Context-Free Languages & Grammars
(CFLs & CFGs)

Reading: Chapter 5

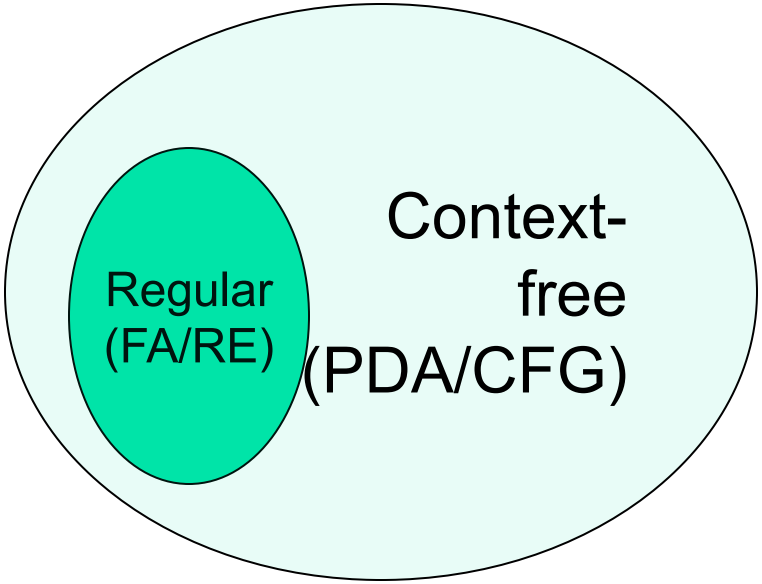
Not all languages are regular

- So what happens to the languages which are not regular?

- Can we still come up with a language recognizer?
  - i.e., something that will accept (or reject) strings that belong (or do not belong) to the language?

Context-Free Languages

- A language class larger than the class of regular languages
- Supports natural, recursive notation called "context-free grammar"
- Applications:
  - Parse trees, compilers
  - XML



An Example

- A palindrome is a word that reads identical from both ends
  - E.g., madam, redivider, malayalam, 010010010
- Let L = { w | w is a binary palindrome}
- Is L regular?
  - No.
  - <u>Proof:</u>
    - Let $w=0^N10^N$ (assuming N to be the p/l constant)
    - By Pumping lemma, w can be rewritten as xyz, such that xy^kz is also L (for any $k\geq 0$)
    - But $|xy|\leq N$ and $y\neq\varepsilon$
    - ==> $y=0^+$
    - ==> xy^kz *will NOT* be in L for k=0
    - ==> Contradiction

But the language of palindromes…

is a CFL, because it supports recursive substitution (in the form of a CFG)

- This is because we can construct a "*grammar*" like this:

1. A ==> ε
2. A ==> 0
3. A ==> 1
4. A ==> 0A0
5. A ==> 1A1

Productions

Terminal

Variable or non-terminal

Same as:
A => 0A0 | 1A1 | 0 | 1 | ε

How does this grammar work?

How does the CFG for palindromes work?

An input string belongs to the language (i.e., accepted) iff it can be generated by the CFG

<u>G:</u>
A => 0A0 | 1A1 | 0 | 1 | ε

- <u>Example:</u> w=01110
- G can generate w as follows:

1. A => 0A0
2. => 01A10
3. => 01110

<u>Generating a string from a grammar:</u>

1. Pick and choose a sequence of productions that would allow us to generate the string.
2. At every step, substitute one variable with one of its productions.

Context-Free Grammar: Definition

- A context-free grammar $G=(V,T,P,S)$, where:
  - V: set of variables or non-terminals
  - T: set of terminals (= alphabet U $\{\varepsilon\}$)
  - P: set of *productions,* each of which is of the form
    $V \Longrightarrow \alpha_1 \mid \alpha_2 \mid \ldots$
    - Where each α_i is an arbitrary string of variables and terminals
  - S ==> start variable

CFG for the language of binary palindromes:
$G=(\{A\},\{0,1\},P,A)$
P: $A \Longrightarrow 0\,A\,0 \mid 1\,A\,1 \mid 0 \mid 1 \mid \varepsilon$

More examples

- Parenthesis matching in code
- Syntax checking
- In scenarios where there is a general need for:
  - Matching a symbol with another symbol, or
  - Matching a count of one symbol with that of another symbol, or
  - Recursively substituting one symbol with a string of other symbols

Example #2

- Language of balanced paranthesis

  e.g., ()((((()))))((())))….

- CFG?

G:

S => (S) | SS | ε

How would you "interpret" the string "(((()))()())" using this grammar?

S→ SS → (S) → (SS)→((S)SS)→ (((S))(S)(S))→ (((()))()())

Example #3

- A grammar for $L = \{0^m1^n \mid m \geq n\}$

- CFG?

G:
S => 0S1 | A
A => 0A | ε

How would you interpret the string "00000111" using this grammar?
S→0S1→ 0 0S1 1→ 0 00S11 1→ 0 000A11 1→ 0 0000A11 1→ 000001111

Example #4

A program containing **if-then(-else)** statements

if *Condition* **then** *Statement* **else** *Statement*

(Or)

if *Condition* **then** *Statement*

CFG?

More examples

- $L_1 = \{0^n \mid n \geq 0\}$
- $L_2 = \{0^n \mid n \geq 1\}$
- $L_3=\{0^i1^j2^k \mid i=j \text{ or } j=k, \text{ where } i,j,k \geq 0\}$
- $L_4=\{0^i1^j2^k \mid i=j \text{ or } i=k, \text{ where } i,j,k \geq 1\}$

Applications of CFLs & CFGs

- Compilers use parsers for syntactic checking
- Parsers can be expressed as CFGs
  1. Balancing paranthesis:
     - B ==> BB | (B) | *Statement*
     - *Statement* ==> …
  2. If-then-else:
     - S ==> SS | *if Condition then Statement else Statement* | *if Condition then Statement* | *Statement*
     - *Condition* ==> …
     - *Statement* ==> …
  3. C paranthesis matching { … }
  4. Pascal *begin-end* matching
  5. YACC (Yet Another Compiler-Compiler)

More applications

- Markup languages
  - Nested Tag Matching
    - HTML
      - <html> …<p> … … </p> … </html>
    - XML
      - <PC> … <MODEL> … </MODEL> .. <RAM> … </RAM> … </PC>

Tag-Markup Languages

Roll ==> <ROLL> Class Students </ROLL>

Class ==> <CLASS> Text </CLASS>

Text ==> Char Text | Char

Char ==> a | b | … | z | A | B | .. | Z

Students ==> Student Students | ε

Student ==> <STUD> Text </STUD>

Here, the left hand side of each production denotes one non-terminals (e.g., “Roll”, “Class”, etc.)

Those symbols on the right hand side for which no productions (i.e., substitutions) are defined are terminals (e.g., ‘a’, ‘b’, ‘|’, ‘<‘, ‘>’, “ROLL”, etc.)

Structure of a production

head derivation body

$$A \text{ =======> } \alpha_1 \mid \alpha_2 \mid \ldots \mid \alpha_k$$

The above is same as:

1. $A \text{ ==> } \alpha_1$
2. $A \text{ ==> } \alpha_2$
3. $A \text{ ==> } \alpha_3$

…

K. $A \text{ ==> } \alpha_k$

CFG conventions

- Terminal symbols <== a, b, c…
- Non-terminal symbols <== A,B,C, …
- Terminal <u>or</u> non-terminal symbols <== X,Y,Z
- Terminal strings <== w, x, y, z
- Arbitrary strings of terminals and non-terminals <== α, β, γ, ..

Syntactic Expressions in Programming Languages

*result = a*b + score + 10 * distance + c*

terminals variables Operators are also terminals

Regular languages have only terminals

- Reg expression = [a-z][a-z0-1]*
- If we allow only letters a & b, and 0 & 1 for constants (for simplification)
  - Regular expression = (a+b)(a+b+0+1)*

String membership

How to say if a string belong to the language defined by a CFG?

1. Derivation
   - Head to body
2. Recursive inference
   - Body to head

Both are equivalent forms

Example:

- w = 01110
- Is w a palindrome?

G:
A => 0A0 | 1A1 | 0 | 1 | ε

A => 0A0
=> 01A10
=> 01110

Simple Expressions…

- We can write a CFG for accepting simple expressions
- G = (V,T,P,S)
  - V = {E,F}
  - T = {0,1,a,b,+,*,(,)}
  - S = {E}
  - P:
    - E ==> E+E | E*E | (E) | F
    - F ==> aF | bF | 0F | 1F | a | b | 0 | 1

Generalization of derivation

- Derivation is *head* ==> *body*

- A==>X (A derives X in a single step)
- A ==>$^*_G$ X (A derives X in a multiple steps)

- <u>Transitivity:</u>

  IF A ==>$^*_G$B, and B ==>$^*_G$C, THEN A ==>$^*_G$ C

Context-Free Language

- The language of a CFG, G=(V,T,P,S), denoted by L(G), is the set of terminal strings that have a derivation from the start variable S.
  - $L(G) = \{ w \text{ in } T^* \mid S \Rightarrow^*_G w \}$

Left-most & Right-most Derivation Styles

G:
$E \Rightarrow E+E \mid E*E \mid (E) \mid F$
$F \Rightarrow aF \mid bF \mid 0F \mid 1F \mid \varepsilon$

Derive the string a*(ab+10) from G:

$E \overset{*}{\Longrightarrow}_G a*(ab+10)$

Left-most derivation:

Always substitute leftmost variable

- E
- ==> E * E
- ==> F * E
- ==> aF * E
- ==> a * E
- ==> a * (E)
- ==> a * (E + E)
- ==> a * (F + E)
- ==> a * (aF + E)
- ==> a * (abF + E)
- ==> a * (ab + E)
- ==> a * (ab + F)
- ==> a * (ab + 1F)
- ==> a * (ab + 10F)
- ==> a * (ab + 10)

Right-most derivation:

Always substitute rightmost variable

- E
- ==> E * E
- ==> E * (E)
- ==> E * (E + E)
- ==> E * (E + F)
- ==> E * (E + 1F)
- ==> E * (E + 10F)
- ==> E * (E + 10)
- ==> E * (F + 10)
- ==> E * (aF + 10)
- ==> E * (abF + 0)
- ==> E * (ab + 10)
- ==> F * (ab + 10)
- ==> aF * (ab + 10)
- ==> a * (ab + 10)

Leftmost vs. Rightmost derivations

Q1) For every leftmost derivation, there is a rightmost derivation, and vice versa. True or False?

True - will use parse trees to prove this

Q2) Does every word generated by a CFG have a leftmost and a rightmost derivation?

Yes – easy to prove (reverse direction)

Q3) Could there be words which have more than one leftmost (or rightmost) derivation?

Yes – depending on the grammar

How to prove that your CFGs are correct?

(using induction)

CFG & CFL

G_{pal}:
A => 0A0 | 1A1 | 0 | 1 | ε

- Theorem: A string w in $(0+1)^*$ is in $L(G_{pal})$, if and only if, w is a palindrome.

- Proof:
  - Use induction
    - on string length for the IF part
    - On length of derivation for the ONLY IF part

Parse trees

Parse Trees

- Each CFG can be represented using a *parse tree:*
  - Each <u>internal node</u> is labeled by a variable in V
  - Each <u>leaf</u> is terminal symbol
  - For a production, $A ==> X_1X_2\ldots X_k$, then any internal node labeled A has k children which are labeled from $X_1,X_2,\ldots X_k$ from left to right

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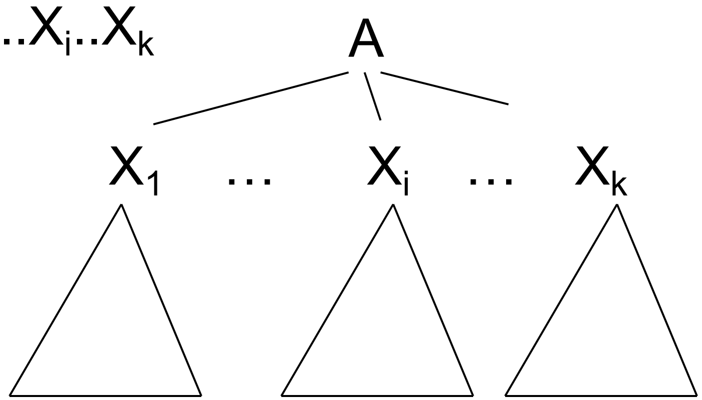
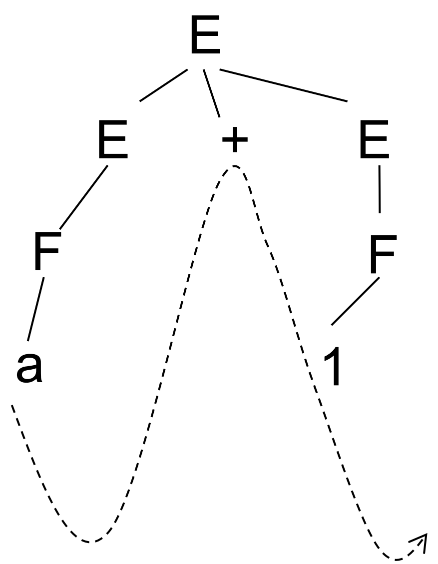
<u>Parse tree for production and all other subsequent productions:</u>

$A ==> X_1..X_i..X_k$

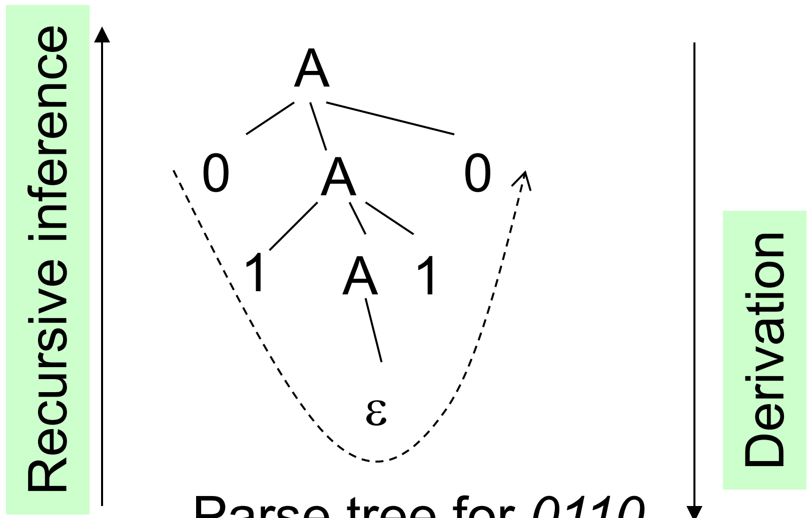


Examples



Parse tree for *a + 1*

G:
E => E+E | E*E | (E) | F
F => aF | bF | 0F | 1F | 0 | 1 | a | b



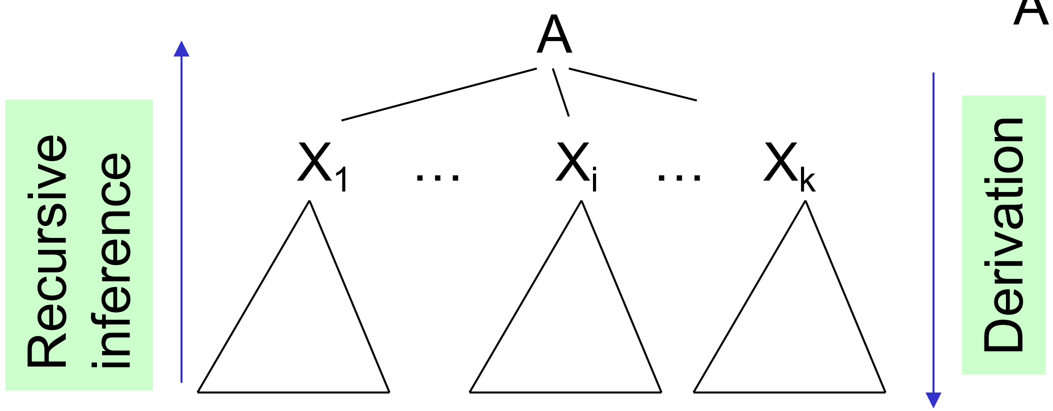
Parse tree for *0110*

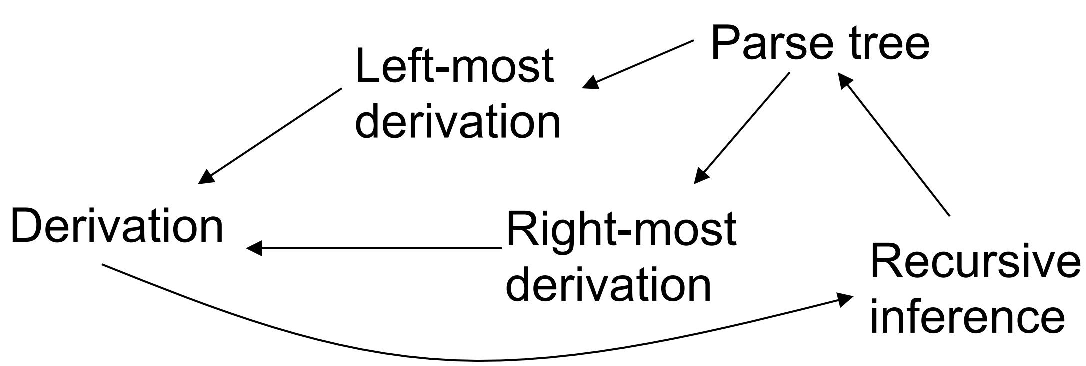
G:
A => 0A0 | 1A1 | 0 | 1 | ε

Parse Trees, Derivations, and Recursive Inferences

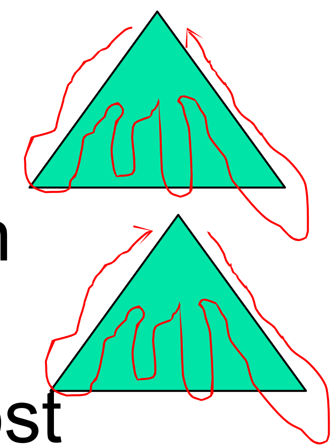
Production:

$A ==> X_1..X_i..X_k$





Interchangeability of different CFG representations



- Parse tree ==> left-most derivation
  - DFS left to right
- Parse tree ==> right-most derivation
  - DFS right to left
- ==> left-most derivation == right-most derivation
- Derivation ==> Recursive inference
  - Reverse the order of productions
- Recursive inference ==> Parse trees
  - bottom-up traversal of parse tree

Connection between CFLs and RLs

What kind of grammars result for regular languages?

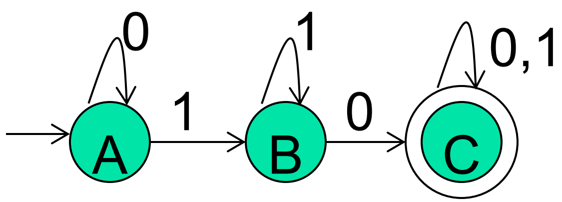
CFLs & Regular Languages

- A CFG is said to be *right-linear* if all the productions are one of the following two forms: *A ==> wB (or) A ==> w*

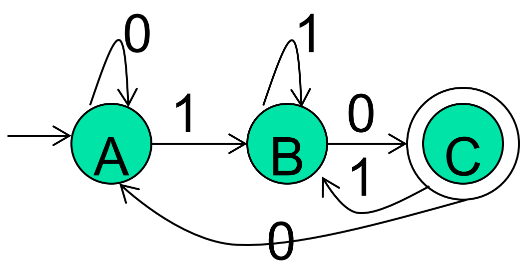
Where:
- A & B are variables,
- w is a string of terminals

- Theorem 1: Every right-linear CFG generates a regular language
- Theorem 2: Every regular language has a right-linear grammar
- Theorem 3: Left-linear CFGs also represent RLs

Some Examples



Right linear CFG?



Right linear CFG?

- A => 01B | C
  B => 11B | 0C | 1A
  C => 1A | 0 | 1

Finite Automaton?

Ambiguity in CFGs and CFLs

Ambiguity in CFGs

- A CFG is said to be *ambiguous* if there exists a string which has more than one left-most derivation

Example:

S ==> AS | ε

A ==> A1 | 0A1 | 01

LM derivation #1:

S => AS

=> 0A1S

=>0A11S

=> 00111S

=> 00111

LM derivation #2:

S => AS

=> A1S

=> 0A11S

=> 00111S

=> 00111

Input string: 00111

Can be derived in two ways

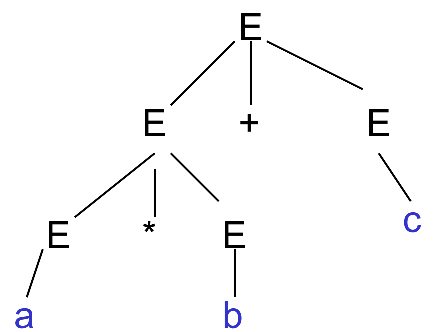
Why does ambiguity matter?

E ==> E + E | E * E | (E) | a | b | c | 0 | 1

Values are different !!!

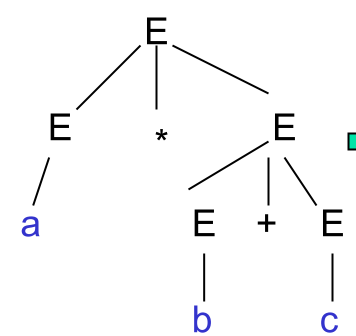
*string = a * b + c*

- <u>LM derivation #1:</u>
  - •E => E + E => E * E + E ==>* a * b + c



(a*b)+c

- <u>LM derivation #2</u>
  - •E => E * E => a * E => a * E + E ==>* a * b + c



a*(b+c)

The calculated value depends on which of the two parse trees is actually used.

Removing Ambiguity in Expression Evaluations

- It MAY be possible to remove ambiguity for some CFLs
  - E.g.,, in a CFG for expression evaluation by imposing rules & restrictions such as precedence
  - This would imply rewrite of the grammar

- <u>Precedence:</u> (), * , +

<u>Modified unambiguous version:</u>

E => E + T | T
T => T * F | F
F => I | (E)
I => a | b | c | 0 | 1

<u>Ambiguous version:</u>

E ==> E + E | E * E | (E) | a | b | c | 0 | 1

How will this avoid ambiguity?

Inherently Ambiguous CFLs

- However, for some languages, it may not be possible to remove ambiguity

- A CFL is said to be *inherently ambiguous* if every CFG that describes it is ambiguous

Example:

- $L = \{ a^n b^n c^m d^m \mid n,m \geq 1\} \cup \{a^n b^m c^m d^n \mid n,m \geq 1\}$
- L is inherently ambiguous
- Why?

Input string: $a^n b^n c^n d^n$

Summary

- Context-free grammars
- Context-free languages
- Productions, derivations, recursive inference, parse trees
- Left-most & right-most derivations
- Ambiguous grammars
- Removing ambiguity
- CFL/CFG applications
  - parsers, markup languages