2.3.16

EE25BTECH11013 - Bhargav

Question:
If $\mathbf{p}$ is a unit vector and $(\mathbf{x} - \mathbf{p}) \cdot (\mathbf{x} + \mathbf{p}) = 80$, then find $\|\mathbf{x}\|$.
Solution:

$$(\mathbf{x} - \mathbf{p})^\top (\mathbf{x} + \mathbf{p}) \tag{0.1}$$

$$= \mathbf{x}^\top (\mathbf{x} + \mathbf{p}) - \mathbf{p}^\top (\mathbf{x} + \mathbf{p}) \tag{0.2}$$

$$= \mathbf{x}^\top \mathbf{x} + \mathbf{x}^\top \mathbf{p} - \mathbf{p}^\top \mathbf{x} - \mathbf{p}^\top \mathbf{p}. \tag{0.3}$$

Since $\mathbf{x}^\top \mathbf{p} = \mathbf{p}^\top \mathbf{x}$, the mixed terms cancel:

$$= (\mathbf{x} - \mathbf{p})^\top (\mathbf{x} + \mathbf{p}) \tag{0.4}$$

$$= \|\mathbf{x}\|^2 - \|\mathbf{p}\|^2 \tag{0.5}$$

Given $\mathbf{p}$ is a unit vector, $\|\mathbf{p}\|^2 = 1$. Also, it is given that

$$(\mathbf{x} - \mathbf{p})^\top (\mathbf{x} + \mathbf{p}) = 80. \tag{0.6}$$

Thus,

$$\|\mathbf{x}\|^2 - 1 = 80 \tag{0.7}$$

$$\|\mathbf{x}\|^2 = 81 \tag{0.8}$$

$$\|\mathbf{x}\| = 9. \tag{0.9}$$

The theoretical solution can be verified by example.

Assume that $\mathbf{p}$ is the unit vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Then from the code we get a possible vector $\mathbf{x}$ would be $\begin{pmatrix} 9 \\ 0 \end{pmatrix}$.

The magnitude of the $\mathbf{x}$ is verified to be 9.