

More Number Systems

Overview

- **Hexadecimal numbers**
 - Related to binary and octal numbers
- **Conversion between hexadecimal, octal and binary**
- **Value ranges of numbers**
- **Representing positive and negative numbers**
- **Creating the complement of a number**
 - Make a positive number negative (and vice versa)
- **Why binary?**

Understanding Binary Numbers

- Binary numbers are made of binary digits (bits):
 - 0 and 1
- How many items does an binary number represent?
 - $(1011)_2 = 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = (11)_{10}$
- What about fractions?
 - $(110.10)_2 = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 + 1 \times 2^{-1} + 0 \times 2^{-2}$
- Groups of eight bits are called a *byte*
 - $(11001001)_2$
- Groups of four bits are called a *nibble*.
 - $(1101)_2$

Understanding Hexadecimal Numbers

- Hexadecimal numbers are made of 16 digits:
 - (0,1,2,3,4,5,6,7,8,9,A, B, C, D, E, F)
- How many items does an hex number represent?
 - $(3A9F)_{16} = 3 \times 16^3 + 10 \times 16^2 + 9 \times 16^1 + 15 \times 16^0 = 14999_{10}$
- What about fractions?
 - $(2D3.5)_{16} = 2 \times 16^2 + 13 \times 16^1 + 3 \times 16^0 + 5 \times 16^{-1} = 723.3125_{10}$
- Note that *each* hexadecimal digit can be represented with four bits.
 - $(1110)_2 = (E)_{16}$
- Groups of four bits are called a *nibble*.
 - $(1110)_2$

Putting It All Together

Decimal	Binary	Octal	Hexadecimal
0	0	0	0
1	1	1	1
2	10	2	2
3	11	3	3
4	100	4	4
5	101	5	5
6	110	6	6
7	111	7	7
8	1000	10	8
9	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

- **Binary, octal, and hexadecimal similar**
- **Easy to build circuits to operate on these representations**
- **Possible to convert between the three formats**

Converting Between Base 16 and Base 2

$$3A9F_{16} = \begin{array}{cccc} \underline{0011} & \underline{1010} & \underline{1001} & \underline{1111} \\ 3 & A & 9 & F \end{array}_2$$

- **Conversion is easy!**
 - **Determine 4-bit value for each hex digit**
- **Note that there are $2^4 = 16$ different values of four bits**
- **Easier to read and write in hexadecimal.**
- **Representations are equivalent!**

Converting Between Base 16 and Base 8

$$3A9F_{16} = \underline{0011} \ \underline{1010} \ \underline{1001} \ \underline{1111}_2$$

3 A 9 F



$$35237_8 = \underline{011} \ \underline{101} \ \underline{010} \ \underline{011} \ \underline{111}_2$$

3 5 2 3 7

1. Convert from Base 16 to Base 2
2. Regroup bits into groups of three starting from right
3. Ignore leading zeros
4. Each group of three bits forms an octal digit.

How To Represent Signed Numbers

- Plus and minus sign used for decimal numbers: 25 (or +25), -16, etc.
- For computers, desirable to represent everything as *bits*.
- Three types of signed binary number representations: **signed magnitude, 1's complement, 2's complement.**
- In each case: **left-most bit indicates sign: positive (0) or negative (1).**

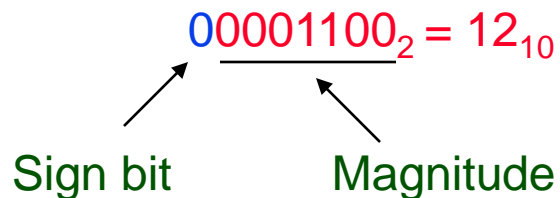
Consider *signed magnitude*:

$$\begin{array}{c} \text{00001100}_2 = 12_{10} \\ \swarrow \quad \nwarrow \\ \text{Sign bit} \quad \text{Magnitude} \end{array}$$

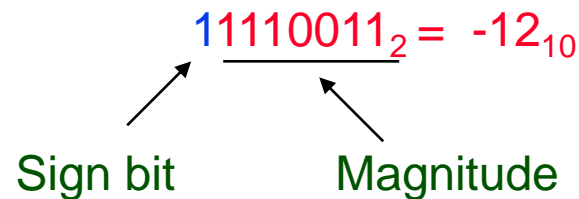
$$\begin{array}{c} \text{10001100}_2 = -12_{10} \\ \swarrow \quad \nwarrow \\ \text{Sign bit} \quad \text{Magnitude} \end{array}$$

One's Complement Representation

- The one's complement of a binary number involves inverting all bits.
 - 1's comp of 00110011 is **11001100**
 - 1's comp of 10101010 is **01010101**
- For an n bit number N the 1's complement is $(2^n - 1) - N$.
- Called diminished radix complement by Mano since 1's complement for base (radix 2).
- To find negative of 1's complement number take the 1's complement.


$$\text{00001100}_2 = 12_{10}$$

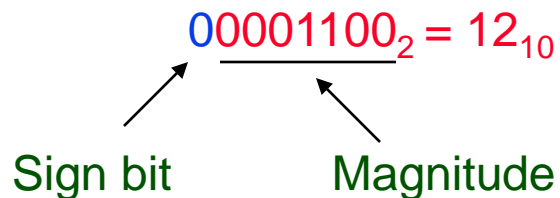
Sign bit Magnitude


$$\text{11110011}_2 = -12_{10}$$

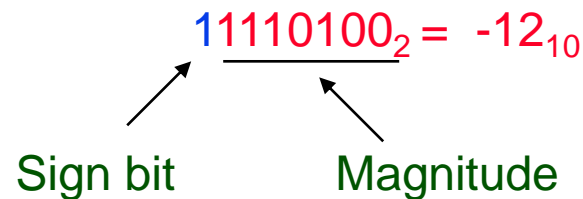
Sign bit Magnitude

Two's Complement Representation

- The two's complement of a binary number involves inverting all bits **and adding 1**.
 - 2's comp of 00110011 is **11001101**
 - 2's comp of 10101010 is **01010110**
- For an n bit number **N** the 2's complement is $(2^n - 1) - N + 1$.
- Called radix complement by Mano since 2's complement for base (radix 2).
- To find negative of 2's complement number take the 2's complement.


$$\text{00001100}_2 = 12_{10}$$

Sign bit Magnitude


$$\text{11110100}_2 = -12_{10}$$

Sign bit Magnitude

Two's Complement Shortcuts

- Algorithm 1 – **Simply complement each bit and then add 1 to the result.**

- Finding the 2's complement of $(01100101)_2$ and of its 2's complement...

N	=	01100101	[N]	=	10011011
		10011010			01100100
+		1	+		1
		-----			-----
		10011011			01100101

- Algorithm 2 – **Starting with the least significant bit, copy all of the bits up to and including the first 1 bit and then complementing the remaining bits.**

- $N = 01100101$
 $[N] = 10011011$

Finite Number Representation

- Machines that use 2's complement arithmetic can represent integers in the range

$$-2^{n-1} \leq N \leq 2^{n-1}-1$$

where n is the number of bits available for representing N . Note that $2^{n-1}-1 = (011\dots11)_2$ and $-2^{n-1} = (100\dots00)_2$

- For 2's complement more negative numbers than positive.
- For 1's complement two representations for zero.
- For an n bit number in base (radix) z there are z^n different unsigned values.

$$(0, 1, \dots, z^{n-1})$$

1's Complement Addition

- Using 1's complement numbers, adding numbers is easy.
- For example, suppose we wish to add $+(1100)_2$ and $+(0001)_2$.
- Let's compute $(12)_{10} + (1)_{10}$.
 - $(12)_{10} = +(1100)_2 = 01100_2$ in 1's comp.
 - $(1)_{10} = +(0001)_2 = 00001_2$ in 1's comp.

Step 1: Add binary numbers

Step 2: Add carry to low-order bit

Comp.

		0	1	1	0	0
Add	+	0	0	0	0	1
<hr/>						
		0	0	1	1	0
Add carry		0	→ 0			
<hr/>						
Final Result		0	1	1	0	1

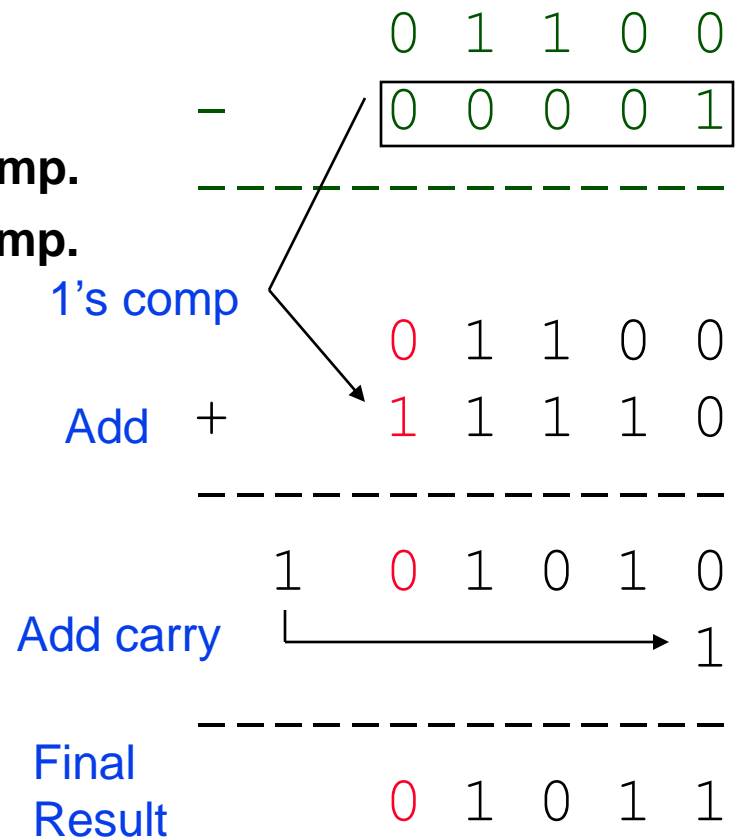
1's Complement Subtraction

- Using 1's complement numbers, subtracting numbers is also easy.
- For example, suppose we wish to **subtract** $+(0001)_2$ from $+(1100)_2$.
- Let's compute $(12)_{10} - (1)_{10}$.
 - $(12)_{10} = +(1100)_2 = \textcolor{red}{0}1100_2$ in 1's comp.
 - $(-1)_{10} = -(0001)_2 = \textcolor{red}{1}1110_2$ in 1's comp.

Step 1: Take 1's complement of 2nd operand

Step 2: Add binary numbers

Step 3: Add carry to low order bit



2's Complement Addition

- Using 2's complement numbers, adding numbers is easy.
- For example, suppose we wish to add $+(1100)_2$ and $+(0001)_2$.
- Let's compute $(12)_{10} + (1)_{10}$.
 - $(12)_{10} = +(1100)_2 = 01100_2$ in 2's comp.
 - $(1)_{10} = +(0001)_2 = 00001_2$ in 2's comp.

Step 1: Add binary numbers

Step 2: Ignore carry bit

Add

+

0 1 1 0 0

0 0 0 0 1

Final Result

0	0	1	1	0	1
---	---	---	---	---	---

Ignore

2's Complement Subtraction

- Using 2's complement numbers, follow steps for subtraction
 - For example, suppose we wish to **subtract** $+(0001)_2$ from $+(1100)_2$.
 - Let's compute $(12)_{10} - (1)_{10}$.
 - $(12)_{10} = +(1100)_2 = \mathbf{0}1100_2$ in 2's comp.
 - $(-1)_{10} = -(0001)_2 = \mathbf{1}1111_2$ in 2's comp.
- ```

 0 1 1
 - 0 0 1

 0 1 0 1 1

```

## Step 1: Take 2's complement of 2<sup>nd</sup> operand

## Step 2: Add binary numbers

### Step 3: Ignore carry bit

Diagram illustrating the addition of a 2's complement number to a decimal number:

Initial values:

- 10 (01100)
- 2's comp (11111)

Operation:

Add +

Final Result:

1 0101

Ignore Carry



## 2's Complement Subtraction: Example #2

- Let's compute  $(13)_{10} - (5)_{10}$ .

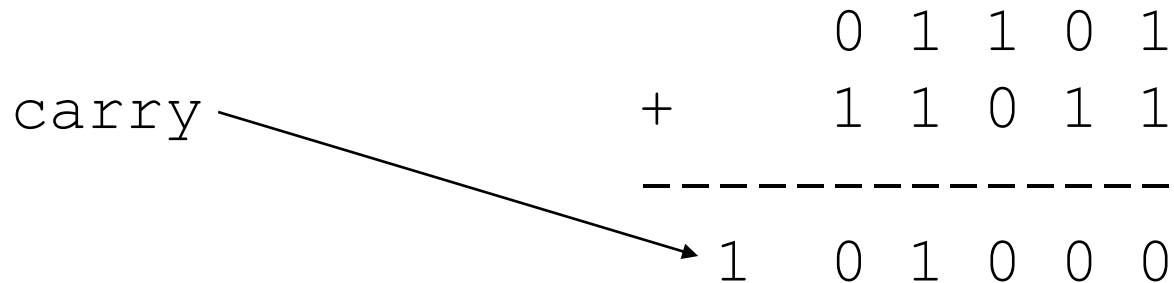
- $(13)_{10} = +(1101)_2 = (01101)_2$

- $(-5)_{10} = -(0101)_2 = (11011)_2$

- Adding these two 5-bit codes...

carry

|  |   |  |       |   |   |   |   |
|--|---|--|-------|---|---|---|---|
|  |   |  | 0     | 1 | 1 | 0 | 1 |
|  | + |  | 1     | 1 | 0 | 1 | 1 |
|  |   |  | ----- |   |   |   |   |
|  |   |  | 1     | 0 | 1 | 0 | 0 |



- Discarding the carry bit, the sign bit is seen to be zero, indicating a correct result. Indeed,

$$(01000)_2 = +(1000)_2 = +(8)_{10}.$$

## 2's Complement Subtraction: Example #3

- Let's compute  $(5)_{10} - (12)_{10}$ .
  - $(-12)_{10} = -(1100)_2 = (10100)_2$
  - $(5)_{10} = +(0101)_2 = (00101)_2$
- Adding these two 5-bit codes...

$$\begin{array}{r} \phantom{+} \phantom{0} \phantom{0} \phantom{1} \phantom{0} \phantom{1} \\ + \phantom{0} \phantom{0} \phantom{1} \phantom{0} \phantom{0} \phantom{0} \\ \hline \phantom{0} \phantom{0} \phantom{1} \phantom{0} \phantom{0} \phantom{1} \end{array}$$

- Here, there is no carry bit and the sign bit is 1. This indicates a negative result, which is what we expect.  $(11001)_2 = -(7)_{10}$ .

# Summary

- **Binary numbers can also be represented in octal and hexadecimal**
- **Easy to convert between binary, octal, and hexadecimal**
- **Signed numbers represented in signed magnitude, 1's complement, and 2's complement**
- **2's complement most important (only 1 representation for zero).**
- **Important to understand treatment of sign bit for 1's and 2's complement.**

