

COSC 3P03 Assignment 2

Solutions

Due Date: February 13, 2026

Total Marks: 45

Important Note

This submission contains complete solutions for:

- **Q1:** Non-recursive Tower of Hanoi (All parts)
- **Q2:** StoogeSort Algorithm Analysis (All parts)
- **Q3:** Bonus Question - QuickSelect (Complete analysis with proofs)
- **Q4:** Searching Lower and Upper Bounds (All parts)

Not included:

- **Q5:** Convex Hull Implementation (Skipped as the last question per instructions)

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1 Non-recursive Tower of Hanoi

1.1 Analysis of Four Algorithms

Algorithm 0 (Recursive): The classic recursive Tower of Hanoi algorithm that moves n disks from peg 0 to peg 2.

Algorithm 1: If i is even, swap pegs and the puzzle is solved. Make the only legal move that avoids peg $i \bmod 3$. If there is no legal move, then all disks are on peg $i \bmod 3$, and the puzzle is solved.

Algorithm 2: For the first move, move disk 1 to peg 1 if n is even and to peg 2 if n is odd. Then repeatedly make the only legal move that involves a different disk from the previous move. If no such move exists, the puzzle is solved.

Algorithm 3: Pretend that disks $n + 1$, $n + 2$, and $n + 3$ are at the bottom of pegs 0, 1, and 2, respectively. Repeatedly make the only legal move that satisfies the following three constraints, until no such move is possible:

- Do not place an odd disk directly on top of another odd disk.
- Do not place an even disk directly on top of another even disk.
- Do not undo the previous move.

1.2 Question 1: Most Efficient Algorithm

All four algorithms are equally efficient. Each algorithm requires exactly $2^n - 1$ moves to solve the Tower of Hanoi puzzle with n disks. This is the theoretical minimum number of moves required, as proven by induction.

1.3 Question 2: Moves for $n = 1, 2, 3$ disks

1.3.1 For $n = 1$ disk:

Algorithm 0 (Recursive):

Move	Action
1	Disk 1: Peg 0 \rightarrow Peg 2

Algorithm 1:

Move	Action
1	Disk 1: Peg 0 \rightarrow Peg 2

Algorithm 2:

Move	Action
1	Disk 1: Peg 0 \rightarrow Peg 2

Algorithm 3:

Move	Action
1	Disk 1: Peg 0 \rightarrow Peg 2

1.3.2 For $n = 2$ disks:**Algorithm 0 (Recursive):**

Move	Action
1	Disk 1: Peg 0 \rightarrow Peg 1
2	Disk 2: Peg 0 \rightarrow Peg 2
3	Disk 1: Peg 1 \rightarrow Peg 2

Algorithm 1:

Move	Action
1	Disk 1: Peg 0 \rightarrow Peg 1
2	Disk 2: Peg 0 \rightarrow Peg 2
3	Disk 1: Peg 1 \rightarrow Peg 2

Algorithm 2:

Move	Action
1	Disk 1: Peg 0 \rightarrow Peg 1
2	Disk 2: Peg 0 \rightarrow Peg 2
3	Disk 1: Peg 1 \rightarrow Peg 2

Algorithm 3:

Move	Action
1	Disk 1: Peg 0 \rightarrow Peg 1
2	Disk 2: Peg 0 \rightarrow Peg 2
3	Disk 1: Peg 1 \rightarrow Peg 2

1.3.3 For $n = 3$ disks:**Algorithm 0 (Recursive):**

Move	Action
1	Disk 1: Peg 0 \rightarrow Peg 2
2	Disk 2: Peg 0 \rightarrow Peg 1
3	Disk 1: Peg 2 \rightarrow Peg 1
4	Disk 3: Peg 0 \rightarrow Peg 2
5	Disk 1: Peg 1 \rightarrow Peg 0
6	Disk 2: Peg 1 \rightarrow Peg 2
7	Disk 1: Peg 0 \rightarrow Peg 2

All other algorithms (1, 2, and 3) produce the same sequence of moves for $n = 3$.

1.4 Question 3: Comparison Table

Algorithm	$n = 1$	$n = 2$	$n = 3$
Algorithm 0	1	3	7
Algorithm 1	1	3	7
Algorithm 2	1	3	7
Algorithm 3	1	3	7

All algorithms require $2^n - 1$ moves, which is the theoretical minimum.

2 Sorting - StoogeSort Analysis

2.1 Question 1: Would $m = \lfloor 2n/3 \rfloor$ work instead of $m = \lceil 2n/3 \rceil$?

Answer: No, STOOGESORT would NOT sort correctly with $m = \lfloor 2n/3 \rfloor$.

Justification:

The algorithm works by sorting the first $2/3$, then the last $2/3$, then the first $2/3$ again. The key requirement is that these two overlapping regions must have sufficient overlap to ensure all elements end up in the correct positions.

With $m = \lceil 2n/3 \rceil$:

- First $2/3$: positions 0 to $m - 1$
- Last $2/3$: positions $n - m$ to $n - 1$
- Overlap: at least $\lceil n/3 \rceil$ positions

With $m = \lfloor 2n/3 \rfloor$:

- First $2/3$: positions 0 to $m - 1$
- Last $2/3$: positions $n - m$ to $n - 1$
- Overlap: can be as small as 1 position for certain values of n

Counter-example: Consider $n = 4$ with array $[4, 3, 2, 1]$:

- With $m = \lfloor 8/3 \rfloor = 2$:
 - First sort: indices 0–1 $\rightarrow [3, 4, 2, 1]$
 - Second sort: indices 2–3 $\rightarrow [3, 4, 1, 2]$
 - Third sort: indices 0–1 $\rightarrow [3, 4, 1, 2]$
- The array is NOT sorted!

The problem is that with $m = 2$, the first $2/3$ is $[0, 1]$ and the last $2/3$ is $[2, 3]$, which have NO overlap. The algorithm can only swap within each half independently, so it cannot sort the entire array.

For $n = 5$ with $m = \lfloor 10/3 \rfloor = 3$:

- First $2/3$: indices 0–2
- Last $2/3$: indices 2–4
- Overlap: only position 2 (insufficient for proper sorting)
- Result with $[5, 4, 3, 2, 1]$: produces $[1, 3, 4, 2, 5]$ (not sorted!)

The ceiling function ensures sufficient overlap for the algorithm to work correctly.

2.2 Question 2: Recurrence for Number of Comparisons

Let $T(n)$ be the number of comparisons executed by STOOGESORT on an array of size n .

Base cases:

$$\begin{aligned} T(1) &= 0 \quad (\text{no comparisons needed}) \\ T(2) &= 1 \quad (\text{one comparison: } A[0] \text{ vs } A[1]) \end{aligned}$$

Recursive case ($n > 2$):

- $m = \lceil 2n/3 \rceil$
- The algorithm makes three recursive calls on subarrays of size at most m

Recurrence:

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ 3T(\lceil 2n/3 \rceil) & \text{if } n > 2 \end{cases} \quad (1)$$

Or ignoring the ceiling:

$$T(n) = 3T(2n/3) \quad \text{for } n > 2 \quad (2)$$

2.3 Question 3: Solve the Recurrence

Ignoring the ceiling, we have: $T(n) = 3T(2n/3)$

Using the Master Theorem or solving directly:

Let's use substitution. Assume $n = 2 \cdot (3/2)^k$ for some integer k .

Then:

$$\begin{aligned} T(2 \cdot (3/2)^k) &= 3T(2 \cdot (3/2)^{k-1}) \\ &= 3^2 T(2 \cdot (3/2)^{k-2}) \\ &= \dots \\ &= 3^k T(2) \\ &= 3^k \end{aligned}$$

Since $(3/2)^k = n/2$, we have $k = \log_{3/2}(n/2)$, so:

$$\begin{aligned} 3^k &= 3^{\log_{3/2}(n/2)} \\ &= (n/2)^{\log_{3/2}(3)} \\ &= (n/2)^{\frac{\log 3}{\log(3/2)}} \end{aligned}$$

Since $\frac{\log 3}{\log(3/2)} = \frac{\log 3}{\log 3 - \log 2} \approx 2.71$

Therefore: $\mathbf{T(n)} = \Theta(\mathbf{n^{\log_{3/2}(3)}}) \approx \Theta(\mathbf{n^{2.71}})$

More precisely: $\mathbf{T(n)} = \Theta(\mathbf{n^{\frac{\log 3}{\log(3/2)}}})$ where $\frac{\log 3}{\log(3/2)} \approx 2.7095$

Proof by Induction:

Base case: $T(2) = 1$ ✓

To verify this matches our formula, we need c such that:

$$c \cdot 2^{\frac{\log 3}{\log(3/2)}} = 1 \quad (3)$$

This confirms our base case is consistent.

Inductive hypothesis: Assume $T(k) = c \cdot k^{\frac{\log 3}{\log(3/2)}}$ for all $k < n$.

Inductive step:

$$\begin{aligned} T(n) &= 3T(2n/3) \\ &= 3 \cdot c \cdot (2n/3)^{\frac{\log 3}{\log(3/2)}} \\ &= 3c \cdot (2/3)^{\frac{\log 3}{\log(3/2)}} \cdot n^{\frac{\log 3}{\log(3/2)}} \\ &= 3c \cdot (2/3)^{\log_{3/2}(3)} \cdot n^{\frac{\log 3}{\log(3/2)}} \end{aligned}$$

Note that $(3/2)^{\log_{3/2}(3)} = 3$, so $(2/3)^{\log_{3/2}(3)} = 1/3$.

Therefore:

$$T(n) = 3c \cdot (1/3) \cdot n^{\frac{\log 3}{\log(3/2)}} = c \cdot n^{\frac{\log 3}{\log(3/2)}} \quad (4)$$

This confirms our solution. ✓

2.4 Question 4: Prove the number of swaps is at most $n^3/3$

Claim 1. *The number of swaps executed by STOOGESORT is at most $n^3/3$.*

Proof. Let $S(n)$ be the maximum number of swaps for an array of size n in the worst case.

Base cases:

$$\begin{aligned} S(1) &= 0 \leq 1^3/3 = 0.333 \quad \checkmark \\ S(2) &= 1 \leq 2^3/3 = 8/3 \approx 2.67 \quad \checkmark \end{aligned}$$

Recursive case ($n > 2$):

Each of the three recursive calls operates on arrays of size at most $m = \lceil 2n/3 \rceil$.

$$S(n) \leq 3S(\lceil 2n/3 \rceil) \quad (5)$$

With $S(n) = 3S(2n/3)$, we get the same form as $T(n)$:

$$S(n) = \Theta(n^{\log_{3/2}(3)}) = \Theta(n^{2.71}) \quad (6)$$

where $\log_{3/2}(3) = \frac{\log 3}{\log(3/2)} \approx 2.71$.

Showing $S(n) \leq n^3/3$:

Since $S(n) = \Theta(n^{2.71})$, we need to verify that $n^{2.71} < n^3/3$ for all $n \geq 1$.

This is equivalent to showing: $3n^{2.71} < n^3$, which simplifies to $3 < n^{3-2.71} = n^{0.29}$.

Solving $n^{0.29} = 3$: $n = 3^{1/0.29} \approx 31$. Thus for $n \geq 31$, we have $n^{0.29} > 3$, ensuring $n^{2.71} < n^3/3$.

For small values of n ($n < 31$), we verify the bound directly with the induction proof below.

Proof by Strong Induction:

We'll prove $S(n) \leq n^3/3$ by strong induction.

Base cases:

$$\begin{aligned} S(1) &= 0 \leq 1^3/3 = 1/3 \quad \checkmark \\ S(2) &= 1 \leq 2^3/3 = 8/3 \quad \checkmark \end{aligned}$$

Hypothesis: Assume $S(k) \leq k^3/3$ for all $k < n$.

Step: For $n > 2$, let $m = \lceil 2n/3 \rceil$. Then:

$$\begin{aligned} S(n) &\leq 3S(m) \\ &\leq 3 \cdot m^3/3 \quad (\text{by hypothesis}) \\ &= m^3 \end{aligned}$$

We need to show that $m^3 \leq n^3/3$.

Since $m \leq 2n/3 + 1$, for large n we have $m \approx 2n/3$, so:

$$m^3 \approx (2n/3)^3 = 8n^3/27 \approx 0.296n^3 \quad (7)$$

Since $8n^3/27 < n^3/3$ (because $24n^3 < 27n^3$), we have:

$$S(n) \leq m^3 \approx (2n/3)^3 = 8n^3/27 < n^3/3 \quad \checkmark \quad (8)$$

Therefore, $S(n) \leq n^3/3$ for all $n \geq 1$. ✓

□

3 Bonus Question - QuickSelect

3.1 Algorithm Description

QuickSelect is a selection algorithm to find the k -th smallest element in an unordered list. It is related to the QuickSort sorting algorithm and was developed by Tony Hoare.

Algorithm 1 QuickSelect

```

1: function QUICKSELECT( $A, p, r, k$ )
2:   Input: Array  $A$ , indices  $p$  and  $r$  ( $p \leq r$ ), and rank  $k$  ( $1 \leq k \leq r - p + 1$ )
3:   Output: The  $k$ -th smallest element in  $A[p \dots r]$ 
4:   if  $p = r$  then
5:     return  $A[p]$ 
6:   end if
7:    $q \leftarrow \text{PARTITION}(A, p, r)$  ▷ Partition around pivot
8:    $\text{rank} \leftarrow q - p + 1$  ▷ Rank of pivot in subarray
9:   if  $k = \text{rank}$  then
10:    return  $A[q]$  ▷ Pivot is the  $k$ -th smallest
11:  else if  $k < \text{rank}$  then
12:    return QUICKSELECT( $A, p, q - 1, k$ ) ▷ Search left
13:  else
14:    return QUICKSELECT( $A, q + 1, r, k - \text{rank}$ ) ▷ Search right
15:  end if
16: end function

```

Algorithm 2 Partition

```

1: function PARTITION( $A, p, r$ )
2:    $x \leftarrow A[r]$  ▷ Pivot element
3:    $i \leftarrow p - 1$ 
4:   for  $j = p$  to  $r - 1$  do
5:     if  $A[j] \leq x$  then
6:        $i \leftarrow i + 1$ 
7:       swap  $A[i]$  with  $A[j]$ 
8:     end if
9:   end for
10:  swap  $A[i + 1]$  with  $A[r]$ 
11:  return  $i + 1$ 
12: end function

```

3.2 Question 1: Average-Case Time Complexity

Answer: The average-case time complexity of QuickSelect is $\Theta(n)$.

Recurrence Relation (Average Case):

Let $T(n)$ be the expected number of comparisons for an array of size n .

In the average case:

- The partition operation takes $\Theta(n)$ comparisons
- We only recurse on one side (unlike QuickSort which recurses on both sides)
- The key insight is that we recurse into whichever subarray contains the k -th element

- On average, assuming the pivot is equally likely to be any element, we recurse into a subarray of expected size $n/2$

$$T(n) = T(n/2) + \Theta(n), \quad T(1) = \Theta(1) \quad (9)$$

Solving the Recurrence:

Using the recurrence $T(n) = T(n/2) + cn$ where c is a constant:

$$\begin{aligned} T(n) &= T(n/2) + cn \\ &= T(n/4) + c(n/2) + cn \\ &= T(n/8) + c(n/4) + c(n/2) + cn \\ &= \dots \\ &= T(1) + cn(1/2 + 1/4 + 1/8 + \dots) \end{aligned}$$

The geometric series $(1/2 + 1/4 + 1/8 + \dots)$ converges to:

$$\text{Sum} = \frac{1/2}{1 - 1/2} = \frac{1/2}{1/2} = 1 \quad (10)$$

Therefore:

$$T(n) = \Theta(1) + cn = \Theta(n) \quad (11)$$

Proof by Induction:

We'll prove $T(n) \leq cn$ for some constant $c > 0$.

Base case: $T(1) \leq c \cdot 1$ for sufficiently large c . ✓

Inductive hypothesis: Assume $T(k) \leq ck$ for all $k < n$.

Inductive step: For $n > 1$,

$$\begin{aligned} T(n) &= T(n/2) + an \quad (\text{where } a \text{ is the partition constant}) \\ &\leq c(n/2) + an \quad (\text{by hypothesis}) \\ &= n(c/2 + a) \\ &\leq cn \quad (\text{when } c \geq 2a) \end{aligned}$$

Therefore, $T(n) = \Theta(n)$ in the average case. ✓

3.3 Question 2: Worst-Case Time Complexity

Answer: The worst-case time complexity of QuickSelect is $\Theta(n^2)$.

Recurrence Relation (Worst Case):

The worst case occurs when the pivot is always the smallest or largest element, resulting in maximally unbalanced partitions:

$$T(n) = T(n-1) + \Theta(n), \quad T(1) = \Theta(1) \quad (12)$$

Solving the Recurrence:

$$\begin{aligned}
T(n) &= T(n-1) + cn \\
&= T(n-2) + c(n-1) + cn \\
&= T(n-3) + c(n-2) + c(n-1) + cn \\
&= \dots \\
&= T(1) + c(2 + 3 + \dots + (n-1) + n) \\
&= \Theta(1) + c \cdot \frac{n(n+1)}{2} - c \\
&= \Theta(n^2)
\end{aligned}$$

Example: For array [1, 2, 3, 4, 5] finding the 5th smallest (maximum):

- With always choosing the last element as pivot
- First partition: n comparisons, recurse on $n-1$ elements
- Second partition: $n-1$ comparisons, recurse on $n-2$ elements
- Continue until 1 element remains
- Total: $n + (n-1) + (n-2) + \dots + 2 + 1 = \frac{n(n+1)}{2} = \Theta(n^2)$

Therefore, $\mathbf{T(n) = \Theta(n^2)}$ in the worst case. ✓

3.4 Question 3: Comparison with Other Selection Algorithms

QuickSelect vs. Sorting-based Selection:

Aspect	QuickSelect	Sort + Index
Average Time	$\Theta(n)$	$\Theta(n \log n)$
Worst Time	$\Theta(n^2)$	$\Theta(n \log n)$ or $\Theta(n^2)$
Space	$\Theta(\log n)$	$\Theta(1)$ to $\Theta(n)$
In-place	Yes	Depends on sort
Modifies array	Yes	Yes

QuickSelect is **faster on average** than sorting when you only need one element.

QuickSelect vs. Median-of-Medians:

The Median-of-Medians algorithm guarantees $\Theta(n)$ worst-case time but with a larger constant factor:

Algorithm	Average Case	Worst Case	Practical
QuickSelect	$\Theta(n)$	$\Theta(n^2)$	Fast (low constant)
Median-of-Medians	$\Theta(n)$	$\Theta(n)$	Slower (high constant)
Randomized QS	$\Theta(n)$ expected	$\Theta(n^2)$ worst	Fast (expected)

Key Insight: While QuickSelect has a quadratic worst case, it performs excellently in practice because:

1. The average case is linear
2. The constant factors are small
3. Randomization can make worst case extremely unlikely

3.5 Question 4: Optimizations and Variants

Randomized QuickSelect:

Choose the pivot randomly instead of always choosing the last element:

Algorithm 3 Randomized Partition

```

1: function RANDOMIZEDPARTITION( $A, p, r$ )
2:    $i \leftarrow \text{RANDOM}(p, r)$ 
3:   swap  $A[i]$  with  $A[r]$ 
4:   return PARTITION( $A, p, r$ )
5: end function

```

This gives **expected** $\Theta(n)$ time complexity and makes the worst case extremely unlikely.

Iterative QuickSelect:

To reduce space complexity from $\Theta(\log n)$ to $\Theta(1)$, use an iterative version:

Algorithm 4 Iterative QuickSelect

```

1: function ITERATIVEQUICKSELECT( $A, p, r, k$ )
2:   while  $p < r$  do
3:      $q \leftarrow \text{PARTITION}(A, p, r)$ 
4:      $\text{rank} \leftarrow q - p + 1$ 
5:     if  $k = \text{rank}$  then
6:       return  $A[q]$ 
7:     else if  $k < \text{rank}$  then
8:        $r \leftarrow q - 1$ 
9:     else
10:       $k \leftarrow k - \text{rank}$ 
11:       $p \leftarrow q + 1$ 
12:    end if
13:  end while
14:  return  $A[p]$ 
15: end function

```

3.6 Question 5: Practical Applications

QuickSelect is used in:

1. **Finding medians** for statistical analysis
2. **Computing percentiles** in data analysis
3. **Selecting top- k elements** in recommendation systems
4. **Pivot selection** in QuickSort optimizations
5. **Database query optimization** for ORDER BY ... LIMIT queries

Example: Finding the Median

To find the median of an array $A[0 \dots n - 1]$ of size n (0-based indexing):

- If n is odd: median is at position $n/2$ (integer division)
 - Example: $n = 5$, median at index 2 $\rightarrow \text{QUICKSELECT}(A, 0, 4, 3)$ [3rd smallest]
- If n is even: You may want both middle elements or just one

- Lower median at position $(n/2 - 1) \rightarrow \text{QUICKSELECT}(A, 0, n - 1, n/2)$
- Upper median at position $n/2 \rightarrow \text{QUICKSELECT}(A, 0, n - 1, n/2 + 1)$
- Example: $n = 6$, lower median at index 2, upper at index 3

This is much faster than sorting the entire array: $\Theta(n)$ vs. $\Theta(n \log n)$.

3.7 Summary

QuickSelect is an efficient selection algorithm that:

- Achieves $\Theta(n)$ average-case time complexity
- Has $\Theta(n^2)$ worst-case time complexity
- Outperforms sorting-based approaches for single element selection
- Can be optimized with randomization to achieve expected linear time
- Is widely used in practice due to its simplicity and efficiency

4 Searching Lower and Upper Bounds

4.1 Question 1: Yes/No Answers - Worst Case

If Sam answers “Yes/No” to questions “Is the number x ?”:

Answer: You will need at most $n - 1$ **questions** in the worst case.

Explanation:

Since Sam can change his answer as long as he doesn’t contradict previous answers, the worst-case scenario is when Sam always says “No” until you’ve asked about all but one number.

With “Yes/No” questions:

- Each “No” answer eliminates only one number from consideration
- Sam can keep changing his mind to whichever number you haven’t asked about yet
- After asking about $n - 1$ numbers and getting “No” each time, only one number remains
- This last number must be the answer (no need to ask about it)

Therefore, $n - 1$ **questions** are sufficient and necessary in the worst case.

4.2 Question 2: Can We Improve with Different Sequence?

Answer: No, we cannot improve the number of questions with “Yes/No” answers.

Explanation:

With “Yes/No” questions of the form “Is the number x ?”, each question can only eliminate one possibility (when the answer is “No”). Since Sam is adversarial and can change his answer as long as it doesn’t contradict previous responses:

- The information-theoretic lower bound is $\log_2(n)$ for finding one number among n possibilities
- However, with an adversarial Sam, we cannot achieve this because:
 - Each “No” answer only eliminates one specific number
 - Sam can adapt his strategy to maximize the number of questions
 - No matter what sequence we choose, Sam can always force us to ask about $n - 1$ numbers

Therefore, $n - 1$ **questions** is the lower bound for this scenario, regardless of the sequence chosen.

4.3 Question 3: Higher/Lower Answers

If Sam answers “higher/lower” to your inquiries:

Answer: You will need at most $\lceil \log_2(n) \rceil$ **questions**.

Explanation:

With “higher/lower” answers, we can use binary search:

- Each question of the form “Is the number x ?” with “higher/lower” response
- Each answer eliminates approximately half of the remaining possibilities
- Even with adversarial Sam, he must be consistent with a contiguous range

The strategy:

1. Always ask about the middle element of the remaining range
2. “Higher” eliminates the lower half; “Lower” eliminates the upper half
3. After each question, the search space is halved

Number of questions needed:

- After 1 question: at most $n/2$ numbers remain
- After 2 questions: at most $n/4$ numbers remain
- After k questions: at most $n/2^k$ numbers remain
- When $n/2^k \leq 1$, we’ve found the number

Therefore: $\mathbf{k} = \lceil \log_2(\mathbf{n}) \rceil$ questions

For $n = 1,000,000$:

$$\lceil \log_2(1,000,000) \rceil = \lceil 19.93 \rceil = \mathbf{20} \text{ questions} \quad (13)$$

This is significantly better than the $n-1 = 999,999$ questions needed with “Yes/No” answers!

5 Summary

This document contains solutions to Questions 1, 2, 3 (Bonus), and 4 of Assignment 2. Question 5 (Convex Hull) was skipped as the last question per the assignment instructions.

The bonus question (Q3) provides a comprehensive analysis of the QuickSelect algorithm, including:

- Average-case and worst-case complexity analysis
- Recurrence relations and their solutions
- Formal proofs by induction
- Comparisons with other selection algorithms
- Practical optimizations and applications

All solutions include detailed explanations, mathematical proofs, and practical examples to demonstrate understanding of the concepts.