WORK OF WALDSPURGER

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1. Introduction

The work of Waldspurger is devoted to a very deep study of the automorphic form on $\overline{SL}_2$, the main tool is the correspondence between the autormophic forms on $\overline{SL}_2$ and automorphic forms on PGL_2, this correspndence was first discovered by Shintani and Niwa using the Weil representation, an ealier approach to this correspondence, based on L-functions, was suggested by Shimura.

R.Howe has outlined a general theory of duality correspondence based on the use of Weil representation. He has introduced the general notion of a dual reductive pair and has defined both a local and global duality correspondence. R.Howe has obtained many deep results in the general situation but many important problems remains.

A systematic study of the duality correspondence for the simplest dual reductive pair $\overline{SL}_2, PGL_2$ from the poitn of view of representation theory has been carried out by Rallis and Schiffmann, Waldspurger refers in many places to Rallis and Schiffman, and in a way Waldspurger's work is a continuation of that of Rallis and Schiffman. However, Waldspurger's work contains many fundamental new ideas especially in the global case.

Flicker has studied a correspondence between the automorphic forms of GL_2 and those of $\overline{GL}_2$ using the trace formula. He has in fact obtain a complete description of this correspondence, since $\overline{SL}_2$ is a subgroup of $\overline{GL}_2$, there is a close connection between the automorphic forms of these two groups. Waldspurger has used Flicker's results in a substantial way to obtain his own results, however Waldspurger's result for $\overline{SL}_2$ are quite surprising and were not predicted from the results for $\overline{GL}_2$. It remains a mystery why the automorphic forms on $\overline{SL}_2$ and $\overline{GL}_2$ behave so differently, for example the strong multiplicity one is true for $\overline{GL}_2$ but not for $\overline{SL}_2$. Also the descent correspondence of automrophic forms from GL_2 to $\overline{GL}_2$ has only a local obstruction while the correspondence from PGL_2 to $\overline{SL}_2$ has a global obstruction but no local obstruction.

In this note, following the article of Piatetski-Shapiro [PS83],we would like to summarize Waldspurger's work in the framework of representation theory, we will explain all of Waldspurger's work except the one deals with the Fourier coefficients of automorphic forms of half-integral weight.

2. Automorphic forms on $\overline{SL}_2(\mathbb{A})$

Let k be a global field, the adele group $SL_2(\mathbb{A})$ has a unique non-trivial two-fold covering $\overline{SL_2(\mathbb{A})}$:

$$1 \to \{\pm 1\} \to \overline{SL_2(\mathbb{A})} \to SL_2(\mathbb{A}) \to 1$$

there is a unique embedding of $SL_2(k)$ into $\overline{SL_2(\mathbb{A})}$ such that it is compatible with the covering map $\overline{SL_2(\mathbb{A})} \to SL_2(\mathbb{A})$. Similarly, there is a embedding of $N(\mathbb{A})$ into $\overline{SL_2(\mathbb{A})}$, where N is the upper unipotent subgroup of SL_2.

Let $\mathcal{A}_0$ denote the space of genuine cuspidal functions on $\overline{SL_2(\mathbb{A})}$, in particular if $f \in \mathcal{A}_0$, then

- $f(\xi\gamma g) = \xi f(g)$, $\xi \in \{\pm 1\},\ \gamma \in SL_2(k),\ g \in \overline{SL_2(\mathbb{A})}$.
- $\int_{k\backslash\mathbb{A}} f(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} g)\ dz = 0$.

Under the right translation, $\mathcal{A}_0$ decomposes into a countable number of irreducible subspaces, an irreducible representation of $\overline{SL_2(\mathbb{A})}$ which occurs in $\mathcal{A}_0$ is called a genuine automorphic cuspidal representation. Let $\mathcal{A}_{00}$ denote the subspace of forms in $\mathcal{A}_0$ orthogonal to the Weil representations of $\overline{SL_2(\mathbb{A})}$.

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Theorem 2.1. *The multiplicity of an irreducible genuine automorphic cuspidal representation of $\overline{SL_2(\mathbb{A})}$ in $\mathcal{A}_{00}$ is one.*

If ψ is a character of $k\backslash\mathbb{A}$ and $f \in \mathcal{A}_{00}$, the ψ-Fourier coefficient of f is defined to be

$$f_\psi(g) = \int_{k\backslash\mathbb{A}} \psi(z) f(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} g)\ dz,\ \ g \in \overline{\mathrm{SL}_2(\mathbb{A})}$$

The multiplicity result follows from the uniqueness of Whittaker models for $\overline{SL_2(\mathbb{A})}$ and the following result of Waldspueger

Theorem 2.2. *Let (σ, V) be a genuine irreducible automrophic cuspidal representation of $\overline{SL_2(\mathbb{A})}$, if $v \to \varphi(v)$ for $v \in V,\ \varphi(v) \in \mathcal{A}_{00}$ is an embedding of (σ, V) into $\mathcal{A}_{00}$, then the vanishing of the ψ-Fourier coefficient $\varphi(v)_\psi$ depends only on (π, V) as an abstract representation, and not on the embedding ψ.*

Two irreducible genuine automorphic cuspidal representations of $\overline{SL_2(\mathbb{A})}$, $\sigma = \otimes_v \sigma_v$ and $\sigma' = \otimes_v \sigma'_v$ are said to be nearly equivalent if $\sigma_v = \sigma'_v$ for almost places v, let $\ell(\sigma)$ denote the set of irreducible genuine automorphic cuspidal representations nearly equivalent to σ. In order to determine the set $\ell(\sigma)$, Waldspurger has defined an involution $\sigma \to \sigma^W$ whenever σ is a discrete series representation of $\overline{SL_2(k_v)}$. Define

$$\Sigma = \{v \mid \sigma_v \text{ is a discrete series representation}\}$$

If $M \subset \Sigma$ and $|M|$ is even, put $\sigma^M = \otimes \sigma_v^M$ and denote

$$\sigma_v^M = \begin{cases} \sigma_v & v \notin M \\ \sigma_v^W & v \in M \end{cases}$$

the relationship of σ^M and $\ell(\sigma)$ is given in the following theorem

Theorem 2.3. *Any representation in $\ell(\sigma)$ is of the form σ^M for some $M \subseteq \Sigma$.*

3. The oscillator representation over a local field

Let k be a local field and let X be a $2n$-dimensional vector space over k with a symplectic structure $\langle\ ,\ \rangle$. If $\mathrm{X} = \mathrm{X}_1 \oplus \mathrm{X}_2$ is a polarization of X, let P be the subgroup of $\mathrm{Sp}(\langle\ ,\ \rangle)$ which preserves X_2, if ψ is a non-trivial character of k, let ω_ψ be the oscillator representation of $\overline{\mathrm{Sp}_{2n}}(k)$, the double cover of $\mathrm{Sp}_{2n}(k)$.

Let us consider a 3-dimensional vector space $\mathrm{M} = \{m \in \mathrm{M}_2(k) \mid \mathrm{tr}(m) = 0\}$, PGL_2 acts on M by conjugation, this conjugation preserves the symplectic form $q(x) = -\det(x)$, let Y be a 2-dimensional vector space over k with symplectic form $\langle\ ,\ \rangle$, define a symplectic vector space X by $X = M \otimes_k Y$

$$\langle m_1 \otimes y_1,\ m_2 \otimes y_2 \rangle = (m_1, m_2)\langle y_1, y_2 \rangle$$

since PGL_2 and SL_2 preserve the forms $(\ ,\ )$ and $\langle\ ,\ \rangle$, there is a natural embedding of $\mathrm{PGL}_2 \times \mathrm{SL}_2$ into Sp_6. Our aim is to use the oscillator representation of $\overline{\mathrm{Sp}_6}$ to define a correspondence between certain irreducible representation of PGL_2 and certain irreducible representations of $\overline{\mathrm{SL}_2}$. Waldspurger has given a different definition of the correspondence based on explicit integral formulas.

Let T be a subgroup of $\mathrm{G} = \mathrm{PGL}_2$, and let N be a subgroup of $\overline{\mathrm{SL}_2}$, let α and β be characters of T and N. Let $\mathrm{X} = \mathrm{X}_1 \oplus \mathrm{X}_2$ be a polarization of X such that $\mathrm{T} \times \mathrm{N} \subset \mathrm{P}$.

Let us suppose that $x_1 \in \mathrm{X}_1$ is a vector such that $\phi \mapsto \phi(x_1)$ transforms $\mathrm{T} \times \mathrm{N}$ under $\alpha \times \beta$

$$\omega_\psi(t, n) \cdot \phi(x_1) = \alpha(t)\beta(n)\phi(x_1)$$

Let (π, V) be an irreducible admissible representation of PGL_2 and let us assume that ℓ is a linear functional on V such that $\ell(\pi(t)v) = \alpha^{-1}(t)\ell(v)$, if the integral

$$\mathrm{F}(h) = \int_{\mathrm{T}\backslash\mathrm{G}} (\omega_\psi(g, h) \cdot \phi)(x_1)\ell(\pi(g)v)\ dg \tag{3.1}$$

converges, then $F(nh) = \beta(n)F(h)$. Let W be the space of all the functions F obtained in this fashion by varying ϕ and v, then $\overline{\mathrm{SL}_2}$ acts on W by right translation. We shall denote this function by $\theta(\pi, \psi)$. Conversely, given an irreducible admissible representation σ of $\overline{\mathrm{SL}_2}$, it is possible to define a representation $\theta(\sigma, \psi)$ of PGL_2, which may be zero.

Waldspurger has proved the following theorem

Theorem 3.1. *Let T and N as above. If (π, V) (respectively (σ, V)) is an irreducible admissible representation of PGL_2 (respectively $\overline{SL_2}$), then the representation of $\overline{SL_2}$ (respectively PGL_2) obtained from the above integral formula is irreducible admissible and depends only on the additive character ψ. It is independent of the choice of the subgroups T and N and the characters α and β.*

4. The θ-correspondence

Let k be a global field, we shall use the same notation globally as was previously introduced locally. The global Weil representation ω_ψ acts on $\mathrm{S}(\mathrm{X}_1(\mathbb{A}))$, let $\mathrm{X} = \mathrm{X}_1 \oplus \mathrm{X}_2$ be the standard polarization of X and identify X_1 with M, for $\phi \in \mathrm{S}(\mathrm{X}_1(\mathbb{A}))$

$$\theta_\psi^\phi(g,h) = \sum_{x \in \mathrm{X}_1(k)} \omega_\psi(g,h) \cdot \phi(x) \ g \in G(\mathbb{A}), \ h \in \overline{\mathrm{SL}_2(\mathbb{A})}$$

here G is either PGL_2 or $\mathrm{PD}^\times$, it is well known that θ_ψ^ϕ is an automorphic function on $\mathrm{G}(\mathbb{A}) \times \overline{\mathrm{SL}_2(\mathbb{A})}$ of moderate growth.

Let π be an irreducible automorphic cuspidal representation of $\mathrm{G}(\mathbb{A})$, if $f \in \pi \subset \mathcal{A}_0$, put

$$\varphi(h) = \int_{\mathrm{G}(k)\backslash\mathrm{G}(\mathbb{A})} \theta_\psi^\phi(g,h) \ f(g) \ dg$$

the fact that θ^{ϕ_ψ} is a function of moderate growth means that the integral is well-defined, and φ is a function on $\mathrm{SL}_2(k)\backslash\overline{\mathrm{SL}_2(\mathbb{A})}$. In the case $\mathrm{G} = \mathrm{PD}^\times$, we also assume that $\int_{\mathrm{G}(k)\backslash\mathrm{G}(\mathbb{A})} f(g) \ dg = 0$.

Proposition 4.1. *φ is a cusp form.*

This follows from the definition.

Theorem 4.2. *The θ-correspondence $\pi \to \theta(\pi, \psi)$ is compatible with the local correspondence.*

Proof. Let π be an irreducible automorphic cuspidal representation of $\mathrm{G}(\mathbb{A})$, for $f \in \pi$ and $\phi \in \mathrm{S}(\mathrm{X}, \mathbb{A})$, let φ be the cusp form

$$\varphi(h) = \int_{\mathrm{G}(k)\backslash\mathrm{G}(\mathbb{A})} \theta_\psi^\phi(g,h) \ f(g) \ dg$$

if $a \in k^\times$, then a calculation similar to the one used to φ is a cusp form shows

$$\begin{aligned}\varphi_a(1) &:= \int_{k\backslash\mathbb{A}} \varphi(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix})\overline{\psi(az)} \ dz \\ &= \int_{\mathrm{T}^a(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})} \omega_\psi(g) \cdot \phi(x_a) \int_{\mathrm{T}^a(k)\backslash\mathrm{T}^a(\mathbb{A})} f(tg) \ dtdg\end{aligned}$$

here x_a is any element in X such that $q(x_a) = a$, T^a is the stabilizer of $x_a \cdot \mathrm{T}^a$ is a torus in G. Put

$$\mathrm{U}(f,g) = \int_{\mathrm{T}^a(t)\backslash\mathrm{T}^a(\mathbb{A})} f(tg) \ dt$$

then the function $\mathrm{U}(f,-)$ satisfies $\mathrm{U}(f,tg) = \mathrm{U}(f,g)$ and the linear function $\ell : \ f \to \mathrm{U}(f,1)$ is a linear functional for which $\ell(\pi(t)f) = \ell(f)$. Locally such a functional is unique, hence we have

$$\mathrm{U}(f,-) = \otimes \mathrm{U}_v(-)$$

where U_v is a function on G_v such that $\mathrm{U}_v(t_v g_v) = \mathrm{U}_v(g_v)$, under the right translation of G_v on $\mathrm{T}_v^a\backslash\mathrm{G}_v$, U_v generates a representation equivalent to π_v, we have the following formula

$$\varphi_a(h) = \int_{\mathrm{T}^a(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})} \omega_\psi(g,h) \cdot \phi(x_a)\mathrm{U}(f,g) \ dg$$

if U is an element in the space generated by U_v and if

$$\mathrm{W}_{\psi^a}(h) := \int_{\mathrm{T}^a\backslash\mathrm{G}_v} \omega_{\phi,v}(g,h) \cdot \phi(x_a)\mathrm{U}(g) \ dg$$

then $\mathrm{W}_{\psi^a}(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} h) = \psi_v(za)\mathrm{W}_{\psi^a}(h)$, compare this with the formula for F (3.1) where we used to define the local correspondence gives the result we need. $\square$

Theorem 4.3. *The θ-correspondence is a $1-1$ correspondence between certain automorphic cuspidal irreducible representation of $G(\mathbb{A})$ and certain genuine automorphic cuspidal irreducible representations of $\overline{SL_2(\mathbb{A})}$.*

Theorem 4.4. *Let $G = PGL_2$, suppose $\sigma \in \mathcal{A}_{00}$ and π is an automorphic cuspidal representation of $PGL_2(\mathbb{A})$, then*

- *$\theta(\sigma, \psi^{-1}) \neq 0$ if and only if σ possesses a nonvanishing ψ-Fourier coefficient.*
- *$\theta(\pi, \psi) \neq 0$ if and only if $L(\pi, \frac{1}{2})$.*

Proof. In order to prove this theorem, we must use a polarization for which the usual unipotent subgroups of $\mathrm{PGL}_2(\mathbb{A})$ and $\overline{\mathrm{SL}_2(\mathbb{A})}$ lie inside P. Let M be the trace zero element of $\mathrm{M}_2(k)$, let Y be a symplectic vector space over k with form $\langle\ ,\ \rangle$ and symplectic basis $y_1,\ y_2$. Let $e_1,\ e_2,\ e_3$ be a basis of M such that q has the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, put $\mathrm{X}_1 = e_1 \otimes \mathrm{Y} + e_2 \otimes ky_1$, $x_2 = e_3 \otimes \mathrm{Y} + e_2 \otimes ky_2$. Suppose σ is an irredicible genuine representation of $\overline{\mathrm{SL}_2(\mathbb{A})}$ lying in $\mathcal{A}_{00}$, for $\varphi \in \sigma$, we let

$$f(g) = \int_{\mathrm{SL}_2(k)\backslash \mathrm{SL}_2(\mathbb{A})} \theta_\psi^\phi(g,h)\ \varphi(h)\ dh$$

we can identify X_1 with $\mathrm{Y} \oplus k$, and we can choose ϕ with the form $\phi = \phi_1 \cdot \phi_2$, for $\phi_1 \in \mathrm{S}(\mathrm{Y}(\mathbb{A}))$, $\phi_2 \in \mathrm{S}(\mathbb{A})$, then

$$\theta_\psi^\phi(1,h) = \mathrm{F}_1(h)\mathrm{F}_2(h)$$

where

$$\mathrm{F}_1(h) = \sum_{y \in \mathrm{Y}_k} \phi_1(yh)$$

and

$$\mathrm{F}_2(h) = \sum_{t \in k} \omega'_\psi(h) \cdot \phi_2(t)$$

we have

$$f(1) = \sum_{t \in k} \int_{\mathrm{N}_\mathbb{A}\backslash \mathrm{SL}_2(\mathbb{A})} \phi_1(y_2 h)\omega'_\psi(h) \cdot \phi_2(t)\varphi_{\psi^{t^2}}(h)\ dh$$

if $\theta(\sigma, \psi^{-1}) \neq 0$, then there exists a t for which $\varphi_{\psi^{t^2}} \neq 0$, this means σ possesses a non-zero ψ-Fourier coefficient.

Now suppose σ possesses a non-zero ψ-Fourier coefficient, then we let

$$f_t(1) = \int_{\mathrm{N}_\mathbb{A}\backslash \mathrm{SL}_2(\mathbb{A})} \omega_\psi(h)\phi(y_2, t)\ \varphi_{t^2}(h)\ dh$$

Let Z be the upper unipotent subgroup of PGL_2 then for $z \in \mathrm{Z}$

$$\omega_\psi(z) \cdot \phi(y_2, t) = \psi(tz)\phi(y_2, t)$$

it follows from this formula that $f_t(1)$ is a Fourier coefficient of f, therefore if $\varphi_\psi \neq 0$ then $f_t(1) \neq 0$ and so $\theta(\sigma, \psi^{-1}) \neq 0$.

To prove the second part of the theorem, we use the standard polarization, if $\sigma = \theta(\pi, \psi) \neq 0$ then $\theta(\sigma, \psi^{-1}) = \pi$, since σ has a non-zero Fourier coefficient, the formula for $\varphi_a(1)$ in 4.2 shows that

$$\int_{\mathrm{T}_k\backslash \mathrm{T}_\mathbb{A}} f(t)\ dt \neq 0$$

From the Jacquet-Langlands theory of L-functions, it is known that for appropriate choice of f

$$\mathrm{L}(\pi, s) = \int_{\mathrm{T}_k\backslash \mathrm{T}_\mathbb{A}} f(t)\ |t|^{s-\frac{1}{2}}\ dt$$

hence $\mathrm{L}(\pi, \frac{1}{2}) = \int_{\mathrm{T}_k \backslash \mathrm{T}_{\mathbb{A}}} f(t)\ dt \neq 0$. Conversely, if $\mathrm{L}(\pi, \frac{1}{2}) \neq 0$, then $\int_{\mathrm{T}_k \backslash \mathrm{T}_{\mathbb{A}}} f(t)\ dt \neq 0$ and hence $\theta(\pi, \psi) \neq 0$. $\square$

5. Non-vanishing of a Fourier coefficient

We have the following theorem determine when $\sigma \in \mathcal{A}_{00}$ admits a nonzero ψ-Fourier coefficient

Theorem 5.1. *Let $\sigma = \otimes_v \sigma_v \subset \mathcal{A}_{00}$, then σ admits a non-zero ψ-Fourier coefficient if and only if*

- *at each place of v, there is a linear functional ℓ_v on the space W of σ_v such that*
$$\ell_v(\sigma \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \omega) = \psi_v(t) \ell_v(\omega)$$
- *$L_\psi(\sigma, \frac{1}{2}) \neq 0$.*

References

[PS83] Ilya Piatetski-Shapiro. Work of Waldspurger. In *Lie Group Representations II: Proceedings of the Special Year held at the University of Maryland, College Park 1982–1983*, pages 280–302. Springer, 1983.