For each $p \in \Omega$ there exists an open neighbourhood U_p and some $\epsilon > 0$ such that the map

$$\begin{aligned}\varphi :& (-\epsilon, \epsilon) \times U \longrightarrow \Omega, \\ & (t,p) \longrightarrow \varphi(t,p) = \varphi_t(p) = \gamma_p(t).\end{aligned}$$

For all $s, t \in \mathbb{R}$ we have $\varphi_s \circ \varphi_t = \varphi_{s+t}$ wherever this expression is meaningful. It is worth noting that domain of $\varphi_s \circ \varphi_t$ is defined by $D_{s\circ t} = \{p|\ p \in \mathcal{D}_t,\ \varphi_t(p) \in \mathcal{D}_s\}$. In addition we have, in general $D_{s\circ t} \subset \mathcal{D}_{s+t}$. The equality holds whenever $st > 0$.

We have the following theorem for completeness of vector fields

Theorem 2.8.4 (Chilingworth, 1976). *The differential equation $\dot{x} = f(x)$, for $x \in M$ with a compact set M and $f \in C^1$ has integral curves defined for all t.*

2.8.4 Linear systems

We first consider the linear dynamical systems $\dot{x} = Ax$ for $x \in \mathbb{R}^n$. It is straightforward to see that the solution of such system is of the form $x(t) = x_0 e^{At}$. One can realises the power e^{At} as a convergent series

$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I_n + A + \frac{1}{2} A^2 + \frac{1}{3!} A^3 + \dots$$

Nevertheless, we can take another more constructive steps using linear algebra.

Theorem 2.8.5. *Suppose that A is a square matrix of dimension n with the characteristic polynomial f and the minimal polynomial g*

$$f(x) = (x - c_1)^{d_1} \dots (x - c_k)^{d_k}, \quad g(x) = (x - c_1)^{r_1} \dots (x - c_k)^{r_k}.$$

Let $W_i = \ker(A - c_i I)^{r_i}$ for each i. Then we obviously have $\dim W_i = d_i$ and $n = \sum_{i=1}^k d_i$ and are nonzero vectors $\beta_{i1}, \dots \beta_{ir_i}$ such that

$$\begin{aligned}&(A - c_i I)\beta_{i1} = 0, \\ &(A - c_i I)\beta_{i2} = \beta_{i1}, \\ &\vdots \\ &(A - c_i I)\beta_{ir_i} = \beta_{ir_{i-1}},\end{aligned} \tag{2.8.8}$$

so that all sets $B_i = \{\beta_{i1} \dots, \beta_{ir_i}\}$ and their union $\cup_{i=1}^k B_i$ are linearly independent.

Proof. Consider the chain $\{\ker(A - c_i I)^t\}_{t=1}^{r_i}$. This chain is strictly increasing because one can easily show $\ker(A - c_i I)^t = \ker(A - c_i I)^{t+1}$ iff for all $t' > t$ the equality $\ker(A - c_i I)^{t'} = \ker(A - c_i I)^t$. Let $W_i' = \ker(A - c_i I)^{r_i'}$ for $r_i' < r_i$, then W_i' is invariant under A and $A_i' = A|_{W_i'}$ is nilpotent. Therefore the minimal polynomial of A_i' is $(A-c_iI)^{r_i''}$ for some $r_i'' \le r_i'$ and this contradicts $W_i' = W_i$. We take the nonzero vector $\beta_{ir_i} \in \ker(A-c_iI)^{r_i}$ and let

$$\begin{aligned}
\beta_{ir_{i-1}} &= (A-c_iI)\beta_{ir_i},\\
\beta_{ir_{i-2}} &= (A-c_iI)^2\beta_{ir_i},\\
&\vdots\\
\beta_{i1} &= (A-c_iI)^{r_i-1}\beta_{ir_i},
\end{aligned}$$

It is clear that $(A-c_iI)\beta_{i1} = 0$.

In general, we know that each c_i is an eigenvalue for A^*. In addition the characteristic and minimal polynomial of A^* are exactly f and g respectively. Here we state the following result, although we will not use it.

Corollary 2.8.5. *suppose α is a nonzero vector orthogonal to all vectors in* $\ker(A-c_iI)^{r_i-1}$ *unless $W_i \setminus \ker(A-c_iI)^{r_i-1}$. Assume that for all $j \neq i$ the vector α is orthogonal to W_j. Then α is an eigenvector of A^* corresponding to c_i.*

Proof. The proof is straightforward and is left as an exercises.

Corollary 2.8.6. *According to the Theorem 2.8.5, the solution of $\dot{x} = Ax$ is explicitly closely given by*

$$x(t) = \sum_{j=1}^{k}\sum_{m=1}^{r_j} a_{jm} \sum_{i=1}^{m} \beta_{ij} \frac{t^{m-i}}{(m-i)!} e^{c_jt} .$$

Here a_{jm} are some constants.

Proof. We have

$$\dot{x} = \sum_{j=1}^{k}\sum_{m=1}^{r_j} a_{jm} \left[\sum_{i=1}^{m-1} \beta_{ji} \frac{t^{m-i-1}}{(m-i-1)!} e^{c_jt} + c_j \sum_{i=1}^{m} \beta ij \frac{t^{m-i}}{(m-i)!} e^{c_jt} \right] .$$

An also

$$\begin{aligned}
Ax(t) &= \sum_{j=1}^{k}\sum_{m=1}^{r_j} a_{jm} \left[A\beta_{j1} \frac{t^{m-1}}{(m-1)!} e^{c_jt} + \sum_{i=2}^{m} A\beta_{ji} \frac{t^{m-i}}{(m-i)!} e^{c_jt} \right]\\
&= \sum_{j=1}^{k}\sum_{m=1}^{r_j} a_{jm} \left[c_1\beta_{j1} \frac{t^{m-1}}{(m-1)!} e^{c_1t} + \sum_{i=2}^{m} (\beta_{ji-1} + c_j\beta_{ji}) \frac{t^{m-i}}{(m-i)!} e^{c_jt} \right]\\
&\vdots\\
&= \sum_{j=1}^{k}\sum_{m=1}^{r_j} a_{jm} \left[\sum_{i=2}^{m} \beta_{ji-1} \frac{t^{m-i}}{(m-i)! e^{c_jt}} + c_j \sum_{i=1}^{m} (\beta_{ji} \frac{t^{m-i}}{(m-i)!} ec_jt \right]\\
&= \sum_{j=1}^{k}\sum_{m=1}^{r_j} a_{jm} \left[\sum s = 1^{m-1} \beta js \frac{t^{m-s-1}}{(m-s-1)!} e^{c_jt} + c_j \sum_{s=1}^{m} \beta_{js} \frac{t^{m-s}}{(m-s)!} e^{c_jt} \right] .
\end{aligned}$$