Seminar Report

Penrose Incompleteness Theorem

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Introduction

This is the placeholder for introduction.

1 Double Null Foliations and Hypersurfaces

This is the first chapter

2 Null Structure Equations

2.1 Connection Coefficients

The normalized frame (e_1, e_2, e_3, e_4) is chosen for simplicity of calculations. We now define (without proof) the connection coefficients with respect to this frame to be smooth function $\Gamma^{\lambda}{}_{\mu\nu}$. The connection coefficients, as we know, are defined as:

$$\nabla_{e_\mu} e_\nu = \Gamma^{\lambda}{}_{\mu\nu} e_\lambda, \quad \lambda, \mu, \nu \in \{1,2,3,4\}.$$

Here, ∇ denotes the connection of the spacetime metric g. Since we are studying the geometry of the null hypersurfaces (that are generated by the null geodesics, L and $\underline{L}$), we want at least one of the indices λ, μ, ν is either 3 or 4 (otherwise, we obtain Christoffel symbols with respect to the induced metric $g\!\!\!/$). Following are the connection coefficients relevant to the dicussion that follows:

- $\chi_{AB} = g(\nabla_A e_4, e_B), \qquad \underline{\chi}_{AB} = g(\nabla_A e_3, e_B)$ (1)
- $\eta_A = g(\nabla_3 e_4, e_A), \qquad \underline{\eta}_A = g(\nabla_4 e_3, e_A)$ (2)
- $\omega = -g(\nabla_4 e_4, e_3), \qquad \underline{\omega} = -g(\nabla_3 e_3, e_4)$ (3)
- $\zeta_A = g(\nabla_A e_4, e_3), \qquad \underline{\zeta}_A = -\zeta_A$ (4)

Here, $A, B \in \{1,2\}$ and $\nabla_\mu = \nabla_{e_\mu}$. These covariant tensor fields are defined only on $T_x S_{u,v}$. However, these can be extended to $T_x \mathscr{M}$ by letting their values to zero if they act on e_3 or e_4. Such tensor fields are called **S-tensor fields**. A vector is an S-tensor field if it is a tangent to the spheres

$S_{u,\underline{u}}$.

Important Remarks:

(i) $\chi, \underline{\chi}$ are referred to as the **second null fundamental forms** of $S_{u,\underline{u}}$ with respect to null hypersurfaces C_u, $\underline{C}_u$, and are symmetric $(0,2)$ S-tensor fields. They can be decomposed into their trace and traceless parts as follows:

$$\chi = \hat{\chi} + \frac{1}{2}(\mathrm{tr}\chi)\not{g}, \quad \text{and} \quad \underline{\chi} = \hat{\underline{\chi}} + \frac{1}{2}(\mathrm{tr}\underline{\chi})\not{g}. \tag{5}$$

The trace of the S-tensor fields $\chi, \underline{\chi}$ is taken with respect to the induced metric $\not{g}$. The $\mathrm{tr}\chi$ is known as *expansion* and the component $\hat{\chi}$ is called the *shear* of $S_{u,\underline{u}}$ with respect to C_u.

(ii) $\omega = \nabla_4(\log\Omega)$, and $\underline{\omega} = \nabla_3(\log\Omega)$.

(iii) Let X be a vector tangential to $S_{u,\underline{u}}$ at a point x. Then, extending X along the null generator γ of C_u passing through x according to the Jacobi equation $[L, X] = [\Omega e_4, X] = 0$, then we obtain an S-tangent vector field along γ. In this case,

$$\nabla_4 X = \nabla_X e_4 + (\nabla_X \log\Omega)\, e_4.$$

However, extending X such that $[e_4, X] = 0$ keeps X tangential to C_u but it is no more tangential to the section $S_{u,\underline{u}}$ of C_u. This is because the sections in the optical functions $u, \underline{u}$ are the affine parameters of the vector fields $L, \underline{L}$ respectively.

(iv) The 1-form ζ is known as the *torsion*. The 1-form $\eta, \underline{\eta}$ is defined using ζ and the exterior derivative $\not{d}$ on $S_{u,\underline{u}}$ via

$$\eta = \zeta + \not{d}(\log\Omega), \quad \text{and} \quad \underline{\eta} = -\zeta + \not{d}(\log\Omega).$$

2.2 Curvature Components

The only components of the Riemann curvature R relevant to our discussion are:

- $\alpha_{\mathbf{AB}} = \mathbf{R}_{\mathbf{A4B4}}$, $\quad \underline{\alpha}_{\mathbf{AB}} = \mathbf{R}_{\mathbf{A3B3}}$

where, $R_{\mu\nu\rho\eta} \equiv R\left(e_\mu, e_\nu, e_\rho, e_\eta\right) = \left\langle R\left(e_\mu, e_\nu\right) e_\rho,\, e_\eta\right\rangle$.

2.3 Null Structure Equations

To understand the geometry $\not g$ of the spheres $S_{u,\underline{u}}$ and how they embedded on the null hypersurfaces. To do this, we derive connection coefficient (1) — (4) with respect to the coordinates specific to the null generators, i.e., e_3 and e_4. This yields the following equations:

(i) The first variational formulas:

$$\not{\mathcal{L}}_4 \not g = 2\chi, \tag{6}$$

$$\not{\mathcal{L}}_3 \not g = 2\underline{\chi}. \tag{7}$$

(ii) The second variational formulas:

$$\not{\mathcal{L}}_4 \chi = -\chi \times \chi - \alpha + \omega\chi \tag{8}$$

$$\not{\mathcal{L}}_3 \underline{\chi} = -\underline{\chi} \times \underline{\chi} - \underline{\alpha} + \underline{\omega\chi} \tag{9}$$

(iii) Raychaudhari Equations: In view of the decomposition of $\chi, \underline{\chi}$ from equations (5) and equations (8) — (9), we have:

$$\not\nabla_4 \mathrm{tr}\chi = -|\chi|^2 + \omega \mathrm{tr}\chi \tag{10}$$

$$\not\nabla_3 \mathrm{tr}\underline{\chi} = -|\underline{\chi}|^2 + \underline{\omega} \mathrm{tr}\underline{\chi} \tag{11}$$

and,

$$\not\nabla_4 \hat{\chi} = \omega\hat{\chi} - (\mathrm{tr}\chi)\hat{\chi} - \alpha \tag{12}$$

$$\not\nabla_3 \hat{\underline{\chi}} = \underline{\omega}\hat{\underline{\chi}} - (\mathrm{tr}\underline{\chi})\hat{\underline{\chi}} - \underline{\alpha} \tag{13}$$

where we used the fact $[\not\nabla_3, \mathrm{tr}] = [\not\nabla_4, \mathrm{tr}] = 0$ to arrive at the former two equations, and for the latter two we used that

$$|\chi|^2 = \frac{1}{2}(\mathrm{tr}\chi)^2 + |\hat{\chi}|^2, \quad \text{and} \quad \hat{\chi} \times \hat{\chi} = \frac{1}{2}|\hat{\chi}|^2 \not g$$

The equations (10) — (11) are known as *Raychaudhari equations*.

3 Regularity of Null Hypersurfaces

For further discussion, we restrict ourselves to one hypersurface C in $(\mathscr{M}, g)$, hence $\Omega = 1$.

$$\omega = \nabla_4 \log \Omega = 0 \quad \Longrightarrow \quad \eta = -\underline{\eta} = \zeta$$

In this case, the geodesic, the normalized, and the equivariant coordinates all coincide on C. Therefore, L is tangential to C and $\nabla_L L = 0$ holds. Let S_τ be the level set of the affine parameter of L on C such that $S_\tau = \mathscr{F}_\tau(S)$. Additionally, $\mathscr{S} = \cup_\tau S_\tau$ is the affine foliation of C. In other words, S_τ are sections of C.

3.1 Jacobi Fields and Tidal Forces

Definition 3.1. *A vector field X on C is called a normal Jacobi field if $\mathscr{L}_L X = 0$. This implies, $[L, X] = 0$.*

The definition above implies X is tangent to the section S_τ of the affine foliation of C. The reason X is so called because it represents the infinitesimal displacement of the null generators around a fixed null generator. Mathematically, this can be expressed as:
If γ is a curve on S, such that: such that

$$\gamma : (-\varepsilon, \varepsilon) \to S_\tau \quad \gamma(0) = x\,, \quad \dot{\gamma}(0) = E_i(\tau)\,.$$

where, $E_i(\tau)$, $i = \{1,2\}$ represent an orthonormal frame which is tangent to the section S_τ. It follows that for a fixed point (on a section for a chosen value of τ), E_i represents an orthonormal basis for X.Then, we have a mapping:

$$\Phi : (-\varepsilon, \varepsilon) \times [0, \infty) \to C\,, \quad (s, \tau) \mapsto G_{\gamma(s)}(\tau)\,.$$

Here, $G_{\gamma(s)}(\tilde{\tau})$ represent the displacement field of $G_x(\tau)$ along null generators. Thus,

$$(E_i)_{G_x(\tau)} = d\mathscr{F}_\tau\big((E_i)_x\big)\,.$$

For a fixed $\tau = \tilde{\tau}$, we have $\Phi(s, \tilde{\tau}) = \alpha(s) = G_{\gamma(s)}(\tilde{\tau})$, which represent the displacement field of $G_x(\tau)$ along null generators, such that:

$$\left.\frac{d}{ds}\right|_{s=0} G_{\gamma_i(s)}(\tilde{\tau}) = E_i\,.$$

Fermi Frame

We consider an orthonormal frame, $E_i(0)$ of $T_x S_0$. To understand how these frames evolve as we move along a null generator G_x such that at each point $G_x(\tau)$ we obtain an orthonormal frame $T_{G_x(\tau)} S_\tau$. One way to do this is by considering a *Fermi frame*, constructed as:

$$\nabla_L E_i = -\zeta_i L, \quad \text{where } \zeta_i = \zeta(E_i).$$

This implies, $L(g(E_i, K)) = 0$ where $K \in \{E_j, L, \underline{L}\}$, $j = 1, 2$. Such a construction ensures that if $E_i(0)$, $i = 1, 2$ is an orthonormal frame of $T_x S_0$ then $E_i(\tau)$, $i = 1, 2$, is an orthonormal frame of $T_{G_x(\tau)} S_\tau$.

Propagation equation of Jacobi Fields

Starting from the fact that $E_i(\tau)$, $i = 1, 2$ is an orthonormal basis, we can express X as:

$$X = X^i(\tau) \cdot E_i(\tau). \tag{14}$$

Starting by computing $\nabla_L X$ and using $g\left(\nabla_L X, E_j\right)$ in the Fermi frame, we find X^i satisfy the following 1st order linear system of equations:

$$\frac{dX^i(\tau)}{d\tau} = \chi_{ij} X^j \tag{15}$$

However, the connection coefficients χ_{ij} depend on the null hypersurface regularity and not on the regularity of the ambient metric g. Therefore, we compute the second order equations to find propagation equations for X^i which only depend on the Riemannian curvature of g. Starting from the Jacobi equation given by

$$J_L X = \nabla_L \nabla_L X - R(L, X) L = 0, \tag{16}$$

where, J_L is the tidal operator, we find $g\left(\nabla_L \nabla_L X,\, E_j\right) = \alpha_{ij} X^i$. Here, α is the curvature component introduced insert section number. Upon taking the covariant derivative of $\nabla_L X$ and of equation (15) with respect to L and equation, we get the following desired 2nd order propagation equation of X^i:

$$\frac{d^2 X^i(\tau)}{d\tau^2} = \alpha_{ij} X^j. \tag{17}$$

The right hand side of equation (17) tells as long as the spacetime metric g remains smooth, the components of X^i of the normal Jacobi vector field satisfy linear second order system of equations. Assuming a well-defined initial data $X^i(0)$, $\exists$ smooth curve of matrices $M(\tau)$ such that the solution of $X^i(\tau)$ take the form:

$$X^i(\tau) = M^i_j(\tau) X^j(0). \tag{18}$$

For $\tau = 0$, $M(0) = I$, hence $\det M(0) = 1$. The matrix M measures the rate of change of the normal Jacobi field X along G_x with respect to its initial value $X(0) = X_x$. Thus, M is also referred to as the *deformation matrix*.

The propagation equation for M, obtained from equation (15), comes out to be:

$$\frac{dM(\tau)}{d\tau} = \chi \times M. \tag{19}$$

3.2 Focal Points

The deformation matrix M is invertible as long as the flow $\mathscr{F}_\tau$ of L induces isomorphisms between $T_x S_0$ and $T_{G_x(\tau)} S_\tau$. Nevertheless, the null generators nearby G_x may converge to G_x such that $X = 0$ at some point $G_x\left(\tau_f\right)$ along G_x. Points such as $G_x\left(\tau_f\right)$[††] are called *focal point* of C. For these focal points, the regularity of null hypersurfaces break down and the extension of geodesics along these null generators beyond the focal points is not possible.

To be able to mathematically interpret this, we express the vanishing of normal Jacobi field X in terms of deformation matrix M. From equation (19), $\forall \tau \in [0, \tau_f)$, $\tau \neq \tau_f$, we can write:

$$\chi = \frac{dM}{d\tau} \times M^{-1}.$$

Computing the derivative of $\det M$ with respect to τ using Jacobi formula for derivative of determinant, we get[¶]:

$$\mathrm{tr}\chi = \frac{1}{\det M} \frac{d \det M}{d\tau}. \tag{20}$$

[††] Assume τ_f to be the first time for which we have a focal point along G_x.

[¶] Recall, that χ is the second null fundamental form on S_τ and is calculated with respect to the induced metric $\not{g}$. $\mathrm{tr}\chi$ with respect to the Fermi frame is simply the identity matrix.

Since the null generators near G_x get closer to G_x along the flow $\mathscr{F}_\tau$, we can say as $\tau \to \tau_f$:

$$\frac{d\det M}{d\tau} \leq 0, \quad \text{and} \quad \det M \to 0^+.$$

Therefore,

$$\mathrm{tr}\chi = -\infty. \tag{21}$$

Equation (21) shows the geometry of the null hypersurface C breaks down at the focal points.

4 Causality for Spacetimes with Trapped Surfaces

Recall the geometric construction in the previous sections. If S is a closed 2-dimensional surface in a globally hyperbolic time-orientable spacetime $(\mathcal{M}, g)$ and C and $\underline{C}$ be future outgoing and incoming null geodesic congruence normal to S respectively. Then:

$$\partial \mathcal{J}^+(S) \subset C \cup \underline{C}. \tag{22}$$

We always have:

$$C \cup \underline{C} \subset \mathcal{J}^+(S). \tag{23}$$

However, it is not always the case that:

$$C \cup \underline{C} \subset \partial \mathcal{J}^+(S). \tag{24}$$

The reason for this is in case $C \cup \underline{C}$ lies in the interior of the future of S, i.e., $\mathcal{J}^+(S)$ implies $\exists$ timelike curves connecting points on $C \cup \underline{C}$ to S and also to a neighborhood of S which is interior of $\mathcal{J}^+(S)$.

Insert connecting text here

Proposition 4.1. *Let S be a closed 2-dimensional manifold surface in a globally hyperbolic time oriented spacetime $(\mathcal{M}, g)$. Let C and $\underline{C}$ denote the (future) outgoing and incoming null geodesic congruence normal to S. Let C^* and $\underline{C}^*$ be parts of C and $\underline{C}$ that do not contain any focal point. Then, $\partial \mathcal{J}^+(S) \subset C^* \cup \underline{C}^*$.*

4.1 Trapped Surface

Assume $\Omega = 1$ on $C \cup \underline{C}$. Consider L and $\underline{L}$ to be the geodesic vector field of C and $\underline{C}$ with τ and $\underline{\tau}$ the respective affine parameters such that $S \equiv \{\tau = 0\} = \{\underline{\tau} = 0\} \equiv S_0$.

Relationsihp between the area of sections S_τ and the second fundamental forms χ and $\underline{\chi}$:

The area relates, as we shall see later, the $\det g$ to $\mathrm{tr}\chi$ and helps deduce the regularity of the null hypersurfaces. We present the solutions only corresponding to the hypersurface C. The computations are same for $\underline{C}$.
We assume the canonical coordinates $(\tau, \theta^1, \theta^2)$ on C, where $(\theta^1, \theta^2) \in \mathcal{U} \subset \mathbb{R}^2$ and g is the induced metric on the sections S_τ of C.

In the following, we use the first variational formula and the Jacobi formula for the derivative of a matrix determinant:

$$\begin{aligned}\nabla_L\left(\sqrt{\det g}\right) &= \frac{1}{2\sqrt{\det g}}\nabla_L(\det g) = \frac{1}{2\sqrt{\det g}}(\det g)\mathrm{tr}\left(\left(g^{-1}\right)^{AB}\nabla_L g_{AB}\right) \\ &= \frac{\sqrt{\det g}}{2}\mathrm{tr}\left(\left(g^{-1}\right)^{AB}\mathscr{L}_L g_{AB}\right) = \frac{\sqrt{\det g}}{2}\mathrm{tr}\left(\left(g^{-1}\right)^{AB}2\chi_{AB}\right) \\ &= \sqrt{\det g}\ \mathrm{tr}(\chi)\end{aligned}$$

We know,

$$\mathrm{Area}(S_\tau) = \int_{\mathcal{U}} \sqrt{\det g(\tau)}\, d\theta^1 d\theta^2$$

Thus,

$$\nabla_L(\mathrm{Area}(S_\tau)) = \int_{\mathcal{U}} \mathrm{tr}\chi\, d\mu_g.$$

More generally,

$$\nabla_{fL}(\mathrm{Area}(S_\tau)) = \int_{\mathcal{U}} f\cdot \mathrm{tr}\chi\, d\mu_g, \quad \forall f \geq 0 \text{ and } f \in C^\infty(S_\tau). \tag{25}$$

Interpretation: The equations above represent the rate of change of the second fundamental form χ and the rate of change of the area of S_τ under infinitesimal displacement along the null generators. Therefore, $\mathrm{tr}\chi$ is also called expansion of S_τ.

Definition 4.1 (Trapped Surfaces). *A 2-dimensional surface S in $(\mathscr{M}, g)$ for which the area decreases under infinitesimal (arbitrary) displacements along the null generators of both null geodesics congruences $C \cup \underline{C}$ normal to S.*

If $(\mathscr{M}, g)$ is globally hyperbolic with a trapped surface, implies that $C \cup C$ bounds the future of S, i.e., $\partial \mathscr{J}^{+}(S) \subset C \cup \underline{C}$. Hence, it cannot expand in its future. Thus the term *trapped*. It is for this reason (24) does not always hold true.

In view of this definition and (25), for a trapped surface, the following statements holds:

$$\int_{\mathscr{U}} f \cdot \mathrm{tr}\chi \, d\mu_{g} < 0, \quad \int_{\mathscr{U}} f \cdot \mathrm{tr}\underline{\chi} \, d\mu_{g} < 0 \quad \forall f \geq 0 \text{ and } f \in C^{\infty}(S_{\tau})$$

Equivalent definition for Trapped Surfaces: A trapped surface is a closed 2-dimensional surface S in a Lorentzian manifold $(\mathscr{M}, g)$ such that:

$$\mathrm{tr}\chi < 0, \quad \mathrm{tr}\underline{\chi} < 0$$

Trapped surfaces and focal points:

A null generators on an incoming null hypersurface $\underline{C}$ contain focal points. The null expansion of an incoming null hypersurface, in our case $\underline{C}$ is negative. The following proposition shows that these two properties are related:

Proposition 4.2. *Assume S to be a closed two-dimensional surface (not necessarily trapped) in a Lorentzian manifold $(\mathscr{M}, g)$ which satisfies the Einstein equation $\mathrm{Ric}(g) = 0$. If $\mathrm{tr}\chi < 0$ at some point $x \in S$, then $\exists$ a focal point on the null generator G_x of C emanating from point x. A similar result holds for $\underline{C}$.*

Proof to Proposition 4.2. Using Raychaudhari equation[¶],

$$\nabla_4(\mathrm{tr}\chi) = -|\chi|^2 + \omega \mathrm{tr}\chi$$

and the other equation:

$$|\chi|^2 = \frac{1}{2}(\mathrm{tr}\chi)^2 + |\hat{\chi}|^2$$

[¶] For our case, the null lapse function, $\Omega = 1$, hence $\omega = 0$.

we get, the following Riccati-type equation[§]:

$$\nabla_L(\mathrm{tr}\chi) = -\frac{1}{2}(\mathrm{tr}\chi)^2 - |\hat{\chi}|^2 \leq 0 \tag{26}$$

If $T = \mathrm{tr}\chi_x = \mathrm{tr}\chi(0) < 0$, then $\mathrm{tr}\chi(\tau) < 0, \quad \forall \tau \geq 0$.
Ignoring the term $|\hat{\chi}|$, yields:

$$\nabla_L\left(-\frac{1}{\mathrm{tr}\chi}\right) \leq -\frac{1}{2}$$

Thus,

$$\Rightarrow \nabla_L\left(-\frac{1}{\mathrm{tr}\chi}\right) \leq -\frac{1}{2}$$

$$\Rightarrow -\frac{1}{\mathrm{tr}\chi} \leq -\frac{1}{T} - \frac{\tau}{2}$$

$$\Rightarrow \tau_* = \frac{2}{-T} = \frac{2}{-\mathrm{tr}\chi_x}$$

We obtain $\mathrm{tr}\chi(\tau_*) = -\infty$ and hence $G_x(\tau_*)$ is the first focal point on G_x. □

Remarks: The equality in equation (26) holds for $\mathrm{Ric}(L, L) = \mathrm{tr}\,\alpha = 0$. From second variational formula insert equation no., we see that Riccati-type equation similar to equation26 can be obtained for a more relaxed condition, $\mathrm{tr}\alpha = \mathrm{Ric}(L, L) \geq 0$. This is also called as the *positive null energy condition*, which is weaker than the Einstein equations. Hence, the proposition 4.2 also holds for the positive null energy condition.

5 Penrose Incompleteness Theorem

5.1 Motivation

Theorem 5.1 (Penrose Incompleteness Theorem). *Let $(\mathcal{M}, g)$ be a globally hyperbolic time-orientable (Hausdorff) spacetime with a non-compact Cauchy Hypersurface $\mathcal{H}$ such that $\mathcal{M}$ contains a trapped surface S. If, in addition, $(\mathcal{M}, g)$ satisfies $\mathrm{Ric}(L, L) \geq 0\ \forall$ null vector fields L, then $\mathcal{M}$ is future geodesically incomplete. In fact, $\exists$ a null generator of $C \cup \underline{C}$, the future null geodesic congruences normal to S, that cannot be extended $\forall \tau \geq 0$ in $\mathcal{M}$.*

[§] Riccati-type equations have a finite-time blow-up.

Corollary 5.1.1. *There are no trapped surfaces in Minkowski spacetime as it is geodesic complete.*

Definition 5.1 (Geodesic Complete). *A geodesic complete manifold implies that all its causal geodesics can be extended to arbitrary values of their affine parameters. Formally, a manifold $(\mathscr{M}, g)$ is geodesically complete if every timelike (or null) geodesic $\gamma : \mathbb{R} \supset (-\varepsilon, \varepsilon) \to \mathscr{M}$, such that $\gamma(0) = p$, $\forall p$ can be extended to $\tilde{\gamma} : \mathbb{R} \to \mathscr{M}$. That is, the timelike or null geodesics can be extended from a real interval to the entire real line.*

Proof to Corollary 5.1.1. In the cartesian coordinate frame for the spatial coordinates, the Minkowski metric, η, takes the form, $\eta = -dt^2 + dx^2 + dy^2 + dz^2$. The Christoffel symbols, $\Gamma^{\sigma}_{\alpha\beta}$, $\alpha, \beta, \sigma = \{0, 1, 2, 3\}$, for this coordinate choice are all identically zero. The geodesic equation for an affinely parameterized curve, $\gamma(\tau)$, with τ as the affine parameter, takes the form:

$$\frac{d^2 x^{\sigma}}{d\tau^2} + \Gamma^{\sigma}_{\alpha\beta} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau} = 0$$

, which reduces to $\frac{d^2 x^{\sigma}}{d\tau^2} = 0$. The general solution of such a differential equation is given by $x^{\mu}(\tau) = A^{\mu}\tau + B^{\mu}$, where the constants A^{μ} and B^{μ} can be determined based on the initial data. We see that $x^{\mu}(\tau)$ is linear in the affine parameter and can be extended to any arbitrary value of τ. Since this is true for any arbitrary curve in Minkowski spacetime, this completes our proof. $\square$

The proof of the Penrose Incompleteness theorem relies on two important theorems/results from differential topology. These are presented below and proof to only the second one is provided:

Ingredient theorem (i) A bijective continuous map from compact spaces to a Hausdorff space is a homeomorphism.

Ingredient theorem (ii) Let $\mathscr{N} \subset \mathscr{M}$ be an (injective) immersed topological submanifold with $\dim(\mathscr{N}) = \dim(\mathscr{M}) = n$ such that $\mathscr{N}$ is compact and $\mathscr{M}$ is a Hausdorff connected non-compact topological manifold. Then $\partial_{\text{mani}}\mathscr{N} \neq \varnothing$, where $\partial_{\text{mani}}\mathscr{N}$ denotes the boundary of $\mathscr{N}$ in the sense of (topological) manifolds.

*Proof to **Ingredient theorem (ii)** (By contradiction).* Assume $\partial_{\text{mani}}\mathscr{N} = \varnothing$
$\implies \forall x \in \mathscr{N}, \exists \delta > 0$, such that $\mathrm{B}_{\delta}(x)$ homeomorphic to $\mathbb{R}^n$.
Since, $\dim(\mathscr{M}) = \dim(\mathscr{N})$, $\mathrm{B}_{\delta}(x)$ is also open in $\mathscr{M}$. By compactness, we can say that $\mathscr{N} \subset \mathscr{M}$

is open. However, $\mathcal{N}$ is compact and is a subset of a Hausdorff space, hence, $\mathcal{N}$ is closed. By connectedness, $\mathcal{M} = \mathcal{N}$, which is a contradiction. $\square$

We move on to the proof of the Penrose Incompleteness theorem.

Proof to the Penrose Incompleteness Theorem 5.1. From the results of Section - look into the previous section, we know that if $\mathrm{tr}\chi_x = -k_x < 0$, $x \in S = \{\tau = 0\}$, then the first focal point of the generator $G_x \subset C$ appears at time $\tau = \frac{2}{k_x}$. In view of compactness of S[†], we have:

$$\sup_S \mathrm{tr}\chi = k_C < 0\,, \qquad \sup_S \mathrm{tr}\underline{\chi} = k_{\underline{C}} < 0\,, \qquad \sup\{k_C, k_{\underline{C}}\} = k < 0 \tag{27}$$

We assume $(\mathcal{M}, g)$ to be future null geodesically complete. We define $\mathcal{V}$ to be the union of all the null generators of $C \cup \underline{C}$ for which $0 \le \tau \le 2/k$, i.e.,

$$\mathcal{V} = \bigcup_{\substack{\tau \in \left[0, \frac{2}{k}\right], \\ x \in S}} \left(G_x(\tau),\ \underline{G}_x(\tau)\right) \tag{28}$$

(Check this bit) Based on this construction, $\mathcal{V} \subset \mathcal{M}$. The null generators, G and $\underline{G}$, can be viewed as the following continuous maps:

$$G : S \times \left[0, \frac{2}{k}\right] \to C\,, \qquad (x, \tau) \mapsto G_x(\tau)$$
$$\underline{G} : S \times \left[0, \frac{2}{k}\right] \to \underline{C}\,, \qquad (x, \tau) \mapsto \underline{G}_x(\tau)$$

Continuity implies that the null generators are mappings between compact sets. This implies that $\mathcal{V}$ is itself compact.

The trace condition implies every null generator in $\mathcal{V}$ contains at least one focal point. Therefore, from the previous section on causality, we can say that:

$$\partial \mathcal{J}^+(S) \subseteq \mathcal{V} \subset \mathcal{M}. \tag{29}$$

The topological boundary $\partial \mathcal{J}^+(S)$ is closed by definition[‡]. Since $\mathcal{V}$ is compact, by 29, we conclude $\partial \mathcal{J}^+(S)$ is compact[**].

We now show that global topological argument leads to a contradiction, and hence $(\mathcal{M}, g)$ cannot be future null geodesically complete.

[†] A continuous function on a compact set is bounded and attains its maximum

[‡] Topological boundary of subset A of a topological space X, ∂A, is given by $\partial A = \overline{A} \cap \overline{X \backslash A}$.

[**] Closed subset of a compact set is itself compact.

Since $\mathscr{M}$ is time-orientable, $\exists$ global timelike vector field T whose integral curves are timelike foliate of $\mathscr{M}$ and intersect the Cauchy Hypersurface exactly once. Futhermore, the integral curves intersect $\partial\mathscr{J}^+(S)$ exactly once, since $\mathscr{J}^+(S)$ is future set and the topological boundary of a future set is a closed achronal three-dimensional Lipschitz submanifold without boundary[‡‡] where ∂_{mani} denotes the boundary in the sense of (topological) manifolds. Projection of $\partial\mathscr{J}^+(S)$ on $\mathscr{H}$ via the integral curve T is a continuous injective mapping from $\partial\mathscr{J}^+(S)$ onto a subset of $\mathcal{T} \subset \mathscr{H}$.

From **Ingredient theorem (i)**, we can say that $\partial\mathscr{J}^+(S)$ is homeomorphic to $\mathcal{T} \subset \mathscr{H}$. This implies that $\mathcal{T}$ is a Lipschitz three-dimensional compact submanifold without a boundary, i.e., $\partial_{\text{mani}}\mathcal{T} \neq \varnothing$, in the compact three-dimensional manifold $\mathscr{H}$. However, from **Ingredient theorem (ii)**, $\mathcal{T}$ must have a non-empty boundary. This is a contradiction. Thus, our assumption that $(\mathscr{M}, g)$ is future null geodesically complete is false. $\square$

[‡‡]**Proposition:** Let $\mathscr{F}$ be a future set in a Lorentzian manifold $\mathscr{M}$. Then the topological boundary $\partial\mathscr{F}$ is closed achronal three-dimensional locally Lipschitz submanifold of $\mathscr{M}$ such that $\partial_{\text{mani}}\partial\mathscr{F} \neq \varnothing$