LANGLANDS DUAL GROUP FOR SPHERICAL VARIETIES

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Contents

1. Introduction

This is my personal study note for Raphael Beuzart-Plessis's report for IHES summer school, in this study note, we present a definition of the Langlands dual group for spherical varieties after Gaitsgory-Nadler[1], Sakellaridis-Venkatesh [2] and Knop-Shalke[3], which is compatible with the relative Langlands conjecture proposed by Prasad in the Galois symmetric variety case and is also believe to be the correct Langlands dual group for the study of smooth and L^2 unramified spectrum for unramified p-adic groups.

2. Notation

We will fix k a characteristic zero field, and G a connected reductive group over k. By a pinning of G, we mean a tuple $(B, A, (X_\alpha)_{\alpha\in\Delta})$, where A is a maximal torus, B a Borel subgroup contains A, Δ the set of simple roots attached to B and A, $X_\alpha \in \mathfrak{g}_\alpha - \{0\}$ for all α.

For X a G-spherical variety over k, we will denote A_X the maximal torus of X, which is a quotient of the maximal torus A, $\mathfrak{a}_X := X_*(A_X) \otimes \mathbb{Q}$.

3. Prasad's conjecture

In this section, we will fix ℓ/k a quadratic extension of local fields and G a quasi-split group over k. We recall Prasad's original construction of another quasi-split group G^{op} which is the conjectural Langlands dual group for Galois symmetric variety $\mathrm{Res}_{\ell/k}G/G$.

Recall that given a reductive algebraic group G over $\overline{k}$ of k, quasi-split form of G over k are described by homomorphism $\mathrm{Gal}(\overline{k}/k) \to \mathrm{Out}G(\overline{k})$, for every reductive group G over $\overline{k}$ there is an automorphism ι that takes every irreducible algebraic representation of G to its contragredient, the automorphism ι defines a well-defined element of $\mathrm{Out}(G(\overline{k}))$, such automorphism ι is called the Chevalley involution.

We will let G^{op} be the quasi-split group over k obtained by twisting G by the Chevalley involution, i.e. the homomorphisms from $\mathrm{Gal}(\overline{k}/k)$ to $\mathrm{Out}(\overline{k}/k)$ associated to G^{op} is obtained the homomorphism $\mathrm{Gal}(\overline{k}/k) \to \mathrm{Out}(G(\overline{k}))$ times the homomorphism $\mathrm{Gal}(\overline{k}/k) \to \mathrm{Out}(G(\overline{k}))$ which sends the nontrivial element of $\mathrm{Gal}(\ell/k)$ to the Chevalley involution ι.

Example 3.1. For $G = GL_n, G^{op} = U_n$ and for $G = U_n, G^{op} = GL_n$.

Date: December 2023.

4. Complex dual group

Following [2], we recall the work of Gaitsgory and Nadler on dual group of spherical varieties.

Let X be a quasiaffine spherical variety over $k = \overline{k}$, an algebraically closed field of characteristic zero. We fix an affine embedding X^a of X, let's denote $\check{A}_{X,GN}$ the image of $\check{A}_X$ in $\check{A}$, and $2\rho_{L(X)}$ the sum of positive roots of $L(X)$ which defines a character $2\rho_{L(X)} : \mathbb{G}_m \to \check{A}$.

Theorem 4.1. *(Gaitsgory and Nadler) To every affine spherical variety X^a one can associate a connected reductive group $\check{G}_{X,GN}$ of $\check{G}$ with maximal torus $\check{A}_{X,GN}$ and the group $\check{G}_{X^a,GN}$ is canonical up to $\check{A}$-conjugacy.*

This theorem is not very informative as stated, to study the root system of the Gaitsgory-Nadler dual group, Sakellaridis and Venkatesh foemulated the following hypotheses, roughly speaking they want the dual group to be compatible with boundary degeneration and has a root system the same as the one from the theory of Luna's spherical systems (certain normalization of spherical roots is needed).

To state the hypotheses, we denote X^a_Θ for every $\Theta \subset \Sigma_X$ the affine embedding of X_Θ, from the definition of X_Θ as a boundary degeneration.

(GN1) The image of $\check{G}_{X^a,GN}$ commutes with $2\rho_{L(X)}(\mathbb{C}^\times) \subset \check{A}$.

(GN2) The Weyl group of $\check{G}_{X^a,GN}$ equals W_X.

(GN3) For any $\Theta \subset \Sigma_X$ the dual group of X^a_Θ is canonically a subgroup of $\check{G}_{X^a,GN}$.

(GN4) If the open G-orbit $X \subset X^a$ is parabolically induced $X = X_L \times^{P^-} G$, where X_L is spherical for the reductive group L of P^-, then the dual group $\check{G}_{X^a,GN}$ belongs to the standard Levi subgroup $\check{L}$ of $\check{G}$ corresponding to the class of parabolic subgroups opposite to P^-. If a connected normal subgroup L_1 of L acts trivially on X_L, then $\check{G}_{X^a,GN}$ belongs to the dual group of L/L_1.

(GN5) If X_1^+ is a spherical homogeneous G-variety and T a torus inside the G-automorphisms and $X_2^+ = X_1^+/T$, if X_1, X_2 are affine embeddings of X_1^+, X_2^+ with $X_2 = \text{spec } k[X_1]^T$, then there is a canonical inclusion $\check{G}_{X_2,GN} \hookrightarrow \check{G}_{X_1,GN}$ which restricts to the natural inclusion of maximal tori $\check{A}_{X_2,GN} \hookrightarrow \check{A}_{X_1,GN}$.

Definition 4.2. We define Δ_X to be a set of generators of intersections of extremal rays of $\mathscr{V}_X^\perp$ with the root lattice $\mathbb{Z}\Delta$ of G. We will write $\Delta_X^\vee \subset \mathfrak{a}_X$ the corresponding set of spherical coroots.

Remark 4.3. In the literature of classification theory of spherical roots, the set of spherical roots Σ_X is taken to be the generators of the intersections of extremal rays of $\mathscr{V}_X^\perp$ with $X^*(A_X)$, the normalization of the spherical root comes from the representation theoretic point of view as we will see later .

We can compare the spherical root for X with the root for G

Proposition 4.4. *For every spherical root $\alpha \in \Delta_X$, exactly one of the following holds:*

(a) α is a positive root for G, i.e. $\alpha \in \Phi^+$.

(b) there exist two positive roots $\gamma_1, \gamma_2 \in \Phi^+$ that are strongly orthogonal and $\alpha = \gamma_1 + \gamma_2$.

We will call a spherical root γ in case (b) a root of type G. The decomposition $\alpha = \gamma_1 + \gamma_2$ is not unique, but we can add more conditions to make this decomposition unique, this is [2] corollary 3.1.4.

The important consequence of the definition of set Δ_X is the following: let $\check{A}$ and $\check{A}_X$ be the complex dual tori of A and A_X, then the surjection $A \to A_X$ dualize to $\check{A}_X \to \check{A}$ and we have

$$\Delta_X^\vee \subset \Phi^\vee|_{\check{A}_X}$$

and this is compatible with the embedding of the dual group $\check{G}_X \to \check{G}$ that we want to establish.

Proposition 4.5. *When X has no spherical root of type N, the quadruple*

$$\mathscr{R}_X := (X^*(A_X), \Delta_X, X_*(A_X), \Delta_X^\vee)$$

is a based root datum.

Definition 4.6. We let $\check{G}_X$ be a complex connected reductive group equipped with a pinning Pin_X and an isomorphism

$$\mathscr{R}_{\check{G}_X} \cong \mathscr{R}_X^\vee$$

here $\mathscr{R}_X^\vee$ is the dual based root datum

$$\mathscr{R}_X^\vee = (X_*(A_X), \Delta_X^\vee, X^*(A_X), \Delta_X)$$

Motivated by the study of L^2-problem, Sakellaridis and Venkatesh also introduced the following algebraic morphism

$$\iota_X^{SL_2} : SL_2(\mathbb{C}) \to \check{G} \tag{4.1}$$

depending only on the parabolic type $P(X)$ of X. The dual group $\check{G}$ comes with a pinning $\mathrm{Pin}_G = (B, A, (X_{\alpha^\vee})_{\alpha\in\Delta})$, the parabolic subgroup $P(X)$ determines a standard Levi subgroup $\check{L}(X)$ of $\check{G}$ with respect to the Borel pair (B, A) equipped with its owning pinning $(B, A, (X_{\alpha^\vee})_{\alpha\in\Delta(L(X))})$. This gives a principal SL_2 morphism $SL_2 \to \check{L}(X)$ whose differential sends $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to the sum $\sum_{\alpha\in\Delta(L(X))} X_{\alpha^\vee}$. Then the map ι_X is the composition of this morphism with the inclusion $\check{L}(X) \subset \check{G}$.

The surjection $A \to A_X$ induces a morphism between dual tori $\check{A}_X \to \check{A}$. The morphism $\check{A}_X \to \check{A}$ has image in the center $Z(\check{L}(X))$ of $\check{L}(X)$ and hence commutes with the image of ι_X.

The resulting morphism $\check{A}_X \times SL_2(\mathbb{C}) \to \check{G}$ can be extended to $\check{G}_X \times SL_2(\mathbb{C}) \to \check{G}$ thanks to the following theorem of Knop-Schlake [3]

Theorem 4.7. *(Knop-Schalke) Assume X has no spherical roots of type N. Then there exists an algebraic morphism*

$$\iota_X : \check{G}_X \times SL_2(\mathbb{C}) \to \check{G}$$

such that

- *The restriction of ι_X to $SL_2(\mathbb{C})$ is the morphism $\iota_X^{SL_2}$ (4.1) we introduced before.*
- *The restriction of ι_X to the maximal torus $\check{A}_X \subset \check{G}_X$ is the morphism $\check{A}_X \to \check{A}$ dual to the surjection $A \to A_X$.*

We also have the following characterization on the image of the root spaces by its differential $d\iota_X$:

- *For every root $\alpha \in \Delta_X$ of type T, i.e. $\alpha \in \Phi$, we have*
$$d\iota_X(\check{\mathfrak{g}}_{X,\alpha^\vee}) = \check{\mathfrak{g}}_{\alpha^\vee}$$
- *For every $\alpha \in \Delta_X$ of type G, we have*
$$d\iota_X(\check{\mathfrak{g}}_{X,\alpha^\vee}) \subset \check{\mathfrak{g}}_{\gamma_1^\vee} \oplus \check{\mathfrak{g}}_{\gamma_2^\vee}$$
where $\gamma_1, \gamma_2 \in \Phi$ are the associated roots of α.

A morphism ι_X satisfies the conditions of the previous theorem is called a distinguished morphism.

Proposition 4.8. *The distinguished morphism is unique up to $Z(\check{L}(X))$-conjugacy.*

Proof. The union of the set of spherical roots of type T and the set of associated roots to spherical roots of type G form a linearly independent set of roots of G and a morphism $\check{G}_X \to \check{G}$ satisfies the last three conditions of theorem 4.7 is unique up to $\check{A}_X$-conjugation and the centralizer of $\iota_X^{SL_2}$ in $\check{A}_X$ is precisely $Z(\check{L}(X))$. □

Let's also explain the Sakellaridis-Venkatesh approach to the existence of distinguished map, their basic idea is to reduce the argument to low rank spherical varieties in the Wasserman table. We will assume that X has no spherical roots of type N, and we denote $\check{G}_X$ the complex reductive group with based root datum $\mathcal{R}_X^\vee$.

Proposition 4.9. *Assume axioms (GN2)-(GN5), for spherical varieties of rank one there is an isomorphism of Gaitsgory-Nadler dual group with $\check{G}_X$ inducing identity on $\check{A}_{X,GN}$, and for any such isomorphism the embedding $\check{G}_X \cong \check{G}_{X,GN} \hookrightarrow \check{G}$ is distinguished.*

By looking at the Wasserman table and assuming the axiom $(GN1)-(GN5)$ one can show that $\check{G}_{X,GN} = \check{G}_{X_\gamma}$.

Corollary 4.10. *Assume axioms $(GN2)-(GN5)$, then there exists a distinguished embedding $\check{G}_X \hookrightarrow \check{G}$, the group $\check{G}_{X,GN}$ is canonically isomorphic to $\check{G}_X$ up to $\check{A}_{X,GN}$-conjugacy.*

Note that $(GN2)$ tells us there is a system of positive coroots for $\check{G}_{X,GN}$ each of which is proportional to one of the $\gamma \in \Delta_X$. On the other hand, by $(GN3)$, the group $\check{G}_{X_\gamma,GN}$ is contained in $\check{G}_{X,GN}$, from the uniqueness of the distinguished morphism for $\check{G}_\gamma \to \check{G}$, one deduce the coroot of $\check{G}_{X,GN}$ proportional to $\gamma \in \Delta_X$ is actually equal to γ, hence the coroots of $\check{G}_{X,GN}$ are precisely $W_X\Delta_X$, the root data of $\check{G}_{X,GN}$ and $\check{G}_X$ coincide canonically and so the two are canonically isomorphic up to $\check{A}_{X,GN}$-conjugacy.

Based on the principal map $SL_2 \to \check{L}(X)$ for X a spherical variety of rank one and $(GN4)$, Sakellaridis and Venkatesh proved

Theorem 4.11. *Assume that there is a distinguished embedding $\check{G}_X \to \check{G}$, then there is a principal $SL_2 \to \check{L}(X)$ and whose image commutes with $\check{G}_X$.*

Example 4.12. We fix $G = SL_3$ and B the upper triangular matrices, A the diagonal maximal torus and α_1, α_2 two positive simple roots for SL_3.

$X = SL_3/SL_2$ is a spherical variety, it has spherical root $\Delta_X = \{\alpha_1 + \alpha_2\}$. We can calculate that the stabilizer of the open Borel orbit is trivial, hence $A_X \cong A$, we see $X^*(A_X) \cong \mathbb{Z}^2$.

The quadruple

$$\mathscr{R}_X^\vee = (X_*(A_X), \Delta_X^\vee, X^*(A_X), \Delta_X)$$

is the root datum of a rank 1 group with a maximal torus of rank two, from the classification of rank one reductive groups, we see that $\check{G}_X \cong GL_2$.

5. L-group

When the spherical variety X is defined over a field k which is not algebraically closed, it is natural to extend the absolute Galois group $\Gamma = \mathrm{Gal}(\overline{k}/k)$ on the dual group $\check{G}_X$, the natural attempt is to fix a suitable Galois stable embedding $\check{G}_X \to \check{G}$ and extends it to a distinguished map ι_X between L-groups

$$\iota_X : {}^LG_X \times SL_2(\mathbb{C}) \longrightarrow {}^LG$$

In [4] motivated by the functorial properties of the dual group for spherical varieties, Knop proposed more conditions on the image of the root space $\check{\mathfrak{g}}_{X,\alpha^\vee}$ by $d\iota_X$ when $\alpha \in \Delta_X$ is a root of type G.

More precisely, he described a line $L_{\alpha^\vee} \subset \check{\mathfrak{g}}_{\gamma_1^\vee} \oplus \check{\mathfrak{g}}_{\gamma_2^\vee}$ where γ_1 and γ_2 are the associated roots of α. Actually, the condition that $\iota_X|_{\check{G}_X}$ commutes with $\iota_X^{SL_2}$ already characterizes the line $L_{\alpha^\vee}$ except in the case when γ_1, γ_2 are also simple roots for G (Galois symmetric variety is such an example). Such spherical roots are called roots of type D_2 by Knop as this is equivalent to root system generated by the support of $\alpha \in \Delta$ being D_2, then for Δ_X of type D_2, Knop's definition is

$$L_{\alpha^\vee} = \mathbb{C}(X_{\gamma_1^\vee} - X_{\gamma_2^\vee})$$

Following Beuzart-Plessis, we will call a distinguished morphism $\iota_X : \check{G}_X \times SL_2(\mathbb{C}) \to \check{G}$ satisfies Knop requirement a *strong distinguished morphism*. The strong distinguished morphisms are unique up to $\check{A}_X$-conjugation.

Proposition 5.1. *The strong distinguished morphisms have the same images in $\check{G}$.*

Proposition 5.2. *The Galois action on $\check{G}$ preserves the image of ι_X.*

Proof. The subspaces $L_{\alpha^\vee}$ for $\alpha \in \Delta_X$ of type D_2 are permuted by Γ hence the composition of ι_X with the action of any $\sigma \in \Gamma$ is again a strong distinguished morphism. □

Example 5.3. Let ℓ/k be a quadratic extension and let $\sigma \in \mathrm{Gal}(\ell/k)$ be the nontrivial element. For any k-algebra R, we consider the three dimensional quasi-split special unitary group attached to the extension ℓ/k:

$$SU_3(R) := \{g \in SL_3(\ell \otimes_k R) : \ \sigma(g)^t Jg = J \ \}$$

for J the antidiagonal matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, denote SU_3 by G, then G has a two dimensional special unitary group SU_2 as a subgroup. We have ${}^LG \cong PGL_3(\mathbb{C}) \rtimes \Gamma$, with the Γ action factors through $\mathrm{Gal}(\ell/k)$ and the nontrivial element $\sigma \in \mathrm{Gal}(\ell/k)$ is sent to $g \mapsto J^{-1}g^{-t}J$.

Over algebraic closure, the spherical variety $X = SU_3/SU_2$ is isomorphic to SL_3/SL_2, hence from the calculation in example 4.12, there is a single spherical root $\gamma = \alpha_1 + \alpha_2$ of type T, then theorem 4.7 tells us that the embedding is given by $GL_2(\mathbb{C}) \hookrightarrow PGL_3(\mathbb{C})$ with $\check{\mathfrak{g}}_{X,\gamma^\vee} = \check{\mathfrak{g}}_{\check{\alpha}_1+\check{\alpha}_2}$, with the induced Galois action from ${}^LG$ one sees that the Langlands dual group of X is isomorphic to ${}^LU_2$, for U_2 the two-dimensonal quasi-split unitary group.

Example 5.4. Let ℓ/k be a quadratic extension and let $X = \mathrm{Res}_{\ell/k}GL_2/GL_2$ be a spherical variety over k, there is a single spherical root of type D_2, from the discussion about the image of the root space $L_{\alpha^\vee}$, we see that the embedding of the L-group is given by $\check{G}_X(\mathbb{C}) \rtimes \Gamma \to \check{G}(\mathbb{C}) \rtimes \Gamma$ with restriction to complex dual group $GL_2(\mathbb{C}) \to GL_2(\mathbb{C}) \times GL_2(\mathbb{C}), g \mapsto (g, g^c)$, for c the Chevalley involution $g \mapsto \omega_\ell g^{-t}\omega_\ell$.

Example 5.5. Let ℓ/k be a cubic extension and let $X = \mathrm{Res}_{\ell/k}GL_2/GL_2 = G/H$, over algebraic closure X is isomorphic to GL_2^3/GL_2, which satisfies $\check{G}_X = \check{G}$, hence we deduce that ${}^LG_X = {}^LG$.

6. Galois symmetric variety

Let us take $X = \mathrm{Res}_{\ell/k}H_\ell/H$ for ℓ/k a quadratic extension, H a connected reductive group over k.

Let's first describe the strong distinguished morphism in the group case, $X = H \times H/H$, then we have

$$\check{G} = \check{H} \times \check{H}, \quad \check{G}_X = \check{H}$$

and all spherical roots are of type D_2, a particular strong distinguished morphism is

$$\iota_X : \check{H} \longrightarrow \check{G}$$

is given by $h \mapsto (h, h^c)$ where $\check{H} \to \check{H} : \ h \mapsto h^c$, is the unique automorphism of $\check{H}$ which sending the pinning $\mathrm{Pin}_H = (B, A, (X_{\alpha^\vee})_{\alpha\in\Delta})$ of $\check{H}$ to its opposite $\mathrm{Pin}_{\check{H}} := (B, A, (-X_{\alpha^\vee})_{\alpha\in\Delta})$ and acting by $t \mapsto \omega_\ell t^{-1}\omega_\ell$ on T where ω_ℓ is the longest Weyl group element. this is the *Chevalley involution* of $\check{H}$.

As the Galois action preserves the pinning Pin_H, it commutes with the Chevalley involution. This let us to extend to an automorphism ${}^LH \to {}^LH : \ h \mapsto h^c$. We now let $X = \mathrm{Res}_{\ell/k}H/H$ be a Galois symmetric variety, we have an isomorphism

$$(X_{\overline{k}}, G_{\overline{k}}) \cong (H_{\overline{k}}, H_{\overline{k}} \times H_{\overline{k}})$$

with the group variety with $\iota_X = \iota_H$. However, these identifications not Galois equivariant, let's denote $\sigma \in \Gamma \mapsto \sigma_H \in \mathrm{Aut}(\check{H})$ the ususal Galois action on $\check{H}$ (i.e. that gives rise to the L-group), the Galois action on $\check{G}$ is given by

$$\sigma_G(h_1, h_2) = \begin{cases} (\sigma_H(h_1), \sigma_H(h_2)) & \text{if } \sigma \in \Gamma_\ell \\ (\sigma_H(h_2), \sigma_H(h_1)) & \text{if } \sigma \in \Gamma_k \backslash \Gamma_\ell \end{cases}$$

for $\sigma \in \Gamma$ and $(h_1, h_2) \in \check{G}$. It induces the following Galois action on $\check{G}_X$

$$\sigma_X(h) = \begin{cases} \sigma_H(h) & \text{if } \sigma \in \Gamma_\ell \\ \sigma_H(h)^\vee & \text{if } \sigma \in \Gamma_k \backslash \Gamma_\ell \end{cases}$$

for $\sigma \in \Gamma$ and $h \in \check{G}_X$. It can be shown that it doesn't preserve any pinning. By definition, ι_X extends to a morphism

$${}^L\iota_X : {}^LG_X = \check{G}_X \rtimes \Gamma \to {}^LG = \check{G} \rtimes \Gamma$$

Note this definition coincides with Prasad's construction of the Langlands dual group G^{op} for the Galois symmetric varieties.

References

[1] Dennis Gaitsgory and David Nadler. Spherical varieties and langlands duality. *Moscow Mathematical Journal*, 10(1):65–137, 2010.

[2] Yiannis Sakellaridis, Akshay Venkatesh, and Akshay Venkatesh. *Periods and harmonic analysis on spherical varieties.* Société mathématique de France, 2017.

[3] Friedrich Knop and Barbara Schalke. The dual group of a spherical variety. *Transactions of the Moscow Mathematical Society*, 78:187–216, 2017.

[4] Friedrich Knop. Functoriality properties of the dual group. *Documenta Mathematica*, 24:47–64, 2019.