

1 Differential Equations

Ordinary differential equations (ODE's)

Given F , a function of x, y , and derivatives of y . Then an equation of the form

$$F(x, y, y', \dots, y^{(n)}) = 0$$

is an implicit ODE of order n . Order is determined by the highest derivative. Implicit means the equation equals 0.

Linear ODE's

A differential equation is said to be linear if F can be written as a linear combination of the derivatives of y :

$$b(x) = \sum_{i=0}^n a_i(x) \cdot y^{(i)}$$

where $a_i(x)$ and $b(x)$ are continuous functions. Why is this called linear?

$$D = \frac{d^{(k)}}{dx^k} + a_{k-1} \frac{d^{(k-1)}}{dx^{k-1}} + \dots + a_0$$

$$D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)$$

$$Df_1 = b_1, Df_2 = b_2 \Rightarrow D(f_1 + f_2) = b_1 + b_2$$

Homogenous

A linear ODE is homogenous when $b(x) = 0$. Inhomogenous otherwise.

Solution Space

Let $I \subset \mathbb{R}$ be an open interval and $k \geq 1$ an integer, and let

$$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = 0$$

be a linear ODE over I with continuous coefficients.

- (1) The set S of k -times differentiable solutions $f : I \rightarrow \mathbb{C}$ of the equation is a complex vector space

which is a subspace of the space of complex valued functions on I . (Analogous for real numbers, if all a_i are real valued)

- (2) The dimension of S is k and for any choice of $x_0 \in I$ and any $(y_0, \dots, y_{k-1}) \in \mathbb{C}^k$ there exists a unique f such that

$$f(x_0) = y_0, f'(x_0) = y_1, \dots, f^{(k-1)}(x_0) = y_{k-1}$$

(Analogous for real numbers, if all a_i are real)

- (3) For an arbitrary b the solution set is $S_b = \{f + f_p \mid f \in S_0\}$ where f_p is a "particular" solution.
- (4) For any initial condition there is a unique solution.

Solving linear ODE's of order 1

Let us consider

$$y' + ay = b.$$

Here a, b are constant functions.

- (1) Find solutions of the corresponding homogenous equation $y' + ay = 0$. Note that if f is a solution so is $z \cdot f \quad \forall z \in \mathbb{C}$. Example:

$$y' + ay = 0$$

$$y' = -ay$$

$$\frac{y'}{y} = -a$$

$$\ln(y) = -\int a + C = -A + C$$

$$y = e^{-A+C} = z \cdot e^{-A} \quad z \in \mathbb{C}$$

- (2) Find a particular solution $f_p : I \rightarrow \mathbb{C}$ such that $f_p' + af_p = b$. Use educated guess or variation of constants.

Educated Guess

Note

- (1) If $b(x)$ is a linear combination of basic functions listed here try the linear combination of educated guesses
- (2) If the educated guess is the same as the solution of the homogenous problem, then try multiplying by x^m where m denotes the multiplicity of the root λ .

$b(x)$	Guess
$a \cdot e^{\alpha x}$	$b \cdot e^{\alpha x}$
$a \sin(\beta x)$	$c \sin(\beta x) + d \cos(\beta x)$
$b \cos(\beta x)$	$c \sin(\beta x) + d \cos(\beta x)$
$ae^{\alpha x} \sin(\beta x)$	$e^{\alpha x} (c \sin(\beta x) + d \cos(\beta x))$
$be^{\alpha x} \cos(\beta x)$	$e^{\alpha x} (c \sin(\beta x) + d \cos(\beta x))$
$P_n(x)e^{\alpha x}$	$R_n(x) \cdot e^{\alpha x}$
$P_n(x)e^{\alpha x} \sin(\beta x)$	$e^{\alpha x} (R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))$
$P_n(x)e^{\alpha x} \cos(\beta x)$	$e^{\alpha x} (R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))$

Variation of constants

- (1) Assume $f_p = z(x)e^{-A(x)}$ for some function $z : I \rightarrow \mathbb{C}$
- (2) We plug this into the equation and see what it forces z to satisfy

$$f_p = z(x)e^{-A(x)} = y$$

$$y' = z'(x)e^{-A(x)} - z(x)A'(x)e^{-A(x)}$$

$$y' = e^{-A(x)} (z'(x) - z(x)a(x))$$

$$ay = a \cdot z(x)e^{-A(x)}$$

$$y' + ay = z'(x)e^{-A(x)} = b(x)$$

$$b(x) = z'(x)e^{A(x)}$$

$$z(x) = \int \frac{e^{A(x)} b(x)}{b(x)} dx$$

$$y_p = z(x)e^{-A(x)}$$

or for degree two

- (1) Assume the homogenous solution is $f = z_1 f_1 + z_2 f_2$
- (2) We will try $f_p = z_1(x) f_1 + z_2(x) f_2$
- (3) Solve the following system

$$\begin{aligned} z_1'(x) f_1 + z_2'(x) f_2 &= 0 \\ z_1'(x) f_1' + z_2'(x) f_2' &= b \end{aligned}$$

Solving Linear ODE's with constant coefficients

We want to solve

$$y^{(k)} + a_{k-1} y^{(k-1)} + \dots + a_0 y = b(x)$$

We assume our solution is $e^{\lambda x}$.

$$\begin{aligned} P(\lambda) &= \lambda^k e^{\lambda x} + a_{k-1} \lambda^{k-1} e^{\lambda x} + \dots + a_0 e^{\lambda x} = 0 \\ &= e^{\lambda x} (\lambda^k + a_{k-1} \lambda^{k-1} + \dots + a_0) = 0 \\ \Rightarrow 0 &= \lambda^k + a_{k-1} \lambda^{k-1} + \dots + a_0 \end{aligned}$$

Which can then be solved for λ . Keep in mind that $\lambda \in \mathbb{C}$ and we might be able to simplify with Euler's formula.

$$e^{ix} = \cos(x) + i \sin(x)$$

If there is a multiple root α of multiplicity j we have

$$\text{Solutions: } e^{\alpha x}, x e^{\alpha x}, \dots, x^{j-1} e^{\alpha x}$$

Complex roots

If $\alpha = \beta + \gamma i$ is a complex root of $P(\lambda)$, then so is $\bar{\alpha} = \beta - \gamma i$. Hence $f_1 = e^{\alpha x}$ and $f_2 = e^{\bar{\alpha} x}$ are solutions and can be replaced by a linear combination of $\tilde{f}_1 = e^{\beta x} \cos(\gamma x)$ and $\tilde{f}_2 = e^{\beta x} \sin(\gamma x)$.

Further if $y^{(k)} + a_{k-1} y^{(k-1)} + \dots + a_0 y = 0$ has real

coefficients, then each pair of complex conjugate roots $\beta_j \pm \gamma_j i$ with multiplicity m_j leads to solution

$$x^l e^{\beta_j x} (\cos(\gamma_j x) + i \sin(\gamma_j x)) \quad \text{for } 0 \leq l \leq m_j$$

Separation of variables

A differential equation of order 1 is separable if it is of the form

$$\begin{aligned} y' &= b(x)g(y) \\ \frac{dy}{dx} &= b(x)g(y) \\ \frac{dy}{g(y)} &= b(x)dx \\ \int \frac{dy}{g(y)} &= \int b(x)dx \end{aligned}$$

2 Differentials in \mathbb{R}^n

Monomial

A monomial of degree e is a function

$$(x_1, \dots, x_n) \mapsto \alpha x_1^{d_1} \cdot \dots \cdot x_n^{d_n} \\ e = d_1 + \dots + d_n$$

Polynomial

A polynomial in n variables of degree $\leq d$ is a finite sum of monomials of degree $e \leq d$

Convergence

Let $(x_k)_{k \in \mathbb{N}}$, $x_k \in \mathbb{R}^n$ and $x_k = (x_{k,1}, x_{k,2}, \dots, x_{k,n})$. The following equivalently define $\lim_{k \rightarrow \infty} x_k = y$.

- (1) $\forall \varepsilon > 0 \exists N \geq 1$ s.t. $\forall k \geq N \quad \|x_k - y\| < \varepsilon$
- (2) For each i , $1 \leq i \leq n$ the sequence $(x_{k,i})_k$ of real numbers converges to y_i .
- (3) The sequence of real numbers $\|x_k - y\|$ converges to 0.

Let $f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $x_0 \in \mathcal{X}$, $y \in \mathbb{R}^m$. We say f has a limit to y as $x \rightarrow x_0$ where $x \neq x_0$ if any of the following apply

- (1) $\forall \varepsilon > 0 \exists \delta > 0$ s.t. $\forall x \in \mathcal{X}$, $x \neq x_0$ such that $\|x - x_0\| < \delta$ we have $\|f(x) - y\| < \varepsilon$.
- (2) \forall sequences (x_k) in \mathcal{X} such that $\lim x_k = x_0$ and $x_k \neq x_0$ the sequence $f(x_k)$ converges to y .

Continuity

Let $f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $x_0 \in \mathcal{X}$. We say f is continuous at x_0 if any of the following apply

- (1) $\forall \varepsilon > 0 \exists \delta > 0$ s.t. if $x \in \mathcal{X}$ satisfies $\|x - x_0\| < \delta$ then $\|f(x) - f(x_0)\| < \varepsilon$.
- (2) \forall sequences (x_k) in \mathcal{X} s.t. $\lim x_k = x_0$ we have $\lim f(x_k) = f(\lim x_k)$.

f is continuous in \mathcal{X} if f is continuous in every point $x_0 \in \mathcal{X}$.

The following statements also hold

- (1) $f(x = x_1, \dots, x_n) \mapsto (f_1(x), \dots, f_m(x))$ and $f_i : \mathbb{R}^n \mapsto \mathbb{R}$ is continuous $\Leftrightarrow f_i \forall i = 1, \dots, m$ are continuous.
- (2) Linear functions $x \mapsto Ax$ are continuous.
- (3) Polynomials are continuous.
- (4) Sums, products of continuous functions are continuous.
- (5) Functions of separated variables are continuous if the factors are continuous.
- (6) Composition of continuous functions are continuous.

Sandwich lemma

If $f, g, h : \mathbb{R}^n \rightarrow \mathbb{R}$ where $f(x) < g(x) < h(x) \quad \forall x \in \mathbb{R}^n$. Let $a \in \mathbb{R}^n$.

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L \Rightarrow \lim_{x \rightarrow a} g(x) = L$$

Polar Coordinates

It is sometimes helpful to use polar coordinates, especially with rational functions $f : \mathbb{R} \rightarrow \mathbb{R}$. $f(x, y) = f(r \cos(\theta), r \sin(\theta))$

Bounded set

A set $\mathcal{X} \subset \mathbb{R}^n$ is bounded if the set $\{\|x\| \mid x \in \mathcal{X}\}$ is bounded in \mathbb{R} .

Closed set

A set $\mathcal{X} \subset \mathbb{R}^n$ is closed if for every sequence $(x_k)_{k \in \mathbb{N}} \subset \mathcal{X}$ that converges in \mathbb{R}^n , converges to a point $y \in \mathcal{X}$.

Here it is often helpful to consider a ball. Counterexamples often include $\frac{1}{k}$ and $<$.

Compact set

A compact set is a closed and bounded set.

Continuous and closed

If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous, then for every $Y \subset \mathbb{R}^m$ that is closed the set $f^{-1}(Y) = \{x \in \mathbb{R}^n \mid f(x) \in Y\} \subset \mathbb{R}^n$ is closed. Careful: Does not imply bounded or compact!

Min-Max theorem

Let $\mathcal{X} \subset \mathbb{R}^n$ be a compact set, $f : \mathcal{X} \rightarrow \mathbb{R}$ a continuous function. Then f is bounded and attains its max and min.

$$f(x^+) = \sup_{x \in \mathcal{X}} f(x) f(x^-) = \inf_{x \in \mathcal{X}} f(x)$$

Open set

A set $\mathcal{X} \subset \mathbb{R}^n$ is called open if its complement $\mathbb{R}^n \setminus \mathcal{X}$ is closed. This is equivalent to $\forall x \in \mathcal{X} \exists r > 0$ s.t. the set $\{y \in \mathbb{R}^n \mid \|y - x\| < r\} = B_r(x) \subset \mathcal{X}$.

Here are some examples

- (1) $(a, b) \subset \mathbb{R}$ is open.
- (2) $[a, b) \subset \mathbb{R}$ is neither open nor closed.
- (3) \mathbb{R}^n and \emptyset are both open.
- (4) $(a_1, b_1) \times (a_2, b_2) \subset \mathbb{R}^2$ is open.
- (5) Inverse image of open sets under continuous maps are open.

Derivative

Given $f : \mathbb{R} \rightarrow \mathbb{R}^n$ the derivative is

$$f'(x_0) = \begin{pmatrix} f'_1(x_0) \\ \vdots \\ f'_n(x_0) \end{pmatrix}$$

Partial derivatives

A partial derivative of a function $f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is obtained by pretending all but one variable are constants and then differentiating the one variable.

$$\frac{\partial f}{\partial x_{0,j}} = \lim_{h \rightarrow 0} \frac{f(x_{0,1}, \dots, x_{0,j} + h, \dots, x_{0,n}) - f(x_0)}{h}$$

If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ for $x_0 \in \mathbb{R}^n$ then

$$\frac{\partial f(x_0)}{\partial x_j} := \begin{pmatrix} \partial f_1(x_0)/\partial x_j \\ \vdots \\ \partial f_m(x_0)/\partial x_j \end{pmatrix}$$

Properties include (assuming partial derivatives for f, g exist w.r.t. x_j)

- (1) $\frac{\partial f+g}{\partial x_j} = \frac{\partial f}{\partial x_j} + \frac{\partial g}{\partial x_j}$
- (2) $\frac{\partial f \cdot g}{\partial x_j} = \frac{\partial f}{\partial x_j} \cdot g + \frac{\partial g}{\partial x_j} \cdot f$
- (3) if $g \neq 0$: $\frac{\partial f/g}{\partial x_j} = \frac{\frac{\partial f}{\partial x_j} \cdot g - \frac{\partial g}{\partial x_j} \cdot f}{g^2}$

Jacobi Matrix

A Matrix with m rows and n columns where

$$J_f = \left(\frac{\partial f_i}{\partial x_j} \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$$

Gradient

The Jacobian of a function $f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R}$. Is often denoted as ∇f . The geometric interpretation is that it indicates the direction and rate of fastest increase.

Directional derivative

Let direction $v = (a, b) \neq (0, 0)$. Instead of adding $+h$ to one component we add $+ah$, $+bh$ and so on to all components and find that derivative as we normally would with a limit. Further, we have

$$\frac{df(x_0 + t\vec{v})}{dt} = J_f(x_0) \cdot \vec{v}$$

Differentiability

Let $\mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R}^p$ be function and $x_0 \in \mathcal{X}$. We say f is differentiable at x_0 if a linear map $u : \mathbb{R}^n \rightarrow \mathbb{R}^p$ exists such that

$$\lim_{x \rightarrow x_0, x \neq x_0} \frac{f(x) - f(x_0) - u(x - x_0)}{\|x - x_0\|} = 0$$

and u is called the total differential of f at x_0 .

Further, if f, g are differentiable at $x_0 \in \mathcal{X}$ we have

- (1) f is continuous at x_0
- (2) f has all partial derivatives at x_0 and the matrix represents the linear map $df(x_0) : x \mapsto Ax$ in the canonical basis is given by the Jacobi Matrix of f at x_0 , i.e. $A = J_f(x_0)$
- (3) $d(f+g)(x_0) = df(x_0) + dg(x_0)$
- (4) If $m = 1$ and $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ differentiable in x_0 then so is $f \cdot g$ and if $g \neq 0$ f/g as well.

Lastly we have

All partial derivatives \exists and cont. $\Rightarrow f$ is differentiable

Tangent space

The approximation of the function at x_0 using one derivative.

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = f(x_0) + u(x - x_0)\}$$

An example:

$$\begin{aligned} f(x, y) &= \sqrt{x^2 + y^2} \\ J_f &= \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right) \\ J_f(3, 4) &= \left(\frac{3}{5}, \frac{4}{5} \right) \\ \Rightarrow g(x, y) &= 5 + \left(\frac{3}{5}, \frac{4}{5} \right) \begin{pmatrix} x-3 \\ y-4 \end{pmatrix} \end{aligned}$$

Chain rule

Let $\mathcal{X} \subset \mathbb{R}^n$ be open, $\mathcal{Y} \subset \mathbb{R}^m$ be open and let $f : \mathcal{X} \rightarrow \mathcal{Y}$, $g : \mathcal{Y} \rightarrow \mathbb{R}^p$ be differentiable functions. Then $g \circ f = g(f) : \mathcal{X} \rightarrow \mathbb{R}^p$ is differentiable in \mathcal{X} . In particular

$$\begin{aligned} d(g \circ f)(x_0) &= dg(f(x_0)) \circ df(x_0) \\ J_{g \circ f}(x_0) &= J_g(f(x_0)) \cdot J_f(x_0) \end{aligned}$$

Change of variables

We say f is a change of variables around x_0 if there is a radius $\rho > 0$ s.t. the restriction of f to the Ball $B = \{x \in \mathbb{R}^n \mid \|x - x_0\| < \rho\}$ so that the image $Y = f(B)$ is open in \mathbb{R}^n and a differentiable map $g : Y \rightarrow B$ exists, such that $f \circ g = \text{id}_Y$ and $g \circ f = \text{id}_B$. I.e.

$$f|_{B(x_0)} \text{ is a bijection to the image with a differentiable inverse } g$$

Inverse function theorem

Let $\mathcal{X} \subseteq \mathbb{R}^n$ be open and $f : \mathcal{X} \rightarrow \mathbb{R}^n$ differentiable. If $x_0 \in \mathcal{X}$ is such that $\det(J_f(x_0)) \neq 0$, i.e. $J_f(x_0)$ is invertible, then f is a change of variables around x_0 . Moreover the Jacobian of g at x_0 is defined by

$$J_g(f(x_0)) = J_f(x_0)^{-1}$$

Higher derivatives

Let $\mathcal{X} \subset \mathbb{R}^n$, $f : \mathcal{X} \rightarrow \mathbb{R}^m$. We say f is of class C' if f is differentiable on \mathcal{X} and all of its partial derivatives are continuous.

We say $f \in C^k$ for $k \geq 2$ if it is differentiable and each $\partial_{x_i} f : \mathcal{X} \rightarrow \mathbb{R}^m$ is of class C^{k-1} . Further, f is smooth

or C^∞ if $f \in C^k \quad \forall k$. Lastly: mixed partials (up to order k) commute:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

Hessian

The $n \times n$ symmetric matrix

$$\text{Hess}_f(x_0) := \left(\frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} \right)$$

Taylor Polynomial

The Taylor polynomial of f at x_0 of order 1 is

$$T_1 f(y; x_0) := f(x_0) + \nabla f(x_0) \cdot y$$

while the first order approximation of f at x_0 is

$$T_1 f(x - x_0; x_0) := f(x_0) + \nabla f(x_0) \cdot (x - x_0)$$

and the second order

$$T_2 f(y; x_0) := f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2} \cdot (x - x_0)^t \cdot \text{Hess}_f(x_0) \cdot (x - x_0)$$

Finally, the general form is

$$T_k f(y; x_0) = f(x_0) + \dots + \sum_{m_1 + \dots + m_n = k} \frac{1}{m_1! \dots m_n!} \frac{\partial^k f(x_0) \cdot y_1^{m_1} \dots y_n^{m_n}}{\partial x_1^{m_1} \dots \partial x_n^{m_n}}$$

Lastly if $f \in C^k$ for $x_0 \in \mathcal{X}$ we have

$$f(x) = T_k(x - x_0; x_0) + E_k(f, x, x_0)$$

$$\lim_{x \rightarrow x_0} \frac{E_k(f, x, x_0)}{\|x - x_0\|^k} \rightarrow 0$$

Local max/min

Let $f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable. We say $x_0 \in \mathcal{X}$

is a local maximum (minimum) if we can find a neighborhood $B_r(x_0) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\} \subset \mathcal{X}$

$$\forall x \in B_r(x_0) \quad f(x) \leq (\geq) f(x_0)$$

We also have

$$x_0 \in \mathcal{X} \text{ is a local extrema} \Rightarrow \nabla f(x_0) = 0$$

Critical point

A point $x_0 \in \mathcal{X}$ where $\nabla f(x_0) = 0$.

Saddle point

A critical point which is not a local min or max.

Global extrema

If $f : \mathcal{X} \rightarrow \mathbb{R}$ is differentiable on the interior of \mathcal{X} and \mathcal{X} is closed and bounded, then a global extrema of f exists and it is either at a critical point or the boundary of \mathcal{X} .

$$\text{Check} = \text{int}(\mathcal{X}) \cup \text{bd}(\mathcal{X})$$

Definite

We have the following

- (1) Positive Definite: All eigenvalues are Positive
- (2) Negative Definite: All eigenvalues are Negative
- (3) Indefinite: Positive and Negative Eigenvalues

Eigenvalues can be found with the characteristic polynomial:

$$\det \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} \right) = \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}$$

$$\Rightarrow \lambda^2 - 1 = 0$$

Calculating determinates

For 2 dimensions we have

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot d - c \cdot b$$

For 3 dimensions we have

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \cdot \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \cdot \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

Test critical point

Let $f : \mathcal{X} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ and $f \in C^2$. Let x_0 be a non-degenerate critical point of f . Then

- (1) If $\text{Hess}_f(x_0)$ pos def. then x_0 is a local minimum
- (2) If $\text{Hess}_f(x_0)$ neg def. then x_0 is a local maximum
- (3) If $\text{Hess}_f(x_0)$ is Indefinite then x_0 is a saddle point

We cannot use this theorem when x_0 is a degenerate critical point ($\det(\text{Hess}_f(x_0)) = 0$) and must decide on a case by case basis!

3 Integrals in \mathbb{R}^n

Simple integral

For $f : \mathbb{R} \rightarrow \mathbb{R}^n$ the integral is

$$\int_a^b f(t) dt = \begin{pmatrix} \int_a^b f_1(t) dt \\ \vdots \\ \int_a^b f_n(t) dt \end{pmatrix}$$

Curve

The image of a function $\gamma : [a, b] \rightarrow \mathbb{R}^n$ where the function γ is continuous and piecewise $\in C^1$.

Line integral

Let $\gamma : [a, b] \rightarrow \mathbb{R}^n$ be a parametrization of a curve and let $\mathcal{X} \subset \mathbb{R}^n$ be a set which contains the image of γ . Further, let $f : \mathcal{X} \rightarrow \mathbb{R}^n$ be a continuous function. A line integral then is

$$\int_{\gamma} f(s) d\vec{s} = \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt$$

The line integral has the following properties

- (1) It is independent of orientation preserving reparametrization, i.e.

$$\begin{aligned} \gamma &: [a, b] \rightarrow \mathbb{R}^n \\ \tilde{\gamma} &: [c, d] \rightarrow \mathbb{R}^n \\ \Phi &: [c, d] \rightarrow [a, b] \\ \tilde{\gamma} &= \gamma \circ \Phi = \gamma(\Phi) \\ \Rightarrow \int_{\gamma} f ds &= \int_{\tilde{\gamma}} f ds \end{aligned}$$

- (2) Let $\gamma_1 + \gamma_2$ be the path formed by the concatenate

tion of the two curves. Then

$$\gamma_1 + \gamma_2 := \begin{cases} \gamma_1(t) & t \in [a, b] \\ \gamma_2(t) & t \in [b, d + b - c] \end{cases}$$

$$\int_{\gamma_1 + \gamma_2} f ds = \int_{\gamma_1} f ds + \int_{\gamma_2} f ds$$

- (3) If $\gamma : [a, b] \rightarrow \mathbb{R}^n$ is a path, let $-\gamma$ be the path traced in the opposite direction, i.e. $(-\gamma)(t) := \gamma(a + b - t)$. Then

$$\int_{-\gamma} f ds = - \int_{\gamma} f ds$$

Vector Field

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Potential

A differentiable scalar field $g : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\nabla g = f$, $f : \mathcal{X} \rightarrow \mathbb{R}^n$ is called a potential for f . This can make stuff easier:

$$\begin{aligned} \int_{\gamma} f ds &= \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt \\ &= \int_a^b \nabla g(\gamma(t)) \cdot \gamma'(t) dt \\ &= \int_a^b \frac{d}{dt} (g \circ \gamma) dt \\ &= (g \circ \gamma)(b) - (g \circ \gamma)(a) \end{aligned}$$

It should be noted that not every function has a potential! Example:

$$\begin{aligned} f(x, y) &= (2xy^2, 2x) \\ \frac{\partial g}{\partial x} &= 2xy^2 \Rightarrow g(x, y) = x^2 y^2 + h(y) \\ \frac{\partial g}{\partial y} &= 2x \neq 2x^2 y + h'(y) \end{aligned}$$

Conservative vector field

Let $f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous vector field. The following are equivalent.

- (1) If for any $x_1, x_2 \in \mathcal{X}$ the line integral $\int_{\gamma} f ds$ is independent of the curve in \mathcal{X} from x_1 to x_2 , then the vector field f is conservative.
- (2) Any line integral of f around a closed curve is 0.
- (3) A potential for f exists.

We also have the following necessary but not sufficient condition

$$f \text{ is conservative} \Rightarrow \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

Path connected

Let $\mathcal{X} \subset \mathbb{R}^n$ be open. \mathcal{X} is said to be path connected if for every pair of points $x, y \in \mathcal{X}$ a C^1 path $\gamma : (0, 1] \rightarrow \mathcal{X}$ exists with $\gamma(0) = x$, $\gamma(1) = y$.

Star shaped

A subset $\mathcal{X} \subset \mathbb{R}^n$ is called star shaped if $\exists x_0 \in \mathcal{X}$ such that $\forall x \in \mathcal{X}$ the line segment joining x_0 to x is contained in \mathcal{X} . Note

$$\text{Convex} \Rightarrow \text{Star shaped}$$

Further if \mathcal{X} is a star shaped open set of \mathbb{R}^n and $f \in C^1$ is a vector field s.t.

$$\begin{aligned} \frac{\partial f_i}{\partial x_j} &= \frac{\partial f_j}{\partial x_i} \quad \forall i, j \Rightarrow f \text{ is conservative} \\ \text{curl}(f) &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow f \text{ is conservative} \end{aligned}$$

Curl

Let $\mathcal{X} \subset \mathbb{R}^3$ be open and $f : \mathcal{X} \rightarrow \mathbb{R}^3$ be a C^1 vector

field. Then the curl of f is the vector field on \mathcal{X} defined by

$$\text{curl}(f) := \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

Partition

A partition P of a closed rectangle $Q = I_1 \times \cdots \times I_n$ where $I_k = [a_k, b_k]$ is a subcollection of rectangular boxes $Q_1, \dots, Q_k \subset Q$ such that

$$(1) \quad Q = \bigcup_{j=1}^k Q_j$$

$$(2) \quad \text{Int } Q_i \cap \text{int } Q_j = \emptyset \quad \forall i \neq j$$

and $\text{Norm}(P) = \delta_P := \max(\text{diam } Q_j)$ while $\text{vol}(Q) = \prod_{i=1}^n (b_i - a_i)$

Riemann Sum

Riemann sum of f , for partition P , interlude point $\{\xi_i\}$ is the sum

$$R(f, P, \xi) = \sum_{j=1}^k f(\xi_j) \cdot \text{vol}(Q_j)$$

For the lower sum instead of $f(\xi_i)$ use $\inf_{x \in Q_j} f(x)$ and for upper sum $\sup_{x \in Q_j} f(x)$

Integrable

The lower Riemann sum equals the upper Riemann sum. We have for $f : \mathbb{R}^n \rightarrow \mathbb{R}$, Q rectangular boxes in \mathbb{R}^n

$$(1) \quad f \text{ is continuous on } Q \Rightarrow f \text{ is integrable}$$

$$(2) \quad f, g : Q \subset \mathbb{R}^n \rightarrow \mathbb{R} \text{ integrable, } \alpha, \beta \in \mathbb{R} \Rightarrow \alpha f + \beta g \text{ is integrable and equals}$$

$$\int_Q (\alpha f + \beta g) dx = \alpha \int_Q f dx + \beta \int_Q g dx$$

$$(3) \quad \text{If } f(x) \leq g(x) \quad \forall x \in Q \text{ then}$$

$$\int_Q f(x) dx \leq \int_Q g(x) dx$$

$$(4) \quad \text{if } f(x) \geq 0 \text{ then}$$

$$\int_Q f(x) dx \geq 0$$

$$(5) \quad \text{We have}$$

$$\begin{aligned} \left| \int_Q f(x) dx \right| &\leq \int_Q |f(x)| dx \\ &\leq \left(\sup_Q |f(x)| \right) \cdot \text{vol}(Q) \end{aligned}$$

$$(6) \quad \text{If } f = 1 \text{ then}$$

$$\int_Q 1 dx = \text{vol}(Q)$$

Fubini's theorem

Let $Q = I_1 \times \cdots \times I_n$ and f be continuous on Q . Then

$$\begin{aligned} &\int_Q f(x_1, \dots, x_n) dx_1 \dots dx_n \\ &= \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \dots, x_n) dx_n \dots dx_1 \end{aligned}$$

Should the domain of integration be of the type $D_1 := \{(x, y) \mid a \leq x \leq b \text{ and } g(x) < y < h(x)\}$, then

$$\int_D f(x, y) dx dy = \int_a^b \int_{g(x)}^{h(x)} f(x, y) dy dx$$

If on the other hand $D_2 := \{(x, y) \mid c \leq y \leq d \text{ and } G(y) < x < H(y)\}$, then

$$\int_D f(x, y) dx dy = \int_c^d \int_{G(y)}^{H(y)} f(x, y) dx dy$$

Changing the order of integration

Changing the order of integration is sometimes necessary, as in the following example.

$$\begin{aligned} \int_0^1 \int_x^1 e^{y^2} dy dx &= \int_0^1 \int_0^y e^{y^2} dx dy \\ &= \int_0^1 \left(x \cdot e^{y^2} \Big|_{x=0}^{x=y} \right) dy \\ &= \int_0^1 y \cdot e^{y^2} dy \\ &= \frac{e^{y^2}}{2} \Big|_0^1 \end{aligned}$$

Negligible sets in \mathbb{R}^n

If for $1 \leq m \leq n$ a parametrized m -set in \mathbb{R}^n is a continuous function

$$\varphi : [a_1, b_1] \times \cdots \times [a_m, b_m]$$

which is C^1 on $(a_1, b_1) \times \cdots \times (a_m, b_m)$, then a subset $Y \subset \mathbb{R}^n$ is negligible if there exist finitely many parametrized m_i -sets $\varphi_i : \mathcal{X}_i \rightarrow \mathbb{R}^n$ with $m_i < n$ such that

$$Y \subset \bigcup \varphi_i(\mathcal{X}_i)$$

This means in practice that when we split up an integral we don't need to worry about counting the bounds twice. We also have:

If $Y \subset \mathbb{R}^n$ closed, bounded and negligible

$$\Rightarrow \int_Y f dx_1 \dots dx_n = 0 \text{ for any } f$$

Improper Integrals

Let $f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a non compact set and f a function such that $\int_K f dx$ exists for every compact set $K \subset \mathcal{X}$ and suppose $f \geq 0$. Finally we have a sequence of regions $\mathcal{X}_k \quad k = 1, 2, \dots$ s.t.

(1) Each region \mathcal{X}_k is closed and bounded

(2) $\mathcal{X}_k \subset \mathcal{X}_{k+1}$

(3) $\bigcup_{k=1}^{\infty} \mathcal{X}_k = \mathcal{X}$

then

$$\int_{\mathcal{X}} f \, dx := \lim_{n \rightarrow \infty} \int_{\mathcal{X}_n} f \, dx$$

Change of variables

Let $\varphi : \mathcal{X} \rightarrow Y$ be a continuous map, where $\mathcal{X} = \mathcal{X}_0 \cup B$, $Y = Y_0 \cup C$ are closed and bounded sets with \mathcal{X}_0 , Y_0 open, B , C negligible subsets of \mathbb{R}^n . Suppose $\varphi : \mathcal{X}_0 \rightarrow Y_0$ is C^1 and bijective with $\det J_{\varphi}(x) \neq 0 \, \forall x \in \mathcal{X}_0$. Let $Y = \varphi(\mathcal{X})$. Suppose $f : Y \rightarrow \mathbb{R}$ is continuous, then

$$\int_Y f(y) \, dy = \int_{\varphi^{-1}(Y)} f(\varphi(x)) \cdot |\det J_{\varphi}(x)| \, dx$$

Here an example with polar coordinates on a quarter circle:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \end{pmatrix}$$

$$J = \begin{pmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{pmatrix}$$

$$\det(J) = r$$

$$dx \, dy = r \, dr \, d\theta$$

$$\begin{aligned} \int_{\mathcal{X}} \frac{dx \, dy}{1+x^2+y^2} &= \int_0^{\frac{\pi}{2}} \int_0^1 \frac{1}{1+r^2} \cdot r \, dr \, d\theta \\ &= \frac{\log(1+r^2)}{2} \Big|_0^1 \end{aligned}$$

We have the following shortcuts

(1) Polar coordinates: $dx \, dy = r \, dr \, d\theta$

(2) Cylindrical coordinates: $dx \, dy \, dz = r \, dr \, d\theta \, dz$

(3) Spherical coordinates: $dx \, dy \, dz = r^2 \sin(\varphi) \, dr \, d\theta \, d\varphi$ We also have

Green's formula

Let \mathcal{X} be a closed and bounded region in \mathbb{R}^2 . Let γ be a curve forming the boundary of \mathcal{X} .

$$\int \int_{\mathcal{X}} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx \, dy = \int_{\gamma} f \, ds$$

where $f : (x, y) \rightarrow \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix}$.

There are implicit assumptions.

(1) We assume that the vector field $f = (f_1, f_2)$ has components f_1, f_2 s.t. $\frac{\partial f_2}{\partial x}, \frac{\partial f_1}{\partial y}$ exist in the region \mathcal{X} . The usual assumption is that if $f \in C^1$, then $\frac{\partial f_i}{\partial x}, \frac{\partial f_i}{\partial y} \, i = 1, 2$ exist and are continuous so that $\text{curl}(f)$ is continuous. Thus the integral on the left side exists.

(2) The region \mathcal{X} needs to be closed and bounded and that its boundary is a simple closed parametrized curve $\gamma : [a, b] \rightarrow \mathbb{R}^2$. (closed: $\gamma(a) = \gamma(b)$, simple: no knots)

(3) \mathcal{X} is always to the left hand side of a tangent vector to the boundary (corners no problem).

(4) Unions of simple closed curves also work (eg. doughnut). Then we would have

$$\int \int_{\mathcal{X}} \text{curl}(f) \, dx \, dy = \sum_{i=1}^k \int_{\gamma_i} f \, ds$$

If we wanted to calculate the area of a set, then handy functions with $\text{curl}(f) = 1$ are

$$f = (0, x) \text{ or } f = (-y, 0)$$

$$\int_{\gamma} f \, ds = \int_{\gamma_1} f \, ds + \int_{\gamma_2} f \, ds$$

Divergence

For a vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $f \in C^1$, $f = (f_1, \dots, f_n)$ the divergence of f is defined by

$$\text{div} f = \frac{\partial f_1}{\partial x_1} + \dots + \frac{\partial f_n}{\partial x_n}$$

which for $n = 2$ we can calculate using Green's formula.

$$\tilde{f}(x, y) = (-f_2, f_1)$$

$$\text{curl}(\tilde{f}) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = \text{div}(f)$$

$$\begin{aligned} \int \int_{\mathcal{X}} \text{div}(f) \, dx \, dy &= \int \int_{\mathcal{X}} \text{curl}(\tilde{f}) \, dx \, dy = \int_{\partial \mathcal{X}} \tilde{f} \, ds \\ &= \int_a^b (f_1(\gamma(t)), f_2(\gamma(t))) \cdot (\gamma'_2(t), -\gamma'_1(t)) \, dt \\ &= \int_a^b (f_1(\gamma(t)), f_2(\gamma(t))) \cdot n(t) \, dt \end{aligned}$$

Here $n(t)$ is called the exterior normal to the curve and $\gamma'(t) \cdot n(t) = 0$.

Divergence-flux

The form or the normal form of Green's theorem.

$$f : (f_1, f_2) : \mathcal{X} \rightarrow \mathbb{R}^2$$

$$\int \int_{\mathcal{X}} \text{div}(f) \, dx \, dy = \int_{\partial \mathcal{X}} f \, d\vec{n}$$

or

$$\int \int_{\mathcal{X}} \text{curl}(f) \, dx \, dy = \int_{\partial \mathcal{X}} \tilde{f} \, d\vec{s}$$

4 Other

Injektiv/ Surjektiv

Injektiv: $\forall a, b \in X, f(a) = f(b) \Rightarrow a = b$

Surjektiv: $\forall y \in Y, \exists x \in X, f(x) = y$

Supremum

Sei $A \subseteq \mathbf{R}$, $A \neq \emptyset$ und A nach oben beschränkt. Dann gibt es eine kleinste obere Schranke von A . Es gibt also ein $c \in \mathbf{R}$ so dass:

1. $\forall a \in A \quad a \leq c$
2. Falls $\forall a \in A \quad a \leq x$ ist $c \leq x$

Man bezeichnet $c := \sup A$

Infimum

Analog zum Supremum die grösste untere Schranke.

Dreiecksungleichung

$$\forall x, y \in \mathbf{R} : ||x| - |y|| \leq |x \pm y| \leq |x| + |y|$$

Bernoulli Ungleichung

$$\forall x \in \mathbf{R} \geq -1 \text{ und } n \in \mathbf{N} : (1+x)^n \geq 1+nx$$

Exponentialfunktion

Für ein $z \in \mathbf{C}$ berechnet man die Exponentialfunktion wie folgt:

$$\exp(z) := 1 + z + \frac{z^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

und es gilt:

$$\exp(z) = \lim_{n \rightarrow \infty} \left(1 + \frac{z}{n}\right)^n$$

Die reelle Exponentialfunktion $\exp : \mathbf{R} \rightarrow]0, \infty[$ ist streng monoton wachsend, stetig und surjektiv.

Es gelten weiter folgende Rechenregeln:

$$(1) \exp(x+y) = \exp(x) * \exp(y)$$

$$(2) x^a := \exp(a * \ln(x))$$

$$(3) \exp(iz) = \cos(z) + i * \sin(z) \quad \forall z \in \mathbf{C}$$

Merke: e^x entspricht $\exp(x)$.

Natürliche Logarithmus

Der natürliche Logarithmus wird als $\ln :]0, \infty[\rightarrow \mathbf{R}$ bezeichnet und ist eine streng monoton wachsende stetige Funktion. Es gilt auch, dass

$$(1) \ln(a * b) = \ln(a) + \ln(b)$$

$$(2) \ln(a/b) = \ln(a) - \ln(b)$$

$$(3) \ln(x^a) = a * \ln(x)$$

$$(4) (x^a)^b = x^{a*b}$$

$$(5) \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \quad (-1 < x \leq 1)$$

Rechenregeln der Ableitung

$$(1) (f+g)'(x_0) = f'(x_0) + g'(x_0)$$

$$(2) (f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)$$

$$(3) \left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0) \cdot g(x_0) - f(x_0) \cdot g'(x_0)}{(g(x_0))^2}$$

$$(4) (g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

Faktorisierungs Lemma

$$a^n - b^n = (a-b)(a^{n-1} + ba^{n-2} + \dots + b^{n-2}a + b^{n-1})$$

Kompaktes Intervall

Ein Intervall $I \subset \mathbf{R}$ ist kompakt, wenn es von der Form $I = [a, b]$, $a \leq b$ ist.

Funktionenfolge

Eine Funktionenfolge ist eine Abbildung:

$$f : \mathbf{N} \rightarrow \mathbf{R}^{\mathbf{D}} = \{f : \mathbf{D} \rightarrow \mathbf{R}\}$$

$$n \mapsto f_n$$

wobei $f_n : \mathbf{D} \rightarrow \mathbf{R}$ eine Funktion ist. Für jedes $x \in \mathbf{D}$ erhält man eine Folge $(f_n(x))_{n \geq 1}$ reeller Zahlen.

Trigonometrische Funktionen

$$\sin(z) := z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\cos(z) := 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

$$\tan(z) := \frac{\sin(z)}{\cos(z)} \quad \forall z \notin \left\{\frac{\pi}{2} + \pi k\right\}$$

welche alle stetige Funktionen sind. Es gilt weiter:

$$(1) \cos(z) = \cos(-z)$$

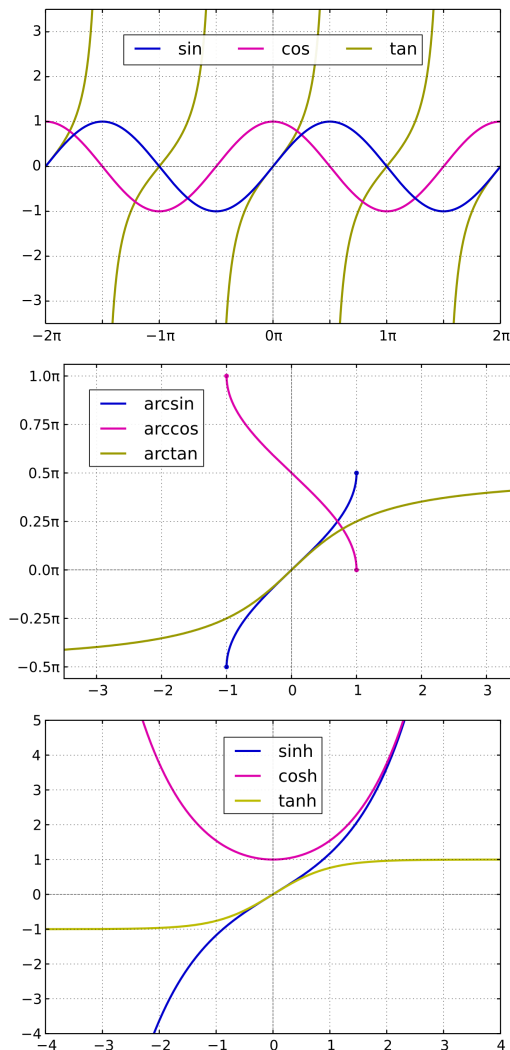
$$(2) \sin(-z) = -\sin(z)$$

$$(3) \cos^2(z) + \sin^2(z) = 1 \quad \forall z \in \mathbf{C}$$

$$(4) \sin(2x) = 2 \cdot \sin(x) \cos(x)$$

$$(5) \cos(2x) = \cos^2(x) - \sin^2(x)$$

α	0°	30°	45°	60°	90°
α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(\alpha)$	$\frac{1}{2}\sqrt{0}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{4}$
$\cos(\alpha)$	$\frac{1}{2}\sqrt{4}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{0}$



Typische Ableitungen/ Stammfunktionen

Es gibt folgende typischen Ableitungen/ Stammfunktionen:

F	$F' = f$
c	0
$x^a + C$	$a \cdot x^{a-1}$
$\frac{x^{\alpha+1}}{\alpha+1} + C$	$x^\alpha, \alpha \neq -1$
$e^x + C$	e^x
$\ln(x) + C$	$\frac{1}{x}$
$\sin(x) + C$	$\cos(x)$
$\cos(x) + C$	$-\sin(x)$
$\tan(x) + C$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$
$\arcsin(x) + C$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x) + C$	$\frac{-1}{\sqrt{1-x^2}}$
$\arctan(x) + C$	$\frac{1}{1+x^2}$
$\sinh(x) + C$	$\cosh(x)$
$\cosh(x) + C$	$\sinh(x)$
$\tanh(x) + C$	$\frac{1}{\cosh^2(x)} = 1 - \tanh^2(x)$
$\operatorname{arcsinh}(x) + C$	$\frac{1}{\sqrt{1+x^2}}$
$\operatorname{arccosh}(x) + C$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{arctanh}(x) + C$	$\frac{1}{1-x^2}$

Kompliziertere Integrale

- $\int \sin^2(x) dx = \frac{1}{2}(x - \sin(x) \cos(x)) + C$
- $\int \cos^2(x) dx = \frac{1}{2}(x + \sin(x) \cos(x)) + C$
- $\int \tan^2(x) dx = \tan(x) - x + C$

$$(4) \int \sin(x) \cdot \cos(x) dx = \frac{\sin^2(x)}{2} + C$$

$$(5) \int \sin^2(x) \cdot \cos(x) dx = \frac{\sin^3(x)}{3} + C$$

$$(6) \int \sin(x) \cdot \cos^2(x) dx = -\frac{\cos^3(x)}{3} + C$$

$$(7) \int \sin^3(x) dx = \frac{\cos^3(x)}{3} - \cos(x) + C$$

$$(8) \int \cos^3(x) dx = \sin(x) - \frac{\sin^3(x)}{3} + C$$

$$(9) \int \frac{1}{\sin(x)} dx = \ln(|\tan(\frac{x}{2})|) + C$$

$$(10) \int \frac{1}{\cos(x)} dx = \ln(|\tan(\frac{x}{2} + \frac{\pi}{4})|) + C$$

$$(11) \int \frac{1}{\tan(x)} dx = \ln(|\sin(x)|) + C$$

Potenzen der Winkelfunktion

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

Funktionen Verknüpfung

$$x \mapsto (g \circ f)(x) := g(f(x))$$

Häufungspunkt

$x_0 \in \mathbf{R}$ ist ein **Häufungspunkt** der Menge \mathbf{D} , falls $\forall \delta > 0 \quad (]x_0 - \delta, x_0 + \delta[\setminus \{x_0\}) \cap \mathbf{D} \neq \emptyset$

Kritische Stelle

Eine **kritische Stelle** einer Funktion ist ein x_0 an der $f'(x_0)$ null oder undefiniert ist.

Hyperbol Funktionen

$$(1) \cosh(x) := \frac{e^x + e^{-x}}{2} : \mathbf{R} \rightarrow [1, \infty]$$

$$(2) \sinh(x) := \frac{e^x - e^{-x}}{2} : \mathbf{R} \rightarrow \mathbf{R}$$

$$(3) \tanh(x) := \frac{e^x - e^{-x}}{e^x + e^{-x}} : \mathbf{R} \rightarrow [-1, 1]$$

und es gilt $\cosh^2(x) - \sinh^2(x) = 1$