Seminar Report

Penrose Incompleteness Theorem

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Introduction

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1 Double Null Hypersurfaces and Foliations

This is the first chapter

2 Null Structure Equations

This is the second chapter

3 Condition for Regularity of Null Hypersurfaces

This is the third chapter

4 Causality for Spacetimes with Trapped Surfaces

Recall the geometric construction in the previous sections. If S is a closed 2-dimensional surface in a globally hyperbolic time-orientable spacetime $(\mathcal{M}, g)$ and C and $\underline{C}$ be future outgoing and incoming null geodesic congruence normal to S respectively. Then:

$$\partial \mathcal{J}^+(S) \subset C \cup \underline{C}. \tag{1}$$

We always have:

$$C \cup \underline{C} \subset \mathcal{J}^+(S). \tag{2}$$

However, it is not always the case that:

$$C \cup \underline{C} \subset \partial \mathcal{J}^+(S). \tag{3}$$

The reason for this is in case $C \cup \underline{C}$ lies in the interior of the future of S, i.e., $\mathcal{J}^+(S)$ implies $\exists$ timelike curves connecting points on $C \cup \underline{C}$ to S and also to a neighborhood of S which is interior of $\mathcal{J}^+(S)$.

Insert connecting text here

Proposition 4.1. *Let S be a closed 2-dimensional manifold surface in a globally hyperbolic time oriented spacetime $(\mathcal{M}, g)$. Let C and $\underline{C}$ denote the (future) outgoing and incoming null geodesic congruence normal to S. Let C^* and $\underline{C}^*$ be parts of C and $\underline{C}$ that do not contain any focal point. Then, $\partial\mathcal{J}^+(S) \subset C^* \cup \underline{C}^*$.*

4.1 Trapped Surface

Assume $\Omega = 1$ on $C \cup \underline{C}$. Consider L and $\underline{L}$ to be the geodesic vector field of C and $\underline{C}$ with τ and $\underline{\tau}$ the respective affine parameters such that $S \equiv \{\tau = 0\} = \{\underline{\tau} = 0\} \equiv S_0$.

Relationsihp between the area of sections S_τ and the second fundamental forms χ and $\underline{\chi}$:

The area relates, as we shall see later, the $\det g$ to $\mathrm{tr}\chi$ and helps deduce the regularity of the null hypersurfaces. We present the solutions only corresponding to the hypersurface C. The computations are same for $\underline{C}$.
We assume the canonical coordinates $(\tau, \theta^1, \theta^2)$ on C, where $(\theta^1, \theta^2) \in \mathcal{U} \subset \mathbb{R}^2$ and g is the induced metric on the sections S_τ of C.

In the following, we use the first variational formula and the Jacobi formula for the derivative of a matrix determinant:

$$
\begin{aligned}
\nabla_L\left(\sqrt{\det g}\right) &= \frac{1}{2\sqrt{\det g}}\nabla_L(\det g) = \frac{1}{2\sqrt{\det g}}(\det g)\,\mathrm{tr}\left(\left(g^{-1}\right)^{AB}\nabla_L g_{AB}\right) \\
&= \frac{\sqrt{\det g}}{2}\mathrm{tr}\left(\left(g^{-1}\right)^{AB}\mathcal{L}_L g_{AB}\right) = \frac{\sqrt{\det g}}{2}\mathrm{tr}\left(\left(g^{-1}\right)^{AB}2\chi_{AB}\right) \\
&= \sqrt{\det g}\ \mathrm{tr}(\chi)
\end{aligned}
$$

We know,

$$\text{Area}(S_\tau) = \int\limits_{\mathscr{U}} \sqrt{\det \not{g}(\tau)}\, d\theta^1 d\theta^2$$

Thus,

$$\nabla_L(\text{Area}(S_\tau)) = \int\limits_{\mathscr{U}} \text{tr}\chi\, d\mu_{\not{g}}.$$

More generally,

$$\nabla_{fL}(\text{Area}(S_\tau)) = \int\limits_{\mathscr{U}} f \cdot \text{tr}\chi\, d\mu_{\not{g}}, \quad \forall f \geq 0 \text{ and } f \in C^\infty(S_\tau). \tag{4}$$

Interpretation: The equations above represent the rate of change of the second fundamental form χ and the rate of change of the area of S_τ under infinitesimal displacement along the null generators. Therefore, trχ is also called expansion of S_τ.

Definition 4.1 (Trapped Surfaces). *A 2-dimensional surface S in $(\mathscr{M}, g)$ for which the area decreases under infinitesimal (arbitrary) displacements along the null generators of both null geodesics congruences $C \cup \underline{C}$ normal to S.*

If $(\mathscr{M}, g)$ is globally hyperbolic with a trapped surface, implies that $C \cup C$ bounds the future of S, i.e., $\partial\mathscr{J}^+(S) \subset C \cup \underline{C}$. Hence, it cannot expand in its future. Thus the term *trapped*. It is for this reason (3) does not always hold true.

In view of this definition and (4), for a trapped surface, the following statements holds:

$$\int\limits_{\mathscr{U}} f \cdot \text{tr}\chi\ d\mu_{\not{g}} < 0, \quad \int\limits_{\mathscr{U}} f \cdot \text{tr}\underline{\chi}\ d\mu_{\not{g}} < 0 \quad \forall f \geq 0 \text{ and } f \in C^\infty(S_\tau)$$

Equivalent definition for Trapped Surfaces: A trapped surface is a closed 2-dimensional surface S in a Lorentzian manifold $(\mathscr{M}, g)$ such that:

$$\text{tr}\chi < 0, \quad \text{tr}\underline{\chi} < 0$$

Trapped surfaces and focal points:

A null generators on an incoming null hypersurface $\underline{C}$ contain focal points. The null expansion of an incoming null hypersurface, in our case $\underline{C}$ is negative. The following proposition shows that these two properties are related:

Proposition 4.2. *Assume S to be a closed two-dimensional surface (not necessarily trapped) in a Lorentzian manifold $(\mathcal{M}, g)$ which satisfies the Einstein equation* $\mathrm{Ric}(g) = 0$*. If* $\mathrm{tr}\chi < 0$ *at some point $x \in S$, then $\exists$ a focal point on the null generator G_x of C emanating from point x. A similar result holds for $\underline{C}$.*

Proof to Proposition 4.2. Using Raychaudhari equation[¶],

$$\not\nabla_4(\mathrm{tr}\chi) = -|\chi|^2 + \omega \mathrm{tr}\chi$$

and the other equation:

$$|\chi|^2 = \frac{1}{2}(\mathrm{tr}\chi)^2 + |\hat{\chi}|^2$$

we get, the following Riccati-type equation[§]:

$$\nabla_L(\mathrm{tr}\chi) = -\frac{1}{2}(\mathrm{tr}\chi)^2 - |\hat{\chi}|^2 \leq 0 \tag{5}$$

If $T = \mathrm{tr}\chi_x = \mathrm{tr}\chi(0) < 0$, then $\mathrm{tr}\chi(\tau) < 0, \quad \forall \tau \geq 0$.
Ignoring the term $|\hat{\chi}|$, yields:

$$\nabla_L\left(-\frac{1}{\mathrm{tr}\chi}\right) \leq -\frac{1}{2}$$

Thus,

$$\Longrightarrow \nabla_L\left(-\frac{1}{\mathrm{tr}\chi}\right) \leq -\frac{1}{2}$$

$$\Longrightarrow -\frac{1}{\mathrm{tr}\chi} \leq -\frac{1}{T} - \frac{\tau}{2}$$

$$\Longrightarrow \tau_* = \frac{2}{-T} = \frac{2}{-\mathrm{tr}\chi_x}$$

We obtain $\mathrm{tr}\chi(\tau_*) = -\infty$ and hence $G_x(\tau_*)$ is the first focal point on G_x. □

Remarks: The equality in equation (5) holds for $\mathrm{Ric}(L, L) = \mathrm{tr}\,\alpha = 0$. From second variational formula insert equation no., we see that Riccati-type equation similar to equation5 can be obtained for a more relaxed condition, $\mathrm{tr}\alpha = \mathrm{Ric}(L, L) \geq 0$. This is also called as the *positive null energy condition*, which is weaker than the Einstein equations. Hence, the proposition 4.2 also holds for the positive null energy condition.

¶ For our case, the null lapse function, $\Omega = 1$, hence $\omega = 0$.

§ Riccati-type equations have a finite-time blow-up.

5 Penrose Incompleteness Theorem

5.1 Motivation

Theorem 5.1 (Penrose Incompleteness Theorem). *Let $(\mathcal{M}, g)$ be a globally hyperbolic time-orientable (Hausdorff) spacetime with a non-compact Cauchy Hypersurface $\mathcal{H}$ such that $\mathcal{M}$ contains a trapped surface S. If, in addition, $(\mathcal{M}, g)$ satisfies $Ric(L, L) \geq 0 \; \forall$ null vector fields L, then $\mathcal{M}$ is future geodesically incomplete. In fact, $\exists$ a null generator of $C \cup \underline{C}$, the future null geodesic congruences normal to S, that cannot be extended $\forall \tau \geq 0$ in $\mathcal{M}$.*

Corollary 5.1.1. *There are no trapped surfaces in Minkowski spacetime as it is geodesic complete.*

Definition 5.1 (Geodesic Complete). *A geodesic complete manifold implies that all its causal geodesics can be extended to arbitrary values of their affine parameters. Formally, a manifold $(\mathcal{M}, g)$ is geodesically complete if every timelike (or null) geodesic $\gamma : \mathbb{R} \supset (-\varepsilon, \varepsilon) \to \mathcal{M}$, such that $\gamma(0) = p$, $\forall p$ can be extended to $\tilde{\gamma} : \mathbb{R} \to \mathcal{M}$. That is, the timelike or null geodesics can be extended from a real interval to the entire real line.*

Proof to Corollary 5.1.1. In the cartesian coordinate frame for the spatial coordinates, the Minkowski metric, η, takes the form, $\eta = -dt^2 + dx^2 + dy^2 + dz^2$. The Christoffel symbols, $\Gamma^{\sigma}_{\alpha\beta}$, $\alpha, \beta, \sigma = \{0, 1, 2, 3\}$, for this coordinate choice are all identically zero. The geodesic equation for an affinely parameterized curve, $\gamma(\tau)$, with τ as the affine parameter, takes the form:

$$\frac{d^2 x^{\sigma}}{d\tau^2} + \Gamma^{\sigma}_{\alpha\beta} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau} = 0$$

, which reduces to $\frac{d^2 x^{\sigma}}{d\tau^2} = 0$. The general solution of such a differential equation is given by $x^{\mu}(\tau) = A^{\mu}\tau + B^{\mu}$, where the constants A^{μ} and B^{μ} can be determined based on the initial data. We see that $x^{\mu}(\tau)$ is linear in the affine parameter and can be extended to any arbitrary value of τ. Since this is true for any arbitrary curve in Minkowski spacetime, this completes our proof. $\square$

The proof of the Penrose Incompleteness theorem relies on two important theorems/results from differential topology. These are presented below and proof to only the second one is provided:

Ingredient theorem (i) A bijective continuous map from compact spaces to a Hausdorff space is a homeomorphism.

Ingredient theorem (ii) Let $\mathcal{N} \subset \mathcal{M}$ be an (injective) immersed topological submanifold with $\dim(\mathcal{N}) = \dim(\mathcal{M}) = n$ such that $\mathcal{N}$ is compact and $\mathcal{M}$ is a Hausdorff connected non-compact topological manifold. Then $\partial_{\text{mani}}\mathcal{N} \neq \varnothing$, where $\partial_{\text{mani}}\mathcal{N}$ denotes the boundary of $\mathcal{N}$ in the sense of (topological) manifolds.

*Proof to **Ingredient theorem (ii)** (By contradiction).* Assume $\partial_{\text{mani}}\mathcal{N} = \varnothing$
$\implies \forall x \in \mathcal{N}$, $\exists \delta > 0$, such that $\mathrm{B}_\delta(x)$ homeomorphic to $\mathbb{R}^n$.
Since, $\dim(\mathcal{M}) = \dim(\mathcal{N})$, $\mathrm{B}_\delta(x)$ is also open in $\mathcal{M}$. By compactness, we can say that $\mathcal{N} \subset \mathcal{M}$ is open. However, $\mathcal{N}$ is compact and is a subset of a Hausdorff space, hence, $\mathcal{N}$ is closed. By connectedness, $\mathcal{M} = \mathcal{N}$, which is a contradiction. □

We move on to the proof of the Penrose Incompleteness theorem.

Proof to the Penrose Incompleteness Theorem 5.1. From the results of Section - look into the previous section, we know that if $\mathrm{tr}\chi_x = -k_x < 0$, $x \in S = \{\tau = 0\}$, then the first focal point of the generator $G_x \subset C$ appears at time $\tau = \frac{2}{k_x}$. In view of compactness of S[†], we have:

$$\sup_S \mathrm{tr}\chi = k_C < 0\,, \qquad \sup_S \mathrm{tr}\underline{\chi} = k_{\underline{C}} < 0\,, \qquad \sup\{k_C, k_{\underline{C}}\} = k < 0 \tag{6}$$

We assume $(\mathcal{M}, g)$ to be future null geodesically complete. We define $\mathcal{V}$ to be the union of all the null generators of $C \cup \underline{C}$ for which $0 \leq \tau \leq 2/k$, i.e.,

$$\mathcal{V} = \bigcup_{\substack{\tau \in \left[0, \frac{2}{k}\right], \\ x \in S}} \left(G_x(\tau),\ \underline{G}_x(\tau)\right) \tag{7}$$

(Check this bit) Based on this construction, $\mathcal{V} \subset \mathcal{M}$. The null generators, G and $\underline{G}$, can be viewed as the following continuous maps:

$$G : S \times \left[0, \frac{2}{k}\right] \to C\,, \qquad (x, \tau) \mapsto G_x(\tau)$$
$$\underline{G} : S \times \left[0, \frac{2}{k}\right] \to \underline{C}\,, \qquad (x, \tau) \mapsto \underline{G}_x(\tau)$$

Continuity implies that the null generators are mappings between compact sets. This implies that $\mathcal{V}$ is itself compact.

[†] A continuous function on a compact set is bounded and attains its maximum

The trace condition implies every null generator in $\mathcal{V}$ contains at least one focal point. Therefore, from the previous section on causality, we can say that:

$$\partial\mathcal{J}^+(S) \subseteq \mathcal{V} \subset \mathcal{M}. \tag{8}$$

The topological boundary $\partial\mathcal{J}^+(S)$ is closed by definition[‡]. Since $\mathcal{V}$ is compact, by 8, we conclude $\partial\mathcal{J}^+(S)$ is compact[**].

We now show that global topological argument leads to a contradiction, and hence $(\mathcal{M}, g)$ cannot be future null geodesically complete.

Since $\mathcal{M}$ is time-orientable, $\exists$ global timelike vector field T whose integral curves are timelike foliate of $\mathcal{M}$ and intersect the Cauchy Hypersurface exactly once. Futhermore, the integral curves intersect $\partial\mathcal{J}^+(S)$ exactly once, since $\mathcal{J}^+(S)$ is future set and the topological boundary of a future set is a closed achronal three-dimensional Lipschitz submanifold without boundary[‡‡] where ∂_{mani} denotes the boundary in the sense of (topological) manifolds. Projection of $\partial\mathcal{J}^+(S)$ on $\mathcal{H}$ via the integral curve T is a continuous injective mapping from $\partial\mathcal{J}^+(S)$ onto a subset of $\mathcal{T} \subset \mathcal{H}$.

From **Ingredient theorem (i)**, we can say that $\partial\mathcal{J}^+(S)$ is homeomorphic to $\mathcal{T} \subset \mathcal{H}$. This implies that $\mathcal{T}$ is a Lipschitz three-dimensional compact submanifold without a boundary, i.e., $\partial_{\text{mani}}\mathcal{T} \neq \varnothing$, in the compact three-dimensional manifold $\mathcal{H}$. However, from **Ingredient theorem (ii)**, $\mathcal{T}$ must have a non-empty boundary. This is a contradiction. Thus, our assumption that $(\mathcal{M}, g)$ is future null geodesically complete is false. □

[‡]Topological boundary of subset A of a topological space X, ∂A, is given by $\partial A = \overline{A} \cap \overline{X \backslash A}$.

[**]Closed subset of a compact set is itself compact.

[‡‡]**Proposition:** Let $\mathcal{F}$ be a future set in a Lorentzian manifold $\mathcal{M}$. Then the topological boundary $\partial\mathcal{F}$ is closed achronal three-dimensional locally Lipschitz submanifold of $\mathcal{M}$ such that $\partial_{\text{mani}}\partial\mathcal{F} \neq \varnothing$