Insurance: Mathematics and Economics 113 (2023) 274–292

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Optimal risk management with reinsurance and its counterparty risk hedging

Yichun Chi [a], Tao Hu [b], Yuxia Huang [b,*]

[a] China Institute for Actuarial Science, Central University of Finance and Economics, Beijing 102206, China
[b] School of Insurance, Central University of Finance and Economics, Beijing 102206, China

ARTICLE INFO

Article history:
Received February 2023
Received in revised form September 2023
Accepted 12 September 2023
Available online 19 September 2023

JEL classification:
D81
G22
G33

Keywords:
Risk management
Counterparty risk
Hedging
Mossin's theorem
No-sabotage condition

ABSTRACT

In this paper, we revisit the study of an optimal risk management strategy for an insurer who wants to maximize the expected utility by purchasing reinsurance and managing reinsurance counterparty risk with a default-free hedging instrument, where the reinsurance premium is calculated by the expected value principle and the price of the hedging instrument equals the expected payoff plus a proportional loading. Different to previous studies, we exclude ex post moral hazard by imposing the no-sabotage condition on reinsurance contracts and derive the optimal strategy analytically. We find that the stop-loss reinsurance is always optimal, but the form of the optimal hedging payoff depends on the cost difference between reinsurance and hedging instrument. We further show that full risk transfer is optimal if and only if both reinsurance pricing and the hedging price are fair. Finally, numerical analyses are conducted to illustrate the effects of some interesting factors on the optimal risk management strategy.

1. Introduction

In practice reinsurance is a critical tool of a primary insurer to reduce risk exposure and maintain financial stability. Since the seminal work of Borch (1960), the study of optimal reinsurance design has drawn significant attention from practitioners and academics. Under the criterion of minimizing the variance of an insurer's risk exposure, Borch (1960) shows that a stop-loss reinsurance contract, which is full reinsurance above a deductible, is optimal when the reinsurance premium is calculated by the expected value principle. That is, the reinsurance premium equals the expected indemnity plus a proportional loading. Arrow (1963) obtains a similar result in favor of the stop-loss reinsurance when the optimization criterion changes to maximizing the expected utility of a risk-averse insurer's final wealth. Their studies have been generalized by using more sophisticated optimality criteria and/or more realistic premium principles; see, for example, Van Heerwaarden et al. (1989), Gajek and Zagrodny (2000), Kaluszka (2001), Cai and Tan (2007), Cai et al. (2008), Cheung (2010), Balbás et al. (2009), Chi and Tan (2011), Zhuang et al. (2016), Cheung and Lo (2017), Boonen and Jiang (2022), and the references therein.

In the afore-mentioned studies, optimal reinsurance design has been analyzed from the perspective of an insurer under an implicit assumption that the reinsurance contract is under full performance. In other words, reinsurance is default-free and an insurer can always receive full indemnity from a reinsurer when the loss occurs. Unfortunately, bankruptcy events of reinsurance companies happen periodically and the reinsurance contract is subject to counterparty risk. For instance, GIO Re and NewCap Re suddenly exited the market in 1999 and Reinsurance Australia Corporation was downgraded in 2000. When reinsurance contracts default, the insurer may only get part of indemnity or cannot timely receive the indemnity. We refer to Gatumel and Lemoyne de Forges (2013) and Bodoff (2013) for more discussions on the reinsurance counterparty risk. Thus, it is quite necessary to incorporate the nonperformance risk of reinsurance

* Corresponding author.
E-mail addresses: yichun@cufe.edu.cn (Y. Chi), TaoHu971@gmail.com (T. Hu), yxhuang1216@163.com (Y. Huang).

https://doi.org/10.1016/j.insmatheco.2023.09.003
0167-6687/

contracts into the design of an optimal reinsurance contract for an insurer. Admittedly, optimal reinsurance design with nonperformance risk has been studied in many papers such as Bernard and Ludkovski (2012), Asimit et al. (2013) and Cai et al. (2014).

Recently, Reichel et al. (2022) suggest managing the reinsurance counterparty risk with a default-free hedging instrument and investigate an optimal risk management strategy with reinsurance and hedging for a risk-averse insurer. Similar to Arrow (1963), they consider maximizing the expected utility of the insurer's final wealth and calculate the reinsurance premium by the expected value principle. They further assume that the hedging instrument is priced as the expected payoff plus a proportional loading. They show that the stop-loss reinsurance is optimal only when the hedging instrument is cheaper than reinsurance (i.e. the loading coefficient in hedging price is smaller than that of reinsurance pricing), and otherwise the marginal reinsurance indemnity may become strictly larger than 1. For the latter case, the reinsurance contract will incentivize the insurer to misreport the realized loss. Therefore, this reinsurance solution can hardly be used in practice because it may lead to ex post moral hazard.

To reduce ex post moral hazard, Huberman et al. (1983) suggest that the reinsurance contract should satisfy the incentive-compatible condition, which is also called no-sabotage condition in Carlier and Dana (2003). This condition asks both the reinsurer and the insurer to pay more for a larger realization of the loss. Equivalently, the marginal indemnity should always be non-negative but less than 1. Recently, this condition has been widely used in the optimal reinsurance design; see, e.g., Asimit et al. (2013), Cai et al. (2014), Zhuang et al. (2016), Xu et al. (2019), Chi and Wei (2020), and Boonen and Jiang (2022).

In this paper, we revisit the optimal risk management problem analyzed in Reichel et al. (2022) by imposing the no-sabotage condition on reinsurance contracts in order to preclude ex post moral hazard. The introduction of such condition brings great technical challenges and classical approaches become unavailable to solve the problem. We develop a new approach which combines a constructive method and a variational argument, and then derive the optimal solution analytically. Same with Arrow (1963), we show that the stop-loss reinsurance is always optimal. The optimal form of the hedging payoff, however, relies heavily on the cost comparison between reinsurance and the hedging instrument. More precisely, if reinsurance is relatively cheaper, the optimal hedging payoff is in a type of change loss; otherwise, the hedging instrument provides coverage of tail loss with a fixed proportion and all the middle risk when the reinsurer defaults. We further show that full risk is transferred if and only if both reinsurance and the hedging instrument are fairly priced (i.e. loading coefficients are zero). This result extends Mossin's theorem (Mossin, 1968) by incorporating the reinsurance counterparty risk hedging. Finally, we carry out numerical analyses to illustrate the effects of some interesting factors such as the loss given default (LGD) rate, the reinsurer's default probability and the hedging cost on the optimal risk management strategy.

The rest of the paper is organized as follows. We formulate a mathematical model of optimal risk management with reinsurance and its counterparty risk hedging in Section 2. In Section 3, a necessary and sufficient condition for an optimal solution is derived, by which optimal forms of reinsurance and hedging are obtained explicitly. Optimal solutions are then derived analytically and numerical analyses are conducted to illustrate the effects of some key factors on the optimal risk management strategy in Section 4. Section 5 concludes the paper. All the proofs are relegated to the Appendix.

2. Model formulation

Suppose that an insurer endowed with an initial wealth w faces an amount of risk X in a fixed time period, where X is non-negative, bounded and defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote the support of X by

$$S(X) = \Big\{x \ge 0 : \mathbb{P}\{X \in (x-\epsilon, x+\epsilon)\} > 0 \text{ for all } \epsilon > 0\Big\},$$

and the essential supremum of X by M, i.e., $M = \sup S(X) < \infty$.

In order to reduce the risk exposure, the insurer seeks to reach a reinsurance agreement with a reinsurer, in which an amount of risk $r(X)$ is ceded to the reinsurer. In the insurance economics literature, $r(x)$ is usually called the ceded loss function and is asked to satisfy the principle of indemnity, i.e., $0 \le r(x) \le x$. However, this condition is insufficient to rule out ex post moral hazard such that the insurer has an incentive to misreport the true realization of the loss. To reduce ex post moral hazard, Huberman et al. (1983) and Xu et al. (2019) suggest imposing an incentive-compatible condition on $r(x)$, which is also called no-sabotage condition in Carlier and Dana (2003). More specifically, both $r(x)$ and $x - r(x)$ are assumed to be increasing.[1] This condition together with the principle of indemnity is equivalent to absolute continuity of r with $0 \le r'(x) \le 1$ almost everywhere and $r(0) = 0$. Thus, in this paper we assume the set of admissible reinsurance contracts to be

$$\mathfrak{C} = \Big\{r(x) : r(0) = 0 \text{ and } 0 \le r'(x) \le 1 \text{ almost everywhere}\Big\}.$$

Unfortunately, in practice reinsurance companies may default such that insurers do not get full indemnity when the loss occurs (Bodoff, 2013). Therefore, it is quite necessary for the insurer to take into consideration the counterparty risk of the reinsurance contract. Following Mahul and Wright (2004) and Reichel et al. (2022), we assume that the reinsurer's default event D, which appears with probability p, is stochastically independent with the insurable risk X.[2] The reimbursement of the reinsurance contract with counterparty credit risk can then be given by $(1 - \tau\mathbb{I}_D)r(X)$, where $\mathbb{I}_A$ is the indicator function of an event A and $\tau \in (0, 1]$ is the LGD rate. To cover the reimbursement, the reinsurer has to collect reinsurance premiums from insurers. Same with Borch (1960) and Reichel et al. (2022), we assume that the reinsurance premium $\pi_R(r)$ is calculated by the expected value principle, i.e.,

$$\pi_R(r) = (1+\rho_R)\mathbb{E}[(1-\tau\mathbb{I}_D)r(X)] = (1+\rho_R)(1-p\tau)\mathbb{E}[r(X)],$$

where $\rho_R \ge 0$ is a safety loading coefficient.

[1] Throughout the paper, "increasing" and "decreasing" mean "non-decreasing" and "non-increasing", respectively.

[2] This assumption is met when the reinsurer's default is mainly induced by investment risks while its underwriting risk is well managed and fully diversified globally. The meltdown of AIG during the 2008 financial crisis is a typical example.

To manage the counterparty risk of the reinsurance contract, Reichel et al. (2022) suggest using a financial hedging instrument with payoff $h(X)\mathbb{I}_D$ for some non-negative measurable function h.[3] This hedging instrument can be a credit default swap (CDS) or a letter of credit issued by another legally unaffiliated financial institution such as an investment bank. With this hedging, the insurer can obtain an amount of $h(X)$ once the reinsurer defaults. Following Reichel et al. (2022), we assume that the hedging instrument is default-free[4] and priced by

$$\pi_H(h) = (1+\rho_H)\mathbb{E}[h(X)\mathbb{I}_D] = p(1+\rho_H)\mathbb{E}[h(X)]$$

for some nonnegative loading coefficient $\rho_H \geq 0$.

In the presence of reinsurance coverage $r(X)$ and hedging $h(X)$, the insurer's final wealth becomes

$$W_{r,h}(X,D) = w - X + (1-\tau\mathbb{I}_D)r(X) + h(X)\mathbb{I}_D - \pi_R(r) - \pi_H(h). \tag{2.1}$$

In this paper, we assume that the insurer is risk-averse and endowed with a utility function $u(\cdot)$ satisfying $u'(\cdot) > 0$ and $u''(\cdot) < 0$, then the expected utility of the insurer's final wealth is

$$\mathcal{L}(r,h) = \mathbb{E}\left[u(W_{r,h}(X,D))\right] = p\mathbb{E}[u(W^d_{r,h}(X))] + (1-p)\mathbb{E}[u(W^s_{r,h}(X))]$$

where

$$\begin{cases} W^d_{r,h}(X) := w - X + (1-\tau)r(X) + h(X) - \pi_R(r) - \pi_H(h); \\ W^s_{r,h}(X) := w - X + r(X) - \pi_R(r) - \pi_H(h). \end{cases}$$

The objective of this paper is to design an optimal risk management strategy with reinsurance and hedging for the insurer who wants to maximize the expected utility of the final wealth. Mathematically, this problem can be formulated as

$$\max_{r\in\mathfrak{C}, h\geq 0} \mathcal{L}(r,h). \tag{2.2}$$

Especially when $p = 0$, there is no counterparty risk in the reinsurance contract, then the hedging instrument becomes useless and Problem (2.2) reduces to the classical Arrow's model, which has already been studied by Arrow (1963). The other extreme case of $p = 1$ seldom appears in practice because few insurers would like to negotiate an agreement with a reinsurer who definitely defaults. Thus, in this paper we only focus on the case of $p \in (0,1)$.

Notably, the only difference between Problem (2.2) and the optimal reinsurance-hedging problem studied in Section 3 of Reichel et al. (2022) is that the reinsurance contract $r(x)$ is asked to satisfy the no-sabotage condition in this paper. The introduction of this condition brings great technical challenges and their approach becomes unavailable. In the next section, we will attempt to derive an explicit form of reinsurance and hedging by developing a new technical approach.

3. Optimal reinsurance-hedging forms

In this section, we attempt to derive forms of optimal reinsurance-hedging strategies. To proceed, we analyze the uniqueness of the solution to Problem (2.2).

Proposition 3.1. *The solution is unique in the sense that $(r_1(X), h_1(X)) = (r_2(X), h_2(X))$ almost surely for any two solutions $\{(r_i(x), h_i(x)) : i = 1, 2\}$ to Problem (2.2) if one of the following conditions is satisfied: (i) $0 \in S(X)$; (ii) $\rho_R + \rho_H \neq 0$.*

In practice, the loss X usually possesses a positive probability at zero, and the reinsurance premium or the price of hedging instrument often contains a positive loading coefficient. Therefore, we can safely conclude the uniqueness of the solution in most situations of practical interest.

Next, we derive an optimal form of reinsurance-hedging strategies, and present the result in the following theorem.

Theorem 3.2.

(i) *For $\tau \in (0,1)$, an optimal solution (r^*, h^*) to Problem (2.2) can be given by*

$$\begin{cases} r^*(x) = (x-l)_+ - (x-m)_+ + (x-t)_+; \\ h^*(x) = (x-c)_+ - (1-\tau)(x-t)_+ \end{cases} \tag{3.1}$$

for some parameters l, m, c, t satisfying $0 \leq l \leq m \leq c \leq t \leq M$, where $(x)_+ := \max\{x, 0\}$.

(ii) *For $\tau = 1$, an optimal solution to Problem (2.2) is*

$$\begin{cases} r^*(x) = r_a(x) := (x-a)_+; \\ h^*(x) = (x-b)_+ \end{cases} \tag{3.2}$$

for some $a, b \in [0, M]$.

[3] To reduce the moral hazard in hedging, it may be necessary to require $(1-\tau)r'(x) + h'(x) \in [0,1]$ and $h'(x) \geq 0$ almost everywhere. However, these constraints will be found redundant in the subsequent analysis.

[4] It is worth noting that this default-free assumption is a little strict and the insurer should be concerned about the counterparty risk of hedging instruments. Luckily, Bodoff (2013) points out that the counterparty risk of CDS can be effectively reduced by posting collateral and the current regulatory initiatives after the passage of the Dodd-Frank Act.

In the appendix, we prove Theorem 3.2 by showing that any admissible reinsurance-hedging strategy (r, h) is suboptimal to a strategy with the form of (3.1) or (3.2). The proof provides a way to improve any suboptimal reinsurance-hedging strategy and reduces the infinite-dimensional maximization problem (2.2) to an optimization problem with a small number of variables. Furthermore, noting that the objective function $\mathcal{L}(r^*, h^*)$ is continuous in parameters a, b, c, l, m and t, we can conclude by Weierstrass theorem that the optimal solution to Problem (2.2) must exist.

In the absence of the hedging instrument and the incentive-compatible condition, Mahul and Wright (2004) find that the optimal indemnity offers more than full coverage above a positive deductible. Bernard and Ludkovski (2012) extend Mahul and Wright (2004)' model by assuming that the LGD rate is stochastically increasing in the loss, and obtain a similar optimal reinsurance form. This optimality of disappearing deductible is also obtained in Reichel et al. (2022) when the hedging instrument is introduced. Unfortunately, the disappearing deductible is seldom used in practice because it may lead to ex post moral hazard (Huberman et al., 1983).

In this paper, we impose the incentive-compatible condition on the ceded loss function $r(x)$, and find surprisingly that the optimal hedging payoff also satisfies this condition, i.e., $h^* \in \mathfrak{C}$. Furthermore, the optimal total marginal payoff with reinsurance and hedging $(1 - \tau \mathbb{I}_D) r^{*\prime}(X) + h^{*\prime}(X)\mathbb{I}_D$ is less than one, and it equals 1 when $X \geq t$. In other words, the insurer wouldn't like to over-hedge the insurance risk, but cedes all the tail risk regardless of whether the reinsurer defaults. This result is quite similar to Arrow (1963)' finding.

In contrast to the optimality of stop loss for the case of total default (i.e., $\tau = 1$), the optimal reinsurance-hedging strategy form in Theorem 3.2 appears a little complicated for partial default. It is useful to simplify this optimal form. To achieve this goal, we need the following theorem, which presents a necessary and sufficient condition to the optimal strategy (r^*, h^*).

Theorem 3.3. *Admissible strategy (r^*, h^*) with $h^*(x) \in \mathfrak{C}$ is a solution to Problem (2.2) if and only if*

$$r^{*\prime}(x) = \begin{cases} 1, & \Phi_1(x; r^*, h^*) > \gamma_R; \\ 0, & \Phi_1(x; r^*, h^*) < \gamma_R \end{cases} \tag{3.3}$$

and

$$\Phi_2(x; r^*, h^*) \leq \gamma_H, \quad h^{*\prime}(x) = 0 \text{ if } \Phi_2(x; r^*, h^*) < \gamma_H \tag{3.4}$$

for all $x \in [0, M)$ except a set with Lebesgue measure zero, where $\gamma_R := (1 - p\tau)(1 + \rho_R)$, $\gamma_H := p(1 + \rho_H)$ and functions $\Phi_i(x; r^*, h^*)$ are given by

$$\Phi_1(x; r^*, h^*) = \frac{p(1-\tau)\mathbb{E}[u'(W^d_{r^*,h^*}(X))|X > x] + (1-p)\mathbb{E}[u'(W^s_{r^*,h^*}(X))|X > x]}{p\mathbb{E}[u'(W^d_{r^*,h^*}(X))] + (1-p)\mathbb{E}[u'(W^s_{r^*,h^*}(X))]} \tag{3.5}$$

and

$$\Phi_2(x; r^*, h^*) = \frac{p\mathbb{E}[u'(W^d_{r^*,h^*}(X))|X > x]}{p\mathbb{E}[u'(W^d_{r^*,h^*}(X))] + (1-p)\mathbb{E}[u'(W^s_{r^*,h^*}(X))]}. \tag{3.6}$$

Similar to Chi and Wei (2020), the above theorem derives a necessary and sufficient condition for an optimal strategy by using some variational arguments. In contrast to the optimization problem with only one decision variable in Chi and Wei (2020), our problem contains decision variables of reinsurance contract r as well as hedging strategy h. As a consequence, our necessary and sufficient condition becomes much more complicated. Although optimal solutions seem very challenging to be derived directly from this condition, it is very helpful to judge whether a reinsurance-hedging strategy is optimal. The result on the optimality of full risk transfer or no risk transfer is presented in the following corollary.

Corollary 3.4.

(i) $(r^*, h^*) = (x, \tau x)$ *if and only if* $\rho_R = 0$ *and* $\rho_H = 0$*;*
(ii) $(r^*, h^*) = (0, 0)$ *if and only if* $\frac{u'(w-M)}{\mathbb{E}[u'(w-X)]} \leq 1 + \rho_R \wedge \rho_H$, *where* $x \wedge y := \min\{x, y\}$.

Notably, the optimal strategy $(x, \tau x)$ is a special case of (r^*, h^*) in Theorem 3.2 with $0 = l = m = c = t$ for $\tau \in (0, 1)$ and $a = b = 0$ for $\tau = 1$. Likewise, the strategy $(0, 0)$ corresponds to the parameters with $l = m = c = t = M$ for $\tau \in (0, 1)$ and $a = b = M$ for $\tau = 1$. Under the strategy of $(r, h) = (x, \tau x)$, the insurer cedes all the insurable risk and the default risk as

$$W^d_{r,h}(X) = W^s_{r,h}(X) = w - (\gamma_R + \tau\gamma_H)\mathbb{E}[X].$$

In the classical Arrow's model without default, Mossin (1968) demonstrates that it is optimal for an cedent to cede all the risk if and only if the contract price is fair. The above corollary extends Mossin's result by showing that full risk transfer is optimal if and only if both reinsurance and the hedging instrument are fairly priced. Corollary 3.4 also shows that no risk will be ceded if and only if both reinsurance and hedging becomes very costly.

In order to exclude these two trivial cases and simplify the analysis, we make the following assumption:

Assumption 3.1. *(i) The loading coefficients ρ_R and ρ_H satisfy*

$$(1 + \rho_R)(1 + \rho_H) \neq 1 \quad \text{and} \quad 1 + \rho_R \wedge \rho_H < \frac{u'(w - M)}{\mathbb{E}[u'(w - X)]};$$

(ii) $S(X) = [0, M]$.

The assumption of $(1+\rho_R)(1+\rho_H) \neq 1$ means that $\rho_R \neq 0$ or $\rho_H \neq 0$ or both, and it excludes the optimality of full risk transfer. Furthermore, the assumption of $S(X) = [0, M]$ allows the probability jump at *zero* and is satisfied by many loss distribution functions used in actuarial science. This assumption together with Theorem 3.3 can simplify the optimal form of reinsurance-hedging strategies obtained in Theorem 3.2.

Theorem 3.5. *Under Assumption 3.1,*

(i) *If $\rho_R > \rho_H$, then*

$$\begin{cases} r^*(x) = (x-t)_+; \\ h^*(x) = (x-c)_+ - (1-\tau)(x-t)_+ \end{cases} \tag{3.7}$$

for some $0 < c < t \leq M$.

(ii) *If $\rho_R < \rho_H$, the optimal solution (r^*, h^*) can be given by*

$$\begin{cases} r^*(x) = (x-l)_+; \\ h^*(x) = \tau(x-t)_+ \end{cases} \tag{3.8}$$

for some $0 \leq l < t \leq M$. Especially when $\mathbb{P}\{X > 0\} > \frac{1-\tau}{\gamma_R}$, the optimal deductible level l must be positive.

(iii) *If $\rho_R = \rho_H$, the optimal strategy (r^*, h^*) is given by*

$$\begin{cases} r^*(x) = (x-d^*)_+; \\ h^*(x) = \tau(x-d^*)_+, \end{cases} \tag{3.9}$$

where $d^* := \sup\left\{d \in [0, M] : \frac{\mathbb{E}[u'(w - X\wedge d - (1+\rho_R)\mathbb{E}[(X-d)_+])]}{u'(w-d-(1+\rho_R)\mathbb{E}[(X-d)_+])} \geq \frac{1}{1+\rho_R}\right\} \in (0, M)$.

Similar to Arrow (1963)' result, we can find from the above theorem that in the presence of reinsurance counterparty risk, the insurer would also like to purchase the stop-loss reinsurance and cede all the tail risk for any level of LGD rate when the hedging instrument is introduced. This finding is quite different from Mahul and Wright (2004) and Bernard and Ludkovski (2012), who obtain the optimality of more than full coverage above a deductible when hedging counterparty risk is not considered. It is also necessary to point out that Reichel et al. (2022) derive the same optimal strategy form as ours for the case of $\rho_R \geq \rho_H$. However, for $\rho_R < \rho_H$, our optimal solution form is quite different to theirs. In other words, the incentive-compatible condition plays an important role in the design of an optimal reinsurance-hedging strategy.

Interestingly, regardless of the value of the LGD rate, the above theorem indicates that both optimal reinsurance and optimal hedging payoff often contain a positive deductible even if only one contract is unfairly priced. This greatly extends the Mossin's theorem. In addition, when $\rho_R = \rho_H$, the problem collapses to Arrow's model whose optimal deductible level has been derived explicitly by Chi (2019). Further comparing the insurer's final wealth without and with default, we can get from the above theorem that

$$W^s_{r^*,h^*}(X) - W^d_{r^*,h^*}(X) = \tau r^*(X) - h^*(X) = \begin{cases} (x-t)_+ - (x-c)_+, & \rho_R > \rho_H; \\ \tau\left\{(x-l)_+ - (x-t)_+\right\}, & \rho_R < \rho_H \end{cases}$$

for some $0 < c < t$ and $0 \leq l < t$. For the case of $\rho_R < \rho_H$, the insurer's final wealth is reduced by the default. However, once reinsurance becomes more costly than the hedging instrument, the above equation shows that the insurer has a larger final wealth in the event of default. Consequently, the insurer may prefer a less creditworthy reinsurer, if reinsurance is very costly and there is a default-free financial instrument to hedge the reinsurer's counterparty risk with very cheap price. It may lead to an increase of systemic risk once this happens over a sufficiently large part of the reinsurance market.

4. Optimal risk management strategies

In Corollary 3.4 and Theorem 3.5, the optimal solution to Problem (2.2) has been derived explicitly for two special cases:

$$\rho_R = \rho_H \quad \text{and} \quad \frac{u'(w-M)}{\mathbb{E}[u'(w-X)]} \leq 1 + \rho_R \wedge \rho_H.$$

In this section, we continue to use Theorem 3.5 and Assumption 3.1 to derive optimal solutions for two remaining cases: $\rho_R > \rho_H$ and $\rho_R < \rho_H$.

4.1. The case of $\rho_R > \rho_H$

By Theorem 3.5, we know that the optimal solution can be given by

$$r^*(x) = r_t(x) = (x-t)_+ \quad \text{and} \quad h^*(x) = r_c(x) - (1-\tau)r_t(x) = (x-c)_+ - (1-\tau)(x-t)_+$$

for some $0 < c < t \leq M$. Thus, $W^d_{r^*,h^*}(X)$ and $W^s_{r^*,h^*}(X)$ can be rewritten as

$$\begin{cases} W^d_X(c,t) := W^d_{r^*,h^*}(X) = w - X\wedge c - \pi_R(r_t) - \pi_H(r_c - (1-\tau)r_t); \\ W^s_X(c,t) := W^s_{r^*,h^*}(X) = w - X\wedge t - \pi_R(r_t) - \pi_H(r_c - (1-\tau)r_t). \end{cases}$$

Proposition 4.1. *Assume $\rho_R > \rho_H$ and let Assumption 3.1 be satisfied. Define*

$$v_\rho = \inf\{x \geq 0 : \mathbb{P}(X > x) \leq \rho\}, \forall \rho \geq 0 \quad \text{and} \quad c_0^* = \inf\left\{c \in [v_{\frac{1}{\gamma_H}}, M) : \psi_0(c) \geq 1 + \rho_H\right\},$$

where

$$\psi_0(c) = \frac{u'(w - c - \pi_H(r_c))}{p\mathbb{E}[u'(W_X^d(c, M))] + (1-p)\mathbb{E}[u'(W_X^s(c, M))]}$$

is strictly increasing over $[v_{1/\gamma_H}, M]$ and strictly less than $1 + \rho_H$ for any $c < v_{1/\gamma_H}$.

The strategy $(0, (x - c_0^*)_+)$ is an optimal solution to Problem (2.2) if and only if

$$\frac{u'(w - M - \pi_H(r_{c_0^*}))}{u'(w - c_0^* - \pi_H(r_{c_0^*}))} \leq 1 + \frac{(\rho_R - \rho_H)(1 - p\tau)}{(1 + \rho_H)(1 - p)}. \tag{4.1}$$

If condition (4.1) is not met, parameters c and t in the optimal solution (r^*, h^*) should satisfy

$$\begin{cases} 1 + \rho_H & = \frac{u'(w - c - \pi_R(r_t) - \pi_H(r_c - (1-\tau)r_t))}{p\mathbb{E}[u'(W_X^d(c,t))] + (1-p)\mathbb{E}[u'(W_X^s(c,t))]}; \\ 1 + \frac{(\rho_R - \rho_H)(1 - p\tau)}{(1+\rho_H)(1-p)} & = \frac{u'(w - t - \pi_R(r_t) - \pi_H(r_c - (1-\tau)r_t))}{u'(w - c - \pi_R(r_t) - \pi_H(r_c - (1-\tau)r_t))}, \end{cases} \quad 0 < c < t < M. \tag{4.2}$$

From the above proposition, we can see that c_0^* does not depend on the safety loading coefficient ρ_R and the LGD rate τ. Therefore, given a ρ_H, (4.1) implies that there exists a threshold $\bar{\rho}_R > \rho_H$ such that $(0, (x - c_0^*)_+)$ is an optimal solution if and only if $\rho_R \geq \bar{\rho}_R$. In other words, if the reinsurance is very costly, the insurer wouldn't like to purchase any reinsurance and may transfer some tail risk through the financial hedging instrument. On the other hand, a smaller τ makes condition (4.1) to be more easily met. That is, a lower LGD rate incentivizes the insurer not to purchase reinsurance. It is quite counterintuitive at first glance. However, it can be explained as follows: when the reinsurance is more costly than the hedging instrument, a lower LGD rate pushes the insurer to pay a higher premium for the same contract. If the loading coefficient is higher for reinsurance than for the hedging instrument, then this effect overcomes the gain in effective reinsurance protection, thereby making reinsurance less attractive.

Following, we will analyze numerically the effects of some factors such as the loss distribution, the LGD rate, the default probability and the hedging cost on the optimal reinsurance-hedging strategy.

Example 4.1. We make the following benchmark assumptions:

(i) The risk X follows a truncated exponential distribution with the probability density function

$$f_X(x) = \frac{0.7e^{-0.7x}}{1 - e^{-7}} \mathbb{I}_{\{0 \leq x \leq 10\}};$$

(ii) The insurer has a power utility function $u(w) = w^{\frac{1}{2}}$;
(iii) The parameters are set by $\rho_R = 0.3$, $\rho_H = 0.1$, $p = 0.1$, $\tau = 0.8$ and $w \in \{20, 25\}$.

It is easy to verify that Assumption 3.1 is satisfied under these settings.

When $w = 25$, we can obtain $c_0^* = 5.57$ and condition (4.1) is satisfied. Thus, the reinsurance-hedging strategy $(0, (x - 5.57)_+)$ is an optimal solution to Problem (2.2). When the insurer's initial wealth w changes to 20, condition (4.1) is not met, and the optimal solution becomes $((x - 9.13)_+, (x - 4.71)_+ - 0.2(x - 9.13)_+)$ by solving Eq (4.2). Thus, the insurer's initial wealth plays an important role in the risk transfer.

We also carry out the numerical analyses to illustrate the effects of the LGD rate, the hedging cost and the default probability on the optimal solution by setting $w = 20$. See Figs. 1, 3 and 5 for more details. Notice that the optimal coefficients c and t are decreasing in the LGD rate τ. It means that more reinsurance coverage is needed for a larger LGD rate. This numerical result is consistent with the comments following Proposition 4.1. However, the optimal parameter t has an indeterminate effect with respect to the change of ρ_H. Surprisingly for very small ρ_H, the reinsurance demand decreases even though the hedging becomes more costly. In regard to the effect of the default probability, we can find that the optimal c is decreasing in the default probability p but the optimal coefficient t is increasing. It means that the insurer will reduce the reinsurance demand and transfer more risk by using the hedging instrument when reinsurance becomes more likely to default. It is quite consistent with the intuition.

Finally, we test the robustness of the above numerical results by changing the exponential loss distribution to Pareto distributed one with

$$f_X(x) = \frac{36}{35} \frac{10^3}{(x + 10)^4} \mathbb{I}_{\{x \in (0, 10]\}}$$

and $\mathbb{P}\{X = 0\} = 0.7$. Comparing Figs. 1, 3 and 5 with Figs. 2, 4, and 6, we can easily see that numerical results are robust with respect to this change of the loss distribution.

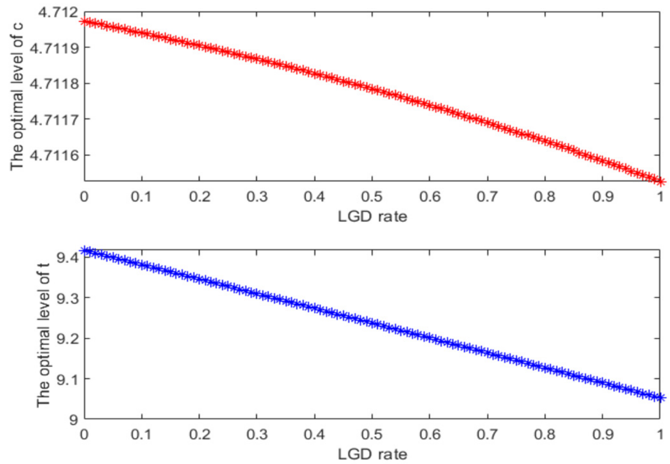

Fig. 1. The effect of the LGD rate on the optimal parameters c and t under the exponential loss distribution.

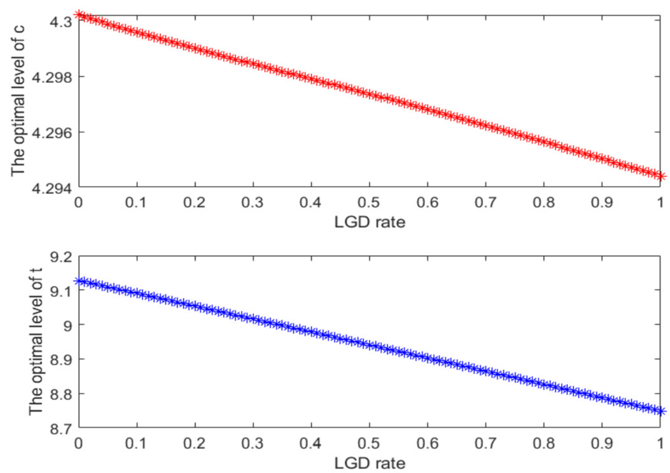

Fig. 2. The effect of the LGD rate on the optimal parameters c and t under the Pareto loss distribution.

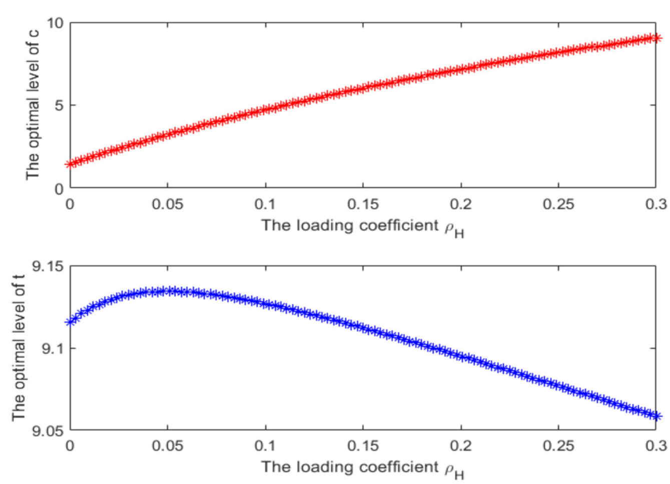

Fig. 3. The effect of ρ_H on the optimal parameters c and t under the exponential loss distribution.

4.2. The case of $\rho_R < \rho_H$

Under the assumption of $\rho_R < \rho_H$, Theorem 3.5 indicates that the optimal solution to Problem (2.2) can be given by

$$r^*(x) = r_l(x) = (x - l)_+ \quad \text{and} \quad h^*(x) = \tau(x - t)_+$$

for some $0 \le l < t \le M$. In this subsection, we attempt to derive the optimal parameters l and t explicitly.

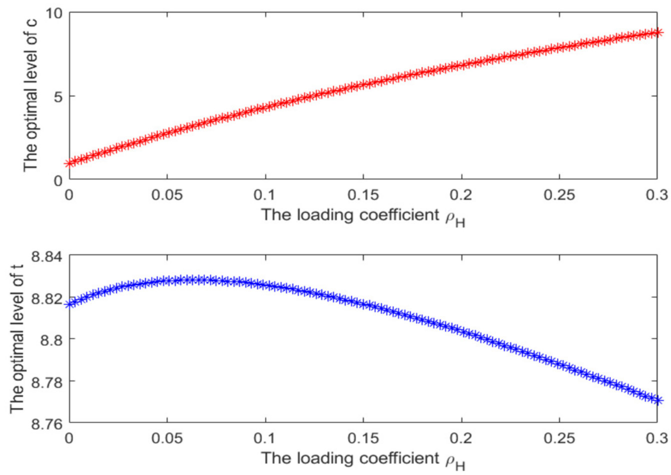

Fig. 4. The effect of ρ_H on the optimal parameters c and t under the Pareto loss distribution.

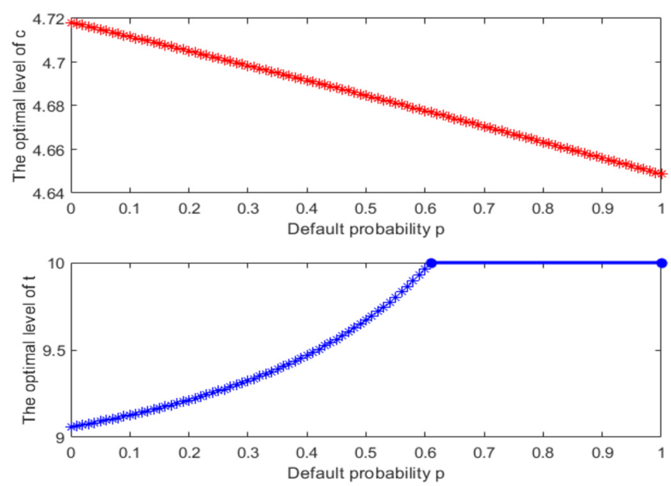

Fig. 5. The effect of the default probability on the optimal parameters c and t under the exponential loss distribution.

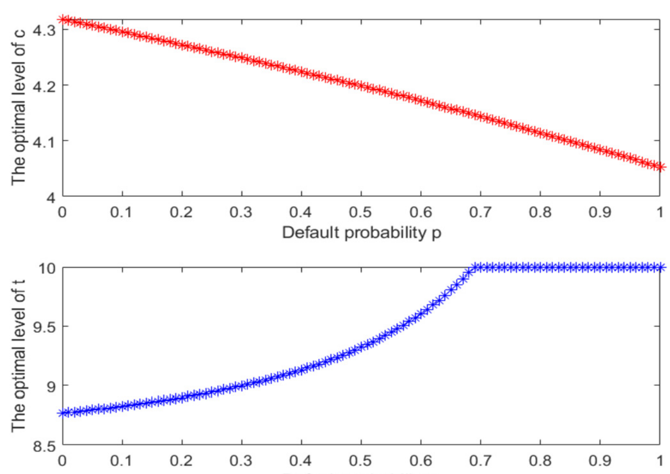

Fig. 6. The effect of the default probability on the optimal parameters c and t under the Pareto loss distribution.

To proceed, we define

$$\psi_1(t) = \frac{u'(w - \tau t - \kappa(t))}{p\mathbb{E}[u'(w - \tau \times X \wedge t - \kappa(t))] + (1-p)u'(w - \kappa(t))}, \tag{4.3}$$

where $\kappa(t) = \pi_R(r_0) + \pi_H(\tau \times r_t)$. It is trivial that $\psi_1(t) < \frac{1}{p\mathbb{P}\{X>t\}} \le 1 + \rho_H$ for any $t < v_{1/\gamma_H}$. Furthermore, $\psi_1(t)$ is strictly increasing over $[v_{1/\gamma_H}, M]$ because

$$\begin{aligned}\psi_1'(t) &= \frac{\tau u''(w-\tau t-\kappa(t))(-1+\gamma_H\mathbb{P}\{X>t\})(1-p\psi_1(t)\mathbb{P}\{X>t\})}{p\mathbb{E}[u'(w-\tau\times X\wedge t-\kappa(t))]+(1-p)u'(w-\kappa(t))} \\ &\quad -\frac{\tau\gamma_H\psi_1(t)\mathbb{P}\{X>t\}\left(p\mathbb{E}[u''(w-\tau X-\kappa(t))\mathbb{I}_{\{X\le t\}}]+(1-p)u''(w-\kappa(t))\right)}{p\mathbb{E}[u'(w-\tau\times X\wedge t-\kappa(t))]+(1-p)u'(w-\kappa(t))} \\ &> 0,\end{aligned}$$

where the inequality is derived by the facts $1-p\psi_1(t)\mathbb{P}\{X>t\}>0$ and $-1+\gamma_H\mathbb{P}\{X>t\}<0$ for any $t>v_{1/\gamma_H}$. Therefore, we can define

$$t_1=\sup\{t\in[0,M]:\psi_1(t)<1+\rho_H\}\ge v_{1/\gamma_H}. \tag{4.4}$$

Proposition 4.2. *Under Assumption 3.1 and the assumption of $\rho_R<\rho_H$, $r^*(x)=x$ if and only if*

$$\gamma_R\le\frac{p(1-\tau)\mathbb{E}[u'(w-\tau\times X\wedge t_1-\kappa(t_1))|X>0]+(1-p)u'(w-\kappa(t_1))}{p\mathbb{E}[u'(w-\tau\times X\wedge t_1-\kappa(t_1))]+(1-p)u'(w-\kappa(t_1))}. \tag{4.5}$$

Furthermore, if condition (4.5) is met, $(x,\tau(x-t_1)_+)$ is an optimal solution to Problem (2.2).

The above proposition provides a necessary and sufficient condition for the optimality of full reinsurance when $\rho_R<\rho_H$. While this condition is not met for the case of $\mathbb{P}\{X>0\}>\frac{1-\tau}{\gamma_R}$ according to Theorem 3.5, it is satisfied by the following example.

Example 4.2. In this example, we assume $w=20$, $p=0.7$, $\rho_R=0.01$, $\rho_H=0.1$ and use other benchmark assumptions in Example 4.1. Under these settings, Assumption 3.1 is satisfied and we have $t_1=5.041$. Then, it is easy to verify that condition (4.5) is satisfied. Thus, Proposition 4.2 indicates that the optimal reinsurance-hedging strategy can be given by $(x,0.8(x-5.041)_+)$.

The above example indicates that full reinsurance can still be optimal even if the reinsurance pricing is unfair. By comparing this result with Mossin's theorem (Mossin, 1968), it must be admitted that incorporating the reinsurance default risk greatly affects the insurer's risk transfer decision.

If condition (4.5) is not satisfied, then the analysis of Problem (2.2) with $\rho_R<\rho_H$ can be simplified to solving the following optimization problem

$$\min_{0<l<t\le M}\Gamma(l,t):=\mathcal{L}\left((x-l)_+,\tau(x-t)_+\right). \tag{4.6}$$

Similar to Proposition 4.1, we can justify whether the optimal solution appears at the boundary $\{(l,M):0<l<M\}$ or not. The following proposition can be derived directly from Theorem 3.3 and Theorem 3.5, and its proof is neglected.

Proposition 4.3. *Assume $\rho_R<\rho_H$, $1+\rho_R<\frac{u'(w-M)}{\mathbb{E}[u'(w-X)]}$, condition (4.5) fails to satisfy and the loss X has a continuous strictly increasing distribution function on $(0,M]$. The strategy $((x-l^*)_+,0)$ for some $l^*\in(0,M)$ is a solution to Problem (2.2) if and only if*

$$\Phi_1(l^*;(x-l^*)_+,0)=\gamma_R\quad\text{and}\quad\lim_{t\uparrow M}\Phi_2(t;(x-l^*)_+,0)\le\gamma_H. \tag{4.7}$$

If the above condition fails to satisfy for any $l^*\in(0,M)$, then the optimal solution to Problem (2.2) can be given by $((x-l)_+,\tau(x-t)_+)$ for some l and t satisfying $0<l<t<M$ and

$$\begin{cases}1+\rho_H &= \frac{u'(w-\tau t-(1-\tau)l-\pi_R(r_l)-\pi_H(\tau\times r_t))}{p\mathbb{E}[u'(\omega_d(X;l,t))]+(1-p)\mathbb{E}[u'(\omega_s(X;l,t))]};\\ \frac{\gamma_R}{1+\rho_H} &= \frac{p(1-\tau)\mathbb{E}[u'(\omega_d(X;l,t))|X>l]+(1-p)u'(w-l-\pi_R(r_l)-\pi_H(\tau\times r_t))}{u'(w-\tau t-(1-\tau)l-\pi_R(r_l)-\pi_H(\tau\times r_t))},\end{cases}$$

where

$$\begin{cases}\omega_d(X;l,t) &= w-\tau\times X\wedge t-(1-\tau)X\wedge l-\pi_R(r_l)-\pi_H(\tau\times r_t);\\ \omega_s(X;l,t) &= w-X\wedge l-\pi_R(r_l)-\pi_H(\tau\times r_t).\end{cases}$$

The above proposition provides a necessary and sufficient condition for the optimality of $t=M$. Obviously, this condition relies heavily upon whether the equation $\Phi_1(l;(x-l)_+,0)=\gamma_R$ for $l\in(0,M)$ has a solution. It is often challenging to verify because the function $\Phi_1(l;(x-l)_+,0)$ may not be monotonic in l. Fortunately, we have

$$\Phi_1(M;(x-M)_+,0)=(1-p\tau)\frac{u'(w-M)}{\mathbb{E}[u'(w-X)]}>\gamma_R$$

due to Assumption 3.1. If we further have $t_1=M$, then

$$\Phi_1(0;(x-0)_+,0)=\frac{p(1-\tau)\mathbb{E}[u'(w-\tau X-\pi_R(r_0))|X>0]+(1-p)u'(w-\pi_R(r_0))}{p\mathbb{E}[u'(w-\tau X-\pi_R(r_0))]+(1-p)u'(w-\pi_R(r_0))}<\gamma_R$$

when condition (4.5) fails to satisfy. Thus, the equation $\Phi_1(l;(x-l)_+,0)=\gamma_R$ must have a solution over $l\in(0,M)$ if $t_1=M$. If $t_1<M$, we have to seek numerical analysis.

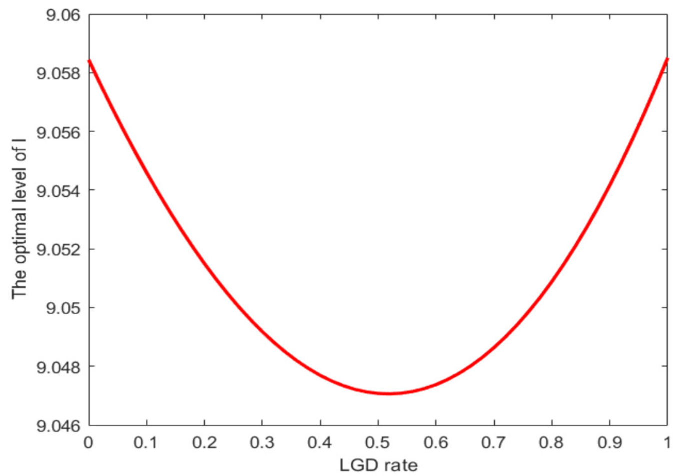

Fig. 7. The effect of the LGD rate on the optimal parameter l under the exponential distributed loss.

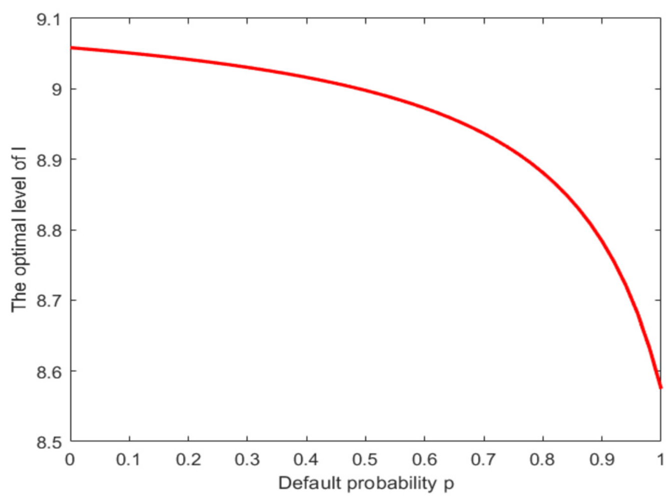

Fig. 8. The effect of the default rate p on the optimal level l under the exponential distributed loss.

Example 4.3. In this example, we assume $w = 20$ and $\rho_H = 0.4$ and adopt other benchmark assumptions in Example 4.1. It is easy to get that Assumption 3.1 is met and $\mathbb{P}\{X > 0\} > \frac{1-\tau}{\gamma_R}$. Thus, condition (4.5) fails to satisfy. By a direct calculation, we obtain $l^* = 9.05$ and find that condition (4.7) is met. Thus, the strategy $((x - 9.05)_+, 0)$ is an optimal solution to Problem (2.2).

In what follows, we investigate the effects of the LGD rate τ, the loading coefficient ρ_H and the default probability p on the optimal strategy (r^*, h^*). Under the above settings, numerical results show that no hedging is optimal regardless of the changes of the LGD rate or the default probability. Thus, we only focus on the effects of these two factors on the optimal parameter l. Surprisingly, Fig. 7 illustrates that the demand for reinsurance increases first and decreases afterwards in the LGD rate. In other words, the LGD rate has an indeterminate effect on the optimal reinsurance design. In contrast, Fig. 8 indicates that the insurer would purchase more reinsurance coverage for a larger default probability. Furthermore, the optimal parameter l is decreasing in the hedging cost ρ_H but the optimal parameter t increases in this factor, as illustrated in Fig. 9. It means that when the hedging becomes more costly, the insurer would purchase more reinsurance and reduce risk transfer by the hedging instrument. Especially for $\rho_H > 0.347$, no hedging is beneficial to the insurer and the strategy $((x - 9.05)_+, 0)$ is optimal. This finding is quite consistent with intuition.

Finally, we obtain the similar numerical findings when the loss distribution is changed to Pareto distribution as described in Example 4.1. Due to the length limit of the paper, we don't present the results. In this sense, we can say that the numerical results are robust with respect to the change of loss distribution.

5. Concluding remarks

In this paper, we design an optimal risk management strategy for an insurer with reinsurance and a default-free financial instrument to hedge reinsurance counterparty risk, imposing the no-sabotage condition on reinsurance contracts. We develop a new approach to overcome technical challenges and derive optimal solutions analytically. The comparison with Reichel et al. (2022)'s optimal solutions illustrates that when the price of hedging instrument is more costly than reinsurance, the optimal reinsurance changes from disappearing deductible to the stop loss once the no-sabotage condition is imposed. More precisely, with the no-sabotage condition, the stop-loss reinsurance is always optimal. This finding generalizes Arrow (1963)'s result with a default-free reinsurer and emphasizes the importance of this condition in the design of a reinsurance contract for an insurer.

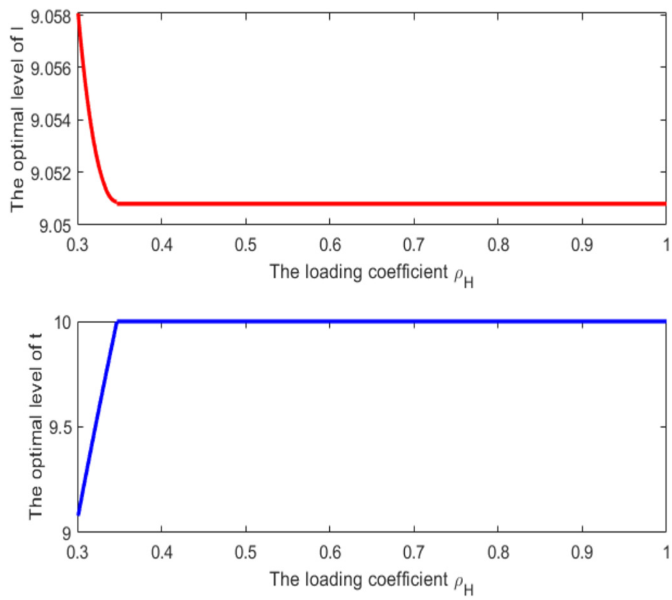

Fig. 9. The effect of loading coefficient ρ_H on the optimal parameters l and t under the exponential distributed loss.

Another key difference between this paper and Reichel et al. (2022) is the technique used to solve the problem. Due to the no-sabotage condition, their approach fails to solve our optimization problems. We develop a new approach by combining the technique of state-wise optimization, construction and variational argument. Luckily, we succeed in deriving optimal solutions analytically, which have a relatively simpler form than Reichel et al. (2022) when reinsurance is cheaper than hedging. It facilitates us to conduct numerical analyses and investigate the effects of some key factors on the optimal risk management strategy.

This work can be extended in a couple of directions. First, we assume in the paper that the reinsurance premium is calculated by the expected value principle and the hedging price equals the expected payoff plus a proportional loading. It is interesting to study this problem with other premium principles or other pricing mechanisms. Second, we ask the hedging payoff to be a non-negative function of the insurable loss. Analyzing the optimal risk management strategy with a generalized hedging payoff may be of interest.

Declaration of competing interest

The authors declare no competing interests.

Data availability

No data was used for the research described in the article.

Acknowledgements

We would like to thank an editor and two anonymous reviewers for their valuable comments, which greatly improve the presentation of this paper. Chi's work was supported by grants from the MOE (China) Project of Key Research Institute of Humanities and Social Sciences at Universities (Grant no. 22JJD790090), National Natural Science Foundation of China (Grant no. 11971505), and the Program for Innovation Research in Central University of Finance and Economics.

Appendix A. Proofs

Proof of Proposition 3.1. Let (r_1, h_1) and (r_2, h_2) be solutions to Problem (2.2), then

$$\mathcal{L}(r_1, h_1) = \mathcal{L}(r_2, h_2).$$

Note that $\mathcal{L}(r, h)$ is strictly concave in r and h. We must have $W^d_{r_1,h_1}(X) = W^d_{r_2,h_2}(X)$ and $W^s_{r_1,h_1}(X) = W^s_{r_2,h_2}(X)$ almost surely. Equivalently,

$$(1-\tau)r_1(X) + h_1(X) - \pi_R(r_1) - \pi_H(h_1) = (1-\tau)r_2(X) + h_2(X) - \pi_R(r_2) - \pi_H(h_2)$$

and

$$r_1(X) - \pi_R(r_1) - \pi_H(h_1) = r_2(X) - \pi_R(r_2) - \pi_H(h_2). \tag{A.1}$$

They further imply

$$\tau r_1(X) - h_1(X) = \tau r_2(X) - h_2(X). \tag{A.2}$$

If $0 \in S(X)$, then the fact $r_1(0) = r_2(0) = 0$ together with (A.1) implies $\pi_R(r_1) + \pi_H(h_1) = \pi_R(r_2) + \pi_H(h_2)$ and $r_1(X) = r_2(X)$ almost surely. Thus, we can get from (A.2) that $h_1(X) = h_2(X)$ almost surely.

Otherwise, if $0 \notin S(X)$, taking the expectations to both sides of (A.1) and (A.2) yields $\tau\mathbb{E}[r_1(X) - r_2(X)] = \mathbb{E}[h_1(X) - h_2(X)]$ and

$$\{1 - (1+\rho_R)(1-p\tau)\}\,\mathbb{E}[r_1(X) - r_2(X)] = p(1+\rho_H)\mathbb{E}[h_1(X) - h_2(X)] = p\tau(1+\rho_H)\mathbb{E}[r_1(X) - r_2(X)]. \tag{A.3}$$

If $\rho_R + \rho_H \neq 0$, then $p\tau\rho_H \neq -\rho_R(1-p\tau)$ because it is assumed that $p \in (0,1)$ and $\tau \in (0,1]$, which is equivalent to $1-(1+\rho_R)(1-p\tau) \neq p\tau(1+\rho_H)$. Thus, it follows from (A.3) that $\mathbb{E}[r_1(X) - r_2(X)] = 0$, then we have $\pi_R(r_1) = \pi_R(r_2)$ and $\pi_H(h_1) = \pi_H(h_2)$, which together with (A.1) imply $r_1(X) = r_2(X)$ almost surely. The final result follows directly from (A.2) and the proof is finally completed. $\square$

Proof of Theorem 3.2. (i) We first analyze Problem (2.2) for $\tau \in (0,1)$. For any admissible strategy (r,h), if $\pi_H(h) = 0$, then $h(X) = 0$ almost surely. For this case, we have

$$\mathcal{L}(r,h) = p\mathbb{E}[u(w - X + (1-\tau)r(X) - \pi_R(r))] + (1-p)\mathbb{E}[u(w - X + r(X) - \pi_R(r))].$$

Noting that $r'(x) \in [0,1]$ almost everywhere and $r(0) = 0$, we can get a stop-loss reinsurance contract $r_d(x) = (x-d)_+$ for some $d \in [0,M]$ such that $\pi_R(r) = \pi_R(r_d)$. Furthermore, it is easy to verify that $r_d(x)$ up-crosses function $r(x)$.[5] Using Lemma 3 in Ohlin (1969), we have

$$X - (1-\alpha)r(X) \geq_{cx} X - (1-\alpha)r_d(X)$$

for any $\alpha \in [0,1]$, where $Z_1 \geq_{cx} Z_2$ means that Z_1 dominates Z_2 in the sense of convex order, i.e.,

$$\mathbb{E}[g(Z_1)] \geq \mathbb{E}[g(Z_2)]$$

for any convex function $g(\cdot)$ provided that expectations exist. As a result, we can get $\mathcal{L}(r,h) \leq \mathcal{L}(r_d,h)$. In other words, when $\pi_H(h) = 0$, any admissible reinsurance strategy is dominated by the stop-loss contract with the same reinsurance premium. Obviously, the strategy $(r_d, 0)$ is a special case of (r^*, h^*) in (3.1) with $l = d$ and $m = c = t = M$.

Next, we consider the case of $\pi_H(h) > 0$. For this case, given the reinsurance contract $r(x)$, we introduce an auxiliary function

$$\varphi(x,y,\lambda) = pu(w - x + (1-\tau)r(x) + y - \pi_R(r) - \pi_H(h)) - \lambda y, \quad x, y \geq 0$$

for some $\lambda \in \mathbb{R}$. It is easy to see that $\varphi(x,y,\lambda)$ is strictly concave in y with the partial derivative

$$\frac{\partial\varphi(x,y,\lambda)}{\partial y} = pu'(w - x + (1-\tau)r(x) + y - \pi_R(r) - \pi_H(h)) - \lambda, \quad y \geq 0.$$

Thus, the maximum value of $\varphi(x,y,\lambda)$ for any non-negative x is attainable at

$$y_\lambda(x) = \inf\{y \geq 0 : pu'(w - x + (1-\tau)r(x) + y - \pi_R(r) - \pi_H(h)) \leq \lambda\}, \tag{A.4}$$

where $\inf\emptyset = \infty$ by convention. It is easy to see that $y_\lambda(x) = \infty$ if $\lambda \leq 0$ and that $y_\lambda(x)$ is decreasing and continuous in λ over the interval $[0,\infty)$. Extremely, we have $\lim_{\lambda\to\infty} y_\lambda(x) = 0$ for each x. As a result, there must exist a $\lambda_h > 0$ such that $\pi_H(h) = \pi_H(y_{\lambda_h})$, which in turn implies

$$\begin{aligned}
\mathcal{L}(r,h) &= p\mathbb{E}[u(w - X + (1-\tau)r(X) + h(X) - \pi_R(r) - \pi_H(h))] \\
&\quad + (1-p)\mathbb{E}[u(w - X + r(X) - \pi_R(r) - \pi_H(h))] \\
&= \mathbb{E}[\varphi(X, h(X), \lambda_h)] + \lambda_h\mathbb{E}[h(X)] + (1-p)\mathbb{E}[u(w - X + r(X) - \pi_R(r) - \pi_H(h))] \\
&\leq \mathbb{E}[\varphi(X, y_{\lambda_h}(X), \lambda_h)] + \lambda_h\mathbb{E}[y_{\lambda_h}(X)] + (1-p)\mathbb{E}[u(w - X + r(X) - \pi_R(r) - \pi_H(y_{\lambda_h}))] \\
&= \mathcal{L}(r, y_{\lambda_h}).
\end{aligned}$$

Noting that $u'(w - x + (1-\tau)r(x) - \pi_R(r) - \pi_H(h))$ is strictly increasing because of $u''(\cdot) < 0$ and $\tau \in (0,1]$, we can define

$$x_1 := \inf\left\{x \in [0,M] : pu'(w - x + (1-\tau)r(x) - \pi_R(r) - \pi_H(h)) \geq \lambda_h\right\} \wedge M.$$

We can conclude $x_1 < M$ because otherwise $x_1 = M$ leads to $y_{\lambda_h}(x) = 0$ for any $x \in [0,M]$ which contradicts the fact $\pi_H(y_{\lambda_h}) = \pi_H(h) > 0$. If $x_1 = 0$, we have

$$y_{\lambda_h}(x) = x - (1-\tau)r(x) - \beta$$

for any $x \in [0,M]$, where $\beta := w - \pi_R(r) - \pi_H(h) - (u')^{-1}(\frac{\lambda_h}{p}) \leq 0$ and $(u')^{-1}(t)$ is the inverse function of $u'(t)$. For this case, we have

[5] Function $g_1(x)$ is said to up-cross function $g_2(x)$ if

$$\begin{cases} g_1(x) \leq g_2(x), & x \leq x_0; \\ g_1(x) \geq g_2(x), & x > x_0 \end{cases}$$

for some $x_0 \in \mathbb{R}$.

$$\begin{aligned}
&\mathcal{L}(r, y_{\lambda_h})\\
&= pu\left(w - \beta(1 - p(1+\rho_H)) - \pi_R(r) - p(1+\rho_H)\mathbb{E}[X - (1-\tau)r(X)]\right)\\
&\qquad +(1-p)\mathbb{E}\left[u(w - X + r(X) - \pi_R(r) - p(1+\rho_H)\mathbb{E}[X-(1-\tau)r(X)] + p(1+\rho_H)\beta)\right],
\end{aligned}$$

which leads to

$$\begin{aligned}
&\frac{\partial \mathcal{L}(r, y_{\lambda_h})}{\partial \beta}\\
&= -p(1-p(1+\rho_H))u'\left(w - \beta(1-p(1+\rho_H)) - \pi_R(r) - p(1+\rho_H)\mathbb{E}[X-(1-\tau)r(X)]\right)\\
&\quad +p(1-p)(1+\rho_H)\mathbb{E}\left[u'(w - X + r(X) - \pi_R(r) - p(1+\rho_H)\mathbb{E}[X-(1-\tau)r(X)] + p(1+\rho_H)\beta)\right]\\
&\geq \{p(1-p)(1+\rho_H) - p(1-p(1+\rho_H))\}\\
&\qquad\qquad \times u'\left(w - \beta(1-p(1+\rho_H)) - \pi_R(r) - p(1+\rho_H)\mathbb{E}[X-(1-\tau)r(X)]\right)\\
&= p\rho_H u'\left(w - \beta(1-p(1+\rho_H)) - \pi_R(r) - p(1+\rho_H)\mathbb{E}[X-(1-\tau)r(X)]\right) \geq 0,
\end{aligned}$$

where the first inequality follows from the facts $\beta \leq 0 \leq X - r(X)$ and $u''(\cdot) < 0$. In other words, for the case of $x_1 = 0$, the strategy $(r(x), y_{\lambda_h}(x))$ is suboptimal to $(r(x), x - (1-\tau)r(x))$. Therefore, we only need to focus on

$$y_{\lambda_h}(x) = (x - (1-\tau)r(x) - \beta)_+ \tag{A.5}$$

for some non-negative β satisfying $x_1 - (1-\tau)r(x_1) = \beta$.

Following, based on $y_{\lambda_h}(x)$ in (A.5), we consider constructing a more desirable reinsurance contract. More specifically, there must exist $d_u \in [x_1, M]$ and $l \in \left[0, \frac{\beta - \tau x_1}{1-\tau}\right]$ such that

$$\mathbb{E}[\tilde{r}(X)\mathbb{I}_{\{X>x_1\}}] = \mathbb{E}[r(X)\mathbb{I}_{\{X>x_1\}}] \quad \text{and} \quad \mathbb{E}[\tilde{r}(X)\mathbb{I}_{\{X\in[0,x_1]\}}] = \mathbb{E}[r(X)\mathbb{I}_{\{X\in[0,x_1]\}}],$$

where

$$\tilde{r}(x) := \begin{cases} r(x_1) + (x - d_u)_+, & x \in [x_1, M];\\ (x-l)_+ - (x - l - \frac{x_1-\beta}{1-\tau})_+, & x \in [0, x_1). \end{cases}$$

Thus, we have $\tilde{r} \in \mathfrak{C}$, $\tilde{r}(x_1) = r(x_1)$ and $\pi_R(r) = \pi_R(\tilde{r})$. Using the similar arguments as the previous analysis, we can get that $\tilde{r}(x)$ up-crosses $r(x)$ over $[x_1, M]$ and hence

$$[X - \tilde{r}(X)|X > x_1] \leq_{cx} [X - r(X)|X > x_1]. \tag{A.6}$$

Similarly, we can get that $\tilde{r}(x)$ up-crosses $r(x)$ on the interval $[0, x_1]$ and hence

$$[X - \alpha\tilde{r}(X)|X \in [0, x_1]] \leq_{cx} [X - \alpha r(X)|X \in [0, x_1]] \tag{A.7}$$

for any $\alpha \in [0, 1]$.

Based upon $\tilde{r}(x)$, we define $\tilde{h}(x) = (x - (1-\tau)\tilde{r}(x) - \beta)_+ = (x - x_1)_+ - (1-\tau)(x - d_u)_+$, then $\pi_H(\tilde{h}) = \pi_H(y_{\lambda_h})$ and $\tilde{h}(x) = y_{\lambda_h}(x) = 0$ for any $x \leq x_1$. Noting that

$$\begin{aligned}
&w - x + (1-\tau)r(x) + y_{\lambda_h}(x) - \pi_R(r) - \pi_H(y_{\lambda_h})\\
&= \begin{cases} w - x + (1-\tau)r(x) - \pi_R(r) - \pi_H(\tilde{h}), & x \leq x_1;\\ w - \beta - \pi_R(r) - \pi_H(\tilde{h}), & x > x_1, \end{cases}
\end{aligned}$$

we have

$$\begin{aligned}
&\mathcal{L}(r, y_{\lambda_h})\\
&= p\mathbb{E}[u(w - X + (1-\tau)r(X) - \pi_R(r) - \pi_H(y_{\lambda_h}))\mathbb{I}_{\{X\in[0,x_1]\}}]\\
&\qquad + pu(w - \beta - \pi_R(r) - \pi_H(y_{\lambda_h}))\mathbb{P}(X > x_1)\\
&\qquad + (1-p)\mathbb{E}[u(w - X + r(X) - \pi_R(r) - \pi_H(y_{\lambda_h}))\mathbb{I}_{\{X\in[0,x_1]\}}]\\
&\qquad + (1-p)\mathbb{E}[u(w - X + r(X) - \pi_R(r) - \pi_H(y_{\lambda_h}))\mathbb{I}_{\{X>x_1\}}]\\
&\leq p\mathbb{E}[u(w - X + (1-\tau)\tilde{r}(X) - \pi_R(\tilde{r}) - \pi_H(\tilde{h}))\mathbb{I}_{\{X\in[0,x_1]\}}]\\
&\qquad + pu(w - \beta - \pi_R(\tilde{r}) - \pi_H(\tilde{h}))\mathbb{P}(X > x_1)\\
&\qquad + (1-p)\mathbb{E}[u(w - X + \tilde{r}(X) - \pi_R(\tilde{r}) - \pi_H(\tilde{h}))\mathbb{I}_{\{X\in[0,x_1]\}}]\\
&\qquad + (1-p)\mathbb{E}[u(w - X + \tilde{r}(X) - \pi_R(\tilde{r}) - \pi_H(\tilde{h}))\mathbb{I}_{\{X>x_1\}}]\\
&= \mathcal{L}(\tilde{r}, \tilde{h}),
\end{aligned}$$

where the inequality follows from (A.6) and (A.7). In other words, (r, y_{λ_h}) is suboptimal to $(\tilde{r}, \tilde{h})$, which is a special case of (r^*, h^*) in (3.1) with $m = l + \frac{x_1-\beta}{1-\tau}$, $c = x_1$ and $t = d_u$.

(ii) For the case of $\tau = 1$, we have

$$\mathcal{L}(r,h) = p\mathbb{E}[u(w - X + h(X) - \pi_R(r) - \pi_H(h))] + (1-p)\mathbb{E}[u(w - X + r(X) - \pi_R(r) - \pi_H(h))].$$

In the previous proof of case (i), we have shown that $X - r(X) \geq_{cx} X - r_d(X)$, which in turn implies $\mathcal{L}(r,h) \leq \mathcal{L}(r_d,h)$. Taking the state-wise optimization approach as used in the proof of case (i), we can get $h_1(x) = (x - \tilde{\beta})_+$ for some $\tilde{\beta} \in \mathbb{R}$ such that $\pi_H(h) = \pi_H(h_1)$ and $\mathcal{L}(r_d,h) \leq \mathcal{L}(r_d,h_1)$. If $\tilde{\beta} \leq 0$, then taking the derivatives of $\mathcal{L}(r_d,h_1)$ with respect to $\tilde{\beta}$ yields

$$\begin{aligned}
&\frac{\partial \mathcal{L}(r_d,h_1)}{\partial \tilde{\beta}}\\
&= p(1-p)(1+\rho_H)\mathbb{E}\left[u'(w - X\wedge d - \pi_R(r_d) - (1+\rho_H)p(\mathbb{E}[X] - \tilde{\beta}))\right]\\
&\quad + p(-1 + (1+\rho_H)p)u'(w - \tilde{\beta} - \pi_R(r_d) - (1+\rho_H)p(\mathbb{E}[X] - \tilde{\beta}))\\
&\geq p\rho_H u'(w - \tilde{\beta} - \pi_R(r_d) - (1+\rho_H)p(\mathbb{E}[X] - \tilde{\beta})) \geq 0.
\end{aligned}$$

Therefore, the optimal solution can be in the form of $(r_d, r_{\tilde{\beta}})$ with $d, \tilde{\beta} \in [0, M]$. The proof is finally completed. □

Proof of Theorem 3.3. The proof is a slight modification to that of Theorem 3.3 in Chi and Wei (2020).

Necessity. We assume that the admissible strategy (r^*, h^*) satisfying $h^* \in \mathfrak{C}$ is a solution to Problem (2.2). For any $r(x) \in \mathfrak{C}$ and $h(x)$ satisfying $h(0) = 0$ and $h'(x) \geq 0$, we define $r^{(q)}(x) = qr^*(x) + (1-q)r(x)$ and $h_q(x) = qh^*(x) + (1-q)h(x)$ for any $q \in [0,1]$. Note that $r^{(q)} \in \mathfrak{C}$ and the optimality of (r^*, h^*) implies that $\frac{\partial \mathcal{L}(r^{(q)},h_q)}{\partial q}|_{q=1} \geq 0$, which is equivalent to

$$\begin{aligned}
0 &\leq p\mathbb{E}\Big[u'(W^d_{r^*,h^*}(X))\big\{(1-\tau)(r^*(X) - r(X)) - \gamma_R\mathbb{E}[r^*(X) - r(X)] + h^*(X) - h(X) - \gamma_H\mathbb{E}[h^*(X) - h(X)]\big\}\Big]\\
&\quad + (1-p)\mathbb{E}\Big[u'(W^s_{r^*,h^*}(X))\big\{r^*(X) - r(X) - \gamma_R\mathbb{E}[r^*(X) - r(X)] - \gamma_H\mathbb{E}[h^*(X) - h(X)]\big\}\Big]\\
&= \int_0^\infty \Big\{p\mathbb{E}[u'(W^d_{r^*,h^*}(X))((1-\tau)\mathbb{I}_{\{X>x\}} - \gamma_R\mathbb{P}(X>x))]\\
&\qquad + (1-p)\mathbb{E}[u'(W^s_{r^*,h^*}(X))(\mathbb{I}_{\{X>x\}} - \gamma_R\mathbb{P}(X>x))]\Big\}(r^{*\prime}(x) - r'(x))dx\\
&\quad + \int_0^\infty \Big\{p\mathbb{E}[u'(W^d_{r^*,h^*}(X))(\mathbb{I}_{\{X>x\}} - \gamma_H\mathbb{P}(X>x))]\\
&\qquad - \gamma_H(1-p)\mathbb{E}[u'(W^s_{r^*,h^*}(X))]\mathbb{P}(X>x)\Big\}(h^{*\prime}(x) - h'(x))dx\\
&= \mathbb{E}[u'(W_{r^*,h^*}(X,D))]\int_0^M \mathbb{P}(X>x)\{\Phi_1(x;r^*,h^*) - \gamma_R\}(r^{*\prime}(x) - r'(x))dx\\
&\quad + \mathbb{E}[u'(W_{r^*,h^*}(X,D))]\int_0^M \mathbb{P}(X>x)\{\Phi_2(x;r^*,h^*) - \gamma_H\}(h^{*\prime}(x) - h'(x))dx,
\end{aligned} \tag{A.8}$$

where $W_{r,h}(X,D)$ is defined in (2.1), $\Phi_1(x;r^*,h^*)$ and $\Phi_2(x;r^*,h^*)$ are given by (3.5) and (3.6) respectively, and the first equality follows from the facts $r^*(x) = \int_0^\infty r^{*\prime}(t)\mathbb{I}_{\{x>t\}}dt$ and $h^*(x) = \int_0^\infty h^{*\prime}(t)\mathbb{I}_{\{x>t\}}dt$. It should be emphasized that the above inequality holds for any $r \in \mathfrak{C}$ and $h(x)$ satisfying $h(0) = 0$ and $h'(x) \geq 0$.

To proceed, we will prove (3.3) by contradiction. If (3.3) were not satisfied, then there would exist a set $E_1 \subset [0, M)$ with positive Lebesgue measure such that either $r^{*\prime}(x) < 1$ and $\Phi_1(x;r^*,h^*) > \gamma_R$ for any $x \in E_1$ or $r^{*\prime}(x) > 0$ and $\Phi_1(x;r^*,h^*) < \gamma_R$ for any $x \in E_1$. In the former case, we can construct a reinsurance contract r such that $r'(x) = 1$ for $x \in E_1$ and $r'(x) = r^{*\prime}(x)$ otherwise. Thus, the first integral on the right-hand side of (A.8) becomes

$$\mathbb{E}[u'(W_{r^*,h^*}(X,D))]\int_{E_1} \mathbb{P}(X>x)\left\{\Phi_1(x;r^*,h^*) - \gamma_R\right\}(r^{*\prime}(x) - 1)dx < 0.$$

For the latter case, we can set $r(x)$ to satisfy $r'(x) = 0$ for any $x \in E_1$ and $r'(x) = r^{*\prime}(x)$ otherwise, then the above integral is

$$\mathbb{E}[u'(W_{r^*,h^*}(X,D))]\int_{E_1} \mathbb{P}(X>x)\left\{\Phi_1(x;r^*,h^*) - \gamma_R\right\}r^{*\prime}(x)dx < 0.$$

These two inequalities contradict (A.8) if $h(x) = h^*(x)$.

In addition, we can conclude $\Phi_2(x; r^*, h^*) \le \gamma_H$ almost everywhere because otherwise a contradiction can be led by letting $h'(x) = h^{*\prime}(x) + 0.1\mathbb{I}_{\{\Phi_2(x;r^*,h^*)>\gamma_H\}}$. We can further show $h^{*\prime}(x) = 0$ if $\Phi_2(x; r^*, h^*) < \gamma_H$. More specifically, if there were existing a set $E_2 \subset [0, M)$ with positive Lebesgue measure such that $h^{*\prime}(x) > 0$ and $\Phi_2(x; r^*, h^*) < \gamma_H$ for any $x \in E_2$, we could find a hedging strategy such that $h'(x) = 0$ for $x \in E_2$ and $h'(x) = h^{*\prime}(x)$ otherwise, then the second integral on the right-hand side of (A.8) becomes

$$\mathbb{E}[u'(W_{r^*,h^*}(X, D))] \int_{E_2} \mathbb{P}(X > x)(\Phi_2(x; r^*, h^*) - \gamma_H)h^{*\prime}(x)dx < 0,$$

which contradicts (A.8).

Sufficiency. If admissible strategy (r^*, h^*) with $h^* \in \mathfrak{C}$ satisfies Eqs. (3.3) and (3.4), we will show $\mathcal{L}(r^*, h^*) \ge \mathcal{L}(\tilde{r}, \tilde{h})$ for any admissible strategy $(\tilde{r}, \tilde{h})$. More specifically, it has been shown in Theorem 3.2 that $\mathcal{L}(\tilde{r}, \tilde{h}) \le \mathcal{L}(r, h)$ for some strategy (r, h) with $r, h \in \mathfrak{C}$. Recalling that $u''(\cdot) < 0$, we further have

$$\begin{aligned}
&\mathcal{L}(r^*, h^*) - \mathcal{L}(r, h)\\
\ge& p\mathbb{E}\Big[u'(W^d_{r^*,h^*}(X))\{(1-\tau)(r^*(X) - r(X)) - \gamma_R\mathbb{E}[r^*(X) - r(X)] + (h^*(X) - h(X) - \gamma_H\mathbb{E}[h^*(X) - h(X)])\}\Big]\\
&+ (1-p)\mathbb{E}\Big[u'(W^s_{r^*,h^*}(X))\big\{(r^*(X) - r(X)) - \gamma_R\mathbb{E}[r^*(X) - r(X)] - \gamma_H\mathbb{E}[h^*(X) - h(X)]\big\}\Big]\\
=& \int_0^\infty \Big\{p\mathbb{E}[u'(W^d_{r^*,h^*}(X))((1-\tau)\mathbb{I}_{\{X>x\}} - \gamma_R\mathbb{P}(X > x))]\\
&\qquad + (1-p)\mathbb{E}[u'(W^s_{r^*,h^*}(X))(\mathbb{I}_{\{X>x\}} - \gamma_R\mathbb{P}(X > x))]\Big\}(r^{*\prime}(x) - r'(x))dx\\
&+ \int_0^\infty \Big\{p\mathbb{E}[u'(W^d_{r^*,h^*}(X))(\mathbb{I}_{\{X>x\}} - \gamma_H\mathbb{P}(X > x))] - \gamma_H(1-p)\mathbb{E}[u'(W^s_{r^*,h^*}(X))]\mathbb{P}(X > x)\Big\}(h^{*\prime}(x) - h'(x))dx\\
=& \mathbb{E}[u'(W_{r^*,h^*}(X, D))] \int_0^M \mathbb{P}(X > x)\{\Phi_1(x; r^*, h^*) - \gamma_R\}(r^{*\prime}(x) - r'(x))dx\\
&+ \mathbb{E}[u'(W_{r^*,h^*}(X, D))] \int_0^M \mathbb{P}(X > x)\{\Phi_2(x; r^*, h^*) - \gamma_H\}(h^{*\prime}(x) - h'(x))dx\\
\ge& 0.
\end{aligned}$$

As a result, (r^*, h^*) is an optimal solution to Problem (2.2). □

Proof of Corollary 3.4. (i) Note that $W^d_{x,\tau x}(X) = W^s_{x,\tau x}(X) = w - \pi_R(x) - \pi_H(\tau x)$, which implies

$$\Phi_1(t; x, \tau x) = 1 - p\tau \quad \text{and} \quad \Phi_2(t; x, \tau x) = p$$

for any $t \in (0, M)$. Thus, it follows from Theorem 3.3 that $(x, \tau x)$ is an optimal solution to Problem (2.2) if and only if

$$1 - p\tau = \Phi_1(t; x, \tau x) \ge \gamma_R = (1 - p\tau)(1 + \rho_R) \quad \text{and} \quad p = \Phi_2(t; x, \tau x) = \gamma_H = p(1 + \rho_H),$$

for $t \in (0, M)$. They are exactly equivalent to $\rho_R = \rho_H = 0$.

(ii) Note that $W^d_{0,0}(X) = W^s_{0,0}(X) = w - X$, which implies

$$\Phi_1(t; 0, 0) = (1 - p\tau)\frac{\mathbb{E}[u'(w - X)|X > t]}{\mathbb{E}[u'(w - X)]} \quad \text{and} \quad \Phi_2(t; 0, 0) = p\frac{\mathbb{E}[u'(w - X)|X > t]}{\mathbb{E}[u'(w - X)]}$$

for all $t \in (0, M)$. Notice that $\mathbb{E}[u'(w - X)|X > t]$ is increasing in t. Thus, we can get from Theorem 3.3 that $(0, 0)$ is an optimal solution to Problem (2.2) if and only if

$$\gamma_R \ge \lim_{t\to M} \Phi_1(t; 0, 0) = (1 - p\tau)\frac{u'(w - M)}{\mathbb{E}[u'(w - X)]} \quad \text{and} \quad \gamma_H \ge \lim_{t\to M} \Phi_2(t; 0, 0) = p\frac{u'(w - M)}{\mathbb{E}[u'(w - X)]}.$$

This necessary and sufficient condition is equivalent to $\frac{u'(w-M)}{\mathbb{E}[u'(w-X)]} \le 1 + \rho_R \wedge \rho_H$. □

Proof of Theorem 3.5. (i) We first assume $\tau \in (0, 1)$. For (r^*, h^*) in (3.1), both function

$$u'(W^d_{r^*,h^*}(z)) = u'\left(w - z + (1-\tau)r^*(z) + h^*(z) - \pi_R(r^*) - \pi_H(h^*)\right)$$

and function $u'(W^s_{r^*,h^*}(z))$ are increasing. So are $\Phi_1(z; r^*, h^*)$ and $\Phi_2(z; r^*, h^*)$ defined in (3.5) and (3.6). If the parameters l, m and t in $r^*(x)$ satisfy $l < m < t$, then it follows from Theorem 3.3 that

$$\gamma_R \le \Phi_1(x; r^*, h^*) \le \Phi_1(y; r^*, h^*) \le \gamma_R$$

for any $l < x < m < y < t$, where the second inequality is derived by the increasing property of $\Phi_1(z; r^*, h^*)$. In other words, $\Phi_1(z; r^*, h^*)$ is constant over the interval $[l, t]$. Since it is assumed that $S(X) = [0, M]$, then the definition of $\Phi_1(z; r^*, h^*)$ implies that $u'(W^d_{r^*,h^*}(z))$ and $u'(W^s_{r^*,h^*}(z))$ are constant over this interval as well. Equivalently, both $z - (1-\tau)r^*(z) - h^*(z)$ and $z - r^*(z)$ are constant over $[l, t]$. Unfortunately, they are impossible according to (3.1). As a consequence, $r^*(z)$ with $l < m < t$ cannot be optimal, and m must equal either l or t. Recalling that $c \in [m, t]$, the optimal strategy (r^*, h^*) to Problem (2.2) can be given in (3.7) or (3.8).
In fact, the optimal solution form relies heavily on the value comparison between ρ_R and ρ_H. Thus, we consider two cases separately: $\rho_R > \rho_H$ and $\rho_R \le \rho_H$.
(1) We first consider the case of $\rho_R > \rho_H$ and divide the analysis into three steps.
Step 1: We show by contradiction that (r^*, h^*) in (3.8) with $0 \le l < t \le M$ cannot be optimal for this case. If it were, Theorem 3.3 together with the increasing property of $\Phi_1(x; r^*, h^*)$ implies

$$\Phi_1(t; r^*, h^*) \ge \gamma_R \quad \text{and} \quad \Phi_2(t; r^*, h^*) \le \gamma_H,$$

where $\Phi_i(M; r^*, h^*)$ is defined as the limit of $\Phi_i(x; r^*, h^*)$ as x approaches M. Notice that $u'(W^d_{r^*,h^*}(x))$ is a constant for any $x \ge t$, then $\Phi_2(t; r^*, h^*) \le \gamma_H$ implies

$$\frac{u'(w - (1-\tau)l - \tau t - \pi_R(r^*) - \pi_H(h^*))}{1+\rho_H} \le p\mathbb{E}[u'(W^d_{r^*,h^*}(X))] + (1-p)\mathbb{E}[u'(W^s_{r^*,h^*}(X))].$$

It together with the fact $u'(w - l - \pi_R(r^*) - \pi_H(h^*)) < u'(w - (1-\tau)l - \tau t - \pi_R(r^*) - \pi_H(h^*))$ can lead to

$$\begin{aligned}
\gamma_R &\le \Phi_1(t; r^*, h^*) \\
&= (1-\tau)\Phi_2(t; r^*, h^*) + \frac{(1-p)u'(w - l - \pi_R(r^*) - \pi_H(h^*))}{p\mathbb{E}[u'(W^d_{r^*,h^*}(X))] + (1-p)\mathbb{E}[u'(W^s_{r^*,h^*}(X))]} \\
&\le (1-\tau)\gamma_H + (1-p)(1+\rho_H)\frac{u'(w - l - \pi_R(r^*) - \pi_H(h^*))}{u'(w - (1-\tau)l - \tau t - \pi_R(r^*) - \pi_H(h^*))} \\
&< (1-\tau)p(1+\rho_H) + (1-p)(1+\rho_H) \\
&= (1+\rho_H)(1-p\tau) < \gamma_R.
\end{aligned}$$

A contradiction is led. Therefore, if $\rho_R > \rho_H$, then (r^*, h^*) cannot be written as in (3.8) with $0 \le l < t \le M$.
Step 2: We further claim that $(r^*, h^*) \ne ((x-t)_+, \tau(x-t)_+)$ for any $t \in [0, M]$. This claim for cases $t = 0$ and $t = M$ has been shown by Corollary 3.4 under Assumption 3.1. Thus, we only need to show this result by contradiction for $t \in (0, M)$. More specifically, if $(r^*, h^*) = ((x-t)_+, \tau(x-t)_+)$ for some $t \in (0, M)$, we have $W^d_{r^*,h^*}(X) = W^s_{r^*,h^*}(X)$, then it follows from Theorem 3.3 that

$$\gamma_R = \Phi_1(t; r^*, h^*) = \frac{1-p\tau}{p}\Phi_2(t; r^*, h^*) = \frac{1-p\tau}{p} \times \gamma_H = (1-p\tau)(1+\rho_H) < \gamma_R,$$

where the last inequality follows from the assumption of $\rho_R > \rho_H$ and the first equality is derived by

$$\begin{aligned}
\gamma_R &\ge \lim_{z\uparrow t}\Phi_1(z; r^*, h^*) = \lim_{z\uparrow t}\frac{(1-p\tau)\mathbb{E}[u'(W^s_{r^*,h^*}(X))|X > z]}{\mathbb{E}[u'(W^s_{r^*,h^*}(X))]} \\
&\ge \lim_{z\uparrow t}\frac{(1-p\tau)u'(w - z - \pi_R(r^*) - \pi_H(h^*))}{\mathbb{E}[u'(W^s_{r^*,h^*}(X))]} = \Phi_1(t; r^*, h^*) \ge \gamma_R.
\end{aligned}$$

A contradiction is thus led.
By Step 1 and Step 2, we know that (r^*, h^*) can be written as in (3.7) with $0 \le c < t \le M$.
Step 3: We can further show $c > 0$. If it were not, then $c = 0$ can imply

$$W^d_{r^*,h^*}(X) = w - \pi_R(r^*) - \pi_H(h^*) \ge w - X \wedge t - \pi_R(r^*) - \pi_H(h^*) = W^s_{r^*,h^*}(X)$$

and this inequality is strict for $X > 0$. Thus, we have $\mathbb{E}[u'(W^s_{r^*,h^*}(X))] > u'(W^d_{r^*,h^*}(X))$ and hence

$$p \le \gamma_H = \Phi_2(t; r^*, h^*) = \frac{pu'(W^d_{r^*,h^*}(X))}{pu'(W^d_{r^*,h^*}(X)) + (1-p)\mathbb{E}[u'(W^s_{r^*,h^*}(X))]} < p,$$

where the first equality follows from Theorem 3.3. A contradiction is led.
(2) The problem for the case of $\rho_R \le \rho_H$ can be analyzed similarly. More specifically, if (r^*, h^*) can be given in (3.7) with $c < t$, then

$$\gamma_H = \Phi_2(x; r^*, h^*) = \frac{pu'(w - c - \pi_R(r^*) - \pi_H(h^*))}{p\mathbb{E}[u'(W^d_{r^*,h^*}(X))] + (1-p)\mathbb{E}[u'(W^s_{r^*,h^*}(X))]}, \forall x \in [c, M]$$

and hence

$$
\begin{aligned}
\gamma_R &\ge \Phi_1(c;r^*,h^*)\\
&= (1-\tau)\Phi_2(c;r^*,h^*) + \frac{(1-p)\mathbb{E}[u'(W^s_{r^*,h^*}(X))|X>c]}{p\mathbb{E}[u'(W^d_{r^*,h^*}(X))]+(1-p)\mathbb{E}[u'(W^s_{r^*,h^*}(X))]}\\
&> p(1-\tau)(1+\rho_H)+(1-p)(1+\rho_H)\frac{u'(W^s_{r^*,h^*}(c))}{u'(w-c-\pi_R(r^*)-\pi_H(h^*))}\\
&= (1+\rho_H)(1-p\tau)\ge \gamma_R,
\end{aligned}
$$

where the second inequality follows from the fact that $u'(W^d_{r^*,h^*}(x))$ is strictly increasing over the interval $[c,t]$ and the last inequality is derived by the assumption of $\rho_R \le \rho_H$. A contradiction is thus led, and the optimal strategy (r^*,h^*) can be given in (3.8) with $0 \le l \le t \le M$ for this case.

Especially, if $\rho_R = \rho_H$, we can show by contradiction that $l = t$ in (3.8). Otherwise, if $l < t$, then $u'(W^d_{r^*,h^*}(x))$ is strictly increasing over $[l,t]$ and constant afterwards. It follows from Theorem 3.3 that $\Phi_1(t;r^*,h^*) > \gamma_R$ and $\Phi_2(t;r^*,h^*) \le \gamma_H$, which in turn imply

$$
\begin{aligned}
\gamma_R &< \Phi_1(t;r^*,h^*)\\
&= (1-\tau)\Phi_2(t;r^*,h^*) + \frac{(1-p)u'(w-l-\pi_R(r^*)-\pi_H(h^*))}{p\mathbb{E}[u'(W^d_{r^*,h^*}(X))]+(1-p)\mathbb{E}[u'(W^s_{r^*,h^*}(X))]}\\
&\le (1-\tau)\gamma_H+(1-p)(1+\rho_H)\frac{u'(w-l-\pi_R(r^*)-\pi_H(h^*))}{u'(w-(1-\tau)l-\tau t-\pi_R(r^*)-\pi_H(h^*))}\\
&< (1-\tau)p(1+\rho_H)+(1-p)(1+\rho_H)\\
&= (1+\rho_H)(1-p\tau)=\gamma_R.
\end{aligned}
$$

A contradiction is led. Thus, if $\rho_R = \rho_H$, we have $r^*(x) = (x-t)_+$ and $h^*(x) = \tau(x-t)_+$ for some $t \in [0,M]$, then $\mathcal{L}(r^*,h^*) = \mathbb{E}[u(w - X\wedge t - (1+\rho_R)\mathbb{E}[(X-t)_+])]$. It is exactly the objective of classical Arrow's model with a stop-loss contract, whose optimal deductible level has been derived explicitly by Chi (2019). That is $t = d^*$. Using Corollary 3.4 and Assumption 3.1, we further have $t \in (0,M)$.

Otherwise, if $\rho_R < \rho_H$, using the similar arguments as Step 2 in the above proof to the case of $\rho_R > \rho_H$, we can get $l < t$. We can further conclude $l > 0$ if $S_X(0) > \frac{1-\tau}{\gamma_R}$ where $S_X(x)$ is the survival distribution function of X. More specifically, if $l = 0$, we have $r^*(x) = x$, then $W^s_{r^*,h^*}(X) = w - \pi_R(r^*) - \pi_H(h^*) \ge W^d_{r^*,h^*}(X)$ and this inequality is strict for $X > 0$. Thus, we have $\mathbb{E}[u'(W^d_{r^*,h^*}(X))|X>0] > u'(W^s_{r^*,h^*}(X))$. It together with Theorem 3.3 can imply

$$
\begin{aligned}
\gamma_R &\le \Phi_1(0;r^*,h^*)\\
&= \frac{p(1-\tau)\mathbb{E}[u'(W^d_{r^*,h^*}(X))|X>0]+(1-p)u'(W^s_{r^*,h^*}(X))}{pS_X(0)\mathbb{E}[u'(W^d_{r^*,h^*}(X))|X>0]+(1-pS_X(0))\,u'(W^s_{r^*,h^*}(X))}\\
&= \gamma_R + \frac{p(1-\tau-\gamma_R S_X(0))\mathbb{E}[u'(W^d_{r^*,h^*}(X))|X>0]}{pS_X(0)\mathbb{E}[u'(W^d_{r^*,h^*}(X))|X>0]+(1-pS_X(0))\,u'(W^s_{r^*,h^*}(X))}\\
&\quad + \frac{(1-p-\gamma_R(1-pS_X(0)))u'(W^s_{r^*,h^*}(X))}{pS_X(0)\mathbb{E}[u'(W^d_{r^*,h^*}(X))|X>0]+(1-pS_X(0))\,u'(W^s_{r^*,h^*}(X))}\\
&< \gamma_R + \left\{p(1-\tau-\gamma_R S_X(0))+(1-p-\gamma_R(1-pS_X(0)))\right\}\\
&\qquad \times \frac{u'(W^s_{r^*,h^*}(X))}{pS_X(0)\mathbb{E}[u'(W^d_{r^*,h^*}(X))|X>0]+(1-pS_X(0))\,u'(W^s_{r^*,h^*}(X))}\\
&= \gamma_R - \frac{(1-p\tau)\rho_R u'(W^s_{r^*,h^*}(X))}{pS_X(0)\mathbb{E}[u'(W^d_{r^*,h^*}(X))|X>0]+(1-pS_X(0))\,u'(W^s_{r^*,h^*}(X))}\\
&\le \gamma_R,
\end{aligned}
$$

where the second inequality follows from the assumption of $S_X(0) > \frac{1-\tau}{\gamma_R}$. Thus, a contradiction is led, and hence we must have $0 < l < t$ when $\rho_R < \rho_H$ and $S_X(0) > \frac{1-\tau}{\gamma_R}$.

(ii) Now we assume $\tau = 1$. Theorem 3.2 has shown that $r^*(x) = (x-a)_+$ and $h^*(x) = (x-b)_+$ for some $0 \le a, b \le M$. Under Assumption 3.1, Corollary 3.4 demonstrates $a > 0$ or $b > 0$ or both.

Next, we show by contradiction that $a > 0$ and $b > 0$. If $a = 0$, then we must have $b > 0$. Using the previous analysis, we have

$$
(1-p)(1+\rho_R) = \gamma_R \le \Phi_1(x;r^*,h^*) = \frac{(1-p)u'(W^s_{r^*,h^*}(X))}{p\mathbb{E}[u'(W^d_{r^*,h^*}(X))]+(1-p)u'(W^s_{r^*,h^*}(X))} < 1-p
$$

for any $x \in (0,M)$, where the first inequality follows from Theorem 3.3 and the last inequality is derived by the fact $W^d_{r^*,h^*}(X) - W^s_{r^*,h^*}(X) = -X\wedge b$. A contradiction is led. Similarly, if $b = 0$, we have $a > 0$, then a contradiction is also led because

$$
p(1+\rho_H) = \gamma_H = \Phi_2(x;r^*,h^*) < p.
$$

Following, we show that $a=b$ if $\rho_R=\rho_H$. Otherwise, if $0<a<b$, then Theorem 3.3 implies

$$\gamma_R=\Phi_1(a;r^*,h^*) \quad \text{and} \quad \Phi_2(b;r^*,h^*)\le\gamma_H,$$

which in turn implies

$$u'(w-a-\pi_R(r^*)-\pi_H(h^*))\ge u'(w-b-\pi_R(r^*)-\pi_H(h^*)).$$

Thus, a contradiction is led. Similarly, if $0<b<a$, then

$$\gamma_R\ge\Phi_1(a;r^*,h^*) \quad \text{and} \quad \Phi_2(b;r^*,h^*)=\gamma_H,$$

which will lead to a contradiction as

$$u'(w-b-\pi_R(r^*)-\pi_H(h^*))\ge u'(w-a-\pi_R(r^*)-\pi_H(h^*)).$$

Therefore, if $\rho_R=\rho_H$, then $r^*(x)=h^*(x)$ and we can get $a=b=d^*$ from the classical Arrow's model.

Finally, we consider the case of $\rho_R\neq\rho_H$ and show $(\rho_R-\rho_H)(a-b)>0$ by contradiction. Otherwise, there are two possibilities: (1) $\rho_R<\rho_H$ and $a\ge b>0$; (2) $\rho_R>\rho_H$ and $b\ge a>0$. For the former case, if $a=M$, it follows from Assumption 3.1 that $M=a>b>0$. The previous analysis has shown $\Phi_2(b;r^*,h^*)=\gamma_H$ but $(1-p)(1+\rho_H)>\gamma_R\ge\Phi_1(a;r^*,h^*)$, which will lead to

$$u'(w-b-\pi_R(r^*)-\pi_H(h^*))>u'(w-a-\pi_R(r^*)-\pi_H(h^*)).$$

It is contradicted to the assumption of $a\ge b$. Otherwise, if $a<M$, we can get from Theorem 3.3 that $\Phi_2(b;r^*,h^*)=\gamma_H$ and $(1-p)(1+\rho_H)>\gamma_R=\Phi_1(a;r^*,h^*)$, which leads to the same contradiction. Thus, the former case is impossible to appear. The proof of the latter case is quite similar and we neglect the details. □

Proof of Proposition 4.1. For the function $\psi_0(y)$, it is easy to see that $\psi_0(y)<\frac{1}{pS_X(y)}\le 1+\rho_H$ for any $y\le v_{\frac{1}{\gamma_H}}$. We can further show that $\psi_0(y)$ is strictly increasing over the interval $[v_{\frac{1}{\gamma_H}},M]$. More specifically, for any $y>v_{\frac{1}{\gamma_H}}$, we have

$$\begin{aligned}\psi_0{}'(y)=&\frac{u''(w-y-\pi_H(r_y))(-1+\gamma_H S_X(y))(1-p\psi_0(y)S_X(y))}{p\mathbb{E}[u'(W_X^d(y,M))]+(1-p)\mathbb{E}[u'(W_X^s(y,M))]}\\&-\frac{\psi_0(y)\gamma_H S_X(y)\mathbb{E}[u''(w-X-\pi_H(r_y))(p\mathbb{I}_{\{X\le y\}}+1-p)]}{p\mathbb{E}[u'(W_X^d(y,M))]+(1-p)\mathbb{E}[u'(W_X^s(y,M))]}\\&>0,\end{aligned}$$

where the inequality follows from the facts $1>p\psi_0(y)S_X(y)$, $u''(t)<0$ and $\gamma_H S_X(y)<1$ for any $y>v_{\frac{1}{\gamma_H}}$. Furthermore, we have $\psi_0(M)=\frac{u'(w-M)}{\mathbb{E}[u'(w-X)]}>1+\rho_H$, where the inequality follows from Assumption 3.1 and the assumption of $\rho_R>\rho_H$.

Next, we consider the case of $t=M$. For this case, noting that $\mathbb{E}[u'(W_X^d(c,t))|X>x]$ and $\mathbb{E}[u'(W_X^s(c,t))|X>x]$ are increasing, it follows from Theorem 3.3 that $(0,(x-c)_+)$ is an optimal solution to Problem (2.2) if and only if

$$\lim_{x\uparrow M}\Phi_1(x;0,r_c)\le\gamma_R \quad \text{and} \quad \Phi_2(c;0,r_c)=\gamma_H,$$

where $r_c(x)=(x-c)_+$, which are equivalent to

$$\psi_0(c)=1+\rho_H \quad \text{and} \quad \frac{u'(w-M-\pi_H(r_c))}{u'(w-c-\pi_H(r_c))}\le 1+\frac{(\rho_R-\rho_H)(1-p\tau)}{(1-p)(1+\rho_H)}.$$

According to the properties of $\psi_0(y)$ analyzed previously, we can get from the above equation that $c=c_0^*$.

Finally, if condition (4.1) is not met, then we must have $t<M$. It follows directly from Theorem 3.3 that

$$\Phi_1(t;r^*,h^*)=\gamma_R \quad \text{and} \quad \Phi_2(c;r^*,h^*)=\gamma_H.$$

The result follows naturally by rearranging the above equations. □

Proof of Proposition 4.2. The proof will be divided into two cases: $t_1=M$ and $t_1<M$, where t_1 is defined in (4.4).

(i) If $t_1=M$, then the strictly increasing property of $\psi_1(t)$ implies $\Phi_2(t;x,\tau(x-t)_+)=p\psi_1(t)<\gamma_H$ for any $t<M$. By Theorem 3.3, we know that $(x,\tau(x-t)_+)$ for any $t<M$ cannot be an optimal solution. Using Theorem 3.3 again, we can find that $(x,0)$ is an optimal solution to Problem (2.2) if and only if

$$\gamma_R\le\Phi_1(0;x,0) \quad \text{and} \quad \lim_{t\uparrow M}\Phi_2(t;x,0)\le\gamma_H.$$

The latter condition is naturally held because $\lim_{t\uparrow M}\Phi_2(t;x,0)=\lim_{t\uparrow M}p\psi_1(t)\le\gamma_H$. Furthermore, the former condition in the above equation is exactly equivalent to (4.5) with $t_1=M$.

(ii) Otherwise, if $t_1 < M$, then we have $\psi_1(t) < 1 + \rho_H$ for any $t < t_1$ and $\psi_1(t) > 1 + \rho_H$ for $t > t_1$. Due to $\lim_{t\uparrow M} \Phi_2(t; x, 0) = \lim_{t\uparrow M} p\psi_1(t) > \gamma_H$, we can get from Theorem 3.3 that $(x, 0)$ cannot be an optimal solution. Theorem 3.3 further implies that $(x, \tau(x-t)_+)$ for some $t \in (0, M)$ is an optimal solution if and only if

$$\gamma_R \le \Phi_1(0; x, \tau(x-t)_+) \quad \text{and} \quad p\psi_1(t) = \Phi_2(t; x, \tau(x-t)_+) = \gamma_H.$$

Since $\psi_1(t)$ is strictly increasing over $[v_{1/\gamma_H}, M]$, then we must have $t = t_1$ and hence the former condition is (4.5). □

References

Arrow, K.J., 1963. Uncertainty and the welfare economics of medical care. American Economic Review 53 (5), 941–973.
Asimit, A.V., Badescu, A.M., Cheung, K.C., 2013. Optimal reinsurance in the presence of counterparty default risk. Insurance: Mathematics and Economics 53 (3), 690–697.
Balbás, A., Balbás, B., Heras, A., 2009. Optimal reinsurance with general risk measures. Insurance: Mathematics and Economics 44 (3), 374–384.
Bernard, C., Ludkovski, M., 2012. Impact of counterparty risk on the reinsurance market. North American Actuarial Journal 16 (1), 87–111.
Bodoff, N., 2013. Reinsurance credit risk: a market-consistent paradigm for quantifying the cost of risk. Variance: Advancing the Science of Risk 7 (1), 11–28.
Boonen, T., Jiang, W., 2022. A marginal indemnity function approach to optimal reinsurance under the Vajda condition. European Journal of Operational Research 303 (2), 928–944.
Borch, K., 1960. An attempt to determine the optimum amount of stop loss reinsurance. In: Transactions of the 16th International Congress of Actuaries, vol. I, pp. 597–610.
Cai, J., Lemieux, C., Liu, F., 2014. Optimal reinsurance with regulatory initial capital and default risk. Insurance: Mathematics and Economics 57 (1), 13–24.
Cai, J., Tan, K.S., 2007. Optimal retention for a stop-loss reinsurance under the VaR and CTE risk measures. ASTIN Bulletin 37 (1), 93–112.
Cai, J., Tan, K.S., Weng, C., Zhang, Y., 2008. Optimal reinsurance under VaR and CTE risk measures. Insurance: Mathematics and Economics 43 (1), 185–196.
Carlier, G., Dana, R.-A., 2003. Pareto efficient insurance contracts when the insurers cost function is discontinuous. Economic Theory 21 (4), 871–893.
Cheung, K.C., 2010. Optimal reinsurance revisited—a geometric approach. ASTIN Bulletin 40 (1), 221–239.
Cheung, K.C., Lo, A., 2017. Characterizations of optimal reinsurance treaties: a cost-benefit approach. Scandinavian Actuarial Journal 2017 (1), 1–28.
Chi, Y., 2019. On the optimality of a straight deductible under belief heterogeneity. ASTIN Bulletin 49 (1), 243–262.
Chi, Y., Tan, K.S., 2011. Optimal reinsurance under VaR and CVaR risk measures: a simplified approach. ASTIN Bulletin 41 (2), 487–509.
Chi, Y., Wei, W., 2020. Optimal insurance with background risk: an analysis of general dependence structures. Finance and Stochastics 24 (4), 903–937.
Gajek, L., Zagrodny, D., 2000. Insurer's optimal reinsurance strategies. Insurance: Mathematics and Economics 27, 105–112.
Gatumel, M., Lemoyne de Forges, S., 2013. Understanding and monitoring reinsurance counterparty risk. Bulletin Français d'Actuariat 13 (26), 121–138.
Huberman, G., Mayers, D., Smith Jr, C.W., 1983. Optimal insurance policy indemnity schedules. Bell Journal of Ecoonomics 14 (2), 415–426.
Kaluszka, M., 2001. Optimal reinsurance under mean-variance premium principles. Insurance: Mathematics and Economics 28 (1), 61–67.
Mahul, O., Wright, B.D., 2004. Implications of incomplete performance for optimal insurance. Economica 71 (284), 661–670.
Mossin, J., 1968. Aspects of rational insurance purchasing. Journal of Political Economy 76 (4), 553–568.
Ohlin, J., 1969. On a class of measures of dispersion with application to optimal reinsurance. ASTIN Bulletin 5 (2), 249–266.
Reichel, L., Schmeiser, H., Schreiber, F., 2022. On the optimal management of counterparty risk in reinsurance contracts. Journal of Economic Behavior & Organization 201, 374–394.
Van Heerwaarden, A.E., Kaas, R., Goovaerts, M.J., 1989. Optimal reinsurance in relation to ordering of risks. Insurance: Mathematics and Economics 8 (1), 11–17.
Xu, Z.Q., Zhou, X.Y., Zhuang, S.C., 2019. Optimal insurance with rank-dependent utility and increasing indemnities. Mathematical Finance 29, 659–692.
Zhuang, S.C., Weng, C., Tan, K.S., Assa, H., 2016. Marginal indemnification function formulation for optimal reinsurance. Insurance: Mathematics and Economics 67, 65–76.