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Time-consistent investment and reinsurance strategies for mean–variance insurers with jumps[☆]

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HIGHLIGHTS

- An investment and reinsurance problem with jumps is discussed in a game framework.
- A corresponding verification theorem is provided.
- Explicit expressions for time-consistent strategies and value function are derived.
- Some special cases and interesting phenomena of our model are presented.
- Some sensitivity analysis and numerical simulations for our results are provided.

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ABSTRACT

This paper studies an optimal investment and reinsurance problem incorporating jumps for mean–variance insurers within a game theoretic framework and aims to seek the corresponding time-consistent strategies. Specially, the insurers are allowed to purchase proportional reinsurance, acquire new business and invest in a financial market, where the surplus of the insurers is assumed to follow a jump–diffusion model and the financial market consists of one risk-free asset and one risky asset whose price process is modeled by a geometric Lévy process. By solving an extended Hamilton–Jacobi–Bellman system, the closed-form expressions for the time-consistent investment and reinsurance strategies and the optimal value function are derived. Moreover, some special cases of our model and results are presented, and some numerical illustrations and sensitivity analysis for our results are provided.

1. Introduction

Mean–variance portfolio selection theory, first proposed by Markowitz (1952), has long been regarded as the milestone of the modern finance theory. It has inspired many extensions and applications in both academia and industry. Breakthrough works on mean–variance analysis by Li and Ng (2000) and Zhou and Li (2000) extend the single period mean–variance model to multi-period and continuous-time versions, respectively, where the explicit solutions are derived by employing an embedding technique and the stochastic dynamic programming approach. Thereafter, many scholars focus on dynamic mean–variance portfolio selection problems. For more detailed discussion, readers are referred to Çakmak and Özekici (2006), Çelikyurt and Özekici (2007), Wu and Li (2011, 2012) and references therein for the multi-period case, and Bielecki et al. (2005), Xia and Yan (2006), Fu et al. (2010) and references therein for the continuous-time case.

However, it is a well-known fact that the mean–variance criterion lacks the iterated-expectation property. As a result, multi-period and continuous-time mean–variance problems cannot be dealt with directly by the traditional dynamic programming ap-

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proach and are time-inconsistent in the sense that Bellman optimality principle does not hold. Strotz (1956) proposes that time-inconsistent problems can be tackled by (i) ignoring the conflict as a spendthrifty, (ii) a strategy of precommitment, or (iii) a strategy of consistent planning. The optimal strategies in the above mentioned literature on multi-period and continuous-time mean–variance problems are the precommitment strategies, which are time-inconsistent. To derive such precommitment strategies, investors are assumed to precommit themselves to adopting the optimal strategies derived at the initial time in the future.

It is very difficult to require investors not to deviate from the optimal strategy chosen at the initial time during the entire investment horizon. Meanwhile, time consistency of strategies is a basic requirement of rational decision making, such as making an economic policy. Hence, it is important to utilize the third way proposed by Strotz (1956) to handle time-inconsistent problems, i.e. seeking the corresponding time-consistent strategies. The time-consistent strategies mean that investors sitting at time t would consider that, starting from $t+\varepsilon$, they will follow the strategies that are optimal sitting at time $t+\varepsilon$, that is, $\pi_t^*(s) = \pi_{t+\varepsilon}^*(s)$ for all $s \geq t+\varepsilon$, where ε is an arbitrary positive constant, $\pi_t^*(s)$ and $\pi_{t+\varepsilon}^*(s)$ are the corresponding optimal strategies sitting at time t and $t+\varepsilon$, respectively.

Recently, research on time-consistent strategies for time-inconsistent problems is a focus and attracts many scholars' attention. Basak and Chabakauri (2010) consider a dynamic mean–variance portfolio problem and derive the closed-form expression for its time-consistent strategy by using the dynamic programming approach. Björk and Murgoci (2010) study the time-inconsistent problems with a general controlled Markov process and a fairly general objective function and seek the corresponding subgame perfect Nash equilibrium strategies, which are time-consistent. They derive an extended Hamilton–Jacobi–Bellman (HJB) equation and provide a verification theorem. Wang and Forsyth (2011) develop a numerical scheme to determine the time-consistent strategy and the precommitment strategy for a continuous-time mean–variance portfolio problem with constraints. Björk et al. (2012) further study the time-consistent strategy for a continuous-time mean–variance portfolio problem within a game theoretic framework, where the risk aversion is assumed to depend dynamically on current wealth. In addition, there are also some literature investigating the time-consistent strategies for the time-inconsistent problems with the objectives of maximizing investors' utilities from consumption and terminal wealth; see Ekeland and Pirvu (2008), Ekeland and Lazrak (2008, 2010), Marín-Solano and Navas (2010), Marín-Solano and Patxot (2012), De-Paz et al. (2012), Ekeland et al. (2012) and references therein.

On the other hand, investment is an increasingly important element and reinsurance is an effective way to control risk for insurers. Many scholars pay much attention to the optimal investment and reinsurance problems for insurers under the mean–variance criterion in recent years. Bäuerle (2005) assumes that the surplus process of an insurer is described by Cramér–Lundberg (C–L) model and solves an optimal proportional reinsurance/new business problem under the mean–variance criterion explicitly by adopting the stochastic dynamic programming approach. Delong and Gerrard (2007) suppose that the price process of the risky asset is driven by a Lévy process and that the claim process of an insurer is modeled by a compound Cox process, whose intensity is modeled by a drifted Brownian motion. They solve one classical mean–variance portfolio problem and one problem with the mean–variance criterion involving a running cost. Bai and Zhang (2008) consider an optimal investment and reinsurance problem for insurers under the mean–variance criterion, and derive the corresponding viscosity solutions by the stochastic dynamical programming approach and the dual method. Zeng et al. (2010) solve a benchmark problem and a mean–variance problem for an insurer by the stochastic maximum principle, where the surplus of the insurer is described by a jump–diffusion model and the insurer is allowed to invest in a financial market consisting of one risk-free asset and one risky asset. However, in the literature mentioned above, the optimal strategies are the precommitment strategies. As we know, time-consistent investment and reinsurance strategies are very important for insurers. Zeng and Li (2011) are the first to present the optimal time-consistent investment and reinsurance strategies for mean–variance insurers, where the surplus of the insurer is modeled by the diffusion model and the price processes of the risky assets are described by geometric Brownian motions. Li et al. (2012) investigate the same problem as that in Zeng and Li (2011), where the price process of the risky asset follows Heston's stochastic volatility model.

Although Zeng and Li (2011) and Li et al. (2012) consider the time-consistent investment and reinsurance strategies for insurers, the surplus of insurers is only modeled by the diffusion model and the price processes of the risky assets are only driven by geometric Brownian motions and Heston's stochastic volatility model, whose paths are continuous. In fact, the surplus process and the price processes of the risky assets are often discontinuous and have jumps. To the best of our knowledge, except Bai and Zhang (2008), under mean–variance criterion there is no other literature considering investment and reinsurance incorporating jumps at the same time. However, in Bai and Zhang (2008), they only consider the case of the risky asset's price process following a geometric Brownian motion and the surplus process evolving according to the C–L model or a diffusion model. Moreover, they only derive the precommitment strategies. In the present paper, we consider the same problem as Zeng and Li (2011) and Li et al. (2012) but relax the assumptions in them. Specifically, we consider an optimal investment and reinsurance problem for mean–variance insurers and aim to seek the corresponding time-consistent strategy; the surplus of the insurers is described by a more general jump–diffusion process. The financial market consists of one risk-free asset and one risky asset; the insurers are allowed to invest in the financial market, purchase proportional reinsurance and acquire new business. For making our optimization problem tractability and deriving the corresponding explicit solution, the logarithm of the risky asset's price process in our paper is assumed to follow a jump–diffusion process, which is a simple Lévy process and has been already widely adopted in the existing literature, such as Merton (1976). It should be pointed out that there are many other popular Lévy processes, such as variance Gamma (VG) processes, normal inverse Gaussian (NIG) processes, which have been used to model the logarithm of the risky assets' price processes. For more information on Lévy processes and their applications in finance, readers are referred to Applebaum (2009), Schoutens (2003), Cont and Tankov (2004), Sato (1999) and references therein. Similar to Björk and Murgoci (2010), in this paper we take time inconsistency seriously and formulate our problem in a game theoretic framework. As a result, this paper extends the models of Zeng and Li (2011) and Li et al. (2012) to the case with jumps and also derive the explicit solutions. Moreover, some special cases of our model are provided in this paper, which include the results of Zeng and Li (2011). In addition, it is not difficult to extend our model to the case of multiple risky assets.

In the next section, the assumptions and model are described. Section 3 formulates the optimization problem we are going to consider in a game theoretic framework and provides a verification theorem. Section 4 solves the optimization problem and derives explicitly the corresponding time-consistent investment and reinsurance strategies and the optimal value function. Some special cases of our model are presented in Section 5. Section 6 provides some numerical illustrations and sensitivity analysis for our results. Section 7 concludes this paper.

2. Assumptions and model

Let $(\Omega, \mathcal{F}, P)$ be a given complete probability space with a filtration $\{\mathscr{F}_t\}_{t\in[0,T]}$ satisfying the usual conditions, i.e. the filtration contains all P-null sets and is right continuous, where $T \in (0, +\infty)$ is a finite constant and represents the time horizon; $\mathscr{F}_t$ stands for the information available up to time t and any decision made at time t is based on this information. All stochastic processes in this paper are assumed to be well defined and adapted processes in this probability space.

2.1. Surplus process

Consider an insurer whose surplus process is described by the following jump–diffusion model:

$$dR(t) = cdt + \sigma_0 dw_0(t) - d\sum_{i=1}^{N_1(t)} Y_i, \tag{1}$$

where $c > 0$ is the premium rate; σ_0 is a positive constant; $\{w_0(t)\}$ is a one-dimensional standard Brownian motion; $\sigma_0 w_0(t)$ can be regarded as the uncertainty from the premium income of the insurer; $N_1(t)$, representing the number of claims occurring during time interval $[0, t]$, is a homogeneous Poisson process with intensity $\lambda_1 > 0$; Y_i is the size of the ith claim and $Y_i,\ i = 1, 2, \dots$ are assumed to be independent and identically distributed (i.i.d.) positive random variables with finite first-order moment $\mu_y = \mathrm{E}[Y_i]$ and second-order moment $\sigma_y^2 = \mathrm{E}[Y_i^2]$. Here, the premium rate is supposed to be calculated according to the expected value principle, i.e. $c = (1 + \theta)\lambda_1\mu_y$, in which θ is the relative safety loading of the insurer.

Suppose that the insurer can purchase proportional reinsurance and acquire new business to control her or his insurance business risk. Denote by $\alpha(t)$ the value of risk exposure at time t, which represents the retention level of reinsurance or new business. $\alpha(t) \in [0, 1]$ corresponds to a proportional reinsurance cover; in this case the cedent should divert part of the premium to the reinsurer at the rate of $(1-\alpha(t))(1+\eta)\lambda_1\mu_y$, where η is the relative safety loading of the reinsurer satisfying $\eta \ge \theta$; in return, for each claim occurring at time t, the reinsurer pays $100(1 - \alpha(t))\%$ of the claim, and the cedent pays the rest. $\alpha(t) \in (1, +\infty)$ corresponds to acquiring new business. Incorporating purchasing proportional reinsurance and acquiring new business into the surplus process, (1) becomes

$$\begin{aligned} dR(t) &= [\theta - \eta + (1 + \eta)\alpha(t)]\,\lambda_1\mu_y dt \\ &\quad + \sigma_0\alpha(t)dw_0(t) - \alpha(t)d\sum_{i=1}^{N_1(t)} Y_i. \end{aligned} \tag{2}$$

2.2. Financial market

Consider a financial market with two available assets: a risk-free asset (e.g., a bond or a bank account) and a risky asset (e.g., a stock). The price process of the risk-free asset $S_0(t)$ is modeled by

$$dS_0(t) = r_0(t)S_0(t)dt, \qquad S_0(0) = s_0 > 0, \tag{3}$$

where $r_0(t) > 0$, representing the risk-free interest rate, is a continuous bounded deterministic function. The price process of the risky asset $S_1(t)$ satisfies the following stochastic differential equation:

$$\begin{aligned} dS_1(t) &= S_1(t)\left[r_1(t)dt + \sigma_1(t)dw_1(t) + d\sum_{i=1}^{N_2(t)} Z_i\right], \\ S_1(0) &= s_1 > 0, \end{aligned} \tag{4}$$

where $r_1(t)$ and $\sigma_1(t) > 0$ are continuous bounded deterministic functions; $\{w_1(t)\}$ is a one-dimensional standard Brownian motion; $N_2(t)$, representing the number of the jumps of the risky asset's price occurring during time interval [0,t], is a homogeneous Poisson process with intensity λ_2; Z_i is the ith jump amplitude of the risky asset's price and $Z_i, i = 1, 2, \dots$ are i.i.d. random variables with finite first-order moment $\mu_z = \mathrm{E}[Z_i]$ and second-order moment $\sigma_z^2 = \mathrm{E}[Z_i^2]$. We assume that $\{w_0(t)\}, \{w_1(t)\}, \{\sum_{i=1}^{N_1(t)} Y_i\}$ and $\{\sum_{i=1}^{N_2(t)} Z_i\}$ are independent and that $P\{Z_i \ge -1 \text{ for all } i \ge 1\} = 1$ to ensure that the risky asset's price remains positive. Generally, the expected return of the risky asset is larger than the risk-free interest rate, so we assume that $r_1(t) + \lambda_2\mu_z > r_0(t)$.

2.3. Wealth process

The insurer, starting from an initial capital x_0 at time 0, is allowed to dynamically purchase proportional reinsurance, acquire new business and invest in the financial market described in the previous subsection. A trading strategy is denoted by a pair of stochastic processes $\pi = \{(\alpha^\pi(t), \beta^\pi(t))\}_{t\in[0,T]}$, where $\alpha^\pi(t)$ and $\beta^\pi(t)$ are the value of the risk exposure and the dollar amount invested in the risky asset at time t, respectively. The dollar amount invested in the risk-free asset at time t is $X^\pi(t) - \beta^\pi(t)$, where $X^\pi(t)$ is the wealth process associated with strategy π. Then the evolution of $X^\pi(t)$ can be described as

$$\begin{aligned} dX^\pi(t) &= \{r_0(t)X^\pi(t) + [\delta + (1 + \eta)\alpha^\pi(t)]\lambda_1\mu_y + r(t)\beta^\pi(t)\}dt \\ &\quad + \sigma_0\alpha^\pi(t)dw_0(t) + \sigma(t)\beta^\pi(t)dw_1(t) \\ &\quad - \alpha^\pi(t)d\sum_{i=1}^{N_1(t)} Y_i + \beta^\pi(t)d\sum_{i=1}^{N_2(t)} Z_i, \end{aligned} \tag{5}$$

where $\delta = \theta - \eta$ and $r(t) = r_1(t) - r_0(t)$.

Definition 1 (*Admissible Strategy*)**.** Let $\mathcal{Q} := [0, T] \times \mathbb{R}$. For any fixed $t \in [0, T]$, a trading strategy $\pi = \{(\alpha^\pi(s), \beta^\pi(s))\}_{s\in[t,T]}$ is said to be admissible if it satisfies that

(i) $\alpha^\pi(s)$ and $\beta^\pi(s)$ are predictable mappings with respect to (w.r.t.) $\mathscr{F}_s$;

(ii) for all $s \in [t, T]$, $\alpha^\pi(s) \ge 0$ and $\mathrm{E}\left[\int_t^T \left(\alpha^\pi(s)^2 + \beta^\pi(s)^2\right) ds\right] < +\infty$;

(iii) (X^π, π) is the unique solution to the stochastic differential equation (5).

In addition, let $\Pi(t, x)$ denote the set of all admissible strategies w.r.t. initial condition $(t, x) \in \mathcal{Q}$ and $\tilde{\Pi}(t, x) := \{\pi \in \Pi(t, x) : \alpha^\pi(s) \equiv 1,\ \forall s \in [t, T]\}$.

3. Problem formulation in a game theoretic framework

In this paper, we consider the mean–variance criterion. Due to that this criterion lacks the iterated-expectation property, time inconsistency, in the sense that the Bellman optimality principle does not hold, is engendered. One way proposed by Strotz (1956) to deal with the time-inconsistent problem is to take the time inconsistency seriously and formulate the problem in game theoretic terms. Similar to Björk and Murgoci (2010), we formulate an investment–reinsurance problem for the insurer in a game theoretic framework. Compared with the problems studied in Zeng and Li (2011) and Li et al. (2012), the difference is that we incorporate jumps into the surplus process of the insurer and the price process of the risky asset.

Specifically, we take our problem as a non-cooperate game, with one player for each time t, where player t can be regarded as the future incarnation of the insurer at time t. For any fixed $(t, x) \in \mathcal{Q}$, the objective of the insurer is to find

$$\sup_{\pi\in\Pi(t,x)} J(t, x, \pi) = \sup_{\pi\in\Pi(t,x)} \left\{ \mathrm{E}_{t,x}[X^{\pi}(T)] - \frac{\gamma}{2}\mathrm{Var}_{t,x}[X^{\pi}(T)] \right\}, \quad (6)$$

and the corresponding equilibrium strategy π^*, where γ is a pre-specified risk aversion coefficient, $\mathrm{E}_{t,x}[\cdot] = \mathrm{E}[\cdot|X^{\pi}(t) = x]$ and $\mathrm{Var}_{t,x}[\cdot] = \mathrm{Var}[\cdot|X^{\pi}(t) = x]$. This problem can be viewed as a dynamic mean–variance problem, since the objective of the insurer updates as state (t, x) changes.

Subsequently, we provide the definition of the equilibrium strategy and a verification theorem for problem (6). Denote

$$\begin{aligned} C^{1,2}(\mathcal{Q}) = \{&\phi(t, x)|\phi(t, \cdot) \text{ is once continuously differentiable} \\ &\text{on } [0, T], \text{ and } \phi(\cdot, x) \text{ is twice continuously} \\ &\text{differentiable on } \mathbb{R}\}, \end{aligned}$$

and for any $\phi(t, x) \in C^{1,2}(\mathcal{Q})$

$$\begin{aligned} \mathcal{A}^{\pi}\phi(t, x) = {}& \phi_t(t, x) + \phi_x(t, x)[r_0(t)x + \delta\lambda_1\mu_y + (1 + \eta)\lambda_1\mu_y \\ &\times \alpha^{\pi}(t) + r(t)\beta^{\pi}(t)] + \frac{1}{2}\phi_{xx}(t, x)[\sigma_0^2\alpha^{\pi}(t)^2 \\ &+ \sigma(t)^2\beta^{\pi}(t)^2] + \lambda_1\mathrm{E}[\phi(t, x - y_1\alpha^{\pi}(t)) \\ &- \phi(t, x)] + \lambda_2\mathrm{E}[\phi(t, x + z_1\beta^{\pi}(t)) - \phi(t, x)]. \end{aligned}$$

Definition 2 (*Equilibrium Strategy*)**.** For any fixed chosen initial state $(t, x) \in \mathcal{Q}$, consider an admissible strategy $\pi^*(t, x)$. Choose three fixed real numbers $\tilde{\alpha} > 0, \tilde{\beta}$, and $\epsilon > 0$ and define the following strategy:

$$\pi^{\epsilon}(s, \tilde{x}) = \begin{cases} (\tilde{\alpha}, \tilde{\beta}), & \text{for } (s, \tilde{x}) \in [t, t + \epsilon) \times \mathbb{R}, \\ \pi^*(s, \tilde{x}), & \text{for } (s, \tilde{x}) \in [t + \epsilon, T] \times \mathbb{R}. \end{cases}$$

If

$$\liminf_{\epsilon\to 0} \frac{J(t, x, \pi^*) - J(t, x, \pi^{\epsilon})}{\epsilon} \geq 0,$$

for all $(\tilde{\alpha}, \tilde{\beta}) \in \mathbb{R}^+ \times \mathbb{R}$ and $(t, x) \in \mathcal{Q}$,

then $\pi^*(t, x)$ is called an equilibrium strategy, and the corresponding equilibrium value function is defined by

$$V(t, x) = J(t, x, \pi^*) = \mathrm{E}_{t,x}[X^{\pi^*}(T)] - \frac{\gamma}{2}\mathrm{Var}_{t,x}[X^{\pi^*}(T)]. \quad (7)$$

By Definition 2, we know that the equilibrium strategy is time-consistent, i.e. sitting at time t, the optimal strategy derived at time t agrees with the optimal strategy at time $t + \Delta t$. Hereafter, equilibrium strategy π^* and equilibrium value function $V(t, x)$ are called the optimal time-consistent strategy and the optimal value function for problem (6).

Theorem 1 (*Verification Theorem*)**.** *For problem (6), if there exist two real value functions $W(t, x), g(t, x) \in C^{1,2}(\mathcal{Q})$ satisfying the following extended HJB system: $\forall(t, x) \in \mathcal{Q}$,*

$$\sup_{\pi\in\Pi(t,x)} \left\{ \mathcal{A}^{\pi}W(t, x) - \mathcal{A}^{\pi}\frac{\gamma}{2}g(t, x)^2 + \gamma g(t, x)\mathcal{A}^{\pi}g(t, x) \right\} = 0, \quad (8)$$

$$W(t, x) = x, \quad (9)$$

$$\mathcal{A}^{\pi^*}g(t, x) = 0, \quad (10)$$

$$g(T, x) = x, \quad (11)$$

where

$$\begin{aligned} \pi^* := \arg\sup_{\pi\in\Pi(t,x)} \Bigg\{ &\mathcal{A}^{\pi}W(t, x) - \mathcal{A}^{\pi}\frac{\gamma}{2}g(t, x)^2 \\ &+ \gamma g(t, x)\mathcal{A}^{\pi}g(t, x) \Bigg\}, \end{aligned} \quad (12)$$

then $V(t, x) = W(t, x)$, $\mathrm{E}_{t,x}[X^{\pi^*}(T)] = g(t, x)$, and π^* is the optimal time-consistent strategy.

The proof of this theorem is similar to Theorem 4.1 of Björk and Murgoci (2010), Theorem 1 of Kryger and Steffensen (2010) and Theorem 1 of Li et al. (2012). So we omit it here.

4. Solution to the optimization problem

In this section, we derive the optimal time-consistent strategy and the optimal value function for problem (6). We will reduce the results obtained in this section to some special cases in the next section.

Suppose that there exist two functions $W(t, x)$ and $g(t, x)$ satisfying the conditions given in Theorem 1 such that the function inside of the bracket of (8), i.e. $\mathcal{A}^{\pi}W(t, x) - 0.5\gamma\mathcal{A}^{\pi}g(t, x)^2 + \gamma g(t, x)\mathcal{A}^{\pi}g(t, x)$, is concave w.r.t. $\alpha^{\pi}(t)$ and $\beta^{\pi}(t)$ for any admissible strategy π. According to Theorem 1, (8) can be rewritten as

$$\begin{aligned} \sup_{\pi\in\Pi(t,x)} \Bigg\{ & W_t(t, x) + W_x(t, x)[r_0(t)x + \delta\lambda_1\mu_y + (1 + \eta)\lambda_1\mu_y \\ &\times \alpha^{\pi}(t) + r(t)\beta^{\pi}(t)] + \frac{1}{2}\left(W_{xx}(t, x) - \gamma g_x(t, x)^2\right) \\ &\times \left(\sigma_0^2\alpha^{\pi}(t)^2 + \sigma(t)^2\beta^{\pi}(t)^2\right) + \lambda_1\mathrm{E}\Big[W(t, x - y_1\alpha^{\pi}(t)) \\ &- \frac{\gamma}{2}g(t, x - y_1\alpha^{\pi}(t))(g(t, x - y_1\alpha^{\pi}(t)) - 2g(t, x))\Big] \\ &+ \lambda_2\mathrm{E}\Big[W(t, x + z_1\beta^{\pi}(t)) - \frac{\gamma}{2}g(t, x + z_1\beta^{\pi}(t)) \\ &\times (g(t, x + z_1\beta^{\pi}(t)) - 2g(t, x))\Big] \\ &- (\lambda_1 + \lambda_2)\left[W(t, x) + \frac{\gamma}{2}g(t, x)^2\right] \Bigg\} = 0. \end{aligned} \quad (13)$$

Since the linear structure of (10) and (13), and the boundary conditions of $W(t, x)$ and $g(t, x)$ given by (9) and (11) are linear in x, similar to Björk and Murgoci (2010) it is natural to assume

$$W(t, x) = A(t)x + B(t), \qquad A(T) = 1, \qquad B(T) = 0, \quad (14)$$

$$g(t, x) = a(t)x + b(t), \qquad a(T) = 1, \qquad b(T) = 0. \quad (15)$$

The corresponding partial derivatives are

$$W_t(t, x) = \dot{A}(t)x + \dot{B}(t), \qquad W_x(t, x) = A(t),$$
$$W_{xx}(t, x) = 0,$$
$$g_t(t, x) = \dot{a}(t)x + \dot{b}(t), \qquad g_x(t, x) = a(t), \qquad g_{xx} = 0,$$

where $\dot{A}(t) = dA(t)/dt, \dot{B}(t) = dB(t)/dt, \dot{a}(t) = da(t)/dt$ and $\dot{b}(t) = db(t)/dt$. Substituting $W(t, x), g(t, x)$ and the above derivatives into (13) yields

$$\begin{aligned} \sup_{\pi\in\Pi(t,x)} \Bigg\{ & \dot{A}(t)x + \dot{B}(t) + A(t)[r_0(t)x + \delta\lambda_1\mu_y + \eta\lambda_1\mu_y\alpha^{\pi}(t) \\ &+ (r(t) + \lambda_2\mu_z)\beta^{\pi}(t)] - \frac{\gamma}{2}a(t)^2[(\sigma_0^2 + \lambda_1\sigma_y^2)\alpha^{\pi}(t)^2 \\ &+ (\sigma(t)^2 + \lambda_2\sigma_z^2)\beta^{\pi}(t)^2] \Bigg\} = 0. \end{aligned} \quad (16)$$

By the first-order condition and differentiating the expression insider of the bracket of the above equation w.r.t. $\alpha^{\pi}(t)$ and $\beta^{\pi}(t)$, respectively, we obtain the following equations for $\pi^* = (\alpha^{\pi^*}(t), \beta^{\pi^*}(t))$:

$$\eta\lambda_1\mu_y A(t) - \gamma(\sigma_0^2 + \lambda_1\sigma_y^2)a(t)^2\alpha^{\pi^*}(t) = 0,$$
$$(r(t) + \lambda_2\mu_z)A(t) - \gamma(\sigma^2 + \lambda_2\sigma_z^2)a(t)^2\beta^{\pi^*}(t) = 0.$$

From the above equations, we can get

$$\alpha^{\pi^*}(t) = \frac{\eta\lambda_1\mu_y A(t)}{\gamma(\sigma_0^2 + \lambda_1\sigma_y^2)a(t)^2}, \quad \beta^{\pi^*}(t) = \frac{(r(t) + \lambda_2\mu_z)A(t)}{\gamma(\sigma(t)^2 + \lambda_2\sigma_z^2)a(t)^2}. \tag{17}$$

Inserting (17) into (16) and (10), we have

$$(\dot{A}(t) + r_0(t)A(t))x + \dot{B}(t) + \delta\lambda_1\mu_y A(t) + \frac{\xi(t)A(t)^2}{2\gamma a(t)^2} = 0, \tag{18}$$

$$(\dot{a}(t) + r_0(t)a(t))x + \dot{b}(t) + \delta\lambda_1\mu_y a(t) + \frac{\xi(t)A(t)}{\gamma a(t)} = 0, \tag{19}$$

where

$$\xi(t) = \frac{\eta^2\lambda_1^2\mu_y^2}{\sigma_0^2 + \lambda_1\sigma_y^2} + \frac{(r(t) + \lambda_2\mu_z)^2}{\sigma(t)^2 + \lambda_2\sigma_z^2}. \tag{20}$$

To ensure (18) and (19) hold, we require

$$\dot{A}(t) + r_0(t)A(t) = 0, \qquad A(T) = 1,$$
$$\dot{B}(t) + \delta\lambda_1\mu_y A(t) + \frac{\xi(t)A(t)^2}{2\gamma a(t)^2} = 0, \qquad B(T) = 0,$$
$$\dot{a}(t) + r_0(t)a(t) = 0, \qquad a(T) = 1,$$
$$\dot{b}(t) + \delta\lambda_1\mu_y a(t) + \frac{\xi(t)A(t)}{\gamma a(t)} = 0, \qquad b(T) = 0.$$

Solving the above equations, we obtain

$$A(t) = e^{\int_t^T r_0(s)ds}, \tag{21}$$

$$B(t) = \delta\lambda_1\mu_y \int_t^T e^{\int_u^T r_0(s)ds}du + \frac{1}{2\gamma}\int_t^T \xi(s)ds, \tag{22}$$

$$a(t) = e^{\int_t^T r_0(s)ds}, \tag{23}$$

$$b(t) = \delta\lambda_1\mu_y \int_t^T e^{\int_u^T r_0(s)ds}du + \frac{1}{\gamma}\int_t^T \xi(s)ds. \tag{24}$$

Substituting (21) and (23) into (17), we have

$$\alpha^{\pi^*}(t) = \frac{\eta\lambda_1\mu_y e^{-\int_t^T r_0(s)ds}}{\gamma(\sigma_0^2 + \lambda_1\sigma_y^2)}, \quad \beta^{\pi^*}(t) = \frac{(r(t) + \lambda_2\mu_z)e^{-\int_t^T r_0(s)ds}}{\gamma(\sigma(t)^2 + \lambda_2\sigma_z^2)}. \tag{25}$$

In addition, by (14)–(15) and (21)–(24), we can also derive the explicit expressions for $W(t,x)$ and $g(t,x)$. By Theorem 1, the above derivation can be summarized as the following theorem.

Theorem 2. *For problem (6), the optimal time-consistent strategy is $\pi^* = (\alpha^{\pi^*}(t), \beta^{\pi^*}(t))$, where $\alpha^{\pi^*}(t)$ and $\beta^{\pi^*}(t)$ are given by (25), and the optimal value function is given by*

$$V(t,x) = W(t,x) = xe^{\int_t^T r_0(s)ds} + \delta\lambda_1\mu_y \times \int_t^T e^{\int_u^T r_0(s)ds}du + \frac{1}{2\gamma}\int_t^T \xi(s)ds, \tag{26}$$

and

$$\mathrm{E}_{t,x}[X^{\pi^*}(T)] = g(t,x) = xe^{\int_t^T r_0(s)ds} + \delta\lambda_1\mu_y \times \int_t^T e^{\int_u^T r_0(s)ds}du + \frac{1}{\gamma}\int_t^T \xi(s)ds. \tag{27}$$

According to Theorem 2 and the definition of the optimal value function given by (7), we have

$$\mathrm{Var}_{t,x}[X^{\pi^*}(T)] = \frac{2}{\gamma}(\mathrm{E}_{t,x}[X^{\pi^*}(T)] - V(t,x)) = \frac{1}{\gamma^2}\int_t^T \xi(s)ds,$$

which leads that $1/\gamma = \sqrt{\mathrm{Var}_{t,x}[X^{\pi^*}(T)]/\int_t^T \xi(s)ds}$. Then substituting the expression for $1/\gamma$ into (27) yields

$$\mathrm{E}_{t,x}[X^{\pi^*}(T)] = xe^{\int_t^T r_0(s)ds} + \delta\lambda_1\mu_y \int_t^T e^{\int_u^T r_0(s)ds}du + \sqrt{\mathrm{Var}_{t,x}[X^{\pi^*}(T)]\int_t^T \xi(s)ds}, \tag{28}$$

that is, for all $(t,x) \in \mathcal{Q}$, the relationship between $\mathrm{E}_{t,x}[X^{\pi^*}(T)]$ and $\mathrm{Var}_{t,x}[X^{\pi^*}(T)]$ is given by (28). The relationship of (28) is also called the efficient frontier of problem (6) at initial state (t,x) in the modern portfolio theory. In addition, by Theorem 2, we find that the optimal reinsurance strategy $\alpha^{\pi^*}(t)$ is independent of the parameters of the risky asset and the optimal investment strategy $\beta^{\pi^*}(t)$ is independent of the parameters of the insurance business.

5. Special cases

In this section, we consider some special cases of our model in the previous sections, which is called the original model hereafter. The results of the original model will be reduced to the following special cases.

Special case 1: *C–L model.* Suppose that the surplus of the insurer follows the classical C–L model, where $\sigma_0 = 0$. In this case, the optimal time-consistent reinsurance strategy $\alpha^{\pi^*}(t)$ given by the first expression in (25) changes to

$$\alpha^{\pi^*}(t) = \frac{\eta\mu_y}{\gamma\sigma_y^2}e^{-\int_t^T r_0(s)ds}, \tag{29}$$

and the optimal time-consistent investment strategy $\beta^{\pi^*}(t)$ given by the second expression in (25) does not change; for any initial state $(t,x) \in \mathcal{Q}$, the optimal value function and the efficient frontier can be written as

$$V(t,x) = xe^{\int_t^T r_0(s)ds} + \delta\lambda_1\mu_y \int_t^T e^{\int_u^T r_0(s)ds}du + \frac{1}{2\gamma}\int_t^T \xi_1(s)ds \tag{30}$$

and

$$\mathrm{E}_{t,x}[X^{\pi^*}(T)] = xe^{\int_t^T r_0(s)ds} + \delta\lambda_1\mu_y \int_t^T e^{\int_u^T r_0(s)ds}du + \sqrt{\mathrm{Var}_{t,x}[X^{\pi^*}(T)]\int_t^T \xi_1(s)ds}, \tag{31}$$

respectively, where $\xi_1(t) = \frac{\eta^2\lambda_1\mu_y^2}{\sigma_y^2} + \frac{(r(t)+\lambda_2\mu_z)^2}{\sigma(t)^2+\lambda_2\sigma_z^2}$.

On comparing the results of this special case with those of the original model, we find the following facts: (a) the insurer will purchase less reinsurance or acquire more new business; (b) the optimal value function will become larger; (c) the efficient frontier will move up in the mean–standard deviation plan. The reason for (a)–(c) is that the risk of the insurance business decreases.

Special case 2: *diffusion model*. Suppose that the surplus of the insurer is modeled by a diffusion model, i.e. let $\sigma_0 = 0$ and Eq. (2) be approximated by

$$dR(t) = (\delta + \eta\alpha(t))\lambda_1\mu_y dt + \sqrt{\lambda_1}\sigma_y\alpha(t)dw_2(t), \tag{32}$$

where $\{w_2(t)\}$ is a one-dimensional standard Brownian motion, which is independent of $\{w_1(t)\}$ and $\{\sum_{i=1}^{N_2(t)} Z_i\}$. One is referred to Grandell (1991), Asmussen et al. (2000) and Zeng and Li (2012) for such approximation. Then the dynamics of the wealth process $X^\pi(t)$ corresponding to an admissible strategy $\pi \in \Pi(t,x)$ can be described by

$$\begin{aligned} dX^\pi(t) &= [(\delta + \eta\alpha^\pi(t))\lambda_1\mu_y + r_0(t)X^\pi(t) + r(t)\beta^\pi(t)]dt \\ &\quad + \sigma(t)\beta^\pi(t)dw_1(t) + \sqrt{\lambda_1}\sigma_y\alpha^\pi(t)dw_2(t) \\ &\quad + \beta^\pi(t)d\sum_{i=1}^{N_2(t)} Z_i. \end{aligned} \tag{33}$$

Similar to the derivation of the original model, we can also obtain the explicit solution to problem (6). By some calculations, it is surprised to find that the results of this model are the same as those of case 1. That is, in this case, the optimal time-consistent reinsurance strategy $\alpha^{\pi^*}(t)$ is given by (29); the optimal time-consistent investment strategy $\beta^{\pi^*}(t)$ is given by the second expression in (25); for any initial state $(t,x) \in \mathcal{Q}$, the optimal value function and the efficient frontier are given by (30) and (31), respectively. In some sense, this finding demonstrates that the diffusion model of the surplus process is a good approximation of C–L model for this problem.

Special case 3: *investment-only model*. Consider the investment-only case. In this case, we assume that the insurer is not allowed to purchase reinsurance or acquire new business, i.e. $\alpha^\pi(t) \equiv 1$, $\forall t \in [0,T]$. Then for any initial state $(t,x) \in \mathcal{Q}$, the set of admissible strategies is $\tilde{\Pi}(t,x)$ given in Definition 1, and by adopting an admissible strategy π the dynamics of the wealth process $X^\pi(t)$ is

$$\begin{aligned} dX^\pi(t) &= [r_0(t)X^\pi(t) + (1+\theta)\lambda_1\mu_y + r(t)\beta^\pi(t)]dt \\ &\quad + \sigma_0 dw_0(t) + \sigma(t)\beta^\pi(t)dw_1(t) - d\sum_{i=1}^{N_1(t)} Y_i \\ &\quad + \beta^\pi(t)d\sum_{i=1}^{N_2(t)} Z_i. \end{aligned} \tag{34}$$

Similar to the derivation of the original model, for problem (6) we have that the optimal time-consistent investment strategy $\beta^{\pi^*}(t)$ is given by the second expression in (25); for any initial state $(t,x) \in \mathcal{Q}$, the optimal value function and the efficient frontier can be written as

$$\begin{aligned} V(t,x) &= xe^{\int_t^T r_0(s)ds} + \theta\lambda_1\mu_y \int_t^T e^{\int_u^T r_0(s)ds}du \\ &\quad - \frac{\gamma(\sigma_0^2 + \lambda_1\sigma_y^2)}{2}\int_t^T e^{\int_u^T 2r_0(s)ds}du \\ &\quad + \frac{1}{2\gamma}\int_t^T \xi_2(s)ds \end{aligned} \tag{35}$$

and provided $\mathrm{Var}_{t,x}[X^{\pi^*}(T)] \geq (\sigma_0^2 + \lambda_1\sigma_y^2)\int_t^T e^{\int_u^T 2r_0(s)ds}du$,

$$\begin{aligned} \mathrm{E}_{t,x}[X^{\pi^*}(T)] &= xe^{\int_t^T r_0(s)ds} + \theta\lambda_1\mu_y \int_t^T e^{\int_u^T r_0(s)ds}du \\ &\quad + \sqrt{\left[\mathrm{Var}_{t,x}[X^{\pi^*}(T)] - (\sigma_0^2 + \lambda_1\sigma_y^2)\int_t^T e^{\int_u^T 2r_0(s)ds}du\right]\int_t^T \xi_2(s)ds}, \end{aligned} \tag{36}$$

respectively, where $\xi_2(t) = \frac{(r(t)+\lambda_2\mu_z)^2}{\sigma(t)^2+\lambda_2\sigma_z^2}$.

Special case 4: *risky asset without jump model*. Assume that the price process of the risky asset has no jumps, i.e. $\lambda_2 = 0$. Then we can find that the optimal time-consistent reinsurance strategy $\alpha^{\pi^*}(t)$ given by the first expression in (25) does not change since it is independent of the parameters of the risky asset; the optimal time-consistent investment strategy is

$$\beta^{\pi^*}(t) = \frac{r(t)}{\gamma\sigma(t)^2}e^{-\int_t^T r_0(s)ds}; \tag{37}$$

for any initial state $(t,x) \in \mathcal{Q}$, the optimal value function and the efficient frontier can be written as

$$\begin{aligned} V(t,x) &= xe^{\int_t^T r_0(s)ds} + \delta\lambda_1\mu_y \int_t^T e^{\int_u^T r_0(s)ds}du \\ &\quad + \frac{1}{2\gamma}\int_t^T \xi_3(s)ds \end{aligned} \tag{38}$$

and

$$\begin{aligned} \mathrm{E}_{t,x}[X^{\pi^*}(T)] &= xe^{\int_t^T r_0(s)ds} + \delta\lambda_1\mu_y \int_t^T e^{\int_u^T r_0(s)ds}du \\ &\quad + \sqrt{\mathrm{Var}_{t,x}[X^{\pi^*}(T)]\int_t^T \xi_3(s)ds}, \end{aligned} \tag{39}$$

respectively, where $\xi_3(t) = \frac{\eta^2\lambda_1^2\mu_y^2}{\sigma_0^2+\lambda_1\sigma_y^2} + \frac{r(t)^2}{\sigma(t)^2}$. In addition, by simple calculations, comparing with the original model we have that if $\frac{\mu_z}{\sigma_z^2} > \frac{r(t)}{\sigma(t)^2}$, the insurer will invest less money in the risky asset, and vice versa.

Subsequently, under the assumption that the price process of the risky asset has no jumps, we further consider the following three situations:

(i) If the surplus process of the insurer is modeled by the C–L model and the diffusion model, the optimal solutions to problem (6) are the same. Specially, the optimal time-consistent reinsurance strategy $\alpha^{\pi^*}(t)$ is given by (29); the optimal time-consistent investment strategy $\beta^{\pi^*}(t)$ is given by (37); for any initial state $(t,x) \in \mathcal{Q}$, the optimal value function and the efficient frontier can be written as

$$\begin{aligned} V(t,x) &= xe^{\int_t^T r_0(s)ds} + \delta\lambda_1\mu_y \int_t^T e^{\int_u^T r_0(s)ds}du \\ &\quad + \frac{1}{2\gamma}\int_t^T \xi_4(s)ds, \end{aligned} \tag{40}$$

and

$$\begin{aligned} \mathrm{E}_{t,x}[X^{\pi^*}(T)] &= xe^{\int_t^T r_0(s)ds} + \delta\lambda_1\mu_y \int_t^T e^{\int_u^T r_0(s)ds}du \\ &\quad + \sqrt{\mathrm{Var}_{t,x}[X^{\pi^*}(T)]\int_t^T \xi_4(s)ds}, \end{aligned} \tag{41}$$

respectively, where $\xi_4(t) = \frac{\eta^2\lambda_1\mu_y^2}{\sigma_y^2} + \frac{r(t)^2}{\sigma(t)^2}$. It is worth pointing out that the results obtained in this case are the same as those of the investment–reinsurance problem with one risky asset in Zeng and Li (2011).

(ii) If the insurer is not allowed to purchase reinsurance or acquire new business, i.e. $\alpha^\pi(t) \equiv 1$ $\forall t \in [0,T]$, for problem (6) the optimal investment strategy $\beta^{\pi^*}(t)$ is also given by (37); for any initial state $(t,x) \in \mathcal{Q}$, the optimal value function and the efficient frontier can be written respectively as

$$V(t,x) = xe^{\int_t^T r_0(s)ds} + \theta\lambda_1\mu_y \int_t^T e^{\int_u^T r_0(s)ds}du$$

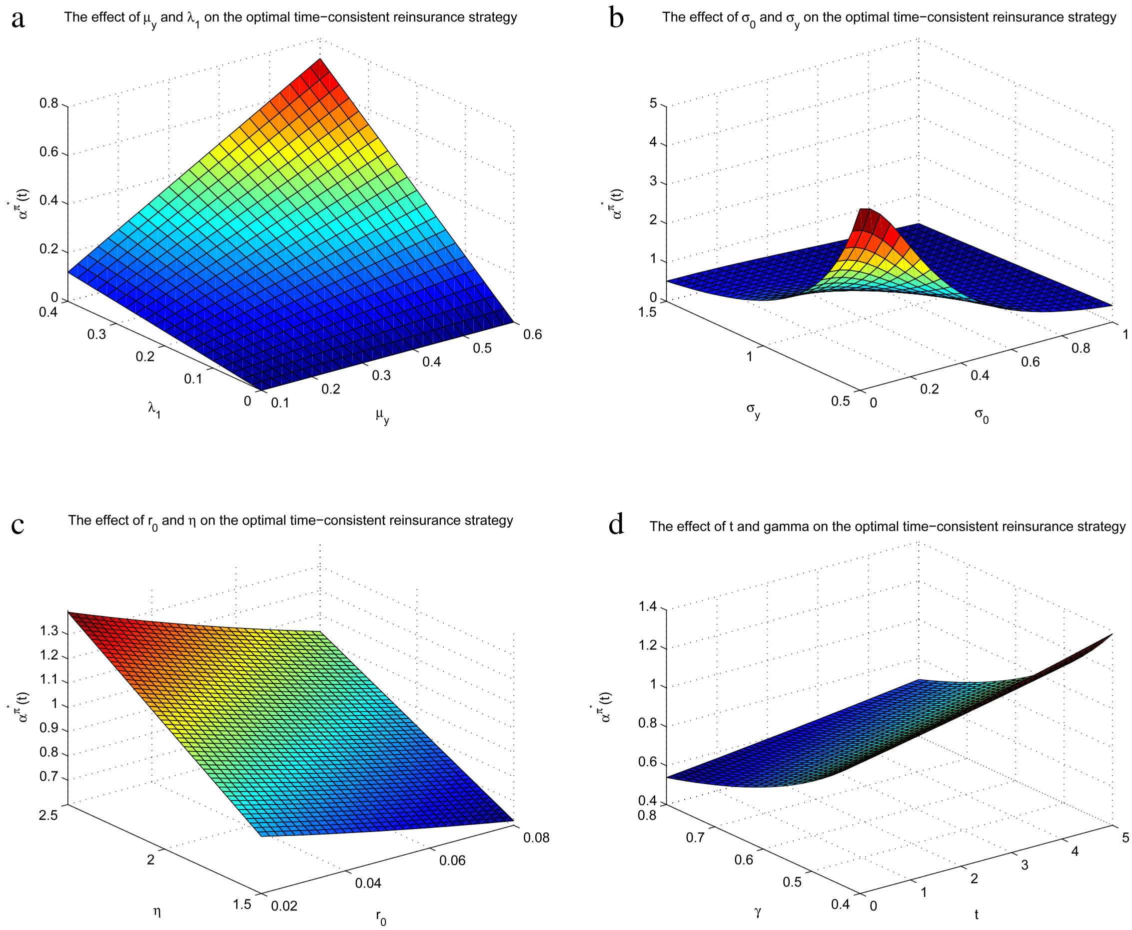

Fig. 1. The effect of parameters on the optimal time-consistent reinsurance strategy.

$$- \frac{\gamma(\sigma_0^2+\lambda_1\sigma_y^2)}{2}\int_t^T e^{\int_u^T 2r_0(s)ds}du + \frac{1}{2\gamma}\int_t^T \left(\frac{r(s)}{\sigma(s)}\right)^2 ds \tag{42}$$

and for $\mathrm{Var}_{t,x}[X^{\pi^*}(T)] \ge (\sigma_0^2+\lambda_1\sigma_y^2)\int_t^T e^{\int_u^T 2r_0(s)ds}du$,

$$\mathrm{E}_{t,x}[X^{\pi^*}(T)] = xe^{\int_t^T r_0(s)ds} + \theta\lambda_1\mu_y\int_t^T e^{\int_u^T r_0(s)ds}du + \sqrt{\left[\mathrm{Var}_{t,x}[X^{\pi^*}(T)] - (\sigma_0^2+\lambda_1\sigma_y^2)\int_t^T e^{\int_u^T 2r_0(s)ds}du\right]\int_t^T\left(\frac{r(s)}{\sigma(s)}\right)^2 ds}. \tag{43}$$

(iii) If the surplus process of the insurer is modeled by the C–L model and the diffusion model and the insurer is not allowed to purchase reinsurance or acquire new business, for problem (6) the optimal investment strategy $\beta^{\pi^*}(t)$ is also given by (37); for any initial state $(t,x)\in\mathcal{Q}$, the optimal value function and the efficient frontier can be written respectively as

$$V(t,x) = xe^{\int_t^T r_0(s)ds} + \theta\lambda_1\mu_y\int_t^T e^{\int_u^T r_0(s)ds}du - \frac{\gamma\lambda_1\sigma_y^2}{2}\int_t^T e^{\int_u^T 2r_0(s)ds}du + \frac{1}{2\gamma}\int_t^T\left(\frac{r(s)}{\sigma(s)}\right)^2 ds, \tag{44}$$

and for $\mathrm{Var}_{t,x}[X^{\pi^*}(T)] \ge \lambda_1\sigma_y^2\int_t^T e^{\int_u^T 2r_0(s)ds}du$,

$$\mathrm{E}_{t,x}[X^{\pi^*}(T)] = xe^{\int_t^T r_0(s)ds} + \theta\lambda_1\mu_y\int_t^T e^{\int_u^T r_0(s)ds}du + \sqrt{\left[\mathrm{Var}_{t,x}[X^{\pi^*}(T)] - \lambda_1\sigma_y^2\int_t^T e^{\int_u^T 2r_0(s)ds}du\right]\int_t^T\left(\frac{r(s)}{\sigma(s)}\right)^2 ds}. \tag{45}$$

The results in (iii) are the same as those of the investment-only problem with one risky asset in Zeng and Li (2011).

6. Numerical illustrations and sensitivity analysis

In this section, we provide some numerical illustrations and sensitivity analysis for our theoretical results derived in the previous sections via an example. For convenience, but without loss of generality, we only analyze the results of the original model with $r_0(t)\equiv r_0$, $r_1(t)\equiv r_1$ and $\sigma(t)\equiv\sigma$ for all $t\in[0,T]$. Unless stated otherwise, the basis parameters are given by: $x_0 = 1$, $T = 5$, $\gamma = 0.6$, $r_0 = 0.05$, $\theta = 1$, $\eta = 1.5$, $\lambda_1 = 0.4$, $\mu_y = 0.6$, $\sigma_0 = 0.5$, $\sigma_y = 1$, $\lambda_2 = 0.5$, $\mu_z = 0.3$, $\sigma = 0.3$ and $\sigma_z = 0.5$.

Due to that the parameters are assumed to be constants in this section, the optimal time-consistent reinsurance strategy $\alpha^{\pi^*}(t)$, the optimal time-consistent investment strategy $\beta^{\pi^*}(t)$ and the optimal value function $V(t,x)$ for the original model (6) can be rewritten as follows:

$$\alpha^{\pi^*}(t) = \frac{\eta\lambda_1\mu_y e^{r_0(t-T)}}{\gamma(\sigma_0^2+\lambda_1\sigma_y^2)}, \tag{46}$$

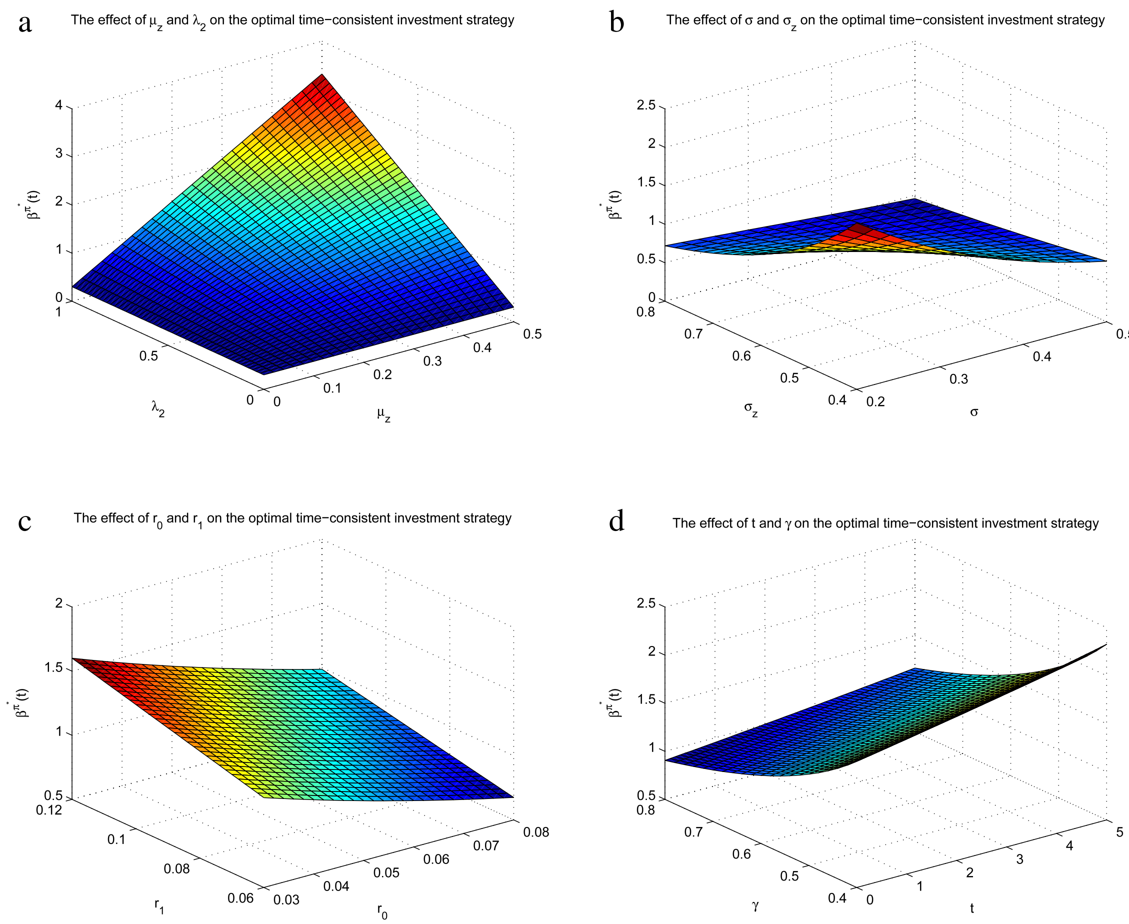

Fig. 2. The effect of parameters on the optimal time-consistent investment strategy.

$$\beta^{\pi^*}(t) = \frac{(r_1 - r_0 + \lambda_2\mu_z)e^{r_0(t-T)}}{\gamma(\sigma^2 + \lambda_2\sigma_z^2)}, \tag{47}$$

$$V(t,x) = xe^{r_0(T-t)} + \frac{\theta - \eta}{r_0}\lambda_1\mu_y\left(e^{r_0(T-t)} - 1\right) + \frac{T-t}{2\gamma}\left(\frac{\eta^2\lambda_1^2\mu_y^2}{\sigma_0^2 + \lambda_1\sigma_y^2} + \frac{(r_1 - r_0 + \lambda_2\mu_z)^2}{\sigma^2 + \lambda_2\sigma_z^2}\right). \tag{48}$$

Without loss of generality, we only picture the cases at $t = 0$.

6.1. Analysis of the optimal time-consistent strategies

Some numerical illustrations and sensitivity analysis for the optimal time-consistent reinsurance strategy and the optimal time-consistent investment strategy are presented in this subsection. According to (46) and (47), we have the following findings:

(a) $\frac{\partial\alpha^{\pi^*}(t)}{\partial\mu_y} > 0$ and $\frac{\partial\alpha^{\pi^*}(t)}{\partial\lambda_1} > 0$, which indicate that when the expectation of the size of each claim μ_y or the intensity of the claims λ_1 increases, the insurer will purchase less reinsurance or acquire more new business. This is showed in Fig. 1(a). In addition, $\frac{\partial\alpha^{\pi^*}(t)}{\partial\sigma_0^2} < 0$ and $\frac{\partial\alpha^{\pi^*}(t)}{\partial\sigma_y^2} < 0$, which mean that the insurer will purchase more reinsurance or acquire less new business when the risk component of the insurance business σ_0^2 or σ_y^2 increases. This is demonstrated in Fig. 1(b).

(b) $\frac{\partial\alpha^{\pi^*}(t)}{\partial r_0} < 0$ and $\frac{\partial\alpha^{\pi^*}(t)}{\partial\eta} > 0$, namely as the risk-free interest rate r_0 increases or as the relative safety loading of the reinsurer η decreases, the insurer will reduce her or his risk exposure by purchasing more reinsurance or acquiring less new business, as showed in Fig. 1(c). Moreover, we find that the relative safety loading of the insurer θ has no impact on the optimal time-consistent reinsurance strategy $\alpha^{\pi^*}(t)$.

(c) $\frac{\partial\beta^{\pi^*}(t)}{\partial\mu_z} > 0$, that is, the insurer will invest more money in the risky asset when the expectation of each jump amplitude of the risky asset's price μ_z becomes larger. In addition, if $\frac{\mu_z}{\sigma_z^2} > (<) \frac{r_1 - r_0}{\sigma^2}$, then $\frac{\partial\beta^{\pi^*}(t)}{\partial\lambda_2} > (<)0$, namely when $\frac{\mu_z}{\sigma_z^2}$ is large (small) enough, the insurer will invest more (less) money in the risky asset as the intensity of the jumps of the risky asset's price λ_2 increases. This is showed in Fig. 2(a). Moreover, $\frac{\partial\beta^{\pi^*}(t)}{\partial\sigma^2} < 0$ and $\frac{\partial\beta^{\pi^*}(t)}{\partial\sigma_z^2} < 0$, which reveal that the insurer will invest less money in the risky asset if the risk component of the risky asset σ^2 or σ_z^2 becomes higher; see Fig. 2(b).

(d) $\frac{\partial\beta^{\pi^*}(t)}{\partial r_0} < 0$ and $\frac{\partial\beta^{\pi^*}(t)}{\partial r_1} > 0$, which illustrate that the insurer will invest less money in the risky asset as the risk-free interest rate r_0 increases or as r_1 decreases. This is illustrated in Fig. 2(c).

(e) $\frac{\partial\alpha^{\pi^*}(t)}{\partial t} > 0$, $\frac{\partial\beta^{\pi^*}(t)}{\partial t} > 0$, $\frac{\partial\alpha^{\pi^*}(t)}{\partial\gamma} < 0$ and $\frac{\partial\beta^{\pi^*}(t)}{\partial\gamma} < 0$. In other words, as time lapses or as the insurer becomes less risk aversion, she or he will preserve more insurance business by purchasing less reinsurance or acquiring more new business and will invest more money in the risky asset, as showed in Figs. 1(d) and 2(d).

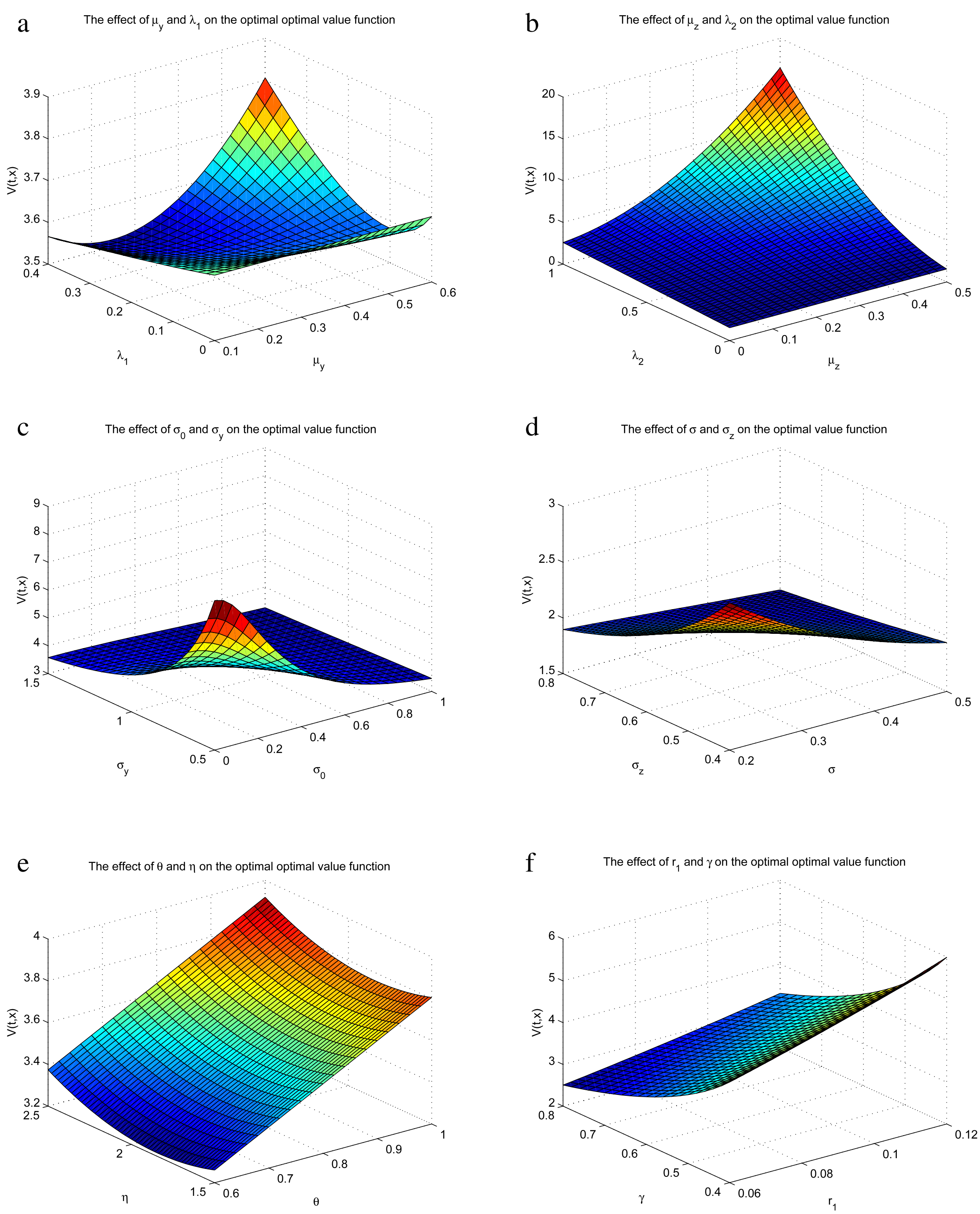

Fig. 3. The effect of parameters on the optimal value function.

6.2. *Analysis of the optimal value function*

This subsection provides some numerical illustrations and sensitivity analysis for the optimal value function. Denote

$$I_1 = \frac{\gamma(\sigma_0^2 + \lambda_1\sigma_y^2)(\eta-\theta)}{\lambda_1\eta^2 r_0(T-t)}(e^{r_0(T-t)} - 1),$$

$$I_2 = \frac{\gamma(\sigma_0^2 + \lambda_1\sigma_y^2)}{\lambda_1\mu_y r_0(T-t)}(e^{r_0(T-t)} - 1).$$

By (48) and some simple calculations, the following results can be derived:

(a) $\frac{\partial V(t,x)}{\partial \mu_y} < (>)0$ if $0 < \mu_y < I_1(\mu_y > I_1)$ and $\frac{\partial V(t,x)}{\partial \lambda_1} > (<)0$ if $(\mu_y - 2I_1)(\lambda_1\sigma_y^2 + \sigma_0^2) + \mu_y\sigma_0^2 > (<)0$. In Fig. 3(a), we can find that as the expectation of the size of each claim μ_y or the intensity of the claims λ_1 increases, the optimal value function $V(t,x)$ first decreases, then increases up to the expectation of the size of each claim μ_y or the intensity of the claims λ_1 larger than some certain point.

(b) $\frac{\partial V(t,x)}{\partial \mu_z} > 0$ and $\frac{\partial V(t,x)}{\partial \lambda_2} < (>)0$ if $r_1 - r_0 - \lambda_2\mu_z > (<) \frac{2\mu_z\sigma^2}{\sigma_z^2}$, which tell us that the optimal value function $V(t,x)$ becomes larger when the expectation of each jump amplitude of the risky asset's price μ_z increases or when $\mu_z \geq 0$ and the intensity of the jumps of the risky asset's price λ_2 increases; see Fig. 3(b).

(c) $\frac{\partial V(t,x)}{\partial \sigma_0^2} < 0$, $\frac{\partial V(t,x)}{\partial \sigma_y^2} < 0$, $\frac{\partial V(t,x)}{\partial \sigma^2} < 0$ and $\frac{\partial V(t,x)}{\partial \sigma_z^2} < 0$, that is, the optimal value function $V(t, x)$ becomes smaller when the risk component of the insurance business σ_0^2 or σ_y^2 increases, or when the risk component of the risky asset σ^2 or σ_z^2 increases, as showed in Fig. 3(c) and (d).

(d) $\frac{\partial V(t,x)}{\partial \eta} < (>)0$ if $\theta \le \eta < I_2(\eta \ge I_2)$, that is, when the relative safety loading of the reinsurer η is small (large) enough, the optimal value function $V(t, x)$ decreases (increases) as the relative safety loading of the reinsurer η becomes larger, as showed in Fig. 3(e). In addition, $\frac{\partial V(t,x)}{\partial \theta} > 0$, which implies that the optimal value function becomes larger as the insurer's relative safety loading θ increases. This is also demonstrated in Fig. 3(e).

(e) $\frac{\partial V(t,x)}{\partial r_1} > 0$ and $\frac{\partial V(t,x)}{\partial \gamma} < 0$, which state that the optimal value function $V(t, x)$ increases as r_1 increases or as the risk aversion coefficient γ decreases; see Fig. 3(f).

7. Concluding remarks

In this paper, a continuous-time dynamic investment and reinsurance optimization problem for mean–variance insurers has been studied within a game theoretic framework. Compared with Zeng and Li (2011) and Li et al. (2012), we introduce jumps by assuming that the surplus of insurers follows a jump–diffusion model and that the price process of the risky asset evolves according to a geometric Lévy motion. We have provided a verification theorem without proof and derived the time-consistent investment and reinsurance strategies and the corresponding value function explicitly. In addition, some special cases of our model have been discussed and the explicit expressions of the corresponding solutions have been derived. Furthermore, we have presented some numerical illustrations and sensitivity analysis to demonstrate the results we have derived. Moreover, this paper extends the model and results of Zeng and Li (2011).

However, our paper also has some limits: (i) for balancing the trade-off between model generality and tractability, we only consider the case that the logarithm of the risky asset's price process follows a jump–diffusion process; (ii) the risk aversion coefficient is assumed to be a constant; (iii) the investment time-horizon is also a constant. In future works, we are going to consider more general cases, such as using VG processes, NIG processes or other more general Lévy processes to model the logarithm of the risky asset's price process, adopting a wealth-dependent risk aversion coefficient, and/or assuming an uncertain time-horizon. These will make the corresponding optimization problems more complicated. To solve such problems, we need to adopt much more sophisticated techniques.

References

Applebaum, D., 2009. Lévy Processes and Stochastic Calculus, second ed. In: Cambridge Studies in Advanced Mathematics, vol. 116. Cambridge University Press, Cambridge.

Asmussen, S., Højgaard, B., Taksar, M., 2000. Optimal risk control and dividend distribution policies: example of excess-of loss reinsurance for an insurance corporation. Finance and Stochastics 4, 299–324.

Bai, L.H., Zhang, H.Y., 2008. Dynamic mean–variance problem with constrained risk control for the insurers. Mathematical Methods of Operations Research 68, 181–205.

Basak, S., Chabakauri, G., 2010. Dynamic mean–variance asset allocation. Review of Financial Studies 23, 2970–3016.

Bäuerle, N., 2005. Benchmark and mean–variance problems for insurers. Mathematical Methods of Operations Research 62, 159–165.

Bielecki, T.R., Jin, H.J., Pliska, S.P., Zhou, X.Y., 2005. Continuous-time mean–variance portfolio selection with bankruptcy prohibition. Mathematical Finance 15, 213–244.

Björk, T., Murgoci, A., 2010. A general theory of Markovian time inconsistent stochastic control problems. In: Working Paper, Stockolm School of Economics. Available at: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1694759.

Björk, T., Murgoci, A., Zhou, X.Y., 2012. Mean variance portfolio optimization with state dependent risk aversion. Mathematical Finance. http://dx.doi.org/10.1111/j.1467-9965.2011.00515.x.

Çakmak, U., Özekici, S., 2006. Portfolio optimization in stochastic markets. Mathematical Methods of Operations Research 63, 151–168.

Çelikyurt, U., Özekici, S., 2007. Multiperiod portfolio optimization models in stochastic markets using the mean–variance approach. European Journal of Operational Research 179, 186–202.

Cont, R., Tankov, P., 2004. Financial Modelling with Jump Processes. Chapman and Hall, London.

Delong, Ł, Gerrard, R., 2007. Mean–variance portfolio selection for a non-life insurance company. Mathematical Methods of Operations Research 66, 339–367.

De-Paz, A., Marín-Solano, J., Navas, J., Roch, O., 2012. Consumption, investment and life insurance strategies with heterogeneous discounting. Working Paper. http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2026127.

Ekeland, I., Lazrak, A., 2008. Equilibrium policies when preferences are time-inconsistent. Working Paper. http://arxiv.org/pdf/0808.3790v1.pdf.

Ekeland, I., Lazrak, A., 2010. The golden rule when preferences are time-inconsistent. Mathematics and Financial Economics 4, 29–55.

Ekeland, I., Mbodji, O., Pirvu, T.A., 2012. Time-consistent portfolio management. SIAM Journal on Financial Mathematics 3, 1–32.

Ekeland, I., Pirvu, T.A., 2008. Investment and consumption without commitment. Mathematics and Financial Economics 2, 57–66.

Fu, C.P., Lari-Lavassani, A., Li, X., 2010. Dynamic mean–variance portfolio selection with borrowing constraint. European Journal of Operational Research 200, 312–319.

Grandell, J., 1991. Aspects of Risk Theory. Springer-Verlag, New York.

Kryger, E.M., Steffensen, M., 2010. Some solvable portfolio problems with quadratic and collective objectives. Working Paper. Available at: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1577265.

Li, D., Ng, W.L., 2000. Optimal dynamic portfolio selection: multiperiod mean–variance formulation. Mathematical Finance 10, 387–406.

Li, Z.F., Zeng, Y., Lai, Y.Z., 2012. Optimal time-consistent investment and reinsurance strategies for insurers under Heston's SV model. Insurance: Mathematics and Economics 51, 191–203.

Marín-Solano, J., Navas, J., 2010. Consumption and portfolio rules for time-inconsistent investors. European Journal of Operational Research 201, 860–872.

Marín-Solano, J., Patxot, C., 2012. Heterogeneous discounting in economic problems. Optimal Control Applications and Methods 33, 32–50.

Markowitz, H.M., 1952. Portfolio selection. Journal of Finance 7, 77–91.

Merton, R., 1976. Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics 3, 125–144.

Sato, K., 1999. Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press, Cambridge.

Schoutens, W., 2003. Lévy Processes in Finance: Pricing Financial Derivatives. In: Wiley Series in Probability and Statistics, Wiley, New York.

Strotz, R.H., 1956. Myopia and inconsistentcy in dynamic utility maximization. Review of Economic Studies 23, 165–180.

Wang, J., Forsyth, P.A., 2011. Continuous time mean variance asset allocation: a time-consistent strategy. European Journal of Operational Research 209, 184–201.

Wu, H.L., Li, Z.F., 2011. Multi-period mean–variance portfolio selection with Markov regime switching and uncertain time-horizon. Journal of Systems Science and Complexity 24, 140–155.

Wu, H.L., Li, Z.F., 2012. Multi-period mean–variance portfolio selection with regime switching and a stochastic cash flow. Insurance: Mathematics and Economics 50, 371–384.

Xia, J.M., Yan, J.A., 2006. Markowitz's portfolio optimization in an incomplete market. Mathematical Finance 16, 203–216.

Zeng, Y., Li, Z.F., 2011. Optimal time-consistent investment and reinsurance policies for mean–variance insurers. Insurance: Mathematics and Economics 49, 145–154.

Zeng, Y., Li, Z.F., 2012. Optimal reinsurance-investment strategies for insurers under mean-CaR criteria. Journal of Industrial and Management Optimization 8 (3), 673–690.

Zeng, Y., Li, Z.F., Liu, J.J., 2010. Optimal strategies of benchmark and mean–variance portfolio selection problems for insurers. Journal of Industrial and Management Optimization 6, 483–496.

Zhou, X.Y., Li, D., 2000. Continuous-time mean–variance portfolio selection: a stochastic LQ framework. Applied Mathematics and Optimization 42, 19–33.