CLASSIFICATION OF SPHERICAL VARIETIES

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1. Introduction

This is my study note for the classification of spherical varieties over $\mathbb{C}$ based on the papers [1], [2].

There is another approach studied by Cupit-Foutou [3] by means of a suitable class of invariant Hilbert schemes.

2. Notation

We will fix G a connected reudctive group over $\mathbb{C}$, A a maximal torus of G, B contains A a Borel subgroup, S set of simple roots of G determined by B, the root datum will be denoted by $\mathcal{R} = (\chi^*, \Phi, \chi_*, \Phi^\vee)$ with $\chi^* = X^*(A)$.

X will be a spherical G-variety over $\mathbb{C}$.

3. Invariants of spherical varieties

In this section, we will introduce some invariants for spherical varieties.

We will denote the characters of B-semiinvariant functions on X by $\chi(X)$, the associated parabolic subgroup of X is the standard parabolic subgroup

$$P(X) := \{g \in G | \mathring{X} \cdot g = \mathring{X}\}$$

From the local structure theorem, we have an isomorphism $\mathring{X} \cong A_X \times U_{P(X)}$, and it can be shown that $\chi = X^*(A_X)$.

We will denote

$$\Lambda(X) = \chi(X)^*, \ \mathfrak{a}_X = \Lambda(X) \otimes \mathbb{Q}$$

we can think $\Lambda(X)$ as the cocharacter lattice of X. An B-invariant, $\mathbb{Q}$-valued valuation on $\mathbb{C}(X)$ which is trivial on $\mathbb{C}^\times$ will induce an element of $\Lambda(X)$ via restriction to $\mathbb{C}(X)^B$ and we will denote $\mathscr{V} \subset \mathfrak{a}_X$ the cone generated by the images of G-invariant valuations. $\mathscr{V}$ contains the image of negative Weyl chamber under the natural map $\mathfrak{a} \to \mathfrak{a}_X$. $\mathscr{V}$ contains the image of the negative Weyl chamber under the natural map $\mathfrak{a} \to \mathfrak{a}_X$. We will denote by $\Lambda(X)^+ = \Lambda(X) \cap \mathscr{V}$. The cone $\mathscr{V} = \mathfrak{a}_X^+$ is the fundamental domain for a finite reflection group $W_X \subset \mathrm{End}(\mathfrak{a}_X)$, called the *little Weyl group* of X.

Consider the strictly convex cone negative dual to $\mathscr{V}$:

$$\mathscr{V}^\perp = \{\chi \in \chi(X) \otimes \mathbb{R} | \ \langle \chi, v \rangle \leq 0 \text{ for every } v \in \mathscr{V}\}$$

The generators of the intersections of the extremal rays with $\chi(X)$ are called the *spherical roots* of X.

The spherical roots are known to form the set of simple roots of a based root system with Weyl group W_X. This root system will be called the *spherical root system* of X, following the notation of [4], we will denote the set of simple roots by Σ_X.

Remark 3.1. There is also a different normalization of spherical roots proposed in [5], the normalized spherical roots which is aimed for application to representation theory.

Date: January 2024.

4. Wonderful varieties

Wonderful varieties is a class of spherical varieties which arises in the embedding theory of spherical varieties.

Definition 4.1. An algebraic G-variety X is *wonderful* of rank r if:

- X is smooth and complete.
- G has a dense orbit in X whose complement is the union of r smooth prime divisors $D_i,\ i = 1, \cdots r$ with normal crossings.
- the intersection of the divisors D_i is nonempty and for all $I \subseteq \{1, \cdots, r\}$

$$(\cap_{i \in I} D_i) \backslash (\cup_{i \notin I} D_i)$$

  is a G-orbit.

A wonderful G-variety is always projective and spherical, this is proved in [4].

Definition 4.2. A spherical variety $H \backslash G$ is called *wonderful* if $H \backslash G$ admits an embedding which is a wonderful variety.

Next, we will fix X a wonderful variety for G. The following proposition can be viewed as a localization principle

Proposition 4.3. *Let $z \in X$ be the unique fixed point of B^- and consider the orbit $Z = G \cdot z$ which is the unique closed orbit in X, then the spherical roots are the T-weights appearing in $T_z X / T_z Z$.*

One can associate to each spherical root γ a G-stable prime divisor D^γ such that γ is the T-weight of $T_z X / T_z D^\gamma$. Consider the intersection of all G-invariant prime divisors of X different from D^γ, this intersection is a wonderful variety of rank 1, and having γ as its spherical root.

If H is wonderful then H has finite index in $N_G(H)$, and if $H = N_G(H)$ then it is wonderful.

We will denote the set of spherical roots of all wonderful G-varieties of rank 1 by $\Sigma(G)$, for G of adjoint type the elements of $\Sigma(G)$ are always linear combinations of simple roots with nonnegative integer coefficients.

Now let's recall some lemmas on colors: Let X be a wonderful G-variety, S the set of simple roots associated to B, for $\alpha \in S$, we let P_α be the standard parabolic subgroup associated to α. Let $\Delta_X(\alpha)$ denote the set of non P_α-stable colors, we will say that α moves the colors in $\Delta_X(\alpha)$, and a color is always moved by some simple roots.

Lemma 4.4. *([4]) For all $\alpha \in S$, $\Delta_X(\alpha)$ has at most two elements and only the following four cases can appear:*

(1) $\Delta_X(\alpha) = \emptyset$, this happens when the open Borel orbit $\mathring{X}$ is stable under P_α, and the set of all such α will be denote by S^p_X.
(2) $\Delta_X(\alpha)$ has two elements, this happens exactly when $\alpha \in \Sigma_X$, the two colors in $\Delta_X(\alpha)$ will be denoted by D^+_α, D^-_α and we have

$$\langle \rho(D^+_\alpha), \gamma \rangle + \langle \rho(D^-_\alpha), \gamma \rangle = \langle \alpha^\vee, \gamma \rangle$$

for every $\gamma \in \Sigma_X$. We will denote by $\mathcal{A}_X$ the union of all $\Delta_X(\alpha)$ for every $\alpha \in S \cap \Sigma_X$.
(3) $\Delta_X(\alpha)$ has one element and $2\alpha \in \Sigma_X$, the color in $\Delta_X(\alpha)$ is denoted by D'_α and we have:

$$\langle \rho(D'_\alpha), \gamma \rangle = \frac{1}{2} \langle \alpha^\vee, \gamma \rangle$$

(4) The remaining case, i.e. $\Delta_X(\alpha)$ has one element but $2\alpha \notin \Sigma_X$. In this case, the color in $\Delta_X(\alpha)$ is denoted by D_α and

$$\langle \rho(D_\alpha), \gamma \rangle = \langle \alpha^\vee, \gamma \rangle$$

for every $\gamma \in \Sigma_X$.

Lemma 4.5. *([6]) For all $\alpha, \beta \in S$, the condition $\Delta_X(\alpha) \cap \Delta_X(\beta) \neq \emptyset$ occurs only in the following two cases:*

(1) if $\alpha, \beta \in S \cap \Sigma_X$ then it can happen that the cardinality of $\Delta_X(\alpha) \cup \Delta_X(\beta)$ is equal to 3.
(2) if α and β are orthogonal and $\alpha + \beta$ or $\frac{1}{2}(\alpha + \beta)$ belongs to Σ_X, then $D_\alpha = D_\beta$.

The relations in these two lemmas come from the study of some analysis of the cases in rank 1 and rank 2, they will appear in the next section as the axioms for *spherical systems*. The spherical systems for a wonderful variety X consists of S^p_X the simple roots moving no colors, Σ_X the set of spherical roots, and $\mathcal{A}_X$ a subset of colors.

5. Spherical systems

The following definition comes from the classification of wonderful varieties of rank less or equal to 2 and some geometric properties of colors studied by Luna 4.4, 4.5.

Definition 5.1. Given a root datum $\mathcal{R} = (\chi^*, \Phi, \chi_*, \Phi^\vee)$ of a connected reductive algebraic group G and a set of positive roots S, a triple $\mathcal{S} = (S^p, \Sigma, \mathcal{A})$ such that $S^p \subseteq S$, $\Sigma \subset \Sigma(G)$, $\mathcal{A}$ is a finite set endowed with a map $\rho : \mathcal{A} \longrightarrow \chi^\vee$, where $\chi = \langle \Sigma \rangle$, $\mathcal{S}$ will be called a *spherical systems* if the following axioms are satisfied:

(A1) $\forall D \in \mathcal{A}$, $\rho(D)(\alpha) \leq 1$ for all $\alpha \in \Sigma$, equality holds if and only if $\alpha \in S \cap \Sigma$.
(A2) $\forall \alpha \in S \cap \Sigma$, $\mathcal{A}(\alpha) := \{D \in \mathcal{A} |\ \rho(D)(\alpha) = 1\} = \{D^+_\alpha,\ D^-_\alpha\}$, and $\rho(D^+_\alpha) + \rho(D^-_\alpha) = \alpha^\vee$.
(A3) $\mathcal{A} = \cup_{\alpha \in S \cap \Sigma} \mathcal{A}(\alpha)$.
(Σ1) If $2\alpha \in \Sigma \cap 2S$, then $\frac{1}{2}\langle \alpha^\vee, \beta \rangle$ is a non-positive integer, $\forall \beta \in \Sigma \backslash \{2\alpha\}$, furthermore $\alpha \notin \chi$ and $\frac{1}{2}\langle \alpha^\vee, \beta \rangle$ is an integer for all $\beta \in \chi$.
(Σ2) If $\alpha, \beta \in S$ are orthogonal and $\alpha + \beta$ belongs to Σ or 2Σ, then $\langle \alpha^\vee, \gamma \rangle = \langle \beta^\vee, \gamma \rangle$, $\forall \gamma \in \chi$.
(S) For all $\alpha \in \Sigma$, there is a wonderful G-variety X of rank 1 with $S^p_X = S^p$, and $\Sigma_X = \{\alpha\}$.

The cardinality of Σ will be called the *rank* of the spherical system.

Let's note that for the spherical systems of a wonderful variety X, the spherical root system (Φ_X, Σ_X) is not part of the axiom.

The definition of the spherical system is such that the following lemmas holds:

Lemma 5.2. *For every wonderful G-variety X the triple $(S^p_X, \Sigma_X, \mathcal{A}_X)$ is a spherical system.*

Let's sketch the proof for this lemma: axioms $(A2), (A3)$ correspond to lemma 4.4 (2), axiom (Σ1) correspond to lemma 4.4 (3), axiom (Σ2) corresponds to 4.5 (2) and axiom (S) follows from the definition of Σ_X and $\Sigma(G)$.

Lemma 5.3. *The map $X \mapsto (S^p_X, \Sigma_X, \mathcal{A}_X)$ is a bijection between rank one (resp. rank two) wonderful varieties (up to G-isomorphisms) and rank one (resp. rank two) spherical systems.*

This lemma is a reformulation of the result of Wasserman [7].

In [8] it is proven that spherical systems classify wonderful G-varieties for G adjoint of type A and he conjectured that wonderful varieties are classified by spherical systems, this program is completed in [2]

Theorem 5.4. *([2]) There is a bijection $X \leftrightarrow (S^p_X, \Sigma_X, \mathcal{A}_X)$ between wonderful G-varieties and spherical systems.*

It is obvious that two G-isomorphic wonderful varieties have the same spherical systems, however, if two wonderful varieties are isomorphic, namely G-isomorphic up to outer automorphism of G, their spherical systems are equal up to *a permutation of of the set S of simple roots.*

References

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