18.03 Symmetric Matrices

Jeremy Orloff

Symmetric matices are very important. Our ultimate goal is to prove the following theorem.

Spectral Theorem: A real $n \times n$ symmetric matrix has n orthogonal eigenvectors with real eigenvalues.

The generalization of this theorem to infinite dimensions is widely used in math and science.

Symmetric Matrices
We can understand symmetric matrices better if we discuss them in terms of their properties instead of their coordinates.
To avoid being too abstract we will rely on coordinates for the following two definitions.
Definition 1. For column vectors v, w the *inner product* is defined in terms of transpose and matrix multiplication: $\langle v,\, w\rangle = v^{\mathrm{T}} w$.
(In 18.02 you called this the *dot product*.)
Definition 2. The matrix A is *symmetric* if $A^{\mathrm{T}} = A$.
Property 3. $(Av)^{\mathrm{T}} = v^{\mathrm{T}} A^{\mathrm{T}}$. (To see this just do the multiplication.)
Property 4. If A is symmetric then $\langle Av,\, w\rangle = \langle v,\, Aw\rangle$.
Proof: $\langle Av,\, w\rangle = (Av)^{\mathrm{T}} w = v^{\mathrm{T}} A^{\mathrm{T}} w = \langle v,\, A^{\mathrm{T}} w\rangle$. If A is symmetric then $A^{\mathrm{T}} = A$. QED.
Everything will follow from this property.
Property 5. If A is symmetric and v, w are eigenvectors with different eigenvalues then $\langle v,\, w\rangle = 0$.
proof: Suppose $Av = l_1 v$ and $Aw = l_2 w$. Then using property (4) we see

$$l_1\langle v,\, w\rangle = \langle l_1 v,\, w\rangle = \langle Av,\, w\rangle = \langle v,\, Aw\rangle = \langle v,\, l_2 w\rangle = l_2\langle v,\, w\rangle.$$

Look at the first and last terms of this chain and use $l_1 \neq l_2$ to conclude $\langle v,\, w\rangle = 0$. QED.
Property 6. If A is real and symmetric then A has a real eigenvalue.
Algebraic proof: If we had defined the term I could just wave the word Hermitian. As it is, I will do the same thing but in a hidden way. For non-zero complex vectors, note $\langle v,\, \overline{v}\rangle = \sum v_k \overline{v}_k > 0$. Also, since A is real, $\overline{Av} = A\overline{v}$.
If v is an eigenvector with eigenvalue l then

$$l\langle v,\, \overline{v}\rangle = \langle lv,\, \overline{v}\rangle = \langle Av,\, \overline{v}\rangle = \langle v,\, A\overline{v}\rangle = \langle v,\, \overline{Av}\rangle = \langle v,\, \overline{lv}\rangle = \overline{l}\langle v,\, \overline{v}\rangle.$$

Looking at the first and last terms in this chain we see $l = \overline{l}$. Which proves l is real. (We have just showed all eigenvalues are real).
Analytic proof: This proof is more involved than the algebaic one but it produces an eigenvector and some geometric insight. First we need to prove the following fact.
Fact. 6.1: Let f be a continuous function on $\mathbf{R}^n$. If S is a closed and bounded set

in $\mathbf{R}^n$ then there is a point $v_1 \in S$ such that $f(v_1)$ is the maximum value of f on S. (We say f achieves its maximum on S.)
Proof: No way. I haven't even defined what closed means. I'll try to make it plausible with the following examples. Suppose f is a continuous function on a closed interval $[a, b]$ then its image is another closed interval $[c, d]$. The point $v_1 \in [a, b]$ such that $f(v_1) = d$ is where f achieves its maximum.
Example 1: $f(x) = \sin x$, $[a, b] = [0, \pi]$, $[c, d] = [0, 1]$, $v_1 = \pi/2$.
Example 2: (Showing closedness is necessary.) The set $(0, 1]$ is bounded but not closed and $f(x) = 1/x$ fails to achieve its maximum on it.
Example 3: (Showing boundedness is necessary.) The set $[0, \infty)$ is closed but not bounded and $f(x) = x$ fails to achieve its maximum on it.
The general topological statement is that a continuous function on a compact set achieves its maximum.
Analytic proof of Property 6: Let S be the unit sphere, i.e. the set of all vectors of length 1. S is closed and bounded so the function $f(v, v) = \langle Av, v\rangle$ achieves its maximum at some vector v_1.
We will show that v_1 is the eigenvector we seek.
Using the symmetry of A we get that $\mathrm{grad} f = 2Av$.
Using Lagrange multipliers (with the constraint $\langle v\, v\rangle = 1$) we find critical points for f satisfy $2Av = l2v$. That is, they are eigenvectors of A with real eigenvalues.
By its definition v_1 is a critical point of f. QED.
Spectral Theorem: Suppose the $n \times n$ matrix A is symmetric. Then it has n orthogonal (hence independent) eigenvectors with real eigenvalues.
proof: Property (6) provides one eigenvector, v_1 with eigenvector l_1.
Let W be the set of all vectors orthogonal to v_1. First we show that if $w \in W$ then $Aw \in W$. To do this we must show $\langle v_1, Aw\rangle = 0$ for all $w \in W$:

$$\langle v_1, Aw\rangle = \langle Av_1, w\rangle = l_1\langle v_1, w\rangle = 0.$$

But now the same argument as in Property (6) (replacing $\mathbf{R}^n$ with W) shows that A has a real eigenvector $v_2 \in W$.

End of Symmetric Matrices.