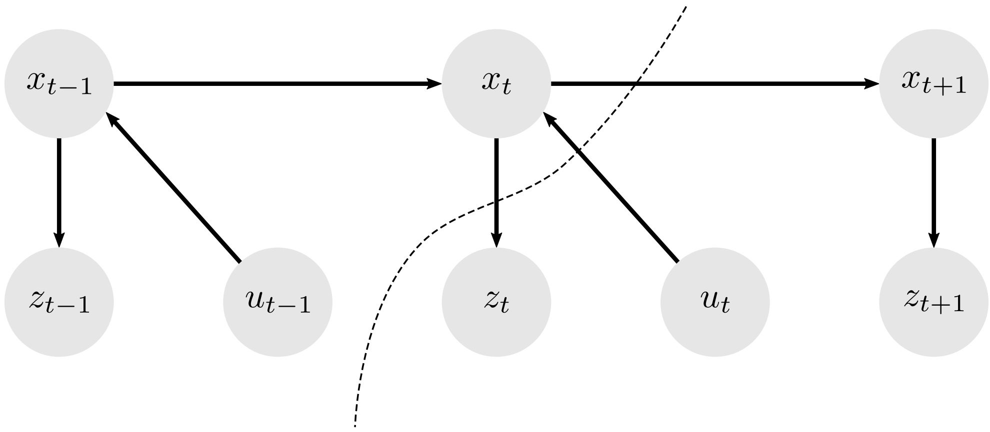
THE KALMAN FILTER



1 DIMENSION

$$\begin{aligned} x_t &= a_t \cdot x_{t-1} + b_t \cdot u_t + \epsilon_{R_t} \\ z_t &= c_t \cdot x_t + \epsilon_{Q_t} \end{aligned}$$

The term ϵ_{R_t} is the system noise:

$$p\left(\epsilon_{R_t}\right) = \mathcal{N}\left(0,\, \sigma^2_{R_t}\right)$$

The term ϵ_{Q_t} is the measurement noise:

$$p\left(\epsilon_{Q_t}\right) = \mathcal{N}\left(0,\, \sigma^2_{Q_t}\right)$$

1. **PREDICTION:**

$$
\begin{aligned}
p\left(x_t \,|\, x_{t-1},\, u_t\right) &= \mathcal{N}\left(a_t \cdot x_{t-1} + b_t \cdot u_t,\, \sigma^2_{R_t}\right) \\
bel\left(x_{t-1}\right) &= p\left(x_{t-1} \,|\, z_{t-1}\right) = \frac{p\left(z_{t-1} \,|\, x_{t-1}\right) \cdot p\left(x_{t-1}\right)}{p\left(z_{t-1}\right)} = \alpha \cdot p\left(z_{t-1} \,|\, x_{t-1}\right) \cdot \overline{bel}\left(x_{t-1}\right) = \\
&= \mathcal{N}(\mu_{t-1},\, \sigma^2_{t-1}) \\
\overline{bel}\left(x_t\right) &= \int_{x_{t-1}=-\infty}^{+\infty} p\left(x_t \,|\, x_{t-1},\, u_t\right) \cdot bel\left(x_{t-1}\right) \cdot dx_{t-1} = \\
&= \int_{x_{t-1}=-\infty}^{+\infty} \mathcal{N}\left(a_t \cdot x_{t-1} + b_t \cdot u_t,\, \sigma^2_{R_t}\right) \cdot \mathcal{N}\left(\mu_{t-1},\, \sigma^2_{t-1}\right) \cdot dx_{t-1} = \mathcal{N}\left(\overline{\mu_t},\, \overline{\sigma_t}^2\right)
\end{aligned}
$$

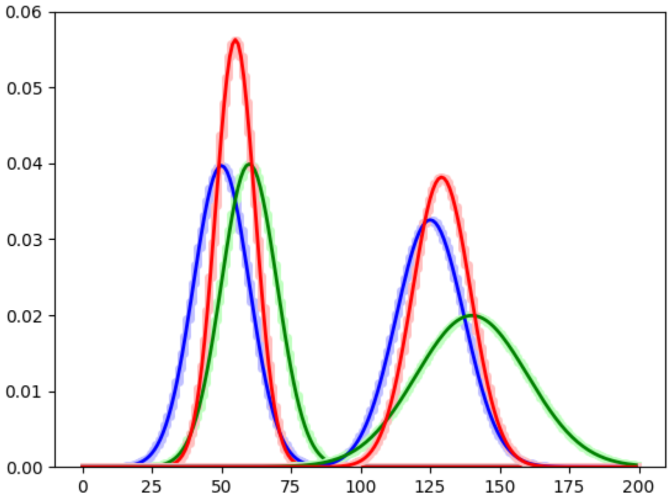
$$
\begin{aligned}
\overline{\mu_t} &= a_t \cdot \mu_{t-1} + b_t \cdot u_t \\
\overline{\sigma_t}^2 &= a_t^2 \cdot \sigma^2_{t-1} + \sigma^2_{R_t}
\end{aligned}
$$

2. **CORRECTION:**

$$
\begin{aligned}
p\left(z_t \,|\, x_t\right) &= \mathcal{N}(c_t \cdot x_t,\, \sigma^2_{Q_t}) \\
\overline{bel}\left(x_t\right) &= \mathcal{N}(\overline{\mu_t},\, \overline{\sigma_t}^2) \\
bel\left(x_t\right) &= p\left(x_t \,|\, z_t\right) = \alpha \cdot p\left(z_t \,|\, x_t\right) \cdot \overline{bel}\left(x_t\right) = \\
&= \alpha \cdot \mathcal{N}(c_t \cdot x_t,\, \sigma^2_{Q_t}) \cdot \mathcal{N}(\overline{\mu_t},\, \overline{\sigma_t}^2) = \mathcal{N}(\mu_t,\, \sigma^2_t)
\end{aligned}
$$

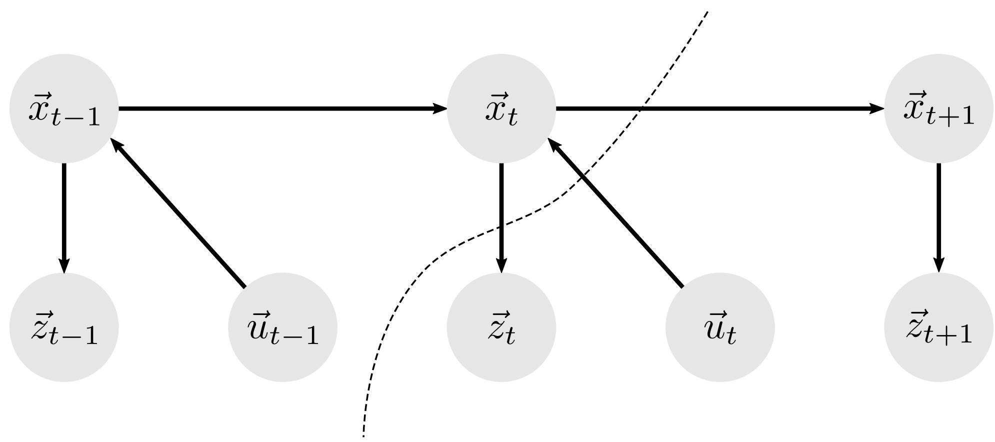
$$
\begin{aligned}
K_t &= \frac{c_t \cdot \overline{\sigma_t}^2}{c_t^2 \cdot \overline{\sigma_t}^2 + \sigma^2_{Q_t}} \\
\mu_t &= \overline{\mu_t} + K_t \cdot (z_t - c_t\, \overline{\mu_t}) \\
\sigma^2_t &= (1 - K_t \cdot c_t) \cdot \overline{\sigma_t}^2
\end{aligned}
$$

$$
\begin{aligned}
K_t = 0 &\longrightarrow \begin{array}{l} \mu_t = \overline{\mu_t} \\ \sigma^2_t = \overline{\sigma_t}^2 \end{array} \\
K_t > 0 &\longrightarrow \sigma^2_t < \overline{\sigma_t}^2
\end{aligned}
$$



0.06
0.05
0.04
0.03
0.02
0.01
0.00
0
25
50
75
100
125
150
175
200

N DIMENSIONS



$$\begin{aligned}
\vec{x}_t &= A_t \cdot \vec{x}_{t-1} + B_t \cdot U_t + \epsilon_{R_t} \\
\vec{z}_t &= C_t \cdot \vec{x}_t + \epsilon_{Q_t}
\end{aligned}$$

The term ϵ_{R_t} is the system noise:

$$p\left(\epsilon_{R_t}\right) = \mathcal{N}\left(0,\, R_t\right)$$

The term ϵ_{Q_t} is the measurement noise:

$$p\left(\epsilon_{Q_t}\right) = \mathcal{N}\left(0,\, Q_t\right)$$

1. **PREDICTION:**

$$\begin{aligned}
p\left(\vec{x}_t \,|\, \vec{x}_{t-1},\, U_t\right) &= \mathcal{N}\left(A_t \cdot \vec{x}_{t-1} + B_t \cdot U_t,\, R_t\right) \\
bel\left(\vec{x}_{t-1}\right) &= p\left(\vec{x}_{t-1} \,|\, \vec{z}_{t-1}\right) = \frac{p\left(\vec{z}_{t-1} \,|\, \vec{x}_{t-1}\right) \cdot p\left(\vec{x}_{t-1}\right)}{p\left(\vec{z}_{t-1}\right)} = \alpha \cdot p\left(\vec{z}_{t-1} \,|\, \vec{x}_{t-1}\right) \cdot \overline{bel}\left(\vec{x}_{t-1}\right) = \\
&= \mathcal{N}(\vec{\mu}_{t-1},\, \Sigma_{t-1}) \\
\overline{bel}\left(\vec{x}_t\right) &= \int_{\vec{x}_{t-1}=-\infty}^{+\infty} p\left(\vec{x}_t \,|\, \vec{x}_{t-1},\, U_t\right) \cdot bel\left(\vec{x}_{t-1}\right) \cdot d\vec{x}_{t-1} = \\
&= \int_{\vec{x}_{t-1}=-\infty}^{+\infty} \mathcal{N}\left(A_t \cdot \vec{x}_{t-1} + B_t \cdot U_t,\, R_t\right) \cdot \mathcal{N}\left(\vec{\mu}_{t-1},\, \Sigma_{t-1}\right) \cdot d\vec{x}_{t-1} = \mathcal{N}\left(\vec{\overline{\mu}}_t,\, \overline{\Sigma}_t\right)
\end{aligned}$$

$$
\begin{aligned}
\vec{\overline{\mu}}_t \,&=\, A_t \,\cdot\, \vec{\mu}_{t-1} \,+\, B_t \,\cdot\, U_t \\
\overline{\Sigma}_t \,&=\, A_t \,\cdot\, \Sigma_{t-1} \,\cdot\, A_t^T \,+\, R_t
\end{aligned}
$$

2. **CORRECTION:**

$$
\begin{aligned}
p\left(\vec{z}_t \,|\, \vec{x}_t\right) \,&=\, \mathcal{N}(C_t \,\cdot\, \vec{x}_t,\, Q_t) \\
\overline{bel}\left(\vec{x}_t\right) \,&=\, \mathcal{N}(\vec{\overline{\mu}}_t,\, \overline{\Sigma}_t) \\
bel\left(\vec{x}_t\right) \,&=\, p\left(\vec{x}_t \,|\, \vec{z}_t\right) \,=\, \alpha \,\cdot\, p\left(\vec{z}_t \,|\, \vec{x}_t\right) \,\cdot\, \overline{bel}\left(\vec{x}_t\right) \,= \\
&=\, \alpha \,\cdot\, \mathcal{N}(C_t \,\cdot\, \vec{x}_t,\, Q_t) \,\cdot\, \mathcal{N}(\vec{\overline{\mu}}_t,\, \overline{\Sigma}_t) \,=\, \mathcal{N}(\vec{\mu}_t,\, \Sigma_t)
\end{aligned}
$$

$$
\begin{aligned}
K_t \,&=\, \overline{\Sigma}_t \,\cdot\, C_t^T \,\cdot\, \left(C_t \,\cdot\, \overline{\Sigma}_t \,\cdot\, C_t^T \,+\, Q_t\right)^{-1} \\
\vec{\mu}_t \,&=\, \vec{\overline{\mu}}_t \,+\, K_t \,\cdot\, \left(\vec{z}_t \,-\, C_t \,\cdot\, \vec{\overline{\mu}}_t\right) \\
\Sigma_t \,&=\, (I \,-\, K_t \,\cdot\, C_t) \,\cdot\, \overline{\Sigma}_t
\end{aligned}
$$

$$
\begin{aligned}
K_t \,=\, 0 \,\longrightarrow\, \vec{\mu}_t \,&=\, \vec{\overline{\mu}}_t \\
\Sigma_t \,&=\, \overline{\Sigma}_t
\end{aligned}
$$

Belonging to $\mathbb{R}^{Nx1}$: $\vec{x}_t,\, \vec{x}_{t-1},\, \epsilon_{R_t},\, \vec{\mu}_t,\, \vec{\mu}_{t-1},\, \vec{\overline{\mu}}_t$.
Belonging to $\mathbb{R}^{NxN}$: $A_t,\, R_t,\, \overline{\Sigma}_t,\, \Sigma_t,\, \Sigma_{t-1}$.
Belonging to $\mathbb{R}^{Mx1}$: U_t.
Belonging to $\mathbb{R}^{NxM}$: B_t.
Belonging to $\mathbb{R}^{Lx1}$: $\vec{z}_t,\, \epsilon_{Q_t}$
Belonging to $\mathbb{R}^{LxN}$: C_t.
Belonging to $\mathbb{R}^{LxL}$: Q_t.
Belonging to $\mathbb{R}^{NxL}$: K_t