Math 450 Quiz 4

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0.1 Problem 1

a) If we use the Maclaurin series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad (|z| < \infty), \tag{1}$$

then we can take the reciprocal in the sum and write the function $f(z) = z^3 e^{1/z}$ as

$$f(z) = z^3 e^{1/z} = z^3 \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{z^{-n+3}}{n!}, \quad (0 < |z| < \infty)}. \tag{2}$$

b) If we manipulate the Maclaurin series

$$\frac{1}{1-z} = \frac{-1}{z-1} = \sum_{n=0}^{\infty} -z^n, \tag{3}$$

then we can use this series to find the Laurent series for $f(z) = \frac{1}{z(z-1)}$ by doing

$$f(z) = \frac{1}{z(z-1)} = \frac{-1}{z(1-z)} = \frac{1}{z}\sum_{n=0}^{\infty} -z^n = \boxed{\sum_{n=0}^{\infty} -z^{n-1}, \quad (0 < |z| < 1)}. \tag{4}$$

c) If we manipulate the Maclaurin series for $\frac{1}{1-z}$, we can get

$$\frac{1}{1-z} = \frac{-1/z}{1-\frac{1}{z}} = \frac{-1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} -z^{-n-1}, \quad (1 < |z| < \infty). \tag{5}$$

Using this, we can manipulate $f(z) = \frac{1}{z(z-1)}$ to get

$$\frac{1}{z(z-1)} = \frac{1/z^2}{1-\left(\frac{1}{z}\right)} = \frac{1}{z^2}\sum_{n=0}^{\infty} z^{-n} = \boxed{\sum_{n=0}^{\infty} z^{-n-2}, \quad (1 < |z| < \infty)} \tag{6}$$

d) We can use a similar manipulation but working with $(z-1)$ instead to get:

$$\frac{1}{1+(z-1)} = \sum_{n=0}^{\infty}(-(z-1))^n = \sum_{n=0}^{\infty}(-1)^n(z-1)^n, \quad (0 < |z-1| < 1). \tag{7}$$

Applying a similar manipulation to our function:

$$\frac{1}{z(z-1)} = \frac{1}{z-1} \times \frac{1}{1+(z-1)} = \frac{1}{z-1}\sum_{n=0}^{\infty}(-1)^n(z-1)^n = \boxed{\sum_{n=0}^{\infty}(-1)^n(z-1)^{n-1}, \quad (0<|z-1|<1)}. \tag{8}$$

0.2 Problem 2

a) Using the theorem of Residues:

$$\int_C f(z)dz = 2\pi i \times \text{Res}[f(z)], \tag{9}$$

we have our Laurent series from Problem 1 as

$$f(z) = z^3 e^{1/z} = z^3 \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{-n+3}}{n!}, \quad (0<|z|<\infty). \tag{10}$$

For the coefficient of the $1/z$ term, we need $-n+3=-1 \rightarrow n=4$. Plugging $n=4$ into the sum for the coefficient, we get $B_0 = \frac{1}{4!}$. Thus,

$$\int_C f(z)dz = \frac{2\pi i}{4!} = \boxed{\frac{\pi i}{12}} \tag{11}$$

b) Using the theorem of Residues for infinities,

$$\int_C f(z)dz = 2\pi i \times \text{Res}_{z=0}[f(z)]. \tag{12}$$

Using this for $f(z) = \frac{z+1}{z(z-1)}$,

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z} \times \frac{z+1}{1-z}. \tag{13}$$

Since $\frac{z+1}{1-z}$ is analytic and has a Maclaurin series at $z=0$, the first term in the series corresponds to the coefficient for the $\frac{1}{z}$ term, so our residue is $\frac{0+1}{1-0} = 1$. Thus,

$$\int_C f(z)dz = \boxed{2\pi i} \tag{14}$$

c) Once again, we use the theory of residues at infinity for the function $f(z) = \frac{z+1}{z^2(z-1)}$.

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z} \times \frac{z+z^2}{1-z}, \tag{15}$$

this polynomial has a Maclaurin series at $z=0$. Looking at how the series compares with the series of $\frac{1}{1-z}$, we can tell that it will not have any reciprocal terms. Thus, the coefficient for the $\frac{1}{z}$ term, the residue, must be 0.

$$\int_C f(z)dz = 2\pi i(0) = \boxed{0} \tag{16}$$

d) Using the theory of residues at infinity for the function $f(z) = \frac{z+1}{z(z-1)^3}$, we have that

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \times \frac{\frac{1}{z}+1}{\frac{1}{z}\left(\frac{1}{z}-1\right)^3} = \frac{1}{z} \times \frac{1+z}{(1-z)^3}. \tag{17}$$

The function $\frac{z^2+z^3}{(1-z)^3}$ has a Maclaurin series. Thus, the first term of this series, which is a coefficient, will be the coefficient of the $\frac{1}{z}$ term being multiplied by it. This results in a residue of 0 since this series has no reciprocal terms in it. Thus,

$$\int_C f(z)dz = 2\pi i(0) = \boxed{0} \tag{18}$$

e) Once again, we use the theory of residues to solve this integral. By expanding $1 - e^{2z}$ to its series form centered at 0:

$$e^{2z} = 1 - \sum_{n=0}^{\infty} \frac{(2z)^n}{(n!)} = -2z - \frac{4z^2}{2!} - \ldots \tag{19}$$

n our function, $f(z) = \frac{1}{1-e^{2z}}$, this series is in the denominator. Looking at the coefficients, it is clear that for the z term in the denominator, which is corresponds to the $\frac{1}{z}$ term, there is a coefficient of $-1/2$, which is our residue. Thus,

$$\int_C f(z)dz = 2\pi i(-1/2) = \boxed{-\pi i} \tag{20}$$

0.3 Problem 3

a) We begin by finding the residues of the function using the theorem that if two function p and q are analytic at a point z_0, where $q(z_0) = 0, p(z_0) \neq 0, q'(z_0) \neq 0$, then

$$\text{Res}\ \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}. \tag{21}$$

Using this, we can see that the residues are of the form $\frac{z^3}{4z^3}$, which is 1/4 for each pole. Thus, using Cauchy's Residue Theorem:

$$\int_C f(z)dz = 2\pi i\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right) = \boxed{2\pi i} \tag{22}$$

b) Using the Residue at Infinity:

$$\int_C f(z)dz = 2\pi i \times \text{Res}_{z=0}[f(z)], \tag{23}$$

we have to find the residue of

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \frac{\frac{1}{z^3}}{\frac{1}{z^4}-1} = \frac{1}{z} \times \frac{1}{1-z^2}. \tag{24}$$

Since $\frac{1}{1-z^2}$ has a Maclaurin series, it will be a polynomial with a coefficient as its first term. This coefficient is the residue because of the factor of $\frac{1}{z}$ making it the coefficient term we are looking for. The first term in the series is simply

$$\frac{1}{1-0^2} = 1, \tag{25}$$

thus,

$$\int_C f(z)dz = 2\pi i(1) = \boxed{2\pi i} \tag{26}$$

0.4 Problem 4

a)Using the theorem of Residues for infinities,

$$\int_C f(z)dz = 2\pi i \times \mathrm{Res}_{z=0}[f(z)]. \tag{27}$$

Using this for $f(z) = \frac{z^5+1}{z^7+1}$,

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\frac{\frac{1}{z^5}+1}{\frac{1}{z^7}+1} = \frac{1}{z} \times \frac{z+z^6}{1+z^7}. \tag{28}$$

Since $\frac{z^5+1}{z^7+1}$ is analytic and has a Maclaurin series at $z = 0$, the first term in the series corresponds to the coefficient for the $\frac{1}{z}$ term, so our residue is substituting in 0 to get $\frac{0+0}{1+0} = 1$. Thus, since our residue is 0:

$$\int_C f(z)dz = 2\pi i(0) = \boxed{0} \tag{29}$$

b) Using Cauchy's residue theorem we can formulate that

$$\int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}_{z_k}[f(z)] - \int_{C_R} f(z)dz. \tag{30}$$

Expanding the denominator and using (21), we can find the residues for each pole to be :

$$\mathrm{Res}\,\frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)} = \frac{z_0}{(z_0^4+2z_0^3+3z_0^2+2z_0+2)'} = \frac{z_0}{(4z_0^3+6z_0^2+6z_0+2)} \tag{31}$$

Thus, we can substitute in the poles, $z = \pm i, -1 \pm i$ to calculate the residues. If we integrate over the upper half of the circle, we will only ever see two of the singularities, $z = i, -1+i$, then we must have the sum:

$$B_0 + B_1 = \frac{1}{10} - \frac{i}{5} - \frac{1}{10} + \frac{3i}{10} = \frac{i}{10} \tag{32}$$

Now, we have

$$\int_{-R}^{R} f(x)dx = 2\pi i(\frac{i}{10}) - \int_{C_R} f(z)dz = \frac{-\pi}{5} - \int_{C_R} f(z)dz \tag{33}$$

which is valid for all values of R greater than unity. Thus, we now must prove that $\int_{C_R} f(z)dz$ approaches 0 as $R \to \infty$. Looking at $f(z)$ and using the triangle inequality, we can see that $|z| \leq |z| = R$ and $|z^4 + 2z^3 + 3z^2 + 2z + 2| \geq ||z^4| - |2z^3| - |3z^2| - |2z| - |2|| = R^4 - 2R^3 - 3R^2 - 2R - 2$. Now, we can state that

$$|f(z)| \leq \frac{R}{R^4 - 2R^3 - 3R^2 - 2R - 2} = M_R. \tag{34}$$

Due to the modulus condition

$$\left|\int_C f(z)dz\right| \leq M_R \times L, \tag{35}$$

where L is the length of the contour, which is just πR in our semicircular case, so

$$\left|\int_C f(z)dz\right| \leq \pi R M_R. \tag{36}$$

Since the denominator of M_R has a higher degree, as $R \to \infty, M_R \to 0$. Thus,

$$\lim_{R\to\infty} \int_C f(z)dz \to 0. \tag{37}$$

Finally, we can now state that

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \frac{-\pi}{5} - \int_{C_R} f(z)dz = \frac{-\pi}{5} - 0 = \boxed{\frac{-\pi}{5}}. \tag{38}$$