6.9 Causality and Invertibility

Invertibility refers to the fact that the MA models can be written as an AR model. Or more generally, if $ARMA$ models can be written as AR models, we say that the time series model is invertible. The essential concept is whether the innovations/noises can be inverted into a representation of past observations.

Theorem

A linear process Y_t is **causal** (strictly, a causal function of ε_t) if there is a

$$\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots$$

with $$\sum_{j=0}^{\infty} |\psi_j| < \infty$$

and $$Y_t = \psi(B)\varepsilon_t$$

A linear process Y_t is **invertible** (strictly, an invertible function of ε_t) if there is a

$$\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \cdots$$

with $$\sum_{j=0}^{\infty} |\pi_j| < \infty$$

and $$\varepsilon_t = \pi(B) Y_t$$

Invertibility of AR Processes

All stationary ARs are invertible. However, not all AR models are stationary.

Theorem

A unique stationary solution to $\phi(B)Y_t = \varepsilon_t$ exists iff

$$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p = 0 \quad \Rightarrow \quad |B| \neq 1.$$

This $AR(p)$ process is causal iff

$$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p = 0 \quad \Rightarrow \quad |B| > 1.$$

Example

$AR(1)$, $\quad Y_t = \phi Y_{t-1} + \delta + \varepsilon_t$

$$Y_t - \phi Y_{t-1} = \varepsilon_t \qquad \text{(Let } \delta = 0\text{)}$$

$$(1-\phi B)Y_t = \varepsilon_t \qquad \text{(Recall that } B \text{ is the backshift operator: } BY_t = Y_{t-1}\text{)}$$

$$\phi(B)Y_t = \varepsilon_t \quad \text{where} \quad \phi(B) = 1 - \phi B$$

This is an $AR(1)$ model only if there is a stationary solution to $\phi(B)Y_t = \varepsilon_t$, which is equivalent to $|\phi| \neq 1$.

Also, we can rewrite

$AR(1)$, $\quad Y_t = \phi Y_{t-1} + \varepsilon_t \qquad$ (Let $\delta = 0$)

$$\begin{aligned}
Y_t &= \phi(\phi Y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\
&= \phi^2 Y_{t-2} + \varepsilon_t + \phi\varepsilon_{t-1} \\
&= \phi^2(\phi Y_{t-3} + \varepsilon_{t-2}) + \varepsilon_t + \phi\varepsilon_{t-1} \\
&= \phi^3 Y_{t-3} + \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} \\
&= \phi^n Y_{t-n} + \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \phi^3\varepsilon_{t-3} + \ldots + \phi^p\varepsilon_{t-p}
\end{aligned}$$

$$Y_t = \sum_{p=0}^{\infty} \phi^p \varepsilon_{t-p} = \sum_{p=0}^{\infty} \phi^p B^p \varepsilon_t$$

If $|\phi| < 1$, $\quad Y_t = \sum_{p=0}^{\infty} \phi^p \varepsilon_{t-p}$ has a unique stationary solution.

This infinite sum converges in mean square, since $|\phi| < 1$ implies $\sum |\phi^j| < \infty$. So, it satisfies the $AR(1)$ recurrence.

Alternatively, $1 - \phi B = 0 \Rightarrow |B| = \dfrac{1}{\phi} > 1$. This means that the roots of the characteristic equation, $\theta(B)$ lies outside of the unit circle.

If $|\phi| > 1$, $\quad \sum_{p=0}^{\infty} \phi^p \varepsilon_{t-p}$ does not converge.

Rearrange $Y_t = \phi Y_{t-1} + \varepsilon_t$ as $Y_{t-1} = \dfrac{1}{\phi} Y_t - \dfrac{1}{\phi} \varepsilon_t$, so that the unique stationary solution becomes $Y_t = -\sum_{p=1}^{\infty} \left(\dfrac{1}{\phi}\right)^p \varepsilon_{t+p} = -\sum_{p=1}^{\infty} \phi^{-p} \varepsilon_{t+p}$.

For this case, Y_t is a linear function of ε_t but it is not causal because Y_t depends on the **future** values of ε_t.

<u>Note:</u>
The condition that $\phi(B)$ have no roots on or inside the unit circle is implied by the requirement that the Yule-Walker equations have a solution.

<u>Invertibility of MA Processes</u>

The general MA is invertible if the equation

$$\theta(B) \equiv 1 - \theta_1 B - \theta_2 B^2 + \cdots + (-\theta_q) B^q = 0$$

has no solution on or inside the unit circle, means that the roots of $\theta(B)$ lies outside the unit circle. For any invertible MA, there is one (or several) non-invertible MA which has the same ACF.

<u>Example</u>
The $MA(1)$ process can be expressed in terms of lagged values of Y_t by substituting repeatedly for lagged values of ε_t. We have

$MA(1)$, $\quad Y_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1}$

$\varepsilon_t = Y_t + \theta_1 \varepsilon_{t-1}$ (Let $\mu = 0$)

$$\begin{aligned}
\varepsilon_t &= Y_t + \theta \varepsilon_{t-1} \\
&= Y_t + \theta(Y_{t-1} + \theta \varepsilon_{t-2}) \\
&= Y_t + \theta Y_{t-1} + \theta^2 \varepsilon_{t-2} \\
&= Y_t + \theta Y_{t-1} + \theta^2 (Y_{t-2} + \theta \varepsilon_{t-3}) \\
&= Y_t + \theta Y_{t-1} + \theta^2 Y_{t-2} + \theta^3 \varepsilon_{t-3} \\
&= Y_t + \theta Y_{t-1} + \theta^2 Y_{t-2} + \theta^3 Y_{t-3} + \ldots + \theta^n \varepsilon_{t-n} \\
&= \sum_{q=0}^{\infty} \theta^q Y_{t-q}
\end{aligned}$$

Also, we can write

$$\varepsilon_t = Y_t + \theta B Y_t + \theta^2 B^2 Y_{t-2} + \theta^3 B^3 Y_{t-3} + \ldots \qquad \text{(Backshift operator)}$$

$$= \frac{Y_t}{1-\theta B}, \qquad |\theta| < 1 \qquad \text{(Geometric series)}$$

This is a representation of another class of models, called infinite AR models. So, we inverted $MA(1)$ to an infinite AR. It was possible due to the assumption that $|\theta| < 1$. Such a process is called an **invertible process**. This is a desired property of time series, so in the example we would choose the model with $\sigma^2 = 25,\ \theta = \frac{1}{5}$.

If $|\theta| > 1$, $\sum_{q=0}^{\infty} \theta^q Y_{t-q}$ diverges.

Rearrange $\varepsilon_t = Y_t + \theta_1 \varepsilon_{t-1}$ as $\varepsilon_{t-1} = \frac{1}{\theta}\varepsilon_t - \frac{1}{\theta} Y_t$.

Unique stationary solution becomes $\varepsilon_t = -\sum_{q=1}^{\infty}\left(\frac{1}{\theta}\right)^q Y_{t+q} = -\sum_{q=1}^{\infty} \theta^{-q} Y_{t+q}$.

So, $MA(1)$ is not invertible.

<u>Note:</u>
All MA processes are stationary but not all are invertible.

<u>Invertibility of $ARMA$ Processes</u>

$ARMA(p,q)$, $\quad Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + \delta + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \ldots - -\theta_q \varepsilon_{t-q}$

$$Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - \ldots - \phi_p Y_{t-p} = \delta + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \ldots - \theta_q \varepsilon_{t-q}$$

$$\phi(B) Y_t = \theta(B) \varepsilon_t \qquad \text{(Let } \delta = 0)$$

The $ARMA(p,q)$ process is invertible, that is there exist constants π_j such that $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and

$$\varepsilon_t = \frac{1}{\theta(B)}\phi(B) Y_t = \pi(B) Y_t = \sum_{j=0}^{\infty} \pi_j B^j Y_t = \sum_{j=0}^{\infty} \pi_j Y_{t-j}$$

if and only if

$$\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q = 0 \text{ only for } |B| > 1.$$

So, the coefficients π_j can be determined by solving

$$\pi(B)\theta(B) = \phi(B)$$

where $\pi(B) = \sum_{j=0}^{\infty} \pi_j B^j$.

<u>Theorem</u>
If ϕ and θ have no common factors, a unique stationary solution to $\phi(B) Y_t = \theta(B) \varepsilon_t$ exists iff the roots of $\phi(B)$ *avoid* the unit circle:

$$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p \neq 0 \quad \Rightarrow \quad |B| = 1.$$

This $ARMA(p,q)$ process is causal iff the roots of $\phi(B)$ are *outside* the unit circle:

$$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p \neq 0 \quad \Rightarrow \quad |B| \leq 1.$$

It is invertible iff the roots of $\theta(B)$ are *outside* the unit circle:

$$\theta(B) = 1 - \theta_1 B - \cdots - \theta_p B^p \neq 0 \quad \Rightarrow \quad |B| \leq 1.$$

Example:
Given the process $Y_t - Y_{t-1} + 0.25Y_{t-2} = \varepsilon_t - \varepsilon_{t-1} + 0.25\varepsilon_{t-2}$. Check the parameter redundancy.

Example:
Consider the process $(1 - 1.5B)Y_t = (1 + 0.2B)\varepsilon_t$. Comment on the stationary, causality and invertibility.

Example:
Consider the process $(1 + 0.25B^2)Y_t = (1 + 2B)\varepsilon_t$. Comment on the stationary, causality and invertibility.

Note:
This notion is very much important if one wants to forecast the future values of the dependent variable, a very relevant issue for many financial practitioners and policy makers. Otherwise, the forecasting task will be impossible when the innovations are not invertible (i.e., the innovations in the past cannot be estimated, as it cannot be observed). If the model is not invertible, the innovations can still be represented by observations of the future, this is not helpful at all for forecasting purpose.