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Article *in* Optimization · February 2017

DOI: 10.1080/02331934.2017.1296837

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Optimization
A Journal of Mathematical Programming and Operations Research

ISSN: 0233-1934 (Print) 1029-4945 (Online) Journal homepage: http://www.tandfonline.com/loi/gopt20

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To cite this article: Peng Yang (2017) Time-consistent mean-variance reinsurance-investment in a jump-diffusion financial market, Optimization, 66:5, 737-758, DOI: 10.1080/02331934.2017.1296837

To link to this article: http://dx.doi.org/10.1080/02331934.2017.1296837

Published online: 25 Feb 2017.

OPTIMIZATION, 2017
VOL. 66, NO. 5, 737–758
http://dx.doi.org/10.1080/02331934.2017.1296837

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ABSTRACT

This paper study an optimal time-consistent reinsurance-investment strategy selection problem in a financial market with jump-diffusion risky asset, where the insurance risk model is modulated by a compound Poisson process. The aggregate claim process and the price process of risky asset are correlated by a common Poisson process. The objective of the insurer is to choose an optimal time-consistent reinsurance-investment strategy so as to maximize the expected terminal surplus while minimizing the variance of the terminal surplus. Since this problem is time-inconsistent, we attack it by placing the problem within a game theoretic framework and looking for subgame perfect Nash equilibrium strategy. We investigate the problem using the extended Hamilton–Jacobi–Bellman dynamic programming approach. Closed-form solutions for the optimal reinsurance-investment strategy and the corresponding value functions are obtained. Numerical examples and economic significance analysis are also provided to illustrate how the optimal reinsurance-investment strategy changes when some model parameters vary.

ARTICLE HISTORY

Received 14 August 2016
Accepted 9 February 2017

KEYWORDS

Time-consistency; proportional reinsurance; investment; jump-diffusion process; common shock

1. Introduction

Due to the facts that reinsurance is an effective way to spread risks and that investment is an increasingly important element in the insurance business, problems concerning optimal investment and reinsurance for insurers have drawn great attentions in recent years. Browne [1] considered a continuous-time diffusion model for the surplus of an insurer and derived an optimal investment strategy by maximizing the expected exponential utility of terminal wealth. Bai and Guo [2] researched a financial market with several risky assets and obtained an optimal reinsurance-investment strategy that maximizes the expected exponential utility of terminal wealth. Baltas et al. [3] studied the optimal investment-reinsurance strategy for an insurer under the effect of inside information. Baltas and Yannacopoulos [4] considered the problem of insurance and reinsurance under the effect of model uncertainty. Yang and Zhang [5], Cao and Wan [6], and Lin and Li [7] studied the optimal investment-reinsurance strategy for an insurer who maximizes the expected utility of the terminal wealth in different situations.

Recently, many scholars have studied the optimal investment and/or reinsurance policies for insurers under the mean-variance criterion, which was pioneered by Markowitz [8] and has long been recognized as the milestone of modern portfolio theory. The dynamic extension of the Markowitz model, especially in continuous time, has been studied extensively. For example, Bäuerle [9] considered the optimal proportional reinsurance/new business problem under the mean-variance criterion, where the surplus process is modelled by the classical Cramér-Lundberg (CL) model, and obtained

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the optimal closed-form strategy. Bai and Zhang [10] studied the optimal investment-reinsurance strategy for an insurer under the mean-variance criterion by the linear quadratic control. Liang et al. [11] considered mean-variance reinsurance and investment in a jump-diffusion financial market with common shock dependence, and obtained the solutions of efficient strategy and efficient frontier explicitly.

The optimal strategies of the above-mentioned literature under the mean-variance criterion are not time-consistent in the sense that, if they are optimal at the initial time, they should be also optimal in any remaining time interval. There are two reasons for this. First, the mean-variance criterion lacks the iterated expectation property. Hence, continuous-time and multi-period mean-variance problems are time-inconsistent in the sense that the Bellman's principle of optimality does not hold. Second, the optimal strategies are derived under the assumption that the investors pre-commit themselves not to deviate from the strategies chosen at the initial time. However, time consistency of insurance policies is a basic requirement for a rational decision made in many situations. A decision-maker sitting at time t would consider that, starting from $t+\Delta t$, he will follow the policy that is optimal sitting at time $t+\Delta t$. Namely, the optimal policy derived at time t should agree with the optimal policy derived at time $t+\Delta t$. Strotz [12] first formalized time inconsistency analytically and worked on time-consistent policies for time-inconsistent problems. He proposed that time-inconsistent problems can be solved either by pre-commitment policies or by time-consistent policies. Björk and Murgoci [13] studied the time-inconsistent problems with a generally controlled Markov process and a fairly general objective function, and sought the corresponding subgame perfect Nash equilibrium strategies, which are time-consistent. They obtained an extended Hamilton–Jacobi–Bellman (HJB) equation and provided a verification theorem. Recently, there has been a renewing interest in time-consistent mean-variance portfolio selection problem. Zeng and Li [14] were the first to present the optimal time-consistent investment and reinsurance strategies for mean-variance insurers, where the surplus of the insurer is modelled by the diffusion model and the price processes of the risky assets are described by geometric Brownian motions. Zeng et al. [15] investigated the same problem as that in Zeng and Li [14], where the price process of the risky asset follows to jump. Li and Li [16] studied Portfolio time-consistent investment strategy with state dependent risk aversion. Zhang and Liang [17] studied the same problem as that in Li and Li [16], where the price process of the risky asset follows jump. Lin and Qian [18] considered time-consistent investment and reinsurance strategies for CEV model. For more detailed discussion, readers can refer to Basak and Chabakauri [19], Björk et al. [20], Czichowsky [21], Ekeland et al. [22], Kronborg and Steffensen [23], Kryger and Steffensen [24] and Wang and Forsyth [25].

In reality, the insurance surplus process and the price processes of the risky assets are often discontinuous and have jumps. Although the number of the literatures on portfolio selection is increasing rapidly, very few papers studied the time-consistent investment and reinsurance strategies with jump. In particular, very few published papers dealt with the optimization problem in the jump risk model with common shock dependence. Zeng et al. [15] considered the time-consistent investment and reinsurance strategies, by incorporating the jumps of insurance surplus process and the risky asset at the same time, however the two jumps are independent. To the best of our knowledge, there is no other literature discussing the aggregate claim and the risky asset price, which are correlated, under mean-variance criterion except Liang et al. [11]. However, in Liang et al. [11] they only derived the pre-commitment strategies.

In the present paper, we consider the same problem as that of Zeng et al. [15], but relax the assumptions involved in them. Specifically, we will study the optimal time-consistent mean-variance reinsurance-investment strategy selection problem for insurers with common shock dependence. That is, the aggregate claim process and the price process of the risky asset are correlated by a common Poisson process. As far as we know, this is the first to study time-consistent reinsurance-investment strategy selection problem with this kind interdependency. Moreover, the insurer can transfer a part of the risks to reinsurance claims via purchasing proportional reinsurance. Besides proportional reinsurance, the insurer can also invest its surplus into a financial market consisting of a risk free asset and a risky asset whose return follows a jump diffusion process. The objective of the insurer is to choose an optimal time-consistent reinsurance-investment strategy so as to maximize the expected terminal surplus while minimizing the variance of the terminal surplus. We will investigate the problem using the extended HJB dynamic programming approach. Closed-form solutions for the optimal reinsurance-investment strategy and the corresponding value functions are obtained. The key contributions of this paper are as follows. First, time-consistent mean-variance reinsurance-investment strategy selection problems for insurers are studied in common shock dependence model. Liang et al. [11] only considered this model in time-inconsistency case. Second, we will study the problem in a financial market with the price of the risky asset governed by a jump-diffusion process model. Some results of the classical Black–Scholes financial market are extended to the jump-diffusion process model. Finally, numerical examples and economic significance analysis are also provided to illustrate how the optimal reinsurance-investment strategy changes with some model parameters vary, and the relationship between the optimal reinsurance strategy and investment strategy is also given. Liang et al. [11] only gave numerical examples to show the impact of model parameters on the efficient frontier.

The rest of the paper is organized as follows. The model and the assumptions are described in Section 2. In Section 3, a general time-consistent mean-variance reinsurance-investment selection problem in a game theoretic framework is formulated. In Section 4, the HJB system and verification theorem are given. In Section 5, the closed-form solution for the time-consistent mean-variance reinsurance-investment strategy selection problem is derived. In Section 6, we provide some numerical examples to illustrate the impacts of some model parameters on the optimal reinsurance-investment strategy. The final section summarizes this paper.

2. Model and assumptions

In this section, we present a continuous-time insurance-financial model. First, we give three mutually independent Poisson processes $N_1(t)$, $N_2(t)$ and $N(t)$, which with intensities $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda > 0$, respectively. We assume that all processes and random variables are defined on a filtered probability space $(\Omega, \mathcal{F}, F, P)$ satisfying the usual conditions, i.e. $F = \{\mathcal{F}_t, t \geq 0\}$ is right continuous and P-complete. A simplified financial market with two securities, namely, a risk-free asset (bond) and a risky asset (stock), is considered. We suppose that these securities are tradable over time in the finite time horizon $[0, T]$ and that fractional units of these securities can be traded, where T is termination time of the trading. In addition, we use the standard assumptions of continuous-time financial models:

- Continuous trading is allowed,
- No transaction cost or tax is involved in trading,
- All assets are infinitely divisible.

2.1. Risk model

The risk process $\{U(t), t \geq 0\}$ of the insurer is defined as follows:

$$U(t) = u_0 + ct - \sum_{i=1}^{K_1(t)} Y_i.$$

Here $u_0 \geq 0$, u_0 denotes the initial capital, and $c > 0$, c is the premium rate per unit of time. Y_i is the size of the ith claim, and $\{Y_i, i = 1, 2, \ldots\}$ is a sequence of independent nonnegative random variables with common distribution. The common distribution function is $F_Y(y)$, and the density function is $f_Y(y)$, and Y_i has finite mean $\mu_{11} = E(Y_i)$ and second moment $\mu_{12} = E(Y_i)^2$. $K_1(t) = N(t) + N_1(t)$ is a counting process, representing the number of claims up to time t. Since $N_1(t)$ and $N(t)$ are mutually independent Poisson processes with intensities λ_1 and λ, $K_1(t)$ is a Poisson process with intensity $\lambda_1 + \lambda$. Thus the compound Poisson process $\sum_{i=1}^{K_1(t)} Y_i$ represents the cumulative amount of claims in the time interval $[0, t]$.

The proportional reinsurance level is associated with the value $1 - a(t)$, where $a(t)$ is the level of risk exposure of the insurer. Here $0 \leq a(t) \leq 1$ corresponds to a proportional reinsurance and $a(t) > 1$ to the acquiring new business, see Bäuerle [9] for details. Set the reinsurance safe load as θ, and the reinsurance premium rate to be calculated by an expected value principle, then the reinsurance premium rate at time t is $1 - a(t)c_1$, with $c_1 = (1+\theta)(\lambda_1+\lambda)\mu_{11}$, and c_1 is the premium of reinsurance per unit time. With the proportional reinsurance being incorporated, the surplus process U_t^a of the insurer becomes

$$dU_t^a = [c - (1 - a(t)c_1)]dt - a(t)d \sum_{i=1}^{K_1(t)} Y_i. \tag{1}$$

2.2. *Financial market*

Without loss of generality, we assume that there are two assets available for the insurance company: one is risk-free asset (bond) and another is risky asset (stock). The price process of risk-free asset (bond) $B(t)$ is given by

$$dB(t) = rB(t)dt,$$

where $r > 0$, and r is the interest rate of the risk-free asset. Let $W(t)$ be a standard Brownian motion on $(\Omega, \mathcal{F}, F, P)$, and we assume that the price process of the risky asset (stock) $S(t)$ evolves over time according to the following jump-diffusion process

$$dS(t) = S(t-) \left[\mu dt + \sigma dW(t) + d \sum_{i=1}^{K_2(t)} Z_i \right]. \tag{2}$$

Here $\mu > r$, μ is the appreciation rate of the risky asset, and $\sigma > 0$, σ is the volatility of risky asset. Z_i is the size of the ith jump, and $\{Z_i, i = 1, 2, \ldots\}$ is a sequence of independent and identically distributed random variables with values in $(-1, +\infty)$, then the assumption that $Z_i > -1$ always leads to positive values of the risky asset price. The common distribution function is $F_Z(z)$, and density function is $f_Z(z)$, and Z_i with finite mean $\mu_{21} = E(Z_i)$ and second moment $\mu_{22} = E(Z_i)^2$. $K_2(t) = N(t) + N_2(t)$ is a counting process, representing the number of jumps to time t. Since $N_2(t)$ and $N(t)$ are mutually independent Poisson processes with intensities λ_2 and λ, $K_2(t)$ is a Poisson process with intensity $\lambda_2 + \lambda$. Thus the compound Poisson process $\sum_{i=1}^{K_2(t)} Z_i$ represents the cumulative amount of jumps in the time interval $[0, t]$.

From the above discuss, we see that the two Poisson processes $\{K_1(t), t \geq 0\}$ and $\{K_2(t), t \geq 0\}$ are correlated in the way that

$$K_1(t) = N_1(t) + N(t) \quad \text{and} \quad K_2(t) = N_2(t) + N(t).$$

It is obvious that the dependence between the aggregate claim process and the price process of the risky asset is due to a common shock governed by the Poisson process $N(t)$. Moreover, assume $\{W(t), t \geq 0\}$, $\{N_1(t), t \geq 0\}$, $\{Y_i, i = 1, 2, \ldots\}$, $\{N_2(t), t \geq 0\}$, $\{Z_i, i = 1, 2, \ldots\}$, and $\{N(t), t \geq 0\}$ are mutually independent.

2.3. *Wealth process*

Assume that the insurer can dynamically purchase proportional reinsurance/acquire new business and invest in the financial market over the time interval $[0, T]$. For each $t \in [0, T]$, let $\pi(t)$ be the amount of the surplus of the insurer invested in the risky asset at time t. The rest of the wealth is then invested in the risk-free asset. Here we suppose that at any time t, the insurer can choose the amount of investment $\pi(t)$ in the risky asset and the risk exposure $a(t)$ of insurance claims based on observable information about the financial price process and the insurance risk process up to time t. In other words, $a(t)$ and $\pi(t)$ are the control parameters of the insurer, and we write them as $u(t) = (a(t), \pi(t))$. For each policy process $\{u(t), t \in [0, T]\}$, the surplus process X_t^u of the insurer with proportional reinsurance and investment evolves over time can be described as

$$\mathrm{d}X_t^u = [c - (1 - a(t)c_1) + rX_t^u + (\mu - r)\pi(t)]\mathrm{d}t + \pi(t)\sigma \mathrm{d}W(t) - a(t)\mathrm{d}\sum_{i=1}^{K_1(t)} Y_i + \pi(t)\mathrm{d}\sum_{i=1}^{K_2(t)} Z_i,$$

with $X_0^u = 0$, and it can also be rewritten as

$$\begin{aligned}\mathrm{d}X_t^u &= [c - (1 - a(t)c_1) + rX_t^u + (\mu - r)\pi(t)]\mathrm{d}t + \pi(t)\sigma \mathrm{d}W(t) - \int_{D_1} a(t)yN_1(\mathrm{d}t, \mathrm{d}y) \\ &\quad + \int_{D_2} \pi(t)zN_2(\mathrm{d}t, \mathrm{d}z) + \iint_D [\pi(t)z - a(t)y]N(\mathrm{d}t, \mathrm{d}y, \mathrm{d}z). \end{aligned} \tag{3}$$

Here $D_1 = \{y|y \geq 0\}$, $D_2 = \{z|z > -1\}$ and $D = \{(y, z)|y \geq 0, z > -1\}$ are measurable sets. $N_1(\mathrm{d}t, \mathrm{d}y)$, $N_2(\mathrm{d}t, \mathrm{d}z)$ and $N(\mathrm{d}t, \mathrm{d}y, \mathrm{d}z)$ are mutually independent Poisson random measures. The compensated Poisson random measures are then given as follows:

$$\begin{aligned}\tilde{N}_1(\mathrm{d}t, \mathrm{d}y) &= N_1(\mathrm{d}t, \mathrm{d}y) - \nu_1(\mathrm{d}t, \mathrm{d}y), \\ \tilde{N}_2(\mathrm{d}t, \mathrm{d}z) &= N_2(\mathrm{d}t, \mathrm{d}z) - \nu_2(\mathrm{d}t, \mathrm{d}z), \\ \tilde{N}(\mathrm{d}t, \mathrm{d}y, \mathrm{d}z) &= N(\mathrm{d}t, \mathrm{d}y, \mathrm{d}z) - \nu(\mathrm{d}t, \mathrm{d}y, \mathrm{d}z),\end{aligned}$$

where $\nu_1(\mathrm{d}t, \mathrm{d}y)$, $\nu_2(\mathrm{d}t, \mathrm{d}z)$ and $\nu(\mathrm{d}t, \mathrm{d}y, \mathrm{d}z)$ are the compensators of $N_1(\mathrm{d}t, \mathrm{d}y)$, $N_2(\mathrm{d}t, \mathrm{d}z)$ and $N(\mathrm{d}t, \mathrm{d}y, \mathrm{d}z)$, respectively.

Definition 1: For any fixed $t \in T$, a reinsurance-investment strategy $u(t) = (a(t), \pi(t))$ is said to be admissible for the insurer if it satisfies the following conditions:

(i) $a(t)$ and $\pi(t)$ are F-progressively measurable and right continuous with left limit;
(ii) $a(t) \geq 0$ and $\int_0^T a(t)\mathrm{d}t < \infty$;
(iii) $\int_0^T \pi(t)\mathrm{d}t < \infty$;
(iv) The stochastic differential Equation (3) for $u(t)$ has a unique strong solution.

The set of all admissible reinsurance-investment strategies of the insurer is denoted by $\mathcal{U}$.

3. Problem formulation

As far as we know, in the existing literature on optimal investment and/or reinsurance strategies for an insurer under mean-variance criterion, most authors assume that the insurer's aim is to solve the following problem:

$$\tilde{J}(0, x, u_0) = \sup_{u \in \mathcal{U}} \left\{ E_{0,x_0}[X_T^u] - \frac{\gamma}{2} Var_{0,x_0}[X_T^u] \right\}, \tag{4}$$

over all admissible reinsurance-investment strategies, where γ is a pre-specified risk aversion parameter of the insurer, $E_{0,x_0}(\cdot) = E[\cdot|X_0^u = x_0]$, $Var_{0,x_0}(\cdot) = Var[\cdot|X_0^u = x_0]$, u_0 is the initial reinsurance-investment strategy and x_0 is the initial capital of the insurer. Problem (4) is a static optimization problem as one determines the optimal strategy at the initial time 0. In Björk and Murgoci [13] and Kronborg and Steffensen [23], Problem (4) is called the mean-variance optimization problem with pre-commitment. The corresponding optimal strategy of (4) is called the optimal pre-commitment strategy.

Time-consistent strategy is a basic requirement of rational decision-making under many situations, and today's preference may be different from tomorrow's preference. However, the problem (4) turn out to be time inconsistent, in the sense that the Bellman optimality principle does not hold. More precisely says that if for some fixed initial point $(0, x_0)$, we determine the strategy u which maximizes $\tilde{J}(0, x, u_0)$, then at some later point (s, X_s) the control law u will no longer be optimal for the functional $\tilde{J}(s, X_s, u)$. The reason for the time inconsistency of the formulation (4) is the conditional variance term. For detailed proof, readers can refer to Czichowsky [21]. Since lack of time consistency, we have both conceptual and computational problems. From a conceptual point of view, it is no longer clear what we mean by the word 'optimal', since a control strategy which is optimal for one choice of starting point in time will generically not be optimal at a later point in time. On the other hand, even with certain precise definition of optimality, we also have a computational problem, since dynamic programming is no longer available. For more detailed discussion, readers can refer to Björk and Murgoci [13].

So our first task is to specify more precisely which problem we are trying to solve. There are two main ways of handling a family of time inconsistent problems. The first way, like the pre-commitment, i.e. we fix one initial point, like for example $(0, x_0)$, and then try to find the control strategy u^* which maximizes $\tilde{J}(0, x_0, u_0)$. We then simply regard as that at a later points in time such as (s, X_s) the control strategy u^* will also be optimal for the functional $\tilde{J}(s, X_s, u)$.

Another way is that we take the time inconsistency seriously and, without pre-commitment mechanism, formulate the problem in game theoretic terms which allow to calculate the optimal time-consistent control strategy. Similar to Björk and Murgoci [13] and Björk et al. [20], in this paper we formulate an investment-reinsurance problem for the insurer in a game theoretic framework.

Specifically, we take our problem as a non-cooperate game, with one player for each time t, where player t can be regarded as the future incarnation of the insurer at time t. For any $(t, x) \in [0, T] \times R$, the objective function the insurer wants to maximize is given as follows:

$$\tilde{J}(t, x, u) = E_{t,x}[X_T^u] - \frac{\gamma}{2} Var_{t,x}[X_T^u], \tag{5}$$

where $[0, T] \times R = \{(t, x) |\, 0 \le t \le T,\ x \in R\}$, $E_{t,x}(\,\cdot\,) = E[\cdot|X_t^u = x]$, $Var_{t,x}(\,\cdot\,) = Var[\cdot|X_t^u = x]$, u is reinsurance-investment strategy at time t, x is the capital of the insurer at time t and γ is a pre-specified risk aversion parameter of the insurer. For convenience, we rewrite the reward function (5) as

$$\tilde{J}(t, x, u) = E_{t,x}[F(X_T^u)] + G(E_{t,x}(X_T^u)), \tag{6}$$

where $F(y) = y - \frac{1}{2}\gamma y^2$ and $G(y) = \frac{1}{2}\gamma y^2$. This problem can be viewed as a dynamic mean-variance problem, since the objective of the insurer updates as state (t, x) changes.

Remark 1: The problem (6) (or problem (5)) is a special case of the following problem (7):

$$\begin{aligned} J(t, x, u) &= E_{t,x}[X_T^u] - \frac{\gamma(x)}{2} Var_{t,x}[X_T^u] \\ &= E_{t,x}[F(x, X_T^u)] + G(x, E_{t,x}(X_T^u)), \end{aligned} \tag{7}$$

where $F(x, y) = y - \frac{\gamma(x)}{2} y^2$, $G(x, y) = \frac{\gamma(x)}{2} y^2$ and $\gamma(x)$ is a state dependent risk aversion function. Obviously, if set $\gamma(x) = \gamma$, we can obtain problem (6) from problem (7). In this paper, we only study the problem (6), and the problem (7) will be studied in the future.

Now, we introduce an equilibrium strategy, which will henceforth be called time-consistent strategy. We introduce a time-consistent strategy from the following three aspects:

- First, we explain what is the meaning of time-consistent strategy.
- Then, we can loosely define the time-consistent strategy from game view.
- Finally, we give the formal definition of the time-consistent strategy.

First, the time-consistent strategy, that is, if it has been applied up to time t, the insurer at time t will apply it as well. Since there is no commitment mechanism to force him to do so, he will only apply strategy u if it is in his own best interests. Denote his current wealth by X_t^u. He has two possibilities: either to stick to the strategy u, or to apply another one. To simplify matters, we will assume that the decision-maker considers only a very short time interval $[t, t+\varepsilon]$, with $\varepsilon \to 0^+$, so short in fact that all strategies can be assumed to be constant on that interval. The decision-maker then just compares the effect of investing π^*, reinsure a^*, as required by the strategy u at time t, with the effect of investing π, reinsure a, for different (constant) values. There will be, as usual, an immediate effect, corresponding to the change in the objective functional $\tilde{J}$ defined above between t and $t+\varepsilon$, and a long-term effect, corresponding to the change in wealth at time $t+\varepsilon$.

Then, we can loosely define the concept of the above idea from game view. This is a control strategy u^* satisfying the following:

- Choose an arbitrary point t in time.
- Suppose that every player s, for all $s > t$, will use the strategy $u^*(s, \cdot)$.
- Then the optimal choice for player t, given the objective functional $\tilde{J}$ defined above, is that he/she also uses the strategy $u^*(t, \cdot)$.

Finally, we present the following formal definition of a time-consistent strategy, i.e. an equilibrium strategy.

Definition 2: For any fixed chosen initial state $(t, x) \in [0, T] \times R$, consider an admissible strategy $u^*(t, x)$. Choose three fixed real numbers $\tilde{a} > 0$, $\tilde{b} \in R$, and $\varepsilon > 0$ and define the following strategy:

$$u^{\varepsilon}(s, \tilde{x}) = \begin{cases} (\tilde{a}, \tilde{b}), \mathit{for}\ (s, \tilde{x}) \in [t, t+\varepsilon] \times R, \\ u^*(s, \tilde{x}), \mathit{for}\ (s, \tilde{x}) \in [t+\varepsilon, T] \times R, \end{cases}$$

If

$$\liminf_{\varepsilon \to 0} \frac{\tilde{J}(t, x, u^*) - \tilde{J}(t, x, u^{\varepsilon})}{\varepsilon} \geq 0, \tag{8}$$

for all $(\tilde{a}, \tilde{b}) \in R^+ \times R$ and $(t, x) \in [0, T] \times R$, then $u^*(t, x)$ is called a time-consistent strategy, i.e. an equilibrium strategy. The value function $V(t, x)$ is defined by

$$V(t, x) = \tilde{J}(t, x, u^*).$$

4. The HJB system and verification theorem

In this section, we first give extended HJB system and verification theorem for problem (7). Then as a special case, it is not difficult to get HJB system and verification theorem for problem (6).

Let $C^{1,2}([0, T] \times R)$ denote the space of $\varphi(t, x)$, such that $\varphi(t, x)$ and its derivatives $\varphi_t(t, x)$, $\varphi_x(t, x)$, $\varphi_{xx}(t, x)$ are continuous on $[0, T] \times R$. For any function $\varphi(t, x) \in C^{1,2}([0, T] \times R)$ and any fixed $u \in \mathcal{U}$, the usual infinitesimal generator $\mathcal{A}^u$ for the jump-diffusion process (3) is defined as

$$\begin{aligned} \mathcal{A}^u \varphi(t, x) = {} & \varphi_t + [c - (1 - a(t)c_1) + rx + (\mu - r)\pi(t)]\varphi_x + \frac{1}{2}\sigma^2 \pi^2(t)\varphi_{xx} \\ & + \lambda_1 E[\varphi(t, x - a(t)Y) - \varphi(t, x)] + \lambda_2 E[\varphi(t, x + \pi(t)Z) - \varphi(t, x)] \\ & + \lambda E[\varphi(t, x - a(t)Y + \pi(t)Z) - \varphi(t, x)]. \end{aligned} \tag{9}$$

Now, we give the following extended HJB system for problem (7), which is similar to Björk and Murgoci [13].

Definition 3: The extended HJB system for the Nash equilibrium problem is defined as follows:

$$\begin{cases} \sup_{u\in\mathcal{U}}\{\mathcal{A}^u V(t,x) - \mathcal{A}^u f(t,x,x) + \mathcal{A}^u f^x(t,x) - \mathcal{A}^u (G\diamond g)(t,x) & \\ +\mathcal{H}^u g(t,x)\} = 0, & 0\le t\le T \\ \mathcal{A}^{u^*} f^y(t,x) = 0, & 0\le t\le T \\ \mathcal{A}^{u^*} g(t,x) = 0, & 0\le t\le T \\ V(T,x) = F(x,x) + G(x,x), & \\ f(T,x,y) = F(y,x), & \\ g(T,x) = x. & \end{cases} \tag{10}$$

Here,

$$u^* = arg\sup_{u\in\mathcal{U}}\{\mathcal{A}^u V(t,x) - \mathcal{A}^u f(t,x,x) + \mathcal{A}^u f^x(t,x) - \mathcal{A}^u (G\diamond g)(t,x) + \mathcal{H}^u g(t,x)\}.$$

$f(t,x,y)$ and $g(t,x)$ have the probabilistic interpretations as follows:

$$\begin{cases} f(t,x,y) = E_{t,x}[F(y,X_T^{u^*})], \\ g(t,x) = E_{t,x}[X_T^{u^*}], \end{cases} \tag{11}$$

and f^y, $G\diamond g$ and $\mathcal{H}^u g$ are defined by

$$\begin{cases} f^y(t,x) = f(t,x,y), \\ (G\diamond g)(t,x) = G(x,g(t,x)), \\ \mathcal{H}^u g(t,x) = G_g(x,g(t,x))\mathcal{A}^u g(t,x). \end{cases}$$

The following theorem shows a connection between the solutions of the extended HJB system and the optimal values.

Theorem 4.1 (Verification Theorem): *Assume that (W,f,g) is a solution of the extended HJB system in Definition 3, and u^* is the control strategy which realizes the supermum in the first equation. Then, u^* is the optimal time-consistent strategy, and $W(t,x)$ coincides with the optimal value function $V(t,x)$. Furthermore, f and g can be interpreted according to (11).*

***Proof*:** The proof which is similar to Björk and Murgoci [13] consists of two steps:

- First, it is started by showing that $W(t,x)$ is the value function corresponding to u^*, i.e. $W(t,x) = J(t,x,u^*)$, and that f and g can be interpreted according to (11). We apply Itô formula to $W(t,x)$. According to Dynkin formula and the boundary conditions of the extended HJB system, we would obtain $W(t,x) = J(t,x,u^*)$.
- Second, by constructing a control strategy and following from Definition 2, we can show that u^* is the optimal time-consistent strategy and thus $W(t,x)$ is the optimal value function.

Step 1 Now, we begin to prove $W(t,x) = J(t,x,u^*)$. First, by applying Itô formula to $W(t,x)$ and by the expression of $\mathcal{A}^{u^*} W(t,x)$, we have

$$\begin{aligned} W(T,X_T^{u^*}) = W(t,x) + \int_t^T \mathcal{A}^{u^*} W(s,X_s^{u^*})\mathrm{d}s + \int_t^T \pi^*(s) W_x \sigma \,\mathrm{d}W(s) \\ + \int_t^T \int_{D_1} [W(s,X_s^{u^*} - a^*(s)y) - W(s,X_s^{u^*})]\tilde{N}_1(\mathrm{d}t,\mathrm{d}y) \end{aligned}$$

$$
\begin{aligned}
&+\int_t^T \int_{D_2} [W(s, X_s^{u^*} + \pi^*(s)Z) - W(s, X_s^{u^*})]\tilde{N}_2(\mathrm{d}t, \mathrm{d}z) \\
&+\int_t^T \iint_D [W(s, X_s^{u^*} - a^*(s)y + \pi^*(s)Z) - W(s, X_s^{u^*})]\tilde{N}(\mathrm{d}t, \mathrm{d}y, \mathrm{d}z). \qquad (12)
\end{aligned}
$$

Because $u^*(t) = (a^*(t), \pi^*(t))$ is an admissible strategy, and $W(t,x) \in C^{1,2}([0,T] \times R)$ implies that $W(t,x)$ is uniformly bounded in $[0,T] \times R$, then we have

$$
\begin{cases}
E\left[\int_t^T \left[\pi^*(s)\sigma W_x\right]^2 \mathrm{d}s\right] < \infty, \\
E\left\{\int_t^T \left[\int_{D_1} \left|W(s, X_s^{u^*} - a^*(s)y) - W(s, X_s^{u^*})\right| \nu_1(\mathrm{d}y)\right.\right. \\
\quad + \int_{D_2} \left|W(s, X_s^{u^*} + \pi^*(s)y) - W(s, X_s^{u^*})\right| \nu_2(\mathrm{d}z) \\
\quad \left.\left. + \iint_D \left|W(s, X_s^{u^*} - a^*(s)y + \pi^*(s)z) - W(s, X_s^{u^*})\right| \nu(\mathrm{d}y, \mathrm{d}z)\right] \mathrm{d}s\right\} < \infty.
\end{cases}
$$

Taking the conditional expectation on both side of (12) yields

$$
E_{t,x}[W(T, X_T^{u^*})] = W(t,x) + E_{t,x}\left[\int_t^T \mathcal{A}^{u^*} W(s, X_s^{u^*})\mathrm{d}s\right].
$$

Notice that $\mathcal{A}^{u^*} f^x(t,x) = 0$ and $\mathcal{A}^{u^*} g(t,x) = 0$, we obtain

$$
\mathcal{A}^{u^*} W(t,x) - \mathcal{A}^{u^*} f(t,x,x) - \mathcal{A}^{u^*}(G \diamond g)(t,x) = 0.
$$

Therefore,

$$
\begin{aligned}
E_{t,x}[W(T, X_T^{u^*})] = W(t,x) &+ E_{t,x}\left[\int_t^T \mathcal{A}^{u^*} f(s, X_s^{u^*}, X_s^{u^*})\mathrm{d}s\right] \\
&+ E_{t,x}\left[\int_t^T \mathcal{A}^{u^*}(G \diamond g)(s, X_s^{u^*})\mathrm{d}s\right]. \qquad (13)
\end{aligned}
$$

According to Dynkin formula, we have

$$
\begin{aligned}
E_{t,x}\left[\int_t^T \mathcal{A}^{u^*} f(s, X_s^{u^*}, X_s^{u^*})\mathrm{d}s\right] &= E_{t,x}[f(T, X_T^{u^*}, X_T^{u^*})] - f(t,x,x), \\
E_{t,x}\left[\int_t^T \mathcal{A}^{u^*}(G \diamond g)(s, X_s^{u^*})\mathrm{d}s\right] &= E_{t,x}[G(X_T^{u^*}, g(T, X_T^{u^*}))] - G(x, g(t,x)).
\end{aligned}
$$

By boundary conditions

$$
\begin{aligned}
W(T, X_T^{u^*}) &= F(X_T^{u^*}, X_T^{u^*}) + G(X_T^{u^*}, X_T^{u^*}), \\
f(T, X_T^{u^*}, X_T^{u^*}) &= F(X_T^{u^*}, X_T^{u^*}), \\
g(T, X_T^{u^*}) &= X_T^{u^*},
\end{aligned}
$$

Equation (13) becomes

$$
\begin{aligned}
&E_{t,x}\left[F(X_T^{u^*}, X_T^{u^*}) + G(X_T^{u^*}, X_T^{u^*})\right] \\
&\quad = W(t,x) + E_{t,x}[F(X_T^{u^*}, X_T^{u^*})] - f(t,x,x) \\
&\quad\quad + E_{t,x}[G(X_T^{u^*}, X_T^{u^*})] - G(x, g(t,x)),
\end{aligned} \tag{14}
$$

which yields

$$
\begin{aligned}
W(t,x)) &= f(t,x,x) + G(x, g(t,x)) \\
&= E_{t,x}[F(x, X_T^{u^*})] + G(x, E_{t,x}(X_T^{u^*})) \\
&= J(t,x,u^*).
\end{aligned}
$$

Step 2 The next step is to prove that u^* is the optimal time-consistent strategy. To that end we construct, for any $\varepsilon > 0$ and an arbitrary $u \in \mathcal{U}$, the control law u^ε defined in Definition 2. Applied to the points t and $t+\varepsilon$, we have

$$
\begin{aligned}
J_n(x,u) = {} & E_{t,x}[J(t+\varepsilon, X_{t+\varepsilon}^u, u)] - E_{t,x}[f^u(t+\varepsilon, X_{t+\varepsilon}^u, X_{t+\varepsilon}^u)] \\
& + E_{t,x}[f^u(t+\varepsilon, X_{t+\varepsilon}^u, x)] - E_{t,x}[G(X_{t+\varepsilon}^u, g^u(t+\varepsilon, X_{t+\varepsilon}^u))] \\
& + G(x, E_{t,x}[g^u(t+\varepsilon, X_{t+\varepsilon}^u)]),
\end{aligned}
$$

where, for ease of notation, we have suppressed the lower index ε of u_ε. By the construction of u we have

$$
\begin{aligned}
J(t+\varepsilon, X_{t+\varepsilon}^u, u) &= W(t+\varepsilon, X_{t+\varepsilon}^u), \\
f^u(t+\varepsilon, X_{t+\varepsilon}^u, x) &= f(t+\varepsilon, X_{t+\varepsilon}^u, x), \\
g^u(t+\varepsilon, X_{t+\varepsilon}^u) &= g(t+\varepsilon, X_{t+\varepsilon}^u),
\end{aligned}
$$

so we obtain

$$
\begin{aligned}
J_n(x,u) = {} & E_{t,x}[W(t+\varepsilon, X_{t+\varepsilon}^u)] - E_{t,x}[f(t+\varepsilon, X_{t+\varepsilon}^u, X_{t+\varepsilon}^u)] \\
& + E_{t,x}[f(t+\varepsilon, X_{t+\varepsilon}^u, x)] - E_{t,x}[G(X_{t+\varepsilon}^u, g(t+\varepsilon, X_{t+\varepsilon}^u))] \\
& + G(x, E_{t,x}[g(t+\varepsilon, X_{t+\varepsilon}^u)]).
\end{aligned}
$$

Furthermore, from the W-equation we have

$$
\mathcal{A}^u W(t,x) - \mathcal{A}^u f(t,x,x) + \mathcal{A}^u f^x(t,x) - \mathcal{A}^u (G \diamond g)(t,x) + \mathcal{H}^u g(t,x) \le 0,
$$

for all $u \in \mathcal{U}$. Discretizing this gives us

$$
\begin{aligned}
& E_{t,x}[W(t+\varepsilon, X_{t+\varepsilon}^u)] - W(t,x) - E_{t,x}[f(t, X_{t+\varepsilon}^u, X_{t+\varepsilon}^u)] + f(t,x,x) \\
& \quad + E_{t,x}[f(t, X_{t+\varepsilon}^u, x)] - f(t,x,x) \\
& \quad - E_{t,x}[G(t+\varepsilon, g(t+\varepsilon, X_{t+\varepsilon}^u))] + G(x, g(t,x)) \\
& \quad + G(x, E_{t,x}[g(t+\varepsilon, X_{t+\varepsilon}^u)]) - G(x, g(t,x) \le o(\varepsilon),
\end{aligned}
$$

or, after simplification,

$$
\begin{aligned}
W(t,x) \ge {} & E_{t,x}[W(t+\varepsilon, X_{t+\varepsilon}^u)] - E_{t,x}[f(t, X_{t+\varepsilon}^u, X_{t+\varepsilon}^u)] \\
& + E_{t,x}[f(t, X_{t+\varepsilon}^u, x)] - E_{t,x}[G(t+\varepsilon, g(t+\varepsilon, X_{t+\varepsilon}^u))] \\
& + G(x, E_{t,x}[g(t+\varepsilon, X_{t+\varepsilon}^u)]) + o(\varepsilon).
\end{aligned} \tag{15}
$$

Using the expression for $J_n(t,x)$ above, and considering the fact that $W(t,x) = J(t,x,u^*)$, we obtain

$$J(t,x,u^*) - J(t,x,u) \geq 0,$$

so

$$\liminf_{\varepsilon\to 0} \frac{J(t,x,u^*) - J(t,x,u)}{\varepsilon} \geq 0.$$

and we are done. □

Before finishing this section, as a special case, we give HJB system and verification theorem for problem (6).

Theorem 4.2: *Let $W(t,x)$, $g(t,x)$ and $h(t,x)$ defined on $[0,T]\times R$, be continuously differentiable in t and twice continuously differentiable in x, that is, $W(t,x)$, $g(t,x)$, $h(t,x) \in C^{1,2}([0,T]\times R)$. If $W(t,x)$, $g(t,x)$ and $h(t,x)$ satisfy the following HJB equations*

$$\begin{aligned}
\sup_{u\in\mathcal{U}} \Bigg\{ & W_t + [c-(1-a(t)c_1)+rx+(\mu-r)\pi(t)]W_x + \frac{1}{2}\sigma^2\pi^2(t)[W_{xx}-\gamma g_x^2] \\
& + \lambda_1(1+\gamma g)E[g(t,x-a(t)Y)-g(t,x)] - \frac{1}{2}\lambda_1\gamma E[h(t,x-a(t)Y)-h(t,x)] \\
& + \lambda_2(1+\gamma g)E[g(t,x+\pi(t)Z)-g(t,x)] - \frac{1}{2}\lambda_2\gamma E[h(t,x+\pi(t)Z)-h(t,x)] \\
& + \lambda(1+\gamma g)E[g(t,x-a(t)Y+\pi(t)Z)-g(t,x)] \\
& -\frac{1}{2}\lambda\gamma E[h(t,x-a(t)Y+\pi(t)Z)-h(t,x)]\Bigg\} = 0,
\end{aligned} \tag{16}$$

$W(T,x) = x$,

$$\begin{aligned}
& g_t + [c-(1-a^*(t)c_1)+rx+(\mu-r)\pi^*(t)]g_x + \frac{1}{2}\sigma^2\pi^{*2}(t)g_{xx} \\
& \quad + \lambda_1 E[g(t,x-a^*(t)Y)-g(t,x)] + \lambda_2 E[g(t,x+\pi^*(t)Z)-g(t,x)] \\
& \quad + \lambda E[g(t,x-a^*(t)Y+\pi^*(t)Z)-g(t,x)] = 0,
\end{aligned} \tag{17}$$

$g(T,x) = x$,

$$\begin{aligned}
& h_t + [c-(1-a^*(t)c_1)+rx+(\mu-r)\pi^*(t)]h_x + \frac{1}{2}\sigma^2\pi^{*2}(t)h_{xx} \\
& \quad + \lambda_1 E[h(t,x-a^*(t)Y)-h(t,x)] + \lambda_2 E[h(t,x+\pi^*(t)Z)-h(t,x)] \\
& \quad + \lambda E[h(t,x-a^*(t)Y+\pi^*(t)Z)-h(t,x)] = 0,
\end{aligned} \tag{18}$$

$h(T,x) = x^2$, and

$$\begin{aligned}
u^*(t) &= (a^*(t),\pi^*(t)) \\
&= arg \sup_{u\in\mathcal{U}} \Bigg\{ W_t + [c-(1-a(t)c_1)+rx+(\mu-r)\pi(t)]W_x + \frac{1}{2}\sigma^2\pi^2(t)[W_{xx}-\gamma g_x^2] \\
& + \lambda_1(1+\gamma g)E[g(t,x-a(t)Y)-g(t,x)] - \frac{1}{2}\lambda_1\gamma E[h(t,x-a(t)Y)-h(t,x)] \\
& + \lambda_2(1+\gamma g)E[g(t,x+\pi(t)Z)-g(t,x)] - \frac{1}{2}\lambda_2\gamma E[h(t,x+\pi(t)Z)-h(t,x)] \\
& + \lambda(1+\gamma g)E[g(t,x-a(t)Y+\pi(t)Z)-g(t,x)] \\
& -\frac{1}{2}\lambda\gamma E[h(t,x-a(t)Y+\pi(t)Z)-h(t,x)]\Bigg\}.
\end{aligned} \tag{19}$$

Then

$$V(t,x) = W(t,x),\ g(t,x) = E_{t,x}(X_T^{u^*}),\ h(t,x) = E_{t,x}(X_T^{u^*})^2, \tag{20}$$

and $u^*(t) = (a^*(t), \pi^*(t))$ is the optimal time-consistent reinsurance-investment strategy.

Proof: For problem (6), it is not difficult to yield

$$\begin{aligned} -\mathcal{A}^u f(t,x,x) + \mathcal{A}^u f^x(t,x) &= 0, \\ \mathcal{A}^u (G \diamond g)(t,x) &= \frac{\gamma}{2} g^2(t,x), \\ \mathcal{H}^u g(t,x) = G_g(x, g(t,x))\mathcal{A}^u g(t,x) &= \gamma g(t,x)\mathcal{A}^u g(t,x). \end{aligned} \tag{21}$$

So the first equation of (10) becomes

$$\sup_{u\in\mathcal{U}} \left\{ \mathcal{A}^u W(t,x) - \mathcal{A}^u \frac{\gamma}{2} g^2(t,x) + \gamma g(t,x)\mathcal{A}^u g(t,x) \right\} = 0. \tag{22}$$

In the following, we give more explicitly form of (22). First, we have the following form,

$$\begin{aligned} & -\mathcal{A}^u \frac{\gamma}{2} g^2(t,x) + \gamma g(t,x)\mathcal{A}^u g(t,x) \\ &= -\frac{1}{2}\sigma^2\pi^2(t)\gamma g_x^2 + \lambda_1 E\left[\gamma g(t,x) g(t,x-a(t)Y) - \frac{\gamma}{2} g^2(t,x-a(t)Y) - \frac{\gamma}{2} g^2(t,x)\right] \\ &\quad + \lambda_2 E\left[\gamma g(t,x) g(t,x+\pi(t)Z) - \frac{\gamma}{2} g^2(t,x+\pi(t)Z) - \frac{\gamma}{2} g^2(t,x)\right] \\ &\quad + \lambda E\left[\gamma g(t,x) g(t,x-a(t)Y+\pi(t)Z) - \frac{\gamma}{2} g^2(t,x-a(t)Y+\pi(t)Z) - \frac{\gamma}{2} g^2(t,x)\right]. \end{aligned} \tag{23}$$

Plugging (23) into (22) and notice that $W(t,x) = g(t,x) - \frac{\gamma}{2}[h(t,x) - g^2(t,x)]$, we obtain after simplification

$$\begin{aligned} \sup_{u\in\mathcal{U}} \Bigg\{ & W_t + [c - (1-a(t)c_1) + rx + (\mu - r)\pi(t)]W_x + \frac{1}{2}\sigma^2\pi^2(t)[W_{xx} - \gamma g_x^2] \\ & + \lambda_1 E[(1+\gamma g) g(t,x-a(t)Y) - (1+\gamma g(t,x)) g(t,x) - \frac{1}{2}\gamma h(t,x-a(t)Y) \\ & + \frac{1}{2}\gamma h(t,x)] + \lambda_2 E[(1+\gamma g) g(t,x+\pi(t)Z) - (1+\gamma g(t,x)) g(t,x) \\ & - \frac{1}{2}\gamma h(t,x+\pi(t)Z) + \frac{1}{2}\gamma h(t,x)] + \lambda E[(1+\gamma g) g(t,x-a(t)Y+\pi(t)Z) \\ & -(1+\gamma g(t,x)) g(t,x) - \frac{1}{2}\gamma h(t,x-a(t)Y+\pi(t)Z) + \frac{1}{2}\gamma h(t,x)] \Bigg\} = 0. \end{aligned} \tag{24}$$

Then, we obtain (16). Equation (17) is more explicitly form of $\mathcal{A}^{u^*} g(t,x) = 0$. The new notation $h(t,x)$ is just for $W(t,x) = g(t,x) - \frac{\gamma}{2}[h(t,x) - g^2(t,x)]$ holds. From (11) and Theorem 4.1, we have $V(t,x) = W(t,x)$ and $g(t,x) = E_{t,x}(X_T^{u^*})$. Therefore must have $h(t,x) = E_{t,x}(X_T^{u^*})^2$ and $\mathcal{A}^{u^*} h(t,x) = 0$ holds, and consequently we complete the proof. □

5. The solution to time-consistent mean-variance problem

In this section, we study the optimal time-consistent mean-variance reinsurance-investment strategy selection problem for an insurer with common shock dependence and obtain the optimal time-consistent reinsurance-investment strategy and the optimal value function. That is, we will give the solution to Theorem 4.2, which is the main result of the paper.

First, we define the following notation:

$$\hat{a}(t) = \frac{[c_1 - \mu_{11}(\lambda_1+\lambda)][(\lambda_2+\lambda)\mu_{22}+\sigma^2] + [\mu - r + \mu_{21}(\lambda_2+\lambda)]\lambda\mu_{11}\mu_{21}}{\mu_{12}(\lambda_1+\lambda)[(\lambda_2+\lambda)\mu_{22}+\sigma^2] - \lambda^2\mu_{11}^2\mu_{21}^2} \times \frac{A(t)}{\gamma m^2(t)}, \tag{25}$$

$$\hat{\pi}(t) = \frac{[\mu - r + (\lambda_2+\lambda)\mu_{21}]\mu_{12}(\lambda_1+\lambda) + [c_1 - \mu_{11}(\lambda_1+\lambda)]\lambda\mu_{11}\mu_{21}}{\mu_{12}(\lambda_1+\lambda)[(\lambda_2+\lambda)\mu_{22}+\sigma^2] - \lambda^2\mu_{11}^2\mu_{21}^2} \times \frac{A(t)}{\gamma m^2(t)}, \tag{26}$$

where $A(t)$ and $m(t)$ will given in Theorem 5.2,

$$l = \mu_{12}(\lambda_1+\lambda)[(\lambda_2+\lambda)\mu_{22}+\sigma^2] - \lambda^2\mu_{11}^2\mu_{21}^2, \tag{27}$$

$$l_1 = [c_1 - \mu_{11}(\lambda_1+\lambda)][(\lambda_2+\lambda)\mu_{22}+\sigma^2] + [\mu - r + \mu_{21}(\lambda_2+\lambda)]\lambda\mu_{11}\mu_{21}, \tag{28}$$

$$l_2 = [\mu - r + (\lambda_2+\lambda)\mu_{21}]\mu_{12}(\lambda_1+\lambda) + [c_1 - \mu_{11}(\lambda_1+\lambda)]\lambda\mu_{11}\mu_{21}, \tag{29}$$

$$l_3 = -\frac{l_1^2\mu_{12}(\lambda_1+\lambda)}{2\gamma l^2} + \frac{[c_1 - \mu_{11}(\lambda_1+\lambda)]l_1}{\gamma l} - \frac{l_2^2[\mu_{22}(\lambda_2+\lambda)+\sigma^2]}{2\gamma l^2} + \frac{[\mu - r + \mu_{21}(\lambda_2+\lambda)]l_2}{\gamma l} + \frac{l_1 l_2 \lambda\mu_{11}\mu_{21}}{\gamma l^2}, \tag{30}$$

$$l_4 = \frac{c_1 l_1 - (\lambda_1+\lambda)\mu_{11}l_1 + (\lambda_2+\lambda)\mu_{21}l_2 + (\mu - r)l_2}{\gamma l}. \tag{31}$$

Lemma 5.1: *Assume $\mathcal{B}(a(t),\pi(t))$ satisfy the following formula,*

$$\begin{aligned}\mathcal{B}(a(t),\pi(t)) = &-0.5\gamma(\lambda_1+\lambda)\mu_{12}m^2(t)a^2(t) + [c_1 - (\lambda_1+\lambda)\mu_{11}]A(t)a(t)\\ &- 0.5\gamma[(\lambda_2+\lambda)\mu_{22}+\sigma^2]m^2(t)\pi^2(t) + [\mu - r\\ &+ (\lambda_2+\lambda)\mu_{21}]A(t)\pi(t) + \lambda\gamma\mu_{11}\mu_{21}m^2(t)a(t)\pi(t).\end{aligned} \tag{32}$$

Then $(\hat{a}(t),\hat{\pi}(t))$ is the unique maximum point of $\mathcal{B}(a(t),\pi(t))$.

Proof: Let

$$\begin{aligned}f_1(a(t),\pi(t)) &= \frac{\partial^2\mathcal{B}(a(t),\pi(t))}{\partial a(t)^2} = -\gamma(\lambda_1+\lambda)\mu_{12}m^2(t),\\ f_2(a(t),\pi(t)) &= \frac{\partial^2\mathcal{B}(a(t),\pi(t))}{\partial a(t)\partial\pi(t)} = \lambda\gamma\mu_{11}\mu_{21}m^2(t),\\ f_3(a(t),\pi(t)) &= \frac{\partial^2\mathcal{B}(a(t),\pi(t))}{\partial\pi(t)^2} = -\gamma[(\lambda_2+\lambda)\mu_{22}+\sigma^2]m^2(t).\end{aligned}$$

Substituting $(\hat{a}(t),\hat{\pi}(t))$ into $f_1(a(t),\pi(t))$, $f_2(a(t),\pi(t))$ and $f_3(a(t),\pi(t))$, respectively, we obtain

$$\begin{aligned}A &= f_1(\hat{a}(t),\hat{\pi}(t)) = -\gamma(\lambda_1+\lambda)\mu_{12}m^2(t) < 0,\\ B &= f_2(\hat{a}(t),\hat{\pi}(t)) = \lambda\gamma\mu_{11}\mu_{21}m^2(t) > 0,\\ C &= f_3(\hat{a}(t),\hat{\pi}(t)) = -\gamma[(\lambda_2+\lambda)\mu_{22}+\sigma^2]m^2(t) < 0.\end{aligned}$$

So $AC - B^2 = \left\{\mu_{12}(\lambda_1+\lambda)[(\lambda_2+\lambda)\mu_{22}+\sigma^2] - \lambda^2\mu_{11}^2\mu_{21}^2\right\}\gamma^2 m^4(t)$.
By the Cauchy–Schwarz inequality, it is not difficult to see that

$$\mu_{12}(\lambda_1+\lambda)[(\lambda_2+\lambda)\mu_{22}+\sigma^2] - \lambda^2\mu_{11}^2\mu_{21}^2 > 0.$$

Therefore $AC-B^2 > 0$, so $(\hat{a}(t), \hat{\pi}(t))$ is the unique maximum point of $\mathcal{B}(a(t), \pi(t))$. We complete the proof. □

Theorem 5.2: *For the wealth process* (3), *the optimal time-consistent reinsurance strategy is given by*

$$a^*(t) = \frac{[c_1 - \mu_{11}(\lambda_1+\lambda)][(\lambda_2+\lambda)\mu_{22}+\sigma^2] + [\mu - r + \mu_{21}(\lambda_2+\lambda)]\lambda\mu_{11}\mu_{21}}{\mu_{12}(\lambda_1+\lambda)[(\lambda_2+\lambda)\mu_{22}+\sigma^2] - \lambda^2\mu_{11}^2\mu_{21}^2}\frac{1}{\gamma e^{r(T-t)}}, \quad (33)$$

and the optimal time-consistent investment strategy is given by

$$\pi^*(t) = \frac{[\mu - r + (\lambda_2+\lambda)\mu_{21}]\mu_{12}(\lambda_1+\lambda) + [c_1 - \mu_{11}(\lambda_1+\lambda)]\lambda\mu_{11}\mu_{21}}{\mu_{12}(\lambda_1+\lambda)[(\lambda_2+\lambda)\mu_{22}+\sigma^2] - \lambda^2\mu_{11}^2\mu_{21}^2}\frac{1}{\gamma e^{r(T-t)}}. \quad (34)$$

The optimal value function is given by

$$W(t,x) = xe^{r(T-t)} + \frac{B(t)}{\gamma}, \quad (35)$$

and the variance of the terminal surplus of the insurer under the optimal strategy is given by

$$Var_{t,x}(X_T^{u^*}) = \frac{2[n(t) - B(t)]}{\gamma^2}, \quad (36)$$

here

$$B(t) = \frac{\gamma(c - c_1)[e^{r(T-t)} - 1]}{r} + (T-t)l_3, \quad (37)$$

and

$$n(t) = \frac{\gamma(c - c_1)[e^{r(T-t)} - 1]}{r} + (T-t)l_4, \quad (38)$$

where l_3 and l_4 are given by (30) *and* (31), *respectively.*

***Proof*:** The proof consists of two steps:

- First, we prove the optimal value function and the variance of the terminal surplus of the insurer. Similar to Zeng et al. [15], we can construct the structure of $W(t,x)$ and $g(t,x)$ satisfy the following (40). Then substitute $W(t,x)$ and $g(t,x)$ into (16) and (17) meanwhile binding Lemma 5.1, we can obtain $W(t,x)$. The variance of the terminal surplus of the insurer is easily to obtain.
- Second, we prove the optimal time-consistent reinsurance and investment strategies. By proving the optimal time-consistent reinsurance strategy $a^*(t) > 0$, and the optimal time-consistent investment strategy $\pi^*(t) > 0$, we complete the proof.

Step 1 We prove the optimal value function and the variance of the terminal surplus of the insurer. From the proof course of Theorem 4.2, we have

$$h(t,x) = g^2(t,x) + \frac{2}{\gamma}[g(t,x) - W(t,x)]. \tag{39}$$

Given the structure of the surplus dynamics, as well as the boundary conditions $W(T,x) = x$ and $g(T,x) = x$, similar to Zeng et al. [15], it is natural to assume

$$\begin{cases} W(t,x) = A(t)x + \frac{B(t)}{\gamma}, \\ g(t,x) = m(t)x + \frac{n(t)}{\gamma}, \end{cases} \tag{40}$$

with boundary conditions

$$\begin{cases} A(T) = 1, \quad B(T) = 0, \\ m(T) = 1, \quad n(T) = 0. \end{cases}$$

Hence the partial derivatives of $W(t,x)$ and $g(t,x)$ are

$$\begin{cases} W_t(t,x) = A'(t)x + \frac{B'(t)}{\gamma}, \ W_x(t,x) = A(t), \ W_{xx}(t,x) = 0, \\ g_t(t,x) = m'(t)x + \frac{n'(t)}{\gamma}, \ g_x(t,x) = m(t), \ g_{xx}(t,x) = 0. \end{cases} \tag{41}$$

Plugging the above (39)–(41) into (16), we obtain after simplification

$$A'(t) + \frac{B(t)}{\gamma} + (c - c_1 + rx)A(t) + \sup_{u\in\mathcal{U}}\{\mathcal{B}(a(t),\pi(t))\} = 0, \tag{42}$$

where $\mathcal{B}(a(t),\pi(t))$ is given by (32).

Setting $\frac{\partial\mathcal{B}(a(t),\pi(t))}{\partial a(t)} = 0$, we obtain

$$-\gamma(\lambda_1+\lambda)\mu_{12}m^2(t)a(t) + [c_1 - (\lambda_1+\lambda)\mu_{11}]A(t) + \lambda\gamma\mu_{11}\mu_{21}m^2(t)\pi(t) = 0. \tag{43}$$

Setting $\frac{\partial\mathcal{B}(a(t),\pi(t))}{\partial \pi(t)} = 0$, we have

$$-\gamma[(\lambda_2+\lambda)\mu_{22}+\sigma^2]m^2(t)\pi(t) + [\mu - r + (\lambda_2+\lambda)\mu_{21}]A(t) + \lambda\gamma\mu_{11}\mu_{21}m^2(t)a(t) = 0. \tag{44}$$

From (43) and (44) we can obtain $\hat{a}(t)$ and $\hat{\pi}(t)$, which given by (25) and (26), respectively. From Lemma 5.1, we know $(\hat{a}(t),\hat{\pi}(t))$ is the unique maximum point of $\mathcal{B}(a(t),\pi(t))$. Therefore substituting $(\hat{a}(t),\hat{\pi}(t))$ into (42), we obtain

$$[A'(t) + rA(t)]x + \frac{B(t)}{\gamma} + (c - c_1)A(t) + \frac{l_3A^2(t)}{m^2(t)} = 0, \tag{45}$$

here l_3 satisfy (30).

By separating the variables with and without x from (45), we can derive the following system of ODEs

$$A'(t) + rA(t) = 0, \quad A(T) = 1, \tag{46}$$

and

$$\frac{B(t)}{\gamma} + (c - c_1)A(t) + \frac{l_3A^2(t)}{m^2(t)} = 0, \quad B(T) = 0. \tag{47}$$

By substituting $(\hat{a}(t), \hat{\pi}(t))$ into (17), we have after simplification

$$[m'(t) + rm(t)]x + \frac{n(t)}{\gamma} + (c - c_1)m(t) + \frac{l_4 A^2(t)}{m^2(t)} = 0, \tag{48}$$

here l_4 satisfy (31).

By separating the variables with and without x from (48), we can derive the following system of ODEs

$$m'(t) + rm(t) = 0, \quad m(T) = 1, \tag{49}$$

and

$$\frac{n(t)}{\gamma} + (c - c_1)n(t) + \frac{l_4 A^2(t)}{m^2(t)} = 0, \quad n(T) = 0. \tag{50}$$

First solving Equations (46) and (49), respectively, we obtain

$$A(t) = m(t) = e^{r(T-t)}. \tag{51}$$

By substituting (51) into (47), then the solution to (47) is $B(t)$, and $B(t)$ satisfies (37). By substituting (51) into (50), then the solution to (50) is $n(t)$, and $n(t)$ satisfies (38). Therefore

$$W(t,x) = A(t)x + \frac{B(t)}{\gamma} = xe^{r(T-t)} + \frac{B(t)}{\gamma}.$$

From (5) we have

$$\begin{aligned} Var_{t,x}(X_T^{u^*}) &= \frac{2}{\gamma}[\tilde{J}(t,x,u^*) - E_{t,x}(X_T^{u^*})] \\ &= \frac{2}{\gamma}[W(t,x) - g(t,x)] \\ &= \frac{2[B(t) - n(t)]}{\gamma^2}. \end{aligned}$$

Step 2 Now, we prove the optimal time-consistent reinsurance and investment strategies. From (26) and (51), we obtain

$$\pi^*(t) = \frac{[\mu - r + (\lambda_2 + \lambda)\mu_{21}]\mu_{12}(\lambda_1 + \lambda) + [c_1 - \mu_{11}(\lambda_1 + \lambda)]\lambda\mu_{11}\mu_{21}}{\mu_{12}(\lambda_1 + \lambda)[(\lambda_2 + \lambda)\mu_{22} + \sigma^2] - \lambda^2\mu_{11}^2\mu_{21}^2} \frac{1}{\gamma e^{r(T-t)}}.$$

By the Cauchy–Schwarz inequality, it is not difficult to see that

$$\mu_{12}(\lambda_1 + \lambda)[(\lambda_2 + \lambda)\mu_{22} + \sigma^2] - \lambda^2\mu_{11}^2\mu_{21}^2 > 0.$$

Because

$$[\mu - r + (\lambda_2 + \lambda)\mu_{21}]\mu_{12}(\lambda_1 + \lambda) > 0,$$

and

$$[c_1 - \mu_{11}(\lambda_1 + \lambda)]\lambda\mu_{11}\mu_{21} > 0,$$

therefore $\pi^*(t) > 0$, so $\pi^*(t)$ is optimal time-consistent investment strategy.

Finally, we prove the optimal time-consistent reinsurance. From (25) and (51), we can obtain

$$a^*(t) = \frac{[c_1 - \mu_{11}(\lambda_1+\lambda)][(\lambda_2+\lambda)\mu_{22}+\sigma^2] + [\mu - r + \mu_{21}(\lambda_2+\lambda)]\lambda\mu_{11}\mu_{21}}{\mu_{12}(\lambda_1+\lambda)[(\lambda_2+\lambda)\mu_{22}+\sigma^2] - \lambda^2\mu_{11}^2\mu_{21}^2}\frac{1}{\gamma e^{r(T-t)}}.$$

Let

$$\begin{aligned} f(\mu_{21}) &= [c_1 - \mu_{11}(\lambda_1+\lambda)][(\lambda_2+\lambda)\mu_{22}+\sigma^2] + [\mu - r + \mu_{21}(\lambda_2+\lambda)]\lambda\mu_{11}\mu_{21} \\ &= (\lambda_2+\lambda)\lambda\mu_{11}(\mu_{21})^2 + (\mu - r)\lambda\mu_{11}\mu_{21} + [c_1 - \mu_{11}(\lambda_1+\lambda)] \\ &\quad \times [(\lambda_2+\lambda)\mu_{22}+\sigma^2]. \end{aligned} \tag{52}$$

Because $(\lambda_2+\lambda)\lambda\mu_{11} > 0$, $(\mu - r)\lambda\mu_{11} > 0$ and $[c_1 - \mu_{11}(\lambda_1+\lambda)][(\lambda_2+\lambda)\mu_{22}+\sigma^2] > 0$, so $f(\mu_{21})$ is always greater than zero or $f(\mu_{21}) = 0$ only has positive root. In the case that $f(\mu_{21}) = 0$ only has positive root, we substitute μ_{21} into $f(\mu_{21})$ and then obtain $f(\mu_{21}) > 0$. Therefore $f(\mu_{21})$ is always greater than zero. And because $\mu_{12}(\lambda_1+\lambda)[(\lambda_2+\lambda)\mu_{22}+\sigma^2] - \lambda^2\mu_{11}^2\mu_{21}^2 > 0$, $\gamma > 0$ and $r > 0$, so $a^*(t) > 0$, i.e. $a^*(t)$ is optimal time-consistent reinsurance strategy. This is the end of proof. □

6. Sensitivity and economic significance analysis

In this section, we will illustrate how the optimal time-consistent reinsurance-investment strategy derived in Section 5 changes when some model parameters vary. In particular, we focus on how the optimal time-consistent proportional reinsurance strategy and optimal time-consistent investment strategy changes with some model parameters, such as the rate of reinsurance premium, the risk aversion parameter, the intensity of claim, the risk-free interest rate, the expected rate of return and the volatility of the risky stock. We also study relationship between the optimal time-consistent proportional reinsurance strategy and the optimal time-consistent investment strategy.

Differentiating (33) with respect to γ, we obtain

$$\begin{aligned} \frac{\partial a^*(t)}{\partial \gamma} &= -\frac{[c_1 - \mu_{11}(\lambda_1+\lambda)][(\lambda_2+\lambda)\mu_{22}+\sigma^2] + [\mu - r + \mu_{21}(\lambda_2+\lambda)]\lambda\mu_{11}\mu_{21}}{\mu_{12}(\lambda_1+\lambda)[(\lambda_2+\lambda)\mu_{22}+\sigma^2] - \lambda^2\mu_{11}^2\mu_{21}^2} \\ &\quad \times \frac{1}{\gamma^2 e^{r(T-t)}} < 0. \end{aligned} \tag{53}$$

Remark 2: We see that $a^*(t)$ is a decreasing function of γ. This is intuitive, as γ is the risk aversion parameter of the insurer. The larger γ is, the less aggressive the insurer will be, and hence the less retention level the insurer will hold.

According to (33), we can obtain

$$\frac{\partial a^*(t)}{\partial c_1} = \frac{(\lambda_2+\lambda)\mu_{22}+\sigma^2}{\mu_{12}(\lambda_1+\lambda)[(\lambda_2+\lambda)\mu_{22}+\sigma^2] - \lambda^2\mu_{11}^2\mu_{21}^2}\frac{1}{\gamma e^{r(T-t)}} > 0. \tag{54}$$

Remark 3: The optimal time-consistent reinsurance strategy is increasing in c_1. As c_1 is the premium of reinsurance per unit time, the larger the value of c_1 is, the larger the value of the premium of reinsurance will be, therefore the insurer should retain a great share of each claim.

From (33), we have

$$\frac{\partial a^*(t)}{\partial t} = \frac{[c_1 - \mu_{11}(\lambda_1+\lambda)][(\lambda_2+\lambda)\mu_{22}+\sigma^2] + [\mu - r + \mu_{21}(\lambda_2+\lambda)]\lambda\mu_{11}\mu_{21}}{\mu_{12}(\lambda_1+\lambda)[(\lambda_2+\lambda)\mu_{22}+\sigma^2] - \lambda^2\mu_{11}^2\mu_{21}^2} \times \frac{r}{\gamma^2 e^{r(T-t)}} > 0. \tag{55}$$

Remark 4: The optimal time-consistent reinsurance strategy is increasing in t. This simply states that as the maturity time is approached, the insurer will keep more insurance business.

Differentiating (34) with respect to γ, we obtain

$$\frac{\partial \pi^*(t)}{\partial \gamma} = -\frac{[\mu - r + (\lambda_2+\lambda)\mu_{21}]\mu_{12}(\lambda_1+\lambda) + [c_1 - \mu_{11}(\lambda_1+\lambda)]\lambda\mu_{11}\mu_{21}}{\mu_{12}(\lambda_1+\lambda)[(\lambda_2+\lambda)\mu_{22}+\sigma^2] - \lambda^2\mu_{11}^2\mu_{21}^2} \times \frac{1}{\gamma^2 e^{r(T-t)}} < 0. \tag{56}$$

Remark 5: We see that $\pi^*(t)$ is a decreasing function of γ. This is intuitive, as γ is the risk aversion parameter of the insurer. The larger γ is, the less aggressive the insurer will be, and hence the insurer will reduce investments in risky assets.

According to (34), we can obtain

$$\frac{\partial \pi^*(t)}{\partial c_1} = \frac{\lambda\mu_{11}\mu_{21}}{\mu_{12}(\lambda_1+\lambda)[(\lambda_2+\lambda)\mu_{22}+\sigma^2] - \lambda^2\mu_{11}^2\mu_{21}^2} \frac{1}{\gamma e^{r(T-t)}} > 0. \tag{57}$$

Remark 6: The optimal time-consistent investment strategy is increasing in c_1. As c_1 is the premium of reinsurance per unit time, the larger the value of c_1 is, the larger the value of the premium of reinsurance will be, therefore the insurer should increase the investment on risky assets.

From (34), we have Differentiating (34) with respect to t, we obtain

$$\frac{\partial \pi^*(t)}{\partial t} = \frac{[\mu - r + (\lambda_2+\lambda)\mu_{21}]\mu_{12}(\lambda_1+\lambda) + [c_1 - \mu_{11}(\lambda_1+\lambda)]\lambda\mu_{11}\mu_{21}}{\mu_{12}(\lambda_1+\lambda)[(\lambda_2+\lambda)\mu_{22}+\sigma^2] - \lambda^2\mu_{11}^2\mu_{21}^2} \times \frac{r}{\gamma e^{r(T-t)}} > 0. \tag{58}$$

Remark 7: The optimal time-consistent investment strategy is increasing in t. This simply states that as the maturity time approaches, the insurer will keep more investment on risky assets.

In the following, we will by numerical computation illustrate how the optimal reinsurance-investment strategy changes when other model parameters vary.

Example 1: Let $c_1 = 0.06$, $\mu_{11} = 0.01$, $\mu_{12} = 0.002$, $\mu_{21} = 0.006$, $\mu_{22} = 0.001$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu = 0.05$, $r = 0.02$, $\sigma = 0.04$, $\gamma = 0.2$, $\lambda \in [1,5]$, we calculate the values of the optimal time-consistent reinsurance and investment strategies $a^*(t)$ and $\pi^*(t)$ by (33) and (34), respectively, and present the result in Figure 1.

Example 2: Let $c_1 = 0.07$, $\mu_{11} = 0.01$, $\mu_{12} = 0.002$, $\mu_{21} = 0.006$, $\mu_{22} = 0.001$, $\lambda = 1$, $\lambda_2 = 2$, $\mu = 0.05$, $r = 0.02$, $\sigma = 0.04$, $\gamma = 0.2$, $\lambda_1 \in [2,6]$, we calculate the values of the optimal time-consistent reinsurance and investment strategies $a^*(t)$ and $\pi^*(t)$ by (33) and (34), respectively, and present the result in Figure 2.

Example 3: Let $c_1 = 0.03$, $\mu_{11} = 0.01$, $\mu_{12} = 0.002$, $\mu_{21} = 0.006$, $\mu_{22} = 0.001$, $\lambda = 1$, $\lambda_1 = 2$, $\mu = 0.05$, $r = 0.02$, $\sigma = 0.05$, $\gamma = 0.2$, $\lambda_2 \in [1,4]$, we calculate the values of the optimal time-consistent reinsurance and investment strategies $a^*(t)$ and $\pi^*(t)$ by (33) and (34), respectively, and present the result in Figure 3.

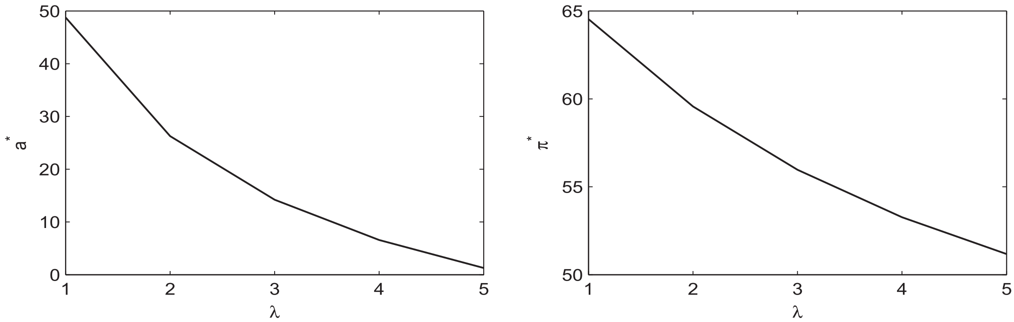

Figure 1. Effects of λ on the optimal time-consistent reinsurance-investment strategy.

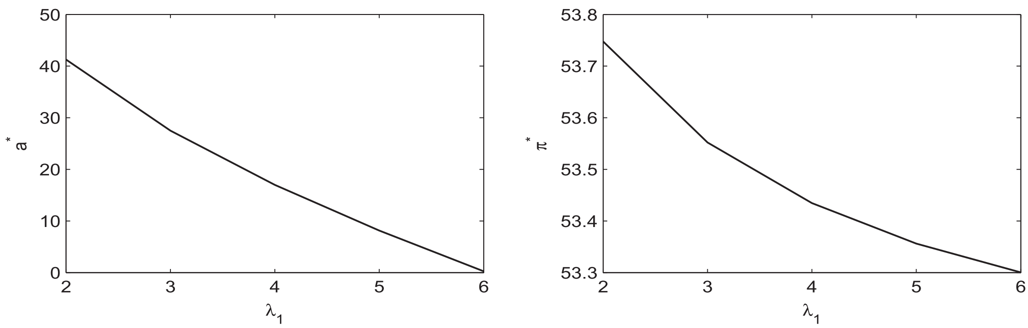

Figure 2. Effects of λ_1 on the optimal time-consistent reinsurance-investment strategy.

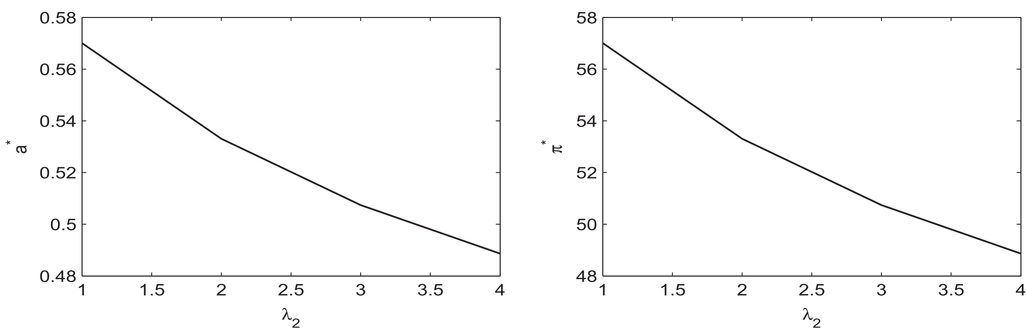

Figure 3. Effects of λ_2 on the optimal time-consistent reinsurance-investment strategy.

From Figures 1–3, we see that $a^*(t)$ and $\pi^*(t)$ are both decreasing functions of λ, λ_1 and λ_2, which reveals that when the claim number or the jump intensity of the risky asset increases, the insurer will reduce the retention (i.e. increase the proportion of reinsurance) and the investment on risky assets. Claim increase suggests that increase in the risk for insurance business and the jump intensity of the risky asset would increase the investment risk. No matter what kind of situation is, the risk can be reduced by increasing the proportion of reinsurance transfer risk and reducing the investment amount in risky assets.

Example 4: Let $c_1 = 0.03$, $\mu_{11} = 0.01$, $\mu_{12} = 0.002$, $\mu_{21} = 0.006$, $\mu_{22} = 0.001$, $\lambda = 2$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu = 0.05$, $r = 0.02$, $\gamma = 0.2$, $\sigma \in [0.01, 0.05]$, we calculate the values of the optimal

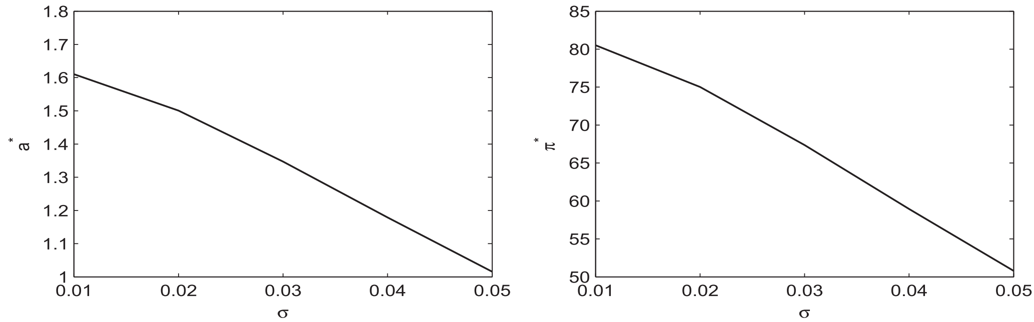

Figure 4. Effects of σ on the optimal time-consistent reinsurance-investment strategy.

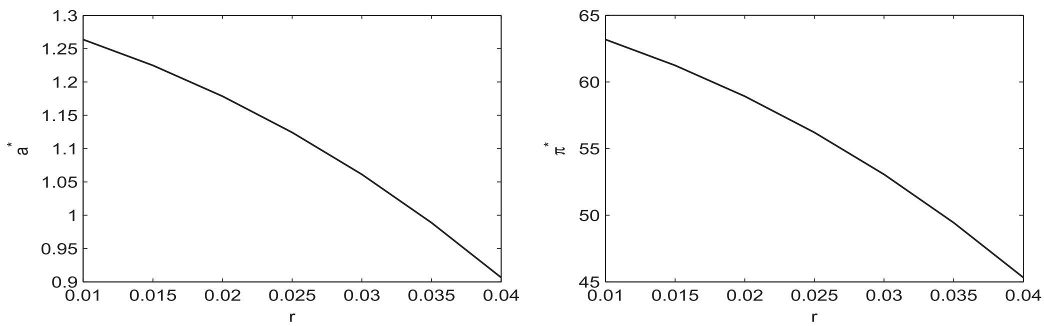

Figure 5. Effects of r on the optimal time-consistent reinsurance-investment strategy.

time-consistent reinsurance and investment strategies $a^*(t)$ and $\pi^*(t)$ by (33) and (34), respectively, and present the result in Figure 4.

From Figure 4, we find that $a^*(t)$ and $\pi^*(t)$ are decreasing in σ, which is the volatility of the risky asset. The larger σ is, the riskier the risky asset will be, and hence the less the insurance company will wish to invest in the risky asset, and the insurance company also will reduce the retention (i.e. increase the proportion of reinsurance).

Example 5: Let $c_1 = 0.03$, $\mu_{11} = 0.01$, $\mu_{12} = 0.002$, $\mu_{21} = 0.006$, $\mu_{22} = 0.001$, $\lambda = 2$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu = 0.05$, $\sigma = 0.04$, $\gamma = 0.2$, $r \in [0.01, 0.04]$, we calculate the values of the optimal time-consistent reinsurance and investment strategies $a^*(t)$ and $\pi^*(t)$ by (33) and (34), respectively, and present the result in Figure 5.

From Figure 5, it can be seen that $a^*(t)$ and $\pi^*(t)$ are decreasing functions of r. As r is the risk-free interest rate, the larger r is, the greater the expected income of the risk-free asset will be, and hence the more the insurance company will wish to invest in the risk-free asset, and the more risk will be transferred to the reinsurance company.

Example 6: Let $c_1 = 0.03$, $\mu_{11} = 0.01$, $\mu_{12} = 0.002$, $\mu_{21} = 0.006$, $\mu_{22} = 0.001$, $\lambda = 2$, $\lambda_1 = 1$, $\lambda_2 = 2$, $r = 0.02$, $\sigma = 0.04$, $\gamma = 0.2$, $\mu \in [0.03, 0.06]$, we calculate the values of the optimal time-consistent reinsurance and investment strategies $a^*(t)$ and $\pi^*(t)$ by (33) and (34), respectively, and present the result in Figure 6.

From Figure 6, it is evident that $a^*(t)$ and $\pi^*(t)$ are increasing functions of μ, which describes the rate of the income of the risky asset. The larger μ is, the greater the expected income of the risky asset will be, and hence the more the insurance company will wish to invest in the risky asset, and the more retention the insurance company will wish to undertake.

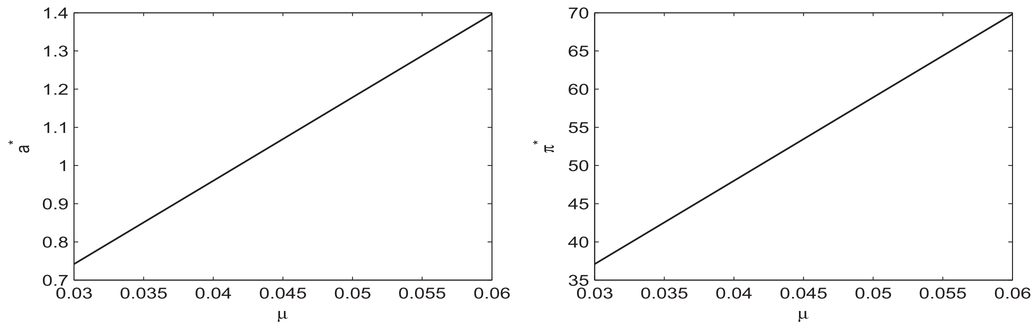

Figure 6. Effects of μ on the optimal time-consistent reinsurance-investment strategy.

Remark 8: The change of optimal time-consistent reinsurance and investment strategies are consistent. This suggests that in practice, the insurer should increase or reduce the amount of investment on risky assets to increase or decrease reinsurance proportion, and the insurer will gain more benefits and face the minimum risks.

7. Conclusions

In this paper, a continuous-time dynamic investment and reinsurance optimization problem for mean-variance insurers has been studied within a game theoretic framework. Compared with Zeng et al. [15] or other papers about time-consistent investment and reinsurance strategies selection, we introduced jumps in risk model and risky asset and meanwhile assume that the aggregate claim and the risky asset price were correlated. Similar to Björk and Murgoci [13], we first provided a verification theorem for the problem (7) which we presented in Section 3; then as a special case of problem (7), we proved the verification theorem for the problem (6) (or problem (5)) which we studied in this paper. Finally, by constructing the structure of value function and applying extended HJB dynamic programming approach, we derived the time-consistent investment and reinsurance strategies and the corresponding value function explicitly. In addition, numerical examples and economic significance analysis have also been provided to illustrate how the optimal reinsurance-investment strategy changes when some model parameters vary.

There are still many problems needed to be investigated in this direction. Firstly, we only consider the case that the risk aversion parameter of the insurer is constant. It may be interesting to consider state dependent risk aversion function, that is, the problem (7) which we presented in Section 3. Secondly, the reinsurance strategy is constrained to be less than or equal to one, that is, we only considered proportional reinsurance strategy. This is a challenging problem because it is difficult to guess the form of the solution of the extended HJB systems and the value function is no longer a twice continuous differential function. Thirdly, the time horizon T is pre-given and fixed. It may be interesting to consider an uncertain time. Although these kinds of problems are challenging, they are meaningful and more realistic to be discussed, also being the directions of our research work in the future.

Acknowledgements

The author would like to thank the anonymous referees for their careful reading and helpful comments on an earlier version of this paper, which led to a considerable improvement of the presentation of the work.

Disclosure statement

No potential conflict of interest was reported by the author.

Funding

This research is supported by the Scientific Research Foundation of Xijing University [grant number XJ160144]; Scientific Research Program Funded by Shaanxi Provincial Education Department [grant number 15JK2183]; National Natural Science Foundation of China [grant number 11271375]; Natural Science Basic Research Plan in Shaanxi Province of China [grant number 2016JM1024], [grant number 2016JM1032].

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