

# Differential Equations

## 1.1 Linear ODE's

A differential equation is said to be linear if  $F$  can be written as a linear combination of the derivatives of  $y$ :

$$b(x) = \sum_{i=0}^n a_i(x) \cdot y^{(i)}$$

where  $a_i(x)$  and  $b(x)$  are continuous functions. Why is this called linear?

$$D = \frac{d^{(k)}}{dx^k} + a_{k-1} \frac{d^{(k-1)}}{dx^{k-1}} + \dots + a_0$$

$$D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)$$

$$Df_1 = b_1, Df_2 = b_2 \Rightarrow D(f_1 + f_2) = b_1 + b_2$$

## 1.2 Solution Space

Let  $I \subset \mathbf{R}$  be an open interval and  $k \geq 1$  an integer, and let

$$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = 0$$

be a linear ODE over  $I$  with continuous coefficients.

- (1) The set  $S$  of  $k$ -times differentiable solutions  $f : I \rightarrow \mathbb{C}$  of the equation is a complex vector space which is a subspace of the space of complex valued functions on  $I$ . (Analogous for real numbers, if all  $a_i$  are real valued)
- (2) The dimension of  $S$  is  $k$  and for any choice of  $x_0 \in I$  and any  $(y_0, \dots, y_{k-1}) \in \mathbb{C}^k$  there exists a unique  $f$  such that

$$f(x_0) = y_0, f'(x_0) = y_1, \dots, f^{(k-1)}(x_0) = y_{k-1}$$

(Analogous for real numbers, if all  $a_i$  are real)

- (3) For an arbitrary  $b$  the solution set is  $S_b = \{f + f_p \mid f \in S_0\}$  where  $f_p$  is a "particular" solution.
- (4) For any initial condition there is a unique solution.

## 1.3 Solving linear ODE's of order 1

$y' + ay = b$ . Here  $a, b$  are constant functions.

- (1) Find solutions of the corresponding homogenous equation  $y' + ay = 0$ . Note that if  $f$  is a solution so is  $z \cdot f \quad \forall z \in \mathbb{C}$ . Example:

$$y' + ay = 0$$

$$y' = -ay$$

$$\frac{y'}{y} = -a$$

$$\ln(y) = -\int a + C = -A + C$$

$$y = e^{-A+C} = z \cdot e^{-A} \quad z \in \mathbb{C}$$

- (2) Find a particular solution  $f_p : I \rightarrow \mathbb{C}$  such that  $f_p' + af_p = b$ . Use educated guess or variation of constants.

## 1.4 Educated Guess

- (1) If  $b(x)$  is a linear combination of basic functions listed here try the linear combination of educated guesses
- (2) If the educated guess is the same as the solution of the homogenous problem, then try multiplying by  $x^m$  where  $m$  denotes the multiplicity of the root  $\lambda$ .

$b(x)$	Guess
$ax^2 + bx$	$cx^2 + dx + e$
$a \cdot e^{\alpha x}$	$b \cdot e^{\alpha x}$
$a \sin(\beta x)$	$c \sin(\beta x) + d \cos(\beta x)$
$b \cos(\beta x)$	$c \sin(\beta x) + d \cos(\beta x)$
$ae^{\alpha x} \sin(\beta x)$	$e^{\alpha x} (c \sin(\beta x) + d \cos(\beta x))$
$be^{\alpha x} \cos(\beta x)$	$e^{\alpha x} (c \sin(\beta x) + d \cos(\beta x))$
$P_n(x)e^{\alpha x}$	$R_n(x) \cdot e^{\alpha x}$
$P_n(x)e^{\alpha x} \sin(\beta x)$	$e^{\alpha x} (R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))$
$P_n(x)e^{\alpha x} \cos(\beta x)$	$e^{\alpha x} (R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))$

### 1.4.1 Variation of constants

- (1) Assume  $f_p = z(x)e^{-A(x)}$  for some function  $z : I \rightarrow \mathbb{C}$
- (2) We plug this into the equation and see what it forces  $z$  to satisfy

$$f_p = z(x)e^{-A(x)} = y$$

$$y' = z'(x)e^{-A(x)} - z(x)A'(x)e^{-A(x)}$$

$$y' = e^{-A(x)} (z'(x) - z(x)a(x))$$

$$ay = a \cdot z(x)e^{-A(x)}$$

$$y' + ay = z'(x)e^{-A(x)} = b(x)$$

$$b(x) = z'(x)e^{A(x)}$$

$$z(x) = \int \frac{e^{A(x)}}{b(x)}$$

$$y_p = z(x)e^{-A(x)}$$

or for degree two

- (1) Assume the homogenous solution is  $f = z_1f_1 + z_2f_2$
- (2) We will try  $f_p = z_1(x)f_1 + z_2(x)f_2$
- (3) Solve the following system

$$z_1'(x)f_1 + z_2'(x)f_2 = 0$$

$$z_1'(x)f_1' + z_2'(x)f_2' = b$$

### 1.4.2 Solving Linear ODE's with constant coefficients

We want to solve

$$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_0y = b(x)$$

We assume our solution is  $e^{\lambda x}$ .

$$P(\lambda) = \lambda^k e^{\lambda x} + a_{k-1} \lambda^{k-1} e^{\lambda x} + \dots + a_0 e^{\lambda x} = 0$$

$$= e^{\lambda x} (\lambda^k + a_{k-1} \lambda^{k-1} + \dots + a_0) = 0$$

$$\Rightarrow 0 = \lambda^k + a_{k-1} \lambda^{k-1} + \dots + a_0$$

Which can then be solved for  $\lambda$ . Keep in mind that  $\lambda \in \mathbb{C}$  and we might be able to simplify with Euler's formula.

$$e^{ix} = \cos(x) + i \sin(x)$$

If there is a multiple root  $\alpha$  of multiplicity  $j$  we have

$$\text{Solutions: } e^{\alpha x}, xe^{\alpha x}, \dots, x^{j-1}e^{\alpha x}$$

## 1.5 Complex roots

If  $\alpha = \beta + \gamma i$  is a complex root of  $P(\lambda)$ , then so is  $\bar{\alpha} = \beta - \gamma i$ . Hence  $f_1 = e^{\alpha x}$  and  $f_2 = e^{\bar{\alpha} x}$  are solutions and can be replaced by a linear combination of  $\tilde{f}_1 = e^{\beta x} \cos(\gamma x)$  and  $\tilde{f}_2 = e^{\beta x} \sin(\gamma x)$ . Further if  $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_0y = 0$  has real coefficients, then each pair of complex conjugate roots  $\beta_j \pm \gamma_j i$  with multiplicity  $m_j$  leads to solution

$$x^l e^{\beta_j x} (\cos(\gamma_j x) + i \sin(\gamma_j x)) \quad \text{for } 0 \leq l \leq m_j$$

## 1.6 Separation of variables

A differential equation of order 1 is separable if it is of the form

$$y' = b(x)g(y)$$

$$\frac{dy}{dx} = b(x)g(y)$$

$$\frac{dy}{g(y)} = b(x)dx$$

$$\int \frac{dy}{g(y)} = \int b(x)dx$$

2    Differentials in  $\mathbf{R}^n$

2.1    Monomial

A monomial of degree  $e$  is a function

$$(x_1, \dots, x_n) \mapsto \alpha x_1^{d_1} \dots x_n^{d_n}$$
$$e = d_1 + \dots + d_n$$

2.2    Polynomial

A polynomial in  $n$  variables of degree  $\leq d$  is a finite sum of monomials of degree  $e \leq d$

2.3    Convergence

Let  $(x_k)_{k \in \mathbf{N}}$ ,  $x_k \in \mathbf{R}^n$  and  $x_k = (x_{k,1}, x_{k,2}, \dots, x_{k,n})$ . The following equivalently define  $\lim_{k \rightarrow \infty} x_k = y$ .

- (1)  $\forall \varepsilon > 0 \exists N \geq 1$  s.t.  $\forall k \geq N \quad \|x_k - y\| < \varepsilon$
- (2) For each  $i$ ,  $1 \leq i \leq n$  the sequence  $(x_{k,i})_k$  of real numbers converges to  $y_i$ .
- (3) The sequence of real numbers  $\|x_k - y\|$  converges to 0.

Let  $f : \mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbf{R}^m$  and  $x_0 \in \mathcal{X}$ ,  $y \in \mathbf{R}^m$ . We say  $f$  has a limit to  $y$  as  $x \rightarrow x_0$  where  $x \neq x_0$  if any of the following apply

- (1)  $\forall \varepsilon > 0 \exists \delta > 0$  s.t.  $\forall x \in \mathcal{X}$ ,  $x \neq x_0$  such that  $\|x - x_0\| < \delta$  we have  $\|f(x) - y\| < \varepsilon$ .
- (2)  $\forall$  sequences  $(x_k)$  in  $\mathcal{X}$  such that  $\lim x_k = x_0$  and  $x_k \neq x_0$  the sequence  $f(x_k)$  converges to  $y$ .

2.4    Continuity

Let  $f : \mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbf{R}^m$  and  $x_0 \in \mathcal{X}$ . We say  $f$  is continuous at  $x_0$  if any of the following apply

- (1)  $\forall \varepsilon > 0 \exists \delta > 0$  s.t. if  $x \in \mathcal{X}$  satisfies  $\|x - x_0\| < \delta$  then  $\|f(x) - f(x_0)\| < \varepsilon$ .
- (2)  $\forall$  sequences  $(x_k)$  in  $\mathcal{X}$  s.t.  $\lim x_k = x_0$  we have  $\lim f(x_k) = f(\lim x_k)$ .

$f$  is continuous in  $\mathcal{X}$  if  $f$  is continuous in every point  $x_0 \in \mathcal{X}$ . The following statements also hold

- (1)  $f(x = x_1, \dots, x_n) \mapsto (f_1(x), \dots, f_m(x))$  and  $f_i : \mathbf{R}^n \mapsto \mathbf{R}$  is continuous  $\Leftrightarrow f_i \forall i = 1, \dots, m$  are continuous.
- (2) Linear functions  $x \mapsto Ax$  are continuous.
- (3) Polynomials are continuous.
- (4) Sums, products of continuous functions are continuous.
- (5) Functions of separated variables are continuous if the factors are continuous.
- (6) Composition of continuous functions are continuous.

2.5    Sandwich lemma

If  $f, g, h : \mathbf{R}^n \rightarrow \mathbf{R}$  where  $f(x) < g(x) < h(x) \quad \forall x \in \mathbf{R}^n$ . Let  $a \in \mathbf{R}^n$ .

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L \Rightarrow \lim_{x \rightarrow a} g(x) = L$$

2.6    Polar Coordinates

It is sometimes helpful to use polar coordinates, especially with rational functions  $f : \mathbf{R} \rightarrow \mathbf{R}$ .  $f(x, y) = f(r \cos(\theta), r \sin(\theta))$

2.7    Bounded set

A set  $\mathcal{X} \subset \mathbf{R}^n$  is bounded if the set  $\{\|x\| \mid x \in \mathcal{X}\}$  is bounded in  $\mathbf{R}$ .

2.8    Closed set

A set  $\mathcal{X} \subset \mathbf{R}^n$  is closed if for every sequence  $(x_k)_{k \in \mathbf{N}} \subset \mathcal{X}$  that converges in  $\mathbf{R}^n$ , converges to a point  $y \in \mathcal{X}$ . Here it is often helpful to consider a ball. Counterexamples often include  $\frac{1}{k}$  and  $<$ .

2.9    Compact set

A compact set is a closed and bounded set.

2.10    Continuous and closed

If  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  is continuous, then for every  $Y \subset \mathbf{R}^m$  that is closed the set  $f^{-1}(Y) = \{x \in \mathbf{R}^n \mid f(x) \in Y\} \subset \mathbf{R}^n$  is closed. Careful: Does not imply bounded or compact!

2.11    Min-Max theorem

Let  $\mathcal{X} \subset \mathbf{R}^n$  be a compact set,  $f : \mathcal{X} \rightarrow \mathbf{R}$  a continuous function. Then  $f$  is bounded and attains its max and min.

$$f(x^+) = \sup_{x \in \mathcal{X}} f(x) f(x^-) = \inf_{x \in \mathcal{X}} f(x)$$

2.12    Open set

A set  $\mathcal{X} \subset \mathbf{R}^n$  is called open if its complement  $\mathbf{R}^n \setminus \mathcal{X}$  is closed. This is equivalent to  $\forall x \in \mathcal{X} \exists r > 0$  s.t. the set  $\{y \in \mathbf{R}^n \mid \|y - x\| < r\} = B_r(x) \subset \mathcal{X}$ . Here are some examples

- (1)  $(a, b) \subset \mathbf{R}$  is open.
- (2)  $[a, b) \subset \mathbf{R}$  is neither open nor closed.
- (3)  $\mathbf{R}^n$  and  $\emptyset$  are both open.
- (4)  $(a_1, b_1) \times (a_2, b_2) \subset \mathbf{R}^2$  is open.
- (5) Inverse image of open sets under continuous maps are open.

2.13    Derivative

Given  $f : \mathbf{R} \rightarrow \mathbf{R}^n$  the derivative is

$$f'(x_0) = \begin{pmatrix} f'_1(x_0) \\ \vdots \\ f'_n(x_0) \end{pmatrix}$$

2.14    Partial derivatives

A partial derivative of a function  $f : \mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbf{R}$  is obtained by pretending all but one variable are constants and then differentiating the one variable.

$$\frac{\partial f}{\partial x_{0,j}} = \lim_{h \rightarrow 0} \frac{f(x_{0,1}, \dots, x_{0,j} + h, \dots, x_{0,n}) - f(x_0)}{h}$$

If  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  for  $x_0 \in \mathbf{R}^n$  then

$$\frac{\partial f(x_0)}{\partial x_j} := \begin{pmatrix} \partial f_1(x_0) / \partial x_j \\ \vdots \\ \partial f_m(x_0) / \partial x_j \end{pmatrix}$$

Properties include (assuming partial derivatives for  $f, g$  exist w.r.t.  $x_j$ )

- (1)  $\frac{\partial f+g}{\partial x_j} = \frac{\partial f}{\partial x_j} + \frac{\partial g}{\partial x_j}$
- (2)  $\frac{\partial f \cdot g}{\partial x_j} = \frac{\partial f}{\partial x_j} \cdot g + \frac{\partial g}{\partial x_j} \cdot f$
- (3) if  $g \neq 0$ :  $\frac{\partial f/g}{\partial x_j} = \frac{\frac{\partial f}{\partial x_j} \cdot g - \frac{\partial g}{\partial x_j} \cdot f}{g^2}$

2.15    Jacobi Matrix

A Matrix with  $m$  rows and  $n$  columns where

$$J_f = \left( \frac{\partial f_i}{\partial x_j} \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$$

2.16    Gradient

The Jacobian of a function  $f : \mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbf{R}$ . Is often denoted as  $\nabla f$ . The geometric interpretation is that it indicates the direction and rate of fastest increase.

2.17    Directional derivative

Let direction  $v = (a, b) \neq (0, 0)$ . Instead of adding  $+h$  to one component we add  $+ah$ ,  $+bh$  and so on to all components and find that derivative as we normally would with a limit. Further, we have

$$\frac{df(x_0 + t\vec{v})}{dt} = J_f(x_0) \cdot \vec{v}$$

## 2.18 Differentiability

Let  $\mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbb{R}^p$  be function and  $x_0 \in \mathcal{X}$ . We say  $f$  is differentiable at  $x_0$  if a linear map  $u : \mathbf{R}^n \rightarrow \mathbf{R}$  exists such that

$$\lim_{x \rightarrow x_0, x \neq x_0} \frac{f(x) - f(x_0) - u(x - x_0)}{\|x - x_0\|} = 0$$

and  $u$  is called the total differential of  $f$  at  $x_0$ .

Further, if  $f, g$  are differentiable at  $x_0 \in \mathcal{X}$  we have

(1)  $f$  is continuous at  $x_0$

(2)  $f$  has all partial derivatives at  $x_0$  and the matrix represents the linear map  $df(x_0) : x \mapsto Ax$  in the canonical basis is given by the Jacobi Matrix of  $f$  at  $x_0$ , i.e.  $A = J_f(x_0)$

(3)  $d(f + g)(x_0) = df(x_0) + dg(x_0)$

(4) If  $m = 1$  and  $f, g : \mathbf{R}^n \rightarrow \mathbf{R}$  differentiable in  $x_0$  then so is  $f \cdot g$  and if  $g \neq 0$   $f/g$  as well.

Lastly we have

All partial derivatives  $\exists$  and cont.  $\Rightarrow f$  is differentiable

## 2.19 Tangent space

The approximation of the function at  $x_0$  using one derivative.

$$\{(x, y) \in \mathbf{R}^n \times \mathbf{R}^m \mid y = f(x_0) + u(x - x_0)\}$$

An example:

$$\begin{aligned} f(x, y) &= \sqrt{x^2 + y^2} \\ J_f &= \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right) \\ J_f(3, 4) &= \left( \frac{3}{5}, \frac{4}{5} \right) \\ \Rightarrow g(x, y) &= 5 + \left( \frac{3}{5}, \frac{4}{5} \right) \begin{pmatrix} x - 3 \\ y - 4 \end{pmatrix} \end{aligned}$$

## 2.20 Chain rule

Let  $\mathcal{X} \subset \mathbf{R}^n$  be open,  $\mathcal{Y} \subset \mathbf{R}^m$  be open and let  $f : \mathcal{X} \rightarrow \mathcal{Y}, g : \mathcal{Y} \rightarrow \mathbf{R}^p$  be differentiable functions. Then  $g \circ f = g(f) : \mathcal{X} \rightarrow \mathbf{R}^p$  is differentiable in  $\mathcal{X}$ . In particular

$$\begin{aligned} d(g \circ f)(x_0) &= dg(f(x_0)) \circ df(x_0) \\ J_{g \circ f}(x_0) &= J_g(f(x_0)) \cdot J_f(x_0) \end{aligned}$$

## 2.21 Change of variables

We say  $f$  is a change of variables around  $x_0$  if there is a radius  $\rho > 0$  s.t. the restriction of  $f$  to the Ball  $B = \{x \in \mathbf{R}^n \mid \|x - x_0\| < \rho\}$  so that the image  $Y = f(B)$  is open in  $\mathbf{R}^n$  and a differentiable map  $g : Y \rightarrow B$  exists, such that  $f \circ g = \text{id}_Y$  and  $g \circ f = \text{id}_B$ . I.e.

$f|_{B(x_0)}$  is a bijection to the image with a differentiable inverse  $g$

## 2.22 Inverse function theorem

Let  $\mathcal{X} \subseteq \mathbf{R}^n$  be open and  $f : \mathcal{X} \rightarrow \mathbf{R}^n$  differentiable. If  $x_0 \in \mathcal{X}$  is such that  $\det(J_f(x_0)) \neq 0$ , i.e.  $J_f(x_0)$  is invertible, then  $f$  is a change of variables around  $x_0$ . Moreover the Jacobian of  $g$  at  $x_0$  is defined by

$$J_g(f(x_0)) = J_f(x_0)^{-1}$$

## 2.23 Higher derivatives

Let  $\mathcal{X} \subset \mathbf{R}^n, f : \mathcal{X} \rightarrow \mathbf{R}^m$ . We say  $f$  is of class  $C'$  if  $f$  is differentiable on  $\mathcal{X}$  and all of its partial derivatives are continuous.

We say  $f \in C^k$  for  $k \geq 2$  if it is differentiable and each  $\partial_{x_i} f : \mathcal{X} \rightarrow \mathbf{R}^m$  is of class  $C^{k-1}$ . Further,  $f$  is smooth or  $C^\infty$  if  $f \in C^k \quad \forall k$ . Lastly: mixed partials (up to order  $k$ ) commute:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

## 2.24 Hessian

The  $n \times n$  symmetric matrix

$$\text{Hess}_f(x_0) := \left( \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} \right)$$

## 2.25 Taylor Polynomial

The Taylor polynomial of  $f$  at  $x_0$  of order 1 is

$$\begin{aligned} T_1(\vec{x}_0, \vec{y}_0) &:= f(\vec{x}_0) + \langle \nabla f(\vec{x}_0), \vec{y} \rangle \\ \vec{y} &= \vec{x} - \vec{x}_0 \\ \vec{x}_0 &= (x_0, y_0) \\ \vec{x} &= (x, y) \end{aligned}$$

and the second order

$$\begin{aligned} T_2(\vec{x}_0, \vec{y}_0) &:= f(\vec{x}_0) + \langle \nabla f(\vec{x}_0), \vec{y} \rangle \\ &\quad + \frac{1}{2} \vec{y} \cdot \text{Hess}_f(\vec{x}_0) \cdot \vec{y} \end{aligned}$$

Finally, the general form is

$$\begin{aligned} T_k f(y; x_0) &= f(x_0) + \dots \\ &+ \sum_{m_1 + \dots + m_n = k} \frac{1}{m_1! \dots m_n!} \frac{\partial^k f(x_0)}{\partial x_1^{m_1} \dots \partial x_n^{m_n}} \end{aligned}$$

Lastly if  $f \in C^k$  for  $x_0 \in \mathcal{X}$  we have

$$\begin{aligned} f(x) &= T_k(x - x_0; x_0) + E_k(f, x, x_0) \\ \lim_{x \rightarrow x_0} \frac{E_k(f, x, x_0)}{\|x - x_0\|^k} &\rightarrow 0 \end{aligned}$$

### 2.25.1 Example

Consider the following function:

$$f(x, y) := e^{x^2 + y^2} + \log(1 + x^2) + \arctan(xy)$$

a) determine the Taylor polynomial of  $f$  at  $(0, 0)$  up to and including third order.

$$\frac{\partial f(x, y)}{\partial x} = 2xe^{x^2 + y^2} + \frac{2x}{1 + x^2} + \frac{y}{1 + x^2 y^2}$$

$$\frac{\partial f(x, y)}{\partial y} = 2ye^{x^2 + y^2} + \frac{x}{1 + x^2 y^2}$$

Direct substitution gives us:

$$df(0, 0) = (0, 0)$$

We now calculate the partial derivatives of second order:

$$\frac{\partial^2 f(x, y)}{\partial x^2} = 2e^{x^2 + y^2} + 4x^2 e^{x^2 + y^2} + \frac{2(1 + x^2) - 4x^2}{(1 + x^2)^2} - \frac{2xy^3}{(1 + x^2 y^2)^2}$$

$$\frac{\partial^2 f(x, y)}{\partial x \partial y} = 4xye^{x^2 + y^2} + \frac{1 + x^2 y^2 - 2x^2 y^2}{(1 + x^2 y^2)^2}$$

$$\frac{\partial^2 f(x, y)}{\partial y^2} = 2e^{x^2 + y^2} + 4y^2 e^{x^2 + y^2} - \frac{2x^3 y}{(1 + x^2 y^2)^2}$$

We need the hessian so we have:

$$\text{Hess}_f(0, 0) = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

We now calculate the partial derivatives of third order. Luckily they all vanish so we have:

$$\begin{aligned} T_3 f((0, 0); (x, y)) &= f(0, 0) + \frac{\partial f(0, 0)}{\partial x} x + \frac{\partial f(0, 0)}{\partial y} y \\ &\quad + \frac{1}{2} \frac{\partial^2 f(0, 0)}{\partial x^2} x^2 + \frac{\partial^2 f(0, 0)}{\partial x \partial y} xy \\ &\quad + \frac{1}{2} \frac{\partial^2 f(0, 0)}{\partial y^2} y^2 + \frac{1}{6} \frac{\partial^3 f(0, 0)}{\partial x^3} x^3 \\ &\quad + \frac{1}{2} \frac{\partial^3 f(0, 0)}{\partial x^2 \partial y} x^2 y \\ &\quad + \frac{1}{2} \frac{\partial^3 f(0, 0)}{\partial x \partial y^2} xy^2 + \frac{1}{6} \frac{\partial^3 f(0, 0)}{\partial y^3} y^3 \\ &= 1 + 2x^2 + xy + y^2 \end{aligned}$$

## 2.26 Local max/min

Let  $f : \mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbf{R}$  be differentiable. We say  $x_0 \in \mathcal{X}$  is a local maximum (minimum) if we can find a neighborhood  $B_r(x_0) = \{x \in$

$$\mathbf{R}^n \mid \|x - x_0\| < r\} \subset \mathcal{X}$$

$$\forall x \in B_r(x_0) \quad f(x) \leq (\geq) f(x_0)$$

We also have

$$x_0 \in \mathcal{X} \text{ is a local extrema} \Rightarrow \nabla f(x_0) = 0$$

## 2.27 Global extrema

If  $f : \mathcal{X} \rightarrow \mathbf{R}$  is differentiable on the interior of  $\mathcal{X}$  and  $\mathcal{X}$  is closed and bounded, then a global extrema of  $f$  exists and it is either at a critical point or the boundary of  $\mathcal{X}$ .

$$\text{Check} = \text{int}(\mathcal{X}) \cup \text{bd}(\mathcal{X})$$

## 2.28 Definite

We have the following

- (1) Positive Definite: All eigenvalues are Positive
- (2) Negative Definite: All eigenvalues are Negative
- (3) Indefinite: Positive and Negative Eigenvalues

Eigenvalues can be found with the characteristic polynomial:

$$\det \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} \right) = \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \\ \Rightarrow \lambda^2 - 1 = 0$$

## 2.29 Calculating determinates

For 2 dimensions we have

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot d - c \cdot b$$

For 3 dimensions we have

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \cdot \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} \\ + c \cdot \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

## 2.30 Test critical point

A point is critical:  $x_0 \in \mathcal{X}$  where  $\nabla f(x_0) = 0$ .

Let  $f : \mathcal{X} \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$  and  $f \in C^2$ . Let  $x_0$  be a non-degenerate critical point of  $f$ . Then

- (1) If  $\text{Hess}_f(x_0)$  pos def. then  $x_0$  is a local minimum
- (2) If  $\text{Hess}_f(x_0)$  neg def. then  $x_0$  is a local maximum
- (3) If  $\text{Hess}_f(x_0)$  is Indefinite then  $x_0$  is a saddle point

We cannot use this theorem when  $x_0$  is a degenerate critical point ( $\det(\text{Hess}_f(x_0)) = 0$ ) and must decide on a case by case basis!

### 3 Integrals in $\mathbf{R}^n$

#### 3.1 Simple integral

For  $f : \mathbf{R} \rightarrow \mathbf{R}^n$  the integral is

$$\int_a^b f(t) dt = \begin{pmatrix} \int_a^b f_1(t) dt \\ \vdots \\ \int_a^b f_n(t) dt \end{pmatrix}$$

#### 3.2 Curve

The image of a function  $\gamma : [a, b] \rightarrow \mathbf{R}^n$  where the function  $\gamma$  is continuous and piecewise  $\in C^1$ .

#### 3.3 Line integral

Let  $\gamma : [a, b] \rightarrow \mathbf{R}^n$  be a parametrization of a curve and let  $\mathcal{X} \subset \mathbf{R}^n$  be a set which contains the image of  $\gamma$ . Further, let  $f : \mathcal{X} \rightarrow \mathbf{R}^n$  be a continuous function. A line integral then is

$$\int_{\gamma} f(s) d\vec{s} = \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt$$

The line integral has the following properties

- (1) It is independent of orientation preserving reparametrization, i.e.

$$\begin{aligned} \gamma &: [a, b] \rightarrow \mathbf{R}^n \\ \tilde{\gamma} &: [c, d] \rightarrow \mathbf{R}^n \\ \Phi &: [c, d] \rightarrow [a, b] \\ \tilde{\gamma} &= \gamma \circ \Phi = \gamma(\Phi) \\ \Rightarrow \int_{\gamma} f ds &= \int_{\tilde{\gamma}} f ds \end{aligned}$$

- (2) Let  $\gamma_1 + \gamma_2$  be the path formed by the concatenation of the two curves. Then

$$\begin{aligned} \gamma_1 + \gamma_2 &:= \begin{cases} \gamma_1(t) & t \in [a, b] \\ \gamma_2(t) & t \in [b, d+b-c] \end{cases} \\ \int_{\gamma_1 + \gamma_2} f ds &= \int_{\gamma_1} f ds + \int_{\gamma_2} f ds \end{aligned}$$

- (3) If  $\gamma : [a, b] \rightarrow \mathbf{R}^n$  is a path, let  $-\gamma$  be the path traced in the opposite direction, i.e.  $(-\gamma)(t) := \gamma(a+b-t)$ . Then

$$\int_{-\gamma} f ds = - \int_{\gamma} f ds$$

##### 3.3.1 Example

$$v(x, y) = \begin{pmatrix} x^2 - 2xy \\ y^2 - 2xy \end{pmatrix} \text{ from } (-1, 1) \text{ to } (1, 1) \text{ along the curve } y = x^2$$

$$\text{The given parametrization of the curve is } \gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

$$\text{and the derivative of } \gamma(t) \text{ is } \gamma'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}. \text{ The vector}$$

$$\text{field } v(\gamma(t)) \text{ is given by } v(\gamma(t)) = \begin{pmatrix} t^2 - 2t^3 \\ t^4 - 2t^3 \end{pmatrix}, \text{ and the dot}$$

product of  $v(\gamma(t))$  and  $\gamma'(t)$  is

$$[v(\gamma(t)) \cdot \gamma'(t) = (t^2 - 2t^3)(1) + (t^4 - 2t^3)(2t) = t^2 - 2t^3 + 2t^5 - 4t^4.]$$

The integral of  $v$  along the curve  $\gamma$  is

$$\begin{aligned} \int_{\gamma} v, d\gamma &= \int_{-1}^1 t^2 - 2t^3 + 2t^5 - 4t^4 dt \\ &= \left[ \frac{t^3}{3} - \frac{t^4}{2} + \frac{t^6}{3} - \frac{4t^5}{5} \right]_{t=-1}^1 \\ &= \frac{1}{3} - \frac{1}{2} + \frac{1}{3} - \frac{4}{5} - \left( -\frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{4}{5} \right) \\ &= \frac{2}{3} - \frac{8}{5} = -\frac{14}{15}. \end{aligned}$$

#### 3.4 Potential

A differentiable scalar field  $g : \mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbf{R}$  such that  $\nabla g = f$ ,  $f : \mathcal{X} \rightarrow \mathbf{R}^n$  is called a potential for  $f$ . This can make stuff easier:

$$\begin{aligned} \int_{\gamma} f ds &= \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt \\ &= \int_a^b \nabla g(\gamma(t)) \cdot \gamma'(t) dt \\ &= \int_a^b \frac{d}{dt} (g \circ \gamma) dt \\ &= (g \circ \gamma)(b) - (g \circ \gamma)(a) \end{aligned}$$

##### 3.4.1 Example

$f(x, y) = (2xy^2 - 5x^4y + 5, -7y^6 - x^5 + 2x^2y)$  is conservative and its potential is:

$$g(x, y) = x^2y^2 - x^5y + 5x - y^7$$

We want to compute  $\int_{\gamma} f \cdot ds$  where  $\gamma$  is the parametrised curve:

$$\gamma : \left[ \frac{\pi}{4}, \frac{5\pi}{4} \right] \rightarrow \mathbf{R}^2$$

$$\phi : \left[ \frac{1}{2} + \frac{1}{\sqrt{2}} \cos(t), \frac{1}{2} + \frac{1}{\sqrt{2}} \sin(t) \right]$$

So we have:

$$g\left(\psi\left(\frac{5\pi}{4}\right)\right) - g\left(\psi\left(\frac{\pi}{4}\right)\right) = g(0, 0) - g(1, 1) = -4$$

It should be noted that not every function has a potential! Example:

$$\begin{aligned} f(x, y) &= (2xy^2, 2x) \\ \frac{\partial g}{\partial x} &= 2xy^2 \Rightarrow g(x, y) = x^2y^2 + h(y) \\ \frac{\partial g}{\partial y} &= 2x \neq 2x^2y + h'(y) \end{aligned}$$

##### 3.4.2 Example

$$f(x, y) = \begin{pmatrix} 3x^2y \\ x^3 \end{pmatrix}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}(3x^2y) = 3x^2 \quad \frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}x^3 = 3x^2$$

If starshaped, integrability is guaranteed. The potential function is

$$\frac{\partial f}{\partial x} = (3x^2y) \quad \frac{\partial f}{\partial y} = x^3$$

We integrate  $\frac{\partial f}{\partial x}$  and we see that the constant can depend on  $y$ .

$$f(x, y) = \int \frac{\partial f}{\partial x} dx = \int 3x^2y dx = x^3y + K(y)$$

With partial differentiation with respect of  $y$  and under consideration of  $\frac{\partial f}{\partial y} = x^3$  we get

$$\frac{\partial f}{\partial y} = x^3 + K'(y) = x^3 \quad K'(y) = 0 \rightarrow K(y) = \text{const.} = C$$

### 3.5 Conservative vector field

Let  $f : \mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a continuous vector field. The following are equivalent.

- (1) If for any  $x_1, x_2 \in \mathcal{X}$  the line integral  $\int_{\gamma} f \, ds$  is independent of the curve in  $\mathcal{X}$  from  $x_1$  to  $x_2$ , then the vector field  $f$  is conservative.
- (2) Any line integral of  $f$  around a closed curve is 0.
- (3) A potential for  $f$  exists.

We also have the following necessary but not sufficient condition

$$f \text{ is conservative} \Rightarrow \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

### 3.6 Path connected

Let  $\mathcal{X} \subset \mathbf{R}^n$  be open.  $\mathcal{X}$  is said to be path connected if for every pair of points  $x, y \in \mathcal{X}$  a  $C^1$  path  $\gamma : (0, 1] \rightarrow \mathcal{X}$  exists with  $\gamma(0) = x, \gamma(1) = y$ .

### 3.7 Star shaped

A subset  $\mathcal{X} \subset \mathbf{R}^n$  is called star shaped if  $\exists x_0 \in \mathcal{X}$  such that  $\forall x \in \mathcal{X}$  the line segment joining  $x_0$  to  $x$  is contained in  $\mathcal{X}$ . Note

$$\text{Convex} \Rightarrow \text{Star shaped}$$

Further if  $\mathcal{X}$  is a star shaped open set of  $\mathbf{R}^n$  and  $f \in C^1$  is a vector field s.t.

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \quad \forall i, j \Rightarrow f \text{ is conservative}$$

$$\text{curl}(f) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow f \text{ is conservative}$$

### 3.8 Curl

Let  $\mathcal{X} \subset \mathbf{R}^3$  be open and  $f : \mathcal{X} \rightarrow \mathbf{R}^3$  be a  $C^1$  vector field. Then the curl of  $f$  is the vector field on  $\mathcal{X}$  defined by

$$\text{curl}(f) := \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

### 3.9 Partition

A partition  $P$  of a closed rectangle  $Q = I_1 \times \cdots \times I_n$  where  $I_k = [a_k, b_k]$  is a subcollection of rectangular boxes  $Q_1, \dots, Q_k \subset Q$  such that

- (1)  $Q = \bigcup_{j=1}^k Q_j$
- (2)  $\text{Int } Q_i \cap \text{int } Q_j = \emptyset \quad \forall i \neq j$

and  $\text{Norm}(P) = \delta_P := \max(\text{diam } Q_j)$  while  $\text{vol}(Q) = \prod_{i=1}^n (b_i - a_i)$

### 3.10 Riemann Sum

Riemann sum of  $f$ , for partition  $P$ , interlude point  $\{\xi_i\}$  is the sum

$$R(f, P, \xi) = \sum_{j=1}^k f(\xi_i) \cdot \text{vol}(Q_j)$$

For the lower sum instead of  $f(\xi_i)$  use  $\inf_{x \in Q_j} f(x)$  and for upper sum  $\sup_{x \in Q_j} f(x)$

### 3.11 Integrable

The lower Riemann sum equals the upper Riemann sum. We have for  $f : \mathbf{R}^n \rightarrow \mathbf{R}$ ,  $Q$  rectangular boxes in  $\mathbf{R}^n$

- (1)  $f$  is continuous on  $Q \Rightarrow f$  is integrable
- (2)  $f, g : Q \subset \mathbf{R}^n \rightarrow \mathbf{R}$  integrable,  $\alpha, \beta \in \mathbf{R} \Rightarrow \alpha f + \beta g$  is integrable and equals

$$\int_Q (\alpha f + \beta g) \, dx = \alpha \int_Q f \, dx + \beta \int_Q g \, dx$$

- (3) If  $f(x) \leq g(x) \quad \forall x \in Q$  then

$$\int_Q f(x) \, dx \leq \int_Q g(x) \, dx$$

- (4) if  $f(x) \geq 0$  then

$$\int_Q f(x) \, dx \geq 0$$

- (5) We have

$$\begin{aligned} \left| \int_G f(x) \, dx \right| &\leq \int_Q |f(x)| \, dx \\ &\leq \left( \sup_Q |f(x)| \right) \cdot \text{vol}(Q) \end{aligned}$$

- (6) If  $f = 1$  then

$$\int_Q 1 \, dx = \text{vol}(Q)$$

### 3.12 Fubini's theorem

Let  $Q = I_1 \times \cdots \times I_n$  and  $f$  be continuous on  $Q$ . Then

$$\begin{aligned} &\int_Q f(x_1, \dots, x_n) \, dx_1 \dots dx_n \\ &= \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \dots, x_n) \, dx_n \dots dx_1 \end{aligned}$$

Should the domain of integration be of the type  $D_1 := \{(x, y) \mid a \leq x \leq b \text{ and } g(x) < y < h(x)\}$ , then

$$\int_D f(x, y) \, dx \, dy = \int_a^b \int_{g(x)}^{h(x)} f(x, y) \, dy \, dx$$

If on the other hand  $D_2 := \{(x, y) \mid c \leq y \leq d \text{ and } G(y) < x < H(y)\}$ , then

$$\int_D f(x, y) \, dx \, dy = \int_c^d \int_{G(y)}^{H(y)} f(x, y) \, dx \, dy$$

### 3.13 Changing the order of integration

Changing the order of integration is sometimes necessary, as in the following example.

$$\begin{aligned} \int_0^1 \int_x^1 e^{y^2} \, dy \, dx &= \int_0^1 \int_0^y e^{y^2} \, dx \, dy \\ &= \int_0^1 \left( x \cdot e^{y^2} \Big|_{x=0}^{x=y} \right) \, dy \\ &= \int_0^1 y \cdot e^{y^2} \, dy \\ &= \frac{e^{y^2}}{2} \Big|_0^1 \end{aligned}$$

### 3.14 Negligible sets in $\mathbf{R}^n$

If for  $1 \leq m \leq n$  a parametrized  $m$ -set in  $\mathbf{R}^n$  is a continuous function

$$\varphi : [a_1, b_1] \times \cdots \times [a_m, b_m]$$

which is  $C^1$  on  $(a_1, b_1) \times \cdots \times (a_m, b_m)$ , then a subset  $Y \subset \mathbf{R}^n$  is negligible if there exist finitely many parametrized  $m_i$ -sets  $\varphi_i : \mathcal{X}_i \rightarrow \mathbf{R}^n$  with  $m_i < n$  such that

$$Y \subset \bigcup \varphi_i(\mathcal{X}_i)$$

This means in practice that when we split up an integral we don't need to worry about counting the bounds twice. We also have:

If  $Y \subset \mathbf{R}^n$  closed, bounded and negligible

$$\Rightarrow \int_Y f \, dx_1 \dots dx_n = 0 \text{ for any } f$$

3.15 Improper Integrals

Let  $f : \mathcal{X} \subset \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a non compact set and  $f$  a function such that  $\int_K f \, dx$  exists for every compact set  $K \subset \mathcal{X}$  and suppose  $f \geq 0$ . Finally we have a sequence of regions  $\mathcal{X}_k \quad k = 1, 2, \dots$  s.t.

- (1) Each region  $\mathcal{X}_k$  is closed and bounded
- (2)  $\mathcal{X}_k \subset \mathcal{X}_{k+1}$
- (3)  $\bigcup_{k=1}^\infty \mathcal{X}_k = \mathcal{X}$

then

∫\_X f dx := lim\_{n→∞} ∫\_{X\_n} f dx

3.16 Change of variables

Let  $\varphi : \mathcal{X} \rightarrow Y$  be a continuous map, where  $\mathcal{X} = \mathcal{X}_0 \cup B, Y = Y_0 \cup C$  are closed and bounded sets with  $\mathcal{X}_0, Y_0$  open,  $B, C$  negligible subsets of  $\mathbf{R}^n$ . Suppose  $\varphi : \mathcal{X}_0 \rightarrow Y_0$  is  $C^1$  and bijective with  $\det J_\varphi(x) \neq 0 \quad \forall x \in \mathcal{X}_0$ . Let  $Y = \varphi(\mathcal{X})$ . Suppose  $f : Y \rightarrow \mathbf{R}$  is continuous, then

∫\_Y f(y) dy = ∫\_{ϕ⁻¹(Y)} f(ϕ(x)) · |detJ\_ϕ(x)| dx

Here an example with polar coordinates on a quarter circle:

(x, y) = (r cos(θ), r sin(θ))
J = (cos(θ) -r sin(θ), sin(θ) r cos(θ))
det(J) = r
dx dy = r dr dθ
∫\_X dx dy / (1+x^2+y^2) = ∫\_0^{π/2} ∫\_0^1 1/(1+r^2) · r dr dθ
= log(1+r^2)/2 | 0^1

Table with 3 columns: Definition, Maximaler Definitionsbereich, Volumenelement. Rows include Polar coordinates, Elliptical coordinates, Cylindrical coordinates, and Spherical coordinates.

3.17 Green's formula

Let X be a closed and bounded region in R^2. Let γ be a curve forming the boundary of X.

∫\_X (∂f\_2/∂x - ∂f\_1/∂y) dx dy = ∫\_γ f ds

where f : (x, y) → (f\_1(x, y), f\_2(x, y)).

There are implicit assumptions.

- (1) We assume that the vector field f = (f1, f2) has components f1, f2 s.t. ∂f2/∂x, ∂f1/∂y exist in the region X. The usual assumption is that if f ∈ C^1, then ∂fi/∂x, ∂fi/∂y i = 1, 2 exist and are continuous so that curl(f) is continuous. Thus the integral on the left side exists.
- (2) The region X needs to be closed and bounded and that its boundary is a simple closed parametrized curve γ : [a, b] → R^2. (closed: γ(a) = γ(b), simple: no knots)
- (3) X is always to the left hand side of a tangent vector to the boundary (corners no problem).
- (4) Unions of simple closed curves also work (eg. doughnut). Then we would have

∫\_X curl(f) dx dy = ∑\_{i=1}^k ∫\_{γ\_i} f ds

If we wanted to calculate the area of a set, then handy functions with curl(f) = 1 are

f = (0, x) or f = (-y, 0)

We also have

∫\_γ f ds = ∫\_{γ1} f ds + ∫\_{γ2} f ds

3.18 Divergence

For a vector field f : R^n → R^n and f ∈ C^1, f = (f1, ..., fn) the divergence of f is defined by

div f = ∂f1/∂x1 + ... + ∂fn/∂xn

which for n = 2 we can calculate using Green's formula.

f-tilde(x, y) = (-f2, f1)
curl(f-tilde) = ∂f1/∂x + ∂f2/∂y = div(f)
∫\_X div(f) dx dy = ∫\_X curl(f) dx dy = ∫\_{∂X} f-tilde ds
= ∫\_a^b (f1(γ(t)), f2(γ(t))) · (γ'2(t), -γ'1(t)) dt
= ∫\_a^b (f1(γ(t)), f2(γ(t))) · n(t) dt

Here n(t) is called the exterior normal to the curve and γ'(t) · n(t) = 0.

3.19 Divergence-flux

The form or the normal form of Green's theorem.

f : (f1, f2) : X → R^2
∫\_X div(f) dx dy = ∫\_{∂X} f dñ
or
∫\_X curl(f) dx dy = ∫\_{∂X} f-tilde dñ

4 Other

4.1 Dreiecksungleichung

forall x, y in R : ||x| - |y|| <= |x +/- y| <= |x| + |y|

4.2 Bernoulli Ungleichung

forall x in R >= -1 und n in N : (1 + x)^n >= 1 + nx

4.3 Exponentialfunktion

exp(z) = lim\_{n -> inf} (1 + z/n)^n

Die reelle Exponentialfunktion exp : R -> ]0, inf[ ist streng monoton wachsend, stetig und surjektiv. Es gelten weiter folgende Rechenregeln:

- 1. exp(x + y) = exp(x) \* exp(y)
- 2. x^a := exp(a \* ln(x))
- 3. x^0 = 1 forall x in R
- 4. exp(iz) = cos(z) + i \* sin(z) forall z in C
- 5. exp(i \* pi/2) = i
- 6. exp(i\*pi) = -1 und exp(2\*pi\*i) = 1
- 7. F#r a > 0 ist ]0, +inf[ -> ]0, +inf[ als x -> x^a eine streng monoton wachsende stetige Bijektion

Merke: e^x entspricht exp(x).

4.4 Nat#rliche Logarithmus

Der nat#rliche Logarithmus wir als ln : ]0, inf[ -> R bezeichnet und ist eine streng monoton wachsende stetige funktion. Es gilt auch, dass

- 1. ln(1) = 0
- 2. ln(e) = 1
- 3. ln(a \* b) = ln(a) + ln(b)
- 4. ln(a/b) = ln(a) - ln(b)
- 5. ln(x^a) = a \* ln(x)
- 6. x^a \* x^b = x^{a+b}
- 7. (x^a)^b = x^{a\*b}
- 8. ln(1 + x) = sum\_{n=1}^inf ((-1)^{n-1} / n) \* x^n (-1 < x <= 1)

4.5 Faktorisierung Lemma

a^n - b^n = (a - b)(a^{n-1} + ba^{n-2} + ... + b^{n-2}a + b^{n-1})

4.6 Sinus Absch#tzung

Es gilt |sin(x)| <= |x| mit folgendem Beweis:

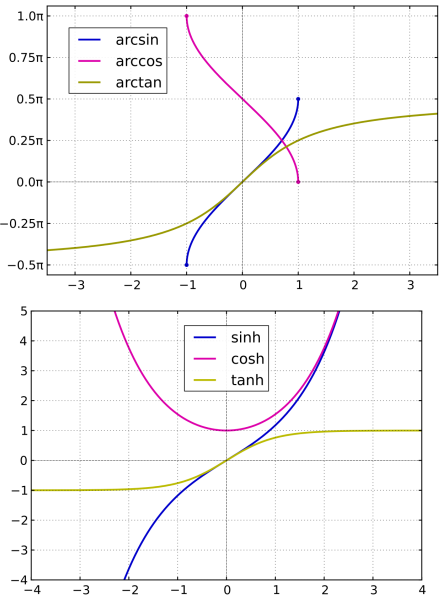
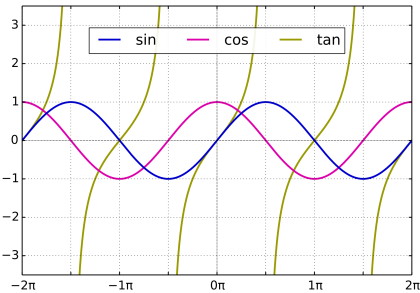
f(x) = x - sin(x), x >= 0  
f'(x) = 1 - cos(x) >= 0

Weil f(0) = 0, f(x) >= 0 f#r x > 0. Dann |sin(x)| <= |x| einfach.

4.7 Trigonometrische Funktionen

exp(x) = sum\_{n=0}^inf x^n / n! r = inf  
sin(x) = sum\_{n=0}^inf (-1)^n x^{2n+1} / (2n+1)! r = inf  
cos(x) = sum\_{n=0}^inf (-1)^n x^{2n} / (2n)! r = inf  
ln(x+1) = sum\_{k=1}^inf (-1)^{k+1} x^k / k r = 1

e^x = 1 + x + x^2/2 + x^3/3! + x^4/4! + O(x^5)  
sin x = x - x^3/3! + x^5/5! + O(x^7)  
sinh(x) = x + x^3/3! + x^5/5! + O(x^7)  
cos(x) = 1 - x^2/2 + x^4/4! - x^6/6! + O(x^8)  
cosh(x) = 1 + x^2/2 + x^4/4! + x^6/6! + O(x^8)  
tan(x) = x + x^3/3 + 2x^5/15 + O(x^7)  
tanh(x) = x - x^3/3 + 2x^5/15 + O(x^7)  
log(1+x) = x - x^2/2 + x^3/3 - x^4/4 + O(x^5)  
(1+x)^alpha = 1 + alpha x + alpha(alpha-1)/2! x^2 + alpha(alpha-1)(alpha-2)/3! x^3 + O(x^4)  
sqrt(1+x) = 1 + x/2 - x^2/8 + x^3/16 - O(x^4)



- 1. cos(z) = cos(-z)
- 2. sin(-z) = -sin(z)
- 3. cos^2(z) + sin^2(z) = 1 forall z in C

4.8 Hyperbol Funktionen

- 1. cosh(x) := (e^x + e^-x) / 2 : R -> [1, inf)
- 2. sinh(x) := (e^x - e^-x) / 2 : R -> R
- 3. tanh(x) := (e^x - e^-x) / (e^x + e^-x) : R -> [-1, 1]
- 4. cos(x) = (e^{ix} + e^{-ix}) / 2
- 5. cos(x) = (e^{ix} - e^{-ix}) / 2i

und es gilt cosh^2(x) - sinh^2(x) = 1

4.9 Funktionen Verkn#pfung

x -> (g o f)(x) := g(f(x))



## 5 Trigonometrie

### 5.1 Regeln

#### 5.1.1 Periodizität

- $\sin(\alpha + 2\pi) = \sin(\alpha) \quad \cos(\alpha + 2\pi) = \cos(\alpha)$
- $\tan(\alpha + \pi) = \tan(\alpha) \quad \cot(\alpha + \pi) = \cot(\alpha)$

#### 5.1.2 Parität

- $\sin(-\alpha) = -\sin(\alpha) \quad \cos(-\alpha) = \cos(\alpha)$
- $\tan(-\alpha) = -\tan(\alpha) \quad \cot(-\alpha) = -\cot(\alpha)$

#### 5.1.3 Ergänzung

- $\sin(\pi - \alpha) = \sin(\alpha) \quad \cos(\pi - \alpha) = -\cos(\alpha)$
- $\tan(\pi - \alpha) = -\tan(\alpha) \quad \cot(\pi - \alpha) = -\cot(\alpha)$

#### 5.1.4 Komplemente

- $\sin(\pi/2 - \alpha) = \cos(\alpha) \quad \cos(\pi/2 - \alpha) = \sin(\alpha)$
- $\tan(\pi/2 - \alpha) = -\tan(\alpha) \quad \cot(\pi/2 - \alpha) = -\cot(\alpha)$

#### 5.1.5 Doppelwinkel

- $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$
- $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 1 - 2\sin^2(\alpha)$
- $\tan(2\alpha) = \frac{2\tan(\alpha)}{1 - \tan^2(\alpha)}$

#### 5.1.6 Addition

- $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$

#### 5.1.7 Subtraktion

- $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$
- $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
- $\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$

#### 5.1.8 Multiplikation

- $\sin(\alpha)\sin(\beta) = -\frac{\cos(\alpha+\beta) - \cos(\alpha-\beta)}{2}$
- $\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha+\beta) + \cos(\alpha-\beta)}{2}$
- $\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2}$

#### 5.1.9 Potenzen

- $\sin^2(\alpha) = \frac{1}{2}(1 - \cos(2\alpha))$
- $\cos^2(\alpha) = \frac{1}{2}(1 + \cos(2\alpha))$
- $\tan^2(\alpha) = \frac{1 - \cos(2\alpha)}{1 + \cos(2\alpha)}$

#### 5.1.10 Diverse

- $\sin^2(\alpha) + \cos^2(\alpha) = 1$
- $\cosh^2(\alpha) - \sinh^2(\alpha) = 1$
- $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$  und  $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$
- $\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \forall x \notin \{\frac{\pi}{2} + \pi k\}$
- $\cot(x) = \frac{\cos(x)}{\sin(x)}$
- $\arcsin(x) = \sin(x)\cos(x)$
- $\cos(\arccos(x)) = x$
- $\sin(\arccos(x)) = \frac{1}{\sqrt{1-x^2}}$
- $\sin(\arctan(x)) = \frac{x}{\sqrt{x^2+1}}$
- $\cos(\arctan(x)) = \frac{1}{\sqrt{x^2+1}}$
- $\sin(x) = \frac{\tan(x)}{\sqrt{1+\tan(x)^2}}$
- $\cos(x) = \frac{1}{\sqrt{1+\tan(x)^2}}$

6 Tabellen

6.1 Ableitungen

<b>F(x)</b>	<b>f(x)</b>	<b>f'(x)</b>
$(x-1)e^x$	$xe^x$	$(x+1)e^x$
$\frac{x^{-a+1}}{-a+1}$	$\frac{1}{x^a}$	$\frac{a}{x^{a+1}}$
$\frac{x^{a+1}}{a+1}$	$x^a \ (a \neq -1)$	$a \cdot x^{a-1}$
$\frac{1}{k \ln(a)} a^{kx}$	$a^{kx}$	$ka^{kx} \ln(a)$
$\ln x $	$\frac{1}{x}$	$-\frac{1}{x^2}$
$\frac{2}{3}x^{3/2}$	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$-\cos(x)$	$\sin(x)$	$\cos(x)$
	$\frac{\sin(x)^2}{2}$	$\sin(x) \cos(x)$
$\sin(x)$	$\cos(x)$	$-\sin(x)$
$\frac{1}{2}(x - \frac{1}{2} \sin(2x))$	$\sin^2(x)$	$2 \sin(x) \cos(x)$
$\tan(x) - x$	$\tan(x)^2$	$2 \sec(x)^2 \tan(x)$
$-\cot(x) - x$	$\cot(x)^2$	$-2 \cot(x) \csc(x)^2$
$\frac{1}{2}(x + \frac{1}{2} \sin(2x))$	$\cos^2(x)$	$-2 \sin(x) \cos(x)$
$-\ln \cos(x) $	$\tan(x)$	$\frac{1}{\cos^2(x)}$ $1 + \tan^2(x)$
$\cosh(x)$	$\sinh(x)$	$\cosh(x)$
$\log(\cosh(x))$	$\tanh(x)$	$\frac{1}{\cosh^2(x)}$
$\ln \sin(x) $	$\cot(x)$	$-\frac{1}{\sin^2(x)}$
$\frac{1}{c} \cdot e^{cx}$	$e^{cx}$	$c \cdot e^{cx}$
$x(\ln x  - 1)$	$\ln x $	$\frac{1}{x}$
$\frac{1}{2}(\ln(x))^2$	$\frac{\ln(x)}{x}$	$\frac{1-\ln(x)}{x^2}$
$\frac{x}{\ln(a)}(\ln x  - 1)$	$\log_a x $	$\frac{1}{\ln(a)x}$

<b>F(x)</b>	<b>f(x)</b>
$\arcsin(x)/\arccos(x)$	$\frac{1/-1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$x \arcsin(x) + \sqrt{1-x^2}$	$\arcsin(x)$
$x \arccos(x) - \sqrt{1-x^2}$	$\arccos(x)$
$x \arctan(x) - \frac{1}{2} \ln(1+x^2)$	$\arctan(x)$
$\ln(\cosh(x))$	$\tanh(x)$
$x^x \ (x > 0)$	$x^x \cdot (1 + \ln x)$
$f(x)^{g(x)}$	$e^{g(x) \ln(f(x))}$
$f(x) = \cos(\alpha)$	$f(x)^n = \sin(x + n \frac{\pi}{2})$
$f(x) = \frac{1}{ax+b}$	$f(x)^n =$ $(-1)^n * a^n * n! * (ax+b)^{-n+1}$
$-\ln(\cos(x))$	$\tan(x)$
$\ln(\sin(x))$	$\cot(x)$
$\ln(\tan(\frac{x}{2}))$	$\frac{1}{\sin(x)}$
$\ln(\tan(\frac{x}{2} + \frac{\pi}{4}))$	$\frac{1}{\cos(x)}$

<b>f(x)</b>	<b>F(x)</b>
$\int f'(x)f(x) \, dx$	$\frac{1}{2}(f(x))^2$
$\int \frac{f'(x)}{f(x)} \, dx$	$\ln f(x) $
$\int_{-\infty}^{\infty} e^{-x^2} \, dx$	$\sqrt{\pi}$
$\int (ax+b)^n \, dx$	$\frac{1}{a(n+1)}(ax+b)^{n+1}$
$\int x(ax+b)^n \, dx$	$\frac{(ax+b)^{n+2}}{(n+2)a^2} - \frac{b(ax+b)^{n+1}}{(n+1)a^2}$
$\int (ax^p+b)^n x^{p-1} \, dx$	$\frac{(ax^p+b)^{n+1}}{ap(n+1)}$
$\int (ax^p+b)^{-1} x^{p-1} \, dx$	$\frac{1}{ap} \ln ax^p+b $
$\int \frac{ax+b}{cx+d} \, dx$	$\frac{ax}{c} - \frac{ad-bc}{c^2} \ln cx+d $
$\int \frac{1}{x^2+a^2} \, dx$	$\frac{1}{a} \arctan \frac{x}{a}$
$\int \frac{1}{x^2-a^2} \, dx$	$\frac{1}{2a} \ln \left  \frac{x-a}{x+a} \right $
$\int \sqrt{a^2+x^2} \, dx$	$\frac{x}{2}f(x) + \frac{a^2}{2} \ln(x+f(x))$

**6.1.1 Potenzen der Winkelfunktion**

$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$   
 $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

**6.1.2 Funktionen Verknüpfung**

$x \mapsto (g \circ f)(x) := g(f(x))$

**6.1.3 Häufungspunkt**

$x_0 \in \mathbf{R}$  ist ein **Häufungspunkt** der Menge  $\mathbf{D}$ , falls  $\forall \delta > 0 \quad (]x_0 - \delta, x_0 + \delta[ \setminus \{x_0\}) \cap \mathbf{D} \neq \emptyset$

**6.1.4 Ordinary differential equations (ODE's)**

Given  $F$ , a function of  $x, y$ , and derivatives of  $y$ . Then an equation of the form

$$F(x, y, y', \dots, y^{(n)}) = 0$$

is an implicit ODE of order  $n$ . Order is determined by the highest derivative. Implicit means the equation equals 0.

**6.1.5 Homogenous**

A linear ODE is homogenous when  $b(x) = 0$ . Inhomogenous otherwise.

**6.1.6 Vector Field**

A function  $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ .

**6.1.7 Kritische Stelle**

Eine **kritische Stelle** einer Funktion ist ein  $x_0$  an der  $f'(x_0)$  null oder undefiniert ist.