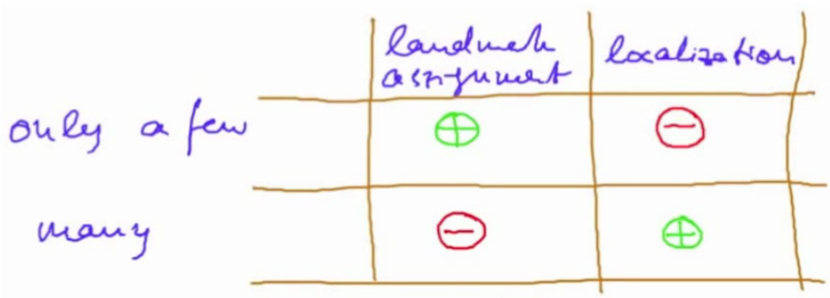
How many landmarks does the ONLINE SLAM algorithm need to have a good performance? There's no simple answer. If there are a few landmarks then the landmark assignment is good because if there's only few it's hard to confuse them. On the other hand it means for the localization that if there are only a few landmarks the algorithm has less observations to work with, so the accuracy is low or even if there are too few landmarks there can be parts in the trajectory without observations at all and the algorithm only runs the prediction, therefore the pose uncertainty will grow quickly. In conversely, if there are many landmarks then localization is good, however, it's easy to confuse them. In order to have many landmarks for good localization and still not to confuse them the algorithm needs distinct landmarks.



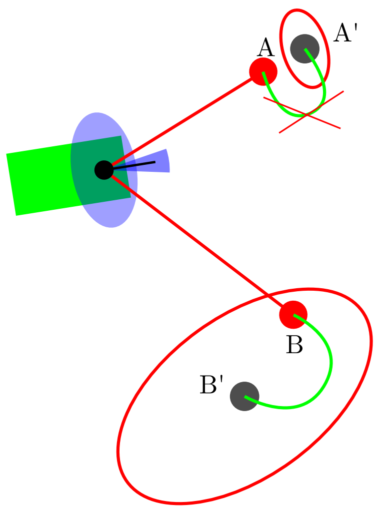
In our case all landmarks were the same because in the real world they actually had exactly the same dimensions. But in practice the landmarks may be different among them in different ways. For example, it is very usual that they differ in dimensions or in color. Therefore, if the algorithm uses the camera in addition it would have been able to detect the different colors of our landmarks and then it could have added the color as a component in the description of the landmarks. But there are other features that can be used to distinguish the landmarks. It can be the dimensions of the landmark, it can be a detector that detects if a surface is flat or that detects the curvature of a surface or that detects if there is a dihedral latch or even if there's a corner. A camera can be used to detect an image of the object, so the algorithm is able to detect feature points for which it can derive a high dimensional vector. In general, the idea is to use landmark signatures, so that each landmark is described by its position and also its signature.

Now our algorithm worked in the following way: The robot takes a scan from the world. Then the algorithm detects the observed cylinders in the scan and then it looks within a certain fixed radius for the closest landmark. If there isn't a match between a cylinder and a landmark the algorithm creates a new landmark. In our experiments we used a radius of 50 cm, then a radius of 40 cm and also a radius of 30 cm and we found out that the algorithm is brittle with respect to this parameter.

Now let's think about how to make the landmark assignment process more robust and reliable. Imagine the following situation. The robot measures the cylinder labeled as A and the cylinder labeled as B. In our map there is a landmark labeled as A' that the robot has observed very often. Consequently, the uncertainty ellipse for the landmark A' is pretty small. On the other hand, there is also a landmark labeled as B', which is a bit further away from cylinder B, than the distance from landmark A' to cylinder A. Besides, the uncertainty ellipse for the landmark B' is really large.
Obviously the cylinder A shouldn't be assigned to landmark A'. Let's explain this. The uncertainty ellipse for the landmark A' is really small, due to the fact that the robot has observed this landmark many times, so it means that the landmark A' is very well positioned in the world. If the cylinder A is not very very close to the landmark A', which is very well positioned in the world, is because, it's very sure that there is another landmark next to the cylinder A.

(Note that we haven't talked about radius to do the assignment yet)

On the other hand, the cylinder B should be assigned to the landmark B'. The uncertainty ellipse for the landmark B' is really large, so it means that the landmark B' couldn't be properly positioned in the world. In this occasion, and due to the large uncertainty ellipse for the landmark B', the cylinder B is considered to be close enough to this landmark to be assigned to it.
The landmark assignment process should take into account the uncertainty of the robot as well.



The new landmark assignment process is called **maximum likelihood landmark** assignment and is based on the Mahalanobis distance. The general expression for the squared Mahalanobis distance between any measurement, $\vec{z}_t$, and the expected measurement to any world landmark, $\vec{\hat{z}}_t = h\left(\vec{x}_t, \vec{p}_W\right)$, is:

$$M_t^2 = \left(\vec{z}_t - \vec{\hat{z}}_t\right)^T \cdot Q_t^{-1} \cdot \left(\vec{z}_t - \vec{\hat{z}}_{jt}\right)$$
$$Q_{jt} = H_t \cdot \overline{\Sigma}_t \cdot H_t^T + Q$$

where the term $\vec{p}_W$ represents the coordinates of a world landmark:

$$\vec{p}_W = \begin{pmatrix} x_W \\ y_W \end{pmatrix}$$

Let's denote M_{jt}^2 to the squared Mahalanobis distance between any measurement, $\vec{z}_t$, and the expected measurement to the world landmark number j, $\vec{\hat{z}}_{jt} = h\left(\vec{\overline{\mu}}_t, \vec{\overline{p}}_{Wj}\right)$:

$$M_{jt}^2 = \left(\vec{z}_t - \vec{\hat{z}}_{jt}\right)^T \cdot Q_{jt}^{-1} \cdot \left(\vec{z}_t - \vec{\hat{z}}_{jt}\right)$$
$$Q_{jt} = H_{jt} \cdot \overline{\Sigma}_t \cdot H_{jt}^T + Q$$

where the term $\vec{\overline{p}}_{Wj}$ represents the predicted coordinates for the world landmark number j:

$$\vec{\overline{p}}_{Wj} = \begin{pmatrix} \overline{\mu}_{x_{Wj}} \\ \overline{\mu}_{y_{Wj}} \end{pmatrix}$$

Finally, let's denote M_{jit}^2 to the squared Mahalanobis distance between the measurement number i, $\vec{z}_{it}$, and the expected measurement to the world landmark number j, $\vec{\hat{z}}_{jt} = h\left(\vec{\overline{\mu}}_t, \vec{\overline{p}}_{Wj}\right)$:

$$M_{jit}^2 = \left(\vec{z}_{it} - \vec{\hat{z}}_{jt}\right)^T \cdot Q_{jt}^{-1} \cdot \left(\vec{z}_{it} - \vec{\hat{z}}_{jt}\right)$$
$$Q_{jt} = H_{jt} \cdot \overline{\Sigma}_t \cdot H_{jt}^T + Q$$

The subscript j is the number of the registered world landmark that the algorithm is currently using. The predicted x coordinate for the world landmark number j, $\overline{\mu}_{x_{Wj}}$, is stored within the predicted state vector, $\vec{\overline{\mu}}_t$, at the index:

$$i = 3 + (2\,j) - 1$$

The predicted y coordinate for the world landmark number j, $\overline{\mu}_{y_{Wj}}$, is stored within the predicted state vector, $\vec{\overline{\mu}}_t$, at the index

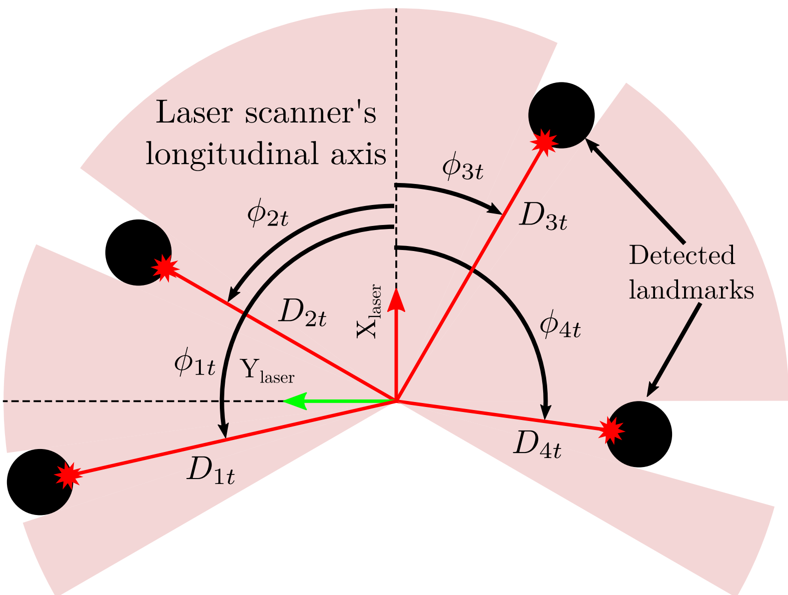
$$i = 3 + (2\,j)$$

$$\vec{\overline{\mu}}_t = \begin{pmatrix} \overline{\mu}_{x_t} \\ \overline{\mu}_{y_t} \\ \overline{\mu}_{\theta_t} \\ \overline{\mu}_{x_{W\,1}} \\ \overline{\mu}_{y_{W\,1}} \\ \vdots \\ \overline{\mu}_{x_{W\,j}} \\ \overline{\mu}_{y_{W\,j}} \\ \vdots \\ \overline{\mu}_{x_{W\,N}} \\ \overline{\mu}_{y_{W\,N}} \end{pmatrix} \begin{array}{l} \longleftarrow i = 0 \\ \longleftarrow i = 1 \\ \longleftarrow i = 2 \\ \\ \\ \\ \longleftarrow \text{Predicted } x \text{ coordinated for the world landmark number } j,\ i = 3 + (2j) - 1 \\ \longleftarrow \text{Predicted } y \text{ coordinated for the world landmark number } j,\ i = 3 + (2j) \\ \\ \\ \\ \end{array}$$

The observation vector, $\vec{z}_t$, is comprises by two terms:

$$\vec{z}_t = \begin{pmatrix} D_t \\ \phi_t \end{pmatrix}$$

The term D_t is the distance from the laser scanner to a detected landmark.
The term ϕ_t is the angle defined between the laser scanner's longitudinal axis and the imaginary line that joints the laser scanner with a detected landmark.



The expected observation vector, $\hat{\vec{z}}_{jt} = h\left(\vec{\overline{\mu}}_t, \vec{p}_{W\,j}\right)$, is comprises by two terms:

$$\hat{\vec{z}}_{jt} = \begin{pmatrix} \hat{D}_{jt} \\ \hat{\phi}_{jt} \end{pmatrix}$$

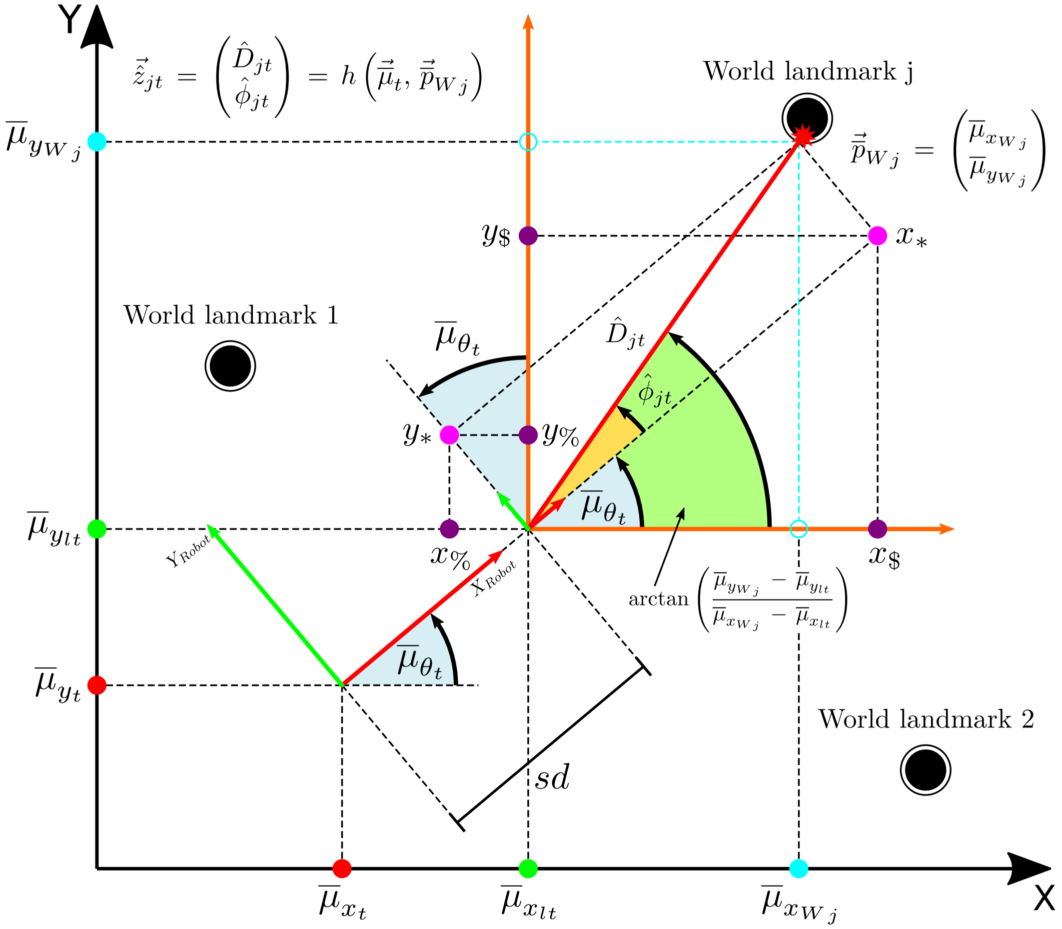
The term $\hat{D}_{jt}$ is the distance from the laser scanner's predicted pose, $\vec{\overline{\mu}}_{lt}^{T} = \left(\overline{\mu}_{x_{lt}}, \overline{\mu}_{y_{lt}}, \overline{\mu}_{\theta_t}\right)$, to the predicted position for the registered world landmark number j, $\left(\overline{\mu}_{xW\,j}, \overline{\mu}_{yW\,j}\right)$.

$$\begin{aligned}
\hat{D}_{jt} &= \sqrt{\left(\overline{\mu}_{xW_j} - \overline{\mu}_{x_{lt}}\right)^2 + \left(\overline{\mu}_{yW_j} - \overline{\mu}_{y_{lt}}\right)^2} \\
\overline{\mu}_{x_{lt}} &= \overline{\mu}_{x_t} + sd\cos\left(\overline{\mu}_{\theta_t}\right) \\
\overline{\mu}_{y_{lt}} &= \overline{\mu}_{y_t} + sd\sin\left(\overline{\mu}_{\theta_t}\right)
\end{aligned}$$

The term $\hat{\phi}_{jt}$ is the angle between:

- The laser scanner's predicted orientation angle, $\overline{\mu}_{\theta_t}$ and
- The orientation angle of the imaginary line that joints the laser scanner's predicted position with the predicted position for the registered world landmark number j, $\arctan\left(\dfrac{\overline{\mu}_{yW_j} - \overline{\mu}_{y_{lt}}}{\overline{\mu}_{xW_j} - \overline{\mu}_{x_{lt}}}\right)$

$$\hat{\phi}_{jt} = \arctan\left(\frac{\overline{\mu}_{yW_j} - \overline{\mu}_{y_{lt}}}{\overline{\mu}_{xW_j} - \overline{\mu}_{x_{lt}}}\right) - \overline{\mu}_{\theta_t}$$



Note:
The difference

$$\vec{z}_t - \vec{\hat{z}}_{jt}$$

is a measurement of how well the detected landmark matches with the registered world landmark number j. The matrix Q is the covariance matrix due to the uncertainty (error) produced in the measure system. The matrix $\overline{\Sigma}_t$ is the uncertainty (error) in the predicted state vector, $\vec{\overline{\mu}}_t$. By multiplying the matrix $\overline{\Sigma}_t$ from the left by the term H_{jt}, and from the right by the term H_{jt}^T, the algorithm is propagating the uncertainty (error) through the function $h\left(\vec{x}_t, \vec{p}_W\right)$. Therefore, the term $H_{jt} \cdot \overline{\Sigma}_t \cdot H_{jt}^T$ represents the uncertainty propagation. Basically this term is due to the uncertainty (error) in the robot's predicted pose and the uncertainty in the measure system.
Then the algorithm uses a maximum distance threshold, ϵ, to accept the match between the detected landmark number i and the registered world landmark number j. The match is accepted or rejected based on the squared Malahanobis distance.

$$M_{jit}^2 \le \epsilon$$

$$\left(\vec{z}_{it} - \vec{\hat{z}}_{jt}\right)^T \cdot Q_{jt}^{-1} \cdot \left(\vec{z}_{it} - \vec{\hat{z}}_{jt}\right) \le \epsilon$$

If $M_{jit}^2 \le \epsilon$ then the algorithm accepts the match between the detected landmark number i and the registered world landmark number j. Otherwise, the match is rejected.
What is happening here is that:
The algorithm is accepting the match between the detected landmark i and the world landmark number j if the measurement $\vec{z}_{it}$ is inside, or in the border, of the projection ellipse defined by $\left(\vec{z}_t - \vec{\hat{z}}_{jt}\right)^T \cdot Q_{jt}^{-1} \cdot \left(\vec{z}_t - \vec{\hat{z}}_{jt}\right) = \epsilon$.
This projection ellipse belongs to the Gaussian distribution $\mathcal{N}\left(\vec{z}_t, \vec{\hat{z}}_{jt}, Q_{jt}\right)$.

General expression for any detected landmark and any world landmark

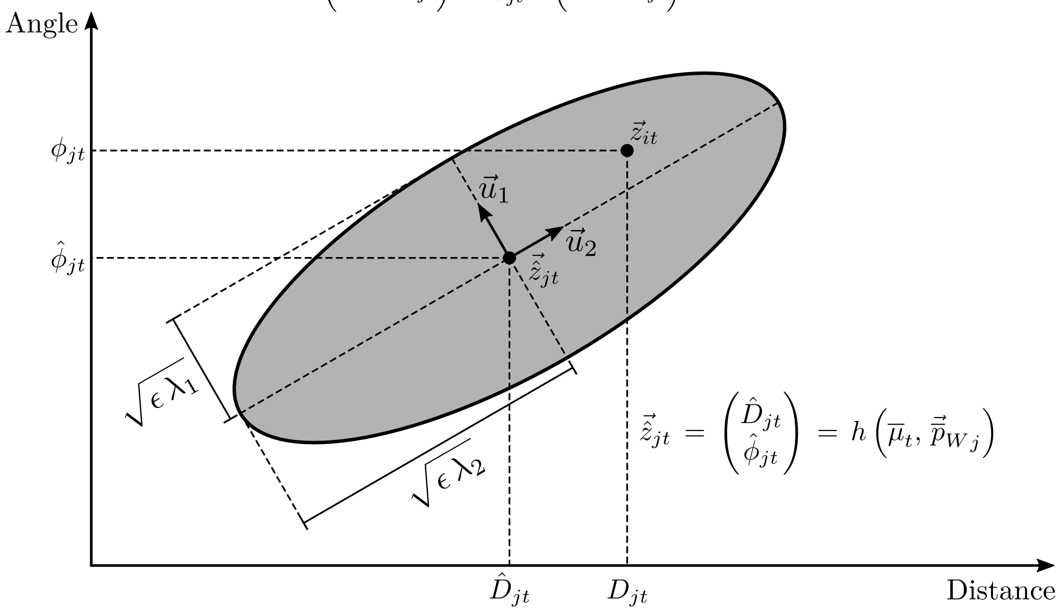
$$\left(\vec{z}_t - \vec{\hat{z}}_t\right)^T \cdot Q_t^{-1} \cdot \left(\vec{z}_t - \vec{\hat{z}}_t\right) \le \epsilon$$

Specific expression for any detected landmark and for the world landmark number j

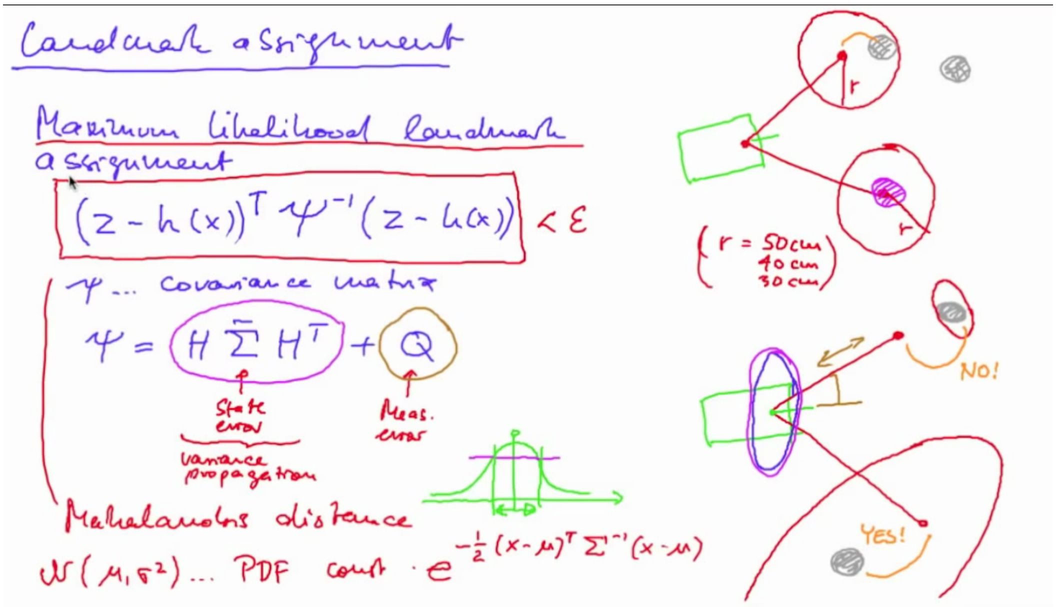
$$\left(\vec{z}_t - \vec{\hat{z}}_{jt}\right)^T \cdot Q_{jt}^{-1} \cdot \left(\vec{z}_t - \vec{\hat{z}}_{jt}\right) \le \epsilon$$

Specific expression for the detected landmark number i and for the world landmark number j

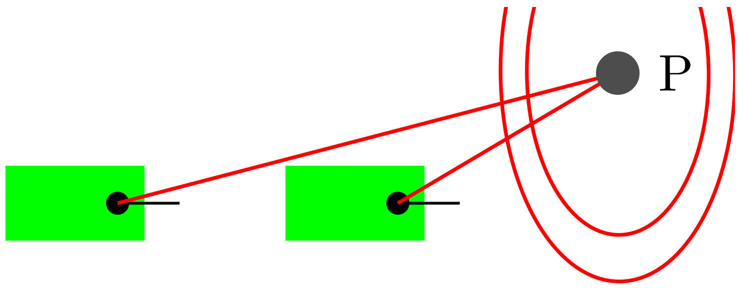
$$\left(\vec{z}_{it} - \vec{\hat{z}}_{jt}\right)^T \cdot Q_{jt}^{-1} \cdot \left(\vec{z}_{it} - \vec{\hat{z}}_{jt}\right) \le \epsilon$$



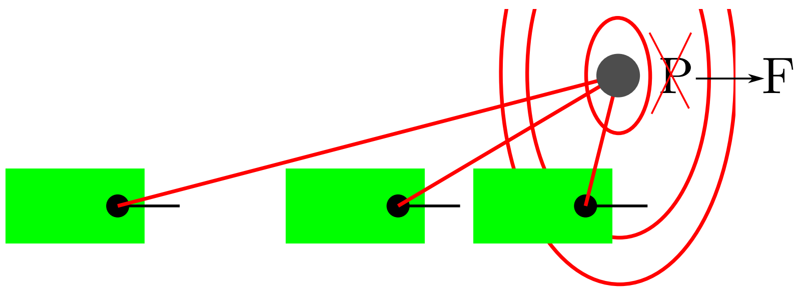
The projection ellipse is centered at the vector $\vec{\hat{z}}_{jt} = h\left(\vec{\overline{\mu}}_t, \vec{\overline{p}}_{Wj}\right)$. The projection ellipse is oriented according to the orthonormal eigenvectors of the covariance matrix Q_{jt}. The principal axes' lengths of the projection ellipse depend on the eigenvalues of the covariance matrix Q_{jt}, $\sqrt{\epsilon\,\lambda_1}$ and $\sqrt{\epsilon\,\lambda_2}$.



Regarding the landmark assignment, now, there is also a provisional landmark list. Let's consider that the robot sees a landmark for the first time. Now the algorithm will not put this landmark into the specific state vector as a standard landmark. Rather the algorithm will store that landmark in the provisional landmark list and it will add to the covariance matrix the landmark's position variances. The robot moves on and it observes the landmark again (for the second time), therefore the landmark uncertainty ellipse will decrease.



The algorithm will put the landmark into the final landmark list only after several observations indicate that this landmark has been found consistently.



In practice, it is relatively easy to handle this. The idea is that any newly observed landmark is considered provisional and it is stored into the specific state vector. But the observation model is modified in a way that for provisional landmarks this model only handles the position of these landmark as variables (concept of mathematics variable) and the robot's position and orientation are not consider as variables. In this way as long as a landmark is considered provisional the robot will influence, by subsequent observations, the accuracy of that provisional landmark's position but that provisional landmark won't influence the robot's pose. So the algorithm adds a flag to know whether a landmark is considered provisional or not, and therefore it can be used a slightly different observation model.

OVERVIEW:

- **FULL SLAM problem**: the algorithm computes the posterior over all states and assignments of landmarks. This problem is usually computationally infeasible.
- **ONLINE SLAM problem**: this problem is easier to handle and it uses a deterministic computation of landmark assignments so instead of providing the full posterior over all states and assignments you sign on the fly using a deterministic algorithm. This is also its main drawback because as we saw it's brittle with respect to landmark confusion. Even though the ONLINE SLAM is less complex than the full slam it still has a huge update complexity that grows with the number of landmarks because the size of the specific state vector grows with a number of landmarks so it is potentially unbound. Nevertheless ONLINE SLAM has been used by many groups with considerable success and as you have seen for a small problem it has worked out perfectly.