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Dirichlet–Jordan test

In mathematics, the **Dirichlet–Jordan test** gives sufficient conditions for a real-valued, periodic function *f* to be equal to the sum of its Fourier series at a point of continuity. Moreover, the behavior of the Fourier series at points of discontinuity is determined as well (it is the midpoint of the values of the discontinuity). It is one of many conditions for the convergence of Fourier series.

The original test was established by Peter Gustav Lejeune Dirichlet in 1829,[1] for piecewise monotone functions. It was extended in the late 19th century by Camille Jordan to functions of bounded variation (any function of bounded variation is the difference of two increasing functions).[2][3]

Dirichlet–Jordan test for Fourier series

The Dirichlet–Jordan test states[4] that if a periodic function $f(x)$ is of bounded variation on a period, then the Fourier series $S_n(f(x))$ converges, as $n \to \infty$, at each point of the domain to

$$\lim_{\varepsilon \to 0} \frac{f(x+\varepsilon)+f(x-\varepsilon)}{2}.$$

In particular, if f is continuous at x, then the Fourier series converges to $f(x)$. Moreover, if f is continuous everywhere, then the convergence is uniform.

Stated in terms of a periodic function of period 2π, the Fourier series coefficients are defined as

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx,$$

and the partial sums of the Fourier series are

$$S_n(f(x)) = \sum_{k=-n}^{n} a_k e^{ikx}$$

The analogous statement holds irrespective of what the period of *f* is, or which version of the Fourier series is chosen.

There is also a pointwise version of the test:[5] if f is a periodic function in L^1, and is of bounded variation in a neighborhood of x, then the Fourier series at x converges to the limit as above

$$\lim_{\varepsilon \to 0} \frac{f(x+\varepsilon)+f(x-\varepsilon)}{2}.$$

Jordan test for Fourier integrals

For the Fourier transform on the real line, there is a version of the test as well.[6] Suppose that $f(x)$ is in $L^1(-\infty,\infty)$ and of bounded variation in a neighborhood of the point x. Then

$$\frac{1}{\pi}\lim_{M\to\infty}\int_0^M du\int_{-\infty}^{\infty} f(t)\cos u(x-t)\,dt = \lim_{\varepsilon\to 0}\frac{f(x+\varepsilon)+f(x-\varepsilon)}{2}.$$

If f is continuous in an open interval, then the integral on the left-hand side converges uniformly in the interval, and the limit on the right-hand side is $f(x)$.

This version of the test (although not satisfying modern demands for rigor) is historically prior to Dirichlet, being due to Joseph Fourier.[7]

Dirichlet conditions in signal processing

In signal processing,[8] the test is often retained in the original form due to Dirichlet: a piecewise monotone bounded periodic function f has a convergent Fourier series whose value at each point is the arithmetic mean of the left and right limits of the function. The condition of piecewise monotonicity is equivalent to having only finitely many local extrema, i.e., that the function changes its variation only finitely many times.[9][7] (Dirichlet required in addition that the function have only finitely many discontinuities, but this constraint is unnecessarily stringent.[10]) Any signal that can be physically produced in a laboratory satisfies these conditions.[11]

As in the pointwise case of the Jordan test, the condition of boundedness can be relaxed if the function is assumed to be absolutely integrable (i.e., L^1) over a period, provided it satisfies the other conditions of the test in a neighborhood of the point x where the limit is taken.[12]

See also

- Dini test

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External links

- "Dirichlet conditions" (https://planetmath.org/DirichletConditions). *PlanetMath*.