WAVE FRONT SETS OF REPRESENTATIONS OF LIE GROUPS

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1. Introduction

This is my study note for Roger Howe's paper[How81], the concept of wave front set has been very useful in analyzing the decomposition of $L^2(X)$, for X a homogeneous space for Lie groups.

2. Generalities

In the past few years, the concept of wave front set has proved fruitful in the theory of distributions, it seems it might also be of use in representation theory.

Let ρ be the representation of the Lie group G, H the Hilbert space on which ρ acts, we denote $J_1 = J_1(H)$ the trace class operators on H. We can define a map

$$\begin{aligned} \mathrm{tr}_\rho(T): \quad J_1 &\longrightarrow \mathbb{C}_b(G) \\ T &\longmapsto \mathrm{Tr}_\rho(T)(\cdot) \end{aligned}$$

we may regard $\mathrm{tr}_\rho(T)$ as a distribution on G via integration

$$\mathrm{tr}_\rho(T)(f) = \int_G f(g)\mathrm{tr}_\rho(T)\ dg$$

Since $\mathrm{tr}_\rho(T)$ is a distribution, we may consider its wave front set $WF(\mathrm{tr}_\rho(T))$, this is closed subset of $T^*(G)$ and is also closed under positive dilations in the fibers.

Definition 2.1. $WF\rho$ is the closure of the union $WF(\mathrm{tr}_\rho(T))$ as T varies over J_1.

it can be shown that $WF\rho$ is closed under the left and right translation of T^*G.

Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}^*$ the dual of $\mathfrak{g}$, let Ad be the adjoint representation of G on $\mathfrak{g}$ and Ad^* the contragredient action on $\mathfrak{g}^*$, we can identify $\mathfrak{g}^*$ with the left invariant exterior 1-forms on G, this leads to an identification

$$T^*G \cong G \times \mathfrak{g}^*$$

A bi-invariant subset of T^*G is identified with a subset of $G \times X$, where X is a Ad^*G invariant subset of $\mathfrak{g}^*$. Thus we can associate $WF^0\rho \subset \mathfrak{g}^*$ to $WF\rho$. It can happen that $WF^0\rho$ could be very uninteresting, it might always be all of $\mathfrak{g}^*$ for example.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, it is well-known that there is a canonical linear isomorphism

$$\sigma:\ U(\mathfrak{g}) \longrightarrow S(\mathfrak{g}) \cong P(\mathfrak{g}^*)$$

here $S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$ and $P(\mathfrak{g}^*)$ is the polynomial algebra on $\mathfrak{g}^*$. σ restricts to a linear isomorphism between $ZU(\mathfrak{g})$ the center of the enveloping algebra and $IP(\mathfrak{g}^*)$ the Ad^*G invariants of $P(\mathfrak{g}^*)$. The map σ has a nice interpretation in terms of PDE.

Let $V(\mathfrak{g}^*)$ be the set of common zeroes of the homogeneous elements of positive degree of $IP(\mathfrak{g}^*)$, we call V the characteristic variety of $\mathfrak{g}^*$.

Proposition 2.2. *Let ρ be an irreducible unitary representation of G, then we have*

$$WF^0\rho \subseteq V(\mathfrak{g}^*)$$

Let $H \subseteq G$ be a Lie subgroup of G, denpte $q : \mathfrak{g}^* \to \mathfrak{h}^*$ the restriction map, using theroem 1.4 from the paper, we can prove

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Proposition 2.3. *We have the inclusion*

$$q(WF^0\rho) \subseteq WF^0(\rho|_H)$$

Let $\mathfrak{h}^\perp$ be the kernel of q, we say H is crosswise to ρ if $\mathfrak{h}^\perp \cap WF^0\rho = \{0\}$. When H is crosswise to ρ, we may restricts $\mathrm{tr}_\rho(T)$ to H, the wave front set of $\mathrm{tr}_\rho(T)_H = \mathrm{tr}_\rho|_H(T)$, hence we have

Proposition 2.4. *If H is crosswise to $WF^0\rho$, then*

$$WF^0(\rho|_H) = q(WF^0\rho)$$

3. Compact groups

Let K be a compact connected Lie group, let T be a maximal torus, let W be the Weyl group of T, let $\mathfrak{t}$ and $\mathfrak{k}$ be the Lie algebra of $\mathfrak{t}$ and $\mathfrak{k}$. If we regard $\mathfrak{t}^*$ as a subsapce of $\mathfrak{k}^*$, then we have

$$Ad^*K(\mathfrak{t}^*) = \mathfrak{k}^*$$

hence any Ad^*K invariant subset of $\mathfrak{k}^*$ is determined by its intersection with $\mathfrak{t}^*$.

Fix a Weyl Chamber C^+ in $\mathfrak{t}^*$, and fix an order of the roots of $\mathfrak{t}$ to make this Weyl chamber be positive, then we have

$$Ad^*W(C^+) = \mathfrak{t}^*$$

so any Ad^*K invariant set of $\mathfrak{k}^*$ is determined by its intersection with C^+.

The irreducible representations of K are described by the celebrated highest weight theory of Cartan and Weyl. Let $\hat{T}$ be the character group of T, we can identify $\hat{T}$ with a lattice in $\mathfrak{t}^*$, the lattice of weights. The intersection $\hat{T}^+ = \hat{T} \cap C^+$ is called the dominant weights and the dominant weights parametrize the set $\hat{K}$ of irreducible unitary representations of K.

Given a set S in a vector space U, we can define $AC(S)$, the asymptotic cone of S. Let's denote the set of hightests of ρ by $\mathrm{supp}(\rho)$, thus

$$\mathrm{supp}(\rho) \subseteq \hat{T}^+ \subseteq C^+$$

Proposition 3.1. *For a unitary representation ρ of K, we have*

$$-WF^0\rho \cap C^+ = AC(supp\rho)$$

4. Semisimple groups

Let G be a semisimple Lie group with finite center and with Iwasawa decomposition

$$G = KAN,\ \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus N$$

and the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

Let $\mathscr{N}$ be the nilpotent set of $\mathfrak{g}$, then it is well known that $\mathscr{N} = V(\mathfrak{g})$ is the characteristic variety of $\mathfrak{g}$, it is also known that there are only finitely many conjugacy classes of nilpotent elements, then we have the following result from 2.2

Proposition 4.1. *If ρ is an irreducible unitary representation of G, then*

$$WF^0\rho \subseteq N$$

In particular, there are only finitely many possibilities for $WF\rho$.

Let ρ be irreducible, and let's consider $\rho|K$, it is a classic result of Harish-Chandra that $\rho|K$ contains each irreducible representations of K a finite number of times, and $\rho|K$ is of strong trace class. From Cartan decomposition $\mathfrak{p}$ consists of semisimple elements, we see $\mathfrak{k}$ is crosswise to $WF^0\rho$ in the sense of proposition 2.4, hence we have

Proposition 4.2. *For irreducible ρ we have*

$$WF^0(\rho|K) = q(WF^0\rho)$$

where q is the orthogonal projection of $\mathfrak{g}$ onto $\mathfrak{k}$, in particular $WF^0(\rho|K)$ is the orthogonal projection on K of certain nilpotent orbits in $\mathfrak{g}$ and is one of only finitely many possibilities.

Let

$$q' : \mathfrak{g} \longrightarrow \mathfrak{g}/(\mathfrak{a} \oplus N)$$

be the natural quotient map.

Proposition 4.3. *We have the inclusion*

$$WF^0(\rho|K) \subseteq q(q'^{-1}(WF^0(\rho|N)))$$

5. Recent applications

References

[How81] Roger Howe. Wave front sets of representations of lie groups. In *Automorphic Forms, Representation Theory and Arithmetic*, pages 117–140, Berlin, Heidelberg, 1981. Springer Berlin Heidelberg.