Green Nested Simulation via Likelihood Ratio: Applications to Longevity Risk Management

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Overview

What is Longevity Risk?

Actuarial Definition

The risk that actual survival rates exceed expectations, leading to:

- Higher-than-expected pension liabilities
- Increased annuity payouts
- Strain on social security systems
- Capital adequacy challenges

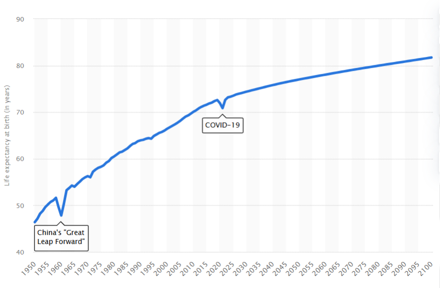

Figure 1: Global life expectancy trends (1950-2100)[1]

[a] Source: https://www.statista.com

Hedging Instruments

Table 1: Longevity risk transfer mechanisms

Instrument	Payout Trigger	Market
Longevity swaps	Actual vs expected deaths	OTC
q-forwards	Mortality index levels	Exchange
Survivor bonds	Population survival rates	Sovereign

Overview

Background

- **Simulation** is crucial for longevity risk analysis:
  - Most stochastic mortality models lack closed-form solutions
  - Needed for risk measures and moment calculations
- Two types of mortality scenarios:
  - **Spot scenarios**: Mortality paths from now ($t = 0$) to future.
    - Used for cash flow hedge analysis
    - Gives present value distributions
  - **Forward scenarios**: Values at future time $\tau > 0$

Forward Mortality Scenarios

- Key quantity:

$$V_\tau = \mathbb{E}[H|\mathcal{F}_\tau] \quad (1)$$

  - H = discounted cash flows
  - $\mathcal{F}_\tau$ = information up to time τ

  We can interpret V_τ as the time-τ value of the mortality-related security, liability or portfolio.
- Risk analysis objectives:
  - Estimate distribution of V_τ given $\mathcal{F}_0$
  - Derive risk measures $\rho(V_\tau)$ (VaR, etc.)

Relevant studies in the operations research domain I

In terms of alleviating the computational burden of nested simulations, operations research has two main research directions:

1. Optimally allocate a fixed simulation budget between the inner- and outer-simulations.
   - Under some assumptions, Gordy and Juneja (2010) show that the optimal asymptotic MSE of the standard nested risk estimator converges at $\mathcal{O}(\Gamma^{-2/3})$ when $M = \mathcal{O}(\Gamma^{2/3}), N = \mathcal{O}(\Gamma^{1/3})$.
   - Broadie et al.(2011) considers sequential simulation procedures whose MSE converges at $\mathcal{O}(\Gamma^{-4/5+\epsilon})$ for any $\epsilon > 0$.

Relevant studies in the operations research domain II

2. Identify efficient alternatives to replace inner simulations.
    - Least Squares Monte Carlo (LSMC) uses a parametric regrssion in place of inner simulations for American option(Longstaff and Schwartz, 2001). Broadie et al. (2015) applied it to risk management problems and showed that MSE converges to the asymptotic error level after $\mathcal{O}(\Gamma^{-1})$, but its performance depends on the choice of basis functions;
    - Hong et al.(2017) proposed a smoothing method based on Nadaraya-Watson kernal estimation, with an MSE convergency speed of $\mathcal{O}(\Gamma^{-\min\{1,4/(d+2)\}})$, where d is the dimension of the problem.The performance of the kernel smoothing approach depends on the bandwidth parameter in the Nadaraya-Watson estimator, which is often difficult and time-consuming to tune.

Contributions

1. Developing and analyzing a simulation method for efficiently estimating $\rho(V_\tau)$, which is suitable for risk measurement of longevity-related securities.
   - Low computational complexity
   - High accuracy
   - Easy implementation
2. Inspired by the "green simulation" idea of Feng and Staum (2016, 2017), the global reuse of the inner loop output is achieved through the likelihood ratio estimator, which improves the estimation accuracy of all outer loop scenarios.

REMARK

Different from American option pricing (stochastic control problem), this study focuses on the nested estimation problem.

Overview

Set-up

1. Let $\{\kappa_t, t = 0, 1, ...\}$ be a discrete-time stochastic series that encodes the time-t value of the underlying period and cohort effects.
2. Assume $\{\kappa_t\}$ is a Markovian process with a natural filtartion $\mathcal{F}_t$.
3. Cosider some mortality-linked securities and liability, whose value $V_t = V_\tau$.
4. Let $\tau \in (0, T]$ be the prescribed risk horizon, and denote the values of period and/or cohort effects beyond τ by $\kappa_{\tau+} = \{\kappa_{\tau+1}, ..., \kappa_T\}$. So the time-$\tau$ value V_τ can be written as

$$\begin{aligned} V_\tau = V(\kappa_\tau) = \mathbb{E}[H(\kappa_{\tau+}|\mathcal{F}_\tau)] = \mathbb{E}[H(\kappa_{\tau+}|\kappa_\tau)] \\ = \int H(\kappa_{\tau+}) f(\kappa_{\tau+}|\kappa_\tau) d\kappa_{\tau+}. \end{aligned} \tag{2}$$

The Lee-Carter model I

Let $m_{x,t}$ be the central death rate at age x and in year t. The Lee-Carter model is specified as

$$\ln m_{x,t} = \alpha_x + \beta_x \kappa_t. \tag{3}$$

The period effect is modeled by a random walk with drift, i.e.,

$$\kappa_t = \kappa_{t-1} + \theta + \epsilon_t,\ \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2) \tag{4}$$

For any $0 \leq s < t \leq T$, the conditional distribution of κ_t given κ_s is given by

$$f(\kappa_t|\kappa_s) \sim \mathcal{N}(\kappa_s + (t-s)\theta, (t-s)\sigma_\epsilon^2). \tag{5}$$

The Lee-Carter model II

Let $S_{x,t}(u)$ be the ex post probability that an individual aged x at time t would have survived to time $t+u, t, u \geq 0$.

$$S_{x,t}(u) = \prod_{s=1}^{u} e^{-m_{x+s-1,t+s}} = \exp\left(-\sum_{s=1}^{u} \exp(\alpha_{x+s-1} + \beta_{x+s-1}\kappa_{t+s})\right). \tag{6}$$

Under equation (6), we can further obtain an expression for the conditional probability that an individual aged x at time $t-1$ dies between time $t-1$ and t:

$$q_{x.t} = 1 - S_{x,t-1}(1) = 1 - \exp(-\exp(\alpha_x + \beta_x \kappa_t)). \tag{7}$$

Mortality-linked securities and liabilities I

K-call options

At time $\tau < T$, its discounted payoff is

$$H(\kappa_{\tau+}) = H(\kappa_T) = e^{-r(T-t)} \cdot (\kappa_T - K)^+. \tag{8}$$

Its time-τ value is $V(\kappa_\tau) = \mathbb{E}[H(\kappa_T)|\kappa_\tau]$.

Mortality-linked securities and liabilities II

Call spreads

A call spread has a payoff of

$$\max\left(\min\left(\frac{x - AP}{EP - AP}, 1\right), 0\right)$$

per \$1 notional at maturity T.
In this study we consider a q-call-spread with $x = q_{x,T} = 1 - S_{x,T-1}(1)$.
The discounted payoff at time τ for a q-call-spread is given by

$$H(\kappa_{\tau+}) = H(\kappa_T) = e^{-r(T-t)} \cdot \max\left(\min\left(\frac{q_{x,T} - AP}{EP - AP}, 1\right), 0\right) \tag{9}$$

Mortality-linked securities and liabilities III

Annuity liabilities

Consider a T-year temporary life annuity-immediate. At time τ, a surviving policyholder will be at age $x+\tau$ and will be holding a $(T-\tau)$-year temporary life annuity-immediate. So the discounted payoff at time-τ is

$$H(\kappa_{\tau+}) = \sum_{u=1}^{T-\tau} e^{-ru} S_{x_0+\tau,\tau}(u) \tag{10}$$

The corresponding time-τ conditional expected value is then

$$V(\kappa_{\tau+}) = \mathbb{E}[H(\kappa_{\tau+})|\kappa_\tau] = \sum_{u=1}^{T-\tau} e^{-ru} \mathbb{E}[S_{x_0+\tau,\tau}(u)|\kappa_\tau]. \tag{11}$$

Closed-form formulas I

Lemma 1

Suppose that the Lee-Carter model specified in equations $(3)-(4)$ *holds. The time-*τ *condition expected value of a K-option is given by*

$$V(\kappa_\tau) = e^{-r(T-t)}\left[\sigma_k \cdot \phi\left(\frac{K-\mu_k}{\sigma_k}\right) + \lambda \cdot (K-\mu_k)\cdot \Phi\left(\frac{\lambda(K-\mu_k)}{\sigma_k}\right)\right], \quad (12)$$

where ϕ, Φ *are the standaed normal density and distribution functions, respectively.* $\mu_k = \tau_k + (T-\tau)\theta, \sigma_k = \sqrt{T-\tau}\sigma_\epsilon$, *and* $\lambda = 1(-1)$ *for put (or call) option.*

Closed-form formulas II

Proof of Lemma1

Consider $K-$Call Option

$$
\begin{aligned}
V(\kappa_\tau) &= \mathbb{E}[H(\kappa_T) \mid \kappa_\tau] \\
&= e^{r(T-t)}\mathbb{E}\left[(\kappa_T - K)^+ \mid \kappa_\tau\right] \\
&= e^{r(T-t)}\int_K^\infty (\kappa_T - K) f(\kappa_T \mid \kappa_\tau) d\kappa_T,
\end{aligned}
$$

where

$f(\kappa_T \mid \kappa_\tau) = \frac{1}{\sqrt{2\pi}\sigma_k}\exp\left\{-\frac{(\kappa_T-\mu_k)^2}{2\sigma_k^2}\right\}, \mu_k = \kappa_\tau + (T-\tau)\theta, \sigma_k^2 = (T-\tau)\sigma_\epsilon^2.$

Then

$$
V(\kappa_\tau) = e^{-r(T-t)}\int_{-d}^\infty (\mu_k + \sigma_k z - K)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz,
$$

$$
\text{where } d = \frac{\mu_k - K}{\sigma_k}, z = \frac{\kappa_T - \mu_k}{\sigma_k} \mid \kappa_\tau \sim \mathcal{N}(0,1)
$$

$$
\begin{aligned}
V(\kappa_\tau) &= e^{-r(T-t)}\left[\int_{-d}^\infty \sigma_k z \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz + \int_{-d}^\infty (\mu_k - K)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\right] \\
&= e^{-r(T-t)}\left\{\sigma_k\phi\left(\frac{K-\mu_k}{\sigma_k}\right) + (\mu_k - K)\Phi\left(\frac{K-\mu_k}{\sigma_k}\right)\right\}.
\end{aligned}
$$

Standard nested simulation

Algorithm 1: Standard Nested Simulation

Input: Mortality model parameters, M (outer scenarios), N (inner paths)
Output: Empirical distribution of $\hat{V}_N^{ns}(\kappa_\tau^{(i)})$

```
Outer Simulation:
for i ← 1 to M do
    Simulate outer scenario κ_τ^(i) from f(κ_τ)
    Inner Simulation:
    for j ← 1 to N do
        Simulate κ_{τ+}^(i,j) from f(κ_{τ+} | κ_τ^(i))
        Compute discounted payoff H(κ_{τ+}^(i,j))
    end
    Compute V̂_N^ns(κ_τ^(i)) = (1/N) Σ_{j=1}^{N} H(κ_{τ+}^(i,j))
end
Construct empirical distribution of {V̂_N^ns(κ_τ^(i))}_{i=1}^{M}
Estimate risk measure ρ(V_τ)
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Approximation methods I

Provided that the required derivatives exists for some anchor scenario $\tilde{\kappa}_\tau$, the Taylor approximation of $V(\kappa_\tau^{(i)})$ fpr the scenario $\kappa_\tau^{(i)}$ is given by

$$\hat{V}^{app}(\kappa_\tau^{(i)}) \approx V(\tilde{\kappa}_\tau) + \frac{V'(\tilde{\kappa}_\tau)}{1!}(\kappa_\tau^{(i)} - \tilde{\kappa}_\tau) + \frac{V''(\tilde{\kappa}_\tau)}{2!}(\kappa_\tau^{(i)} - \tilde{\kappa}_\tau)^2 + \cdots . \quad (13)$$

Approximation methods II

Lemma 2

Suppose that the Lee-Carter model specified in equations (3) − (4) *holds. A first-order Taylor approximation for a K-option around some scenario* $\tilde{\kappa}_\tau$ *is given by*

$$V(\kappa_\tau) \approx V(\tilde{\kappa}_\tau) + e^{-r(T-\tau)} \cdot (\kappa_\tau - \tilde{\kappa}_\tau) \cdot \Phi\left(\frac{\lambda(K - \tilde{\kappa}_\tau - \theta(T-\tau))}{\sigma_\epsilon \sqrt{T-\tau}}\right). \quad (14)$$

Approximation methods III

Proof of Lemma 2

We derive the pathwise gradient estimators for $V(\kappa_\tau)$. Let $\kappa_T = \kappa_\tau + Z_{\tau,T}$,where $Z_{\tau,T} \sim \mathcal{N}((T-\tau)\theta, (T-\tau)\sigma_\epsilon^2)$ and $\tau \leq T$.

$$
\begin{aligned}
V'(\kappa_\tau) &= \frac{d\mathbb{E}\left[e^{-r(T-t)}(\kappa_T - K)^+ \mid \kappa_\tau\right]}{d\kappa_\tau} \\
&= e^{-r(T-t)}\mathbb{E}\left[\frac{d(\kappa_\tau + Z_{\tau,T} - K)^+}{d\kappa_\tau} \mid \kappa_\tau\right] \\
&= e^{-r(T-t)}\mathbb{E}\left[\mathbf{1}_{\left\{\kappa_\tau + Z_{\tau,T} > K\right\}} \mid \kappa_\tau\right],
\end{aligned}
$$

where the equalities hold by verifying the unbiasedness conditions in Section 7.2.2 in Glasserman (2004). One can then estimate $V'(\tilde{\kappa}_\tau)$ via simulation by

$$
\hat{V}_N'(\tilde{\kappa}_\tau) = \frac{e^{-r(T-t)}}{N}\sum_{j=1}^{N}\mathbf{1}_{\left\{\tilde{\kappa}_\tau + Z_{\tau,T}^{(j)} > K\right\}}.
$$

Approximation methods IV

Recall from the equation (6) that the survival probability under the Lee-Carter model is $S_{x,t}(u) = \exp\left(-\sum_{s=1}^{u} \exp(\alpha_{x+s-1} + \beta_{x+s-1}\kappa_{t+s})\right)$. It follows that the derivative of $S_{x,t}(u)$ w.r.t κ_t is

$$\frac{d}{d\kappa_t} S_{x,t}(u) = S_{x,t}(u) \cdot \left(-\sum_{s=1}^{u} \beta_{x+s-1} \cdot \exp(\alpha_{x+s-1} + \beta_{x+s-1}\kappa_{t+s})\right).$$

Then the derivative of $V(\kappa_\tau)$ w.r.t. κ_τ for a q-call-spread is

$$\begin{aligned}
V'(\kappa_\tau) &= \frac{e^{-r(T-t)}}{EP-AP} \frac{d}{d\kappa_\tau} \left(\mathbb{E}[(q_{x,T}-AP)^+ \mid \kappa_\tau] - \mathbb{E}[(q_{x,T}-EP)^+ \mid \kappa_\tau]\right) \\
&= \frac{e^{-r(T-t)}}{EP-AP} \left(\mathbb{E}\left[\frac{d}{d\kappa_\tau}(1-S_{x,T-1}(1)-AP)^+ \mid \kappa_\tau\right] - \mathbb{E}\left[\frac{d}{d\kappa_\tau}(1-S_{x,T-1}(1)-EP)^+ \mid \kappa_\tau\right]\right) \\
&= \frac{e^{-r(T-t)}}{EP-AP} \left(\mathbb{E}\left[\mathbf{1}_{\{1-EP>S_{x,T-1}(u)\}} \frac{dS_{x,T-1}(1)}{d\kappa_\tau} \mid \kappa_\tau\right] - \mathbb{E}\left[\mathbf{1}_{\{1-EP>S_{x,T-1}(u)\}} \frac{dS_{x,T-1}(1)}{d\kappa_\tau} \mid \kappa_\tau\right]\right),
\end{aligned}$$

where $\dfrac{dS_{x,T-1}(1)}{d\kappa_\tau} = -S_{x,T-1}(1) \cdot \beta_x \cdot \exp\left(\alpha_x + \beta_x(\kappa_\tau + Z_{\tau,T})\right)$.

To estimate $V'(\tilde{\kappa}_\tau)$, one can again use simulation to compute the two expections in $V'(\kappa_\tau)$ and in turn obtain $\hat{V}'_N(\tilde{\kappa}_\tau)$.

Approximation methods V

Lemma 3

Suppose that the Lee-Carter model specified in equations $(3)-(4)$ *holds. For any age* x*, time* τ*, and duration* u*, a first-order probit-Taylor approximation for* $\mathbb{E}[S_{x,\tau}(u)|\kappa_\tau]$ *around an anchor scenario* $\tilde{\kappa}_\tau$ *is given by*

$$\mathbb{E}[S_{x,\tau}(u)|\kappa_\tau] \approx \Phi\left(D_{x,\tau}(u) + D'_{x,\tau}(u)(k_\tau - \tilde{\kappa}_\tau)\right) \tag{15}$$

where $D_{x,\tau}(u) = \Phi^{-1}(\mathbb{E}[S_{x,\tau}(u)|\kappa_\tau = \tilde{\kappa}_\tau]), D'_{x,\tau}(u) = \frac{d}{d\kappa_\tau}\Phi^{-1}(\mathbb{E}[S_{x,\tau}(u)|\kappa_\tau])\Big|_{\kappa_\tau=\tilde{\kappa}_\tau}$.

Proof of Lemma 3

Consider a temporary life annuity, whose time-τ value is

$$V(\kappa_\tau) = \sum_{u=1}^{T-\tau} e^{-ru}\mathbb{E}\left[S_{x_0+\tau,\tau}(u) \mid \kappa_\tau\right].$$

To estimate $\mathbb{E}\left[S_{x_0+\tau,\tau}(u) \mid \kappa_\tau\right], u = 1, ..., T - \tau$, the first-order probit-Taylor approximation around an anchor scenario $\tilde{\kappa}_\tau$ is given by

$$\Phi^{-1}\left(\mathbb{E}\left[S_{x_0+\tau,\tau}(u) \mid \kappa_\tau\right]\right) \approx D_{x_0+\tau,\tau}(u) + D'_{x_0+\tau,\tau}(u)(\kappa_\tau - \tilde{\kappa}_\tau),$$

where $D_{x_0+\tau,\tau}(u) := \Phi^{-1}\left(\mathbb{E}\left[S_{x_0+\tau,\tau}(u) \mid \tilde{\kappa}_\tau\right]\right)$

and

$$D'_{x_0+\tau,\tau}(u) = \frac{d}{d\kappa_\tau}\Phi^{-1}\left(\mathbb{E}\left[S_{x_0+\tau,\tau}(u) \mid \kappa_\tau\right]\right)\Big|_{\kappa_\tau=\kappa_{\tilde{\kappa}_\tau}}$$

$$= \frac{1}{\phi\left(\Phi^{-1}\left(\mathbb{E}\left[S_{x_0+\tau,\tau}(u) \mid \tilde{\kappa}_\tau\right]\right)\right)}\left(\frac{d}{d\kappa_\tau}\mathbb{E}\left[S_{x_0+\tau,\tau(u)|\kappa_\tau}\right]\Big|_{\kappa_\tau=\tilde{\kappa}_\tau}\right)$$

$$= \frac{1}{\phi\left(\Phi^{-1}\left(\mathbb{E}\left[S_{x_0+\tau,\tau}(u) \mid \tilde{\kappa}_\tau\right]\right)\right)}\left(\mathbb{E}\left[\frac{d}{d\kappa_\tau}S_{x_0+\tau,\tau(u)|\tilde{\kappa}_\tau}\right]\right),$$

with

$$\frac{d}{d\kappa_\tau}S_{x_0+\tau,\tau}(u) = S_{x_0+\tau,\tau}(u)\cdot\left(-\sum_{s=1}^{u}\beta_{x_0+\tau+s-1}\cdot\exp(\alpha_{x_0+\tau+s-1}+\beta_{x_0+\tau+s-1}\kappa_{\tau+s})\right).$$

Finally, the expectations in $D_{x_0+\tau,\tau}(u)$ and $D'_{x_0+\tau,\tau}(u)$ need to be computed by simulation.

Overview

The GNS procedure I

Note that the time-τ value (2) can be written as

$$\begin{aligned} V(\kappa_\tau) &= \mathbb{E}[H(\kappa_{\tau+}|\mathcal{F}_\tau)] = \mathbb{E}[H(\kappa_{\tau+}|\kappa_\tau)] \\ &= \int H(\kappa_{\tau+}) f(\kappa_{\tau+}|\kappa_\tau) d\kappa_{\tau+} \\ &= \int H(\kappa_{\tau+}) \frac{f(\kappa_{\tau+}|\kappa_\tau)}{g(\kappa_{\tau+})} g(\kappa_{\tau+}) d\kappa_{\tau+} \\ &= \mathbb{E}[H(\kappa_{\tau+}) \cdot W(\kappa_{\tau+}|\kappa_\tau)],\ \kappa_{\tau+} \sim g(\kappa_{\tau+}). \end{aligned} \tag{16}$$

where $g(\kappa_{\tau+})$ is a sampling density of the inner sample paths $\kappa_{\tau+}$ and $W(\kappa_{\tau+}|\kappa_\tau) = \frac{f(\kappa_{\tau+}|\kappa_\tau)}{g(\kappa_{\tau+})}$ is the likelihood ratio.

The GNS procedure II

Algorithm 2: Green Nested Simulation with Importance Sampling

Input: Initial mortality state κ_0, M (outer scenarios), $\tilde{N}$ (inner paths)

Output: Empirical distribution of $\hat{V}^{gns}_{\tilde{N}}(\kappa_\tau^{(i)})$ or $\hat{V}^{sn}_{\tilde{N}}(\kappa_\tau^{(i)})$

Outer Simulation:

for $i \leftarrow 1$ ***to*** M **do**

  Simulate $\kappa_\tau^{(i)} \sim f(\kappa_\tau|\kappa_0)$

  Inner Simulation:

  Simulate $\{\kappa_{\tau+}^{(j)}\}_{j=1}^{\tilde{\Gamma}} \sim g(\kappa_{\tau+})$

  Importance Weighting:

  Compute $W^{(ij)} = \frac{f(\kappa_{\tau+}^{(j)}|\kappa_\tau^{(i)})}{g(\kappa_{\tau+}^{(j)})}$ for $j = 1, ..., \tilde{\Gamma}$

  Estimation:

  if *self-normalized* **then**

    $\hat{V}^{sn}_{\tilde{\Gamma}}(\kappa_\tau^{(i)}) = \frac{\sum_{j=1}^{\tilde{\Gamma}} H(\kappa_{\tau+}^{(j)})\bar{W}^{(ij)}}{\sum_{j=1}^{\tilde{\Gamma}} \bar{W}^{(ij)}}$

    where $\bar{W}^{(ij)} = \frac{W^{(ij)}}{\frac{1}{\tilde{\Gamma}}\sum_{\ell=1}^{\tilde{\Gamma}} W^{(i\ell)}}$

  else

    $\hat{V}^{gns}_{\tilde{\Gamma}}(\kappa_\tau^{(i)}) = \frac{1}{\tilde{\Gamma}}\sum_{j=1}^{\tilde{\Gamma}} H(\kappa_{\tau+}^{(j)})W^{(ij)}$

  end

end

Construct empirical distribution of $\{\hat{V}^{gns}_{\tilde{\Gamma}}(\kappa_\tau^{(i)})\}_{i=1}^{M}$ or $\{\hat{V}^{sn}_{\tilde{\Gamma}}(\kappa_\tau^{(i)})\}_{i=1}^{M}$

Estimate risk measure $\rho(V_\tau)$

The GNS procedure III

Proposition 1

Consider any given outer scenario κ_τ such that $V(\kappa_\tau)$ is defined. Assume that $g(\kappa_{\tau+})$ such that $g(\kappa_{\tau+}) > 0$ whenever $H(\kappa_\tau, \kappa_{\tau+}) f(\kappa_{\tau+}|\kappa_\tau) \neq 0$. Then

$$\mathbb{E}\left[\hat{V}_{\tilde{\Gamma}}^{gns}(\kappa_\tau)\right] = V(\kappa_\tau), \tag{17}$$

$$\hat{V}_{\tilde{\Gamma}}^{gns}(\kappa_\tau) \xrightarrow{p} V(\kappa_\tau) \text{ and } \hat{V}_{\tilde{\Gamma}}^{gns}(\kappa_\tau) \xrightarrow{a.s.} V(\kappa_\tau) \text{ as } \tilde{\Gamma} \to \infty. \tag{18}$$

Moreover, if $Var[H(\kappa_{\tau+})W(\kappa_{\tau+}|\kappa_\tau)] < \infty$, then

$$\frac{\hat{V}_{\tilde{\Gamma}}^{gns}(\kappa_\tau) - V(\kappa_\tau)}{\sqrt{\frac{Var[H(\kappa_{\tau+})W(\kappa_{\tau+}|\kappa_\tau)]}{\tilde{\Gamma}}}} \xrightarrow{d} \mathcal{N}(0,1) \text{ as } \tilde{\Gamma} \to \infty. \tag{19}$$

The GNS procedure IV

Lemma 4

Consider any given outer scenario κ_τ such that $V(\kappa_\tau)$ is defined. Assume that $g(\kappa_{\tau+})$ such that $g(\kappa_{\tau+}) > 0$ whenever $H(\kappa_\tau, \kappa_{\tau+}) f(\kappa_{\tau+}|\kappa_\tau) \neq 0$. Then

$$\hat{V}^{sn}_{\tilde{\Gamma}}(\kappa_\tau) \xrightarrow{p} V(\kappa_\tau) \text{ and } \hat{V}^{sn}_{\tilde{\Gamma}}(\kappa_\tau) \xrightarrow{a.s.} V(\kappa_\tau) \text{ as } \tilde{\Gamma} \to \infty. \tag{20}$$

Lastly, when the risk factor κ_τ is modeled by a Markovian process, the likelihood ratio can be calculated as

$$W(\kappa_{\tau+}|\kappa_\tau) = \frac{f(\kappa_{\tau+}|\kappa_\tau)}{g(\kappa_{\tau+})} = \frac{f(\kappa_{\tau+1}|\kappa_\tau) f(\kappa_{\tau+2}|\kappa_{\tau+1}) \cdots f(\kappa_T|\kappa_{T-1})}{g_{\tau+1}(\kappa_{\tau+1}) f(\kappa_{\tau+2}|\kappa_{\tau+1}) \cdots f(\kappa_T|\kappa_{T-1})} = \frac{f(\kappa_{\tau+1}|\kappa_\tau)}{g_{\tau+1}(\kappa_{\tau+1})}, \tag{21}$$

where $g_{\tau+1}(\kappa_{\tau+1})$ is the sampling distribution for the risk factor at time $\tau+1$. Reducing the product of $(T-\tau)$ likelihood ratio calculations, i.e., $\frac{f(\kappa_{\tau+}|\kappa_\tau)}{g(\kappa_{\tau+})}$ to one, i.e., $\frac{f(\kappa_{\tau+1}|\kappa_\tau)}{g_{\tau+1}(\kappa_{\tau+1})}$, further reduces the comuputations needed fo the GNS procedure.

Mixture sampling distribution I

In the standard nested simulation, for each scenario $\kappa_\tau^{(i)}$, the sampling distribution for the corresponding inner sample path is $f(\kappa_{\tau+}|\kappa_\tau^{(i)})$. When reusing the j-th scenario's inner sample path to estimate the i-th scenario's time-τ value, the likelihood ratio is $\frac{f(\kappa_{\tau+}|\kappa_\tau^{(i)})}{f(\kappa_{\tau+}|\kappa_\tau^{(j)})}$. This is similar to the indiviual likelihood ratio (ILR) estimator in Feng and Staum (2017), which has a poor performance. Intuitively, the ILR estimator has a large variance when the sampling scenario $\kappa_\tau^{(j)}$ and the target scenario $\kappa_\tau^{(i)}$ are significantly different, as then the likelihood ratio may have a large or even infinite variance (despite expectation still being 1).

Mixture sampling distribution II

Specifically, given M outer scenarios $\kappa_\tau^{(1)}, ..., \kappa_\tau^{(M)}$, the collective sample of all M scenarios' inner sample paths, i.e., $\{\kappa_{\tau+}^{(ij)}, j = 1, ..., N, i = 1, ..., M\}$, can be viewed as a stratified sample of the mixture distribution

$$g(\kappa_{\tau+}) = \frac{1}{M} \sum_{i=1}^{M} f(\kappa_{\tau+} | \kappa_\tau^{(i)}). \tag{22}$$

The mixture distribution (22) weights the M conditional distributions equally because the same number of inner sample paths are generated in each outer scenario in a standard nested simulation. This mixture distribution can be employed in the GNS procedure with or without self-normalization.

Mixture sampling distribution III

That is noting that $\kappa_{\tau+}^{(ij)} \overset{i.i.d}{\sim} f(\kappa_{\tau+}|\kappa_{\tau}^{(i)})$ for $j = 1, ..., N, i = 1, ..., M$, for any scenario κ_τ we have

$$
\begin{aligned}
\mathbb{E}[\hat{V}_{\Gamma}^{gns}(\kappa_\tau)] &= \mathbb{E}\left[\frac{1}{MN}\sum_{j=1}^{N}\sum_{i=1}^{M} H(\kappa_{\tau}^{(ij)})\frac{f(\kappa_{\tau+}|\kappa_\tau)}{g(\kappa_{\tau+})}\right] \\
&= \frac{1}{N}\sum_{j=1}^{N}\frac{1}{M}\sum_{i=1}^{M}\mathbb{E}\left[H(\kappa_{\tau}^{(ij)})\frac{f(\kappa_{\tau+}|\kappa_\tau)}{g(\kappa_{\tau+})}\right], \ \kappa_{\tau+}^{(ij)} \sim f(\kappa_{\tau+}|\kappa_{\tau}^{(i)}) \\
&= \frac{1}{N}\sum_{j=1}^{N}\frac{1}{M}\sum_{i=1}^{M}\int H(\kappa_{\tau}^{(ij)})\frac{f(\kappa_{\tau+}|\kappa_\tau)}{g(\kappa_{\tau+})} f(\kappa_{\tau+}|\kappa_{\tau}^{(i)}) d\kappa_{\tau+} \\
&= \frac{1}{N}\sum_{j=1}^{N}\int H(\kappa_{\tau}^{(ij)})\frac{\frac{1}{M}\sum_{i=1}^{M} f(\kappa_{\tau+}|\kappa_{\tau}^{(i)})}{g(\kappa_{\tau+})} f(\kappa_{\tau+}|\kappa_\tau) d\kappa_{\tau+} \\
&= \frac{1}{N}\sum_{j=1}^{N}\int H(\kappa_{\tau}^{(ij)}) f(\kappa_{\tau+}|\kappa_\tau) d\kappa_{\tau+} = V(\kappa_\tau).
\end{aligned}
$$

Overview

Data and Model Specification I

- **Mortality data**: Female population (ages 60-89) from England and Wales (1962-2011), sourced from Human Mortality Database
- **Lee-Carter model**:
  - Estimated via Poisson MLE (Brouhns et al., 2002a)
  - Parameters: α_x, β_x, κ_t (See Figure 2)
  - Risk factor process: $\theta = -0.4962$, $\sigma = 0.8724$, $\kappa_0 = -14.21$
- **Assumptions**: Current time=0, maturity=T, horizon=τ, discount rate $r = 3\%$

Data and Model Specification II

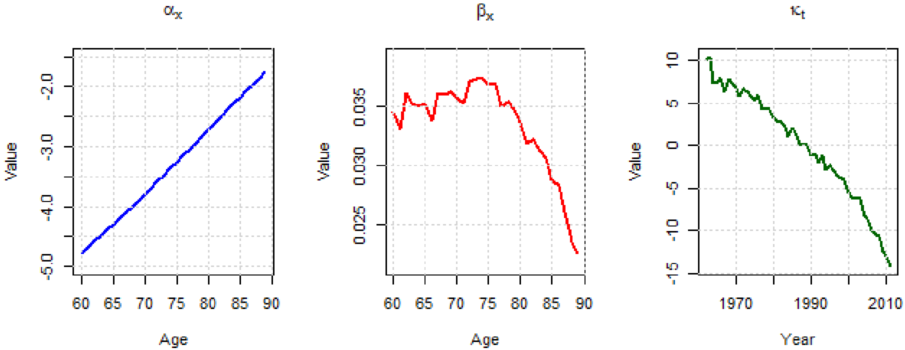

Figure 2: Estimates of α_x, β_x and κ_t in Lee-Carter model

Evaluation Metrics

- **95% CI Width**: Measures variability from inner path randomness
- **MSE**:

$$\mathrm{MSE}(\kappa_\tau) = \mathbb{E}\left[(\hat{V}(\kappa_\tau) - V(\kappa_\tau))^2 \mid \kappa_\tau\right]$$

- **IMSE** (accounts for outer scenario randomness):

$$\mathrm{IMSE} = \mathbb{E}\left[(\hat{V}(\kappa_\tau) - V(\kappa_\tau))^2\right] = \mathbb{E}[\mathrm{MSE}(\kappa_\tau)] \tag{23}$$

  - Decomposable into integrated variance and squared bias

Experimental Design

- **Outer scenarios**:
  - $M = 10^3$ scenarios from $\kappa_\tau^{(i)} \sim \mathcal{N}(\kappa_0 + \tau\theta, \tau\sigma_\epsilon^2)$
  - Implemented via equally-spaced quantiles for uniformity
- **Anchor scenario**: $\tilde{\kappa} = \mathbb{E}[\kappa_\tau] = \kappa_0 + \tau\theta$
- **IMSE estimation**:

$$\widehat{\mathrm{IMSE}} = \frac{1}{M}\sum_{i=1}^{M}\left[\frac{1}{R}\sum_{r=1}^{R}(\hat{V}_{(r)}(\kappa_\tau^{(i)}) - V(\kappa_\tau^{(i)}))^2\right], \tag{24}$$

  - $R = 10^3$ replications
  - $V(\kappa_\tau^{(i)})$ approximated via 10^7 inner paths when closed-form unavailable

Valuation Methods

- **Standard nested simulation**:
  - Inner paths $N \in \{1, 10, 100\}$
  - Total budgets: 10^3, 10^4, 10^5
- **Taylor approximation**:
  - Estimates $V(\tilde{\kappa}_\tau)$ and $V'(\tilde{\kappa}_\tau)$ via 10^3-10^5 inner paths
- **GNS procedure**:
  - Uses mixture distribution (22) as sampling distribution
  - Includes both regular and self-normalized estimators
  - Inner paths generated identically to standard method

Case 1: K-Call Option I

Parameters

- Notional: \$1
- Maturity: $T = 10$ years
- Strike: $K = -19.17$.(Through (5), we can obtain $K = \mathbb{E}[\kappa_T|\kappa_0] = \kappa_0 + T\theta = -14.21 + 10 \times (-0.4962) \approx 19.17$.)
- Risk horizon: $\tau = 5$ years

Method Comparison

- Closed-form available
- Taylor approximation exhibits downward bias

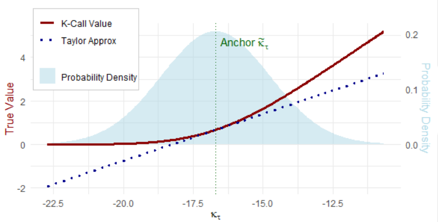

Figure 3: Problem settings for risk measurement of a K-call.

Case 1: K-Call Option II

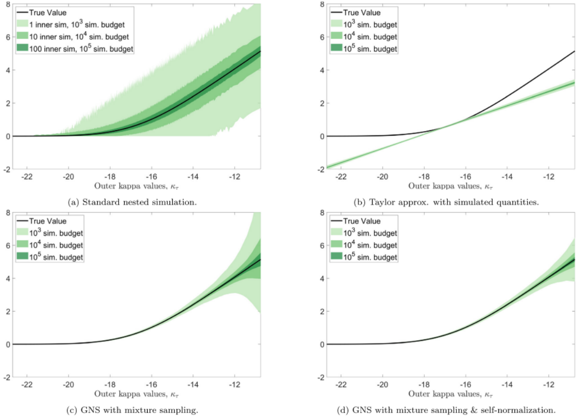

Figure 4: 95% confidence bands for the estimating $V(\kappa_\tau)$ for different valuation methods, K-call example.

Case 1: K-Call Option III

Table 1
Estimated IMSEs for the K-option case study.

Case study	K-option		
Simulation budget	10^3	10^4	10^5
Standard nested sim.	1.08	1.08×10^{-1}	1.08×10^{-2}
Taylor approx.	1.84×10^{-1}	1.83×10^{-1}	1.83×10^{-1}
Green nested sim.	1.55×10^{-2}	1.55×10^{-3}	1.49×10^{-4}
GNS, self-normalized	4.47×10^{-3}	4.92×10^{-4}	4.58×10^{-5}

Case 1: K-Call Option IV

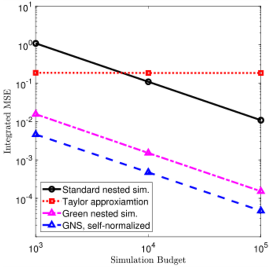

Figure 5: Estimated IMSEs for different valuation methods and simulation budgets when the number of outer simulations is set to $M = 1000$,K-call example.

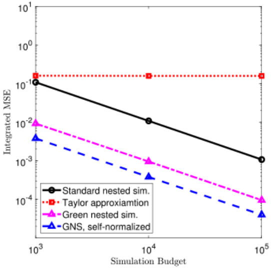

Figure 6: Estimated IMSEs for different valuation methods and simulation budgets when the number of outer simulations is set to $M = 100$,K-call example.

Case 2: q-call-spread I

Parameters

- Reference age: $x = 60$
- Time-to-maturity: $T = 10$ years
- Attachment Point(AP): 3.703×10^{-2}(5th percentile)
- Exhaustion Point (EP): 5.037×10^{-2}(95th percentile)
- Risk horizon: $\tau = 5$years

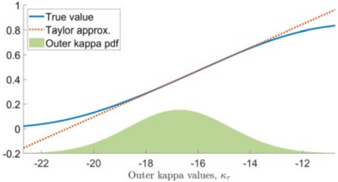

Figure 7: Problem settings for risk measurement of a q-call-spread.

Case 2: q-call-spread II

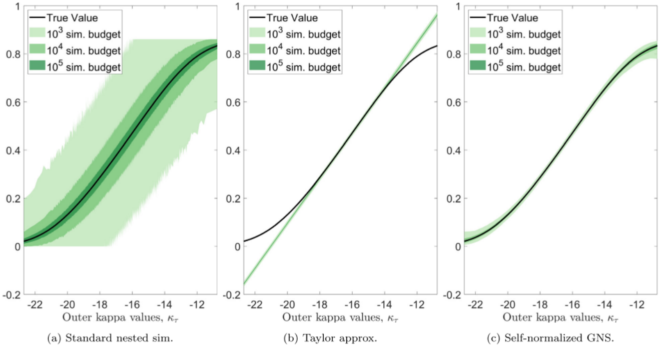

Figure 8: 95% confidence bands for the estimating $V(\kappa_\tau)$ for different valuation methods, q-call-spread example.

Case 2: q-call-spread III

Table 2
Estimated IMSEs for the q-call-spread case study.

Case study	q-Call-Spread		
Simulation budget	10^3	10^4	10^5
Standard nested sim.	2.90×10^{-2}	2.91×10^{-3}	2.90×10^{-4}
Taylor approx	4.01×10^{-4}	3.69×10^{-4}	3.66×10^{-4}
GNS, self-normalized	7.35×10^{-5}	7.27×10^{-6}	7.63×10^{-7}

Case 3: temporary life annuity I

Parameters

- Initial age: $x_0 = 60$
- Policy term: $T = 30$ years
- Risk horizon: $\tau = 10$ years
- Remaining liability at$\tau = 10$: $T - \tau = 20$-year annuity (if survived to age $x_0 + \tau = 70$)

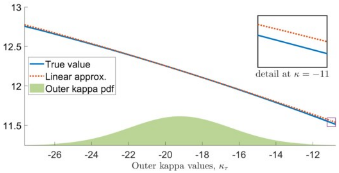

Figure 9: Problem settings for risk measurement of a temporary life annuity.

Case 3: temporary life annuity II

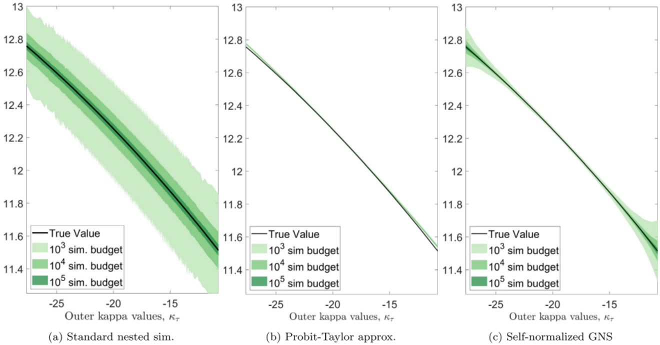

Figure 10: 95% confidence bands for the estimating $V(\kappa_\tau)$ for different valuation methods, temporary life annuity example.

Case 3: temporary life annuity III

Table 3
Estimated IMSEs for the temporary life annuity case study.

Case study	Temporary Life Annuity		
Simulation budget	10^3	10^4	10^5
Standard nested sim.	2.30×10^{-2}	2.30×10^{-3}	2.30×10^{-4}
(Probit-)Taylor approx	3.71×10^{-5}	1.81×10^{-5}	1.63×10^{-5}
GNS, self-normalized	1.27×10^{-4}	1.23×10^{-5}	1.28×10^{-6}

Overview

Conclusion

- Proposed Green Nested Simulation (GNS) for mortality-linked securities:
  - More accurate than standard nested simulation
  - Unbiased/asymptotically unbiased (vs biased approximations)
  - No customization needed for different securities
- Practical advantages:
  - Applicable to portfolios with dispersed age profiles
  - Potential for greater computational savings
- Extendable framework:
  - Can incorporate multiple future time points
  - Potential for output reuse in repeated experiments
- Future research directions:
  - Incorporation of additional risk factors (interest rates, stock indices)
  - Application to broader insurance products

References

[1] Ben Mingbin Feng and Johnny Siu-Hang Li and Kenneth Q. Zhou (2022)
Green nested simulation via likelihood ratio: Applications to longevity risk management
Insurance: Mathematics and Economics 106, 285 – 301.

[2] Paul Glasserman (2004)
Monte Carlo Methods in Financial Engineering
Springer vol. 53.

[3] Ben Mingbin Feng, Jeremy Staum (2017)
Green simulation: Reusing the output of repeated experiments
ACM Transactions on Modeling and Computer Simulation (TOMACS) 27(4),1-28.

Thanks