Seminar Report

Penrose Incompleteness Theorem

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Introduction

Entities composing the nature constantly alter due to the processes of nature and their interactions with the other entities. The concept of conservation, an omnipresent phenomena, is therefore the key to understand these processes as it helps in the determination of parameters such as quantity of interacting material, its velocity, temperature, energy and so forth. This report deals with the nature of connservation laws and by means of mathematical modelling helps in qualitative understanding of these equations as propagating waves of disturbances through a medium. Specifically, this report is an endeavor to understand the challenges that arise when numerically solving Hyperbolic PDEs by throwing light on the discontunuities that arise in its solution space and brings forth key ideas and techniques to better combat the challenges in their computational implementations.

1 Double Null Hypersurfaces and Foliations

This is the first chapter

2 Null Structure Equations

This is the second chapter

3 Null Hypersurface Regularity

This is the third chapter

4 Causality for Spacetimes with Trapped Surfaces

Obviously! This is the fourth chapter

5 Penrose Incompleteness Theorem

5.1 Motivation

Theorem 5.1 (Penrose Incompleteness Theorem). *Let $(\mathscr{M}, g)$ be a globally hyperbolic time-orientable (Hausdorff) spacetime with a non-compact Cauchy Hypersurface $\mathscr{H}$ such that $\mathscr{M}$ contains a trapped surface S. If, in addition, $(\mathscr{M}, g)$ satisfies $Ric(L, L) \geq 0 \; \forall$ null vector fields L, then $\mathscr{M}$ is future geodesically incomplete. In fact, $\exists$ a null generator of $C \cup \underline{C}$, the future null geodesic congruences normal to S, that cannot be extended $\forall \tau \geq 0$ in $\mathscr{M}$.*

Corollary 5.1.1. *There are no trapped surfaces in Minkowski spacetime as it is geodesic complete.*

Definition 1 (Geodesic Complete). *A geodesic complete manifold implies that all its causal geodesics can be extended to arbitrary values of their affine parameters. Formally, a manifold $(\mathscr{M}, g)$ is geodesically complete if every timelike (or null) geodesic $\gamma : \mathbb{R} \supset (-\epsilon, \epsilon) \to \mathscr{M}$, such that $\gamma(0) = p$, $\forall p$ can be extended to $\tilde{\gamma} : \mathbb{R} \to \mathscr{M}$. That is, the timelike or null geodesics can be extended from a real interval to the entire real line.*

Proof to Corollary 5.1.1. In the cartesian coordinate frame for the spatial coordinates, the Minkowski metric, η, takes the form, $\eta = -dt^2 + dx^2 + dy^2 + dz^2$. The Christoffel symbols, $\Gamma^{\sigma}_{\alpha\beta}$, $\alpha, \beta, \sigma = \{0, 1, 2, 3\}$, for this coordinate choice are all identically zero. The geodesic equation for an affinely parameterized curve, $\gamma(\tau)$, with τ as the affine parameter, takes the form:

$$\frac{d^2 x^{\sigma}}{d\tau^2} + \Gamma^{\sigma}_{\alpha\beta} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau} = 0$$

, which reduces to $\frac{d^2 x^{\sigma}}{d\tau^2} = 0$. The general solution of such a differential equation is given by $x^{\mu}(\tau) = A^{\mu}\tau + B^{\mu}$, where the constants A^{μ} and B^{μ} can be determined based on the initial data. We see that $x^{\mu}(\tau)$ is linear in the affine parameter and can be extended to any arbitrary value of τ. Since this is true for any arbitrary curve in Minkowski spacetime, this completes our proof. $\square$