

Differential Equations

1.1 Linear ODE's

A differential equation is said to be linear if F can be written as a linear combination of the derivatives of y :

$$b(x) = \sum_{i=0}^n a_i(x) \cdot y^{(i)}$$

where $a_i(x)$ and $b(x)$ are continuous functions. Why is this called linear?

$$D = \frac{d^{(k)}}{dx^k} + a_{k-1} \frac{d^{(k-1)}}{dx^{k-1}} + \dots + a_0$$

$$D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)$$

$$Df_1 = b_1, Df_2 = b_2 \Rightarrow D(f_1 + f_2) = b_1 + b_2$$

1.2 Solution Space

Let $I \subset \mathbf{R}$ be an open interval and $k \geq 1$ an integer, and let

$$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = 0$$

be a linear ODE over I with continuous coefficients.

- (1) The set S of k -times differentiable solutions $f : I \rightarrow \mathbb{C}$ of the equation is a complex vector space which is a subspace of the space of complex valued functions on I . (Analogous for real numbers, if all a_i are real valued)

- (2) The dimension of S is k and for any choice of $x_0 \in I$ and any $(y_0, \dots, y_{k-1}) \in \mathbb{C}^k$ there exists a unique f such that

$$f(x_0) = y_0, f'(x_0) = y_1, \dots, f^{(k-1)}(x_0) = y_{k-1}$$

(Analogous for real numbers, if all a_i are real)

- (3) For an arbitrary b the solution set is $S_b = \{f + f_p \mid f \in S_0\}$ where f_p is a "particular" solution.
- (4) For any initial condition there is a unique solution.

1.3 Solving linear ODE's of order 1

$y' + ay = b$. Here a, b are constant functions.

- (1) Find solutions of the corresponding homogenous equation $y' + ay = 0$. Note that if f is a solution so is $z \cdot f \quad \forall z \in \mathbb{C}$. Example:

$$y' + ay = 0$$

$$y' = -ay$$

$$\frac{y'}{y} = -a$$

$$\ln(y) = -\int a + C = -A + C$$

$$y = e^{-A+C} = z \cdot e^{-A} \quad z \in \mathbb{C}$$

- (2) Find a particular solution $f_p : I \rightarrow \mathbb{C}$ such that $f_p' + af_p = b$. Use educated guess or variation of constants.

Assume we have $y' + \frac{y}{x} = 2 \cos(x^2)$ The homogenous equation $y' = -\frac{1}{x}y$ has a constant solution $y_h(x) = 0$. Otherwise we have:

$$\log(y) = \int \frac{y'}{y} dx = -\int \frac{1}{x} dx = -\log(x) + c$$

$$y = \frac{e^c}{x}$$

$$y = \frac{C}{x}$$

Our educated guess is $y_p = \frac{C(x)}{x}$

$$\frac{C'(x)x - C(x)}{x^2} + \frac{1}{x} \frac{C(x)}{x} = 2 \cos(x^2)$$

We solve for $C'(x)$

$$C'(x) = \frac{g(x)}{y_1(x)} \rightarrow C(x) = \int \frac{g(x)}{y_1(x)} dx = \int \frac{2 \cos(x^2)}{\frac{1}{x}} dx$$

$$= \int 2x \cos(x^2) = \sin(x^2)y(x) = \frac{c + \sin(x^2)}{x}$$

1.4 Educated Guess

$b(x)$	Guess
$ax^2 + bx$	$cx^2 + dx + e$
$a \cdot e^{\alpha x}$	$b \cdot e^{\alpha x}$
$a \sin(\beta x)$	$c \sin(\beta x) + d \cos(\beta x)$
$b \cos(\beta x)$	$c \sin(\beta x) + d \cos(\beta x)$
$ae^{\alpha x} \sin(\beta x)$	$e^{\alpha x} (c \sin(\beta x) + d \cos(\beta x))$
$be^{\alpha x} \cos(\beta x)$	$e^{\alpha x} (c \sin(\beta x) + d \cos(\beta x))$
$P_n(x)e^{\alpha x}$	$R_n(x) \cdot e^{\alpha x}$
$P_n(x)e^{\alpha x} \sin(\beta x)$	$e^{\alpha x} (R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))$
$P_n(x)e^{\alpha x} \cos(\beta x)$	$e^{\alpha x} (R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))$

1.4.1 Variation of constants

- (1) Assume $f_p = z(x)e^{-A(x)}$ for some function $z : I \rightarrow \mathbb{C}$
- (2) We plug this into the equation and see what it forces z to satisfy

$$f_p = z(x)e^{-A(x)} = y$$

$$y' = z'(x)e^{-A(x)} - z(x)A'(x)e^{-A(x)}$$

$$y' = e^{-A(x)} (z'(x) - z(x)a(x))$$

$$ay = a \cdot z(x)e^{-A(x)}$$

$$y' + ay = z'(x)e^{-A(x)} = b(x)$$

$$b(x) = z'(x)e^{A(x)}$$

$$z(x) = \int \frac{e^{A(x)}}{b(x)} dx$$

$$y_p = z(x)e^{-A(x)}$$

1.4.2 Solving Linear ODE's with constant coefficients

We want to solve

$$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_0y = b(x)$$

We assume our solution is $e^{\lambda x}$.

$$P(\lambda) = \lambda^k e^{\lambda x} + a_{k-1} \lambda^{k-1} e^{\lambda x} + \dots + a_0 e^{\lambda x} = 0$$

$$= e^{\lambda x} (\lambda^k + a_{k-1} \lambda^{k-1} + \dots + a_0) = 0$$

$$\Rightarrow 0 = \lambda^k + a_{k-1} \lambda^{k-1} + \dots + a_0$$

Which can then be solved for λ . Keep in mind that $\lambda \in \mathbb{C}$ and we might be able to simplify with Euler's formula.

$$e^{ix} = \cos(x) + i \sin(x)$$

If there is a multiple root α of multiplicity j we have

$$\text{Solutions: } e^{\alpha x}, xe^{\alpha x}, \dots, x^{j-1}e^{\alpha x}$$

1.5 Complex roots

If $\alpha = \beta + \gamma i$ is a complex root of $P(\lambda)$, then so is $\bar{\alpha} = \beta - \gamma i$. Hence $f_1 = e^{\alpha x}$ and $f_2 = e^{\bar{\alpha} x}$ are solutions and can be replaced by a linear combination of $\tilde{f}_1 = e^{\beta x} \cos(\gamma x)$ and $\tilde{f}_2 = e^{\beta x} \sin(\gamma x)$. Further if $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_0y = 0$ has real coefficients, then each pair of complex conjugate roots $\beta_j \pm \gamma_j i$ with multiplicity m_j leads to solution

$$x^l e^{\beta_j x} (\cos(\gamma_j x) + i \sin(\gamma_j x)) \quad \text{for } 0 \leq l \leq m_j$$

1.6 Separation of variables

A differential equation of order 1 is separable if it is of the form

$$y' = b(x)g(y)$$

$$\frac{dy}{dx} = b(x)g(y)$$

$$\frac{dy}{g(y)} = b(x)dx$$

$$\int \frac{dy}{g(y)} = \int b(x)dx$$

2 Differentials in \mathbf{R}^n

2.1 Monomial

A monomial of degree e is a function

$$(x_1, \dots, x_n) \mapsto \alpha x_1^{d_1} \cdot \dots \cdot x_n^{d_n}$$
$$e = d_1 + \dots + d_n$$

2.2 Polynomial

A polynomial in n variables of degree $\leq d$ is a finite sum of monomials of degree $e \leq d$

2.3 Convergence

Let $(x_k)_{k \in \mathbb{N}}$, $x_k \in \mathbf{R}^n$ and $x_k = (x_{k,1}, x_{k,2}, \dots, x_{k,n})$. The following equivalently define $\lim_{k \rightarrow \infty} x_k = y$.

- (1) $\forall \varepsilon > 0 \exists N \geq 1$ s.t. $\forall k \geq N \quad \|x_k - y\| < \varepsilon$
- (2) For each i , $1 \leq i \leq n$ the sequence $(x_{k,i})_k$ of real numbers converges to y_i .
- (3) The sequence of real numbers $\|x_k - y\|$ converges to 0.

Let $f : \mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbf{R}^m$ and $x_0 \in \mathcal{X}$, $y \in \mathbf{R}^m$. We say f has a limit to y as $x \rightarrow x_0$ where $x \neq x_0$ if any of the following apply

- (1) $\forall \varepsilon > 0 \exists \delta > 0$ s.t. $\forall x \in \mathcal{X}$, $x \neq x_0$ such that $\|x - x_0\| < \delta$ we have $\|f(x) - y\| < \varepsilon$.
- (2) \forall sequences (x_k) in \mathcal{X} such that $\lim x_k = x_0$ and $x_k \neq x_0$ the sequence $f(x_k)$ converges to y .

2.4 Continuity

Let $f : \mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbf{R}^m$ and $x_0 \in \mathcal{X}$. We say f is continuous at x_0 if any of the following apply

- (1) $\forall \varepsilon > 0 \exists \delta > 0$ s.t. if $x \in \mathcal{X}$ satisfies $\|x - x_0\| < \delta$ then $\|f(x) - f(x_0)\| < \varepsilon$.
- (2) \forall sequences (x_k) in \mathcal{X} s.t. $\lim x_k = x_0$ we have $\lim f(x_k) = f(\lim x_k)$.

f is continuous in \mathcal{X} if f is continuous in every point $x_0 \in \mathcal{X}$. The following statements also hold

- (1) $f(x = x_1, \dots, x_n) \mapsto (f_1(x), \dots, f_m(x))$ and $f_i : \mathbf{R}^n \mapsto \mathbf{R}$ is continuous $\Leftrightarrow f_i \forall i = 1, \dots, m$ are continuous.
- (2) Linear functions $x \mapsto Ax$ are continuous.
- (3) Polynomials are continuous.
- (4) Sums, products of continuous functions are continuous.
- (5) Functions of separated variables are continuous if the factors are continuous.
- (6) Composition of continuous functions are continuous.

2.5 Sandwich lemma

If $f, g, h : \mathbf{R}^n \rightarrow \mathbf{R}$ where $f(x) < g(x) < h(x) \quad \forall x \in \mathbf{R}^n$. Let $a \in \mathbf{R}^n$.

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L \Rightarrow \lim_{x \rightarrow a} g(x) = L$$

2.6 Polar Coordinates

It is sometimes helpful to use polar coordinates, especially with rational functions $f : \mathbf{R} \rightarrow \mathbf{R}$. $f(x, y) = f(r \cos(\theta), r \sin(\theta))$

2.7 Bounded set

A set $\mathcal{X} \subset \mathbf{R}^n$ is bounded if the set $\{\|x\| \mid x \in \mathcal{X}\}$ is bounded in \mathbf{R} .

2.8 Closed set

A set $\mathcal{X} \subset \mathbf{R}^n$ is closed if for every sequence $(x_k)_{k \in \mathbb{N}} \subset \mathcal{X}$ that converges in \mathbf{R}^n , converges to a point $y \in \mathcal{X}$. Here it is often helpful to consider a ball. Counterexamples often include $\frac{1}{k}$ and $<$.

2.9 Compact set

A compact set is a closed and bounded set.

2.10 Continuous and closed

If $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is continuous, then for every $Y \subset \mathbf{R}^m$ that is closed the set $f^{-1}(Y) = \{x \in \mathbf{R}^n \mid f(x) \in Y\} \subset \mathbf{R}^n$ is closed. Careful: Does not imply bounded or compact!

2.11 Min-Max theorem

Let $\mathcal{X} \subset \mathbf{R}^n$ be a compact set, $f : \mathcal{X} \rightarrow \mathbf{R}$ a continuous function. Then f is bounded and attains its max and min.

$$f(x^+) = \sup_{x \in \mathcal{X}} f(x) f(x^-) = \inf_{x \in \mathcal{X}} f(x)$$

2.12 Open set

A set $\mathcal{X} \subset \mathbf{R}^n$ is called open if its complement $\mathbf{R}^n \setminus \mathcal{X}$ is closed. This is equivalent to $\forall x \in \mathcal{X} \exists r > 0$ s.t. the set $\{y \in \mathbf{R}^n \mid \|y - x\| < r\} = B_r(x) \subset \mathcal{X}$. Here are some examples

- (1) $(a, b) \subset \mathbf{R}$ is open.
- (2) $[a, b) \subset \mathbf{R}$ is neither open nor closed.
- (3) \mathbf{R}^n and \emptyset are both open.
- (4) $(a_1, b_1) \times (a_2, b_2) \subset \mathbf{R}^2$ is open.
- (5) Inverse image of open sets under continuous maps are open.

2.13 Derivative

Given $f : \mathbf{R} \rightarrow \mathbf{R}^n$ the derivative is

$$f'(x_0) = \begin{pmatrix} f'_1(x_0) \\ \vdots \\ f'_n(x_0) \end{pmatrix}$$

2.14 Partial derivatives

A partial derivative of a function $f : \mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbf{R}$ is obtained by pretending all but one variable are constants and then differentiating the one variable.

$$\frac{\partial f}{\partial x_{0,j}} = \lim_{h \rightarrow 0} \frac{f(x_{0,1}, \dots, x_{0,j} + h, \dots, x_{0,n}) - f(x_0)}{h}$$

If $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ for $x_0 \in \mathbf{R}^n$ then

$$\frac{\partial f(x_0)}{\partial x_j} := \begin{pmatrix} \partial f_1(x_0) / \partial x_j \\ \vdots \\ \partial f_m(x_0) / \partial x_j \end{pmatrix}$$

Properties include (assuming partial derivatives for f, g exist w.r.t. x_j)

- (1) $\frac{\partial f+g}{\partial x_j} = \frac{\partial f}{\partial x_j} + \frac{\partial g}{\partial x_j}$
- (2) $\frac{\partial f \cdot g}{\partial x_j} = \frac{\partial f}{\partial x_j} \cdot g + \frac{\partial g}{\partial x_j} \cdot f$
- (3) if $g \neq 0$: $\frac{\partial f/g}{\partial x_j} = \frac{\frac{\partial f}{\partial x_j} \cdot g - \frac{\partial g}{\partial x_j} \cdot f}{g^2}$

2.15 Jacobi Matrix

A Matrix with m rows and n columns where

$$J_f = \left(\frac{\partial f_i}{\partial x_j} \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$$

2.16 Gradient

The Jacobian of a function $f : \mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbf{R}$. Is often denoted as ∇f . The geometric interpretation is that it indicates the direction and rate of fastest increase. Remember: $\text{curl}(\nabla f) = 0$ is a necessary condition for a vector field to be a gradient!

$$\text{Curl} \neq 0 \rightarrow \text{not a potential}$$

2.17 Directional derivative

Let direction $v = (a, b) \neq (0, 0)$. Instead of adding $+h$ to one component we add $+ah, +bh$ and so on to all components and find that derivative as we normally would with a limit. Further, we have

$$\frac{df(x_0 + t\vec{v})}{dt} = J_f(x_0) \cdot \vec{v}$$

2.18 Differentiability

Let $\mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbf{R}^p$ be function and $x_0 \in \mathcal{X}$. We say f is differentiable at x_0 if a linear map $u : \mathbf{R}^n \rightarrow \mathbf{R}^p$ exists such that

$$\lim_{x \rightarrow x_0, x \neq x_0} \frac{f(x) - f(x_0) - u(x - x_0)}{\|x - x_0\|} = 0$$

and u is called the total differential of f at x_0 . Further, if f, g are differentiable at $x_0 \in \mathcal{X}$ we have

- (1) f is continuous at x_0
- (2) f has all partial derivatives at x_0 and the matrix represents the linear map $df(x_0) : x \mapsto Ax$ in the canonical basis is given by the Jacobi Matrix of f at x_0 , i.e. $A = J_f(x_0)$
- (3) $d(f + g)(x_0) = df(x_0) + dg(x_0)$
- (4) If $m = 1$ and $f, g : \mathbf{R}^n \rightarrow \mathbf{R}$ differentiable in x_0 then so is $f \cdot g$ and if $g \neq 0$ f/g as well.

Lastly we have

All partial derivatives \exists and cont. $\Rightarrow f$ is differentiable

2.19 Tangent space

The approximation of the function at x_0 using one derivative.

$$\{(x, y) \in \mathbf{R}^n \times \mathbf{R}^m \mid y = f(x_0) + u(x - x_0)\}$$

An example:

$$\begin{aligned} f(x, y) &= \sqrt{x^2 + y^2} \\ J_f &= \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right) \\ J_f(3, 4) &= \left(\frac{3}{5}, \frac{4}{5} \right) \\ \Rightarrow g(x, y) &= 5 + \left(\frac{3}{5}, \frac{4}{5} \right) \begin{pmatrix} x - 3 \\ y - 4 \end{pmatrix} \end{aligned}$$

2.20 Chain rule

Let $\mathcal{X} \subset \mathbf{R}^n$ be open, $\mathcal{Y} \subset \mathbf{R}^m$ be open and let $f : \mathcal{X} \rightarrow \mathcal{Y}, g : \mathcal{Y} \rightarrow \mathbf{R}^p$ be differentiable functions. Then $g \circ f = g(f) : \mathcal{X} \rightarrow \mathbf{R}^p$ is differentiable in \mathcal{X} . In particular

$$\begin{aligned} d(g \circ f)(x_0) &= dg(f(x_0)) \circ df(x_0) \\ J_{g \circ f}(x_0) &= J_g(f(x_0)) \cdot J_f(x_0) \end{aligned}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$\frac{d}{dt} f(\gamma(t)) = \nabla f(\gamma(t)) \gamma'(t)$$

2.21 Change of variables

We say f is a change of variables around x_0 if there is a radius $\rho > 0$ s.t. the restriction of f to the Ball $B = \{x \in \mathbf{R}^n \mid \|x - x_0\| < \rho\}$ so that the image $Y = f(B)$ is open in \mathbf{R}^n and a differentiable map $g : Y \rightarrow B$ exists, such that $f \circ g = \text{id}_Y$ and $g \circ f = \text{id}_B$. I.e.

$$f|_{B(x_0)} \text{ is a bijection to the image with a differentiable inverse } g$$

2.22 Inverse function theorem

Let $\mathcal{X} \subseteq \mathbf{R}^n$ be open and $f : \mathcal{X} \rightarrow \mathbf{R}^n$ differentiable. If $x_0 \in \mathcal{X}$ is such that $\det(J_f(x_0)) \neq 0$, i.e. $J_f(x_0)$ is invertible, then f is a change of variables around x_0 . Moreover the Jacobian of g at x_0 is defined by

$$J_g(f(x_0)) = J_f(x_0)^{-1}$$

2.23 Higher derivatives

Let $\mathcal{X} \subset \mathbf{R}^n, f : \mathcal{X} \rightarrow \mathbf{R}^m$. We say f is of class C' if f is differentiable on \mathcal{X} and all of its partial derivatives are continuous. We say $f \in C^k$ for $k \geq 2$ if it is differentiable and each $\partial_{x_i} f : \mathcal{X} \rightarrow \mathbf{R}^m$ is of class C^{k-1} . Further, f is smooth or C^∞ if $f \in C^k \quad \forall k$. Lastly: mixed partials (up to order k) commute:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

2.24 Hessian

The $n \times n$ symmetric matrix

$$\text{Hess}_f(x_0) := \left(\frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} \right)$$

2.25 Taylor Polynomial

Good for approximation \rightarrow affine function The Taylor polynomial of f at x_0 of order 1 is

$$\begin{aligned} T_1(\vec{x}_0, \vec{y}_0) &:= f(\vec{x}_0) + \langle \nabla f(\vec{x}_0), \vec{y} \rangle \\ \vec{y} &= \vec{x} - \vec{x}_0 \\ \vec{x}_0 &= (x_0, y_0) \\ \vec{x} &= (x, y) \end{aligned}$$

and the second order

$$\begin{aligned} T_2(\vec{x}_0, \vec{y}_0) &:= f(\vec{x}_0) + \langle \nabla f(\vec{x}_0), \vec{y} \rangle \\ &\quad + \frac{1}{2} \vec{y} \cdot \text{Hess}_f(\vec{x}_0) \cdot \vec{y} \end{aligned}$$

Finally, the general form is

$$\begin{aligned} T_k f(y; x_0) &= f(x_0) + \dots \\ &\quad + \sum_{m_1 + \dots + m_n = k} \frac{1}{m_1! \dots m_n!} \frac{\partial^k f(x_0)}{\partial x_1^{m_1} \dots \partial x_n^{m_n}} \end{aligned}$$

Lastly if $f \in C^k$ for $x_0 \in \mathcal{X}$ we have

$$\begin{aligned} f(x) &= T_k(x - x_0; x_0) + E_k(f, x, x_0) \\ \lim_{x \rightarrow x_0} \frac{E_k(f, x, x_0)}{\|x - x_0\|^k} &\rightarrow 0 \end{aligned}$$

2.25.1 Example

Consider the following function:

$$f(x, y) := e^{x^2 + y^2} + \log(1 + x^2) + \arctan(xy)$$

a) determine the Taylor polynomial of f at $(0, 0)$ up to and including third order.

$$\frac{\partial f(x, y)}{\partial x} = 2xe^{x^2 + y^2} + \frac{2x}{1 + x^2} + \frac{y}{1 + x^2 y^2}$$

$$\frac{\partial f(x, y)}{\partial y} = 2ye^{x^2 + y^2} + \frac{x}{1 + x^2 y^2}$$

Direct substitution gives us:

$$df(0, 0) = (0, 0)$$

We now calculate the partial derivatives of second order:

$$\frac{\partial^2 f(x, y)}{\partial x^2} = 2e^{x^2 + y^2} + 4x^2 e^{x^2 + y^2} + \frac{2(1 + x^2) - 4x^2}{(1 + x^2)^2} - \frac{2xy^3}{(1 + x^2 y^2)^2}$$

$$\frac{\partial^2 f(x, y)}{\partial x \partial y} = 4xye^{x^2 + y^2} + \frac{1 + x^2 y^2 - 2x^2 y^2}{(1 + x^2 y^2)^2}$$

$$\frac{\partial^2 f(x, y)}{\partial y^2} = 2e^{x^2 + y^2} + 4y^2 e^{x^2 + y^2} - \frac{2x^3 y}{(1 + x^2 y^2)^2}$$

We need the hessian so we have:

$$\text{Hess}_f(0, 0) = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

We now calculate the partial derivatives of third order. Luckily they all vanish so we have:

$$\begin{aligned} T_3 f((0, 0); (x, y)) &= f(0, 0) + \frac{\partial f(0, 0)}{\partial x} x + \frac{\partial f(0, 0)}{\partial y} y \\ &\quad + \frac{1}{2} \frac{\partial^2 f(0, 0)}{\partial x^2} x^2 + \frac{\partial^2 f(0, 0)}{\partial x \partial y} xy \\ &\quad + \frac{1}{2} \frac{\partial^2 f(0, 0)}{\partial y^2} y^2 + \frac{1}{6} \frac{\partial^3 f(0, 0)}{\partial x^3} x^3 \\ &\quad + \frac{1}{2} \frac{\partial^3 f(0, 0)}{\partial x^2 \partial y} x^2 y \\ &\quad + \frac{1}{2} \frac{\partial^3 f(0, 0)}{\partial x \partial y^2} xy^2 + \frac{1}{6} \frac{\partial^3 f(0, 0)}{\partial y^3} y^3 \\ &= 1 + 2x^2 + xy + y^2 \end{aligned}$$

2.26 Local max/min

Let $f : \mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbf{R}$ be differentiable. We say $x_0 \in \mathcal{X}$ is a local maximum (minimum) if we can find a neighborhood $B_r(x_0) = \{x \in \mathbf{R}^n \mid \|x - x_0\| < r\} \subset \mathcal{X}$

$$\forall x \in B_r(x_0) \quad f(x) \leq (\geq) f(x_0)$$

We also have

$$x_0 \in \mathcal{X} \text{ is a local extrema} \Rightarrow \nabla f(x_0) = 0$$

2.27 Global extrema

If $f : \mathcal{X} \rightarrow \mathbf{R}$ is differentiable on the interior of \mathcal{X} and \mathcal{X} is closed and bounded, then a global extrema of f exists and it is either at a critical point or the boundary of \mathcal{X} .

$$\text{Check} = \text{int}(\mathcal{X}) \cup \text{bd}(\mathcal{X})$$

2.28 Definite

We have the following

- (1) Positive Definite: All eigenvalues are Positive
- (2) Negative Definite: All eigenvalues are Negative
- (3) Indefinite: Positive and Negative Eigenvalues

Eigenvalues can be found with the characteristic polynomial:

$$\det \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} \right) = \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \\ \Rightarrow \lambda^2 - 1 = 0$$

2.29 Calculating determinates

For 2 dimensions we have

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot d - c \cdot b$$

For 3 dimensions we have

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \cdot \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} \\ + c \cdot \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

2.30 Test critical point

A point is critical: $x_0 \in \mathcal{X}$ where $\nabla f(x_0) = 0$.

Let $f : \mathcal{X} \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$ and $f \in C^2$. Let x_0 be a non-degenerate critical point of f . Then

- (1) If $\text{Hess}_f(x_0)$ pos def. then x_0 is a local minimum
- (2) If $\text{Hess}_f(x_0)$ neg def. then x_0 is a local maximum
- (3) If $\text{Hess}_f(x_0)$ is Indefinite then x_0 is a saddle point

We cannot use this theorem when x_0 is a degenerate critical point ($\det(\text{Hess}_f(x_0)) = 0$) and must decide on a case by case basis!

3 Integrals in \mathbf{R}^n

3.1 Simple integral

For $f : \mathbf{R} \rightarrow \mathbf{R}^n$ the integral is

$$\int_a^b f(t) dt = \begin{pmatrix} \int_a^b f_1(t) dt \\ \vdots \\ \int_a^b f_n(t) dt \end{pmatrix}$$

3.2 Curve

The image of a function $\gamma : [a, b] \rightarrow \mathbf{R}^n$ where the function γ is continuous and piecewise $\in C^1$.

3.3 Line integral

Let $\gamma : [a, b] \rightarrow \mathbf{R}^n$ be a parametrization of a curve and let $\mathcal{X} \subset \mathbf{R}^n$ be a set which contains the image of γ . Further, let $f : \mathcal{X} \rightarrow \mathbf{R}^n$ be a continuous function. A line integral then is

$$\int_{\gamma} f(s) d\vec{s} = \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt$$

The line integral has the following properties

- (1) It is independent of orientation preserving reparametrization, i.e.

$$\begin{aligned} \gamma &: [a, b] \rightarrow \mathbf{R}^n \\ \tilde{\gamma} &: [c, d] \rightarrow \mathbf{R}^n \\ \Phi &: [c, d] \rightarrow [a, b] \\ \tilde{\gamma} &= \gamma \circ \Phi = \gamma(\Phi) \\ \Rightarrow \int_{\gamma} f ds &= \int_{\tilde{\gamma}} f ds \end{aligned}$$

- (2) Let $\gamma_1 + \gamma_2$ be the path formed by the concatenation of the two curves. Then

$$\gamma_1 + \gamma_2 := \begin{cases} \gamma_1(t) & t \in [a, b] \\ \gamma_2(t) & t \in [b, d+b-c] \end{cases}$$

$$\int_{\gamma_1 + \gamma_2} f ds = \int_{\gamma_1} f ds + \int_{\gamma_2} f ds$$

- (3) If $\gamma : [a, b] \rightarrow \mathbf{R}^n$ is a path, let $-\gamma$ be the path traced in the opposite direction, i.e. $(-\gamma)(t) := \gamma(a+b-t)$. Then

$$\int_{-\gamma} f ds = - \int_{\gamma} f ds$$

3.3.1 Length of curve (Bogenlänge)

The length of a curve (Bogenlänge) from a function f on the interval $[a, b]$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

3.3.2 Example

$$v(x, y) = \begin{pmatrix} x^2 - 2xy \\ y^2 - 2xy \end{pmatrix} \text{ from } (-1, 1) \text{ to } (1, 1) \text{ along the curve } y = x^2$$

$$\text{The given parametrization of the curve is } \gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

$$\text{and the derivative of } \gamma(t) \text{ is } \gamma'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}. \text{ The vector}$$

$$\text{field } v(\gamma(t)) \text{ is given by } v(\gamma(t)) = \begin{pmatrix} t^2 - 2t^3 \\ t^4 - 2t^3 \end{pmatrix}, \text{ and the dot}$$

product of $v(\gamma(t))$ and $\gamma'(t)$ is

$$[v(\gamma(t)) \cdot \gamma'(t) = (t^2 - 2t^3)(1) + (t^4 - 2t^3)(2t) = t^2 - 2t^3 + 2t^5 - 4t^4.]$$

The integral of v along the curve γ is

$$\begin{aligned} \int_{\gamma} v, d\gamma &= \int_{-1}^1 t^2 - 2t^3 + 2t^5 - 4t^4 dt \\ &= \left[\frac{t^3}{3} - \frac{t^4}{2} + \frac{t^6}{3} - \frac{4t^5}{5} \right]_{t=-1}^1 \\ &= \frac{1}{3} - \frac{1}{2} + \frac{1}{3} - \frac{4}{5} - \left(-\frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{4}{5} \right) \\ &= \frac{2}{3} - \frac{8}{5} = -\frac{14}{15}. \end{aligned}$$

3.4 Potential

A differentiable scalar field $g : \mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbf{R}$ such that $\nabla g = f$, $f : \mathcal{X} \rightarrow \mathbf{R}^n$ is called a potential for f . This can make stuff easier:

$$\begin{aligned} \int_{\gamma} f ds &= \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt \\ &= \int_a^b \nabla g(\gamma(t)) \cdot \gamma'(t) dt \\ &= \int_a^b \frac{d}{dt} (g \circ \gamma) dt \\ &= (g \circ \gamma)(b) - (g \circ \gamma)(a) \end{aligned}$$

3.4.1 Example

$f(x, y) = (2xy^2 - 5x^4y + 5, -7y^6 - x^5 + 2x^2y)$ is conservative and its potential is:

$$g(x, y) = x^2y^2 - x^5y + 5x - y^7$$

We want to compute $\int_{\gamma} f \cdot ds$ where γ is the parametrised curve:

$$\gamma : \left[\frac{\pi}{4}, \frac{5\pi}{4} \right] \rightarrow \mathbf{R}^2$$

$$\phi : \left[\frac{1}{2} + \frac{1}{\sqrt{2}} \cos(t), \frac{1}{2} + \frac{1}{\sqrt{2}} \sin(t) \right]$$

So we have:

$$g\left(\psi\left(\frac{5\pi}{4}\right)\right) - g\left(\psi\left(\frac{\pi}{4}\right)\right) = g(0, 0) - g(1, 1) = -4$$

It should be noted that not every function has a potential! Example:

$$\begin{aligned} f(x, y) &= (2xy^2, 2x) \\ \frac{\partial g}{\partial x} &= 2xy^2 \Rightarrow g(x, y) = x^2y^2 + h(y) \\ \frac{\partial g}{\partial y} &= 2x \neq 2x^2y + h'(y) \end{aligned}$$

3.4.2 Example

$$f(x, y) = \begin{pmatrix} 3x^2y \\ x^3 \end{pmatrix}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}(3x^2y) = 3x^2 \quad \frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}x^3 = 3x^2$$

If starshaped, integrability is guaranteed. The potential function is

$$\frac{\partial f}{\partial x} = (3x^2y) \quad \frac{\partial f}{\partial y} = x^3$$

We integrate $\frac{\partial f}{\partial x}$ and we see that the constant can depend on y .

$$f(x, y) = \int \frac{\partial f}{\partial x} dx = \int 3x^2y dx = x^3y + K(y)$$

With partial differentiation with respect of y and under consideration of $\frac{\partial f}{\partial y} = x^3$ we get

$$\frac{\partial f}{\partial y} = x^3 + K'(y) = x^3 \quad K'(y) = 0 \rightarrow K(y) = \text{const.} = C$$

3.5 Conservative vector field

Let $f : \mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a continuous vector field. The following are equivalent.

- (1) If for any $x_1, x_2 \in \mathcal{X}$ the line integral $\int_{\gamma} f \, ds$ is independent of the curve in \mathcal{X} from x_1 to x_2 , then the vector field f is conservative.
- (2) Any line integral of f around a closed curve is 0.
- (3) A potential for f exists.

We also have the following necessary but not sufficient condition

$$f \text{ is conservative} \Rightarrow \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

3.6 Path connected

Let $\mathcal{X} \subset \mathbf{R}^n$ be open. \mathcal{X} is said to be path connected if for every pair of points $x, y \in \mathcal{X}$ a C^1 path $\gamma : (0, 1] \rightarrow \mathcal{X}$ exists with $\gamma(0) = x, \gamma(1) = y$.

3.7 Changing the order of integration

Changing the order of integration is sometimes necessary, as in the following example.

$$\begin{aligned} \int_0^1 \int_x^1 e^{y^2} \, dy \, dx &= \int_0^1 \int_0^y e^{y^2} \, dx \, dy \\ &= \int_0^1 \left(x \cdot e^{y^2} \Big|_{x=0}^{x=y} \right) dy \\ &= \int_0^1 y \cdot e^{y^2} \, dy \\ &= \frac{e^{y^2}}{2} \Big|_0^1 \end{aligned}$$

3.8 Star shaped

A subset $\mathcal{X} \subset \mathbf{R}^n$ is called star shaped if $\exists x_0 \in \mathcal{X}$ such that $\forall x \in \mathcal{X}$ the line segment joining x_0 to x is contained in \mathcal{X} . Note

$$\text{Convex} \Rightarrow \text{Star shaped}$$

Further if \mathcal{X} is a star shaped open set of \mathbf{R}^n and $f \in C^1$ is a vector field s.t.

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \quad \forall i, j \Rightarrow f \text{ is conservative}$$

$$\text{curl}(f) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow f \text{ is conservative}$$

3.9 Curl

Let $\mathcal{X} \subset \mathbf{R}^3$ be open and $f : \mathcal{X} \rightarrow \mathbf{R}^3$ be a C^1 vector field. Then the curl of f is the vector field on \mathcal{X} defined by

$$\text{curl}(f) := \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

3.10 Partition

A partition P of a closed rectangle $Q = I_1 \times \dots \times I_n$ where $I_k = [a_k, b_k]$ is a subcollection of rectangular boxes $Q_1, \dots, Q_k \subset Q$ such that

$$(1) \quad Q = \bigcup_{j=1}^k Q_j$$

$$(2) \quad \text{Int } Q_i \cap \text{int } Q_j = \emptyset \quad \forall i \neq j$$

and $\text{Norm}(P) = \delta_P := \max(\text{diam } Q_j)$ while $\text{vol}(Q) = \prod_{i=1}^n (b_i - a_i)$

3.11 Riemann Sum

Riemann sum of f , for partition P , interlude point $\{\xi_i\}$ is the sum

$$R(f, P, \xi) = \sum_{j=1}^k f(\xi_i) \cdot \text{vol}(Q_j)$$

For the lower sum instead of $f(\xi_i)$ use $\inf_{x \in Q_j} f(x)$ and for upper sum $\sup_{x \in Q_j} f(x)$

3.12 Integrable

The lower Riemann sum equals the upper Riemann sum. We have for $f : \mathbf{R}^n \rightarrow \mathbf{R}$, Q rectangular boxes in \mathbf{R}^n

$$(1) \quad f \text{ is continuous on } Q \Rightarrow f \text{ is integrable}$$

$$(2) \quad f, g : Q \subset \mathbf{R}^n \rightarrow \mathbf{R} \text{ integrable, } \alpha, \beta \in \mathbf{R} \Rightarrow \alpha f + \beta g \text{ is integrable and equals}$$

$$\int_Q (\alpha f + \beta g) \, dx = \alpha \int_Q f \, dx + \beta \int_Q g \, dx$$

$$(3) \quad \text{If } f(x) \leq g(x) \quad \forall x \in Q \text{ then}$$

$$\int_Q f(x) \, dx \leq \int_Q g(x) \, dx$$

$$(4) \quad \text{if } f(x) \geq 0 \text{ then}$$

$$\int_Q f(x) \, dx \geq 0$$

$$(5) \quad \text{We have}$$

$$\begin{aligned} \left| \int_G f(x) \, dx \right| &\leq \int_Q |f(x)| \, dx \\ &\leq \left(\sup_Q |f(x)| \right) \cdot \text{vol}(Q) \end{aligned}$$

$$(6) \quad \text{If } f = 1 \text{ then}$$

$$\int_Q 1 \, dx = \text{vol}(Q)$$

3.13 Fubini's theorem

Let $Q = I_1 \times \dots \times I_n$ and f be continuous on Q . Then

$$\begin{aligned} \int_Q f(x_1, \dots, x_n) \, dx_1 \dots dx_n \\ = \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} f(x_1, \dots, x_n) \, dx_n \dots dx_1 \end{aligned}$$

Should the domain of integration be of the type $D_1 := \{(x, y) \mid a \leq x \leq b \text{ and } g(x) < y < h(x)\}$, then

$$\int_D f(x, y) \, dx \, dy = \int_a^b \int_{g(x)}^{h(x)} f(x, y) \, dy \, dx$$

If on the other hand $D_2 := \{(x, y) \mid c \leq y \leq d \text{ and } G(y) < x < H(y)\}$, then

$$\int_D f(x, y) \, dx \, dy = \int_c^d \int_{G(y)}^{H(y)} f(x, y) \, dx \, dy$$

3.14 Negligible sets in \mathbf{R}^n

If for $1 \leq m \leq n$ a parametrized m -set in \mathbf{R}^n is a continuous function

$$\varphi : [a_1, b_1] \times \dots \times [a_m, b_m]$$

which is C^1 on $(a_1, b_1) \times \dots \times (a_m, b_m)$, then a subset $Y \subset \mathbf{R}^n$ is negligible if there exist finitely many parametrized m_i -sets $\varphi_i : \mathcal{X}_i \rightarrow \mathbf{R}^n$ with $m_i < n$ such that

$$Y \subset \bigcup \varphi_i(\mathcal{X}_i)$$

This means in practice that when we split up an integral we don't need to worry about counting the bounds twice. We also have:

$$\text{If } Y \subset \mathbf{R}^n \text{ closed, bounded and negligible}$$

$$\Rightarrow \int_Y f \, dx_1 \dots dx_n = 0 \text{ for any } f$$

3.15 Improper Integrals

Let $f : \mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a non compact set and f a function such that $\int_K f \, dx$ exists for every compact set $K \subset \mathcal{X}$ and suppose $f \geq 0$. Finally we have a sequence of regions $\mathcal{X}_k \quad k = 1, 2, \dots$ s.t.

$$(1) \quad \text{Each region } \mathcal{X}_k \text{ is closed and bounded}$$

$$(2) \quad \mathcal{X}_k \subset \mathcal{X}_{k+1}$$

$$(3) \quad \bigcup_{k=1}^{\infty} \mathcal{X}_k = \mathcal{X}$$

then

$$\int_{\mathcal{X}} f \, dx := \lim_{n \rightarrow \infty} \int_{\mathcal{X}_n} f \, dx$$

3.16 Change of variables

Let $\varphi : \mathcal{X} \rightarrow Y$ be a continuous map, where $\mathcal{X} = \mathcal{X}_0 \cup B$, $Y = Y_0 \cup C$ are closed and bounded sets with \mathcal{X}_0 , Y_0 open, B , C negligible subsets of \mathbf{R}^n . Suppose $\varphi : \mathcal{X}_0 \rightarrow Y_0$ is C^1 and bijective with $\det J_\varphi(x) \neq 0 \quad \forall x \in \mathcal{X}_0$. Let $Y = \varphi(\mathcal{X})$. Suppose $f : Y \rightarrow \mathbf{R}$ is continuous, then

$$\int_Y f(y) dy = \int_{\varphi^{-1}(Y)} f(\varphi(x)) \cdot |\det J_\varphi(x)| dx$$

Here an example with polar coordinates on a quarter circle:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \end{pmatrix}$$

$$J = \begin{pmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{pmatrix}$$

$$\det(J) = r$$

$$dx dy = r dr d\theta$$

$$\begin{aligned} \int_{\mathcal{X}} \frac{dx dy}{1+x^2+y^2} &= \int_0^{\frac{\pi}{2}} \int_0^1 \frac{1}{1+r^2} \cdot r dr d\theta \\ &= \frac{\log(1+r^2)}{2} \Big|_0^1 \end{aligned}$$

Koordinatentransformationen in \mathbf{R}^2

Polarkoordinaten		
Definition	Maximaler Definitionsbereich	Volumenelement
$x = r \cos \varphi$	$0 \leq r < \infty$	$dx dy = r dr d\varphi$
$y = r \sin \varphi$	$0 \leq \varphi < 2\pi$	

↪ det J

Elliptische Koordinaten		
Definition	Maximaler Definitionsbereich	Volumenelement
$x = ra \cos \varphi$	$0 \leq r < \infty$	$dx dy = ab r dr d\varphi$
$y = rb \sin \varphi$	$0 \leq \varphi < 2\pi$	

Koordinatentransformationen in \mathbf{R}^3

Zylinderkoordinaten		
Definition	Maximaler Definitionsbereich	Volumenelement
$x = r \cos \varphi$	$0 \leq r < \infty$	$dx dy dz = r dr d\varphi dz$
$y = r \sin \varphi$	$0 \leq \varphi < 2\pi$	
$z = z$	$-\infty < z < \infty$	

Kugelkoordinaten		
Definition	Maximaler Definitionsbereich	Volumenelement
$x = r \sin \theta \cos \varphi$	$0 \leq r < \infty$	$dx dy dz = r^2 dr \sin \theta d\theta d\varphi$
$y = r \sin \theta \sin \varphi$	$0 \leq \theta \leq \pi$	
$z = r \cos \theta$	$0 \leq \varphi < 2\pi$	

3.17 Green's formula

Let \mathcal{X} be a closed and bounded region in \mathbf{R}^2 . Let γ be a curve forming the boundary of \mathcal{X} .

$$\int_{\mathcal{X}} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx dy = \int_{\gamma} f ds$$

where $f : (x, y) \rightarrow \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix}$. There are implicit assumptions.

- (1) We assume that the vector field $f = (f_1, f_2)$ has components f_1, f_2 s.t. $\frac{\partial f_2}{\partial x}, \frac{\partial f_1}{\partial y}$ exist in the region \mathcal{X} . The usual assumption is that if $f \in C^1$, then $\frac{\partial f_i}{\partial x}, \frac{\partial f_i}{\partial y} \quad i = 1, 2$ exist and are continuous so that $\text{curl}(f)$ is continuous. Thus the integral on the left side exists.
- (2) The region \mathcal{X} needs to be closed and bounded and that its boundary is a simple closed parametrized curve $\gamma : [a, b] \rightarrow \mathbf{R}^2$. (closed: $\gamma(a) = \gamma(b)$, simple: no knots)
- (3) \mathcal{X} is always to the left hand side of a tangent vector to the boundary (corners no problem).
- (4) Unions of simple closed curves also work (eg. doughnut). Then we would have

$$\int \int_{\mathcal{X}} \text{curl}(f) dx dy = \sum_{i=1}^k \int_{\gamma_i} f ds$$

If we wanted to calculate the area of a set, then handy functions with $\text{curl}(f) = 1$ are

$$f = (0, x) \text{ or } f = (-y, 0) \text{ or } f = \left(\frac{-y}{2}, \frac{x}{2} \right)$$

We also have

$$\int_{\gamma} f ds = \int_{\gamma_1} f ds + \int_{\gamma_2} f ds$$

3.17.1 Example

Straight forward application of Green's formula: if γ is a simple closed param. curve. Calculate

$$\int_{\gamma} f ds = \int_b^a \langle f(\gamma(t)), \gamma'(t) \rangle dt$$

γ simple closed parameter curve. Compute:

$$\int_{\partial A} f(x, y) dx dy \text{ for } f(x, y) = f : (x, y) \rightarrow \begin{pmatrix} \sqrt{1+x^3} \\ 2xy \end{pmatrix}$$

$\partial A = d_1 + d_2 + d_3$ Direct Computation:

$$\int_{\partial} A = \int_{\partial} d_1 + \int_{\partial} d_2 + \int_{\partial} d_3$$

Green's Formula:

$$A = (x, y) | 0 \leq x \leq 1, 0 \leq y \leq 3x$$

$$\partial x f_2 - \partial y f_1 = 2y - 0 = 2y$$

$$\int_{\partial A} f ds = \int_A 2y dx dy = \int_0^1 \int_0^{3x} 2y dy dx = \int_0^1 9x^2 dx = 3$$

3.17.2 Example

Calculate the area of $\Omega := (x, y) \in \mathbf{R}^2 | (x-2)^2 - 1 \leq y \leq 0$ with the Green's formula.

First, calculate intersection points:

$$\begin{aligned} (x-2)^2 - 1 &= 0 \\ &= x^2 - 4x + 3 \\ &= (x-3)(x-1) \end{aligned}$$

We parametrize:

$$\gamma_1 : [1, 3] \rightarrow \mathbf{R}^2 : t \rightarrow (t, (t-2)^2 - 1)$$

$$\gamma_2 : [3, 1] \rightarrow \mathbf{R}^2 : t \rightarrow (t, 0)$$

Note that is counter clockwise. We consider the vectorfield $v : \mathbf{R}^2 \rightarrow \mathbf{R}^2 : (x, y) \rightarrow (0, x)$. It is

$$\text{curl} v(x, y) = \frac{\partial v_y}{\partial x}(x, y) - \frac{\partial v_x}{\partial y}(x, y) = 1$$

$$\begin{aligned} \int \int_{\Omega} 1 dx dy &= \int \int_{\Omega} \text{curl} v dx dy = \int_{\gamma_1} v ds + \int_{\gamma_2} v ds \\ &= \int_1^3 v(\gamma_1(t)) \gamma_1'(t) dt + \int_3^1 v(\gamma_2(t)) \gamma_2'(t) dt \\ &= \int_1^3 (0, t)(1, 2(t-2)) dt + \int_3^1 (0, t)(1, 0) dt \\ &= \int_1^3 2t^2 - 4t dt = \frac{2}{3} t^3 - 2t^2 \Big|_1^3 \\ &= 18 - \frac{2}{3} - 18 + 2 = \frac{4}{3} \end{aligned}$$

3.18 Divergence

For a vector field $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ and $f \in C^1$, $f = (f_1, \dots, f_n)$ the divergence of f is defined by

$$\text{div} f = \frac{\partial f_1}{\partial x_1} + \dots + \frac{\partial f_n}{\partial x_n}$$

which for $n = 2$ we can calculate using Green's formula.

$$\tilde{f}(x, y) = (-f_2, f_1)$$

$$\text{curl}(\tilde{f}) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = \text{div}(f)$$

$$\begin{aligned} \int \int_{\mathcal{X}} \text{div}(f) dx dy &= \int \int_{\mathcal{X}} \text{curl}(\tilde{f}) dx dy = \int_{\partial \mathcal{X}} \tilde{f} ds \\ &= \int_a^b (f_1(\gamma(t)), f_2(\gamma(t))) \cdot (\gamma_2'(t), -\gamma_1'(t)) dt \\ &= \int_a^b (f_1(\gamma(t)), f_2(\gamma(t))) \cdot n(t) dt \end{aligned}$$

Here $n(t)$ is called the exterior normal to the curve and $\gamma'(t) \cdot n(t) = 0$.

3.19 Divergence-flux

The form or the normal form of Green’s theorem.

$$f : (f_1, f_2) : \mathcal{X} \rightarrow \mathbf{R}^2$$
$$\int \int_{\mathcal{X}} \operatorname{div}(f) \, dx \, dy = \int_{\partial \mathcal{X}} f \, d\vec{\mathbf{n}}$$

or

$$\int \int_{\mathcal{X}} \operatorname{curl}(f) \, dx \, dy = \int_{\partial \mathcal{X}} \vec{\mathbf{f}} \, d\vec{\mathbf{s}}$$

4 Other

4.1 Dreiecksungleichung

forall x,y in R : ||x| - |y|| <= |x +/- y| <= |x| + |y|

4.2 Bernoulli Ungleichung

forall x in R >= -1 und n in N : (1 + x)^n >= 1 + nx

4.3 Exponentialfunktion

exp(z) = lim_{n -> inf} (1 + z/n)^n

Die reelle Exponentialfunktion exp : R ->]0,inf[ist streng monoton wachsend, stetig und surjektiv. Es gelten weiter folgende Rechenregeln:

- 1. exp(x + y) = exp(x) * exp(y)
- 2. x^a := exp(a * ln(x))
- 3. x^0 = 1 forall x in R
- 4. exp(iz) = cos(z) + i * sin(z) forall z in C
- 5. exp(i * pi/2) = i
- 6. exp(i*pi) = -1 und exp(2*pi*i) = 1
- 7. F#r a > 0 ist]0,+inf[->]0,+inf[als x -> x^a eine streng monoton wachsende stetige Bijektion

Merke: e^x entspricht exp(x).

4.4 Nat#rliche Logarithmus

Der nat#rliche Logarithmus wir als ln :]0,inf[-> R bezeichnet und ist eine streng monoton wachsende stetige funktion. Es gilt auch, dass

- 1. ln(1) = 0
- 2. ln(e) = 1
- 3. ln(a * b) = ln(a) + ln(b)
- 4. ln(a/b) = ln(a) - ln(b)
- 5. ln(x^a) = a * ln(x)
- 6. x^a * x^b = x^{a+b}
- 7. (x^a)^b = x^{a*b}
- 8. ln(1 + x) = sum_{n=1}^inf ((-1)^{n-1} / n) * x^n (-1 < x <= 1)

4.5 Faktorisierung Lemma

a^n - b^n = (a - b)(a^{n-1} + ba^{n-2} + ... + b^{n-2}a + b^{n-1})

4.6 Sinus Absch#tzung

Es gilt |sin(x)| <= |x| mit folgendem Beweis:

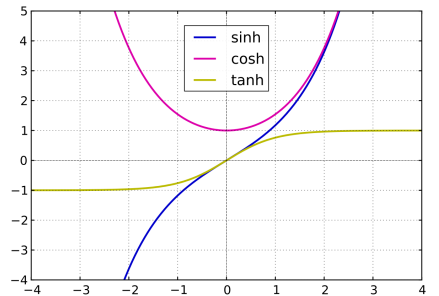
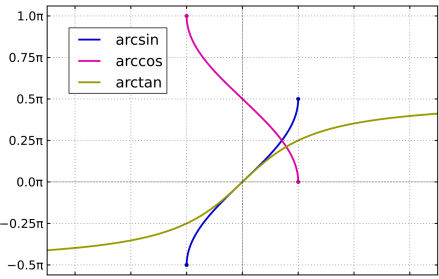
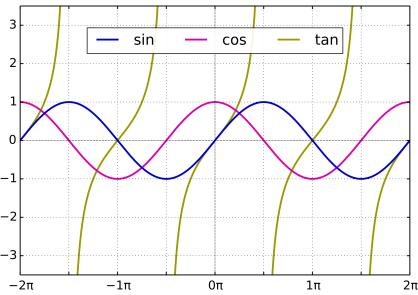
f(x) = x - sin(x), x >= 0
f'(x) = 1 - cos(x) >= 0

Weil f(0) = 0, f(x) >= 0 f#r x > 0. Dann |sin(x)| <= |x| einfach.

4.7 Trigonometrische Funktionen

exp(x) = sum_{n=0}^inf x^n / n! r = inf
sin(x) = sum_{n=0}^inf (-1)^n x^{2n+1} / (2n+1)! r = inf
cos(x) = sum_{n=0}^inf (-1)^n x^{2n} / (2n)! r = inf
ln(x+1) = sum_{k=1}^inf (-1)^{k+1} x^k / k r = 1

e^x = 1 + x + x^2/2 + x^3/3! + x^4/4! + O(x^5)
sin x = x - x^3/3! + x^5/5! + O(x^7)
sinh(x) = x + x^3/3! + x^5/5! + O(x^7)
cos(x) = 1 - x^2/2 + x^4/4! - x^6/6! + O(x^8)
cosh(x) = 1 + x^2/2 + x^4/4! + x^6/6! + O(x^8)
tan(x) = x + x^3/3 + 2x^5/15 + O(x^7)
tanh(x) = x - x^3/3 + 2x^5/15 + O(x^7)
log(1+x) = x - x^2/2 + x^3/3 - x^4/4 + O(x^5)
(1+x)^alpha = 1 + alpha x + alpha(alpha-1)/2! x^2 + alpha(alpha-1)(alpha-2)/3! x^3 + O(x^4)
sqrt(1+x) = 1 + x/2 - x^2/8 + x^3/16 - O(x^4)



- 1. cos(z) = cos(-z)
- 2. sin(-z) = -sin(z)
- 3. cos^2(z) + sin^2(z) = 1 forall z in C

4.8 Hyperbol Funktionen

- 1. cosh(x) := (e^x + e^-x) / 2 : R -> [1,inf]
- 2. sinh(x) := (e^x - e^-x) / 2 : R -> R
- 3. tanh(x) := (e^x - e^-x) / (e^x + e^-x) : R -> [-1,1]

und es gilt cosh^2(x) - sinh^2(x) = 1

4.9 Funktionen Verkn#pfung

x -> (g o f)(x) := g(f(x))

5 Trigonometrie

5.1 Regeln

5.1.1 Periodizit#t

- sin(alpha + 2pi) = sin(alpha) cos(alpha + 2pi) = cos(alpha)
- tan(alpha + pi) = tan(alpha) cot(alpha + pi) = cot(alpha)

5.1.2 Parit#t

- sin(-alpha) = -sin(alpha) cos(-alpha) = cos(alpha)
- tan(-alpha) = -tan(alpha) cot(-alpha) = -cot(alpha)

5.1.3 Ergänzung

- $\sin(\pi - \alpha) = \sin(\alpha) \quad \cos(\pi - \alpha) = -\cos(\alpha)$
- $\tan(\pi - \alpha) = -\tan(\alpha) \quad \cot(\pi - \alpha) = -\cot(\alpha)$

5.1.4 Komplemente

- $\sin(\pi/2 - \alpha) = \cos(\alpha) \quad \cos(\pi/2 - \alpha) = \sin(\alpha)$
- $\tan(\pi/2 - \alpha) = -\tan(\alpha) \quad \cot(\pi/2 - \alpha) = -\cot(\alpha)$

5.1.5 Doppelwinkel

- $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$
- $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 1 - 2\sin^2(\alpha)$
- $\tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$

5.1.6 Addition

- $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\tan(\alpha + \beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$

5.1.7 Subtraktion

- $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$
- $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
- $\tan(\alpha - \beta) = \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$

5.1.8 Multiplikation

- $\sin(\alpha)\sin(\beta) = -\frac{\cos(\alpha+\beta)-\cos(\alpha-\beta)}{2}$
- $\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$
- $\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$

5.1.9 Potenzen

- $\sin^2(\alpha) = \frac{1}{2}(1 - \cos(2\alpha))$
- $\cos^2(\alpha) = \frac{1}{2}(1 + \cos(2\alpha))$
- $\tan^2(\alpha) = \frac{1-\cos(2\alpha)}{1+\cos(2\alpha)}$

5.1.10 Diverse

- $\sin^2(\alpha) + \cos^2(\alpha) = 1$
- $\cosh^2(\alpha) - \sinh^2(\alpha) = 1$
- $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$ und $\cos(z) = \frac{e^{iz}+e^{-iz}}{2i}$
- $\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \forall z \notin \{\frac{\pi}{2} + \pi k\}$
- $\cot(x) = \frac{\cos(x)}{\sin(x)}$
- $\arcsin(x) = \sin(x)\cos(x)$
- $\cos(\arccos(x)) = x$
- $\sin(\arccos(x)) = \frac{1}{\sqrt{1-x^2}}$
- $\sin(\arctan(x)) = \frac{x}{\sqrt{x^2+1}}$
- $\cos(\arctan(x)) = \frac{1}{\sqrt{x^2+1}}$
- $\sin(x) = \frac{\tan(x)}{\sqrt{1+\tan(x)^2}}$
- $\cos(x) = \frac{1}{\sqrt{1+\tan(x)^2}}$

6 Tabellen

6.1 Ableitungen

F(x)	f(x)	f'(x)
$(x-1)e^x$	xe^x	$(x+1)e^x$
$\frac{x^{-a+1}}{-a+1}$	$\frac{1}{x^a}$	$\frac{a}{x^{a+1}}$
$\frac{x^{a+1}}{a+1}$	$x^a \ (a \neq -1)$	$a \cdot x^{a-1}$
$\frac{1}{k \ln(a)} a^{kx}$	a^{kx}	$ka^{kx} \ln(a)$
$\ln x $	$\frac{1}{x}$	$-\frac{1}{x^2}$
$\frac{2}{3}x^{3/2}$	\sqrt{x}	$\frac{1}{2\sqrt{x}}$
$-\cos(x)$	$\sin(x)$	$\cos(x)$
	$\frac{\sin(x)^2}{2}$	$\sin(x)\cos(x)$
$\sin(x)$	$\cos(x)$	$-\sin(x)$
$\frac{1}{2}(x - \frac{1}{2}\sin(2x))$	$\sin^2(x)$	$2\sin(x)\cos(x)$
$\tan(x) - x$	$\tan(x)^2$	$2\sec(x)^2 \tan(x)$
$-\cot(x) - x$	$\cot(x)^2$	$-2\cot(x)\csc(x)^2$
$\frac{1}{2}(x + \frac{1}{2}\sin(2x))$	$\cos^2(x)$	$-2\sin(x)\cos(x)$
$-\ln \cos(x) $	$\tan(x)$	$\frac{1}{\cos^2(x)}$ $1 + \tan^2(x)$
$\cosh(x)$	$\sinh(x)$	$\cosh(x)$
$\log(\cosh(x))$	$\tanh(x)$	$\frac{1}{\cosh^2(x)}$
$\ln \sin(x) $	$\cot(x)$	$-\frac{1}{\sin^2(x)}$
$\frac{1}{c} \cdot e^{cx}$	e^{cx}	$c \cdot e^{cx}$
$x(\ln x - 1)$	$\ln x $	$\frac{1}{x}$
$\frac{1}{2}(\ln(x))^2$	$\frac{\ln(x)}{x}$	$\frac{1-\ln(x)}{x^2}$
$\frac{x}{\ln(a)}(\ln x - 1)$	$\log_a x $	$\frac{1}{\ln(a)x}$

$\mathbf{F(x)}$	$\mathbf{f(x)}$
$\arcsin(x)/\arccos(x)$	$\frac{1/-1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$x \arcsin(x) + \sqrt{1-x^2}$	$\arcsin(x)$
$x \arccos(x) - \sqrt{1-x^2}$	$\arccos(x)$
$x \arctan(x) - \frac{1}{2} \ln(1+x^2)$	$\arctan(x)$
$\ln(\cosh(x))$	$\tanh(x)$
$x^x \ (x > 0)$	$x^x \cdot (1 + \ln x)$
$f(x)^{g(x)}$	$e^{g(x)\ln(f(x))}$
$f(x) = \cos(\alpha)$	$f(x)^n = \sin(x + n\frac{\pi}{2})$
$f(x) = \frac{1}{ax+b}$	$f(x)^n = (-1)^n * a^n * n! * (ax+b)^{-n+1}$
$-\ln(\cos(x))$	$\tan(x)$
$\ln(\sin(x))$	$\cot(x)$
$\ln(\tan(\frac{x}{2}))$	$\frac{1}{\sin(x)}$
$\ln(\tan(\frac{x}{2} + \frac{\pi}{4}))$	$\frac{1}{\cos(x)}$

$\mathbf{f(x)}$	$\mathbf{F(x)}$
$\int f'(x)f(x) \, dx$	$\frac{1}{2}(f(x))^2$
$\int \frac{f'(x)}{f(x)} \, dx$	$\ln f(x) $
$\int_{-\infty}^{\infty} e^{-x^2} \, dx$	$\sqrt{\pi}$
$\int (ax+b)^n \, dx$	$\frac{1}{a(n+1)}(ax+b)^{n+1}$
$\int x(ax+b)^n \, dx$	$\frac{(ax+b)^{n+2}}{(n+2)a^2} - \frac{b(ax+b)^{n+1}}{(n+1)a^2}$
$\int (ax^p+b)^n x^{p-1} \, dx$	$\frac{(ax^p+b)^{n+1}}{ap(n+1)}$
$\int (ax^p+b)^{-1} x^{p-1} \, dx$	$\frac{1}{ap} \ln ax^p+b $
$\int \frac{ax+b}{cx+d} \, dx$	$\frac{ax}{c} - \frac{ad-bc}{c^2} \ln cx+d $
$\int \frac{1}{x^2+a^2} \, dx$	$\frac{1}{a} \arctan \frac{x}{a}$
$\int \frac{1}{x^2-a^2} \, dx$	$\frac{1}{2a} \ln \left \frac{x-a}{x+a} \right $
$\int \sqrt{a^2+x^2} \, dx$	$\frac{x}{2}f(x) + \frac{a^2}{2} \ln(x+f(x))$

6.1.1 Potenzen der Winkelfunktion

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

6.1.2 Funktionen Verknüpfung

$$x \mapsto (g \circ f)(x) := g(f(x))$$

6.1.3 Häufungspunkt

$x_0 \in \mathbf{R}$ ist ein **Häufungspunkt** der Menge \mathbf{D} , falls $\forall \delta > 0 \quad (|x_0 - \delta, x_0 + \delta| \setminus \{x_0\}) \cap \mathbf{D} \neq \emptyset$

6.1.4 Ordinary differential equations (ODE's)

Given F , a function of x, y , and derivatives of y . Then an equation of the form

$$F(x, y, y', \dots, y^{(n)}) = 0$$

is an implicit ODE of order n . Order is determined by the highest derivative. Implicit means the equation equals 0.

6.1.5 Homogenous

A linear ODE is homogenous when $b(x) = 0$. Inhomogenous otherwise.

6.1.6 Vector Field

A function $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$.

6.1.7 Kritische Stelle

Eine **kritische Stelle** einer Funktion ist ein x_0 an der $f'(x_0)$ null oder undefiniert ist.

6.2 Important