4.13.31

EE25BTECH11026-Harsha

Question:
Line L has intercepts a and b on the coordinate axes. When the axes are rotated through a given angle, keeping the origin fixed, line L has intercepts p and q. Then

1) $a^2+b^2 = p^2+q^2$ 2) $\frac{1}{a^2}+\frac{1}{b^2} = \frac{1}{p^2}+\frac{1}{q^2}$ 3) $a^2+p^2 = b^2+q^2$ 4) $\frac{1}{a^2}+\frac{1}{p^2} = \frac{1}{b^2}+\frac{1}{q^2}$

Solution:
Let us solve the given question theoretically and then verify the solution computationally.

According to the question,

$$\text{The equation of line}: \begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 1 \tag{4.1}$$

Let the row coefficient vector of the original line be $\mathbf{m}$ and for the rotated line be $\mathbf{m'}$.

$$\mathbf{m'} = \mathbf{P}\mathbf{m} \tag{4.2}$$

where $\mathbf{P}$ is the rotation matrix

$$\|\mathbf{m'}\|^2 = \mathbf{m'}^\top \mathbf{m'} = \left(\mathbf{m}^\top \mathbf{P}^\top\right)(\mathbf{P}\mathbf{m}) = \mathbf{m}^\top \left(\mathbf{P}^\top \mathbf{P}\right) \mathbf{m} \tag{4.3}$$

Since $\mathbf{P}$ is an orthogonal matrix,

$$\therefore \mathbf{m}^\top \left(\mathbf{P}^\top \mathbf{P}\right) \mathbf{m} = \mathbf{m}^\top \mathbf{m} = \|m\|^2 \tag{4.4}$$

$$\implies \|\mathbf{m'}\|^2 = \|\mathbf{m}\|^2 \tag{4.5}$$

As $\mathbf{m'}$ is given by $\begin{pmatrix} \frac{1}{p} & \frac{1}{q} \end{pmatrix}^\top$,

$$\therefore \frac{1}{p^2} + \frac{1}{q^2} = \frac{1}{a^2} + \frac{1}{b^2} \tag{4.6}$$

From the figure, it is clearly verified that the theoretical solution matches with the computational solution.