Problem 2.2.25. Find the angle between the following pairs of lines:

(a) $\mathbf{r} = 2\hat{\imath} - 5\hat{\jmath} + \hat{k} + \lambda(3\hat{\imath} + 2\hat{\jmath} + 6\hat{k}), \quad \mathbf{r} = 7\hat{\imath} - 6\hat{\jmath} + \mu(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$

(b) $\mathbf{r} = 3\hat{\imath} + \hat{\jmath} - 2\hat{k} + \lambda(\hat{\imath} - \hat{\jmath} - 2\hat{k}), \quad \mathbf{r} = 2\hat{\imath} - \hat{\jmath} - 5\hat{k} + \mu(3\hat{\imath} - 5\hat{\jmath} - 4\hat{k})$

(c) $\dfrac{x-2}{2} = \dfrac{y-1}{5} = \dfrac{z+3}{-3}, \quad \dfrac{x+2}{-1} = \dfrac{y-4}{8} = \dfrac{z-5}{-4}$

(d) $\dfrac{x}{2} = \dfrac{y}{5} = \dfrac{z}{1}, \quad \dfrac{x-5}{4} = \dfrac{y-2}{1} = \dfrac{z-3}{8}$

General Formula:

If two lines are given in vector form, their direction vectors are denoted as

$$\mathbf{d_1}, \quad \mathbf{d_2}.$$

The angle θ between the lines is the angle between these two vectors, given by

$$\cos\theta = \frac{\mathbf{d_1}^T\mathbf{d_2}}{\|\mathbf{d_1}\|\,\|\mathbf{d_2}\|}. \tag{1}$$

Here:

- $\mathbf{d_1}, \mathbf{d_2}$ are the direction vectors of the given lines,
- $\mathbf{d_1}^T\mathbf{d_2}$ is the matrix product (dot product),
- $\|\mathbf{d_i}\| = \sqrt{\mathbf{d_i}^T\mathbf{d_i}}$ is the magnitude (norm) of vector $\mathbf{d_i}$.

Case	$\mathbf{a_1}$	$\mathbf{d_1}$	$\mathbf{a_2}$	$\mathbf{d_2}$
(a)	$\begin{pmatrix}2\\-5\\1\end{pmatrix}$	$\begin{pmatrix}3\\2\\6\end{pmatrix}$	$\begin{pmatrix}7\\-6\\0\end{pmatrix}$	$\begin{pmatrix}1\\2\\2\end{pmatrix}$
(b)	$\begin{pmatrix}3\\1\\-2\end{pmatrix}$	$\begin{pmatrix}1\\-1\\-2\end{pmatrix}$	$\begin{pmatrix}2\\-1\\-5\end{pmatrix}$	$\begin{pmatrix}3\\-5\\-4\end{pmatrix}$
(c)	$\begin{pmatrix}2\\1\\-3\end{pmatrix}$	$\begin{pmatrix}2\\5\\-3\end{pmatrix}$	$\begin{pmatrix}-2\\4\\5\end{pmatrix}$	$\begin{pmatrix}-1\\8\\-4\end{pmatrix}$
(d)	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}2\\5\\1\end{pmatrix}$	$\begin{pmatrix}5\\2\\3\end{pmatrix}$	$\begin{pmatrix}4\\1\\8\end{pmatrix}$

Table 1: Points and direction vectors for Problem 2.2.25

(a) $\mathbf{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k}), \quad \mathbf{r} = 7\hat{i} - 6\hat{j} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

Solution:

$$\mathbf{d_1} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}, \qquad \mathbf{d_2} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{2}$$

$$\mathbf{d_1}^T\mathbf{d_2} = \begin{pmatrix} 3 & 2 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 19 \tag{3}$$

$$\|\mathbf{d_1}\| = \sqrt{\mathbf{d_1}^T\mathbf{d_1}} = \sqrt{49} = 7, \qquad \|\mathbf{d_2}\| = \sqrt{\mathbf{d_2}^T\mathbf{d_2}} = \sqrt{9} = 3 \tag{4}$$

$$\cos\theta = \frac{19}{21} \tag{5}$$

Final Answer:

$$\theta = \cos^{-1}\left(\tfrac{19}{21}\right) \tag{6}$$

(b) $\mathbf{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k}), \quad \mathbf{r} = 2\hat{i} - \hat{j} - 5\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$

Solution:

$$\mathbf{d_1} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \qquad \mathbf{d_2} = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} \tag{7}$$

$$\mathbf{d_1}^T\mathbf{d_2} = \begin{pmatrix} 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} = 16 \tag{8}$$

$$\|\mathbf{d_1}\| = \sqrt{\mathbf{d_1}^T\mathbf{d_1}} = \sqrt{6}, \qquad \|\mathbf{d_2}\| = \sqrt{\mathbf{d_2}^T\mathbf{d_2}} = \sqrt{50} \tag{9}$$

$$\cos\theta = \frac{16}{\sqrt{6}\cdot\sqrt{50}} = \frac{8}{5\sqrt{3}} \tag{10}$$

Final Answer:

$$\theta = \cos^{-1}\left(\tfrac{8}{5\sqrt{3}}\right) \tag{11}$$

(c) $\dfrac{x-2}{2} = \dfrac{y-1}{5} = \dfrac{z+3}{-3}, \quad \dfrac{x+2}{-1} = \dfrac{y-4}{8} = \dfrac{z-5}{-4}$

Solution:

$$\mathbf{d_1} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}, \qquad \mathbf{d_2} = \begin{pmatrix} -1 \\ 8 \\ -4 \end{pmatrix} \tag{12}$$

$$\mathbf{d_1}^T\mathbf{d_2} = \begin{pmatrix} 2 & 5 & -3 \end{pmatrix} \begin{pmatrix} -1 \\ 8 \\ -4 \end{pmatrix} = 50 \tag{13}$$

$$\|\mathbf{d_1}\| = \sqrt{\mathbf{d_1}^T\mathbf{d_1}} = \sqrt{38}, \qquad \|\mathbf{d_2}\| = \sqrt{\mathbf{d_2}^T\mathbf{d_2}} = 9 \tag{14}$$

$$\cos\theta = \frac{50}{9\sqrt{38}} \tag{15}$$

Final Answer:

$$\theta = \cos^{-1}\left(\tfrac{50}{9\sqrt{38}}\right) \tag{16}$$

(d) $\dfrac{x}{2} = \dfrac{y}{5} = \dfrac{z}{1}, \quad \dfrac{x-5}{4} = \dfrac{y-2}{1} = \dfrac{z-3}{8}$

Solution:

$$\mathbf{d_1} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}, \qquad \mathbf{d_2} = \begin{pmatrix} 4 \\ 1 \\ 8 \end{pmatrix} \tag{17}$$

$$\mathbf{d_1}^T\mathbf{d_2} = \begin{pmatrix} 2 & 5 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 8 \end{pmatrix} = 21 \tag{18}$$

$$\|\mathbf{d_1}\| = \sqrt{\mathbf{d_1}^T\mathbf{d_1}} = \sqrt{30}, \qquad \|\mathbf{d_2}\| = \sqrt{\mathbf{d_2}^T\mathbf{d_2}} = 9 \tag{19}$$

$$\cos\theta = \frac{21}{9\sqrt{30}} = \frac{7}{3\sqrt{30}} \tag{20}$$

Final Answer:

$$\theta = \cos^{-1}\left(\tfrac{7}{3\sqrt{30}}\right) \tag{21}$$

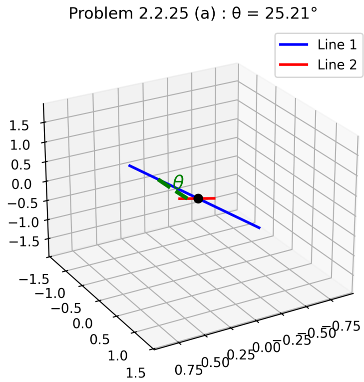

Figure 1

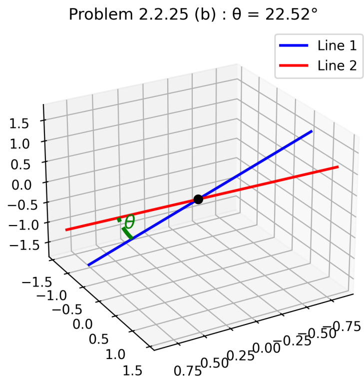

Figure 2

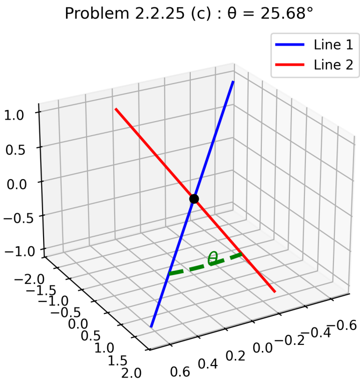

Figure 3

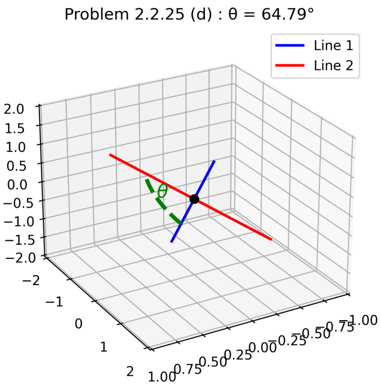

Figure 4