1 Differential Equations

1.1 Linear ODE's

A differential equation is said to be linear if F can be written as a linear combination of the derivatives of y:

$$b(x) = \sum_{i=0}^{n} a_i(x) \cdot y^{(i)}$$

where $a_i(x)$ and $b(x)$ are continuous functions. Why is this called linear?

$$D = \frac{d^{(k)}}{dx^k} + a_{k-1}\frac{d^{(k-1)}}{dx^{k-1}} + \cdots + a_0$$
$$D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)$$
$$Df_1 = b_1, Df_2 = b_2 \Rightarrow D(f_1 + f_2) = b_1 + b_2$$

1.2 Solution Space

Let $I \subset \mathbf{R}$ be an open interval and $k \geq 1$ an integer, and let

$$y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_1 y' + a_0 y = 0$$

be a linear ODE over I with continuous coefficients.

(1) The set S of k-times differentiable solutions $f : I \to \mathbb{C}$ of the equation is a complex vector space wich is a subspace of the space of complex valued funcitons on I. (Analogous for real numbers, if all a_i are real valued)

(2) The dimension of S is k and for any choice of $x_0 \in I$ and any $(y_0, \ldots, y_{k-1}) \in \mathbb{C}^k$ there exists a unique f such that

$$f(x_0) = y_0, f'(x_0) = y_1, \ldots, f^{(k-1)}(x_0) = y_{k-1}$$

(Analogous for real numbers, if all a_i are real)

(3) For an arbitrary b the solution set is $S_b = \{f + f_p \mid f \in S_0\}$ where f_p is a "particular" solution.

(4) For any initial condition there is a unique solution.

1.3 Solving linear ODE's of order 1

$y' + ay = b$. Here a, b are constant functions.

(1) Find solutions of the corresponding homogenous equation $y' + ay = 0$. Note that if f is a solution so is $z \cdot f \quad \forall z \in \mathbb{C}$. Example:

$$y' + ay = 0$$
$$y' = -ay$$
$$\frac{y'}{y} = -a$$
$$ln(y) = -\int a + C = -A + C$$
$$y = e^{-A+C} = z \cdot e^{-A} \quad z \in \mathbb{C}$$

(2) Find a particular solution $f_p : I \to \mathbb{C}$ such that $f_p' + af_p = b$. Use educated guess or variation of constants.

Assume we have $y' + \frac{y}{x} = 2\cos(x^2)$ The homogenous equation $y' = -\frac{1}{x}y$ has a constant solution $y_h(x) = 0$. Otherwise we have:

$$\log(y) = \int \frac{y'}{y}dx = -\int \frac{1}{x}dx = -\log(x) + c$$
$$y = \frac{e^c}{x}$$
$$y = \frac{C}{x}$$

Our educated guess is $y_p = \frac{C(x)}{x}$

$$\frac{C'(x)x - C(x)}{x^2} + \frac{1}{x}\frac{C(x)}{x} = 2\cos(x^2)$$

We solve for $C'(x)$

$$C'(x) = \frac{g(x)}{y_1(x)} \to C(x) = \int \frac{g(x)}{y_1(x)}dx = \int \frac{2\cos(x^2)}{\frac{1}{x}}$$
$$= \int 2x\cos(x^2) = \sin(x^2)$$

1.4 Educated Guess

(1) If $b(x)$ is a linear combination of basic functions listed here try the linear combination of educated guesses

(2) If the educated guess is the same as the solution of the homogenous problem, then try multiplying by x^m where m denotes the multiplicity of the root λ.

$b(x)$	Guess
$ax^2 + bx$	$cx^2 + dx + e$
$a \cdot e^{\alpha x}$	$b \cdot e^{\alpha x}$
$a\sin(\beta x)$	$c\sin(\beta x) + d\cos(\beta x)$
$b\cos(\beta x)$	$c\sin(\beta x) + d\cos(\beta x)$
$ae^{\alpha x}\sin(\beta x)$	$e^{\alpha x}\Big(c\sin(\beta x) + d\cos(\beta x)\Big)$
$be^{\alpha x}\cos(\beta x)$	$e^{\alpha x}\Big(c\sin(\beta x) + d\cos(\beta x)\Big)$
$P_n(x)e^{\alpha x}$	$R_n(x) \cdot e^{\alpha x}$
$P_n(x)e^{\alpha x}\sin(\beta x)$	$e^{\alpha x}\Big(R_n(x)\sin(\beta x) + S_n(x)\cos(\beta x)\Big)$
$P_n(x)e^{\alpha x}\cos(\beta x)$	$e^{\alpha x}\Big(R_n(x)\sin(\beta x) + S_n(x)\cos(\beta x)\Big)$

1.4.1 Variation of constants

(1) Assume $f_p = z(x)e^{-A(x)}$ for some function $z : I \to \mathbb{C}$

(2) We plug this into the equation and see what it forces z to satisfy

$$f_p = z(x)e^{-A(x)} = y$$
$$y' = z'(x)e^{-A(x)} - z(x)A'(x)e^{-A(x)}$$
$$y' = e^{-A(x)}\Big(z'(x) - z(x)a(x)\Big)$$
$$ay = a \cdot z(x)e^{-A(x)}$$
$$y' + ay = z'(x)e^{-A(x)} = b(x)$$
$$b(x) = z'(x)e^{A(x)}$$
$$z(x) = \int \frac{e^{A(x)}}{b(x)}$$
$$y_p = z(x)e^{-A(x)}$$

or for degree two

(1) Assume the homogenous solution is $f = z_1 f_1 + z_2 f_2$

(2) We will try $f_p = z_1(x)f_1 + z_2(x)f_2$

(3) Solve the following system

$$z_1'(x)f_1 + z_2'(x)f_2 = 0$$
$$z_1'(x)f_1' + z_2'(x)f_2' = b$$

1.4.2 Solving Linear ODE's with constant coefficients

We want to solve

$$y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_0 y = b(x)$$

We assume our solution is $e^{\lambda x}$.

$$
\begin{aligned}
P(\lambda) &= \lambda^k e^{\lambda x} + a_{k-1}\lambda^{k-1}e^{\lambda x} + \cdots + a_0 e^{\lambda x} = 0 \\
&= e^{\lambda x}\Big(\lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_0\Big) = 0 \\
\Rightarrow 0 &= \lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_0
\end{aligned}
$$

Which can then be solved for λ. Keep in mind that $\lambda \in \mathbb{C}$ and we might be able to simplify with Euler's formula.

$$e^{ix} = \cos(x) + i\sin(x)$$

If there is a multiple root α of multiplicity j we have

$$\text{Solutions: } e^{\alpha x},\ xe^{\alpha x},\ \ldots,\ x^{j-1}e^{\alpha x}$$

1.5 Complex roots

If $\alpha = \beta + \gamma i$ is a complex root of $P(\lambda)$, then so is $\bar{\alpha} = \beta - \gamma i$. Hence $f_1 = e^{\alpha x}$ and $f_2 = e^{\bar{\alpha} x}$ are solutions and can be replaced by a linear combination of $\tilde{f}_1 = e^{\beta x}\cos(\gamma x)$ and $\tilde{f}_2 = e^{\beta x}\sin(\gamma x)$. Further if $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_0 y = 0$ has real coefficients, then each pair of complex conjugate roots $\beta_j \pm \gamma_j i$ with multiplicity m_j leads to solution

$$x^l e^{\beta_j x}\Big(\cos(\gamma_j x) + i\sin(\gamma_j x)\Big) \quad \text{for } 0 \leq l \leq m_j$$

1.6 Separation of variables

A differential equation of oder 1 is separable if it is of the form

$$
\begin{aligned}
y' &= b(x)g(y) \\
\frac{dy}{dx} &= b(x)g(y) \\
\frac{dy}{g(y)} &= b(x)dx \\
\int \frac{dy}{g(y)} &= \int b(x)dx
\end{aligned}
$$

2 Differentials in $\mathbf{R}^n$

2.1 Monomial

A monomial of degree e is a function

$$(x_1, \ldots, x_n) \mapsto \alpha {x_1}^{d_1} \cdot \ldots \cdot {x_n}^{d_n}$$
$$e = d_1 + \ldots + d_n$$

2.2 Polynomial

A polynomial in n variables of degree $\leq d$ is a finite sum of monomials of degree $e \leq d$

2.3 Convergence

Let $(x_k)_{k\in\mathbb{N}}$, $x_k \in \mathbf{R}^n$ and $x_k = (x_{k,1}, x_{k,2}, \ldots, x_{k,n})$. The following equivalently define $\lim_{k\to\infty} x_k = y$.

(1) $\forall \varepsilon > 0\ \exists N \geq 1$ s.t. $\forall k \geq N \quad \|x_k - y\| < \varepsilon$

(2) For each i, $1 \leq i \leq n$ the sequence $(x_{k,i})_k$ of real numbers converges to y_i.

(3) The sequence of real numbers $\|x_k - y\|$ converges to 0.

Let $f : \mathcal{X} \subset \mathbf{R}^n \to \mathbf{R}^m$ and $x_0 \in \mathcal{X}$, $y \in \mathbf{R}^m$. We say f has a limit to y as $x \to x_0$ where $x \neq x_0$ if any of the following apply

(1) $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x \in \mathcal{X}$, $x \neq x_0$ such that $\|x - x_0\| < \delta$ we have $\|f(x) - y\| < \varepsilon$.

(2) $\forall$ sequences (x_k) in $\mathcal{X}$ such that $\lim x_k = x_0$ and $x_k \neq x_0$ the sequence $f(x_k)$ converges to y.

2.4 Continuity

Let $f : \mathcal{X} \subset \mathbf{R}^n \to \mathbf{R}^m$ and $x_0 \in \mathcal{X}$. We say f is continuous at x_0 if any of the following apply

(1) $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $x \in \mathcal{X}$ satisfies $\|x - x_0\| < \delta$ then $\|f(x) - f(x_0)\| < \varepsilon$.

(2) $\forall$ sequences (x_k) in $\mathcal{X}$ s.t. $\lim x_k = x_0$ we have $\lim f(x_k) = f(\lim x_k)$.

f is continuous in $\mathcal{X}$ if f is continuous in every point $x_0 \in \mathcal{X}$.
The following statements also hold

(1) $f(x = x_1, \ldots, x_n) \mapsto (f_1(x), \ldots, f_m(x))$ and $f_i : \mathbf{R}^n \mapsto \mathbf{R}$ is continuous $\Leftrightarrow f_i\ \forall i = 1, \ldots, m$ are continuous.

(2) Linear functions $x \mapsto Ax$ are continuous.

(3) Polynomials are continuous.

(4) Sums, products of continuous functions are continuous.

(5) Functions of separated variables are continuous if the factors are continuous.

(6) Composition of continuous functions are continuous.

2.5 Sandwich lemma

If $f, g, h : \mathbf{R}^n \to \mathbf{R}$ where $f(x) < g(x) < h(x) \quad \forall x \in \mathbf{R}^n$. Let $a \in \mathbf{R}^n$.

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \Rightarrow \lim_{x\to a} g(x) = L$$

2.6 Polar Coordinates

It is sometimes helpful to use polar coordinates, especially with rational functions $f : \mathbf{R} \to \mathbf{R}$. $f(x, y) = f(r\cos(\theta), r\sin(\theta))$

2.7 Bounded set

A set $\mathcal{X} \subset \mathbf{R}^n$ is bounded if the set $\{\|x\| \mid x \in \mathcal{X}\}$ is bounded in $\mathbf{R}$.

2.8 Closed set

A set $\mathcal{X} \subset \mathbf{R}^n$ is closed if for every sequence $(x_k)_{k\in\mathbb{N}} \subset \mathcal{X}$ that converges in $\mathbf{R}^n$, converges to a point $y \in \mathcal{X}$.
Here it is often helpful to consider a ball. Counterexamples often include $\frac{1}{k}$ and $<$.

2.9 Compact set

A compact set is a closed and bounded set.

2.10 Continuous and closed

If $f : \mathbf{R}^n \to \mathbf{R}^m$ is continuous, then for every $Y \subset \mathbf{R}^m$ that is closed the set $f^{-1}(Y) = \{x \in \mathbf{R}^n \mid f(x) \in Y\} \subset \mathbf{R}^n$ is closed. Careful: Does not imply bounded or compact!

2.11 Min-Max theorem

Let $\mathcal{X} \subset \mathbf{R}^n$ be a compact set, $f : \mathcal{X} \to \mathbf{R}$ a continuous function. Then f is bounded and attains its max and min.

$$f(x^+) = \sup_{x\in\mathcal{X}} f(x) f(x^-) \qquad = \inf_{x\in\mathcal{X}} f(x)$$

2.12 Open set

A set $\mathcal{X} \subset \mathbf{R}^n$ is called open if its complement $\mathbf{R}^n \setminus \mathcal{X}$ is closed. This is equivalent to $\forall x \in \mathcal{X}\ \exists r > 0$ s.t. the set $\{y \in \mathbf{R}^n \mid \|y - x\| < r\} = B_r(x) \subset \mathcal{X}$.
Here are some examples

(1) $(a, b) \subset \mathbf{R}$ is open.

(2) $[a, b) \subset \mathbf{R}$ is neither open nor closed.

(3) $\mathbf{R}^n$ and $\emptyset$ are both open.

(4) $(a_1, b_1) \times (a_2, b_2) \subset \mathbf{R}^2$ is open.

(5) Inverse image of open sets under continuous maps are open.

2.13 Derivative

Given $f : \mathbf{R} \to \mathbf{R}^n$ the derivative is

$$f'(x_0) = \begin{pmatrix} f_1'(x_0) \\ \vdots \\ f_n'(x_0) \end{pmatrix}$$

2.14 Partial derivatives

A partial derivative of a function $f : \mathcal{X} \subset \mathbf{R}^n \to \mathbf{R}$ is obtained by pretending all but one variable are constants and then differentiating the one variable.

$$\frac{\partial f}{\partial x_{0,j}} = \lim_{h\to 0} \frac{f(x_{0,1}, \ldots, x_{0,j} + h, \ldots, x_{0,n}) - f(x_0)}{h}$$

If $f : \mathbf{R}^n \to \mathbf{R}^m$ for $x_0 \in \mathbf{R}^n$ then

$$\frac{\partial f(x_0)}{\partial x_j} := \begin{pmatrix} \partial f_1(x_0)/\partial x_j \\ \vdots \\ \partial f_m(x_0)/\partial x_j \end{pmatrix}$$

Properties include (assuming partial derivatives for f, g exist w.r.t. x_j)

(1) $\frac{\partial f+g}{\partial x_j} = \frac{\partial f}{\partial x_j} + \frac{\partial g}{\partial x_j}$

(2) $\frac{\partial f\cdot g}{\partial x_j} = \frac{\partial f}{\partial x_j} \cdot g + \frac{\partial g}{\partial x_j} \cdot f$

(3) if $g \neq 0$: $\frac{\partial f/g}{\partial x_j} = \frac{\frac{\partial f}{\partial x_j} \cdot g - \frac{\partial g}{\partial x_j} \cdot f}{g^2}$

2.15 Jacobi Matrix

A Matrix with m rows and n columns where

$$J_f = \left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$$

2.16 Gradient

The Jacobian of a function $f : \mathcal{X} \subset \mathbf{R}^n \to \mathbf{R}$. Is often denoted as ∇f. The geometric interpretation is that it indicates the direction and rate of fastest increase.

2.17 Directional derivative

Let direction $v = (a, b) \neq (0, 0)$. Instead of adding $+h$ to one component we add $+ah$, $+bh$ and so on to all components and find that derivative as we normally would with a limit. Further, we have

$$\frac{df(x_0 + t\vec{\mathbf{v}})}{dt} = J_f(x_0) \cdot \vec{\mathbf{v}}$$

2.18 Differentiabiliy

Let $\mathcal{X} \subset \mathbf{R}^n \to \mathbb{R}^>$ be function and $x_0 \in \mathcal{X}$. We say f is differentiable at x_0 if a linear map $u : \mathbf{R}^n \to \mathbf{R}$ exists such that

$$\lim_{x \to x_0, x \neq x_0} \frac{f(x) - f(x_0) - u(x - x_0)}{\|x - x_0\|} = 0$$

and u is called the total differential of f at x_0.
Further, if f, g are differentiable at $x_0 \in \mathcal{X}$ we have

(1) f is continuous at x_0

(2) f has all partial derivatives at x_0 and the matrix represents the linear map $df(x_0) : x \mapsto Ax$ in the canonical basis is given by the Jacobi Matrix of f at x_0, i.e. $A = J_f(x_0)$

(3) $d(f + g)(x_0) = df(x_0) + dg(x_0)$

(4) If $m = 1$ and $f, g : \mathbf{R}^n \to \mathbf{R}$ differentiable in x_0 then so is $f \cdot g$ and if $g \neq 0$ f/g as well.

Lastly we have

All partial derivatives $\exists$ and cont. $\Rightarrow f$ is differentiable

2.19 Tangent space

The approximation of the function at x_0 using one derivative.

$$\{(x, y) \in \mathbf{R}^n \times \mathbf{R}^m \mid y = f(x_0) + u(x - x_0)\}$$

An example:

$$\begin{aligned} f(x, y) &= \sqrt{x^2 + y^2} \\ J_f &= \Big(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}\Big) \\ J_f(3, 4) &= (\frac{3}{5}, \frac{4}{5}) \\ \Rightarrow g(x, y) &= 5 + (\frac{3}{5}, \frac{4}{5}) \begin{pmatrix} x - 3 \\ y - 4 \end{pmatrix} \end{aligned}$$

2.20 Chain rule

Let $\mathcal{X} \subset \mathbf{R}^n$ be open, $\mathcal{Y} \subset \mathbf{R}^m$ be open and let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathbf{R}^p$ be differentiable functions. Then $g \circ f = g(f) : \mathcal{X} \to \mathbf{R}^p$ is differentiable in $\mathcal{X}$. In particular

$$\begin{aligned} d(g \circ f)(x_0) &= dg(f(x_0)) \circ df(x_0) \\ J_{g \circ f}(x_o) &= J_g(f(x_0)) \cdot J_f(x_0) \end{aligned}$$

2.21 Change of variables

We say f is a change of variables around x_0 if there is a radius $\rho > 0$ s.t. the restriction of f to the Ball $B = \{x \in \mathbf{R}^n \mid \|xx_0\| < \rho\}$ so that the image $Y = f(B)$ is open in $\mathbf{R}^n$ and a differentiable map $g : Y \to B$ exists, such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_B$. I.e.

$f\Big|_{B(x_0)}$ is a bijection to the image with a differentiable inverse g

2.22 Inverse function theorem

Let $\mathcal{X} \subseteq \mathbf{R}^n$ be open and $f : \mathcal{X} \to \mathbf{R}^n$ differentiable. If $x_0 \in \mathcal{X}$ is such that $det(J_f(x_0)) \neq 0$, i.e. $J_f(x_0)$ is invertible, then f is a change of variables around x_0. Moreover the Jacobian of g at x_0 is defined by

$$J_g(f(x_0)) = J_f(x_0)^{-1}$$

2.23 Higher derivatives

Let $\mathcal{X} \subset \mathbf{R}^n$, $f : \mathcal{X} \to \mathbf{R}^m$. We say f is of class C' if f is differentiable on $\mathcal{X}$ and all of its partial derivatives are continuous.
We say $f \in C^k$ for $k \geq 2$ if it is differentiable and each $\partial_{x_i} f : \mathcal{X} \to \mathbf{R}^m$ is of class C^{k-1}. Further, f is smooth or C^∞ if $f \in C^k \quad \forall k$. Lastly: mixed partials (up to order k) commute:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

2.24 Hessian

The $n \times n$ symmetric matrix

$$\mathrm{Hess}_f(x_0) := \Big(\frac{\partial^2 f(x_0)}{\partial x_i \partial x_j}\Big)$$

2.25 Taylor Polynomial

The Taylor polynomial of f at x_0 of order 1 is

$$\begin{aligned} T_1(\vec{x_0}, \vec{y_0} &:= f(\vec{x_0}) + \langle \nabla f(\vec{x_0}), \vec{y} \rangle \\ \vec{y} &= \vec{x} - \vec{x_0} \\ \vec{x_0} &= (x_0, y_0) \\ \vec{x} &= (x, y) \end{aligned}$$

and the second order

$$\begin{aligned} T_2(\vec{x_0}, \vec{y_0}) := & f(\vec{x_0}) + \langle \nabla f(\vec{x_0}), \vec{y} \rangle \\ & + \frac{1}{2} \vec{y} \cdot \mathrm{Hess}_f(\vec{x_0}) \cdot \vec{y}^t \end{aligned}$$

Finally, the general form is

$$\begin{aligned} T_k f(y; x_0) &= f(x_0) + \ldots \\ + \sum_{m_1 + \cdots + m_n = k} & \frac{1}{m_1! \ldots m_n!} \frac{\partial^k f(x_0) \cdot y_1^{m_1} \ldots y_n^{m_n}}{\partial x_1^{m_1} \ldots \partial x_n^{m_n}} \end{aligned}$$

Lastly if $f \in C^k$ for $x_0 \in \mathcal{X}$ we have

$$\begin{aligned} f(x) &= T_k(x - x_0; x_0) + E_k(f, x, x_0) \\ \lim_{x \to x_0} & \frac{E_k(f, x, x_0)}{\|x - x_0\|^k} \to 0 \end{aligned}$$

2.25.1 Example

Consider the following function:

$$f(x, y) := e^{x^2 + y^2} + log(1 + x^2) + \arctan(xy)$$

a) determine the Taylor plynomial of f at $(0, 0)$ up to and including third order.

$$\begin{aligned} \frac{\partial f(x, y)}{\partial x} &= 2xe^{x^2 + y^2} + \frac{2x}{1 + x^2} + \frac{y}{1 + x^2 y^2} \\ \frac{\partial f(x, y)}{\partial y} &= 2ye^{x^2 + y^2} + \frac{x}{1 + x^2 y^2} \end{aligned}$$

Direct substitution gives us:

$$df(0, 0) = (0, 0)$$

We now calculate the partial derivatives of second order:

$$\begin{aligned} \frac{\partial^2 f(x, y)}{\partial x^2} &= 2e^{x^2 + y^2} + 4x^2 e^{x^2 + y^2} + \frac{2(1 + x^2) - 4x^2}{(1 + x^2)^2} - \frac{2xy^3}{(1 + x^2 y^2)^2} \\ \frac{\partial^2 f(x, y)}{\partial x \partial y} &= 4xye^{x^2 + y^2} + \frac{1 + x^2 y^2 - 2x^2 y^2}{(1 + x^2 y^2)^2} \\ \frac{\partial^2 f(x, y)}{\partial y^2} &= 2e^{x^2 + y^2} + 4y^2 e^{x^2 + y^2} - \frac{2x^3 y}{(1 + x^2 y^2)^2} \end{aligned}$$

We need the hessian so we have:

$$Hess_f(0, 0) = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

We now calculate the partial derivatives of third order. Luckily they all vanish so we have:

$$\begin{aligned} T_3 f((0, 0); (x, y)) = & f(0, 0) + \frac{\partial f(0, 0)}{\partial x} x + \frac{\partial f(0, 0)}{\partial y} y \\ & + \frac{1}{2} \frac{\partial^2 f(0, 0)}{\partial x^2} x^2 + \frac{\partial^2 f(0, 0)}{\partial x \partial y} xy \\ & + \frac{1}{2} \frac{\partial^2 f(0, 0)}{\partial y^2} y^2 + \frac{1}{6} \frac{\partial^3 f(0, 0)}{\partial x^3} x^3 \\ & + \frac{1}{2} \frac{\partial^3 f(0, 0)}{\partial x^2 \partial y} x^2 y \\ & + \frac{1}{2} \frac{\partial^3 f(0, 0)}{\partial x \partial y^2} xy^2 + \frac{1}{6} \frac{\partial^3 f(0, 0)}{\partial y^3} y^3 \\ & = 1 + 2x^2 + xy + y^2 \end{aligned}$$

2.26 Local max/min

Let $f : \mathcal{X} \subset \mathbf{R}^n \to \mathbf{R}$ be differentiable. We say $x_0 \in \mathcal{X}$ is a local maximum (minimum) if we can find a neighborhood $B_r(x_0) = \{x \in$

$\mathbf{R}^n \mid \|x - x_0\| < r\} \subset \mathcal{X}$

$$\forall x \in B_r(x_0) \quad f(x) \leq (\geq) f(x_0)$$

We also have

$$x_0 \in \mathcal{X} \text{ is a local extrema } \Rightarrow \nabla f(x_0) = 0$$

2.27 Global extrema

If $f : \mathcal{X} \to \mathbf{R}$ is differentiable on the interior of $\mathcal{X}$ and $\mathcal{X}$ is closed and bounded, then a global extrema of f exists and it is either at a critical point or the boundary of $\mathcal{X}$.

$$\text{Check } = int(\mathcal{X}) \cup bd(\mathcal{X})$$

2.28 Definite

We have the following

(1) Positive Definite: All eigenvalues are Positive

(2) Negative Definite: All eigenvalues are Negative

(3) Indefinite: Positive and Negative Eigenvalues

Eigenvalues can be found with the characteristic polynomial:

$$det\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}\right) = det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}$$
$$\Rightarrow \lambda^2 - 1 = 0$$

2.29 Calculating determinates

For 2 dimensions we have

$$det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot d - c \cdot b$$

For 3 dimensions we have

$$det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \cdot det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot det\begin{pmatrix} d & f \\ g & i \end{pmatrix}$$
$$+ c \cdot det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

2.30 Test critical point

A point is critical: $x_0 \in \mathcal{X}$ where $\nabla f(x_0) = 0$.
Let $f : \mathcal{X} \subseteq \mathbf{R}^n \to \mathbf{R}$ and $f \in C^2$. Let x_0 be a non-degenerate critical point of f. Then

(1) If $\text{Hess}_f(x_0)$ pos def. then x_0 is a local minimum

(2) If $\text{Hess}_f(x_0)$ neg def. then x_0 is a local maximum

(3) If $\text{Hess}_f(x_0)$ is Indefinite then x_0 is a saddle point

We cannot use this theorem when x_0 is a degenerate critical point ($det(Hess_f(x_0)) = 0$) and must decide on a case by case basis!

3 Integrals in $\mathbf{R}^n$

3.1 Simple integral

For $f : \mathbf{R} \to \mathbf{R}^n$ the integral is

$$\int_a^b f(t)dt = \begin{pmatrix} \int_a^b f_1(t)dt \\ \vdots \\ \int_a^b f_n(t)dt \end{pmatrix}$$

3.2 Curve

The image of a function $\gamma : [a, b] \to \mathbf{R}^n$ where the function γ is continuous and piecewise $\in C^1$.

3.3 Line integral

Let $\gamma : [a, b] \to \mathbf{R}^n$ be a parametrization of a curve and let $\mathcal{X} \subset \mathbf{R}^n$ be a set which contains the image of γ. Further, let $f : \mathcal{X} \to \mathbf{R}^n$ be a continuous function. A line integral then is

$$\int_\gamma f(s)\, d\vec{s} = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$$

The line integral has the following properties

(1) It is independent of orientation preserving reparametrization, i.e.

$$\begin{aligned} \gamma &: [a,\, b] \to \mathbf{R}^n \\ \tilde{\gamma} &: [c,\, d] \to \mathbf{R}^n \\ \varPhi &: [c,\, d] \to [a,\, b] \\ \tilde{\gamma} &= \gamma \circ \varPhi = \gamma(\varPhi) \\ \Rightarrow \int_\gamma f\, ds &= \int_{\tilde{\gamma}} f\, ds \end{aligned}$$

(2) Let $\gamma_1 + \gamma_2$ be the path formed by the concatenation of the two curves. Then

$$\gamma_1 + \gamma_2 := \begin{cases} \gamma_1(t) & t \in [a,\, b] \\ \gamma_2(t) & t \in [b,\, d + b - c] \end{cases}$$

$$\int_{\gamma_1 + \gamma_2} f\, ds = \int_{\gamma_1} f\, ds + \int_{\gamma_2} f\, ds$$

(3) If $\gamma : [a,\, b] \to \mathbf{R}^n$ is a path, let $-\gamma$ be the path traced in the opposite direction, i.e. $(-\gamma)(t) := \gamma(a + b - t)$. Then

$$\int_{-\gamma} f\, ds = -\int_\gamma f\, ds$$

3.3.1 Length of curve (Bogenlänge)

The length of a curve (Bogenlänge) from a function f on the interval $[a, b]$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

3.3.2 Example

$v(x, y) = \begin{pmatrix} x^2 - 2xy \\ y^2 - 2xy \end{pmatrix}$ from $(-1, 1)$ to $(1, 1)$ along the curve $y = x^2$

The given parametrization of the curve is $\gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$ and the derivative of $\gamma(t)$ is $\gamma'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$. The vector field $v(\gamma(t))$ is given by $v(\gamma(t)) = \begin{pmatrix} t^2 - 2t^3 \\ t^4 - 2t^3 \end{pmatrix}$, and the dot product of $v(\gamma(t))$ and $\gamma'(t)$ is

$[v(\gamma(t)) \cdot \gamma'(t) = (t^2 - 2t^3)(1) + (t^4 - 2t^3)(2t) = t^2 - 2t^3 + 2t^5 - 4t^4.]$

The integral of v along the curve γ is

$$\begin{aligned} \int_\gamma v, d\gamma &= \int_{-1}^1 t^2 - 2t^3 + 2t^5 - 4t^4 dt \\ &= \left[\frac{t^3}{3} - \frac{t^4}{2} + \frac{t^6}{3} - \frac{4t^5}{5}\right]_{t=-1}^1 \\ &= \frac{1}{3} - \frac{1}{2} + \frac{1}{3} - \frac{4}{5} - \left(-\frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{4}{5}\right) \\ &= \frac{2}{3} - \frac{8}{5} = -\frac{14}{15}. \end{aligned}$$

3.4 Potential

A differentiable scalar field $g : \mathcal{X} \subset \mathbf{R}^n \to \mathbf{R}$ such that $\nabla g = f$, $f : \mathcal{X} \to R^n$ is called a potential for f. This can make stuff easier:

$$\begin{aligned} \int_\gamma f\, ds &= \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt \\ &= \int_a^b \nabla g(\gamma(t)) \cdot \gamma'(t)\, dt \\ &= \int_a^b \frac{d}{dt}(g \circ \gamma)\, dt \\ &= (g \circ \gamma)(b) - (g \circ \gamma)(a) \end{aligned}$$

3.4.1 Example

$f(x, y) = (2xy^2 - 5x^4y + 5, -7y^6 - x^5 + 2x^2y)$ is conservative and its potential is:

$$g(x, y) = x^2y^2 - x^5y + 5x - y^7$$

We want to compute $\int_\gamma f \cdot ds$ where γ is the parametrised curve:

$$\gamma : \left[\frac{\pi}{4}, \frac{5\pi}{4}\right] \to \mathbb{R}^2$$

$$\phi : \left[\frac{1}{2} + \frac{1}{\sqrt{2}} \cos(t), \frac{1}{2} + \frac{1}{\sqrt{2}} \sin(t)\right]$$

So we have:

$$g\left(\psi\left(\frac{5\pi}{4}\right)\right) - g\left(\psi\left(\frac{\pi}{4}\right)\right) = g(0, 0) - g(1, 1) = -4$$

It should be noted that not every function has a potential! Example:

$$\begin{aligned} f(x, y) &= (2xy^2, 2x) \\ \frac{\partial g}{\partial x} &= 2xy^2 \Rightarrow g(x, y) = x^2y^2 + h(y) \\ \frac{\partial g}{\partial y} &= 2x \neq 2x^2y + h'(y) \end{aligned}$$

3.4.2 Example

$$f(x, y) = \begin{pmatrix} 3x^2y \\ x^3 \end{pmatrix}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}(3x^2y) = 3x^2 \qquad \frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x} x^3 = 3x^2$$

If starshaped, integrability is guaranteed. The potential function is

$$\frac{\partial f}{\partial x} = (3x^2y) \qquad \frac{\partial f}{\partial y} = x^3$$

We integrate $\frac{\partial f}{\partial x}$ and we see that the consant can depent on y.

$$f(x, y) = \int \frac{\partial f}{\partial x} dx = \int 3x^2y dx = x^3y + K(y)$$

With partiel differentiation with respect of y and under consideration of $\frac{\partial f}{\partial y} = x^3$ we get

$$\frac{\partial f}{\partial y} = x^3 + K'(y) = x^3 \quad K'(y) = 0 \to K(y) = const. = C$$

3.5 Conservative vector field

Let $f : \mathcal{X} \subset \mathbf{R}^n \to \mathbf{R}^n$ be a continuous vector field. The following are equivalent.

(1) If for any $x_1, x_2 \in \mathcal{X}$ the line integral $\int_\gamma f\, ds$ is independent of the curve in $\mathcal{X}$ from x_1 to x_2, then the vector field f is conservative.

(2) Any line integral of f around a closed curve is 0.

(3) A potential for f exists.

We also have the following necessary but not sufficient condition

$$f \text{ is conservative} \quad \Rightarrow \quad \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

3.6 Path connected

Let $\mathcal{X} \subset \mathbf{R}^n$ be open. $\mathcal{X}$ is said to be path connected if for every pair of points $x, y \in \mathcal{X}$ a C^1 path $\gamma : (0, 1] : \to \mathcal{X}$ exists with $\gamma(0) = x$, $\gamma(1) = y$.

3.7 Star shaped

A subset $\mathcal{X} \subset \mathbf{R}^n$ is called star shaped if $\exists x_0 \in \mathcal{X}$ such that $\forall x \in \mathcal{X}$ the line segment joining x_0 to x is contained in $\mathcal{X}$. Note

$$\text{Convex} \quad \Rightarrow \quad \text{Star shaped}$$

Further if $\mathcal{X}$ is a star shaped open set of $\mathbf{R}^n$ and $f \in C^1$ is a vector field s.t.

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \quad \forall i, j \quad \Rightarrow \quad f \text{ is conservative}$$

$$curl(f) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad f \text{ is conservative}$$

3.8 Curl

Let $\mathcal{X} \subset \mathbf{R}^3$ be open and $f : \mathcal{X} \to \mathbf{R}^3$ be a C^1 vector field. Then the curl of f is the vector field on $\mathcal{X}$ defined by

$$curl(f) := \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

3.9 Partition

A partition P of a closed rectangle $Q = I_1 \times \cdots \times I_n$ where $I_k = [a_k, b_k]$ is a subcollection of rectangular boxes $Q_1, \ldots, Q_k \subset Q$ such that

(1) $Q = \bigcup_{j=1}^k Q_j$

(2) Int $Q_i \cap$ int $Q_j = \emptyset \quad \forall i \neq j$

and $Norm(P) = \delta_P := \max(\text{diam } Q_j)$ while $vol(Q) = \prod_{i=1}^n (b_i - a_i)$

3.10 Riemann Sum

Riemann sum of f, for partition P, interlude point $\{\xi_i\}$ is the sum

$$R(f, P, \xi) = \sum_{j=1}^k f(\xi_i) \cdot vol(Q_j)$$

For the lower sum instead of $f(\xi_i)$ use $\inf_{x \in Q_j} f(x)$ and for upper sum $\sup_{x \in Q_j} f(x)$

3.11 Integrable

The lower Riemann sum equals the upper Riemann sum. We have for $f : \mathbf{R}^n \to \mathbf{R}$, Q rectangular boxes in $\mathbf{R}^n$

(1) f is continuous on $Q \Rightarrow f$ is integrable

(2) $f, g : Q \subset \mathbf{R}^n \to \mathbf{R}$ integrable, $\alpha, \beta \in \mathbf{R} \Rightarrow \alpha f + \beta g$ is integrable and equals

$$\int_Q (\alpha f + \beta g)\, dx = \alpha \int_Q f\, dx + \beta \int_Q g\, dx$$

(3) If $f(x) \leq g(x) \quad \forall x \in Q$ then

$$\int_Q f(x)\, dx \leq \int_Q g(x)\, dx$$

(4) if $f(x) \geq 0$ then

$$\int_Q f(x)\, dx \geq 0$$

(5) We have

$$\begin{aligned} \left| \int_G f(x)\, dx \right| &\leq \int_Q |f(x)|\, dx \\ &\leq \left(\sup_Q |f(x)| \right) \cdot vol(Q) \end{aligned}$$

(6) If $f = 1$ then

$$\int_Q 1\, dx = vol(Q)$$

3.12 Fubini's theorem

Let $Q = I_1 \times \cdots \times I_n$ and f be continuous on Q. Then

$$\begin{aligned} &\int_Q f(x_1, \ldots, x_n) dx_1 \ldots dx_n \\ &= \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n) dx_n \ldots dx_1 \end{aligned}$$

Should the domain of integration be of the type $D_1 := \{(x, y) \mid a \leq x \leq b \text{ and } g(x) < y < h(x)\}$, then

$$\int_D f(x, y)\, dx\, dy = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx$$

If on the other hand $D_2 := \{(x, y) \mid c \leq y \leq d \text{ and } G(y) < x < H(y)\}$, then

$$\int_D f(x, y)\, dxdy = \int_c^d \int_{G(y)}^{H(y)} f(x, y)\, dx\, dy$$

3.13 Changing the order of integration

Changing the order of integration is sometimes necessary, as in the following example.

$$\begin{aligned} \int_0^1 \int_x^1 e^{y^2}\, dy\, dx &= \int_0^1 \int_0^y e^{y^2}\, dx\, dy \\ &= \int_0^1 \left(x \cdot e^{y^2} \Big|_{x=0}^{x=y} \right) dy \\ &= \int_0^1 y \cdot e^{y^2}\, dy \\ &= \frac{e^{y^2}}{2} \Big|_0^1 \end{aligned}$$

3.14 Negligible sets in $\mathbf{R}^n$

If for $1 \leq m \leq n$ a parametrized m-set in $\mathbf{R}^n$ is a continuous function

$$\varphi : [a_1, b_1] \times \cdots \times [a_m, b_m]$$

which is C^1 on $(a_1, b_1) \times \cdots \times (a_m, b_m)$, then a subset $Y \subset \mathbf{R}^n$ is negligible if there exist finitely many parametrized m_i-sets $\varphi_i : \mathcal{X}_i \to \mathbf{R}^n$ with $m_i < n$ such that

$$Y \subset \bigcup \varphi_i(\mathcal{X}_i)$$

This means in practice that when we split up an integral we don't need to worry about counting the bounds twice. We also have:

$$\begin{gathered} \text{If } Y \subset \mathbf{R}^n \text{ closed, bounded and negligible} \\ \Rightarrow \int_Y f\, dx_1 \ldots dx_n = 0 \text{ for any } f \end{gathered}$$

3.15 Improper Integrals

Let $f : \mathcal{X} \subset \mathbf{R}^n \to \mathbf{R}^n$ be a non compact set and f a function such that $\int_K f\, dx$ exists for every compact set $K \subset \mathcal{X}$ and suppose $f \geq 0$. Finally we have a sequence of regions $\mathcal{X}_k \quad k = 1, 2, \dots$ s.t.

(1) Each region $\mathcal{X}_k$ is closed and bounded

(2) $\mathcal{X}_k \subset \mathcal{X}_{k+1}$

(3) $\bigcup_{k=1}^{\infty} \mathcal{X}_k = \mathcal{X}$

then

$$\int_{\mathcal{X}} f\, dx := \lim_{n\to\infty} \int_{\mathcal{X}_n} f\, dx$$

3.16 Change of variables

Let $\varphi : \mathcal{X} \to Y$ be a continuous map, where $\mathcal{X} = \mathcal{X}_0 \cup B$, $Y = Y_0 \cup C$ are closed and bounded sets with $\mathcal{X}_0$, Y_0 open, B, C negligible subsets of $\mathbf{R}^n$. Suppose $\varphi : \mathcal{X}_0 \to Y_0$ is C^1 and bijective with $det J_\varphi(x) \neq 0 \quad \forall x \in \mathcal{X}_0$. Let $Y = \varphi(\mathcal{X})$. Suppose $f : Y \to \mathbf{R}$ is continuous, then

$$\int_Y f(y)\, dy = \int_{\varphi^{-1}(Y)} f(\varphi(x)) \cdot |det J_\varphi(x)|\, dx$$

Here an example with polar coordinates on a quarter circle:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix}$$

$$J = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix}$$

$$det(J) = r$$

$$dx\, dy = r\, dr\, d\theta$$

$$\int_{\mathcal{X}} \frac{dx\, dy}{1 + x^2 + y^2} = \int_0^{\frac{\pi}{2}} \int_0^1 \frac{1}{1+r^2} \cdot r\, dr\, d\theta$$

$$= \frac{\log(1+r^2)}{2} \Big|_0^1$$

Koordinatentransformationen in $\mathbb{R}^2$

Polarkoordinaten		
Definition	Maximaler Definitionsbereich	Volumenelement
$x = r\cos\varphi$ $y = r\sin\varphi$	$0 \leq r < \infty$ $0 \leq \varphi < 2\pi$	$dxdy = \underline{r}drd\varphi$

det Jg

Elliptische Koordinaten		
Definition	Maximaler Definitionsbereich	Volumenelement
$x = ra\cos\varphi$ $y = rb\sin\varphi$	$0 \leq r < \infty$ $0 \leq \varphi < 2\pi$	$dxdy = \underline{abr}drd\varphi$

Koordinatentransformationen in $\mathbb{R}^3$

Zylinderkoordinaten		
Definition	Maximaler Definitionsbereich	Volumenelement
$x = r\cos\varphi$ $y = r\sin\varphi$ $z = z$	$0 \leq r < \infty$ $0 \leq \varphi < 2\pi$ $-\infty < z < \infty$	$dxdydz = \underline{r}drd\varphi dz$

Kugelkoordinaten		
Definition	Maximaler Definitionsbereich	Volumenelement
$x = r\sin\theta\cos\varphi$ $y = r\sin\theta\sin\varphi$ $z = r\cos\theta$	$0 \leq r < \infty$ $0 \leq \theta \leq \pi$ $0 \leq \varphi < 2\pi$	$dxdydz = \underline{r^2}dr\,\underline{\sin\theta}d\theta d\varphi$

3.17 Green's formula

Let $\mathcal{X}$ be a closed and bounded region in $\mathbf{R}^2$. Let γ be a curve forming the boundary of $\mathcal{X}$.

$$\int\int_{\mathcal{X}} \Big(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\Big)\, dx\, dy = \int_\gamma f\, ds$$

where $f : (x, y) \to \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix}$.

There are implicit assumptions.

(1) We assume that the vector field $f = (f_1, f_2)$ has components f_1, f_2 s.t. $\frac{\partial f_2}{\partial x}$, $\frac{\partial f_1}{\partial y}$ exist in the region $\mathcal{X}$. The usual assumption is that if $f \in C^1$, then $\frac{\partial f_i}{\partial x}$, $\frac{\partial f_i}{\partial y} \quad i = 1, 2$ exist and are continuous so that $curl(f)$ is continuous. Thus the integral on the left side exists.

(2) The region $\mathcal{X}$ needs to be closed and bounded and that its boundary is a simple closed parametrized curve $\gamma : [a, b] \to \mathbf{R}^2$. (closed: $\gamma(a) = \gamma(b)$, simple: no knots)

(3) $\mathcal{X}$ is always to the left hand side of a tangent vector to the boundary (corners no problem).

(4) Unions of simple closed curves also work (eg. doughnut). Then we would have

$$\int\int_{\mathcal{X}} curl(f)\, dx\, dy = \sum_{i=1}^{k} \int_{\gamma_i} f\, ds$$

If we wanted to calculate the area of a set, then handy functions with $curl(f) = 1$ are

$$f = (0, x) \text{ or } f = (-y, 0)$$

We also have

$$\int_\gamma f\, ds = \int_{\gamma_1} f\, ds + \int_{\gamma_2} f\, ds$$

3.18 Divergence

For a vector field $f : \mathbf{R}^n \to \mathbf{R}^n$ and $f \in C^1$, $f = (f_1, \dots, f_n)$ the divergence of f is defined by

$$div\, f = \frac{\partial f_1}{\partial x_1} + \dots + \frac{\partial f_n}{\partial x_n}$$

which for $n = 2$ we can calculate using Green's formula.

$$\tilde{f}(x, y) = (-f_2, f_1)$$

$$curl(\tilde{f}) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = div(f)$$

$$\int\int_{\mathcal{X}} div(f)\, dx\, dy = \int\int_{\mathcal{X}} curl(f)\, dx\, dy = \int_{\partial\mathcal{X}} \tilde{f}\, ds$$

$$\int_a^b (f_1(\gamma(t)), f_2(\gamma(t))) \cdot (\gamma_2'(t), -\gamma_1'(t))\, dt$$

$$= \int_a^b (f_1(\gamma(t)), f_2(\gamma(t))) \cdot n(t)\, dt$$

Here $n(t)$ is called the exterior normal to the curve and $\gamma'(t) \cdot n(t) = 0$.

3.19 Divergence-flux

The form or the normal form of Green's theorem.

$$f : (f_1, f_2) : \mathcal{X} \to \mathbf{R}^2$$

$$\int\int_{\mathcal{X}} div(f)\, dx\, dy = \int_{\partial\mathcal{X}} f\, d\vec{\mathbf{n}}$$

or

$$\int\int_{\mathcal{X}} curl(f)\, dx\, dy = \int_{\partial\mathcal{X}} \vec{\mathbf{f}}\, d\vec{\mathbf{s}}$$

4 Other

4.1 Dreiecksungleichung

$$\forall x, y \in \mathbf{R} : ||x| - |y|| \leq |x \pm y| \leq |x| + |y|$$

4.2 Bernoulli Ungleichung

$$\forall x \in \mathbf{R} \geq -1 \text{ und } n \in \mathbf{N} : (1+x)^n \geq 1 + nx$$

4.3 Exponentialfunktion

$$exp(z) = \lim_{n\to\infty} (1 + \frac{z}{n})^n$$

Die reelle Exponentialfunktion $exp : \mathbf{R} \to]0, \infty[$ ist streng monoton wachsend, stetig und surjektiv.
Es gelten weiter folgende Rechenregeln:

1. $exp(x+y) = exp(x) * exp(y)$
2. $x^a := exp(a * ln(x))$
3. $x^0 = 1 \quad \forall x \in \mathbf{R}$
4. $exp(iz) = cos(z) + i * sin(z) \quad \forall z \in \mathbf{C}$
5. $exp(i * \frac{\pi}{2}) = i$
6. $exp(i\pi) = -1$ und $exp(2\pi i) = 1$
7. Für $a > 0$ ist $]0, +\infty[\to]0, +\infty[$ als $x \to x^a$ eine streng monoton wachsende stetige Bijektion

Merke: e^x entspricht $exp(x)$.

4.4 Natürliche Logaritmus

Der natürliche Logaritmus wir als $ln :]0, \infty[\to \mathbf{R}$ bezeichnet und ist eine streng monoton wachsende stetige funktion. Es gilt auch, dass

1. $ln(1) = 0$
2. $ln(e) = 1$
3. $ln(a * b) = ln(a) + ln(b)$
4. $ln(a/b) = ln(a) - ln(b)$
5. $ln(x^a) = a * ln(x)$
6. $x^a * x^b = x^{a+b}$
7. $(x^a)^b = x^{a*b}$
8. $ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \quad (-1 < x \leq 1)$

4.5 Faktorisierungs Lemma

$$a^n - b^n = (a-b)(a^{n-1} + ba^{n-2} + \cdots + b^{n-2}a + b^{n-1})$$

4.6 Sinus Abschätzung

Es gilt $|\sin(x)| \leq |x|$ mit folgendem Beweis:

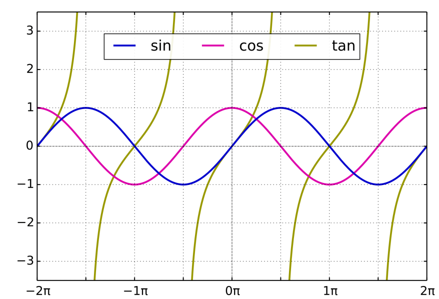
$$f(x) = x - \sin(x), x \geq 0$$
$$f'(x) = 1 - \cos(x) \geq 0$$

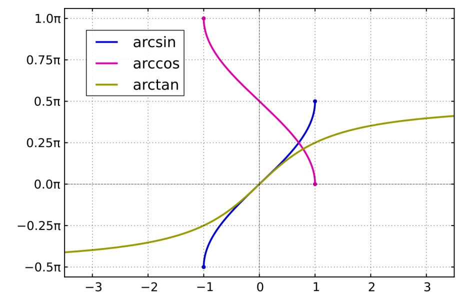
Weil $f(0) = 0$, $f(x) \geq 0$ für $x > 0$. Dann $|\sin(x)| \leq |x|$ einfach.

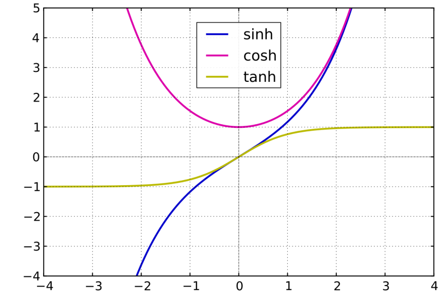
4.7 Trigonometrische Funktionen

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad r = \infty$$
$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad r = \infty$$
$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad r = \infty$$
$$\ln(x+1) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \qquad r = 1$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \mathcal{O}(x^5)$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7)$$
$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7)$$
$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \mathcal{O}(x^8)$$
$$\cosh(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!} + \mathcal{O}(x^8)$$
$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \mathcal{O}(x^7)$$
$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} + \mathcal{O}(x^7)$$
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \mathcal{O}(x^5)$$
$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \mathcal{O}(x^4)$$
$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \mathcal{O}(x^4)$$







1. $\cos(z) = \cos(-z)$
2. $\sin(-z) = -\sin(z)$
3. $\cos^2(z) + \sin^2(z) = 1 \quad \forall z \in \mathbf{C}$

4.8 Hyperbol Funktionen

1. $\cosh(x) := \frac{e^x + e^{-x}}{2} : \mathbf{R} \to [1, \infty]$
2. $\sinh(x) := \frac{e^x - e^{-x}}{2} : \mathbf{R} \to \mathbf{R}$
3. $\tanh(x) := \frac{e^x - e^{-x}}{e^x + e^{-x}} : \mathbf{R} \to [-1, 1]$
4. $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$
5. $\cos(x) = \frac{e^{ix} - e^{-ix}}{2i}$

und es gilt $\cosh^2(x) - \sinh^2(x) = 1$

4.9 Funktionen Verknüpfung

$x \mapsto (g \circ f)(x) := g(f(x))$

5 Trigonometrie

5.1 Regeln

5.1.1 Periodizität

- $\sin(\alpha + 2\pi) = \sin(\alpha) \quad \cos(\alpha + 2\pi) = \cos(\alpha)$
- $\tan(\alpha + \pi) = \tan(\alpha) \quad \cot(\alpha + \pi) = \cot(\alpha)$

5.1.2 Parität

- $\sin(-\alpha) = -\sin(\alpha) \quad \cos(-\alpha) = \cos(\alpha)$
- $\tan(-\alpha) = -\tan(\alpha) \quad \cot(-\alpha) = -\cot(\alpha)$

5.1.3 Ergänzung

- $\sin(\pi - \alpha) = \sin(\alpha) \quad \cos(\pi - \alpha) = -\cos(\alpha)$
- $\tan(\pi - \alpha) = -\tan(\alpha) \quad \cot(\pi - \alpha) = -\cot(\alpha)$

5.1.4 Komplemente

- $\sin(\pi/2 - \alpha) = \cos(\alpha) \quad \cos(\pi/2 - \alpha) = \sin(\alpha)$
- $\tan(\pi/2 - \alpha) = -\tan(\alpha) \quad \cot(\pi/2 - \alpha) = -\cot(\alpha)$

5.1.5 Doppelwinkel

- $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$
- $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 1 - 2\sin^2(\alpha)$
- $\tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$

5.1.6 Addition

- $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\tan(\alpha + \beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$

5.1.7 Subtraktion

- $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$
- $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
- $\tan(\alpha - \beta) = \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$

5.1.8 Multiplikation

- $\sin(\alpha)\sin(\beta) = -\frac{\cos(\alpha+\beta)-\cos(\alpha-\beta)}{2}$
- $\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$
- $\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$

5.1.9 Potenzen

- $\sin^2(\alpha) = \frac{1}{2}(1 - \cos(2\alpha))$
- $\cos^2(\alpha) = \frac{1}{2}(1 + \cos(2\alpha))$
- $\tan^2(\alpha) = \frac{1-\cos(2\alpha)}{1+\cos(2\alpha)}$

5.1.10 Diverse

- $\sin^2(\alpha) + \cos^2(\alpha) = 1$
- $\cosh^2(\alpha) - \sinh^2(\alpha) = 1$
- $\sin(z) = \frac{e^{iz}-e^{-iz}}{2}$ und $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$
- $\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \forall z \notin \{\frac{\pi}{2} + \pi k\}$
- $\cot(x) = \frac{\cos(x)}{\sin(x)}$
- $\arcsin(x) = \sin(x)\cos(x)$
- $\cos(\arccos(x)) = x$
- $\sin(\arccos(x)) = \frac{1}{\sqrt{1-x^2}}$
- $\sin(\arctan(x)) = \frac{x}{\sqrt{x^2+1}}$
- $\cos(\arctan(x)) = \frac{1}{\sqrt{x^2+1}}$
- $\sin(x) = \frac{\tan(x)}{\sqrt{1+\tan(x)^2}}$
- $\cos(x) = \frac{1}{\sqrt{1+\tan(x)^2}}$

6 Tabellen

6.1 Ableitungen

$\mathbf{F(x)}$	$\mathbf{f(x)}$	$\mathbf{f'(x)}$
$(x-1)e^x$	xe^x	$(x+1)e^x$
$\frac{x^{-a+1}}{-a+1}$	$\frac{1}{x^a}$	$\frac{a}{x^{a+1}}$
$\frac{x^{a+1}}{a+1}$	$x^a \ (a \neq -1)$	$a \cdot x^{a-1}$
$\frac{1}{k\ln(a)}a^{kx}$	a^{kx}	$ka^{kx}\ln(a)$
$\ln\lvert x\rvert$	$\frac{1}{x}$	$-\frac{1}{x^2}$
$\frac{2}{3}x^{3/2}$	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$-\cos(x)$	$\sin(x)$	$\cos(x)$
	$\frac{\sin(x)^2}{2}$	$\sin(x)\cos(x)$
$\sin(x)$	$\cos(x)$	$-\sin(x)$
$\frac{1}{2}(x-\frac{1}{2}\sin(2x))$	$\sin^2(x)$	$2\sin(x)\cos(x)$
$\tan(x)-x$	$\tan(x)^2$	$2\sec(x)^2\tan(x)$
$-\cot(x)-x$	$\cot(x)^2$	$-2\cot(x)\csc(x)^2$
$\frac{1}{2}(x+\frac{1}{2}\sin(2x))$	$\cos^2(x)$	$-2\sin(x)\cos(x)$
$-\ln\lvert\cos(x)\rvert$	$\tan(x)$	$\frac{1}{\cos^2(x)}$ $1+\tan^2(x)$
$\cosh(x)$	$\sinh(x)$	$\cosh(x)$
$\log(\cosh(x))$	$\tanh(x)$	$\frac{1}{\cosh^2(x)}$
$\ln\lvert\sin(x)\rvert$	$\cot(x)$	$-\frac{1}{\sin^2(x)}$
$\frac{1}{c}\cdot e^{cx}$	e^{cx}	$c\cdot e^{cx}$
$x(\ln\lvert x\rvert-1)$	$\ln\lvert x\rvert$	$\frac{1}{x}$
$\frac{1}{2}(\ln(x))^2$	$\frac{\ln(x)}{x}$	$\frac{1-\ln(x)}{x^2}$
$\frac{x}{\ln(a)}(\ln\lvert x\rvert-1)$	$\log_a\lvert x\rvert$	$\frac{1}{\ln(a)x}$

$\mathbf{F(x)}$	$\mathbf{f(x)}$
$\arcsin(x)/\arccos(x)$	$\frac{1/-1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$x\arcsin(x)+\sqrt{1-x^2}$	$\arcsin(x)$
$x\arccos(x)-\sqrt{1-x^2}$	$\arccos(x)$
$x\arctan(x)-\frac{1}{2}\ln(1+x^2)$	$\arctan(x)$
$\ln(\cosh(x))$	$\tanh(x)$
$x^x \ (x>0)$	$x^x\cdot(1+\ln x)$
$f(x)^{g(x)}$	$e^{g(x)ln(f(x))}$
$f(x)=cos(\alpha)$	$f(x)^n = sin(x+n\frac{\pi}{2})$
$f(x)=\frac{1}{ax+b}$	$f(x)^n =$ $(-1)^n * a^n * n! * (ax+b)^{-n+1}$
$-\ln(\cos(x))$	$\tan(x)$
$\ln(\sin(x))$	$\cot(x)$
$\ln(\tan(\frac{x}{2}))$	$\frac{1}{\sin(x))}$
$\ln(\tan(\frac{x}{2}+\frac{\pi}{4})$	$\frac{1}{cos(x)}$

$\mathbf{f(x)}$	$\mathbf{F(x)}$
$\int f'(x)f(x)\ \mathrm{d}x$	$\frac{1}{2}(f(x))^2$
$\int \frac{f'(x)}{f(x)}\ \mathrm{d}x$	$\ln\lvert f(x)\rvert$
$\int_{-\infty}^{\infty} e^{-x^2}\ \mathrm{d}x$	$\sqrt{\pi}$
$\int(ax+b)^n\ \mathrm{d}x$	$\frac{1}{a(n+1)}(ax+b)^{n+1}$
$\int x(ax+b)^n\ \mathrm{d}x$	$\frac{(ax+b)^{n+2}}{(n+2)a^2}-\frac{b(ax+b)^{n+1}}{(n+1)a^2}$
$\int(ax^p+b)^n x^{p-1}\ \mathrm{d}x$	$\frac{(ax^p+b)^{n+1}}{ap(n+1)}$
$\int(ax^p+b)^{-1}x^{p-1}\ \mathrm{d}x$	$\frac{1}{ap}\ln\lvert ax^p+b\rvert$
$\int\frac{ax+b}{cx+d}\ \mathrm{d}x$	$\frac{ax}{c}-\frac{ad-bc}{c^2}\ln\lvert cx+d\rvert$
$\int\frac{1}{x^2+a^2}\ \mathrm{d}x$	$\frac{1}{a}\arctan\frac{x}{a}$
$\int\frac{1}{x^2-a^2}\ \mathrm{d}x$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$
$\int\sqrt{a^2+x^2}\ \mathrm{d}x$	$\frac{x}{2}f(x)+\frac{a^2}{2}\ln(x+f(x))$

6.1.1 Potenzen der Winkelfunktion

$sin^2(x) = \frac{1}{2}(1-cos(2x))$
$cos^2(x) = \frac{1}{2}(1+cos(2x))$

6.1.2 Funktionen Verknüpfung

$x \mapsto (g\circ f)(x) := g(f(x))$

6.1.3 Häufungspunkt

$x_0 \in \mathbf{R}$ ist ein **Häufungspunkt** der Menge $\mathbf{D}$, falls $\forall\delta>0 \quad (]x_0-\delta, x_0+\delta[\setminus\{x_0\})\cap\mathbf{D}\neq\emptyset$

6.1.4 Ordinary differential equations (ODE's)

Given F, a function of x, y, and derivatives of y. Then an equation of the form

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

is an implicit ODE of order n. Order is determined by the highest derivative. Implicit means the equation equals 0.

6.1.5 Homogenous

A linear ODE is homogenous when $b(x) = 0$. Inhomogenous otherwise.

6.1.6 Vector Field

A function $f : \mathbf{R}^n \to \mathbf{R}^n$.

6.1.7 Kritische Stelle

Eine **kritische Stelle** einer Funktion ist ein x_0 an der $f'(x_0)$ null oder undefiniert ist.