*Seminar Report*

# Penrose Incompleteness Theorem

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# Introduction

Entities composing the nature constantly alter due to the processes of nature and their interactions with the other entities. The concept of conservation, an omnipresent phenomena, is therefore the key to understand these processes as it helps in the determination of parameters such as quantity of interacting material, its velocity, temperature, energy and so forth. This report deals with the nature of connservation laws and by means of mathematical modelling helps in qualitative understanding of these equations as propagating waves of disturbances through a medium. Specifically, this report is an endeavor to understand the challenges that arise when numerically solving Hyperbolic PDEs by throwing light on the discontunuities that arise in its solution space and brings forth key ideas and techniques to better combat the challenges in their computational implementations.

# 1 Double Null Hypersurfaces and Foliations

This is the first chapter

# 2 Null Structure Equations

This is the second chapter

# 3 Null Hypersurface Regularity

This is the third chapter

# 4 Causality for Spacetimes with Trapped Surfaces

Obviously! This is the fourth chapter

# 5 Penrose Incompleteness Theorem

## 5.1 Motivation

**Theorem 5.1 (Penrose Incompleteness Theorem).** *Let $(\mathscr{M}, g)$ be a globally hyperbolic time-orientable (Hausdorff) spacetime with a non-compact Cauchy Hypersurface $\mathscr{H}$ such that $\mathscr{M}$ contains a trapped surface S. If, in addition, $(\mathscr{M}, g)$ satisfies $Ric(L,L) \geq 0\ \forall$ null vector fields L, then $\mathscr{M}$ is future geodesically incomplete. In fact, $\exists$ a null generator of $C \cup \underline{C}$, the future null geodesic congruences normal to S, that cannot be extended $\forall \tau \geq 0$ in $\mathscr{M}$.*

**Corollary 5.1.1.** *There are no trapped surfaces in Minkowski spacetime as it is geodesic complete.*

**Definition 1 (Geodesic Complete).** *A geodesic complete manifold implies that all its causal geodesics can be extended to arbitrary values of their affine parameters. Formally, a manifold $(\mathscr{M}, g)$ is geodesically complete if every timelike (or null) geodesic $\gamma : \mathbb{R} \supset (-\varepsilon, \varepsilon) \to \mathscr{M}$, such that $\gamma(0) = p$, $\forall p$ can be extended to $\widetilde{\gamma} : \mathbb{R} \to \mathscr{M}$. That is, the timelike or null geodesics can be extended from a real interval to the entire real line.*

*Proof to Corollary 5.1.1.* In the cartesian coordinate frame for the spatial coordinates, the Minkowski metric, $\eta$, takes the form, $\eta = -dt^2 + dx^2 + dy^2 + dz^2$. The Christoffel symbols, $\Gamma^{\sigma}_{\alpha\beta}$, $\alpha, \beta, \sigma = \{0, 1, 2, 3\}$, for this coordinate choice are all identically zero. The geodesic equation for an affinely parameterized curve, $\gamma(\tau)$, with $\tau$ as the affine parameter, takes the form:

$$\frac{d^2 x^{\sigma}}{d\tau^2} + \Gamma^{\sigma}_{\alpha\beta} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau} = 0$$

, which reduces to $\frac{d^2 x^{\sigma}}{d\tau^2} = 0$. The general solution of such a differential equation is given by $x^{\mu}(\tau) = A^{\mu}\tau + B^{\mu}$, where the constants $A^{\mu}$ and $B^{\mu}$ can be determined based on the initial data. We see that $x^{\mu}(\tau)$ is linear in the affine parameter and can be extended to any arbitrary value of $\tau$. Since this is true for any arbitrary curve in Minkowski spacetime, this completes our proof. □

The proof of the Penrose Incompleteness theorem relies on two important theorems/results from differential topology. These are presented below and proof to only the second one is provided:

**Ingredient theorem (i)** A bijective continuous map from compact spaces to a Hausdorff space is a homeomorphism.

**Ingredient theorem (ii)** Let $\mathscr{N} \subset \mathscr{M}$ be an (injective) immersed topological submanifold with $\dim(\mathscr{N}) = \dim(\mathscr{M}) = n$ such that $\mathscr{N}$ is compact and $\mathscr{M}$ is a Hausdorff connected non-compact topological manifold. Then $\partial_{\text{mani}}\mathscr{N} \neq \varnothing$, where $\partial_{\text{mani}}\mathscr{N}$ denotes the boundary of $\mathscr{N}$ in the sense of (topological) manifolds.

*Proof to* ***Ingredient theorem (ii)*** *(By contradiction).* Assume $\partial_{\text{mani}}\mathscr{N} = \varnothing$
$\implies \forall x \in \mathscr{N}$, $\exists \delta > 0$, such that $B_\delta(x)$ homeomorphic to $\mathbb{R}^n$.
Since, $\dim(\mathscr{M}) = \dim(\mathscr{N})$, $B_\delta(x)$ is also open in $\mathscr{M}$. By compactness, we can say that $\mathscr{N} \subset \mathscr{M}$ is open. However, $\mathscr{N}$ is compact and is a subset of a Hausdorff space, hence, $\mathscr{N}$ is closed. By connectedness, $\mathscr{M} = \mathscr{N}$, which is a contradiction. □

We move on to the proof of the Penrose Incompleteness theorem.

*Proof to the Penrose Incompleteness Theorem 5.1.* From the results of Section - look into the previous section, we know that if $\operatorname{tr}\chi_x = -k_x < 0$, $x \in S = \{\tau = 0\}$, then the first focal point of the generator $G_x \subset C$ appears at time $\tau = \frac{2}{k_x}$. In view of compactness of $S$[†], we have:

$$\sup_S \operatorname{tr}\chi = k_C < 0\,, \qquad \sup_S \operatorname{tr}\underline{\chi} = k_{\underline{C}} < 0\,, \qquad \sup\{k_C, k_{\underline{C}}\} = k < 0 \tag{1}$$

We assume $(\mathscr{M}, g)$ to be future null geodesically complete. We define $\mathcal{V}$ to be the union of all the null generators of $C \cup \underline{C}$ for which $0 \le \tau \le 2/k$, i.e.,

$$\mathcal{V} = \bigcup_{\substack{\tau \in \left[0, \frac{2}{k}\right], \\ x \in S}} \left(G_x(\tau),\ \underline{G}_x(\tau)\right) \tag{2}$$

(Check this bit) Based on this construction, $\mathcal{V} \subset \mathscr{M}$. The null generators, $G$ and $\underline{G}$, can be viewed as the following continuous maps:

$$G : S \times \left[0, \frac{2}{k}\right] \to C\,, \qquad (x,\tau) \mapsto G_x(\tau)$$
$$\underline{G} : S \times \left[0, \frac{2}{k}\right] \to \underline{C}\,, \qquad (x,\tau) \mapsto \underline{G}_x(\tau)$$

Continuity implies that the null generators are mappings between compact sets. This implies that $\mathcal{V}$ is itself compact.

[†] A continuous function on a compact set is bounded and attains its maximum

The trace condition implies every null generator in $\mathcal{V}$ contains at least one focal point. Therefore, from the previous section on causality, we can say that:

$$\partial \mathscr{J}^{+}(S) \subseteq \mathcal{V} \subset \mathscr{M}. \tag{3}$$

The topological boundary $\partial \mathscr{J}^{+}(S)$ is closed by definition[‡]. Since $\mathcal{V}$ is compact, by 3, we conclude $\partial \mathscr{J}^{+}(S)$ is compact[**].

We now show that global topological argument leads to a contradiction, and hence $(\mathscr{M}, g)$ cannot be future null geodesically complete.

Since $\mathscr{M}$ is time-orientable, $\exists$ global timelike vector field $T$ whose integral curves are timelike foliate of $\mathscr{M}$ and intersect the Cauchy Hypersurface exactly once. Futhermore, the integral curves intersect $\partial \mathscr{J}^{+}(S)$ exactly once, since $\mathscr{J}^{+}(S)$ is future set and the topological boundary of a future set is a closed achronal three-dimensional Lipschitz submanifold without boundary[‡‡] where $\partial_{\text{mani}}$ denotes the boundary in the sense of (topological) manifolds. Projection of $\partial \mathscr{J}^{+}(S)$ on $\mathscr{H}$ via the integral curve $T$ is a continuous injective mapping from $\partial \mathscr{J}^{+}(S)$ onto a subset of $\mathcal{T} \subset \mathscr{H}$.

From **Ingredient theorem (i)**, we can say that $\partial \mathscr{J}^{+}(S)$ is homeomorphic to $\mathcal{T} \subset \mathscr{H}$. This implies that $\mathcal{T}$ is a Lipschitz three-dimensional compact submanifold without a boundary, i.e., $\partial_{\text{mani}} \mathcal{T} \neq \varnothing$, in the compact three-dimensional manifold $\mathscr{H}$. However, from **Ingredient theorem (ii)**, $\mathcal{T}$ must have a non-empty boundary. This is a contradiction. Thus, our assumption that $(\mathscr{M}, g)$ is future null geodesically complete is false. $\square$

[‡]Topological boundary of subset $A$ of a topological space $X$, $\partial A$, is given by $\partial A = \overline{A} \cap \overline{X \setminus A}$.

[**]Closed subset of a compact set is itself compact.

[‡‡]**Proposition:** Let $\mathscr{F}$ be a future set in a Lorentzian manifold $\mathscr{M}$. Then the topological boundary $\partial \mathscr{F}$ is closed achronal three-dimensional locally Lipschitz submanifold of $\mathscr{M}$ such that $\partial_{\text{mani}} \partial \mathscr{F} \neq \varnothing$