Math 450 Quiz 3

Gabriel Nowaskie

April 1st, 2023

0.1 Problem 1

a)

$$\int_C f(z)dz = \int_0^\pi f(z(\theta))\frac{dz(\theta)}{d\theta}d\theta = \int_0^\pi \frac{e^{2i\theta+2}}{e^{i\theta}}\left(ie^{i\theta}\right)d\theta = i\int_0^\pi e^{2i\pi}+2\ d\theta = i\left[\frac{1}{2i}e^{2i\theta}+2\theta\right|_0^\pi = \boxed{2i\pi} \tag{1}$$

b) We can parameterize this line by the equation $z = x + 2ix$ from $x = 0$ to $x = 1$:

$$\begin{aligned}\int_C f(z)dz = \int_0^1 f(x+2ix)(dx+2idx) = \int_0^1 (x+2x+2xi)(1+2i)dx = (1+2i)\left[\frac{x^2}{2}+x^2+x^2i\right|_0^1 \\ = \boxed{-\frac{1}{2}+4i}\end{aligned} \tag{2}$$

c)

$$\begin{aligned}\int_C f(z)dz = \int_0^\pi f(z(\theta))\frac{dz(\theta)}{d\theta}d\theta = \int_0^\pi \frac{e^{2i\theta}+e^{i\theta}+10}{e^{i\theta}}\left(ie^{i\theta}\right)d\theta = i\int_0^\pi e^{2i\theta}+e^{i\theta}+10\ d\theta \\ = \left[\frac{1}{2}e^{2i\theta}+e^{i\theta}+10i\theta\right|_0^\pi = \boxed{-2+10\pi i}\end{aligned} \tag{3}$$

0.2 Problem 2

a) Since $F(z) = \frac{-3}{z^4}$ is analytic everywhere except at $z = 0$, C lies in a domain which $F'(z) = f(z)$. Thus,

$$\int_C f(z)dz = \int_{-i}^i F'(z)dz = \int_{-i}^i dF(z) = [F(z)|_{-i}^i = \left[\frac{-3}{z^4}\right|_{-i}^i = \boxed{0} \tag{4}$$

b) Since $F'(z) = f(z), F(z) = \frac{1}{\ln i}e^{z\ln i}$. It is seen that $F(z)$ is analytical for all of C. Thus, we can write the integral as:

$$\begin{aligned}\int_C f(z)dz = \int_0^{-2} F'(z)dz = \int_0^{-2} dF(z) = [F(z)|_0^{-2} = \left[\frac{1}{\ln i}e^{z\ln i}\right|_0^{-2} \\ = \frac{2}{i^3\pi} - \frac{-2i}{\pi} = \boxed{\frac{4i}{\pi}}\end{aligned} \tag{5}$$

c) Since $F'(z) = f(z), F(z) = \frac{1}{1+i}e^{(1+i)\ln z}$. It is seen that $F(z)$ is analytical for all of C. Thus, we can write the integral as:

$$\int_C f(z)dz = \int_1^i F'(z)dz = \int_1^i dF(z) = [F(z)|_1^{-i} = \left[\frac{1}{1+i}e^{(1+i)\ln z}\right|_1^i = \frac{i^{1+i}}{1+i} - \frac{1^{1+i}}{1+i} = \boxed{\left(\frac{1}{2}+\frac{i}{2}\right)(i+i^i)} \tag{6}$$

0.3 Problem 3

a) If we have $f(z) = e^{i\pi z}$, $f(z)$ in analytic everywhere. Thus, for each point on or interior to C, $f(z)$ is analytic. Thus, the integral

$$\int_C \frac{e^{i\pi z}\ dz}{z-3} \tag{7}$$

can be evaluated using the Cauchy-Integral Formula:

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\ dz}{z-z_0}. \tag{8}$$

where $z_0 = 3$. Now,

$$I = \int_C \frac{e^{i\pi z}\ dz}{z-3} = 2\pi i f(3) = \boxed{-2\pi i} \tag{9}$$

b) Using the same reasoning as in A, if $f(z) = e^{i\pi z}$, which is analytic everywhere, then the integral

$$\int_C \frac{e^{i\pi z}\ dz}{z+3} \tag{10}$$

Can be solved using the Cauchy-Integral Formula for $z_0 = -3$:

$$I = \int_C \frac{e^{i\pi z}\ dz}{z+3} = 2\pi i f(-3) = \boxed{-2\pi i} \tag{11}$$

c) If we allow $f(z) = e^{\pi z}$, f is analytic everywhere, in other words: f is analytic for all points interior to C and on C. Thus, we can use Cauchy-Integral Formula where $z_0 = -i$:

$$\int_C \frac{e^{\pi z}\ dz}{z+i} = 2\pi i f(-i) = \boxed{-2\pi i} \tag{12}$$

d) Taking the integral

$$I = \int_C \frac{6e^{i\pi z}\ dz}{z^2-5z+4}, \tag{13}$$

we can factor the denominator such that

$$I = \int_C \frac{6e^{i\pi z}\ dz}{(z-1)(z-4)}. \tag{14}$$

To use the Cauchy-Integral Formula, $f(z)$ has to be analytic on C and for every point interior to C. Thus, we look at the denominator to see which discontinuity is not within C.

$$|4-2| = 2 \leq 3, \quad |1-2| = 1 \leq 3. \tag{15}$$

Well, both of our singularities are in C, therefore we cannot simply just manipulate what $f(z)$ is to ignore them. Thus, we create two contours that are clockwise around each singularity, where

$C_0 = |z-4| = R_0$ and $C_1 = |z-1| = R_1$. R_0 and R_1 are arbitrary radii that we can chose such that the contours fit within C and each inner-contour does not overlap each other. These radii could be any value satisfying that condition, and as long as that can be met, they don't provide any other purpose in the calculation since they are only there to show that we can create two simple, non-intersecting contours about the singularities. We will let $f(z) = \frac{6e^{i\pi z}}{(z-1)(z-4)}$, and define $g_0(z) = f(z)(z-4), g_1(z) = f(z)(z-1)$. Using the Cauchy-Gourset theorem, we can write our integral as:

$$\int_C f(z)dz + \sum_{n=0}^{k} \int_{C_n} f(z)dz = \int_C f(z)dz + \int_{C_0} f(z)dz + \int_{C_1} f(z)dz = 0 \tag{16}$$

Considering that C_0, C_1 are clockwise, we can rewrite this equation for if they are redefined to be counter-clockwise as:

$$\int_C f(z)dz = \int_{C_0} f(z)dz + \int_{C_1} f(z)dz = 0. \tag{17}$$

Using the definitions of $g_0(z)$ and $g_1(z)$, this can be written as:

$$\int_C f(z)dz = \int_{C_0} f(z)dz + \int_{C_1} f(z)dz = 0 \tag{18}$$

Since g_0 and g_1 don't include the other singularities and their own singularity because of the multiplied factor when we defined g_0 and g_1, they are both analytic on their respective contours C_0, C_1, on C, and all points within those contours. Thus, we can use the Cauchy-Integral Formula on each of these integrals:

$$\int_C f(z)dz = \int_{C_0} \frac{g_0(z)}{z-4}dz + \int_{C_1} \frac{g_1(z)}{z-1}dz = 2\pi i g_0(4) + 2\pi i g_1(1) = 2(2\pi i) + 2(2\pi i) = \boxed{8\pi i} \tag{19}$$

e) Taking the integral

$$I = \int_C \frac{e^{\pi z}\ dz}{z^2 - 5iz - 6}, \tag{20}$$

we can factor the denominator such that

$$I = \int_C \frac{e^{\pi z}\ dz}{(z-2i)(z-3i)} \tag{21}$$

To use the Cauchy-Integral Formula, $f(z)$ has to be analytic on C and for every point interior to C. Thus, we look at the denominator to see which discontinuity is not within C.

$$|3i - 2| = \sqrt{13} > 3, \tag{22}$$

Thus, $z = 3i$ is not within C. We move this factor to the numerator and take $f(z)$ to be

$$f(z) = e^{\pi z}(-3i + z)^{-1}. \tag{23}$$

$f(z)$ is analytic for all points on or interior to C, thus, we can use the Cauchy-Integral Formula to find that, using $z_0 = 2i$:

$$I = \int_C \frac{e^{\pi z}(-3i+z)^{-1}\ dz}{z-2i} = 2\pi i f(2i) = -2\pi i^2 = \boxed{-2\pi}. \tag{24}$$

f) If we do partial fraction decomposition on the integrand, we get:

$$I = \int_C \frac{e^{i\pi z}\ dz}{z^2(z-1)} = \int_C -\frac{e^{i\pi z}}{z-0} - \frac{e^{i\pi z}}{(z-0)^{1+1}} + \frac{e^{i\pi z}}{z-1}\ dz \tag{25}$$

If we take $f(z) = e^{i\pi z}$ for each integral part, $f(z)$ will be analytic on and within the entirety of C. Thus, we can use the Cauchy-Integral Formula to evaluate each of these integrals. For the second integral part, we will use the differential version of the Cauchy-Integral Formula, where

$$f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)\ dz}{(z-z_0)^{n+1}}. \tag{26}$$

Evaluating each part of the integral, we are thus left with:

$$I = -2\pi i f(0) + 2\pi i f(1) - 2\pi i f'(0) = -2\pi i - 2\pi i - 2\pi i(i\pi) = \boxed{-4\pi i + 2\pi^2} \tag{27}$$

0.4 Problem 4

According to the theorem of moduli for contour integrals: if $|f(z)| \leq M$, then

$$\left|\int_C f(z)dz\right| \leq ML, \tag{28}$$

where L is the length of the contour. For this problem, we have a semicircle with radius R. Thus, the length is simply $L = \pi R$. If we consider that z is a point on C_R, then (using the triangle and reverse triangle inequalities):

$$|z+2| \leq |z| + |2| = R+2, \quad |z^3+3| \geq ||z^3| - |3|| = R^3 - 3. \tag{29}$$

Thus,

$$\left|\frac{z+2}{z^3+3}\right| = \frac{|z+2|}{|z^3+3|} \leq \frac{R+2}{R^3-3} = M_R. \tag{30}$$

Using the theorem of modli for contour integrals, then

$$\left|\int_{C_R} \frac{z+2}{z^3+3}dz\right| \leq M_R L = \frac{(R+2)(\pi R)}{R^3-3}. \tag{31}$$

Taking the limit on both sides, we see that the RHS approaches zero (due to a higher degree polynomial in the denomenator):

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{z+2}{z^3+3}dz\right| \leq \lim_{R\to\infty}\frac{(R+2)(\pi R)}{R^3-3} = 0. \tag{32}$$

Since the modulus must be a real number, it cannot be less than 0. Thus,

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{z+2}{z^3+3}dz\right| = 0, \tag{33}$$

and so the integral must also approach this value. We can then conclude that:

$$\boxed{\lim_{R\to\infty}\int_{C_R} \frac{z+2}{z^3+3}dz = 0} \tag{34}$$

0.5 Problem 5

If the point $|z_0| < 10$, then there are two singularities in C: $z = z_0$ ad $z_0 = 1$. Using the Cauchy-Goursat theorem:

$$\int_C f(z)dz = \int_{C_0} f(z)dz + \int_{C_1} f(z)dz, \tag{35}$$

where we define C_0 to be a contour about the singularity $z = 1$ and C_1 is a contour surrounding the singularity $z = z_0$. We can then define the two functions $g_0(z) = f(z)(z-1), g_1(z) = f(z)(z-z_0)$ which are analytic on C_0 and C_1 respectively because they don't have the singularity within their contour and their contours don't include each other's singularity. Thus,

$$\int_{C_0} f(z)dz + \int_{C_1} f(z)dz = \int_{C_0} \frac{g_0(z)\ dz}{z-1} + \int_{C_1} \frac{g_0(z)\ dz}{z-z_0} = 2\pi i(g_0(1) + g_1(z_0)) = 2\pi i \left(\frac{1}{1-z_0} + \frac{-z_0^2}{z_0-1}\right)$$

$$= \boxed{2\pi i(z_0+1), \quad |z| < 10} \tag{36}$$

For the scenario $|z| > 10$, C only has one singularity because $z = z_0$ will be outside of C. Thus, the function

$$f(z) = \frac{z^2}{z-z_0} \tag{37}$$

is analytic for all points on and in C. We can then use the Cauchy-integral theorem to get that

$$\int_C f(z)dz = 2\pi i f(1) = \boxed{\frac{2\pi i}{1-z_0}, \quad |z| > 10} \tag{38}$$