HOWE DUALITY AND RELATIVE LANGLANDS DUALITY

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Contents

1. Introduction

This is my study note for the paper [GJ23] that establishes the connection between Howe duality and relative Langlands duality.

2. Nilpotent orbits and generalized Whittaker model

2.1. **Classification of nilpotent orbits.** We first talk about the $\mathfrak{sl}_2$ triple. We fix κ an $\mathrm{Ad}(G)$-invariant non-degenerate bilinear form on $\mathfrak{g}$, let $\gamma = \{e,\ h,\ f\} \subset \mathfrak{g}$ be an $\mathfrak{sl}_2$-triple associated to a nilpotent orbit of $\mathfrak{g}$.

Remark 2.1. Recall by the Jacobson-Morosov theorem, there is a bijection between conjugacy classes of $\mathfrak{sl}_2$ triples and nilpotent orbits.

Under the adjoint $\mathfrak{sl}_2$-action, $\mathfrak{g}$ decomposes into $\mathfrak{sl}_2$ weight spaces

$$\mathfrak{g}_j = \{v \in \mathfrak{g} \mid \mathrm{ad}(h)v = jv \ \ \}$$

for $j \in \mathbb{Z}$. We can define the parabolic $\mathfrak{p} = \oplus_{j\geq 0}\mathfrak{g}_j = \mathfrak{l} \oplus \mathfrak{u}$. Set $\mathfrak{u}^+ = \oplus_{j\geq 2}\mathfrak{g}_j$. We get the corresponding subgroups $P = L \ltimes U$ and U^+ of G, note $\mathfrak{l} = \mathfrak{g}_0$, hence L is the stabilizer of h. Denote the centralizer of γ by M_γ, which is reductive.

We define a character $\chi_{\gamma,\psi}$ on U^+ via

$$\chi_{\gamma,\psi}(\exp\ u) := \psi(\kappa(f, u))\ \forall\ u \in \mathfrak{u}^+$$

we also denote $\kappa_f(u) := \kappa(f, u)$.

Suppose G is the isometry group of a n-dimensional vector space V equipped with an orthogonal form B over F. From the $\mathfrak{sl}_2$-triple above, we obtain a decomposition of V as $V = \oplus_{j=1}^{l} V^{(j)}$ with

$$V^{(j)} = W_j \otimes V_j$$

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which is the isotypic component of V for the j-dimensional representation W_j of $\mathfrak{sl}_2$ and V_j the multiplicity space.

From the $\mathfrak{sl}_2$ theory, W_j is symplectic if j is even and it is orthogonal if j is odd. The form B induces a symplectic or orthogonal form B_j on the multiplicity space V_j. B_j is symplectic if j is even.

M_γ is a direct product of the isometry groups

$$M_\gamma \cong \prod_{j=1}^{l} G(V_j, B_j)$$

Proposition 2.2. *We have a parametrization of the nilpotent orbits of G*

- *the partition $\lambda = [l^{a_l}, \cdots, 1^{a_1}]$.*
- *the forms on the multiplicity spaces (V_j, B_j).*

such that we have

$$\oplus_j (V_j, B_j) \otimes (W_j, A_j) \cong (V, B)$$

If G is an orthogonal group, then the even parts must also occur with even multiplicity in λ.

2.2. **Generalized Whittaker models.** We now define the generalized Whittaker models W_γ associated with a nilpotent orbit γ and the associated generalized Whittaker models.

Definition 2.3. We say the nilpotent orbit associated with γ is even if $U = U^+$.

Definition 2.4. When $U = U^+$, we define

$$W_{\gamma,\psi} := \mathrm{ind}_{M_\gamma U}^{G} \chi_\gamma$$

with trivial M_γ-action on χ_γ. Also for $\pi \in \mathrm{Irr}(G)$

$$W_{\gamma,\psi}(\pi) := \mathrm{Hom}_G(\mathrm{ind}_{M_\gamma U}^{G} \chi_\gamma, \pi^\vee)$$

this is called the space of generalized Whittaker functionals of π.

We have a symplectic structure κ_1 on $\mathfrak{g}_1$ as $\kappa_1(v, w) = \kappa(f, [v, w])$ for $v, w \in \mathfrak{g}_1$, hence $\mathfrak{u}/\mathfrak{u}^+ \cong \mathfrak{g}_1$ carries a M_γ-invariant symplectic form κ_1. Since M_γ preserves the symplectic form κ_1, similar to the Weil representation constructed from the representation ω_ψ of the Heisenberg group, we can construct a representation ω_ψ on $\tilde{M}_\gamma$ some central cover of M_γ. For a genuine representation ρ of $\tilde{M}_\gamma$ with trivial U action, the representation $\rho \otimes \omega_\psi$ descends to an actual representation of $M_\gamma U$.

Definition 2.5. We define

$$W_{\gamma,\rho,\psi} := \mathrm{ind}_{M_\gamma U}^{G} \rho \otimes \omega_\psi$$

and $W_{\gamma,\rho,\psi}(\pi) := \mathrm{Hom}_G(\mathrm{ind}_{M_\gamma U}^{G} \rho \otimes \omega_\psi, \pi^\vee)$, the generalized Whittaker model of π associated to γ and ρ. More generally, ρ may be a genuine representation of $\tilde{H}$ for H a reductive subgroup of M_γ.

In the even orbit case, we have a canonical choice of ρ which is the trivial one. In the non-even case, this should be achieved by choosing the smallest (in the sense of Gelfand-Kirillov dimension) possible ρ.

3. Howe duality

3.1. **Theta correspondence.** We will fix a non-trivial unitary character $\psi : F \to \mathbb{C}^\times$.

Suppose (G_1, G_2) is a type I reductive dual pair, if dim V_1 is odd then we have to work with representations of $\mathrm{Mp}(V_2)$. We assume G_1 is the smaller group of the two.

One can restrict the Weil representation ω_ψ of $\mathrm{Mp}(V_1 \otimes V_2)$ to $G_1 \times G_2$ and for each $\pi \in \mathrm{Irr}(G_1)$ define the big theta lift $\Theta(\pi)$ of π as

$$\Theta_\psi(\pi) := (\omega_\psi \otimes \pi^\vee)_{G_1}$$

the maximal G_1-invariant quotient of $\omega_\psi \otimes \pi^\vee$.

Theorem 3.1. *(Howe duality) Let*

$$C = \{(\pi_1, \pi_2) \in Irr(G_1) \times Irr(G_2) \mid \pi_1 \otimes \pi_2 \text{ is a quotient of } \omega_\psi\}$$

then C is the graph of a bijective function between $Irr(G_1)$ and $Irr(G_2)$. Furthermore, we have

$$\dim Hom(\omega_\psi, \pi_1 \otimes \pi_2) \leq 1$$

for all $\pi_1 \in Irr(G_1)$, $\pi_2 \in Irr(G_2)$. We will denote $\theta(\pi)$ the unique irreducible quotient of $\Theta(\pi)$, and call it the small theta lift.

In general, the theta correspondence will not preserve the L-packet and there is the Adams conjecture which describes the effects of theta correspondence on A-parameters when $\dim V_2$ is sufficiently large. There is a characterization of this sufficiently large condition in terms of the "first occurrence indices".

3.2. **Gomez-Zhu's result.** One would like to use the theta correspondence to relate the two generalized Whittaker models on a dual pair, one need a correspondence of nilpotent orbits and this is achieved via the moment map. We replace G_1 and G_2 by G, G'

Proposition 3.2. *One has moment maps*

$$\mathfrak{g} \xleftarrow{\phi} Hom(V, V') \xrightarrow{\phi'} \mathfrak{g}'$$

defined by $\phi(f) = ff^*$ and $\phi'(f) = f^*f$.

Given a nilpotent element e in the image of ϕ corresponds to a $\mathfrak{sl}_2$-triple γ, one can define a nilpotent orbit of $\mathfrak{sl}_2$-triple γ' of $\mathfrak{g}'$ such that

- *e, e' are the images of some common element $f \in Hom(V, V')$.*
- *the form on V' restricts to a nondegenerate form on $ker(f)$.*
- *f sends the k-weight space of V' to the $k+1$-weight space of V for all $k \in \mathbb{Z}$.*

The partitions corresponding to γ, γ' are related in the following way: suppose their corresponding Young tableaux are d, d', then one removes the first column of d and adds suitably many rows of length 1 to obtain d'. In other words, one has $(V_j', B_j') = (V_j, B_j)$ and $V_1' = V_2 \oplus V_{new}$ for V_{new} the newly added rows of length 1 in d'.

We assume that the nilpotent orbit defined by γ is in the image of the moment map ϕ, recall from the discussion of the centralizer of the nilpotent orbit, we have

$$M_\gamma \cong \prod_{k=1}^{j} G(V_k, B_k) \quad M_{\gamma'} \cong \prod_{k=1}^{j} G'(V_k', B_k')$$

we observe that M_γ and $M_{\gamma'}$ contain factors $G(V_1, B_1)$, $G'(V_1', B_1')$ corresponding to the tows of length 1 in d and d'. Furthermore $G'(V_1', B_1')$ contains a subgroup $G'(V_{new})$ which is an isometry subgroup of the subspace $V_{new} \subseteq V_1'$ corresponding to the *newly added* rows of length 1 in d'. We have that $G(V_1, B_1)$ and $G'(V_{new})$ forms a reductive dual pair inside $\mathrm{Sp}(V_1 \otimes V_{new})$.

Example 3.3. For the nilpotent orbit γ_1 of $\mathfrak{so}_{2k}$ corresponds to a regular nilpotent orbit $\gamma_{r,1}$ of $\mathfrak{sp}_{2k-2a}$, it corresponds to the partition $[2k-2a-1, 1^{2a+1}]$ of $\mathfrak{so}_{2k}$.

The following is a result from [GZ14]

Proposition 3.4. *For any $\pi \in Irr(G')$ and for a genuie representation $\tau \in Irr(G(\widetilde{V_1, B_1}))$, one has*

$$W_{\gamma,\tau,\psi}(\Theta_\psi)(\pi) \cong W_{\gamma',\Theta(\tau)^\vee,\psi}(\pi^\vee)$$

here

- *$\Theta(\pi)$ is the big theta lift for the dual pair (G, G').*
- *$\Theta(\tau)^\vee$ is the dual of the big theta lift for the dual pair $(G(V_1, B_1), G'(V_{new}))$.*

Remark 3.5. If the nilpotent orbit defined by γ is not in the image of the moment map ϕ, then one has

$$W_{\gamma,\tau,\psi}(\Theta_\psi(\pi)) = 0$$

for all $\pi \in \mathrm{Irr}(G')$.

4. Hyperspherical variety and geometric quantization

4.1. **Geometric quantization of Whittaker induction.** In this section, G, H will be Lie groups over $\mathbb{C}$, and H is a subgroup of G.

Definition 4.1. Consider any reductive subgroup H of G and a commuting SL_2-factor, S a symplectic H-vector space, we can define the Whittaker induction of S along $H \times SL_2 \to G$ as the symplectic induction of $S \times (\mathfrak{u}/\mathfrak{u}^+)$ from HU to G.

Under the philosophy of quantization, Whittaker induction corresponds to the formation of generalized Whittaker representation 2.5 where

- S corresponds to ρ.
- $\mathfrak{u}/\mathfrak{u}^+$ corresponds to the oscillator representation ω_ψ of U.
- the symplectic reduction corresponds to the induction of representations.

It will be very interesting to make precise this philosophy in a way that unifies the quantization of hyperspherical varieties for the hook-type partitions and exceptional cases.

4.2. **Hyperspherical Whittaker models.** We determine an upper bound for the possible generalized Whittaker models for the orthogonal groups arise from hyperspherical varieties.

Proposition 4.2. *Let M be a hyperspherical variety, then $H\backslash L$ is a smooth affine spherical L-variety, where L is the Levi factor of $P = LU$ associated to the $\mathfrak{sl}_2$ triple γ. In particular, H is a spherical subgroup of M_γ and M_γ is a spherical subgroup of L.*

We consider the case when the nilpotent orbit is even. For $G = O_n$ acting on an n-dimensional vector space V with an orthogonal form B, the nilpotent orbits in G are parametrized by partition $\lambda = [l^{a_l}, \cdots, 1^{a_1}]$, and forms on the multiplicity space (V_j, B_j). For the even nilpotent orbits, all the partition λ have the same parity, we have

$$H = M_\gamma \cong \prod_{j=1}^{l} G(V_j, B_j)$$

By checking the table of [KVS06], one can characterize all the nilpotent orbits γ which allow hyperspherical varieties

Theorem 4.3. *Let G be the orthogonal group O_n and M a hyperspherical variety, it is obtained as the Whittaker induction along a map $H \times SL_2 \to G$, let γ be the nilpotent orbit determined by the SL_2 factor, if γ is even, then it corresponds to a partition of the form*

- $[2^{a_2}]$ *(Shalika).*
- $[n - a_1, 1^{a_1}]$ *(hook-type).*
- *finitely many low rank-exceptions:* $[3, 3], [4, 4], [6, 6]$.

5. Examples of Relative Langlands duality

5.1. **Even orthogonal group.** We determine the expected hyperspherical dual for the hook-type partitions $[n - a_1, 1^{a_1}]$ of O_n. Suppose $n = 2k$ is even, then we must have $a_1 = 2a + 1$ is odd.

Theorem 5.1. *The hyperspherical varieties M_1 and M_2 defined by*

- *the datum $O_{2a+1} \times SL_2 \to O_{2k}$ corresponds to the nilpotent orbit with partition $[2k - 2a - 1, 1^{2a+1}]$ and trivial S.*
- *the datum $O_{2k-2a+1} \times SL_2 \to O_{2k}$ corresponding to the nilpotent orbit with partition $[2a-1, 1^{2k-2a+1}]$ and trivial S.*

they are dual under relative Langlands duality.

Recall that M_1 and M_2 have quantization $W_{\gamma_1,\mathrm{triv}_1,\psi}$ and $W_{\gamma_2,\mathrm{triv}_2,\psi}$ from our discussion on geometric quantization of Whittaker induction.

Theorem 5.2. *We have:*

- *If π is an irreducible representation of O_{2k} occurs as a quotient of $W_{\gamma_1,triv_1,\psi}$ then $\pi=\theta_\psi(\sigma)$ for σ an irreducible representation of Sp_{2k-2a}. Conversely, if σ is an irreducible ψ-generic representation of Sp_{2k-2a}, then $\pi:=\theta_\psi(\sigma)$ is an irreducible representation of O_{2k} which occurs as a quotient of $W_{\gamma_1,triv_1,\psi}$.*
- *If π is an irreducible representation of O_{2k} which occurs as a quotient of $W_{\gamma_2,triv_2,\psi}$ then $\pi=\theta_\psi(\sigma)$ for σ an irreducible representation of Sp_{2a}. Conversely if σ is an irreducible ψ-generic representation of Sp_{2a} then $\pi:=\theta_\psi(\sigma)$ is an irreducible representation of O_{2k} which occurs as a quotient of $W_{\gamma_2,triv_2,\psi}$.*

Proof. We only prove the first one, the second one is similar. From the result 3.4 we have

$$W_{\gamma_{r,1},\mathrm{triv},\psi}(\Theta_\psi(\pi))\cong W_{\gamma_1,\mathrm{triv}_1,\psi}(\pi^\vee)$$

for all $\pi\in\mathrm{Irr}(O_{2k})$.

On one hand if π occurs as a quotient of W_{γ_1} then $W_{\gamma_1}(\pi^\vee)\neq 0$ hence $W_{\gamma_{r,1}}(\Theta(\pi))\neq 0$ in particular $\Theta(\pi)\neq 0$ hence $\theta(\pi)\neq 0$. From theorem, π is the small theta lift of an irreducible representation of Sp_{2k-2a}. Note if $\Theta(\pi)$ is already irreducible hence equal to $\theta(\pi)$, then π is the small theta lift of an irreducible ψ-generic representation of Sp_{2k-2a}.

On the other hand, let σ be an irreducible ψ-generic tempered representation of Sp_{2k-2a}, then $W_{\gamma_{r,1}}(\sigma)\neq 0$, then as $1\le a\le k-1$ and σ generic, we have $\theta(\sigma)\neq 0$. Now we want to show $W_{\gamma_1}(\theta(\sigma))\neq 0$, suppose otherwise $W_{\gamma_1}(\theta(\sigma))=0$ then $W_{\gamma_{r,1}}(\Theta(\theta(\sigma)))=0$ but this means σ as a quotient of $\Theta(\theta(\sigma))$ is not generic, a contradiction. ☐

In other words, the theta-lift realizes the desired functorial lifting via the maps $O_{2k-2a+1}\times SL_2\to O_{2k}$ and $O_{2a+1}\times SL_2\to O_{2k}$. When $a=k-1$, the corresponding nilpotent orbit is trivial and we obtain the case of spherical variety $O_{2k-1}\backslash O_{2k}$.

5.2. **Exceptional partitions.** For the $[3,3]$ partition, since $A_3=D_3$, we can assume our group is GL_4 and the nilpotent orbit is of type [3,1], we have the following theorem:

Theorem 5.3. *Let*

- *M_1 be the hyperspherical variety associated with the datum $GL_1\times SL_2\to GL_4$ corresponding to the nilpotent orbit γ of GL_4 with partition $[3,1]$ and trivial S.*
- *M_2 be the hyperspherical variety associated with the datum $GL_4\times SL_2\to GL_4$ corresponds to the trivial nilpotent orbit and $S=Std\oplus Std^*$, for Std the standard representation of GL_4.*

Then M_1 and M_2 are dual under relative Langlands duality.

The quantization of M_1 is the generalized Whittaker representation $W_{\gamma,\psi}$ and M_2 is the quantization of the pullback of the Weil representation ω_ψ of Sp_8 to the Levi factor GL_4 of its Siegel parabolic subgroup. The decomposition of $W_{\gamma,\psi}$ follows from the result 3.4 and the decomposition of ω_ψ can be viewed as the Adams conjecture for the dual pair $U_1\times U_4\cong GL_4$.

For the $[4,4]$ partition. We may take the group as $G=PGSO_8$ to be the adjoint group. There are three non-conjugate homomorphisms

$$f_j:\ SO_8\to G=PGSO_8$$

and $p_j: G^\vee=\mathrm{Spin}_8\to SO_8$. If we denote SO_7 the stabilizer in SO_8 of a unit vector in the standard representation and

$$\mathrm{Spin}_7^{[j]}:=p_j^{-1}(SO_7)\subset SO_8$$

p_1 is the standard representation and p_2,p_3 are considered as the half-spin representations of Spin_8, this gives three distinct conjugacy classes of embeddings $\mathrm{Spin}_7\to\mathrm{Spin}_8$ and hence three spherical varieties $X_j=\mathrm{Spin}_7^{[j]}\backslash\mathrm{Spin}_8$.

Theorem 5.4. *Let*

- *M_1 be the hyperspherical variety associated with the datum corresponding to a nilpotent orbit of $PGSO_8$ associated to a partition $[4,4]$.*
- *M_2 is the cotangent bundle of the spherical variety*

$$X_2=Spin_7^{[2]}\backslash Spin_8$$

then M_1 and M_2 are dual under relative Langlands duality.

The quantization of M_1 is the generalized Whittaker model associated with the partition $[4,4]$. The quantization of M_2 is $L^2(X_2)$. The triality automorphism θ carries $\mathrm{Spin}_7^{[1]}$ to $\mathrm{Spin}_7^{[2]}$ and it induces

$$C_c^\infty(X_2) \cong C_c^\infty(X_1)^\theta$$

for X_1 we have $SO_7\backslash SO_8 \cong \mathrm{Spin}_7^{[1]}\backslash\mathrm{Spin}_8 = X_1$.

For the $[6,6]$ partition. One expects the following by the result of [WZ21].

Theorem 5.5. *Let*

- *M_1 be the hyperspherical variety associated with the datum corresponding to a nilpotent orbit γ of $PGSO_{12}$ with partition $[6,6]$.*
- *M_2 the half-spin representations S of $Spin_{12}$.*

Then M_1 and M_2 are dual under relative Langlands duality.

As before, M_1 has a quantization $W_{\gamma,\psi}$ and the quantization of M_2 can be obtained from the pullback of the half-spin representation of the Weil representation ω_ψ of Mp_{32}, $(SL_2, H) = (SL_2, \mathrm{Spin}_{12})$ is a dual pair in the exceptional group E_7, where H is the derived subgroup of the Levi factor L of a Heisenberg parabolic $P = LU$ of E_7 and the unipotent U is a Heisenberg group corresponding to a 32-dimensional symplectic vector space on which H acts via half-spin representation. For Π the minimal representation of E_7, one has

$$\omega_\psi \cong \Pi_{N,\psi}$$

as Spin_{12}, where N is a maximal unipotent subgroup of SL_2. The decomposition of ω_ψ can be described in terms of the exceptional theta correspondence.

References

[GJ23] Wee Teck Gan and Bryan Wang Peng Jun. Generalised whittaker models as instances of relative langlands duality. *arXiv preprint arXiv:2309.08874*, 2023.

[GZ14] Raul Gomez and Chen-Bo Zhu. Local theta lifting of generalized whittaker models associated to nilpotent orbits. *Geometric and Functional Analysis*, 24:796–853, 2014.

[KVS06] Friedrich Knop and Bart Van Steirteghem. Classification of smooth affine spherical varieties. *Transformation groups*, 11(3):495–516, 2006.

[WZ21] Chen Wan and Lei Zhang. Periods of automorphic forms associated to strongly tempered spherical varieties. *arXiv preprint arXiv:2102.03695*, 2021.