Matrices in Geometry 2.8.20

EE25BTECH11037 - Divyansh

Question: If $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$, then prove that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

Solution:

$$\|\mathbf{a}+\mathbf{b}\| = \|\mathbf{a}-\mathbf{b}\| \tag{1}$$

$$\text{Two vectors } \mathbf{a} \text{ and } \mathbf{b} \text{ are orthogonal if,} \quad \mathbf{a}^\top\mathbf{b} = 0 \tag{2}$$

$$\|\mathbf{a}+\mathbf{b}\|^2 = \|\mathbf{a}-\mathbf{b}\|^2 \tag{3}$$

We know that $\|\mathbf{a}\|^2 = \mathbf{a}^\top\mathbf{a}$

$$\implies (\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b}) = (\mathbf{a}-\mathbf{b})^\top(\mathbf{a}-\mathbf{b}) \tag{4}$$

$$\implies \mathbf{a}^\top\mathbf{a} + 2\mathbf{a}^\top\mathbf{b} + \mathbf{b}^\top\mathbf{b} = \mathbf{a}^\top\mathbf{a} - 2\mathbf{a}^\top\mathbf{b} + \mathbf{b}^\top\mathbf{b} \tag{5}$$

$$\implies 4\mathbf{a}^\top\mathbf{b} = 0 \tag{6}$$

$$\implies \mathbf{a}^\top\mathbf{b} = 0 \tag{7}$$

This shows that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.
Let us try to show this for an example

$$\text{Let } \mathbf{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{8}$$

$$(\mathbf{a}+\mathbf{b}) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, (\mathbf{a}-\mathbf{b}) = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \tag{9}$$

$$\text{We can clearly see that } \|\mathbf{a}+\mathbf{b}\| = \|\mathbf{a}-\mathbf{b}\| = 2 \tag{10}$$

$$\mathbf{a}^\top\mathbf{b} = \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 - 1 = 0 \tag{11}$$

This property is also proved for an example.
Hence, Proved