Matrices in Geometry 2.10.66

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Question: If vectors $\mathbf{a},\ \mathbf{b},\ \mathbf{c}$ are coplanar, show that

$$\left|\begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \end{pmatrix}\right| = 0$$

Solution: We have to show that

$$\Delta = |\mathbf{A}| = \left|\begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \end{pmatrix}\right| = 0 \tag{1}$$

Since $\mathbf{a},\ \mathbf{b},\ \mathbf{c}$ are coplanar, $\mathbf{a},\ \mathbf{b},\ \mathbf{c}$ are linearly dependent. Therefore, we can express $\mathbf{c}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ as:

$$\mathbf{c} = m\mathbf{a} + n\mathbf{b}, \text{where m, n are scalars.} \tag{2}$$

$$\text{Since, } |\mathbf{A}| = \left|\mathbf{A}^\top\right| \implies \Delta = \left|\mathbf{A}^\top\right| \tag{3}$$

$$\left|\mathbf{A}^\top\right| = \left|\begin{pmatrix} \mathbf{a} & \mathbf{a}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{a} \\ \mathbf{b} & \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} \\ \mathbf{c} & \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \end{pmatrix}\right| \xrightarrow{R_3\to R_3 - mR_1 - nR_2} \left|\begin{pmatrix} \mathbf{a} & \mathbf{a}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{a} \\ \mathbf{b} & \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} \\ \mathbf{c} - m\mathbf{a} - n\mathbf{b} & \mathbf{a}\cdot(\mathbf{c} - m\mathbf{a} - n\mathbf{b}) & \mathbf{b}\cdot(\mathbf{c} - m\mathbf{a} - n\mathbf{b}) \end{pmatrix}\right| \tag{4}$$

Since, $\mathbf{c} = m\mathbf{a} + n\mathbf{b}$, $\mathbf{A}$ becomes

$$\left|\mathbf{A}^\top\right| = \left|\begin{pmatrix} \mathbf{a} & \mathbf{a}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{a} \\ \mathbf{b} & \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} \\ 0 & 0 & 0 \end{pmatrix}\right| \tag{5}$$

$$\implies \Delta = \left|\mathbf{A}^\top\right| = 0, \text{ as one of the rows is zero} \tag{6}$$

Hence, Proved.