BASE CHANGE FOR GL(2)

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1. INTRODUCTION

I summarize the results for the global base change for GL(2) from Langland's book.

2. PROPERTIES OF BASE CHANGE

So far applications of the trace formula to the comparison of automorphic representations of two different groups have been accompanied by local comparison theorems for characters, the typical example being provided by twisted forms of GL(2). Base change for cyclic extensions is no exception, following Shintani, local liftings can be defined by character relations.

Suppose F is a local field, E a cyclic extension of prime degree ℓ, the Galois group $\Gamma = \mathrm{Gal}(E/F)$ acts on $G(E)$ and we introduce the semi-direct product

$$G'(E) = G(E) \times \mathfrak{G}$$

the group $\mathfrak{G}$ operates on irreducible admissible representations of $G(E)$

$$\Pi^\tau : \ g \to \Pi(\tau(g))$$

and Π can be extended to a representation Π' of $G'(E)$ on the same space if and only if $\Pi^\tau \sim \Pi$ for all τ.

If g lies in $G(E)$, we form

$$Ng = g\sigma(g)\cdots\sigma^{\ell-1}(g)$$

this operation introduced by Saito, is easy to study.

The representation Π of $G(E)$ is said to be a lifting the representation π of $G(F)$ if one of the following two conditions is satisfied

- Π is $\pi(\mu', \nu')$ and π is $\pi(\mu, \nu)$, and $\mu'(x) = \mu(N_{E/F}x)$, $\nu'(x) = \nu(N_{E/F}x)$ for $x \in E^\times$.
- Π is fixed by $\mathfrak{G}$ and for some choice of Π' the equality

$$\chi_{\Pi'}(g \times \sigma) = \chi_\pi(h)$$

  is valid whenever $h = Ng$ has distinct eigenvalues.

It should perhaps be underlined that it is understood that π and Π are irreducible and admissible, and that they are sometimes representations, and sometimes classes of equivalent representations. It is at first sight dismaying that liftings cannot universally characterized by character identities, but it is so, and we are meeting here a particular manifestation of a widespread phenomenon.

We have the following results on the local lifting for fields of characteristic zero

(a) Every π has a unique lifting.
(b) Π is a lifting if and only if $\Pi^\tau \cong \Pi$ for all $\tau \in \Gamma$.
(c) Suppose that Π is a lifting of π and of π'. If $\pi = \pi(\mu, \nu)$ then $\pi' = \pi(\mu', \nu')$ where $\mu^{-1}\mu'$ and $\nu^{-1}\nu'$ are characters of $NE^\times/F^\times$. Otherwise $\pi' \cong \omega \otimes \pi$ where ω is a character of $NE^\times\backslash F^\times$. If ω is non-trivial then $\pi \cong \omega \otimes \pi$ if and only if ℓ is 2 and there is a quasi-character θ of $E^\times$ such that $\pi = \pi(\tau)$ with

$$\tau = \mathrm{Ind}(W_{E/F}, W_{E/E}, \theta)$$

(d) If $k \subset F \subset E$ and E/k, F/k are Galois and $\tau \in \mathrm{Gal}(E/k)$ then the lifting of π^τ is Π^τ if the lifting of π is Π.
(e) If ρ is reducible or dihedral and $\pi = \pi(\rho)$ then the lifting of π is $\pi(P)$ if P is the restriction of ρ to $W_{K/E}$.

(f) If Π is the lifting of π and if Π and π have the central characters ω_Π and ω_π respectively, then $\omega_\Pi(z) = \omega_\pi(N_{E/F}z)$.

(g) The notion of local lifting is independent of the choice of σ.

Many of the properties of local liftings will be proved by global means, namely the trace formula.

Let's recall the function of the trace formula. Let F be a global field and E a cyclic extension of prime degree ℓ, let Z be the group of scalar matrices and set

$$Z_E(\mathbb{A}) = Z(F)N_{E/F}Z(\mathbb{A}_E)$$

let ξ be a unitary character of $Z_E(\mathbb{A})$ trivial on $Z(F)$.

We introduce the space $L_s(\xi)$ of measurable functions φ on $G(F)\backslash G(\mathbb{A})$ which satisfy

$$\varphi(zg) = \xi(z)\varphi(g), \quad \int_{Z_E(\mathbb{A})G(F)\backslash G(\mathbb{A})} |\varphi(g)|^2 \, dg < \infty$$

$G(\mathbb{A})$ acts on $L_s(\xi)$ by left translations. The space $L_s(\xi)$ is the direct sum of two mutually orthogonal invariant subspaces: $L_{sp}(\xi)$, the space of square-integrable cusp forms and $L_{se}(\xi)$, its orthogonal complement. The theory of Eisenstein series decomposes $L_{se}(\xi)$ into the sum of $L_{se}^0(\xi)$, the space of the one dimensional invariant subspaces of $L_s(\xi)$ and $L_{se}^1(\xi)$. We denote by r the representation of $G(\mathbb{A})$ on the sum of $L_{sp}(\xi)$ and $L_{se}^0(\xi)$.

Suppose we have a collection of functions f_v one for each place v of F, satisfying the following conditions

- f_v is a function on $G(F_v)$, smooth and compactly supported modulo $Z(F_v)$.
- $f_v(zg) = \xi^{-1}(z)f_v(g)$ for $z \in N_{E_v/F_v}Z(E_v)$.
- For almost all v, f_v is invariant under $G(\mathcal{O}_{F_v})$ is supported on the product

$$G(\mathcal{O}_{F_v})N_{E_v/F_v}Z(E_v)$$

  and satisfies

$$\int_{N_{E_v/F_v}Z(E_v)\backslash G(\mathcal{O}_{F_v})N_{E_v/F_v}Z(E_v)} f_v(g) \, dg = 1$$

we may define a function f on $G(\mathbb{A})$ by

$$f(g) = \prod_v f_v(g_v)$$

where $g = (g_v)$, the operator

$$r(f) = \int_{N_{E/F}Z(\mathbb{A}_E)\backslash G(\mathbb{A})} f(g)r(g) \, dg$$

is defined and is of trace class.

We will consider functions on ϕ on $G(\mathbb{A}_E)$ defined by a collection ϕ_v one for each place v of E satisfying

- ϕ_v is a function on $G(E_v)$, smooth and compactly supported modulo $Z(E_v)$.
- $\phi_v(zg) = \xi_E^{-1}f_v(g)$ for $z \in Z(E_v)$.
- For almost all v, ϕ_v is invariant under $G(\mathcal{O}_{E_v})$, is supported on $Z(E_v)G(\mathcal{O}_{E_v})$ and satisfies

$$\int_{Z(E_v)\backslash Z(E_v)G(\mathcal{O}_{E_v})} \phi(g) \, dg = 1$$

  then $\phi(g) = \prod_v \phi_v(g_v)$ and

$$r(\phi) = \int_{Z(\mathbb{A}_E)\backslash G(\mathbb{A}_E)} \phi(g)r(g) \, dg$$

is defined and of trace class.

we now introduce another representation R of $G'(\mathbb{A}_E)$, if ℓ is odd, then R is the direct sum of ℓ copies of r, the definition of R for ℓ even is more complicated. The function of the trace formula is to show that for compatible choices of ϕ and f

(2.1) $$\text{trace } R(\phi)R(\sigma) = \text{trace } r(f)$$

here σ is the fixed generator of $\text{Gal}(E/F)$. The trace formula for the left side is different from the usual trace formula, and is usually referred as the twisted trace formula. The condition of compatibility means

that for $\phi_v \to f_v$ for all v, when v does not ramify in E, ϕ_v lies in $\mathcal{H}_{E_v}$ and f_v is its image in $\mathcal{H}_{F_v}$ under the homomorphism $\mathcal{H}_{F_v} \to \mathcal{H}_{E_v}$. The homomorphism from $\mathcal{H}_{E_v}$ to $\mathcal{H}_{F_v}$ was just one of many provided by the general theory of spherical functions and the formalism of the L-group.

The definition of the arrow $\phi_v \to f_v$ and the structure of the trace formula together imply immediately that the two sides of (2.1) are almost equal. The difference is made up of terms contributed to the trace formula by the cusps. There is a place for insight and elegance in the proof that it is indeed zero, but note that the proof is regarded as a technical difficulty.

One chooses a finite set of places V, including the infinite places and the places ramifying in E and for each $v \notin V$ an unramified representation Π_v of $G(E_v)$ such that $\Pi_v^\sigma \sim \Pi_v$, let $\mathfrak{U}$ be the set of irreducible constituents Π of R counted with multiplicity, such that Π_v is the given Π_v outside V, by the strong multiplicity one, $\mathfrak{U}$ is either empty or consists of a single repeated element, and if $\Pi \in \mathfrak{U}$ then $\Pi^\sigma \sim \Pi$. If $\Pi^\sigma \sim \Pi$ then $G'(\mathbb{A}_E)$ leaves the space of Π invariant, and so we obtain a representation Π' of $G'(\mathbb{A}_E)$ as well as local representations Π'_v, set

$$A = \sum_{\Pi \in \mathfrak{U}} \prod_{v \in V} \text{trace } \Pi_v(\phi_v) \Pi'_v(\sigma)$$

let $\mathfrak{B}$ be the set of constituents $\pi = \otimes_v \pi_v$ of r such that Π_v is a lifting of π_v for each v outside of V, set

$$B = \sum_{\pi \in \mathfrak{B}} \prod_{v \in V} \text{trace } \pi_v(f_v)$$

elementary functional analysis enables us to deduce from (2.1) that $A = B$.

A word might be in order to explain why we need the more elaborate definition of R when $\ell = 2$. When $[E : F] = 2$ and there are two dimensional representations ρ of the Weil group W_F induced from characters of the Weil group W_E. These representations have several distinctive properties which we might expect to be mirrored by $\pi(\rho)$. If F is global this means that the cuspidal representation $\pi(\rho)$ becomes Eisensteinian upon lifting.

In the course of proving the results on local lifting, we obtain the existence of global liftings, at least for a cyclic extension of prime degree ℓ, if Π is an automorphic representation of $G(\mathbb{A}_E)$, then for each place v of F, Π determines a representation Π_v of $G(E_v)$ and Π is said to be a lifting of π if Π_v is a lifting of π_v for each v. The first properties of global liftings are:

(A) Every π has a unique lifting.

(B) If Π is isobaric, in particular cuspidal, then Π is a lifting if and only if $\Pi^\tau \sim \Pi$ for all $\tau \in \text{Gal}(E/F)$.

(C) Suppose π lifts to Π, if $\pi = \pi(\mu, \nu)$ with two characters of the idele class group, then the only other automorphic representations lifting to Π are $\pi(\mu_1\mu, \nu_1\nu)$ where μ_1, ν_1 are characters of $F^\times N_{E/F} I_E \backslash I_F$. If π is cuspidal then π' lifts to Π if and only if $\pi' = \omega \otimes \pi$ where ω is a character of $F^\times N_{E/F} I_E$. The number of such π' is ℓ unless $\ell = 2$ and $\pi = \pi(\tau)$ where τ is a two-dimensional representation of $W_{E/F}$ induced by a character of $E^\times \backslash I_E$ when it is one, for $\pi \sim \omega \otimes \pi$.

(D) Suppose $k \subset F \subset E$, F/k, E/k are Galois, if $\tau \in \text{Gal}(E/k)$ and Π is a lifting of π, then Π^τ is a lifting of π^τ.

(E) The central character ω_π of π is defined by $\pi(z) = \omega_\pi(z)I$, $z \in Z(\mathbb{A}) = I_F$, and ω_Π is defined by a similar fashion. If Π is a lifting of π then

$$\omega_\Pi(z) = \omega_\pi(N_{E/F}z)$$

If Π is cuspidal then Π is said to be a quasi-lifting of π if Π_v is a lifting of π_v for almost all v.

(F) A quasi-lifting is a lifting.

It is worthwhile to remark that the first five of these properties have analogues for two-dimensional representations of the Weil group W_F of F if lifting is replaced by restriction to W_E.

3. The trace formula

Set

$$Z_E(\mathbb{A}) = Z(F)N_{E/F}Z(\mathbb{A}_E)$$

let ξ be a unitary character of $Z_E(\mathbb{A})$ trivial on $Z(F)$, $L_s(\xi)$ is the space of measurable functions φ on $G(F)\backslash G(\mathbb{A})$ which satisfy

- $\varphi(zg) = \xi(z)\varphi(g)$ for all $z \in Z_E(\mathbb{A})$.
- $\int_{Z_E(\mathbb{A})G(F)\backslash G(\mathbb{A})} |\varphi(g)|^2\ dg < \infty.$

$G(\mathbb{A})$ acts on $L_s(\xi)$ by right translations

$$r(g)\varphi(h) = \varphi(hg)$$

the space $L_s(\xi)$ is the direct sum of three mutually orthogonal subspaces $L_{sp}(\xi)$, $L_{se}^0(\xi)$ and $L_{se}^1(\xi)$, the representation of $G(\mathbb{A})$ on $L_{sp}(\xi) + L_{se}^0(\xi)$ is denoted by r.

Let f be a function on $G(\mathbb{A})$ defined by

$$f(g) = \prod_v\ f_v(g_v)$$

where f_v satisfied certain conditions, recall that

$$r(f)\varphi(h) = \int_{N_{E/F}Z(\mathbb{A}_E)\backslash G(\mathbb{A})} \varphi(hg)f(g)\ dg$$

it is of trace class, we shall express it as a sum of invariant distributions.

The first term of the sum is

$$\sum_\gamma\ \epsilon(\gamma)\mathrm{means}(N_{E/F}Z(\mathbb{A}_E)G_\gamma(F)\backslash G_\gamma(\mathbb{A})) \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})}\ f(g^{-1}\gamma g)\ dg$$

the sum is over conjugacy classes in $G(F)$ for which $G_\gamma(F)$ does not lie in a Borel subgroup taken modulo $N_{E/F}Z(F)$.

4. The comparison

Let V be a finite set of places containing all infinite places and all finite places ramified in E. Suppose that for each $v \notin V$ we are given $r_v = \begin{pmatrix} a_v & 0 \\ 0 & b_v \end{pmatrix}$, where $a_v b_v = \xi(\varpi_v)$ if v is split and $(a_v b_v)^\ell = \xi(\varpi_v^\ell)$ if v is not split. Set

$$A_1 = \sum \prod_{v \in V}\ \mathrm{trace}(\Pi_v(\phi_v)\Pi_v'(\sigma))$$

the sum is taken over all Π occurring in the representation of $G(E)$ on $L_{sp}(\xi_E) \oplus L_{se}^0(\xi_E)$ for which Π_v is unrmaified outside of V and for which

$$\mathrm{trace}\ \Pi_v(\phi_v) = f_v^\vee(r_v)$$

for all $v \notin V$ and all spherical ϕ_v. Observe that by the strong form of the multiplicity one theorem, the sum is either empty or contains a single term.

We set

$$A_2 = \sum \prod_{v \in V}\ \mathrm{trace}\ \tau(\phi_v, \eta_v)\ \tau(\sigma,\ \eta_v)$$

since $\tau(\eta) \sim \tau(\tilde{\eta})$, we take the sum over unordered pairs $(\eta, \tilde{\eta})$ for which

- $\eta^\sigma = \tilde{\eta}$.
- $\eta \neq \tilde{\eta}$.
- $\eta = (\mu, \nu)$ and $\mu\nu = \xi_E$.
- η_v is unramified for $v \notin V$.
- if $\phi_v,\ v \notin V$ is spherical then

$$\mathrm{trace}\ \tau(\phi_v, \eta_v) = \mathrm{trace}\ \rho(\phi_v, \eta_v) = f_v^\vee(r_v)$$

set

$$A = \ell A_1 + A_2$$

Finally set

$$B = \sum \prod_{v \in V}\ \mathrm{trace}\ \pi_v(f_v)$$

4

the sum is taken over all π occuring in the representation r for which π_v is unramified outside of V and for which

$$\text{trace } \pi_v(f_v) = f_v^\vee(r_v)$$

we know that

$$A = B$$

the next lemma follows from $A = B$

Lemma 4.1. *We have*

(a) Suppose E is a quadratic extension of the global field F and $\Pi = \pi(\mu, \mu^\sigma)$ with $\mu^\sigma \neq \mu$, then Π is the lifting of a unique π.

(b) Suppose that E is cyclic of prime degree ℓ and Π is a cuspidal automorphic representation of $G(\mathbb{A}_E)$ with $\Pi^\sigma \cong \Pi$, then Π is the lifting of ℓ cuspidal automorphic representations.

(c) If Π is a cuspidal automorphic representation of $G(\mathbb{A}_E)$ and $\Pi^\sigma \sim \Pi$ then Π is a quasi-lifting of some π.