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Section 2

Bi 1 HW 6

1a)

State	Energy	Multiplicity	Weight
Promoter OFF	$P\varepsilon_{pd}^{NS}$	$\binom{N_{NS}}{P} = \frac{N_{NS}!}{P!\,(N_{NS}-P)!} \approx \frac{(N_{NS})^P}{P!}$	$\frac{(N_{NS})^P}{P!}\text{Exp}[\frac{-P\varepsilon_{pd}^{NS}}{k_bT}]$
Promoter ON	$(P-1)\varepsilon_{pd}^{NS} + \varepsilon_{pd}^{S}$	$\binom{N_{NS}}{P-1} = \frac{N_{NS}!}{(P-1)!\,(N_{NS}-P-1)!} \approx \frac{(N_{NS})^{P-1}}{(P-1)!}$	$\frac{(N_{NS})^{P-1}}{(P-1)!}\text{Exp}\left[\frac{-(P-1)\varepsilon_{pd}^{NS}}{k_bT}\right] * \text{Exp}[\frac{-\varepsilon_{pd}^{S}}{k_bT}]$

- Assumed that $N_{NS} \gg P$, so $\frac{(N_{NS})!}{(N_{NS}-P)!} \approx (N_{NS})^P$ and $\frac{(N_{NS})!}{(N_{NS}-P-1)!} \approx (N_{NS})^{P-1}$

1b)

$$p_{bound} = \frac{W_{on}}{W_{on} + W_{off}}$$

$$= \frac{\frac{(N_{NS})^{P-1}}{(P-1)!}\text{Exp}\left[\frac{-(P-1)\varepsilon_{pd}^{NS}}{k_bT}\right] * \text{Exp}[\frac{-\varepsilon_{pd}^{S}}{k_bT}]}{\frac{(N_{NS})^{P}}{P!}\text{Exp}\left[\frac{-P\varepsilon_{pd}^{NS}}{k_bT}\right] + \frac{(N_{NS})^{P-1}}{(P-1)!}\text{Exp}\left[\frac{-(P-1)\varepsilon_{pd}^{NS}}{k_bT}\right] * \text{Exp}[\frac{-\varepsilon_{pd}^{S}}{k_bT}]}$$

$$= \frac{1}{\frac{N_{NS}}{P}\text{Exp}\left[\frac{-\varepsilon_{pd}^{NS}}{k_bT}\right] * \text{Exp}[\frac{\varepsilon_{pd}^{S}}{k_bT}] + 1} = \frac{1}{\frac{N_{NS}}{P}\text{Exp}\left[\frac{\varepsilon_{pd}^{S} - \varepsilon_{pd}^{NS}}{k_bT}\right] + 1}$$

$$= \frac{\boldsymbol{\frac{P}{N_{NS}}\textbf{Exp}\left[\frac{\varepsilon_{pd}^{NS} - \varepsilon_{pd}^{S}}{k_bT}\right]}}{\boldsymbol{\frac{P}{N_{NS}}\textbf{Exp}\left[\frac{\varepsilon_{pd}^{NS} - \varepsilon_{pd}^{S}}{k_bT}\right] + 1}}$$

1c) $p_{bound} = \frac{\frac{P}{N_{NS}}\text{Exp}\left[\frac{\varepsilon_{pd}^{NS}-\varepsilon_{pd}^{S}}{k_bT}\right]}{\frac{P}{N_{NS}}\text{Exp}\left[\frac{\varepsilon_{pd}^{NS}-\varepsilon_{pd}^{S}}{k_bT}\right]+1}$

Substituting $-\Delta\varepsilon_p = \varepsilon_{pd}^{NS} - \varepsilon_{pd}^{S}$ into the above equation yields

$$\boldsymbol{p_{bound} = \frac{\frac{P}{N_{NS}}\textbf{Exp}\left[\frac{-\Delta\varepsilon_p}{k_bT}\right]}{\frac{P}{N_{NS}}\textbf{Exp}\left[\frac{-\Delta\varepsilon_p}{k_bT}\right]+1}}$$

2 a and 2b -> see Jupyter notebook

3a)

State	Weight	Renormalized weight
Empty	$\frac{(N_{NS})^{P+R}}{P!\,R!}\text{Exp}\left[\frac{-P\varepsilon_{pd}^{NS}}{k_bT}\right] * \text{Exp}\left[\frac{-R\varepsilon_{pd}^{NS}}{k_bT}\right]$	1
Promoter	$\frac{(N_{NS})^{P-1+R}}{(P-1)!\,R!}\text{Exp}\left[\frac{-(P-1)\varepsilon_{pd}^{NS}}{k_bT}\right] *$ $\text{Exp}\left[\frac{-\varepsilon_{pd}^{p}}{k_bT}\right] * \text{Exp}[\frac{-R\varepsilon_{pd}^{NS}}{k_bT}]$	$\frac{P}{N_{NS}}\text{Exp}\left[\frac{\varepsilon_{pd}^{NS}-\varepsilon_{pd}^{p}}{k_bT}\right]$ $= \boldsymbol{\frac{P}{N_{NS}}\textbf{Exp}\left[\frac{-\Delta\varepsilon_{pd}}{k_bT}\right]}$
Repressor	$\frac{(N_{NS})^{P+R-1}}{P!\,(R-1)!}\text{Exp}\left[\frac{-P\varepsilon_{pd}^{NS}}{k_bT}\right] *$ $\text{Exp}\left[\frac{-(R-1)\varepsilon_{pd}^{NS}}{k_bT}\right] * \text{Exp}[\frac{-\varepsilon_{pd}^{r}}{k_bT}]$	$\frac{R}{N_{NS}}\text{Exp}\left[\frac{\varepsilon_{pd}^{NS}-\varepsilon_{pd}^{r}}{k_bT}\right]$ $= \boldsymbol{\frac{R}{N_{NS}}\textbf{Exp}\left[\frac{-\Delta\varepsilon_{rd}}{k_bT}\right]}$

3b)

$$p_{bound} = \frac{w_{promoter}}{w_{promoter} + w_{repressor} + w_{empty}}$$

$$= \frac{\frac{\boldsymbol{P}}{\boldsymbol{N_{NS}}} \textbf{Exp}\left[\frac{-\Delta\boldsymbol{\varepsilon_{pd}}}{\boldsymbol{k_b T}}\right]}{\frac{\boldsymbol{P}}{\boldsymbol{N_{NS}}} \textbf{Exp}\left[\frac{-\Delta\boldsymbol{\varepsilon_{pd}}}{\boldsymbol{k_b T}}\right] + \frac{\boldsymbol{R}}{\boldsymbol{N_{NS}}} \textbf{Exp}\left[\frac{-\Delta\boldsymbol{\varepsilon_{rd}}}{\boldsymbol{k_b T}}\right] + \mathbf{1}}$$

3c) See Jupyter notebook

3d)

At steady state, $\frac{dm}{dt} = 0$, so therefore

$$\frac{dm}{dt} = p_{bound} r - \gamma m = 0$$

$$\boldsymbol{m = \frac{p_{bound} r}{\gamma}}$$

The fold-change is thus

$$F = \frac{m_{ss}(R>0)}{m_{ss}(R=0)} = \frac{\frac{(\boldsymbol{p_{bound}})_{\boldsymbol{R>0}} \boldsymbol{r}}{\boldsymbol{\gamma}}}{\frac{(\boldsymbol{p_{bound}})_{\boldsymbol{R=0}} \boldsymbol{r}}{\boldsymbol{\gamma}}} = \frac{(\boldsymbol{p_{bound}})_{\boldsymbol{R>0}}}{(\boldsymbol{p_{bound}})_{\boldsymbol{R=0}}} = \frac{\frac{\frac{P}{N_{NS}} \text{Exp}\left[\frac{-\Delta\varepsilon_{pd}}{k_b T}\right]}{\frac{P}{N_{NS}} \text{Exp}\left[\frac{-\Delta\varepsilon_{pd}}{k_b T}\right] + \frac{R}{N_{NS}} \text{Exp}\left[\frac{-\Delta\varepsilon_{rd}}{k_b T}\right] + 1}}{\frac{\frac{P}{N_{NS}} \text{Exp}\left[\frac{-\Delta\varepsilon_{pd}}{k_b T}\right]}{\frac{P}{N_{NS}} \text{Exp}\left[\frac{-\Delta\varepsilon_{pd}}{k_b T}\right] + 1}} =$$

$$\frac{\frac{\boldsymbol{P}}{\boldsymbol{N_{NS}}} \textbf{Exp}\left[\frac{-\Delta\boldsymbol{\varepsilon_{pd}}}{\boldsymbol{k_b T}}\right] + \mathbf{1}}{\frac{\boldsymbol{P}}{\boldsymbol{N_{NS}}} \textbf{Exp}\left[\frac{-\Delta\boldsymbol{\varepsilon_{pd}}}{\boldsymbol{k_b T}}\right] + \frac{\boldsymbol{R}}{\boldsymbol{N_{NS}}} \textbf{Exp}\left[\frac{-\Delta\boldsymbol{\varepsilon_{rd}}}{\boldsymbol{k_b T}}\right] + \mathbf{1}}$$

3e) Since $P \ll N_{NS}$, $\frac{P}{N_{NS}} \text{Exp}\left[\frac{-\Delta\varepsilon_{pd}}{k_b T}\right] \ll 1$, and thus

$$F = \frac{\frac{P}{N_{NS}} \mathrm{Exp}\left[\frac{-\Delta\varepsilon_{pd}}{k_b T}\right] + 1}{\frac{P}{N_{NS}} \mathrm{Exp}\left[\frac{-\Delta\varepsilon_{pd}}{k_b T}\right] + \frac{R}{N_{NS}} \mathrm{Exp}\left[\frac{-\Delta\varepsilon_{rd}}{k_b T}\right] + 1} \approx \frac{\mathbf{1}}{\frac{\boldsymbol{R}}{\boldsymbol{N_{NS}}} \mathbf{Exp}\left[\frac{-\boldsymbol{\Delta\varepsilon_{rd}}}{\boldsymbol{k_b T}}\right] + \mathbf{1}}$$

3f) See Jupyter notebook

4a)

Assume there are A activator binding sites

State	Weight	Renormalized weight
Empty	$\frac{(N_{NS})^{P+R+A}}{P!\,R!\,A!} \mathrm{Exp}\left[\frac{-P\varepsilon_{pd}^{NS}}{k_b T}\right] * \mathrm{Exp}\left[\frac{-R\varepsilon_{pd}^{NS}}{k_b T}\right]$ $* \mathrm{Exp}\left[\frac{-A\varepsilon_{pd}^{NS}}{k_b T}\right]$	1
Promoter	$\frac{(N_{NS})^{P-1+R+A}}{(P-1)!\,R!\,A!} \mathrm{Exp}\left[\frac{-(P-1)\varepsilon_{pd}^{NS}}{k_b T}\right] *$ $\mathrm{Exp}\left[\frac{-\varepsilon_{pd}^{p}}{k_b T}\right] * \mathrm{Exp}[\frac{-R\varepsilon_{pd}^{NS}}{k_b T}] * \mathrm{Exp}[\frac{-A\varepsilon_{pd}^{NS}}{k_b T}]$	$\frac{P}{N_{NS}} \mathrm{Exp}\left[\frac{\varepsilon_{pd}^{NS} - \varepsilon_{pd}^{p}}{k_b T}\right]$ $= \frac{\boldsymbol{P}}{\boldsymbol{N_{NS}}} \mathbf{Exp}\left[\frac{-\boldsymbol{\Delta\varepsilon_{pd}}}{\boldsymbol{k_b T}}\right]$
Repressor	$\frac{(N_{NS})^{P+R-1+A}}{P!\,(R-1)!\,A!} \mathrm{Exp}\left[\frac{-P\varepsilon_{pd}^{NS}}{k_b T}\right] *$ $\mathrm{Exp}\left[\frac{-(R-1)\varepsilon_{pd}^{NS}}{k_b T}\right] * \mathrm{Exp}\left[\frac{-\varepsilon_{pd}^{r}}{k_b T}\right]$ $* \mathrm{Exp}\left[\frac{-A\varepsilon_{pd}^{NS}}{k_b T}\right]$	$\frac{R}{N_{NS}} \mathrm{Exp}\left[\frac{\varepsilon_{pd}^{NS} - \varepsilon_{pd}^{r}}{k_b T}\right]$ $= \frac{\boldsymbol{R}}{\boldsymbol{N_{NS}}} \mathbf{Exp}\left[\frac{-\boldsymbol{\Delta\varepsilon_{rd}}}{\boldsymbol{k_b T}}\right]$

Activator binding site	$\frac{(N_{NS})^{P+R+A-1}}{P!\,R!\,(A-1)!}\text{Exp}\left[\frac{-P\varepsilon_{pd}^{NS}}{k_bT}\right]*$ $\text{Exp}\left[\frac{-R\varepsilon_{pd}^{NS}}{k_bT}\right]*\text{Exp}\left[\frac{-(A-1)\varepsilon_{pd}^{NS}}{k_bT}\right]$ $*\text{Exp}\left[\frac{-\varepsilon_{pd}^{A}}{k_bT}\right]$	$\frac{A}{N_{NS}}\text{Exp}\left[\frac{\varepsilon_{pd}^{NS}-\varepsilon_{pd}^{A}}{k_bT}\right]$ $=\boldsymbol{\frac{A}{N_{NS}}}\,\mathbf{Exp}\left[\boldsymbol{\frac{-\Delta\varepsilon_{ad}}{k_bT}}\right]$

4b)

$$p_{bound}=\frac{w_{promoter}}{w_{promoter}+w_{repressor}+w_{act}+w_{empty}}$$

$$=\boldsymbol{\frac{\frac{P}{N_{NS}}\mathbf{Exp}\left[\frac{-\Delta\varepsilon_{pd}}{k_bT}\right]}{\frac{P}{N_{NS}}\mathbf{Exp}\left[\frac{-\Delta\varepsilon_{pd}}{k_bT}\right]+\frac{R}{N_{NS}}\mathbf{Exp}\left[\frac{-\Delta\varepsilon_{rd}}{k_bT}\right]+\frac{A}{N_{NS}}\mathbf{Exp}\left[\frac{-\Delta\varepsilon_{ad}}{k_bT}\right]+1}}$$

4c)

$$F = \frac{m_{ss}(R>0)}{m_{ss}(R=0)} = \frac{\frac{(\boldsymbol{p_{bound}})_{\boldsymbol{R>0}}\boldsymbol{r}}{\boldsymbol{\gamma}}}{\frac{(\boldsymbol{p_{bound}})_{\boldsymbol{R=0}}\boldsymbol{r}}{\boldsymbol{\gamma}}} = \frac{(\boldsymbol{p_{bound}})_{\boldsymbol{R>0}}}{(\boldsymbol{p_{bound}})_{\boldsymbol{R=0}}}$$

$$= \frac{\dfrac{\frac{P}{N_{NS}}\,\mathrm{Exp}\left[\frac{-\Delta\varepsilon_{pd}}{k_bT}\right]}{\frac{P}{N_{NS}}\,\mathrm{Exp}\left[\frac{-\Delta\varepsilon_{pd}}{k_bT}\right] + \frac{R}{N_{NS}}\,\mathrm{Exp}\left[\frac{-\Delta\varepsilon_{rd}}{k_bT}\right] + \frac{A}{N_{NS}}\,\mathrm{Exp}\left[\frac{-\Delta\varepsilon_{ad}}{k_bT}\right] + 1}}{\dfrac{\frac{P}{N_{NS}}\,\mathrm{Exp}\left[\frac{-\Delta\varepsilon_{pd}}{k_bT}\right]}{\frac{P}{N_{NS}}\,\mathrm{Exp}\left[\frac{-\Delta\varepsilon_{pd}}{k_bT}\right] + 1}}$$

$$= \frac{\frac{\boldsymbol{P}}{\boldsymbol{N_{NS}}}\,\mathbf{Exp}\left[\frac{-\Delta\boldsymbol{\varepsilon_{pd}}}{\boldsymbol{k_bT}}\right] + \mathbf{1}}{\frac{\boldsymbol{P}}{\boldsymbol{N_{NS}}}\,\mathbf{Exp}\left[\frac{-\Delta\boldsymbol{\varepsilon_{pd}}}{\boldsymbol{k_bT}}\right] + \frac{\boldsymbol{R}}{\boldsymbol{N_{NS}}}\,\mathbf{Exp}\left[\frac{-\Delta\boldsymbol{\varepsilon_{rd}}}{\boldsymbol{k_bT}}\right] + \frac{\boldsymbol{A}}{\boldsymbol{N_{NS}}}\,\mathbf{Exp}\left[\frac{-\Delta\boldsymbol{\varepsilon_{ad}}}{\boldsymbol{k_bT}}\right] + \mathbf{1}}$$

$$\approx \frac{\mathbf{1}}{\frac{\boldsymbol{A}}{\boldsymbol{N_{NS}}}\,\mathbf{Exp}\left[\frac{-\Delta\boldsymbol{\varepsilon_{ad}}}{\boldsymbol{k_bT}}\right] + \mathbf{1}}$$