2.5.20

EE25BTECH11057 - Rushil Shanmukha Srinivas

Question : Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors with $||\mathbf{a}|| = 1,\ ||\mathbf{b}|| = 2,\ ||\mathbf{c}|| = 3$. If the projection of $\mathbf{b}$ on $\mathbf{a}$ equals the projection of $\mathbf{c}$ on $\mathbf{a}$, and $\mathbf{b} \perp \mathbf{c}$, find

$$\left\|3\mathbf{a} - 2\mathbf{b} + 2\mathbf{c}\right\|. \tag{0.1}$$

Solution (matrix / rank style). : Let us denote the scalar products

$$x = \mathbf{a} \cdot \mathbf{b}, \qquad y = \mathbf{a} \cdot \mathbf{c}, \qquad z = \mathbf{b} \cdot \mathbf{c}. \tag{0.2}$$

Given: projection of $\mathbf{b}$ on $\mathbf{a}$ equals projection of $\mathbf{c}$ on $\mathbf{a}$. Since $||\mathbf{a}|| = 1$, this implies

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \quad \Rightarrow \quad x = y. \tag{0.3}$$

Also $\mathbf{b} \perp \mathbf{c} \Rightarrow z = 0$. Using the magnitudes,

$$\mathbf{a} \cdot \mathbf{a} = 1, \quad \mathbf{b} \cdot \mathbf{b} = 4, \quad \mathbf{c} \cdot \mathbf{c} = 9. \tag{0.4}$$

Form the Gram (inner-product) matrix of $(\mathbf{a}, \mathbf{b}, \mathbf{c})$:

$$G = \begin{pmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{c} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{b} & \mathbf{c} \cdot \mathbf{c} \end{pmatrix} = \begin{pmatrix} 1 & x & x \\ x & 4 & 0 \\ x & 0 & 9 \end{pmatrix}, \tag{0.5}$$

where we used $x = y$ and $z = 0$.

Now denote the coefficient vector of $3\mathbf{a} - 2\mathbf{b} + 2\mathbf{c}$ relative to $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ by

$$\mathbf{u} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}. \tag{0.6}$$

Then the squared norm is the quadratic form

$$\left\|3\mathbf{a} - 2\mathbf{b} + 2\mathbf{c}\right\|^2 = \mathbf{u}^\top G\, \mathbf{u}. \tag{0.7}$$

$$\mathbf{u}^\top G\, \mathbf{u} = \begin{pmatrix} 3 & -2 & 2 \end{pmatrix} \begin{pmatrix} 1 & x & x \\ x & 4 & 0 \\ x & 0 & 9 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = 9 - 6x + 16 + 6x + 36 = 61 \tag{0.8}$$

Therefore

$$\left\|3\mathbf{a} - 2\mathbf{b} + 2\mathbf{c}\right\|^2 = 61 \quad \Longrightarrow \quad \boxed{\left\|3\mathbf{a} - 2\mathbf{b} + 2\mathbf{c}\right\| = \sqrt{61}}. \tag{0.9}$$

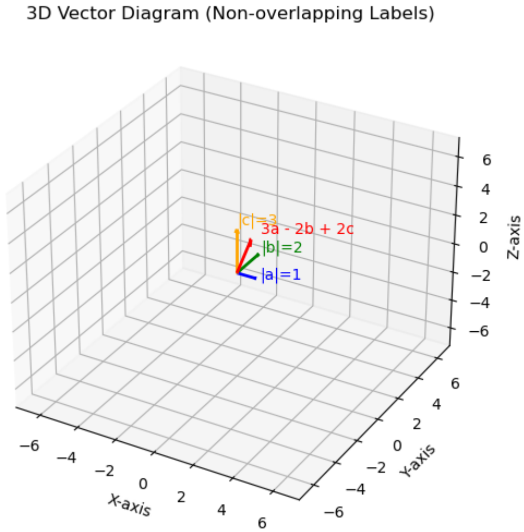

Fig: Representation of vectors