FOURIER TRANSFORM ON BASIC AFFINE SPACE AND DOUBLING METHOD

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1. Introduction

This is a study note for the paper by Shahidi [Sha18].

The Braverman-Kazhdan-Ngo program aims to generalize the method of Godement-Jacquet on principal L-functions for GL(n) to other groups. In the case of standard L-functions for classical groups, a theory of ths nature was developed by Piatetski-Shapiro and Rallis, called the doubling method. It was later by Braverman and Kazhdan using an algebro-geometric approach, introduced a space of Schwartz functions and a Fourier transform which projected onto those from thedoubling method. The purpose of the Shahidi paper is to show that the Fourier transform of Braverman-Kazhdan projects onto that of doubling method.

2. Doubling method

Let E be a p-adic local field of characteristic zero with an automorphism θ of order 1 or 2 with a fixed field F. Let $|\ | = |\ |_E$ be the normalized absolute value of E given by $|\omega_E|_E = q_E^{-1}$, here q_E is the order of $\mathcal{O}_E/P_E$. Let h be semi-linear E-valued form on an n-dimensional vector space V over E such that

$$\theta(h(v,u)) = \epsilon h(u,v)$$

where $\epsilon\theta(\epsilon) = 1$.

We consider the pair (V,h) and let G be the isometry group of this pair as an algebraic group over F, $G(F) \subset GL_n(E)$ or $G(F) \subset GL_n(F)$ depending h is hermitian or not.

Doubling method induces a doubling of our data. We consider the space $W = V \oplus V$ together with the form $h^\square = h \oplus -h$. Let $H = G^\square$ be the isometry group of $h^\square$, we can identify $G \times G$ as a subgroup of H preserving $V_1 = V \oplus \{0\}$ and $V_2 = \{0\} \oplus V$. If $h \neq 0$, then $V^d = \{(v,v) \mid v \in V\}$ is a maximal totally isotropic subspace of $(W, h^\square)$, and hence the stabilizer of v^d will be a maximal parabolic subgroup P whose Levi subgroup $M \cong GL(V^d)$.

Let χ be a character of $E^\times$ and $s \in \mathbb{C}$, $\chi_s = \chi \cdot |\ |_E^s$, define a character of $M(F) = GL(V^d)(F)$ by $\chi_s \cdot \Delta$ where $\Delta = \det$ under the isomorphsim $M \cong GL(V^d)$, set

$$I(s,\chi) := \mathrm{Ind}_{P(F)}^{H(F)} \chi_s \cdot \Delta$$

The main goal of the doubling method was to develop the theory of standard L-functions for classical groups along the lines of Godement-Jacquet, i.e. by means of matrix coefficients.

One of the main motivation for a local theory as proposed by Braverman-Kazhdan and Ngo, is to define the local γ-factor needed to derive the global functional equation. In the case of doubling method these factors are defined in every case. The main tool in definition is a normalized intertwining operator originating from $I(s,\chi)$.

Let ω_0 be the image of $(I, -I) \in G \times G$ in H, it is the long element of the Weyl group of H modulo that of $M \cong GL(V^d)$. We define the intertwining operator

$$(M_{\omega_0} f)(x) = \int_{U_P(F)} f(\omega_0 u x)\, du$$

We write $M(s,\chi) f_s$ to denote $M_{\omega_0} f$. Note that

$$M(s,\chi):\ I(s,\chi) \longrightarrow I(-s, \theta(\chi)^{-1})$$

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We now define the zeta-function that generalizes GJ's zeta function. Let π be an irreducible admissible representation of $G(F)$ and $\tilde{\pi}$ its contragredient. Let $p = p_\pi$ be the standard pairing

$$p : \pi \otimes \tilde{\pi} \longrightarrow \mathbb{C}$$

Given $\alpha \otimes \tilde{\alpha} \in \pi \otimes \tilde{\pi}$, consider the function $p(\pi(g)\alpha \otimes \tilde{\alpha})$, a matrix coefficient of π. We then set

$$Z(f_s, \alpha \otimes \tilde{\alpha}) = \int_{G(F)} f_s(i(g,1))p(\pi(g)\alpha \otimes \tilde{\alpha})\ dg$$

where $i : G(F) \times G(F) \hookrightarrow H(F)$ is the embedding as before. The integral defining the zeta function converges for $\mathrm{Re}(s) >> 0$ and extends to a rational function in q^{-s} on all of $\mathbb{C}$.

There exists a scalar-valued function $\Gamma(s, \pi, \theta)$ such that

$$Z(M(s)f_s, \alpha \otimes \tilde{\alpha}) = \Gamma(s, \pi, \chi)Z(f_s, \alpha \otimes \tilde{\alpha}) \tag{2.1}$$

The point is that $\Gamma(s, \pi, \chi)$ is not the γ-factor

$$\gamma(s, \pi \times \chi, \mathrm{std}, \psi) = \epsilon(s, \pi \times \chi, \mathrm{std}, \psi)\frac{L(1-s, \tilde{\pi} \times \chi^{-1}, \mathrm{std})}{L(s, \pi \times \chi, \mathrm{std})}$$

To understand this discrepancy one has to compute the zeta function for the unramified data. We can define a function $d_H(s, \chi)$ as a product of 1-dimensional L-functions according to h is symplectic or hermitian.

We let $\alpha_0 \in \pi^K,\ \tilde{\alpha}_0 \in \tilde{\pi}^K$ with $p(\alpha_0 \otimes \tilde{\alpha}_0)$, define $f^0 \in I(0, \chi)$ such that $f^0(k) = 1$ for all $k \in K$.

Proposition 2.1. *Let $\alpha_0,\ \tilde{\alpha}_0$ and f_s^0 be the K-fixed data, assume χ is unramified, then*

$$Z(f_s^0, \alpha_0 \otimes \tilde{\alpha}_0) = \frac{L(s+\frac{1}{2}, \pi \times \chi)}{d_H(s, \chi)}$$

To obtain the γ-factor $\gamma(s, \pi \times \chi, \psi)$ attached to the standard L-function $L(s, \pi \times \chi)$ one needs to correct $\Gamma(s, \pi, \chi)$, this will be given by a formal normalization of $M(\overline{s}, \chi)$. We introduce

$$a_H(s) = a_H(s, 1) = d_H(s - \langle \rho, \epsilon_1 \rangle, 1)$$

Write

$$M(s)f_s^0 = m(s)f_{-s}^0$$

where f_{-s}^0 is the normalized spherical function in $I(-s, 1)$ and $m(s)$ is a scalar. Then the calculation in [LR05] shows that

$$m(s) = \frac{a_H(s)}{d_H(s)}$$

for $d_H(s) := d_H(s, 1)$. Applying (3.1) to the unramified case, we get

$$m(s)Z(f_{-s}^0, \alpha_0 \otimes \tilde{\alpha}_0) = \Gamma(s, \pi, 1)Z(f_s^0, \alpha_0 \otimes \tilde{\alpha}_0)$$

Now the unramified computation gives us

$$m(s)d_H(-s)^{-1}L(-s+\frac{1}{2}, \tilde{\pi}) = \Gamma(s, \pi, 1)L(s+\frac{1}{2}, \pi)d_H(s)^{-1}$$

In the unramified case

$$\begin{aligned}\gamma(s+\frac{1}{2}, \pi, \psi) &= \frac{L(1-(s+\frac{1}{2}), \tilde{\pi})}{L(s+\frac{1}{2}, \pi)} \\ &= \Gamma(s, \pi, 1)\frac{d_H(-s)}{d_H(s)}m(s)^{-1} \\ &= \Gamma(s, \pi, 1)\frac{d_H(-s)}{a_H(s)}\end{aligned}$$

we conclude that the correct factor

$$\eta(s) = \frac{d_H(-s)}{a_H(s)}$$

will give the correct γ-factor.

Proposition 2.2. *The unramified γ-factor*

$$\gamma(s,\pi,\psi) := \frac{L(1-s,\tilde{\pi})}{L(s,\pi)}$$

is equal to

$$\gamma(s,\pi,\psi) = \Gamma(s-\frac{1}{2},\pi,1)\eta(s-\frac{1}{2})$$

What generalizes this to the ramified case is an extension of local coefficients to our degenerate induced representation $I(s,\chi)$. Consider the induced representation $I(\psi_A) = \mathrm{Ind}_{U_P(F)}^{H(F)}\psi_A$, one knows that

$$\dim\ (\mathrm{Hom}_{H(F)}(I(s,\chi), I(\psi_A))) \leq 1$$

we have

$$\ell_{\psi_A}(f) = \int_{U_P(F)} f(\omega_0 u)\overline{\psi_A}(u)\ du$$

is a non-zero candidate for this space. One defines simiarly a functional ℓ'_{ψ_A} for $I(-s,\chi^{-1})$.

A degenerate local coefficient $c(s,\chi,A,\psi)$ is defined by

$$\ell'_{\psi_A}(M_{\omega_0}(f)) = c(s,\chi,A,\psi)\ell_{\psi_A}(f)$$

We put

$$M^*_{\omega_0}(s,\chi,A,\psi) = c(s,\chi,A,\psi)^{-1}M_{\omega_0}(s,\chi)$$

Then

$$M^*_{\omega_0}(-s,\chi^{-1},A,\psi)M^*_{\omega_0}(s,\chi,A,\psi) = I$$

One now defines

$$\Gamma(s,\pi,\chi,A,\psi) = \Gamma(s,\pi,\chi)c(s,\chi,A,\psi)^{-1}$$

3. Connection with the Fourier transform

Let F be a p-adic field with $\mathcal{O} = \mathcal{O}_F$ its ring of integers, let H be a split connected reductive group over F, we fix a parabolic subgroup P with a fixed Levi decomposition $P = MU_P$ where P is the unipotent radical of P, set

$$M_{ab} := M/M_{der} \cong P/P_{der}$$

where $M_{der} = [M,M]$ and $P_{der} = [P,P]$, let

$$X_P := P_{der}\backslash H$$

it is a $M_{ab}\times H$-space. And $P \hookrightarrow H$ gives an embedding

$$P \hookrightarrow M_{ab}\times H$$

and one can identify X_P with $P\backslash M_{ab}\times H$, for F-points we have

$$\begin{aligned} X_P(F) &= (P_{der}\backslash H)(F) \\ &= P_{der}(F)\backslash H(F) \end{aligned}$$

if $H^1(P_{der})$ is trivial. This will be the case if H_{der} is simply connected, but it will also be the case for classical group where P is the Siegel parabolic for which $M = GL_n$.

Given a character χ of $M_{ab}(F)$ and a representation π of $H(F)$, Frobenius reciprocity implies

$$\mathrm{Hom}_{M_{ab}\times H(F)}(\chi\otimes\pi, C^\infty(X_P)) = \mathrm{Hom}_{P(F)}(\chi\otimes\pi, \delta_P^{1/2})$$

where

$$\tilde{\chi}:\ M(F) \longrightarrow M_{ab}(F) \longrightarrow \mathbb{C}^\times$$

take π to be the right action of $H(F)$ on $C_c^\infty(X_P)$, we have an $H(F)$-map

$$C_c^\infty(X_P) \longrightarrow \mathrm{Ind}_{P(F)}^{H(F)}\tilde{\chi}$$

We now assume that H_{der} is simply connected and fix a non-trivial additive character ψ of F. We will first define a Schwartz space in this case which turns out to be the ρ-Schwartz space when ρ is the standard representation of an appropriate L-group.

Let P and Q be two parabolic subgroups of H, sharing the same Levi subgroup M, then Braverman and Kazhdan define an intertwining map

$$\mathcal{F}_{Q|P} = \mathcal{F}_{Q|P,\psi} : L^2(X_P) \longrightarrow L^2(X_Q)$$

which is $(M_{ab} \times H)(F)$-equivariant and an isometry. As we range the parabolic subgroup, the family of maps satisfy

$$\mathcal{F}_{R|Q} \cdot \mathcal{F}_{Q|P} = \mathcal{F}_{R|P}$$

and $\mathcal{F}_{P|P} = id$ and thus

$$\mathcal{F}_{P|Q} \cdot \mathcal{F}_{Q|P} = id$$

The Schwartz space $S(X_P)$ is defined as

$$S(X_P) := \sum_{Q} \mathcal{F}_{P|Q}(C_c^\infty(X_Q))$$

where Q runs over prabolic subgroups sharing the same Levi subgroup P. The space is a smooth $(M_{ab} \times H)(F)$-representation through the action on different X_Q of $(M_{ab} \times H)(F)$.

The Fourier transform $\mathcal{F}$ is defined and the following diagram is commutative

$$\begin{array}{ccc} S(X_Q) & \xrightarrow{\mathcal{F}_{P|Q}} & S(X_P) \\ \downarrow & & \downarrow \\ \mathrm{Ind}_{Q(F)}^{H(F)}\tilde{\chi} & \xrightarrow{N_{P|Q}} & \mathrm{Ind}_{P(F)}^{H(F)}\tilde{\chi} \end{array} \tag{3.1}$$

In particular, $\mathcal{F}_{P|Q}$ projects to a normalized standard intertwining operator $N_{P|Q}$, we remark that there are several ways to normalize the standard operator and this one is not the standard one used in the trace formula.

The Fourier transform are defined in two steps:

1. The first Radon map $\mathcal{R}_{P|Q}$

$$\begin{array}{ccc} C_c^\infty(X_Q) & \xrightarrow{\mathcal{R}_{P|Q}} & C^\infty(X_P) \\ \downarrow & & \downarrow \\ \mathrm{Ind}_{Q(F)}^{H(F)}\tilde{\chi} & \xrightarrow{J_{P|Q}} & \mathrm{Ind}_{P(F)}^{H(F)}\tilde{\chi} \end{array} \tag{3.2}$$

here $J_{P|Q}$ is the standard intertwining operator.

2.The normalizing factor. Braverman and Kazhdan define these factors on $C_c^\infty(X_Q(F))$ as a distribution and when projected via diagrams (3.1) and (3.2) becomes a normalizing factor for $J_{P|Q}$. We shall now explain, let T be a split torus in H we have the lattice of cocharacters

$$\Lambda_*(T) = \mathrm{Hom}(\mathbb{G}_m, T) = \Lambda^*(\hat{T})$$

where $\hat{T}$ is the dual group of T.

Let $L = \oplus_{i=1}^k L_i$ be a graded finite dimensional representation of $\hat{T}$ defined by a collection of elements $\lambda_1, \cdots, \lambda_k \in X^*(\hat{T})$, we assign integers $n_1, \cdots, n_k$ to these eigenspaces which we will be more specific in the case of interest. We allow multiplicity among λ_i and n_i.

Considering $\lambda_i : \mathbb{G}_m \to T$ and we now consider its pushforward $(\lambda_i)!$ on the space of distribution on $\mathbb{G}_m$ and thus

$$(\lambda_i!)(\eta)(\varphi) = \eta(\varphi \cdot \lambda_i)$$

for each distribution η on F^* and each function φ on $T(F)$ whenever it makes sense.

We will now restrict ourselves to a specific distribution on F^*, let ψ be a nontrivial additive character of F, fix a self-dual measure dx with respect to ψ, let $s \in \mathbb{C}$, define the distribution

$$\eta := \psi(x)|x|^s|dx|$$

the distribution $\eta^s_\psi = \eta$ can be integrated against a character $\chi \in \hat{F}^*$

$$\langle \eta, \chi \rangle = \int \chi(x)\psi(x)|x|^s|dx|$$

we get a rational function via Laurent power series

$$M_{\eta,\chi}(z) = \sum_{n=-\infty}^{\infty} z^n \int_{|x|=q^{-n}} \chi(x)\psi(x)|dx|$$

setting $z = 1$, we get $M(\eta)(\chi) = M_{\eta,\chi}(1)$, it is clear that

$$M(\eta^s_\psi)(\chi) = \gamma(s, \chi, \psi)$$

Given our graded $\hat{T}$-representation L, one can define a distribution

$$\eta_{L,\psi} := \eta^{s_1,s_2\cdots s_k}_{\lambda_1,\lambda_2\cdots\lambda_k,\psi}$$

Moreover

$$M(\eta_{L,\psi}) = M((\lambda_1!)(\eta^{s_1}_\psi)) \cdots M((\lambda_k!)(\eta^{s_k}_\psi))$$

note that since $\chi \in \widehat{T(F)}$, $\chi \cdot \lambda_i \in \hat{F}^*$ and thus

$$M((\lambda_i!)(\eta^{s_i}_\psi))(\chi) = M((\eta^{s_i}_\psi))(\chi \cdot \lambda_i) = \gamma(s_i, \chi \cdot \lambda_i, \psi)$$

and therefore

$$M(\eta_{L,\psi})(\chi) = \prod_{i=1}^{k} \gamma(s_i, \chi \cdot \lambda_i, \psi)$$

now assume $T = M_{ab}$, one can consider the convolution $\eta_{L,\psi} * \varphi$ of the distribution $\eta_{L,\psi}$ on $M_{ab}(F)$ with any function $\varphi \in C^\infty_c(X_P)$ on $M_{ab}(F)$ to get another distribution on $M_{ab}(F)$.

Proposition 3.1. *The convolution by $\eta_{L,\psi}$ of elements of $C^\infty_c(X_P)$ covers multiplication by the rational function $M(\eta_{L,\psi})(\chi)$ on $Ind^{H(F)}_{P(F)}\tilde{\chi}$ for each $\chi \in \widehat{M_{ab}(F)}$.*

We now turn to the situation for the doubling method, that is G is the isometry group of (V, h), and we consider the space $W = V \oplus V$ together with the form $h^\square = h \oplus -h$. Let $H = G^\square$ be the isometry group of $h^\square$, we can identify $G \times G$ as a subgroup of H preserving $V_1 = V \oplus \{0\}$ and $V_2 = \{0\} \oplus V$. If $h \neq 0$, then $V^d = \{(v, v) \mid v \in V\}$ is a maximal totally isotropic subspace of $(W, h^\square)$, and hence the stabilizer of v^d will be a maximal parabolic subgroup P whose Levi subgroup $M \cong GL(V^d)$.

Proposition 3.2. *The normalizing factor $\eta(s)$ as defined in* (3.2) *is the same as the one defined by $M(\eta_{L,\psi})$ of proposition 3.1 by Braverman-Kazhdan for the data $(\hat{\mathfrak{u}}_{\mathfrak{p}})^e$.*

References

[LR05] Erez M Lapid and Stephen Rallis. On the local factors of representations of classical groups. *Automorphic representations, L-functions and applications: progress and prospects*, 11:309–359, 2005.

[Sha18] Freydoon Shahidi. On generalized Fourier transforms for standard L-functions. In *Geometric Aspects of the Trace Formula 2*, pages 351–404. Springer, 2018.