^T$ and $F_t = S_t(1+r)^{T-t}$ which yields the desired result after substitution.

(b) Differentiating twice with respect to S_t we obtain $\Gamma = \frac{\partial^2 D_t}{\partial S_t^2} = \frac{1}{S_t^2(1+r)^{T-t}}$, and the dollar gamma is thus $\frac{1}{2} \times \frac{1}{(1+r)^{T-t}}$.

(c) When interest rates are zero the dollar gamma of 2 log-contracts is 1. Substituting into Equation (8-2) p.89 yields $\text{P\&L}_{\Delta t} \approx \left(\frac{\Delta S}{S}\right)^2 - \sigma^2 \Delta t$. Summing over 252 trading days then gives the desired result.

Problem 10: Volga

(a) If volatility goes down 1 point the P&L would be about −10,000. If volatility is up 5 points the P&L would be about $-10{,}000 \times 5^2 = -250{,}000$. Our volga position is unfavorable because we lose money whenever the volatility parameter changes.

(b) At the 95% confidence level the maximum loss would be $-10{,}000 \times 3^2 = -90{,}000$. We would lose more 5% of the time.

Chapter 9

Problem 1: Fractional and continuous interest rates

From \ To	Annual gross rate	Fractional every 6 months	Fractional every month	Continuously compounded
Annual gross rate	10%	9.76%	9.57%	9.53%
Fractional every 6 months	$(1+10\%/2)^2-1$ $=10.25\%$	10%	9.80%	9.76%
Fractional every month	$(1+10\%/12)^{12}-1$ $\approx 10.47\%$	10.21%	10%	9.96%
Continuously compounded	$e^{10\%}-1\approx 10.52\%$	10.25%	10.04%	10%

Problem 2: Continuous compounding

(a) Interest is paid annually: $K_t = K(1+R)^t$, $t = 1, 2, \ldots$

(b) Interest is split into m equal payments: $K_t = K\left(1+\frac{R}{m}\right)^{mt}$, $t = \frac{1}{m}, \frac{2}{m}, \ldots$

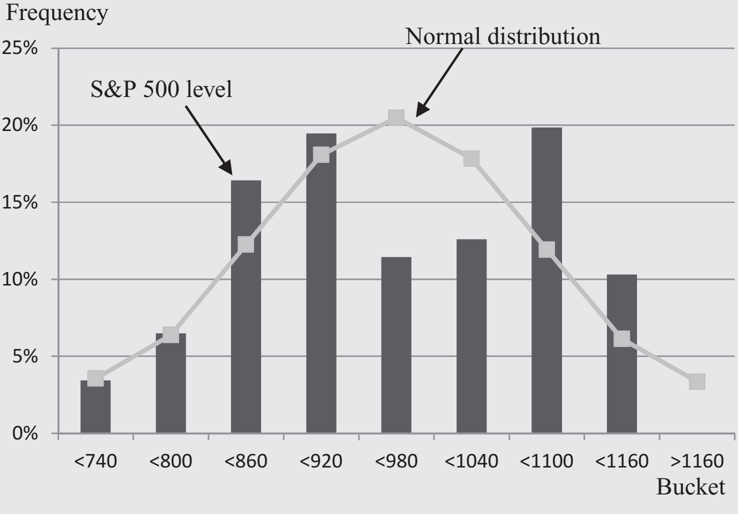
(c) Interest is paid continuously: $K_t = Ke^{Rt}$, $t \geq 0$. Over the period $[t, t+dt]$, the return on $(K_t)_{t\geq 0}$ is $\frac{K_{t+dt}-K_t}{K_t}$. Since interest is paid continuously we can write $\frac{K_{t+dt}-K_t}{K_t} = R \times dt$, which yields the required differential equation. Thus: $\frac{dK_t}{dt} = RK_t$, whose solutions are of the form $K_t = c + K_0e^{Rt}$. Since $K_0 = K$ we obtain $c = 0$, which verifies the previous result.

Problem 3: Some econometrics

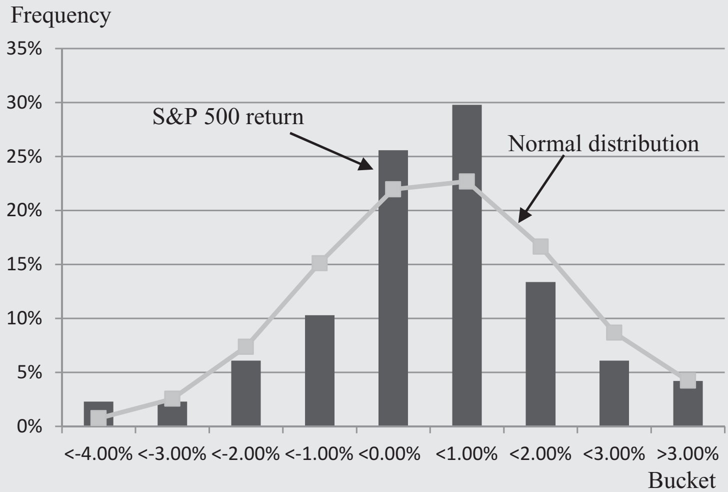
We selected the S&P 500 index published by Standard & Poor's for the year 2009.

(a) Mean price: 948.46. Standard deviation: 115.54.

(b) The empirical distribution of the S&P 500 index for the year 2009 does not quite fit a normal distribution:

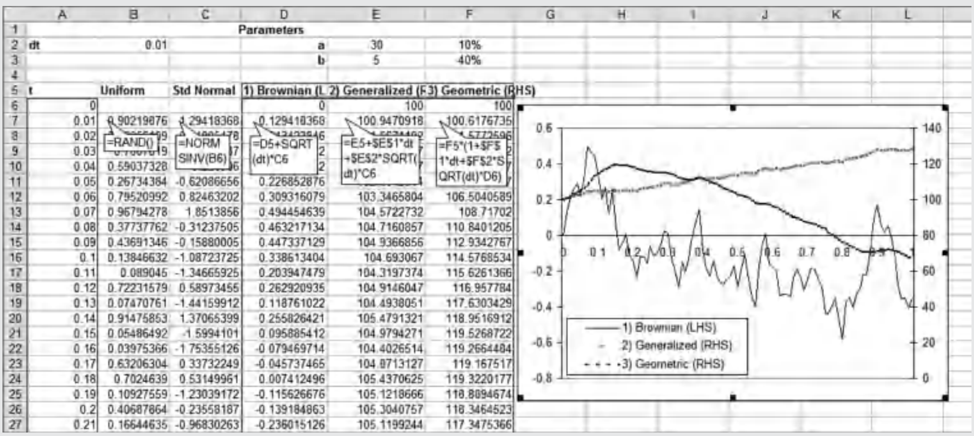


(c) The series of daily returns of the S&P 500 index has a mean of 0.009% and a standard deviation of 1.69%. The empirical distribution is a lot more similar to a normal distribution with higher frequency around the mean.



Problem 4: Simulating Brownian motions

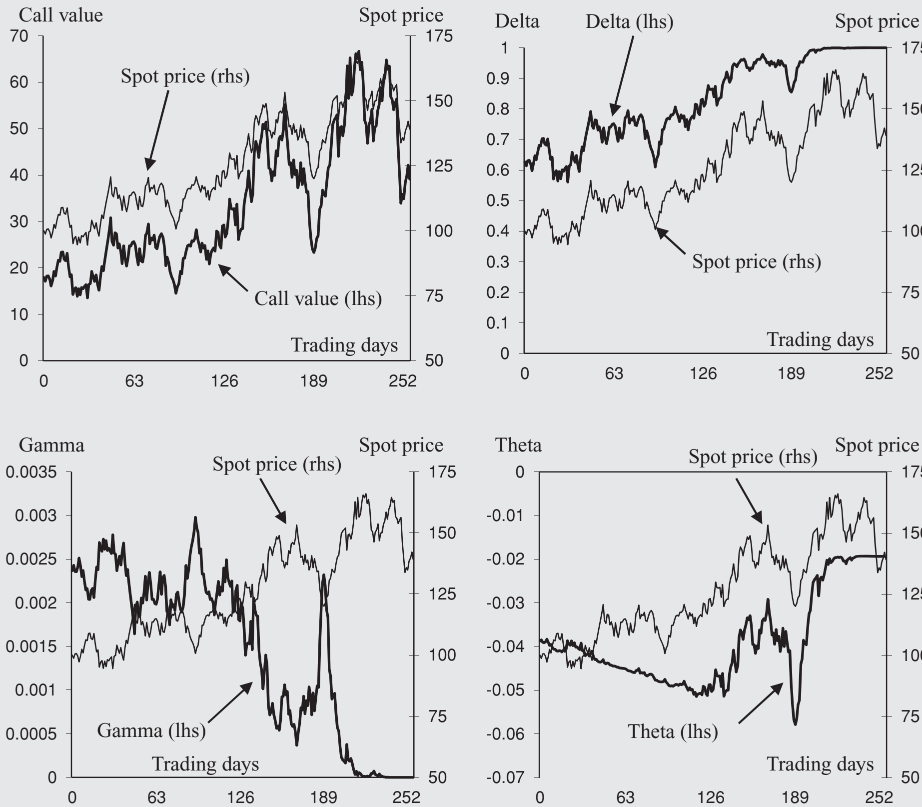
The screen capture below shows how to obtain the desired results:



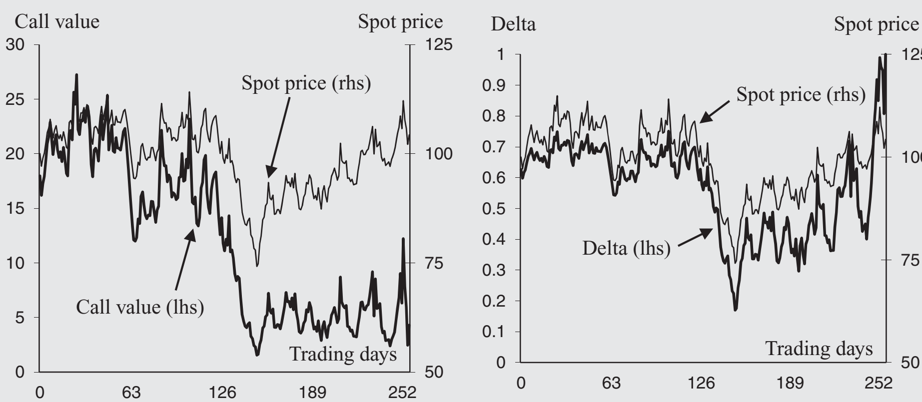
Problem 5: Simulating market prices and option values

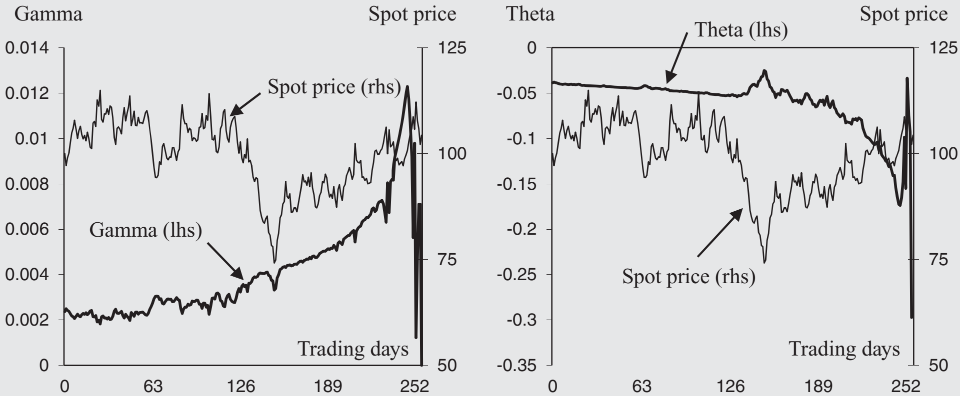
Using the methodology of Problem 4 to simulate the geometric Brownian motion, we obtained the following results:

(a) Scenario 1: 'In-the-money'. In this simulation, the underlying asset grew from its initial price of 100 to 140 after a year. Accordingly, the call value converged to a payoff of 40, following a similar pattern as the spot price. The delta accordingly converged to 1, also following a similar pattern. The gamma, however, followed a reverse pattern: it initially oscillated and then dropped to a low level as the option went well in-the-money. The theta followed a somewhat similar pattern as the underlying spot price.



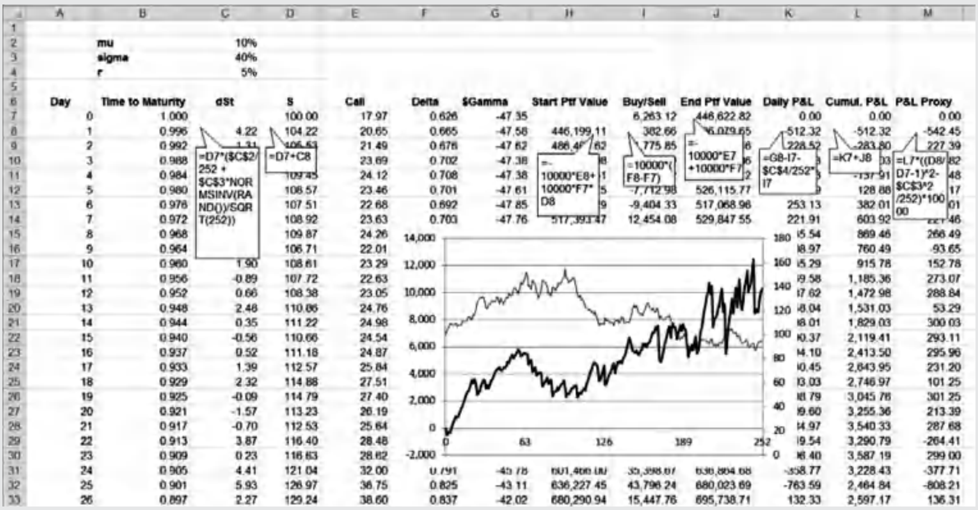
(b) Scenario 2: ‘Pinned at the strike.’ In this simulation, the underlying asset oscillated around the strike throughout the year and finished slightly above 100. The call value followed a similar pattern and converged to a payoff close to 0. The delta oscillated around its initial value of 0.6 with wider swings through time. Accordingly, the gamma and thus theta became larger as time passed. This scenario is the option trader’s nightmare: because the underlying is ‘pinned at the strike’ there is ongoing uncertainty whether the option will end up in- or out-of-the-money and the risk parameters become wild.





Problem 6: Delta-hedging simulation

The screen capture below displays our results. In this case the cumulative P&L is positive, but in other simulations it may be negative. The P&L proxy is quite accurate and produces a cumulative P&L which is only −1.7% away from the true figure.



mu 10%
sigma 40%
r 5%

Day	Time to Maturity	dSt	S	Call	Delta	$Gamma	Start Ptf Value	Buy/Sell	End Ptf Value	Daily P&L	Cumul. P&L	P&L Proxy
0	1.000		100.00	17.97	0.626	-47.35		6,263.12	446,622.82	0.00	0.00	0.00
1	0.996	4.22	104.22	20.65	0.665	-47.58	446,199.11	382.66	[illegible]	-512.32	-512.32	-542.45
2	0.992	[illegible]	[illegible]	21.49	0.676	-47.62	[illegible]	[illegible]	[illegible]	[illegible]	-283.80	227.39
3	0.988	[illegible]	[illegible]	23.69	0.702	-47.38	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4	0.984	[illegible]	109.45	24.12	0.708	-47.38	[illegible]	[illegible]	[illegible]	[illegible]	-137.91	[illegible]
5	0.980	[illegible]	108.57	23.46	0.701	-47.61	[illegible]	-7,712.98	526,115.77	[illegible]	128.88	[illegible]
6	0.976	[illegible]	107.51	22.68	0.692	-47.85	[illegible]	-9,404.33	517,068.98	253.13	382.01	[illegible]
7	0.972	[illegible]	108.92	23.63	0.703	-47.76	517,393.47	12,454.08	529,847.55	221.91	603.92	[illegible]
8	0.968	[illegible]	109.87	24.26	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	869.46	266.49
9	0.964	[illegible]	106.71	22.01	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	760.49	-93.65
10	0.960	1.90	108.61	23.29	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	915.78	152.78
11	0.956	-0.89	107.72	22.63	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1,185.36	273.07
12	0.952	0.66	108.38	23.05	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1,472.98	288.84
13	0.948	2.48	110.86	24.76	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1,531.03	53.29
14	0.944	0.35	111.22	24.98	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1,829.03	300.03
15	0.940	-0.56	110.66	24.54	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2,119.41	293.11
16	0.937	0.52	111.18	24.87	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2,413.50	295.96
17	0.933	1.39	112.57	25.84	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2,643.95	231.20
18	0.929	2.32	114.88	27.51	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2,746.97	101.25
19	0.925	-0.09	114.79	27.40	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3,045.76	301.25
20	0.921	-1.57	113.23	26.19	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3,255.36	213.39
21	0.917	-0.70	112.53	25.64	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3,540.33	287.68
22	0.913	3.87	116.40	28.48	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3,290.79	-264.41
23	0.909	0.23	116.63	28.62	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3,587.19	299.00
24	0.905	4.41	121.04	32.00	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	-358.77	3,228.43	-377.71
25	0.901	5.93	126.97	36.75	0.825	-43.11	636,227.45	43,796.24	680,023.69	-763.59	2,464.84	-808.21
26	0.897	2.27	129.24	38.60	0.837	-42.02	680,290.94	15,447.76	695,738.71	132.33	2,597.17	136.31

Problem 7: Applying the Ito-Doeblin theorem

In both questions we apply the Ito-Doeblin theorem (Equation (9-7) p.102) on a function $f(t, X)$ of the Ito process X with drift $a(t, X) = \mu X$ and volatility $b(t, X) = \sigma X$.

(a) Here $f(t, X) = X^n$ and the Ito-Doeblin theorem yields:

$$
\begin{aligned}
dZ_t = df &= \left(\frac{\partial f}{\partial t} + \mu X_t \frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 f}{\partial X^2}\right) dt + \left(\sigma X_t \frac{\partial f}{\partial X}\right) dW_t \\
&= \left(0 + n\mu X_t X_t^{n-1} + \frac{1}{2} n(n-1)\sigma^2 X_t^2 X_t^{n-2}\right) dt + \left(n\sigma X_t X_t^{n-1}\right) dW_t
\end{aligned}
$$

$$= \left(n\mu + \frac{1}{2}n(n-1)\sigma^2\right) X_t^n dt + n\sigma X_t^n dW_t$$
$$= \left(n\mu + \frac{1}{2}n(n-1)\sigma^2\right) Z_t dt + n\sigma Z_t dW_t$$

Hence $Z_t = X_t^n$ follows a geometric Brownian motion with parameters $(n\mu + n(n-1)\sigma^2/2, n\sigma)$.

(b) Here $f(t, X) = 1/X$ and the Ito-Doeblin theorem yields:

$$\begin{aligned} dH_t = df &= \left(\frac{\partial f}{\partial t} + \mu X_t \frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 f}{\partial X^2}\right) dt - \left(\sigma X_t \frac{\partial f}{\partial X}\right) dW_t \\ &= \left(0 - \mu X_t \frac{1}{X_t^2} + \frac{1}{2}\sigma^2 X_t^2 \frac{2}{X_t^3}\right) dt - \left(\sigma X_t \frac{1}{X_t^2}\right) dW_t \\ &= \left(-\mu + \sigma^2\right) \frac{1}{X_t} dt - \sigma \frac{1}{X_t} dW_t \\ &= \left(-\mu + \sigma^2\right) H_t - \sigma H_t dW_t \end{aligned}$$

Hence H follows a geometric Brownian motion with parameters $(-\mu + \sigma^2, -\sigma)$. Because $(-W)$ is a standard Brownian motion, this is equivalent to say that H follows a geometric Brownian motion with parameters $(-\mu + \sigma^2, \sigma)$.

Problem 8: Standard Brownian motion

(a) From properties (1) and (2) in Section 9-3.1 p.98 we know that $W_T - W_0 = W_T$ follows a normal distribution with zero mean and standard deviation $\sqrt{T}$.

(b) $\mathrm{Cov}(W_t, W_{t'}) = \mathrm{Cov}(W_t, W_{t'} - W_t + W_t) = \mathrm{Cov}(W_t - W_0, W_{t'} - W_t) + \mathbb{V}(W_t) = 0 + t$ (recall that the two increment variables $D = W_t - W_0$ and $\Delta = W_{t'} - W_t$ are independent, which implies that their covariance is zero). The covariance between W_t and $W_{t'}$ is thus non-zero and these two variables are *not* independent.

(c) Using the transfer formula:

$$\begin{aligned} h_t = \mathbb{E}\left(e^{\sigma W_t}\right) = \mathbb{E}\left(e^{\sigma\sqrt{t}\tilde{\varepsilon}}\right) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\sigma\sqrt{t}x} e^{-\frac{x^2}{2}} dx \\ &= e^{\frac{\sigma^2 t}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\sigma\sqrt{t})^2}{2}} dx = e^{\frac{\sigma^2 t}{2}}, \end{aligned}$$

because $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\sigma\sqrt{t})^2}{2}} dx = 1$ as the full integral of the density of a normal distribution. Hence $G_t = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$. Choosing $f(t, X) = e^{\sigma X - \frac{1}{2}\sigma^2 t}$, $X_t = W_t$, $dX_t = dW_t$, $a = 0$, $b = 1$ and applying the Ito-Doeblin theorem we obtain:

$$\begin{aligned} dG_t = df &= \left(\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial X} + \frac{1}{2}b^2 \frac{\partial^2 f}{\partial X^2}\right) dt + \left(b\frac{\partial f}{\partial X}\right) dW_t \\ &= \left(-\frac{1}{2}\sigma^2 G_t + 0 + \frac{1}{2}\sigma^2 G_t\right) dt + (\sigma G_t) dW_t \\ &= \sigma G_t dW_t \end{aligned}$$

Therefore G follows a geometric Brownian motion with parameters $(0, \sigma)$.

Problem 9: Continuity and non-differentiability of Brownian motion

(a) By property (2) of Section 9-3.1 p.98 D_t follows a normal distribution with zero mean and standard deviation $\sqrt{h}$. Thus, δ_t follows a normal distribution with zero mean and standard deviation $1/\sqrt{h}$.

(b) $\mathbb{V}(D_t) = h \to 0$. A random variable with zero variance is a constant, and since D_t has zero mean this result means that D_t converges to zero as the time interval h goes to 0. In other words two points of a Brownian motion taken at infinitesimally close times are also infinitesimally close: the path of a Brownian motion is thus continuous. (Note that this is an interpretation, not a formal proof.)

(c) $\mathbb{V}(\delta_t) = 1/h \to +\infty$. δ_t is the slope of the Brownian path over a time interval of length h. A slope which converges to a finite limit as h goes to zero means that the path is differentiable. Here the variance of the slope goes to infinity, which precludes convergence: the path of a Brownian motion is thus non-differentiable. (Note that this is an interpretation, not a formal proof.)

Problem 10*: Positivity of geometric Brownian motion

(a) For large n, $1/n$ is close to 0 and we have $X_{t+\frac{1}{n}} - X_t \approx dX_t = aX_t dt + bX_t dW_t$ (where $dt = 1/n$). Thus $R_n \approx adt + bdW_t$ follows a normal distribution with mean $adt = a/n$ and standard deviation $b\sqrt{dt} = b/\sqrt{n}$.

(b) Since $-y^2 \leq y$ for $y \leq -1$ we have:

$$p_n = \frac{\sqrt{n}}{b\sqrt{2\pi}} \int_{-\infty}^{-1} \exp\left(-\frac{n(x-\frac{a}{n})^2}{2b^2}\right) dx \leq \frac{\sqrt{n}}{b\sqrt{2\pi}} \int_{-\infty}^{-1} \exp\left(\frac{n(x-\frac{a}{n})}{2b^2}\right) dx.$$

But $\int_{-\infty}^{-1} \exp\left(\frac{n(x-\frac{a}{n})}{2b^2}\right) dx = \left[\frac{2b^2}{n}\exp\left(\frac{n(x-\frac{a}{n})}{2b^2}\right)\right]_{-\infty}^{-1} = \frac{2b^2}{n}\exp\left(\frac{n(-1-\frac{a}{n})}{2b^2}\right)$, which yields after substitution $p_n \leq b\sqrt{\frac{2}{\pi n}}\exp\left(-\frac{n+a}{2b^2}\right) \xrightarrow[n\to+\infty]{} 0$. Hence the probability that $R_\infty < -1$ is nil. If X_t is the price of an asset, R_n is the return on that asset between t and $t+dt$, and the event $R_\infty < -100\%$ means that the asset loses more than 100% of its value, i.e. $X_{t+dt} < 0$. As a result X_t is non-negative with probability 1, which is a useful property of geometric Brownian motions for modeling asset prices.

Problem 11*: Quadratic variation

(a) From property (2) of Section 9-3.1 p.98 we know that

$$\mathbb{E}\left[\left(W_{\frac{k+1}{n}t} - W_{\frac{k}{n}t}\right)^2\right] = \mathbb{V}\left(W_{\frac{k+1}{n}t} - W_{\frac{k}{n}t}\right) = \frac{t}{n}.$$

Hence $\mathbb{E}\left(Q_n(t)\right) = t$.

(b) Rewriting $Q_n(t) = \frac{t}{n}\sum_{k=0}^{n-1}\tilde{\varepsilon}_k^2$ where $(\tilde{\varepsilon}_k)_{k\geq 0}$ are independent standard normals, and noting that each $\tilde{\varepsilon}_k^2$ has a gamma distribution with parameters ($1/2$, 2), we conclude that $Q_n(t)$ has a gamma distribution with parameters ($n/2$, $2t/n$) also known as a chi-square distribution with n degrees of freedom and a t/n scaling factor.

(c) As n goes to infinity the gamma distribution has parameters $(\infty, 0)$, i.e. $Q_\infty(t)$ is a constant equal to $\mathbb{E}\left(Q_n(t)\right) = t$.

(d) As n goes to infinity $Q_n(t)$ intuitively converges to $\int_0^t (dW_s)^2$ (the total sum of squared infinitesimal Brownian increments over the interval $[0,t]$). We may thus write $\int_0^t (dW_s)^2 \equiv t$, and by differentiation with respect to t we get $(dW_t)^2 \equiv dt$.

Problem 12: Stochastic integration

(a) Rewriting $X_n = \sqrt{\frac{T}{n}} \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \tilde{\varepsilon}_k$ where $(\tilde{\varepsilon}_k)_{k\geq 0}$ are independent standard normals, we conclude that X_n is normally distributed with zero mean and variance $\frac{T}{n} \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right)^2 = \frac{T}{n^3} \sum_{k=1}^{n} k^2 = \frac{T(n+1)(2n+1)}{6n^2}$. As n goes to infinity X_n converges to a normal distribution with zero mean and variance $\lim_{n\to+\infty} \frac{T(n+1)(2n+1)}{6n^2} = \frac{T}{3}$.

(b) Using the Ito-Doeblin theorem on (W_t) as the underlying Ito process with $a = 0$ and $b = 1$ we get:

$$\begin{aligned} d\left[(T-t)W_t\right] = df &= \left(\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial x^2}\right)dt + b\frac{\partial f}{\partial x}dW_t \\ &= (-W_t + 0 + 0)\,dt + (T-t)dW_t \\ &= -W_t dt + (T-t)dW_t \end{aligned}$$

Integrating over $[0,\ T]$ and rearranging terms yields $\int_0^T (T-t)dW_t = \int_0^T W_t dt + [(T-t)W_t]_0^T = \int_0^T W_t dt$. Thus $X = \frac{1}{T}\int_0^T W_t dt$ which is the mean value of W over $[0,T]$.

Problem 13: Stock price modeling

(a) (i) Using Equation (9-2) p.99 we may write $X_t = 100 + mt + sW_t$. Hence $\mathbb{E}(X_t) = 100 + mt$, and $\mathbb{V}(X_t) = s^2 t$. Solving $100 + mt = 100 \times (1 + 12\%) = 112$ and $s\sqrt{t} = 100 \times 40\% = 40$ for $t = 1$ we find the parameters $m = 12$ and $s = 40$.

(ii) X_t being normally distributed with parameters $(100 + 12t, 40\sqrt{t})$, the probability that X_t exceeds €120 in a year's time is $\mathbb{P}\{X_1 > 120\} = \mathbb{P}\left\{\frac{X_1 - 112}{40} > 0.2\right\} = 1 - N(0.2) \approx 42\%$. Similarly, in two years' time $\mathbb{P}\{X_2 > 120\} \approx 52\%$.

(iii) X_t being normally distributed there is a positive probability that it takes a negative value, which is counter-intuitive for a stock price.

(b) (i) Using Equation (9-4) p.100 we may write $X_t = 100 \times e^{(\mu-\sigma^2/2)t+\sigma W_t}$. From Problem 8 question (c) we know that $\mathbb{E}(X_t) = 100 \times e^{\mu t}$; solving $\mathbb{E}(X_1) = 112$ for μ yields $\mu = \ln(112/100) \approx 11.33\%$. As for variance:

$$\begin{aligned} \mathbb{V}(X_t) &= 100^2 \times e^{2\mu t} \times \mathbb{V}\left(e^{\sigma W_t - \frac{1}{2}\sigma^2 t}\right) \\ &= 100^2 e^{2\mu t}\left[\mathbb{E}\left(e^{2\sigma W_t - \sigma^2 t}\right) - \left(\mathbb{E}\left(e^{\sigma W_t - \frac{1}{2}\sigma^2 t}\right)\right)^2\right] \\ &= 100^2 e^{2\mu t}\left[e^{\sigma^2 t}\mathbb{E}\left(e^{2\sigma W_t - 2\sigma^2 t}\right) - 1\right] \\ &= 100^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right) \end{aligned}$$

Solving $\sqrt{\mathbb{V}(X_1)} = 40$ for σ we find $\sigma = \sqrt{\ln\left(1 + \left(\frac{40}{112}\right)^2\right)} \approx 34.64\%$.

(ii) The probability that the stock price exceeds €120 at time t is:

$$\mathbb{P}\{X_t > 120\} = \mathbb{P}\left\{\left(0.1133 - \frac{1}{2}0.3664^2\right)t + 0.3664W_t > \ln\frac{120}{100}\right\}$$
$$= 1 - N\left[\frac{1}{0.3664\sqrt{t}}\left(\ln(1.2) - \left(0.1133 - \frac{1}{2}0.3664^2\right)t\right)\right].$$

For $t = 1$ we get $\mathbb{P}\{G_1 > 120\} \approx 35\%$, and for $t = 2$ we get $\mathbb{P}\{G_2 > 120\} \approx 44\%$.

(iii) Since $X_t = 100 \times \exp\left(\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t\right) > 0$ the probability that Winner AG's stock price becomes negative is nil, which is more satisfactory than in the previous model.

Problem 14*: Binomial tree and Brownian motion

(a) There are $n + 1$ terminal nodes in an n-step recombining tree.

(b) To reach the final node $u^i d^j S_0$, the underlying must go up i times and down $j = n - i$ times. The total number of possible paths is thus $\binom{n}{i} = \frac{n!}{i!(n-i)!} = \frac{(i+j)!}{i!j!}$.

(c) S_T is a discrete random variable whose $n + 1$ possible values are S_0u^n, $S_0u^{n-1}d$, $S_0u^{n-2}d^2, \ldots, S_0u^i d^{n-i}, \ldots, S_0d^n$ with respective probabilities $\left(\frac{1}{2}\right)^n, n\left(\frac{1}{2}\right)^n$, $\frac{n(n-1)}{2}\left(\frac{1}{2}\right)^n, \cdots, \binom{n}{i}\left(\frac{1}{2}\right)^n, \cdots, \left(\frac{1}{2}\right)^n$.

(d) (i) Since $S_T = S_0u^{I_T}d^{n-I_T} = (u/d)^{I_T}S_0d^n$, taking the log of both sides and rearranging terms yields $I_T = \frac{\ln(S_T/S_0) - n\ln d}{\ln(u/d)}$. Substituting $u = \exp(\sigma\sqrt{\tau})$, $d = \exp(-\sigma\sqrt{\tau})$, we get: $I_T = \frac{n}{2} + \frac{1}{2\sigma\sqrt{\tau}}\ln\frac{S_T}{S_0}$.

(ii) From question (c) we get that I_T has a binomial distribution with parameters $(n, 1/2)$.

(e) From question (d) we may write $\frac{\ln(S_T/S_0)}{\sigma\sqrt{\tau}} = 2I_T - n$. Since I_T has a binomial distribution with parameters $(n, 1/2)$ it may be rewritten as a sum of independent Bernouilli variables $B_{T,i}$. Substituting and then dividing both sides by $\sqrt{n}$ we get $\frac{\ln(S_T/S_0)}{\sigma\sqrt{n\tau}} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(2B_{T,i} - 1)$.

Noting that $n\tau = T$ and that by the central limit theorem $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(2B_{T,i} - 1)$ converges to a standard normal variable, we conclude that $\ln S_T$ converges to a normal distribution with mean $\ln S_0$ and standard deviation $\sigma\sqrt{T}$.

(f) (W_t) satisfies the three properties of a Brownian motion listed in Section 9-3.1 p.98:

(i) $W_0 = \frac{1}{\sigma}\ln\frac{\tilde{S}_0}{\tilde{S}_0} = 0$.

(ii) Generalizing the results of question (e) for any interval $[t, t']$, the distribution of any increment $D = W_{t'} - W_t = \frac{1}{\sigma}\ln\frac{\tilde{S}_{t'}}{\tilde{S}_t}$ is normal with zero mean and standard deviation $\sqrt{t' - t}$.

(iii) The independence between two increments results from the independence of 'up' and 'down' moves in the tree. (A formal proof would require verification that any finite sequence of increments is independent.)

From question (e) we know that the distribution of $\ln\tilde{S}_T$ is normal with mean $\ln S_0$ and standard deviation $\sigma\sqrt{T}$, thus $\tilde{S}_T$ follows a lognormal distribution with these parameters. This distribution is *not* fully consistent with the lognormal model in which the mean is $\ln S_0 - 1/2\sigma^2 T$ when interest rates are nil. This may easily be fixed by introducing a drift correction by letting $u = \exp\left(-\frac{1}{2}\sigma^2\tau + \sigma\sqrt{\tau}\right)$ and $d = \exp\left(-\frac{1}{2}\sigma^2\tau - \sigma\sqrt{\tau}\right)$.

Chapter 10

Problem 1: True or False?

False: Consider for instance the effect of hedging a call on a technology stock with shares in a supermarket chain: even if they had the same volatility, one can intuitively tell that technology companies and supermarket chains are not exposed to the same sources of risks.

In Black-Scholes theory it is crucial that the derivative and the underlying asset be exposed to the *same* source of risk ($\boldsymbol{W_t}$):

- For the underlying asset: $dS_t = \mu S_t\, dt + \sigma S_t\, \boldsymbol{dW_t}$;
- For the derivative security:

$$dD_t = df = \left(\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2}\right) dt + \left(\sigma S_t \frac{\partial f}{\partial S}\right) \boldsymbol{dW_t}.$$

For example, if $\frac{\partial f}{\partial S} > 0$ the value of the derivative and the price of the underlying asset will both increase whenever $dW_t > 0$. Hedging the risk (W_t) with another asset following a geometric Brownian motion $dX_t = \mu' X_t\, dt + \sigma X_t\, dW'_t$ would be an imperfect hedge exposed to the level of correlation between (W_t) and (W'_t). In particular, for low or zero correlation, the hedge would be totally inefficient.

Problem 2: Limit-cases of the Black-Scholes formulas

(a) Limit as $T \to 0$: $d_1 = \frac{\ln \frac{S_0}{K}}{\sigma\sqrt{T}} + \left(\frac{r}{\sigma} + \frac{\sigma}{2}\right)\sqrt{T} \xrightarrow[T\to 0]{} \begin{cases} +\infty \text{ if } S_0 > K \\ 0 \text{ if } S_0 = K \\ -\infty \text{ if } S_0 < K \end{cases}$

and $d_2 = d_1 - \sigma\sqrt{T}$ has the same limit. Hence:

$$c_0 = S_0 N(d_1) - Ke^{-rT} N(d_2) \xrightarrow[T\to 0]{} \begin{cases} S_0 - K \text{ if } S_0 > K \\ 0 \text{ if } S_0 \le K \end{cases}$$

$$p_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \xrightarrow[T\to 0]{} \begin{cases} K - S_0 \text{ if } S_0 < K \\ 0 \text{ if } S_0 \ge K \end{cases}$$

These are the call and put payoffs at maturity: $\max(S_0 - K, 0)$ and $\max(K - S_0, 0)$.

(b) Limit as $\sigma \to 0$: $d_1 = \frac{\ln \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \xrightarrow[\sigma\to 0]{} \begin{cases} +\infty \text{ if } S_0 e^{rT} > K \\ 0 \text{ if } S_0 e^{rT} = K \\ -\infty \text{ if } S_0 e^{rT} < K \end{cases}$

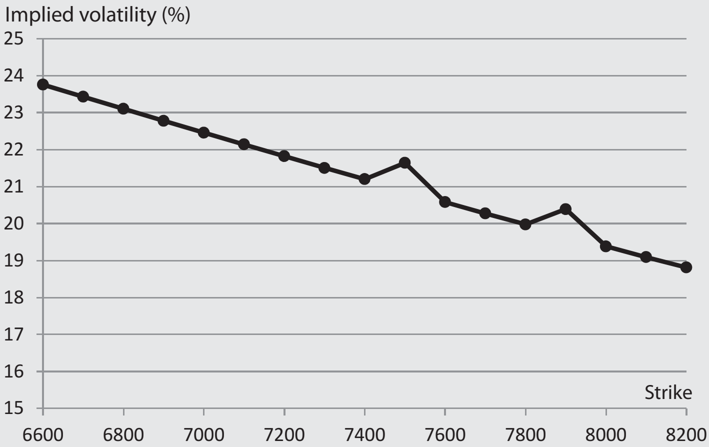
and $d_2 = d_1 - \sigma\sqrt{T}$ has the same limit. Hence:

$$c_0 = S_0 N(d_1) - Ke^{-rT} N(d_2) \xrightarrow[\sigma\to 0]{} \begin{cases} S_0 - Ke^{-rT} \text{ if } S_0 e^{rT} > K \\ 0 \text{ if } S_0 e^{rT} \le K \end{cases}$$

$$p_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \xrightarrow[\sigma\to 0]{} \begin{cases} Ke^{-rT} - S_0 \text{ if } S_0 e^{rT} < K \\ 0 \text{ if } S_0 e^{rT} \ge K \end{cases}$$

When $\sigma \to 0$ the underlying asset has no volatility (i.e. it is riskless) and it must grow at the riskless rate r under penalty of arbitrage. Provided the strike K is lower than the forward price $F_0 = S_0 e^{rT}$, the call is then worth the same as a forward contract, otherwise it is worthless. The results for the put are symmetrical.

Problem 3: Implied volatility 'smile' 'Smile' curve of the DAX:



Problem 4: Black-Scholes and forward contracts

(a) The proof is identical to Section 5-2.2 p.51 using a continuous rather than gross annual interest rate. Let $f(t, S_t) = S_t - Ke^{-r(T-t)}$; then $\frac{\partial f}{\partial t} + rS_t \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2} = -rKe^{-r(T-t)} + rS_t + 0 = rf(t, S_t)$.

(b) At an arbitrary time t the Black-Scholes formulas are given as:

$$c_t = S_t N(d_1) - Ke^{-r(T-t)} N(d_2)$$
$$p_t = Ke^{-r(T-t)} N(-d_2) - S_t N(-d_1)$$

with $d_1 = \frac{\ln \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$. Hence:

$c_t - p_t = S_t\,[N(d_1) + N(-d_1)] - Ke^{-r(T-t)}\,[N(d_2) + N(-d_2)] = S_t - Ke^{-r(T-t)} = \phi_t$,

because $N(-x) = 1 - N(x)$.

Problem 5: Exponential asset

The value of an admissible or tradable derivative must satisfy the Black-Scholes partial differential equation (Equation (10-4) p.111), which is not the case of the "exponential asset":

$$\frac{\partial f}{\partial t} + rS_t \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2} = 0 + rS_t e^{S_t} + \frac{1}{2}\sigma^2 S_t^2 e^{S_t} \neq re^{S_t}.$$

This derivative security cannot exist within the Black-Scholes framework: trading it on the market would lead to arbitrage opportunities.

Problem 6: Stock-options

(a) The only relevant data is the \$10 closing price of MeToo.Com and the implied volatility of the 2-year call struck at \$12. Using the Black-Scholes formula, a stock-option is worth:

$$c_0 = 10\,N(d_1) - 12e^{-5\% \times 2} N(d_2),$$

where $d_1 = \frac{\ln(10/12)+\left(5\%+\frac{1}{2}(30\%)^2\right)\times 2}{30\%\times\sqrt{2}} \approx 0.0181$, $d_2 = d_1 - 30\% \times \sqrt{2} \approx -0.4062$; i.e. it is worth \$13.554, and the 1,000 stock-options' bundle is thus worth about \$13,554.

(b) (i) Using the result from question (a), an increase by $10\% \times 1{,}000{,}000 = 100{,}000$ stock-options should cost about \$1,355,400. This estimate is made 'all other things being equal'. Since the strike is fixed only when the stock-options are distributed, this cost will linearly increase or decrease with the stock price. Additionally, an increase in volatility or rates would also result in a higher cost.

(ii) The offer is only \$4600 higher than the previous estimate, which may look very aggressive. However, we must be careful with the schedule: the estimate is 1 year forward, while the bank's offer is against immediate payment. An equivalent offer with payment in 1 year would be $1{,}360{,}000 \times e^{5\%} = \$1{,}429{,}730$. This is a good offer if we think that the price of MeToo.Com is likely to go up.

Problem 7: Exact relationship between gamma and theta

Substituting Greeks to partial derivatives, the Black-Scholes partial differential equation reads:

$$rf = \Theta + rS_t\Delta + \frac{1}{2}\sigma^2 S_t^2 \Gamma,$$

where f is the value of the derivative. Rearranging terms we get $\Theta + \frac{1}{2}\sigma^2 S_t^2\Gamma = r\,(f - \Delta S_t)$, which is the desired result once we recognize $f - \Delta S_t$ as the value of the delta-hedged portfolio. When r is close to 0 we obtain the gamma-theta proxy relationship $\Theta \approx -\frac{1}{2}\sigma^2 S_t^2\Gamma$.

Problem 8*: Black-Scholes with continuous dividends

(a) When annual dividends are proportional at rate d the forward price of S with a maturity of T whole years is $F_0 = S_0\frac{e^{rT}}{(1+d)^T}$ (see Chapter 5, Problem 8 p.63). If we suppose that dividends are split into m equal payments the forward price becomes $F_0 = S_0\frac{e^{rT}}{(1+d/m)^{mT}}$. Taking the limit as m goes to infinity we get $F_0 = S_0e^{(r-q)T}$ (see Chapter 1, Problem 7 p.9).

(a) Substituting $F_0 = S_0e^{(r-q)T}$ into Equations (7-1) and (7-2) pp.76–77 we obtain:

$$\begin{aligned} c_0 &= S_0e^{-qT}N(d_1) - Ke^{-rT}N(d_2) \\ p_0 &= Ke^{-rT}N(-d_2) - S_0e^{-qT}N(-d_1) \\ d_1 &= \frac{\ln(S_0/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},\ d_2 = d_1 - \sigma\sqrt{T} \end{aligned}$$

(b) (i) Being long δ units of underlying between t and $t + dt$ we collect a dividend amount equal to $q\delta S_t dt$. The change in portfolio value is thus $dP_t = -dD_t + \frac{\partial f}{\partial S}dS_t + q\frac{\partial f}{\partial S}S_t dt$.

(ii) Following the reasoning of Section 10-1 p.109 we have by the Ito-Doeblin theorem $dD_t = df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS_t + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial S^2}dt$. Substituting into the equation for dP_t and cancelling the dS_t terms we obtain $dP_t = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial S^2}\right)dt + q\frac{\partial f}{\partial S}S_t dt$. As before the delta-hedged portfolio is riskless and thus $dP_t = rP_t dt$, which yields after rearranging terms and "dividing by dt": $rf = \frac{\partial f}{\partial t} + (r - q)S_t\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial S^2}$.

Chapter 11

Problem 1: Realized volatility

(a) Before the flash crash the 6-month realized volatility was about $0.8\% \times \sqrt{252} \approx 12.7\%$. After the flash crash it jumped to about $\sqrt{\frac{252}{126}\left(125 \times 0.8\%^2 + 9\%^2\right)} \approx 17.9\%$.

(b) Let x be the level of volatility during the 3-month window between 9 and 12 months. Under zero-mean assumption we have $(\text{1-year vol})^2 = \frac{1}{4}x^2 + \frac{3}{4}(\text{9-month vol})^2$. Solving for x yields $x = \sqrt{4 \times 0.3^2 - 3 \times 0.2^2} \approx 49\%$. The stock had very high volatility for 3 months then moderate volatility for the following 9 months.

Problem 2: Call spread and Black-Scholes

(a) Using a Black-Scholes calculator we find that the at-the-money call at 35% implied volatility is worth \$13.90 while the out-of-the-money call at 25% implied volatility is worth \$2.84. The price of the call spread is thus $13.90 - 2.84 = \$11.06$.

(b) Using $\sigma = 35\%$ for the out-of-the-money call we get a theoretical value of \$6.15, bringing the call spread down to \$7.35. With $\sigma = 25\%$ the at-the-money call has a theoretical value of \$9.95 bringing the call spread down to \$7.11. In both cases the Black-Scholes theoretical value of the call spread is significantly off the \$11.06 market price.

(c) There is no solution, i.e. no single value of σ can reproduce the market price.

Problem 3: Volatility trading with options

(a) If $dS_t = \sigma_{\mathrm{R}} S_t dW_t$ then $r_t^2 \approx \left(\frac{dS_t}{S_t}\right)^2 \equiv \sigma_{\mathrm{R}}^2 dt$ and the cumulative P&L proxy simplifies to:

$$\text{Cumulative P\&L} \approx \left(\sigma_{\mathrm{R}}^2 - \sigma^2\right) \times \Delta t \times \sum_{t=1}^{N} \Gamma_{t-1}^{\$} > 0,$$

because $\sigma_{\mathrm{R}} > \sigma$ and $\Gamma > 0$. We are thus guaranteed to make money in this scenario.

(b) If $dS_t = \sigma_t S_t dW_t$ then the cumulative P&L proxy simplifies to:

$$\text{Cumulative P\&L} \approx \Delta t \times \sum_{t=1}^{N} \Gamma_{t-1}^{\$}(\sigma_t^2 - \sigma^2),$$

which may or may not be positive. There is thus no guarantee to make money in this scenario. However, the expected P&L is positive: $\mathbb{E}(\text{Cumulative P\&L}) \approx \Delta t \times \sum_{t=1}^{N} \mathbb{E}\left(\Gamma_{t-1}^{\$}\right)\left[\mathbb{E}(\sigma_t^2) - \sigma^2\right] > 0$ because σ_t is independent from (S_t) and thus from $\Gamma_{t-1}^{\$}$.

Problem 4: Forward variance swap

(a) Under zero-mean return assumption we may write $2\times(\text{2-year realized vol})^2 = (\text{1-year realized vol})^2 + {\sigma_{\mathrm{R}}}^2$. Thus $K_{\text{fvar}} = \sqrt{2 \times 0.36^2 - 0.4^2} \approx 31.5\%$. The replicating portfolio is long two 2-year variance swaps and short one 1-year variance swap.

(b) Because the payoffs of the 1-year and 2-year variance swaps occur at different times, the replication portfolio is imperfect when interest rates are non-zero. We must indeed take into account the cost of carrying the payoff of the 1-year variance swap at 5% interest rate until the end of the second year:

$$\text{Portfolio payoff} = 2\left[\left(\sigma_R^{2Y}\right)^2 - 0.36^2\right] - \left[\left(\sigma_R^{1Y}\right)^2 - 0.4^2\right] \times (1+5\%).$$

However this does not match the formula for the forward realized variance: $2 \times \sigma_R^{2Y} - 1.05 \times \sigma_R^{1Y} \neq \sigma_{FR}^2$. The correct replicating portfolio is long two 2-year variance swaps and short $1/1.05 \approx 0.9524$ 1-year variance swaps. Remarkably enough this does not change the fair strike K_{fvar}:

$$\sigma_{FR}^2 - K_{\text{fvar}}^2 = 2\left[\left(\sigma_R^{2Y}\right)^2 - 0.36^2\right] - \frac{1}{1.05}\left[\left(\sigma_R^{1Y}\right)^2 - 0.4^2\right] \times (1+5\%)$$
$$\Rightarrow K_{\text{fvar}} = \sqrt{2 \times 0.36^2 - 0.4^2} \approx 31.5\%.$$

Problem 5: Variance swap pricing

The forward price of the DAX is $F_0 = 7484.5 \times \exp(1.4\% \times 1.46) \approx 7640$. We must thus calculate put values for strikes 5,000 to 7,600 and call values for strikes 7,700 to 10,000. The fair strike of the variance swap is then given as $K_{\text{var}} \approx \sqrt{\frac{2(1+r)^T}{T}\left[\sum_{i=1}^{n} \frac{p_0(K_i)}{K_i^2}\Delta K_i + \sum_{i=n+1}^{n+m} \frac{c_0(K_i)}{K_i^2}\Delta K_i\right]} \approx 21.5\%$.

Strike (€)	Implied volatility	Put value (€)	Strike (€)	Implied volatility	Call value (€)
5000	31.0%	145	7700	20.4%	708
5100	30.6%	155	7800	20.1%	655
5200	30.1%	165	7900	19.8%	603
5300	29.6%	176	8000	19.5%	554
5400	29.2%	188	8100	19.2%	507
5500	28.7%	201	8200	18.9%	463
5600	28.3%	214	8300	18.6%	420
5700	27.8%	228	8400	18.3%	380
5800	27.4%	243	8500	18.0%	343
5900	27.0%	259	8600	17.7%	307
6000	26.6%	276	8700	17.4%	274
6100	26.1%	294	8800	17.1%	243
6200	25.7%	313	8900	16.8%	215
6300	25.4%	333	9000	16.6%	188
6400	25.0%	354	9100	16.3%	164
6500	24.6%	377	9200	16.0%	142
6600	24.2%	400	9300	15.8%	123
6700	23.8%	426	9400	15.5%	105
6800	23.5%	452	9500	15.2%	89
6900	23.1%	480	9600	15.0%	75
7000	22.8%	510	9700	14.7%	62
7100	22.4%	542	9800	14.5%	51
7200	22.1%	575	9900	14.2%	42
7300	21.7%	609	10000	14.0%	34
7400	21.4%	646			
7500	21.1%	685			
7600	20.7%	725			

Problem 6: Volatility swap. Suppose $K_{\text{vol}} \geq K_{\text{var}}$. Then buy a variance swap and sell x volatility swaps. At maturity collect a net payoff:

$$\begin{aligned}
100^2 &\times \left(\sigma^2_{\text{Realized}} - K^2_{\text{var}}\right) - 100x \times \left(\sigma_{\text{Realized}} - K_{\text{vol}}\right) \\
&= \left(100\sigma_{\text{Realized}} - x/2\right)^2 - x^2/4 + 100xK_{\text{vol}} - 100^2 K^2_{\text{var}} \\
&\geq \left(100\sigma_{\text{Realized}} - x/2\right)^2 - x^2/4 + 100xK_{\text{var}} - 100^2 K^2_{\text{var}} \\
&\geq \left(100\sigma_{\text{Realized}} - x/2\right)^2 - (x/2 - 100K_{\text{var}})^2
\end{aligned}$$

Choosing $x = 200K_{\text{var}}$ the second term vanishes and we always obtain a nonnegative quantity: we just made an arbitrage.

Problem 7*: Continuous delta-hedging P&L equation

(a) If $P_t > 0$ we need to borrow P_t in cash and the borrow cost between t and $t + dt$ is $rP_t dt$. If $P_t < 0$ we need to lend $|P_t|$ in cash and the lending profit is $r|P_t|dt$.

(b) The total P&L between times t and $t + dt$ is $\pi_t = dP_t - rP_t dt = dD_t - \delta dS_t - rP_t dt$. Using the Ito-Doeblin theorem we get $dD_t = df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS_t)^2$. Replacing partial derivatives with the corresponding Greek letters we have $dD_t = \Theta dt + \delta dS_t + \frac{1}{2}\Gamma(dS_t)^2$ and thus, after cancelling the δdS_t terms, $\pi_t = \Theta dt + \frac{1}{2}\Gamma(dS_t)^2 - rP_t dt$. Substituting the exact gamma-theta relationship from Chapter 10, Problem 7 p.116 and cancelling the Θdt terms we obtain $\pi_t = \frac{1}{2}\Gamma(dS_t)^2 - \frac{1}{2}\Gamma\sigma^2 S_t^2 dt$. Factoring by $\frac{1}{2}\Gamma(S_t)^2$ yields the desired result.

(c) Expanding the square we may write $\left(\frac{dS_t}{S_t}\right)^2 = \mu_t^2 dt^2 + 2\mu_t\sigma_t dt\, dW_t + \sigma_t^2(dW_t)^2$. From Chapter 9, Problem 11 p.105 we know that $(dW_t)^2 \equiv dt$, and from property (2) of Section 9-3.1 p.98 we know that $dW_t \equiv \tilde{\varepsilon}\sqrt{dt}$ where $\tilde{\varepsilon}$ has a standard normal distribution. To leading order in dt we thus have $(dS_t/S_t)^2 \equiv \sigma_t^2 dt$. Substituting this result into the P&L equation from question (b) we get $\pi_t = \frac{1}{2}\Gamma_t S_t^2\left[\sigma_t^2 - \sigma^2\right]dt$, and summing continuously over $[0, T]$ with appropriate capitalization factor $e^{r(T-t)}$ to horizon T we obtain the required result.

Chapter 12

Problem 1: Digital options

(a) From Chapter 7, Problem 9 p.81 we know that the fair value of the digital call is $e^{-rT}N(d_2)$ with $d_2 = \frac{\ln S - \ln K + rT}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}$, thus $\delta = e^{-rT}\frac{\partial}{\partial S}N(d_2) = e^{-rT}N'(d_2)\frac{1}{S\sigma\sqrt{T}}$.

(b) If $S \neq K$ then $d_2 \to \pm\infty$ and thus $\delta \to 0$; if $S = K$ then $d_2 \to 0$ and thus $\delta \to +\infty$. This means that if the spot price is near the strike the number of shares to trade becomes very large as we approach maturity: digital options can be very difficult to delta-hedge.

(c) (*) A quantity $1/\varepsilon$ of a call spread with strikes $K - \varepsilon$ and K converges to a digital call payoff as $\varepsilon \to 0$. Thus:

$$\begin{aligned}
D_0 &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big\{S_0\left[N\left(d_1(K-\varepsilon)\right) - N\left(d_1(K)\right)\right] - Ke^{-rT}\left[N\left(d_2(K-\varepsilon)\right) - N\left(d_2(K)\right)\right] \\
&\quad + \varepsilon e^{-rT}N\left(d_2(K-\varepsilon)\right)\Big\} \\
&= e^{-rT}N\left(d_2(K)\right) + Ke^{-rT}\frac{\partial}{\partial k}N\left(d_2(k)\right)\Big|_{k=K} - S_0\frac{\partial}{\partial k}N\left(d_1(k)\right)\Big|_{k=K} \\
&= e^{-rT}N\left(d_2(K)\right) + Ke^{-rT}N'\left(d_2(K)\right)\frac{\partial d_2}{\partial k}(K) - S_0 N'\left(d_1(K)\right)\frac{\partial d_1}{\partial k}(K)
\end{aligned}$$

From $d_2(k) = d_1(k) - \sigma^*(k)\sqrt{T}$ we get $\frac{\partial d_2}{\partial k}(K) = \frac{\partial d_1}{\partial k}(K) - \frac{\partial \sigma^*}{\partial k}(K)\sqrt{T}$. Substituting and then rearranging terms we get $D_0 = e^{-rT} N(d_2(K)) - Ke^{-rT} N'(d_2(K)) \frac{\partial \sigma^*}{\partial k}(K) + e^{-rT}\left[KN'(d_2(K)) - F_0 N'(d_1(K))\right] \frac{\partial d_1}{\partial k}(K)$. The expression between square brackets vanishes as a result of Chapter 8, Problem 7 p.91 question (a), yielding the desired result. Typically $\frac{\partial \sigma^*}{\partial k} < 0$ (the smile slope downwards) and the fair value of the digital option is thus greater than in Black-Scholes.

Problem 2: Asian options

(a) Because the call payoff function $f(x) = \max(0, x - K)$ is convex we have by Jensen's inequality $f\left(\frac{1}{12}\sum_{t=1}^{12} S_{t/12}\right) \leq \frac{1}{12}\sum_{t=1}^{12} f(S_{t/12})$ which yields the desired result in terms of option values under penalty of arbitrage.

(b) Generalizing the result from question (a) we obtain that the fair value of the Asian call is at most $U = \frac{1}{T}\int_0^T c_0(S_0, t)dt$. From Chapter 7, Problem 8 pp.80–81 we have $c_0(S_0, t) \approx 0.4 S_0 \sigma\sqrt{t}$ and thus $U \approx 0.4\frac{S_0}{T}\sigma\int_0^T \sqrt{t}dt = \frac{2}{3}0.4 S_0\sigma\sqrt{T} \approx \frac{2}{3}c_0(S_0, T)$ as required. The $2/3 \approx 0.667$ factor is a little higher than the $1/\sqrt{3} \approx 0.577$ proxy rule mentioned in Section 12-1.2 p.127.

Problem 3: Single-asset structured products

Note that because the continuous dividend rate is equal to the continuous interest rate, all SPX forward prices are equal to the spot price SPX_{initial}.

(a) This payoff corresponds to a 5-year zero-coupon bond worth $e^{-2.5\%\times 5} \approx 88.25\%$ together with a 5-year at-the-money European call worth about $0.4\sigma\sqrt{T} = 0.4 \times 0.3 \times \sqrt{5} \approx 26.8\%$ by rule of thumb (see Chapter 7, Problem 8 pp.80–81). Solving $88.25\% + p\times 26.8\% = 100\%$ for p yields a participation level of about 43%. (A more accurate calculation of the call value using closed-form formulas would yield a slightly higher participation of about 48%.)

(b) This payoff combines a quantity λ of 10-year zero-coupon bonds worth $e^{-2.5\%\times 10} \approx 77.88\%$ and a 10-year European call with strike λ. Substituting closed-form formulas we must solve:

$$\begin{cases} 77.88\% \left[\lambda + (N(d_1) - \lambda N(d_2))\right] = 100\% \\ d_1 = \dfrac{-\ln\lambda + \frac{1}{2}0.3^2 \times 10}{0.3\sqrt{10}}, \quad d_2 = d_1 - 0.3\sqrt{10} \end{cases}$$

Using a numerical solver we find a floor level $\lambda \approx 87.7\%$.

(c) This payoff may be decomposed as a 3-year zero-strike forward contract worth $e^{-2.5\%\times 3} \approx 92.77\%$ and an in-the-money knock-out put with strike $(1 + c)$ and barrier $60\%\times SPX_{\text{initial}}$. When $c = 0$ the barrier put is worth 3.25%, and a 1% increase in strike costs 0.2%. Assuming linear relationships we must solve $92.77\% + 3.25\% + 0.2\%\times c = 100\%$ for c, which yields a coupon level $c \approx 19.8\%$. (A more accurate calculation using a barrier pricer would find $c \approx 18\%$.)

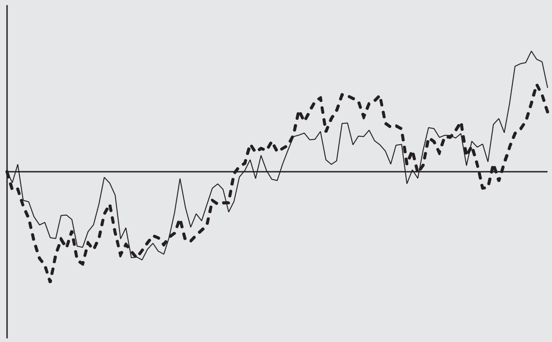
Problem 4: Barrier monitoring

When the barrier is monitored every month rather than continuously there are fewer chances to hit the barrier, hence the option value must decrease. Following the proxy rule from Section 12-1.3 p.128 we may shift the barrier *down* by a factor $\exp(-0.5826\sigma\sqrt{\Delta t}) = \exp(-0.5826 \times$

$0.4 \times \sqrt{1/12}) \approx 0.935$. The new barrier is thus $\$40 \times 0.935 \approx \$37.40$ and the barrier call is worth approximately $3.56 + 0.56 \times (37.40 - 40) \approx \$2.10$.

Problem 5: Correlated Brownian motions

Because (V_t) and (W_t) are uncorrelated we have $(dV_t)(dZ_t) = (dV_t)(\rho dV_t + \sqrt{1-\rho^2}dW_t) = \rho(dV_t)^2 \equiv \rho dt$. Simulated paths:



Problem 6: Spread option

(a) The following screen capture displays our adaptation of the spreadsheet from Chapter 7, Problem 5 p.80, using $S_T^{(2)} = \exp[\sigma_2 Z_T - \frac{1}{2}\sigma_2^2 T] = \exp[\sigma_2(\rho V_T + \sqrt{1-\rho^2}W_T) - \frac{1}{2}\sigma_2^2 T]$ where V_T is the Brownian motion associated to $S_T^{(1)}$ and W_T is an uncorrelated Brownian motion. We find a fair value of 17.58%.

	A	B	C	D	E	F	G
1		Forward	1	1			
2		Maturity	2	years			
3		Risk-free rate	0.00%				
4		Volatility	40%	35%	Price	17.58%	
5		Correl	65%				
6			=$D$1*EXP(SQRT($C$2)*$D$4*(rho*NORMSINV(B10)+SQRT(1-				
7			rho^2)*NORMSINV(C10))-0.5*$D$4^2*$C$2)				
8							
9	Simulation	Uniform1	Uniform1	$S_T^{(1)}$	$S_T^{(2)}$	Payoff	
10	1	0.885956609	0.591201655	1.69	1.42	26.31%	
11	2	0.191180996	0.403667682	0.52	0.61	0.00%	
12	3	0.799886517	0.802533672	1.37	1.60	0.00%	
13	4	0.896807019	0.058303903	1.74	0.74	100.53%	
14	5	0.753063491	0.413652966	1.25	1.02	23.92%	
15	6	0.047869949	0.833627649	0.33	0.75	0.00%	
16	7	0.342095184	0.264241267	0.68	0.61	6.46%	
17	8	0.928128773	0.54618487	1.95	1.48	46.92%	
18	9	0.364010292	0.018051837	0.70	0.36	34.03%	
19	10	0.156409396	0.487104843	0.48	0.63	0.00%	
20	11	0.308008368	0.503689895	0.64	0.76	0.00%	
21	12	0.161508195	0.35157681	0.49	0.56	0.00%	
22	13	0.383307603	0.426469749	0.72	0.75	0.00%	
23	14	0.675298416	0.716801617	1.10	1.27	0.00%	
24	15	0.317074791	0.131477196	0.65	0.50	15.28%	
25	16	0.202852292	0.115887128	0.53	0.43	10.06%	
26	17	0.423689874	0.358392916	0.76	0.73	3.87%	
5005	4996	0.20197664	0.147767349	0.53	0.46	7.51%	
5006	4997	0.82391307	0.076673866	1.44	0.70	74.49%	
5007	4998	0.861587123	0.573466615	1.58	1.35	23.06%	
5008	4999	0.858459816	0.269144845	1.56	0.99	57.25%	
5009	5000	0.145181732	0.774936461	0.47	0.84	0.00%	
5010							

(b) Using a 66% correlation parameter the value goes down to 17.33%, a −0.25% impact.

Problem 7*: Correlation proxy

(a) From $c \leq \rho_{i,j} \leq 1$ we get $c \sum\limits_{i<j} \sigma_i \sigma_j \leq \sum\limits_{i<j} \sigma_i \sigma_j \rho_{i,j} \leq \sum\limits_{i<j} \sigma_i \sigma_j$ which yields the desired result because $c > 0$. When $\rho_{i,j} = c$ for all pairs we have $\bar{\rho} = c$.

(b) From Equation (4-2) p.41 we get $\sigma_{\mathrm{B}}^2 = \frac{1}{n^2} \sum\limits_{(i,j)} \sigma_i \sigma_j \rho_{i,j} = \frac{2}{n^2} \sum\limits_{i<j} \sigma_i \sigma_j \rho_{i,j} + \frac{s^2}{n}$. Expanding the square of $\bar{\sigma} = \frac{1}{n} \sum\limits_{i=1}^{n} \sigma_i$ we get $\bar{\sigma}^2 = \frac{s^2}{n} + \frac{2}{n^2} \sum\limits_{i<j} \sigma_i \sigma_j$. Thus $\frac{\sigma_{\mathrm{B}}^2 - s^2/n}{\bar{\sigma}^2 - s^2/n} = \frac{2 \sum_{i<j} \sigma_i \sigma_j \rho_{i,j}}{2 \sum_{i<j} \sigma_i \sigma_j} = \bar{\rho}$.

(c) Note that $s^2/n \to 0$ since $a^2 \leq s^2 \leq b^2$; thus, if $\sigma_B, \bar{\sigma}$ both happen to converge, then $\bar{\rho} \to \left(\frac{\lim \sigma_{\mathrm{B}}}{\lim \bar{\sigma}}\right)^2$. This is what typically happens in practice, in which case this argument is enough. However it is theoretically conceivable that $\sigma_B, \bar{\sigma}$ may not converge, but even in this case we have $0 \leq \frac{s^2/n}{\sigma_{\mathrm{B}}^2} = \frac{s^2/n}{\frac{1}{n^2} \sum_{(i,j)} \sigma_i \sigma_j \rho_{i,j}} \leq \frac{b^2/n}{a^2 c} \to 0$, and thus $\frac{s^2}{n} = o\left(\sigma_{\mathrm{B}}^2\right)$. As a result $\bar{\rho} = \frac{\sigma_{\mathrm{B}}^2 - s^2/n}{\bar{\sigma}^2 - s^2/n} \sim \left(\frac{\sigma_{\mathrm{B}}}{\bar{\sigma}}\right)^2$.

Problem 8: Zero-cost variance dispersion.

(a) Dispersion payoff: $\sigma_{\mathrm{B}}^2 - \frac{1}{n} \sum_{i=1}^n \sigma_i^2$. Cost: $v_{\mathrm{B}}^2 - \frac{1}{n} \sum_{i=1}^n v_i^2$.

(b) Solving $v_{\mathrm{B}}^2 - \frac{\beta}{n} \sum\limits_{i=1}^{n} v_i^2 = 0$ for β yields $\beta = \frac{v_{\mathrm{B}}^2}{\frac{1}{n} \sum_{i=1}^n v_i^2}$.

(c) Zero-cost dispersion payoff: $\sigma_{\mathrm{B}}^2 - \frac{\beta}{n} \sum\limits_{i=1}^{n} \sigma_i^2 = \sigma_{\mathrm{B}}^2 - \beta V = V \left(\frac{\sigma_B^2}{V} - \beta\right) = V(\hat{\rho} - k)$ since $\beta = k$.

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APPENDICES

A
Probability Review

A-1 States of Nature. Random Variables. Events

A-1.1 States of Nature

The classical representation of uncertainty in finance consists in a collection Ω of all possible future outcomes ω called **states of nature**: $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$. For example, a representation of the future market condition in a year's time could be $\Omega = \{$"Boom," "Bull," "Flat," "Bear," "Bust"$\}$, as illustrated in Figure A-1 below.

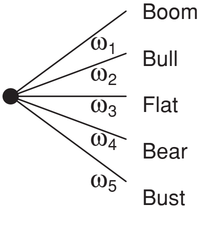


Figure A-1 States of nature for the future market condition

The states of nature form the backbone structure of uncertainty and only have an interpretative meaning. They should not be confused with the numerical values sometimes associated with them (thus one should *not* write $\omega_1 = +30\%$, $\omega_2 = +15\%, \ldots$).

Uncertainty may also be represented with an infinite number of states of nature $\Omega = \{\omega_i \,|\, i \in I\}$, where I is the set of natural integers or real numbers. However, these infinite representations require a more sophisticated mathematical framework which is beyond our scope. We refer the interested reader to Jacod and Protter (2004).

A-1.2 Random Variables

A **random variable** maps each state of nature ω with a real number. For example, the future stock price X of ABC Inc. could be defined as: $X(\omega_1) = 130$, $X(\omega_2) = 115$, $X(\omega_3) = 100$, $X(\omega_4) = 90$ and $X(\omega_4) = 60$, as illustrated in Figure A-2 below.

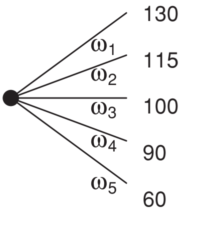


Figure A-2 Future price of ABC Inc.

We can then use X to define other random variables such as the payoff $C = \max(0, X - 100)$ of a 1-year European call option on ABC Inc. struck at \$100. C is also a random variable with $C(\omega_1) = 30$, $C(\omega_2) = 15$, and $C(\omega_3) = C(\omega_4) = C(\omega_5) = 0$.

A-1.3 Events

It is often useful to group several states of nature together into a "meta-state" called an **event**. Every event is thus a subset of Ω. In our example the event "the market is up" would be the subset $U = \{\omega_1, \omega_2\}$, and the event "the payoff of the call is zero" would be the subset $Z = \{\omega_3, \omega_4, \omega_5\}$.

Some frequently used events based on a random variable X have standard notations:

- $\{X = x\}$: all states of nature where X takes the value x (e.g. $Z = \{C = 0\}$);
- $\{X > x\}$: all states of nature where X takes values greater than x;
- $\{x \leq X \leq y\}$, $\{X \geq x\}$ etc.

Events may be assembled or transformed using logical operators:

- $A \cup B$: "A or B" ;
- $A \cap B$: "A and B" ;
- $\bar{A}$: "not A";
- $A \cup \overline{(B \cap C)}$, etc.

A-2 Probability. Expectation. Variance

A-2.1 Probability

Once a suitable representation of uncertainty has been found, the next step is to assign a **probability of occurrence** p_i to each state of nature ω_i. Intuitively, the number p_i is the likelihood level between 0% and 100% of ω_i happening, and the sum of all probabilities must be 100%.

Continuing our example, we could set $p_1 = 15\%$, $p_2 = 20\%$, $p_3 = 35\%$, $p_4 = 25\%$, $p_5 = 5\%$, as shown in Figure A-3 below.

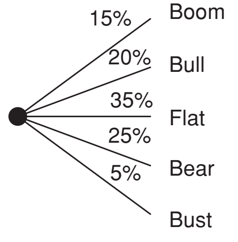


Figure A-3 Probabilities of the states of nature

The **probability of an event** A, denoted $\mathbb{P}(A)$, is then the sum of all probabilities p_i of the states of nature ω_i in A. In our example:

- The probability of the event “the market is up” is $\mathbb{P}(U) = \mathbb{P}(\{\omega_1, \omega_2\}) = p_1 + p_2 = 0.15 + 0.2 = 35\%$.
- The probability of the event $Z = \{C = 0\}$ (“the payoff of the call is zero”) is $\mathbb{P}(\{C = 0\}) = \mathbb{P}(\{\omega_3, \omega_4, \omega_5\}) = 0.35 + 0.25 + 0.05 = 65\%$.

A-2.2 Expectation

The **expectation** or **mean** $\mathbb{E}(X)$ of a random variable X is the average of its values weighted by the corresponding probabilities:

$$\mathbb{E}(X) = \sum_i X(\omega_i) p_i.$$

In our example, the expected future price of ABC Inc. is: $\mathbb{E}(X) = (130 \times 0.15) + (115 \times 0.2) + (100 \times 0.35) + (90 \times 0.25) + (60 \times 0.05) = 103$. This means that “on average” we expect the price of ABC Inc. to grow slightly over the year. Note that this is only a calculation, not an event: there is in fact only a 35% chance that the stock price actually goes up in our model.

A-2.3 Variance

The variance of a random variable measures its level of instability with respect to expectation. Specifically, **variance** is the weighted average of squared deviations from the mean:

$$\mathbb{V}(X) = \sum_i (X(\omega_i) - \mathbb{E}(X))^2 p_i.$$

Standard deviation is then defined as the square root of variance:

$$\sigma(X) = \sqrt{\mathbb{V}(X)}.$$

In our example, the variance and standard deviation of the future price X of ABC Inc. are:

- $\mathbb{V}(X) = (130 - 103)^2 \times 0.15 + (115 - 103)^2 \times 0.25 + (100 - 103)^2 \times 0.35 + (90 - 103)^2 \times 0.25 + (60 - 103)^2 \times 0.05 = 283.2$
- $\sigma(X) = \sqrt{283.2} \approx 16.83$

This means that the price of ABC Inc. in a year’s time will likely deviate by ±\$16.83 from the \$103 mean.

A-3 Distribution. Normal Distribution

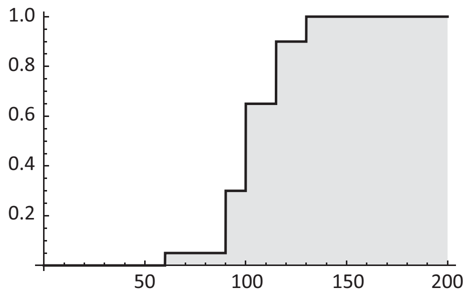
A-3.1 Cumulative Distribution

The **cumulative distribution** F_X of a random variable X is given as:

$$F_X(x) = \mathbb{P}(\{X \leq x\}).$$

Continuing our example, the cumulative distribution for the future price X of ABC Inc. in a year's time is the step function:

$$F_X(x) = \begin{cases} 0 & \text{for } x < 60 \\ 0.05 & \text{for } 60 \leq x < 90 \\ 0.05 + 0.25 = 0.3 & \text{for } 90 \leq x < 100 \\ 0.3 + 0.35 = 0.65 & \text{for } 100 \leq x < 115 \\ 0.65 + 0.25 = 0.9 & \text{for } 115 \leq x < 130 \\ 1 & \text{for } x \geq 130 \end{cases}$$



A-3.2 Continuous Random Variables

A random variable X is said to be **continuous** when its cumulative distribution is continuously differentiable, except perhaps in a finite number of points x. The derivative of F_X is called the **partial distribution** or **distribution density** of X and is often denoted f_X. The formulas for the expectation and variance of X are given in Section A-5 below.

A-3.3 Normal Distribution

A random variable G is said to follow **a normal distribution with mean m and standard deviation** σ when its cumulative distribution is:

$$F_G(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(u-m)^2}{2\sigma^2}} du.$$

In shorthand notation: $G \sim \mathcal{N}(m, \sigma)$.

When $m = 0$ and $\sigma = 1$, G is said to follow a **standard normal distribution**, and its cumulative distribution is often denoted N:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du.$$

Normally distributed variables are examples of continuous random variables. Their distribution density has a well-known bell-shaped curve centered around m, and the probability that their value lies between $m - \sigma$ and $m + \sigma$ is roughly 2/3, as illustrated in Figure A-4 overleaf.

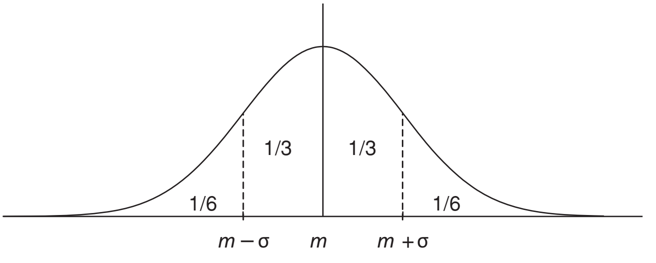


Figure A-4 Density of the normal distribution

A-3.4 Lognormal Distribution

A random variable X is said to follow a **lognormal distribution with parameters m and σ** when the random variable $Y = \ln X$ follows a normal distribution with mean m and standard deviation σ.

The lognormal distribution density is:

$$f_X(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - m)^2}{2\sigma^2}}, \text{ for } x > 0.$$

A-4 Independence. Correlation

A-4.1 Independence

Two events A, B are said to be **independent** when:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B).$$

Two random variables X, Y are said to be independent when any event A_X defined through X is independent from any event B_Y defined through Y. Examples of A_X and B_Y are: $\{X = a\}$, $\{X > c\}, \ldots \{Y \leq b\}, \{d < Y < e\}, \ldots$ and all their logical combinations.

A-4.2 Covariance and Correlation

The **covariance** between two random variables X, Y is the weighted average of their joint deviation from their respective means:

$$\text{Cov}(X, Y) = \sum_i (X(\omega_i) - \mathbb{E}(X))(Y(\omega_i) - \mathbb{E}(Y))\, p_i.$$

The **coefficient of correlation** between X and Y is a normalized measure of covariance between -1 and $+1$:

$$\rho_{X,Y} = \frac{\text{Cov}(X, Y)}{\sigma(X)\,\sigma(Y)}.$$

If X and Y are independent, their covariance and thus correlation is zero. Note that the converse is *not* true: one may find variables with zero correlation which are not mathematically independent.

A correlation coefficient equal to 1 means that there are constants $a > 0$ and b such that $Y = aX + b$ with 100% certainty, and a correlation equal to –1 means that $Y = –aX + b$ with 100% certainty.

A-5 Probability Formulas

A-5.1 Probability of Events

- $\mathbb{P}(\bar{A}) = 1 – \mathbb{P}(A)$
- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) – \mathbb{P}(A \cap B)$

A-5.2 Expectation, Variance, and Standard Deviation of a Discrete Random Variable

- $\mathbb{E}(X) = \sum_i X(\omega_i)p_i$
- $\mathbb{V}(X) = \sum_i [X(\omega_i) - \mathbb{E}(X)]^2 p_i = \mathbb{E}\left(X^2\right) - (\mathbb{E}(X))^2$
- $\sigma(X) = \sqrt{\mathbb{V}(X)}$

A-5.3 Expectation, Variance, and Standard Deviation of a Continuous Random Variable

- $\mathbb{E}(X) = \int_{-\infty}^{+\infty} uf_X(u)du$
- $\mathbb{V}(X) = \int_{-\infty}^{+\infty} (u - \mathbb{E}(X))^2 f_X(u)du = \mathbb{E}\left(X^2\right) - [\mathbb{E}(X)]^2$
- $\sigma(X) = \sqrt{\mathbb{V}(X)}$

A-5.4 Covariance and Correlation between Two Discrete Variables

- $\text{Cov}(X, Y) = \sum_i (X(\omega_i) - \mathbb{E}(X))(Y(\omega_i) - \mathbb{E}(Y))\, p_i = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$
- $\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma(X)\sigma(Y)}$

A-5.5 Properties of Expectation, Variance, Covariance, and Correlation

- $\mathbb{E}(aX + bY) = a\,\mathbb{E}(X) + b\,\mathbb{E}(Y)$ (linearity)
- $\mathbb{V}(X) \geq 0$
- $\mathbb{V}(aX + b) = a^2\,\mathbb{V}(X)$
- $\mathbb{V}(X + Y) = \mathbb{V}(X) + \mathbb{V}(Y) + 2\text{Cov}(X, Y)$
- $\text{Cov}(X, X) = \mathbb{V}(X)$
- $\text{Cov}(X, Y) = \text{Cov}(Y, X)$ (symmetry)
- $\text{Cov}(X, aY + bZ) = a\,\text{Cov}(X, Y) + b\,\text{Cov}(X, Z)$ (bilinearity)
- $-1 \leq \rho_{X,Y} \leq +1$

A-5.6 Transfer Formula ('Law of the Unconscious Statistician')

- For a discrete variable: $\mathbb{E}\left(g(X)\right) = \sum\limits_{i} g\left(X(\omega_i)\right) p_i$. In particular: $\mathbb{E}\left(X^2\right) = \sum\limits_{i \in I} \left(X(\omega_i)\right)^2 p_i$.
- For a continuous variable: $\mathbb{E}\left(g(X)\right) = \int_{-\infty}^{+\infty} g\left(u\right) f_X\left(u\right) du$. In particular: $\mathbb{E}\left(X^2\right) = \int_{-\infty}^{+\infty} u^2 f_X\left(u\right) du$.

A-5.7 Chebyshev's Inequality. Central-Limit Theorem

- Chebyshev's inequality: For all $\varepsilon > 0$, $\mathbb{P}\left(|X - \mathbb{E}(X)| \geq \varepsilon\right) \leq \frac{\mathbb{V}(X)}{\varepsilon^2}$.
- Central-limit theorem: Let $(X_k)_{k \geq 1}$ be a sequence of independent, identically distributed random variables with mean μ and standard deviation σ. Then the cumulative distribution of the random variable $S_n = \frac{1}{\sigma\sqrt{n}} \sum\limits_{k=1}^{n} \left(X_k - \mu\right)$ converges to that of a standard normal as n goes to infinity.

A-6 Further Reading

Jean Jacod and Philip Protter (2004) *Probability Essentials* 2nd edition, Springer.

B
Calculus Review

B-1 Functions of Two Variables *x* and *y*

B-1.1 Continuity

- The **graph** of a function of two variables $f(x, y)$ is a three-dimensional surface $z = f(x, y)$ as illustrated in Figures B-1 and B-2 below.
- A function of two variables is said to be **continuous** at a point M_0 of the graph when, no matter how small, we can always find a sphere centered on M_0 which encloses a portion of the graph. Formally: for any $\varepsilon > 0$ there exists $\rho > 0$ such that for all (x, y) satisfying $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \rho$, we have: $|f(x, y) - f(x_0, y_0)| < \varepsilon$. Figure B-1 below is an example of a continuous function at the origin, and Figure B-2 below is a counter-example.

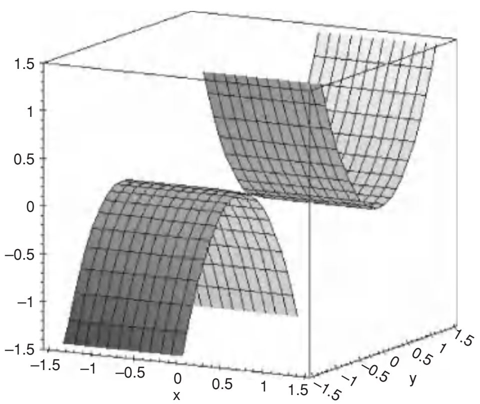


Figure B-1 Graph of $f(x, y) = \dfrac{xy^2}{|x|}$ for $x \neq 0$, $f(0, 0) = 0$

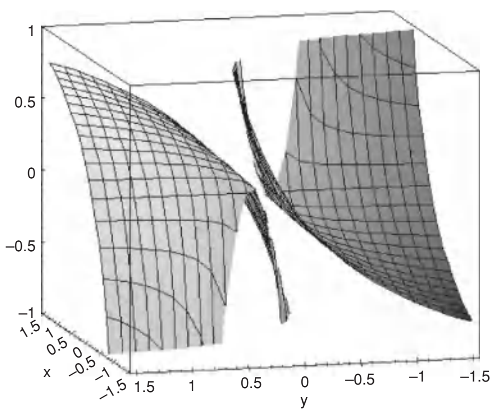


Figure B-2 Graph of $g(x, y) = \dfrac{xy}{x + y}$

B-1.2 Partial Derivatives

- The **partial derivative of *f* with respect to *x***, denoted $\dfrac{\partial f}{\partial x}$, is the function of two variables obtained by differentiating f with respect to x while holding y constant.
- For example, the partial derivatives of $f(x, y) = x + (xy)^2$ are: $\dfrac{\partial f}{\partial x}(x, y) = 1 + 2xy^2$, $\dfrac{\partial f}{\partial y}(x, y) = 2x^2y$.
- The partial derivatives of the partial derivatives are called **second-order partial derivatives** and denoted $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial x \partial y}$, $\dfrac{\partial^2 f}{\partial y \partial x}$, $\dfrac{\partial^2 f}{\partial y^2}$.

- **Schwarz's theorem**: If the partial derivatives all exist and are continuous (in the sense of Section B-1.1 above) then:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

B-2 Taylor Expansions

B-2.1 Taylor Expansions for a Function of a Single Variable *x*

- Let f be a continuously differentiable function of a single variable x, and x_0 a real number. The **first-order Taylor expansion** around x_0 is given as:

$$f(x_0 + h) \underset{h \to 0}{\overset{(1)}{\approx}} f(x_0) + hf'(x_0).$$

- For example, a first-order Taylor expansion of $f(x) = x^2$ around $x_0 = 1$ is: $f(1 + h) \underset{h \to 0}{\overset{(1)}{\approx}} 1 + 2h$.
- Let f be a twice continuously differentiable function of a single variable x, and x_0 a real number. The **second-order Taylor expansion** around x_0 is given as:

$$f(x_0 + h) \underset{h \to 0}{\overset{(2)}{\approx}} f(x_0) + hf'(x_0) + \frac{1}{2}h^2 f''(x_0).$$

- For example, a second-order Taylor expansion of $f(x) = \sqrt{x}$ around $x_0 = 1$ is:

$$f(1 + h) \underset{h \to 0}{\overset{(2)}{\approx}} 1 + \frac{1}{2}h - \frac{1}{8}h^2.$$

B-2.2 Taylor Expansions for a Function of Two Variables *x* and *y*

- Let f be a function of two variables x and y with continuous partial derivatives, and (x_0, y_0) a pair of real numbers. A **first-order Taylor expansion** around (x_0, y_0) is then given as:

$$f(x_0 + h, y_0 + k) \underset{(h,k) \to (0,0)}{\overset{(1)}{\approx}} f(x_0, y_0) + h\frac{\partial f}{\partial x}(x_0, y_0) + k\frac{\partial f}{\partial y}(x_0, y_0).$$

- For example, a first-order Taylor expansion of $f(x, y) = x + (xy)^2$ around $(1, 2)$ is:

$$f(1 + h, 2 + k) \underset{(h,k) \to (0,0)}{\overset{(1)}{\approx}} 5 + 9h + 4k.$$

- Denoting $dx = h$, $dy = k$, $df = f(x_0 + dx, y_0 + dy) - f(x_0, y_0)$, and omitting the dependence on (x_0, y_0), we have the **differential notation**:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

- Intuitively, this means that a small change in f decomposes as the sum of changes in x and y weighted by partial derivatives.

- Let f be a function of two variables x and y with continuous second-order partial derivatives, and (x_0, y_0) a pair of real numbers. A **second-order Taylor expansion** around (x_0, y_0) is given as:

$$f(x_0 + h, y_0 + k) \underset{(h,k)\to(0,0)}{\overset{(2)}{\approx}} f(x_0, y_0) + h\frac{\partial f}{\partial x}(x_0, y_0) + k\frac{\partial f}{\partial y}(x_0, y_0) + \frac{1}{2}\left[h^2\frac{\partial^2 f}{\partial x^2}(x_0, y_0) + 2hk\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) + k^2\frac{\partial^2 f}{\partial y^2}(x_0, y_0)\right]$$

- For example, a second-order Taylor expansion of $f(x, y) = x + (xy)^2$ around $(1, 2)$ is:

$$f(1 + h, 2 + k) \underset{(h,k)\to(0,0)}{\overset{(2)}{\approx}} 5 + 9h + 4k + 4h^2 + 8hk + k^2.$$

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P ubEu Sl s@Omen J o00h @nd i yo&LLe t endbmee i hs&0yed ubEu s r oyn B o&r a Son0 rnde

C

Finance Formulas

C-1 Rates and Yields

- Compounding of capital K at annual interest rate r over t years: $K_t = K(1 + r)^t$ (Section 1-2.2 p.5)
- Interest rate conversion formula: $\left(1 + r^{[\tau_1]}\right)^{\frac{1}{\tau_1}} = \left(1 + r^{[\tau_2]}\right)^{\frac{1}{\tau_2}}$ (Equation (1-1) p.6)
- Compounding of capital K at continuous interest rate $\widehat{r}$ over t years: $K_t = K e^{\widehat{r}t}$ (Section 9-1.2 p.96)
- Gross rate of return: $\text{ROR} = \frac{\text{Earnings}}{\text{Price}} = \frac{P_T - P_0 + I}{P_0}$ (Section 2-1.1 p.11)
- Time of return: $\text{TOR} = \frac{\text{Price}}{\text{Earnings}} = \frac{1}{\text{ROR}}$ (Section 2-1.2 p.11)
- Internal rate of return (IRR): solution r^* to $\text{NPV}(r) = 0$, i.e. $-C_0 + \sum_{i=1}^{n} \frac{F_i}{(1+r)^{t_i}} = 0$ (Section 2-3 p.13)
- Bond yield: solution y^* to $P = \frac{C_{t_1}}{(1+y)^{t_1}} + \frac{C_{t_2}}{(1+y)^{t_2}} + \cdots + \frac{N+C_T}{(1+y)^T}$ (Section 3-3.1 p.24)
- Zero-coupon rate for maturity T: $z(T) = \left(\frac{N}{P}\right)^{\frac{1}{T}} - 1$ (Section 3-4.1 p.27)
- Return of a portfolio of N assets: $R_{\text{P}} = \sum_{i=1}^{N} w_i R_i$ (Section 4-2.2 p.39)

C-2 Present Value. Arbitrage Price

- Present value of a future cash flow C at time T: $\text{PV} = \frac{C}{(1+r)^T}$ (Section 1-3.1 p.7)
- Present value of a cash flow table: $\text{PV} = \sum_{i=1}^{n} \frac{F_i}{(1+r)^{t_i}}$ (Section 2-2 p.12)
- Net present value: $\text{NPV} = -C_0 + \sum_{i=1}^{n} \frac{F_i}{(1+r)^{t_i}}$ (Section 2-2 p.12)
- Arbitrage price (mark-to-market value) of a portfolio of N assets: $P = \sum_{k=1}^{N} q_k p_k$ (Section 3-1.5 p.22 and Section 4-2.1 p.38)
- Arbitrage price of a bond: $P = \frac{C_{t_1}}{(1+z(t_1))^{t_1}} + \frac{C_{t_2}}{(1+z(t_2))^{t_2}} + \cdots + \frac{N+C_T}{(1+z(T))^T}$ (Section 3-4.2 p.28)
- Arbitrage price of a fixed income security: $P = \sum_{i=1}^{n} \frac{F_{t_i}}{(1+z(t_i))^{t_i}}$ (Section 3-4.2 p.28)

C-3 Forward Contracts

- Arbitrage price of a forward contract (Section 5-2 p.50):

	No dividend	Single cash dividend D paid at time t_D	Single proportional dividend at gross rate d paid at time t_d
At $t = 0$	$\phi_0 = S_0 - \frac{K}{(1+r)^T}$	$\phi_0 = S_0 - \frac{D}{(1+z(t_D))^{t_D}} - \frac{K}{(1+z(T))^T}$	$\phi_0 = \frac{S_0}{1+d} - \frac{K}{(1+r)^T}$
At $t \geq 0$	$\phi_t = S_t - \frac{K}{(1+r)^{T-t}}$	$\phi_t = S_t - \frac{D}{(1+z(t_D))^{t_D-t}} - \frac{K}{(1+z(T))^{T-t}}$	$\phi_0 = \frac{S_0}{1+d} - \frac{K}{(1+r)^{T-t}}$
	No dividend	**Continuous annual dividend yield $\widehat{q}$**	
At $t \geq 0$, continuous rates (Section 9-1.2 p.96)	$\phi_t = S_t - Ke^{-\widehat{r}(T-t)}$	$\phi_t = S_t e^{-\widehat{q}(T-t)} - Ke^{-\widehat{r}(T-t)}$	

Note: for successive dividends see Chapter 5, Problem 8 p.63.

- Forward price (Section 5-2 p.50):

	No dividend	Single cash dividend D paid at time t_D	Single proportional dividend at gross rate d paid at time t_d
At $t = 0$	$F_0 = S_0(1+r)^T$	$F_0 = \left[S_0 - \frac{D}{(1+z(t_D))^{t_D}}\right] \times (1+z(T))^T$	$F_0 = S_0 \frac{(1+r)^T}{1+d}$
At $t > 0$	$F_t = S_t(1+r)^{T-t}$	$F_t = \left[S_t - \frac{D}{(1+z(t_D))^{t_D-t}}\right] \times (1+z(T))^{T-t}$	$F_t = S_t \frac{(1+r)^{T-t}}{1+d}$
	No dividend	**Continuous annual dividend yield $\widehat{q}$**	
At $t > 0$, continuous rates (Section 9-1.2 p.96)	$F_t = S_t e^{\widehat{r}(T-t)}$	$F_t = S_t e^{(\widehat{r}-\widehat{q})(T-t)}$	

C-4 Options

- European call payoff: $c_T = \max(0, S_T - K)$ (Section 5-3.1 p.56)
- European put payoff: $p_T = \max(K - S_T, 0)$
- Binomial model: $\Delta = \frac{D_T^{(u)} - D_T^{(d)}}{S_T^{(u)} - S_T^{(d)}}$, $D_0 = \Delta S_0 + \frac{D_T^{(\bullet)} - \Delta S_T^{(\bullet)}}{(1+r)^T}$ (Section 6-1.2 p.67)

- Lognormal model: $D_0 = \mathbb{E}\left(\frac{D_T}{(1+r)^T}\right) = \mathbb{E}\left(\frac{f(T,S_T)}{(1+r)^T}\right)$ (Section 7-1 p.75)
- Closed-form formulas for European vanilla options (Section 7-2 p.76):

At $t = 0$

$$\begin{cases} c_0 = \dfrac{1}{(1+r)^T}\left[F_0 N(d_1) - K N(d_2)\right] \\ p_0 = \dfrac{1}{(1+r)^T}\left[K N(-d_2) - F_0 N(-d_1)\right] \end{cases} \qquad \begin{cases} d_1 = \dfrac{\ln \frac{F_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \\ d_2 = \dfrac{\ln \frac{F_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \end{cases}$$

At $t \geq 0$

$$\begin{cases} c_t = \dfrac{1}{(1+r)^{T-t}}\left[F(t,T) N(d_1) - K N(d_2)\right] \\ p_t = \dfrac{1}{(1+r)^{T-t}}\left[K N(-d_2) - F(t,T) N(-d_1)\right] \end{cases} \qquad \begin{cases} d_1 = \dfrac{\ln \frac{F(t,T)}{K} + \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{T-t}} \\ d_2 = \dfrac{\ln \frac{F(t,T)}{K} - \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{T-t}} \end{cases}$$

- Black-Scholes partial differential equation: $\widehat{r} f = \dfrac{\partial f}{\partial t} + \widehat{r} S_t \dfrac{\partial f}{\partial S} + \dfrac{1}{2}\sigma^2 S_t^2 \dfrac{\partial^2 f}{\partial S^2}$ (Section 10-1 p.109)
- Black-Scholes formulas for European vanilla options (Section 10-2 p.112):

$$\begin{cases} c_0 = S_0 N(d_1) - K e^{-\widehat{r}T} N(d_2) \\ p_0 = K e^{-\widehat{r}T} N(-d_2) - S_0 N(-d_1) \end{cases} \quad \text{with} \quad \begin{cases} d_1 = \dfrac{\ln \frac{S_0}{K} + \left(\widehat{r} + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \\ d_2 = d_1 - \sigma\sqrt{T} \end{cases}$$

(Note that in Black-Scholes the underlying asset S does not pay any dividend.)

C-5 Volatility

- Standard deviation of returns: $\sigma_{\text{periodic}} = \sqrt{\frac{1}{N-1}\sum_{t=1}^{N}(r_t - \bar{r})^2}$ (Section 4-1.1 p.35)
- Risk/Historical volatility/Realized volatility:
  $\sigma_{\text{annual}} = \sigma_{\text{periodic}} \times \sqrt{\text{Number of periods per year}}$ (Section 4-1.1 p.35; Section 10-3.1 p.113; Section 11-1.1 p.117)
- Sharpe ratio: Sharpe $= \dfrac{r_A - r_f}{\sigma_A}$ (Section 4-1.2 p.37)
- Two-asset portfolio volatility: $\sigma_P = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho}$ (Section 4-2.3 p.39)
- Three-asset portfolio volatility:

$$\sigma_P = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_1\sigma_2\rho_{1,2} + 2w_2w_3\sigma_2\sigma_3\rho_{2,3} + 2w_1w_3\sigma_1\sigma_3\rho_{1,3}}$$

- n-asset portfolio volatility:

$$\begin{cases} \sigma_P = \sqrt{\mathbb{V}(R_P)} \\ \mathbb{V}(R_P) = \mathbb{V}\left(\sum_{i=1}^{n} w_i R_i\right) = \sum_{i=1}^{n} w_i^2 \underbrace{\sigma_i^2}_{\mathbb{V}(R_i)} + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} w_i w_j \underbrace{\sigma_i \sigma_j \rho_{i,j}}_{\mathrm{Cov}(R_i, R_j)} \end{cases}$$

- Implied volatility: solution σ^* to $S_0 N(d_1(\sigma)) - Ke^{-\widehat{r}T} N(d_2(\sigma)) = A$ for a European call trading at A and $Ke^{-\widehat{r}T} N(-d_2(\sigma)) - S_0 N(-d_1(\sigma)) = B$ for a European put trading at B (Section 10-3.2 p.114 and Section 11-1.2 p.117).

C-6 Stochastic Processes. Stochastic Calculus

The formulas below are all from Chapter 9:

- Generalized Brownian motion: $dX_t = adt + bdW_t$, $X_t = X_0 + at + bW_t$.
- Geometric Brownian motion: $dX_t = aX_t dt + bX_t dW_t, \quad X_t = X_0 e^{\left(a - \frac{1}{2}b^2\right)t + bW_t}$.
- Ito process: $dX_t = a(t, X_t)dt + b(t, X_t)dW_t$.
- Ito-Doeblin theorem:

$$dY_t = df = \left(\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial X} + \frac{1}{2}b^2 \frac{\partial^2 f}{\partial X^2}\right)dt + \left(b\frac{\partial f}{\partial X}\right)dW_t$$

$$\begin{aligned} dY_t = df(t, X_t) = & \left[\frac{\partial f}{\partial t}(t, X_t) + a(t, X_t)\frac{\partial f}{\partial X}(t, X_t) + \frac{1}{2}b^2(t, X_t)\frac{\partial^2 f}{\partial X^2}(t, X_t)\right]dt \\ & + \left[b(t, X_t)\frac{\partial f}{\partial X}(t, X_t)\right]dW_t \end{aligned}$$

C-7 Greeks etc.

- Greek letters (Section 8-1.2 p.85):

δ **or** Δ **(delta)**	Γ **(gamma)**	Θ **(theta)**	$\mathscr{V}$ **(vega)**	ρ **(rho)**
$\frac{\partial f}{\partial S}$	$\frac{\partial^2 f}{\partial S^2} = \frac{\partial \Delta}{\partial S}$	$\frac{\partial f}{\partial t}$	$\frac{\partial f}{\partial \sigma}$	$\frac{\partial f}{\partial r}$

- Relationship between theta and gamma: $\Theta \approx -\frac{1}{2}\Gamma S^2 \sigma^2$ (Section 8-2.3 p.88)
- Option trading P&L proxy: $\mathrm{P\&L}_{\Delta t} \approx \frac{1}{2}\Gamma S^2 \left[\left(\frac{\Delta S}{S}\right)^2 - \sigma^2 \Delta t\right]$ (Section 8-2.3 p.88)

- Option Greeks at $t = 0$ in the lognormal model (annual interest rate r, no dividends):

δ or Δ (delta)	$\Delta_{\text{call}} = N(d_1), \;\; \Delta_{\text{put}} = -N(-d_1)$
Γ (gamma)	$\Gamma_{\text{call}} = \Gamma_{\text{put}} = \dfrac{N'(d_1)}{\sigma S_0 \sqrt{T}}$
Θ (theta)	$\begin{cases} \Theta_{\text{call}} = -\dfrac{\sigma S_0 N'(d_1)}{2\sqrt{T}} - \dfrac{K \ln(1+r)}{(1+r)^T} N(d_2) \\ \Theta_{\text{put}} = -\dfrac{\sigma S_0 N'(d_1)}{2\sqrt{T}} + \dfrac{K \ln(1+r)}{(1+r)^T} N(-d_2) \end{cases}$
$\mathscr{V}$ (vega)	$\mathscr{V}_{\text{call}} = \mathscr{V}_{\text{put}} = S_0 \sqrt{T} N'(d_1)$
ρ (rho)	$\begin{cases} \rho_{\text{call}} = \dfrac{KT}{(1+r)^{T+1}} N(d_2) \\ \rho_{\text{put}} = -\dfrac{KT}{(1+r)^{T+1}} N(-d_2) \end{cases}$

Note: formulas at an arbitrary time $t \geq 0$ are obtained by substituting $T - t$ for T.

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