THE EXTENDED KALMAN FILTER

$$\vec{x}_t = g\left(\vec{x}_{t-1}, \vec{u}_t\right)$$

$$\left(\begin{array}{l} x_t = g_1\left(x_{t-1}, \vec{u}_t\right) = x_{t-1} + \left(Rs_t + \frac{W}{2}\right)\left(+\sin\left(\theta_{t-1} + \alpha_t\right) - \sin\left(\theta_{t-1}\right)\right) \\ y_t = g_2\left(y_{t-1}, \vec{u}_t\right) = y_{t-1} + \left(Rs_t + \frac{W}{2}\right)\left(-\cos\left(\theta_{t-1} + \alpha_t\right) + \cos\left(\theta_{t-1}\right)\right) \\ \theta_t = g_3\left(\theta_{t-1}, \vec{u}_t\right) = \theta_{t-1} + \alpha_t \end{array}\right)$$

$$\alpha_t = f_1\left(\vec{u}_t\right) = \frac{r_t - l_t}{W}$$

$$Rs_t = f_2\left(\vec{u}_t\right) = \frac{l_t}{\alpha_t} = \left(\frac{l_t}{r_t - l_t}\right) W$$

$$\vec{z}_t = \begin{pmatrix} D_t \\ \phi_t \end{pmatrix}$$

$$\vec{\hat{z}}_t = \begin{pmatrix} \hat{D}_t \\ \hat{\phi}_t \end{pmatrix} = h\left(\vec{x}_t, \vec{p}_W\right)$$

Because the motion model, function $g\left(\cdot\right)$, and the measure model, function $h\left(\cdot\right)$[1] , are non-linear I have to use another version of the N-dimensional Kalman filter called: **the extended Kalman filter (EKF)**.

THE PREDICTION STEP

$$\left(\begin{array}{l} \overline{\mu}_{x_t} = g_1\left(\mu_{x_{t-1}}, \vec{u}_t\right) = \mu_{x_{t-1}} + \left(Rs_t + \frac{W}{2}\right)\left(+\sin\left(\mu_{\theta_{t-1}} + \alpha_t\right) - \sin\left(\mu_{\theta_{t-1}}\right)\right) \\ \overline{\mu}_{y_t} = g_2\left(\mu_{y_{t-1}}, \vec{u}_t\right) = \mu_{y_{t-1}} + \left(Rs_t + \frac{W}{2}\right)\left(-\cos\left(\mu_{\theta_{t-1}} + \alpha_t\right) + \cos\left(\mu_{\theta_{t-1}}\right)\right) \\ \overline{\mu}_{\theta_t} = g_3\left(\mu_{\theta_{t-1}}, \vec{u}_t\right) = \mu_{\theta_{t-1}} + \alpha_t \end{array}\right)$$

$$\overline{\Sigma}_t = G_t \cdot \Sigma_{t-1} \cdot G_t^T + R_t$$

The covariance matrix R_t is due to the propagation of the noise belonging to the control commands through the motion model. The covariance matrix $\Sigma_{control,t}$ is due to the noise belonging to the control commands. $R_t \in \mathbb{R}^{3\times 2}$, $\Sigma_{control,t} \in \mathbb{R}^{2\times 2}$.

$$R_t = V_t \cdot \Sigma_{control,t} \cdot V_t^T$$

[1] In this video lecture the terms $\hat{D}_t$ and $\hat{\phi}_t$ are called r and α, respectively. I use other names just to avoid confusions with other terms that appears during the course.

$$\Sigma_{control,t} = \begin{pmatrix} \sigma_{l_t}^2 & 0 \\ 0 & \sigma_{r_t}^2 \end{pmatrix}$$

The variance in the left control, $\sigma_{l_t}^2$, and the variance in the right control, $\sigma_{r_t}^2$, are defined as[2]:

$$\sigma_{l_t}^2 = (p_1\, l_t)^2 + \big(p_2\,(l_t - r_t)\big)^2$$

$$\sigma_{r_t}^2 = (p_1\, r_t)^2 + \big(p_2\,(l_t - r_t)\big)^2$$

$$0 \le p_1 \le 1$$
$$0 \le p_2 \le 1$$

These variances capture the exactness of the motion on the left and right wheels of the robot. The fact that the covariance matrix $\Sigma_{control,t}$ is diagonal assumes that there is no correlation between the control command l_t and the control command r_t.

The term p_1 takes into account the error in the motion when the robot moves straight forward. Therefore, the value of the term p_1 is normally small, $p1 \approx 0$. The term p_2 takes into account the error when the robot turns. There's a lot of slipping of the wheels/tracks on the ground when the robot turns. Therefore, the value of the term p_2 is normally greater than the value of the term p_1.
The term G_t is the jacobian matrix of the function $g\left(\vec{x}_{t-1},\, \vec{u}_t\right)$ with respect to $\vec{x}_{t-1}$.

$$G_t = \frac{\partial g\left(\vec{x}_{t-1},\, \vec{u}_t\right)}{\partial \vec{x}_{t-1}} = \begin{pmatrix} \frac{\partial g_1}{\partial x_{t-1}} & \frac{\partial g_1}{\partial y_{t-1}} & \frac{\partial g_1}{\partial \theta_{t-1}} \\ \frac{\partial g_2}{\partial x_{t-1}} & \frac{\partial g_2}{\partial y_{t-1}} & \frac{\partial g_2}{\partial \theta_{t-1}} \\ \frac{\partial g_3}{\partial x_{t-1}} & \frac{\partial g_3}{\partial y_{t-1}} & \frac{\partial g_3}{\partial \theta_{t-1}} \end{pmatrix}$$

[2] Terms of a variance add quadratically.

$r_t \neq l_t$

$$
\begin{aligned}
\frac{\partial g_1}{\partial x_{t-1}} &= 1 \\
\frac{\partial g_1}{\partial y_{t-1}} &= 0 \\
\frac{\partial g_1}{\partial \theta_{t-1}} &= \left(Rs_t + \frac{W}{2}\right)\Big(+\cos\left(\theta_{t-1} + \alpha_t\right) - \cos\left(\theta_{t-1}\right)\Big) \\
\frac{\partial g_2}{\partial x_{t-1}} &= 0 \\
\frac{\partial g_2}{\partial y_{t-1}} &= 1 \\
\frac{\partial g_2}{\partial \theta_{t-1}} &= \left(Rs_t + \frac{W}{2}\right)\Big(+\sin\left(\theta_{t-1} + \alpha_t\right) - \sin\left(\theta_{t-1}\right)\Big) \\
\frac{\partial g_3}{\partial x_{t-1}} &= 0 \\
\frac{\partial g_3}{\partial y_{t-1}} &= 0 \\
\frac{\partial g_3}{\partial \theta_{t-1}} &= 1
\end{aligned}
$$

$$
G_t = \begin{pmatrix}
1 & 0 & \left(Rs_t + \frac{W}{2}\right)\Big(+\cos\left(\theta_{t-1} + \alpha_t\right) - \cos\left(\theta_{t-1}\right)\Big) \\
0 & 1 & \left(Rs_t + \frac{W}{2}\right)\Big(+\sin\left(\theta_{t-1} + \alpha_t\right) - \sin\left(\theta_{t-1}\right)\Big) \\
0 & 0 & 1
\end{pmatrix}
$$

In the terms $\frac{\partial g_1}{\partial \theta_{t-1}}$ and $\frac{\partial g_2}{\partial \theta_{t-1}}$ there is a singularity when $l_t = r_t$, because $\alpha_t = 0$ and $Rs_t = \infty$. Therefore, we have to derived new expressions for these terms when the control commands are equal.

$l_t = r_t$

$$
\begin{aligned}
\frac{\partial g_1}{\partial \theta_{t-1}} &\overset{l_t = r_t}{=} \lim_{l_t \to r_t} \frac{\partial g_1}{\partial \theta_{t-1}} = \\
&= \lim_{l_t \to r_t} \left(Rs_t + \frac{W}{2}\right)\Big(\cos\left(\theta_{t-1} + \alpha_t\right) - \cos\left(\theta_{t-1}\right)\Big) = \\
&= \lim_{l_t \to r_t} Rs_t \Big(\cos\left(\theta_{t-1} + \alpha_t\right) - \cos\left(\theta_{t-1}\right)\Big) + \lim_{l_t \to r_t} \frac{W}{2}\Big(\cos\left(\theta_{t-1} + \alpha_t\right) - \cos\left(\theta_{t-1}\right)\Big) = \\
&= \lim_{l_t \to r_t} \frac{l_t}{\alpha_t}\Big(\cos\left(\theta_{t-1} + \alpha_t\right) - \cos\left(\theta_{t-1}\right)\Big) + \lim_{l_t \to r_t} \frac{W}{2}\Big(\cos\left(\theta_{t-1} + \alpha_t\right) - \cos\left(\theta_{t-1}\right)\Big) = \\
&= \frac{l_t}{0}\,(0) + \frac{W}{2}\,(0)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial g_1}{\partial \theta_{t-1}} &\overset{l_t = r_t}{=} \lim_{l_t \to r_t} \frac{\partial g_1}{\partial \theta_{t-1}} = \\
&= \lim_{l_t \to r_t} \frac{l_t \left(\cos\left(\theta_{t-1} + \alpha_t\right) - \cos\left(\theta_{t-1}\right) \right)}{\alpha_t} = \lim_{l_t \to r_t} \frac{num\left(\alpha_t\right)}{den\left(\alpha_t\right)} \overset{\text{L'Hôpital}}{=} \lim_{l_t \to r_t} \frac{num'\left(\alpha_t\right)}{den'\left(\alpha_t\right)} = \\
&= \frac{-\, l_t \, \sin\left(\theta_{t-1} + \alpha_t\right)}{1} = -\, l_t \, \sin\left(\theta_{t-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial g_2}{\partial \theta_{t-1}} &\overset{l_t = r_t}{=} \lim_{l_t \to r_t} \frac{\partial g_2}{\partial \theta_{t-1}} = \\
&= \lim_{l_t \to r_t} \left(Rs_t + \frac{W}{2} \right) \left(\sin\left(\theta_{t-1} + \alpha_t\right) - \sin\left(\theta_{t-1}\right) \right) = \\
&= \lim_{l_t \to r_t} Rs_t \left(\sin\left(\theta_{t-1} + \alpha_t\right) - \sin\left(\theta_{t-1}\right) \right) + \lim_{l_t \to r_t} \frac{W}{2} \left(\sin\left(\theta_{t-1} + \alpha_t\right) - \sin\left(\theta_{t-1}\right) \right) = \\
&= \lim_{l_t \to r_t} \frac{l_t}{\alpha_t} \left(\sin\left(\theta_{t-1} + \alpha_t\right) - \sin\left(\theta_{t-1}\right) \right) + \lim_{l_t \to r_t} \frac{W}{2} \left(\sin\left(\theta_{t-1} + \alpha_t\right) - \sin\left(\theta_{t-1}\right) \right) = \\
&= \frac{l_t}{0} \, (0) + \frac{W}{2} \, (0)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial g_2}{\partial \theta_{t-1}} &\overset{l_t = r_t}{=} \lim_{l_t \to r_t} \frac{\partial g_2}{\partial \theta_{t-1}} = \\
&= \lim_{l_t \to r_t} \frac{l_t \left(\sin\left(\theta_{t-1} + \alpha_t\right) - \sin\left(\theta_{t-1}\right) \right)}{\alpha_t} = \lim_{l_t \to r_t} \frac{num\left(\alpha_t\right)}{den\left(\alpha_t\right)} \overset{\text{L'Hôpital}}{=} \lim_{l_t \to r_t} \frac{num'\left(\alpha_t\right)}{den'\left(\alpha_t\right)} = \\
&= \frac{l_t \, \cos\left(\theta_{t-1} + \alpha_t\right)}{1} = l_t \, \cos\left(\theta_{t-1}\right)
\end{aligned}
$$

$$
G_t = \begin{pmatrix} 1 & 0 & -\, l_t \, \sin\left(\theta_{t-1}\right) \\ 0 & 1 & l_t \, \cos\left(\theta_{t-1}\right) \\ 0 & 0 & 1 \end{pmatrix}
$$

The term V_t is the jacobian matrix of the function $g\left(\vec{x}_{t-1}, \, \vec{u}_t\right)$ with respect to $\vec{u}_t$.

$$
V_t = \frac{\partial g\left(\vec{x}_{t-1}, \, \vec{u}_t\right)}{\partial \vec{u}_t} = \begin{pmatrix} \frac{\partial g_1}{\partial l_t} & \frac{\partial g_1}{\partial r_t} \\ \frac{\partial g_2}{\partial l_t} & \frac{\partial g_2}{\partial r_t} \\ \frac{\partial g_3}{\partial l_t} & \frac{\partial g_3}{\partial r_t} \end{pmatrix}
$$

$$\left(Rs_t + \frac{W}{2}\right) = \left(\frac{l_t}{r_t - l_t} + \frac{1}{2}\right) W = \left(\frac{r_t + l_t}{r_t - l_t}\right) \frac{W}{2}$$

$$\begin{pmatrix} x_t = x_{t-1} + \left(\frac{r_t + l_t}{r_t - l_t}\right) \frac{W}{2} \left(+ \sin\left(\theta_{t-1} + \frac{r_t - l_t}{W}\right) - \sin\left(\theta_{t-1}\right)\right) \\ y_t = y_{t-1} + \left(\frac{r_t + l_t}{r_t - l_t}\right) \frac{W}{2} \left(- \cos\left(\theta_{t-1} + \frac{r_t - l_t}{W}\right) + \cos\left(\theta_{t-1}\right)\right) \\ \theta_t = \theta_{t-1} + \alpha_t \end{pmatrix}$$

$l_t \neq r_t$

$$\begin{aligned} \frac{\partial g_1}{\partial l_t} &= \left(\left(\frac{r_t + l_t}{r_t - l_t}\right) \frac{W}{2}\right)' \left(\sin\left(\theta_{t-1} + \frac{r_t - l_t}{W}\right) - \sin\left(\theta_{t-1}\right)\right) + \\ &+ \left(\left(\frac{r_t + l_t}{r_t - l_t}\right) \frac{W}{2}\right) \left(\sin\left(\theta_{t-1} + \frac{r_t - l_t}{W}\right) - \sin\left(\theta_{t-1}\right)\right)' = \\ &= \frac{W}{2} \left(\frac{1\left(r_t - l_t\right) - (-1)\left(r_t + l_t\right)}{\left(r_t - l_t\right)^2}\right) (\ldots) + \left(\frac{r_t + l_t}{r_t - l_t}\right) \frac{W}{2} cos\left(\theta_{t-1} + \frac{r_t - l_t}{W}\right) \left(-\frac{1}{W}\right) = \\ &= W \frac{r_t}{\left(r_t - l_t\right)^2} \left(\sin\left(\theta_{t-1} + \alpha_t\right) - \sin\left(\theta_{t-1}\right)\right) - \frac{1}{2}\left(\frac{r_t + l_t}{r_t - l_t}\right) \cos\left(\theta_{t-1} + \alpha_t\right) \end{aligned}$$

$$\frac{\partial g_2}{\partial l_t} = W \frac{r_t}{\left(r_t - l_t\right)^2} \left(- \cos\left(\theta_{t-1} + \alpha_t\right) + \cos\left(\theta_{t-1}\right)\right) - \frac{1}{2}\left(\frac{r_t + l_t}{r_t - l_t}\right) \sin\left(\theta_{t-1} + \alpha_t\right)$$

$$\frac{\partial g_3}{\partial l_t} = -\frac{1}{W}$$

$$\frac{\partial g_1}{\partial r_t} = -W \frac{l_t}{\left(r_t - l_t\right)^2} \left(\sin\left(\theta_{t-1} + \alpha_t\right) - \sin\left(\theta_{t-1}\right)\right) + \frac{1}{2}\left(\frac{r_t + l_t}{r_t - l_t}\right) \cos\left(\theta_{t-1} + \alpha_t\right)$$

$$\frac{\partial g_2}{\partial r_t} = -W \frac{l_t}{\left(r_t - l_t\right)^2} \left(- \cos\left(\theta_{t-1} + \alpha_t\right) + \cos\left(\theta_{t-1}\right)\right) + \frac{1}{2}\left(\frac{r_t + l_t}{r_t - l_t}\right) \sin\left(\theta_{t-1} + \alpha_t\right)$$

$$\frac{\partial g_3}{\partial r_t} = \frac{1}{W}$$

In the terms $\frac{\partial g_1}{\partial l_t}$, $\frac{\partial g_2}{\partial l_t}$, $\frac{\partial g_1}{\partial r_t}$ and $\frac{\partial g_2}{\partial r_t}$ there is a singularity when $l_t = r_t$, because $\alpha_t = 0$. Therefore, we have to derived new expressions for these terms when the control commands are equal.

$l_t \ = \ r_t$

$$
\begin{aligned}
\frac{\partial g_1}{\partial l_t} &\overset{l_t = r_t}{=} \lim_{l_t \to r_t} \frac{\partial g_1}{\partial l_t} = \\
&= W \, \frac{r_t}{(0)^2} \, \Big(\sin\left(\theta_{t-1} \, + \, 0\right) \, - \, \sin\left(\theta_{t-1}\right)\Big) \, - \, \frac{1}{2} \left(\frac{2\, r_t}{0}\right) \, \cos\left(\theta_{t-1} \, + \, 0\right) \, = \\
&= W \, \frac{r_t}{(0)^2} \, (0) \, - \, \frac{1}{2} \left(\frac{2\, r_t}{0}\right) \, \cos\left(\theta_{t-1}\right) \\
\frac{\partial g_1}{\partial l_t} &\overset{l_t = r_t}{=} \lim_{l_t \to r_t} \frac{\partial g_1}{\partial l_t} \overset{\text{L'Hôpital}}{=} \frac{1}{2} \left(\cos\left(\theta_{t-1}\right) \, + \, \frac{l_t}{W} \, \sin\left(\theta_{t-1}\right)\right) \\
\frac{\partial g_1}{\partial r_t} &\overset{l_t = r_t}{=} \lim_{l_t \to r_t} \frac{\partial g_1}{\partial r_t} \overset{\text{L'Hôpital}}{=} \frac{1}{2} \left(\cos\left(\theta_{t-1}\right) \, - \, \frac{l_t}{W} \, \sin\left(\theta_{t-1}\right)\right) \\
\frac{\partial g_2}{\partial l_t} &\overset{l_t = r_t}{=} \lim_{l_t \to r_t} \frac{\partial g_2}{\partial l_t} \overset{\text{L'Hôpital}}{=} \frac{1}{2} \left(\sin\left(\theta_{t-1}\right) \, - \, \frac{l_t}{W} \, \cos\left(\theta_{t-1}\right)\right) \\
\frac{\partial g_2}{\partial r_t} &\overset{l_t = r_t}{=} \lim_{l_t \to r_t} \frac{\partial g_2}{\partial r_t} \overset{\text{L'Hôpital}}{=} \frac{1}{2} \left(\sin\left(\theta_{t-1}\right) \, + \, \frac{l_t}{W} \, \cos\left(\theta_{t-1}\right)\right)
\end{aligned}
$$

$$
V_t \ = \ \begin{pmatrix}
\frac{1}{2} \left(\cos\left(\theta_{t-1}\right) \, + \, \frac{l_t}{W} \, \sin\left(\theta_{t-1}\right)\right) & \frac{1}{2} \left(\cos\left(\theta_{t-1}\right) \, - \, \frac{l_t}{W} \, \sin\left(\theta_{t-1}\right)\right) \\
\frac{1}{2} \left(\sin\left(\theta_{t-1}\right) \, - \, \frac{l_t}{W} \, \cos\left(\theta_{t-1}\right)\right) & \frac{1}{2} \left(\sin\left(\theta_{t-1}\right) \, + \, \frac{l_t}{W} \, \cos\left(\theta_{t-1}\right)\right) \\
-\frac{1}{W} & \frac{1}{W}
\end{pmatrix}
$$

THE CORRECTION STEP

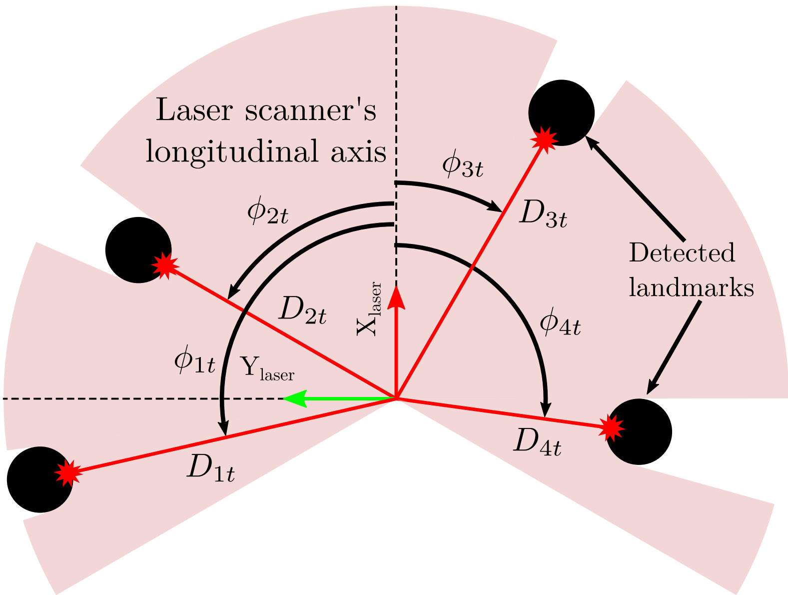
I call *detected landmark* to a landmark that is detected by the algorithm in a scan, which is taken by the robot's laser scanner in the world.
The term

$$\vec{z}_t = \begin{pmatrix} D_t \\ \phi_t \end{pmatrix}$$

is a measurement that the algorithm calculates in a scan. This measurement is comprised by the distance from the laser scanner to a detected landmark, D_t, and the angle formed between the laser scanner's longitudinal axis and the straight line that joints the laser scanner with the detected landmark.
Let's define the term $\vec{z}_{it}$ as the measurement computed for each detected landmark in a scan:

$$\vec{z}_{it} = \begin{pmatrix} D_{it} \\ \phi_{it} \end{pmatrix}$$

where $i = 1, \ldots, N_s$. Example: Let's suppose that the algorithm detects 4 landmark in a scan, $N_s = 4$.



$$\vec{z}_{1t} = \begin{pmatrix} D_{1t} \\ \phi_{1t} \end{pmatrix}, \ \vec{z}_{2t} = \begin{pmatrix} D_{2t} \\ \phi_{2t} \end{pmatrix}, \ \vec{z}_{3t} = \begin{pmatrix} D_{3t} \\ \phi_{3t} \end{pmatrix}, \ \vec{z}_{4t} = \begin{pmatrix} D_{4t} \\ \phi_{4t} \end{pmatrix}$$

I called *world landmark* to a landmark that it's placed in a specific position in the world that the algorithm knows since the beginning of its execution.
The notation to reference in a generic way the position of any landmark is:

$$\vec{p}_W = \begin{pmatrix} x_W \\ y_W \end{pmatrix}$$

The notation to reference to the position of the landmark number j is:

$$\vec{p}_{W\,j} = \begin{pmatrix} x_{W\,j} \\ x_{W\,j} \end{pmatrix}$$

where $j = 1, \ldots, N_w$.
The first task that the algorithm has to accomplish is to match each detected landmark with a world landmark. These matches among the detected landmarks and the world landmarks are based on the proximity among them.
The algorithm calculates the global coordinates of each detected landmark using the robot's predicted pose, $\vec{\overline{\mu}}_t$, computed in the predicted step, and each term $\vec{z}_{it}$.
The global coordinates of each detected landmark are:

$$\begin{aligned}
x_i &= \overline{\mu}_{x_{lt}} + D_{it} \cos\left(\overline{\mu}_{\theta_t} + \phi_{it}\right) = \\
&= \overline{\mu}_{x_t} + sd \cos\left(\overline{\mu}_{\theta_t}\right) + D_{it} \cos\left(\overline{\mu}_{\theta_t} + \phi_{it}\right) \\
y_i &= \overline{\mu}_{y_{lt}} + D_{it} \sin\left(\overline{\mu}_{\theta_t} + \phi_{it}\right) = \\
&= \overline{\mu}_{y_t} + sd \sin\left(\overline{\mu}_{\theta_t}\right) + D_{it} \sin\left(\overline{\mu}_{\theta_t} + \phi_{it}\right)
\end{aligned}$$

where $i = 1, 2, \ldots, N_s$.
The next step for the algorithm is to match each detected landmark with the nearest world landmark. This task is possible since the algorithm now knows the global coordinates of each detected landmark and each world landmark. Let's suppose, that the algorithm has matched detected landmark number 1 with world landmark number 2, detected landmark number 2 with world landmark number 7, detected landmark number 3 with world landmark number 5 and detected landmark number 4 with world landmark number 1.

$$(i = 1, j = 2), (i = 2, j = 7), (i = 3, j = 5), (i = 4, j = 1)$$

If the distance from a detected landmark to its matched world landmark is bigger than a maximum threshold, ϵ, then this pair of detected landmark-world landmark is rejected.

$$\sqrt{\left(x_i - x_{Wj}\right)^2 + \left(y_i - y_{Wj}\right)^2} \leq \epsilon \longrightarrow \text{match accepted}$$

$$\sqrt{\left(x_i - x_{Wj}\right)^2 + \left(y_i - y_{Wj}\right)^2} > \epsilon \longrightarrow \text{match rejected}$$

Let's suppose that the second match in this example is rejected since it doesn't satisfy the threshold condition. Therefore, the resulting matches are:

$$(i = 1, j = 2), (i = 3, j = 5), (i = 4, j = 1)$$

Before continuing it's necessary to define the concept of the expected measurement. The term $\vec{\hat{z}}_t$ is called expected measurement. The expected measurement to a world landmark is comprised by the distance from the laser scanner to a world landmark, $\hat{D}_t$, and the angle defined between the laser scanner's longitudinal axis and the straight line that joints the laser scanner with the world landmark.

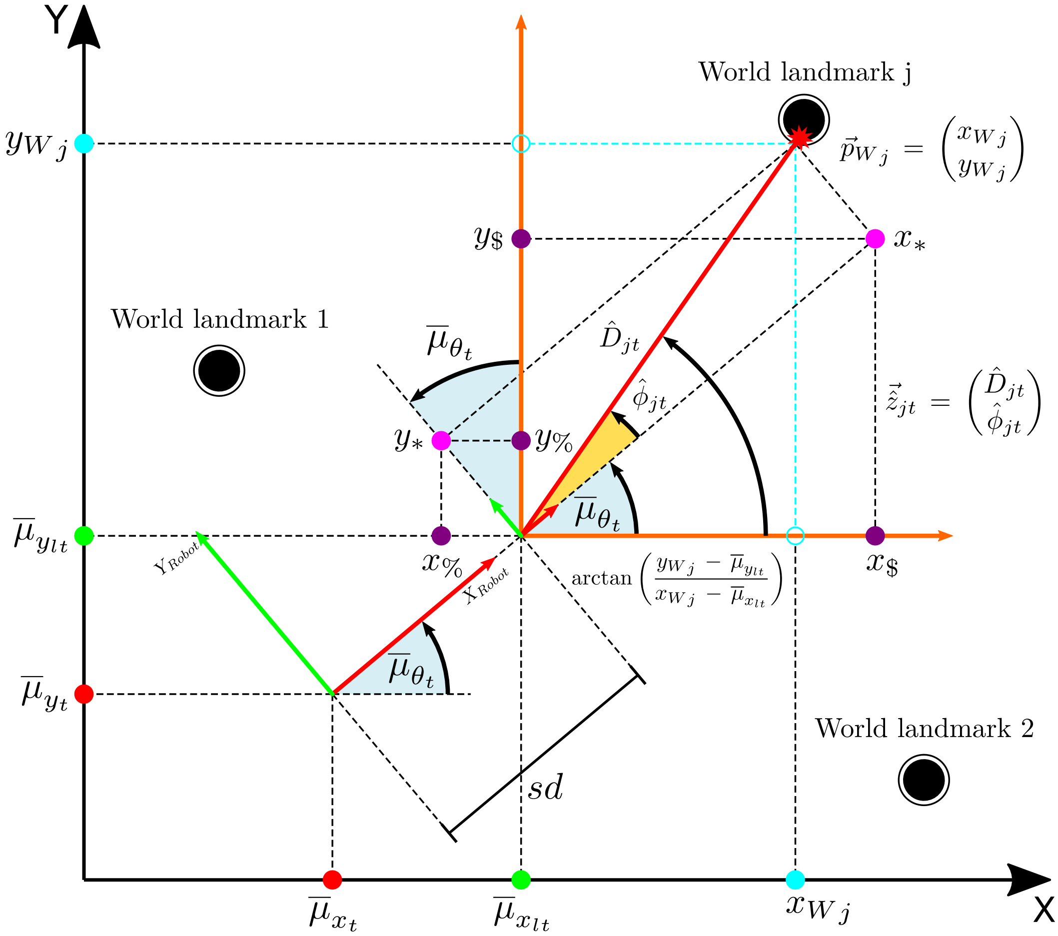
$$\vec{\hat{z}}_t = h\left(\vec{x}_t, \vec{p}_W\right) = \begin{pmatrix} \hat{D}_t = h_1\left(\vec{x}_t, \vec{p}_W\right) = \sqrt{\left(x_W - x_{lt}\right)^2 + \left(y_W - y_{lt}\right)^2} \\ \hat{\phi}_t = h_2\left(\vec{x}_t, \vec{p}_W\right) = \arctan\left(\dfrac{y_W - y_{lt}}{x_W - x_{lt}}\right) - \theta_t \end{pmatrix}$$

$$\begin{pmatrix} x_{lt} = x_t + sd\cos\left(\theta_t\right) \\ y_{lt} = y_t + sd\sin\left(\theta_t\right) \end{pmatrix}$$

The value $\vec{\hat{z}}_{jt}$ indicates the expected measurement between the scanner laser's predicted position, $\vec{\overline{\mu}}_{lt}$, and the position of the landmark number j, $\vec{p}_{Wj}$:

$$\vec{\hat{z}}_{jt} = h\left(\vec{\overline{\mu}}_t, \vec{p}_{Wj}\right)$$

$$\begin{pmatrix} \hat{D}_{jt} = h_1\left(\vec{\overline{\mu}}_t, \vec{p}_{Wj}\right) = \sqrt{\left(x_{Wj} - \overline{\mu}_{x_{lt}}\right)^2 + \left(y_{Wj} - \overline{\mu}_{y_{lt}}\right)^2} \\ \hat{\phi}_{jt} = h_2\left(\vec{\overline{\mu}}_t, \vec{p}_{Wj}\right) = \arctan\left(\dfrac{y_{Wj} - \overline{\mu}_{y_{lt}}}{x_{Wj} - \overline{\mu}_{x_{lt}}}\right) - \overline{\mu}_{\theta_t} \end{pmatrix}$$



Then for each world landmark successfully matched to a detected landmark the algorithm computes the distance from the laser scanner's predicted pose to that world landmark, $\hat{D}_{jt}$, and the angle defined between the laser scanner's predicted longitudinal axis and the imaginary line that joints the laser scanner's predicted position with that world landmark, $\hat{\phi}_{jt}$. These computations are carried out with the function $h\left(\vec{x}_t, \vec{p}_W\right)$. Therefore the algorithm computes:

$$\vec{\hat{z}}_{2t} = \begin{pmatrix} \hat{D}_{2t} \\ \hat{\phi}_{2t} \end{pmatrix} = h\left(\overline{\mu}_t, \vec{p}_{W2}\right)$$

$$\vec{\hat{z}}_{5t} = \begin{pmatrix} \hat{D}_{5t} \\ \hat{\phi}_{5t} \end{pmatrix} = h\left(\overline{\mu}_t, \vec{p}_{W5}\right)$$

$$\vec{\hat{z}}_{1t} = \begin{pmatrix} \hat{D}_{1t} \\ \hat{\phi}_{1t} \end{pmatrix} = h\left(\overline{\mu}_t, \vec{p}_{W1}\right)$$

So, at this moment the algorithm has ended with the matching between detected landmarks and world landmarks.

$$\left(\vec{z}_{1t},\, \hat{\vec{z}}_{2t}\right),\ \left(\vec{z}_{3t},\, \hat{\vec{z}}_{5t}\right),\ \left(\vec{z}_{4t},\, \hat{\vec{z}}_{1t}\right)$$

$$\vec{z}_{1t} = \begin{pmatrix} D_{1t} \\ \phi_{1t} \end{pmatrix},\ \vec{z}_{3t} = \begin{pmatrix} D_{3t} \\ \phi_{3t} \end{pmatrix},\ \vec{z}_{4t} = \begin{pmatrix} D_{4t} \\ \phi_{4t} \end{pmatrix}$$

$$\hat{\vec{z}}_{2t} = \begin{pmatrix} \hat{D}_{2t} \\ \hat{\phi}_{2t} \end{pmatrix},\ \hat{\vec{z}}_{5t} = \begin{pmatrix} \hat{D}_{5t} \\ \hat{\phi}_{5t} \end{pmatrix},\ \hat{\vec{z}}_{1t} = \begin{pmatrix} \hat{D}_{1t} \\ \hat{\phi}_{4t} \end{pmatrix}$$

For each one of the previous matches the algorithm performs a correction step. All these correction steps are carried out in a row, one after the other. In the example there are three matches, so there are three correction steps in a row. In each correction step the algorithm computes the terms H_t, K_t, $\vec{\mu}_t$ and Σ_t. The term H_t is the jacobian matrix of the funcion $h\left(\vec{x}_t, \vec{p}_W\right)$ with respect to $\vec{x}_t$.

$$H_t = \frac{\partial h\left(\vec{x}_t,\, \vec{p}_W\right)}{\partial \vec{x}_t} = \begin{pmatrix} \dfrac{\partial h_1}{\partial x_t} & \dfrac{\partial h_1}{\partial y_t} & \dfrac{\partial h_1}{\partial \theta_t} \\ \dfrac{\partial h_2}{\partial x_t} & \dfrac{\partial h_2}{\partial y_t} & \dfrac{\partial h_2}{\partial \theta_t} \end{pmatrix}$$

Let's give a shorter name to some terms that are going to appear immediately:

$$\Delta x_t \triangleq (x_W - x_{lt})$$
$$\Delta y_t \triangleq (y_W - y_{lt})$$
$$q_t \triangleq (x_W - x_{lt})^2 + (y_W - y_{lt})^2$$

$$
\begin{aligned}
\frac{\partial h_1}{\partial x_t} &= -\left(\frac{x_W - x_{lt}}{\sqrt{q_t}}\right) = -\frac{\Delta x_t}{\sqrt{q_t}} \\
\frac{\partial h_1}{\partial y_t} &= -\left(\frac{y_W - y_{lt}}{\sqrt{q_t}}\right) = -\frac{\Delta y_t}{\sqrt{q_t}} \\
\frac{\partial h_1}{\partial \theta_t} &= \frac{1}{2\sqrt{q_t}}\big(2\,\Delta x_t\, sd\, \sin(\theta_t) - 2\,\Delta y_t\, sd\, \cos(\theta_t)\big) = \\
&= \frac{sd}{\sqrt{q_t}}\big(\Delta x_t \sin(\theta_t) - \Delta y_t \cos(\theta_t)\big) \\
\frac{\partial h_2}{\partial x_t} &= \frac{\Delta y_t}{q_t} \\
\frac{\partial h_2}{\partial y_t} &= -\frac{\Delta x_t}{q_t} \\
\frac{\partial h_2}{\partial \theta_t} &= -\frac{sd}{q_t}\big(\Delta x_t \cos(\theta_t) + \Delta y_t \sin(\theta_t)\big) - 1
\end{aligned}
$$

In the first correction step the terms $\vec{\overline{\mu}}_t$ and $\overline{\Sigma}_t$, computed in the predicted step, are used to compute the terms H_t, K_t, $\vec{\mu}_t$ and Σ_t. If there is more than one correction step, from the second correction step onwards the results obtained at the end of the previous correction step are feed into the current correction step that is about to star. Therefore, It can be said that at the end of each intermediate correction step the algorithm gives a partial corrected result. Only at the end of the last correction step there is a definitive corrected result, μ_t and Σ_t.

Continuing with our example, because of the algorithm could only match three of those detected landmarks with world landmarks only three correction steps are performed, one after the other, i.e, in a row, using a `for` loop, even though four landmarks were detected in the scan.

$$
\begin{aligned}
\vec{\mu}_t &= \vec{\overline{\mu}}_t \\
\Sigma_t &= \overline{\Sigma}_t
\end{aligned}
$$

$\texttt{for}\ k = 1\ \texttt{to}\ N_{\texttt{matches}}:$

$$
\begin{aligned}
& H_t\left(\vec{\mu}_t, \vec{p}_{W_j}^{\,\texttt{(match k)}}\right) \\
K_t &= \Sigma_t \cdot H_t^T \cdot \left(H_t \cdot \Sigma_t \cdot H_t^T + Q\right)^{-1} \\
\vec{\mu}_t &= \vec{\mu}_t + K_t \cdot \left(\vec{z}_{it}^{\,\texttt{(match k)}} - \vec{\hat{z}}_{jt}^{\,\texttt{(match k)}}\right) \\
\Sigma_t &= (I - K_t \cdot H_t) \cdot \Sigma_t
\end{aligned}
$$

The term $\vec{z}_{it}^{\,\texttt{(match k)}} - \vec{\hat{z}}_{jt}^{\,\texttt{(match k)}}$ is called **innovation.**

The term Q is the measurement covariance matrix, defined as:

$$
Q = \begin{pmatrix} \sigma_D^2 & 0 \\ 0 & \sigma_\phi^2 \end{pmatrix}
$$

and appears in the probability density function:

$$p\left(\vec{z}_t \mid \vec{x}_t\right) = \mathcal{N}\left(\hat{\vec{z}}_t,\, Q\right)$$

The first interesting thing that can be observed in the matrix Q is that it doesn't depend on the time, subscript t, therefore, it's a constant matrix. The second thing I observe is that the matrix Q is diagonal, so it means that the random variables D_t and ϕ_t are uncorrelated.

A note about angles:
Sometimes it's necessary to subtract two angles and a bit of attention has to be paid in order to prevent mistakes when computing the difference. For example, when the algorithm computes the innovation it has to subtract the term $\hat{\vec{z}}_t$ from the term $\vec{z}_t$. In this operation there is involved a subtraction of two angles.

$$\vec{z}_t - \hat{\vec{z}}_t = \begin{pmatrix} D_t - \hat{D}_t \\ \phi_t - \hat{\phi}_t \end{pmatrix}$$

Let's suppose that:

$$\begin{aligned} \phi_t &= +180° - \delta \\ \hat{\phi}_t &= -180° + \epsilon \end{aligned}$$

and the angles δ and ϵ are contained in the first quadrant, i.e, $0° \leq \delta \leq 90°$ and $0° \leq \epsilon \leq 90°$. Therefore, ϕ_t is necessarily located in the second quadrant, $90° \leq \phi_t \leq 180°$, and $\hat{\phi}_t$ is necessarily located in the third quadrant, $-180° \leq \hat{\phi}_t \leq -90°$.

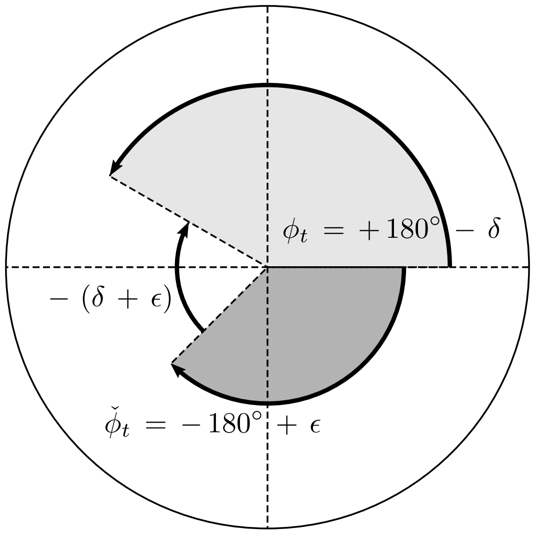
$$\phi_t - \hat{\phi}_t = 180° - \delta - \left(-180° + \epsilon\right) = 360° - (\delta + \epsilon)$$
$$180° \leq \phi_t - \hat{\phi}_t \leq 360°$$

If you are working with angles in the range $[-180°, +180°]$ the angle $\phi_t - \hat{\phi}_t$, which is in the range $[+180°, +360°]$, is not admissible. The angle that you would like is $-(\delta + \epsilon)$, which is in the range $[-180°, +180°]$. So, every time you work with angles that must belong to the range $[-180°, +180°]$ you must do the following correction:

$$\left(\left(\left(\phi_t - \hat{\phi}_t\right) + \pi\right) \bmod 2\pi\right) - \pi$$

Example:

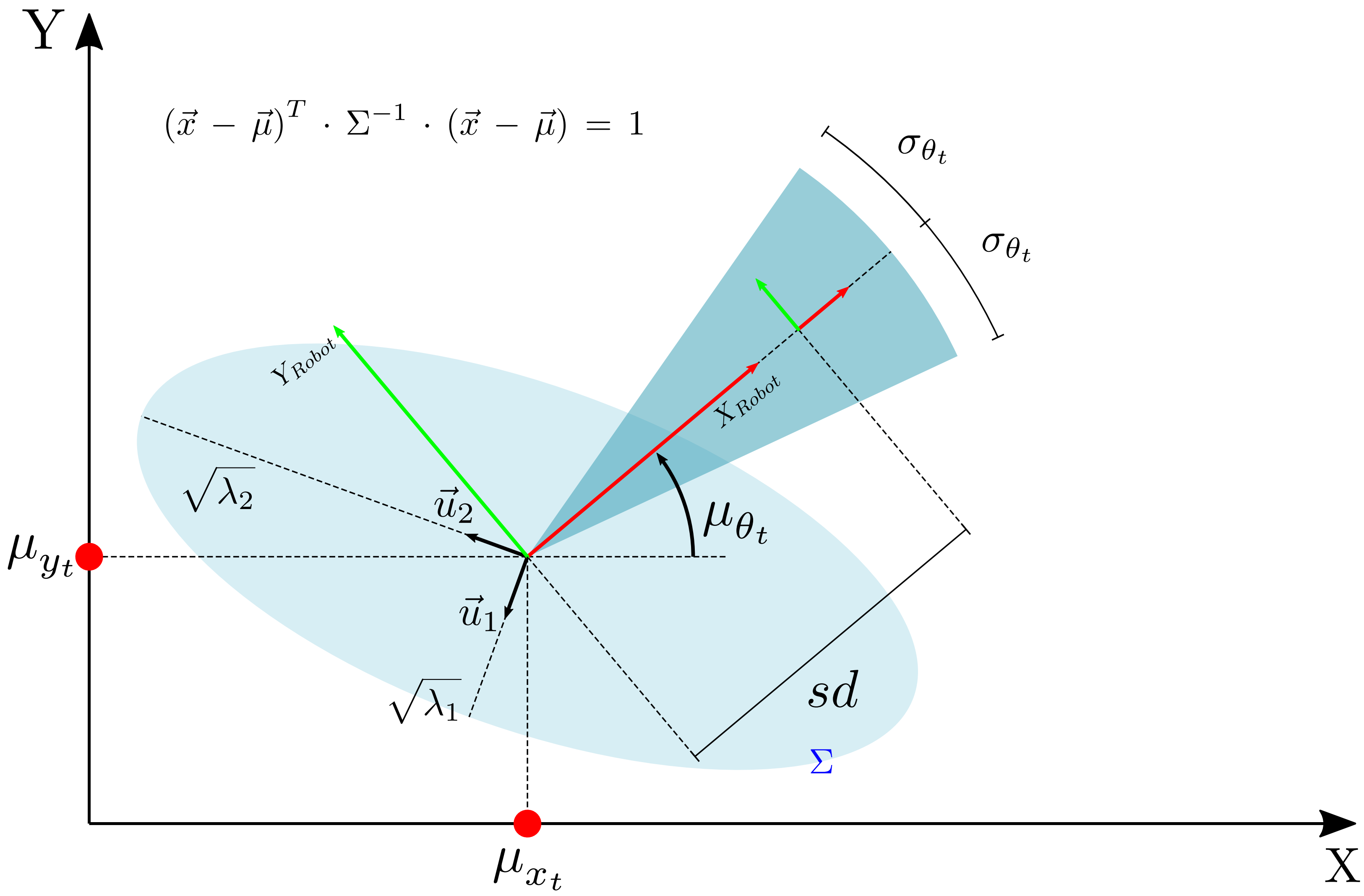
$$
\begin{aligned}
\delta &= 30^\circ \\
\epsilon &= 45^\circ \\
\phi_t &= +180^\circ - \delta = +150^\circ \\
\hat{\phi}_t &= -180^\circ + \epsilon = -135^\circ \\
\phi_t - \hat{\phi}_t &= 150^\circ - \left(-135^\circ\right) = 150^\circ + 135^\circ = 285^\circ \\
\left(\phi_t - \hat{\phi}_t\right) + 180^\circ &= 285^\circ + 180^\circ = 465^\circ \\
\left(\left(\phi_t - \hat{\phi}_t\right) + 180^\circ\right) \bmod 360^\circ &= 465^\circ \bmod 360^\circ = 105^\circ \\
\left(\left(\left(\phi_t - \hat{\phi}_t\right) + 180^\circ\right) \bmod 360^\circ\right) - 180^\circ &= 105^\circ - 180^\circ = -75^\circ = -(\delta + \epsilon)
\end{aligned}
$$



Note:

$$
\vec{\overline{\mu}}_t = \begin{pmatrix} \overline{\mu}_{x_t} \\ \overline{\mu}_{y_t} \\ \overline{\mu}_{\theta_t} \end{pmatrix} = \begin{pmatrix} \widetilde{x}_t \\ \widetilde{y}_t \\ \widetilde{\theta}_t \end{pmatrix} = \vec{\widetilde{x}}_t
$$

$$
\vec{\mu}_t = \begin{pmatrix} \mu_{x_t} \\ \mu_{y_t} \\ \mu_{\theta_t} \end{pmatrix} = \begin{pmatrix} \widehat{x}_t \\ \widehat{y}_t \\ \widehat{\theta}_t \end{pmatrix} = \vec{\widehat{x}}_t
$$



Y
$(\vec{x} - \vec{\mu})^T \cdot \Sigma^{-1} \cdot (\vec{x} - \vec{\mu}) = 1$
σ_{θ_t}
σ_{θ_t}
Y_{Robot}
X_{Robot}
$\sqrt{\lambda_2}$
$\vec{u}_2$
μ_{θ_t}
μ_{y_t}
$\vec{u}_1$
$\sqrt{\lambda_1}$
sd
Σ
μ_{x_t}
X