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Full Length Article

Optimal strategies for collective defined contribution plans when the stock and labor markets are co-integrated

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ABSTRACT

This paper investigates a collective defined contribution (CDC) pension fund scheme in continuous time, where members' contributions are fixed in advance, and benefit payments depend on the final salary rate. We take account of the co-integration between labor income and the stock market by letting the difference between logs of labor and dividends follow a mean-reverting process. Further, labor income is also a product of aggregate labor income and member's idiosyncratic shocks whose constant growth rate is unknown. It can be modeled by a continuous-time two-state hidden Markov chain. Further, the pension fund can be invested in the financial market consisting of one risky asset and one risk-free asset to enhance profits. After using Hamilton-Jacobi-Bellman (HJB) equations, the closed-form solutions for the optimal asset allocation and the benefit payment policies are obtained to maximize the social welfare and the terminal surplus wealth. Numerical examples are also conducted to illustrate the sensitivity of parameters on the optimal strategies.

1. Introduction

The pension fund is one of the critical topics in the financial study. It is an essential financial tool to help people obtain income and sustain consumption after retirement. According to a determination of benefits, two traditional types of pension plan design exist: defined benefit (DB) and defined contribution (DC) plans. DB pension plans are popular among employees since the benefits are predetermined, and all risks of providing fixed benefits are borne by employers, which can be analyzed under static and dynamic models. Regarding dynamic models, [17] first present a dynamic model for a DB occupational pension scheme, in which the contribution and solvency risks are introduced. Then, stochastic control theory has been applied in the literature study of DB pension plans. For example, [19] study optimal risk management in a stochastic DB plan, where a geometric Brownian motion model benefits. [20] extend their work to a mean-variance portfolio selection problem and describe the properties of fund investment and optimal contributions when labor income differs across contributors to the fund.

However, with an increasingly mobile worker, the structure of the industrial labor force composition changes. DB plans are not portable across employers. This means the employee who switches jobs may not get the defined pension benefits. Sponsors may also withdraw their guarantees when the plan is less than fully funded. Employees in DB pension plans face higher insolvency risk in this

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case. Due to this, traditional DB pension plans are gradually losing dominance. There has been a shift from DB to DC pension plans over the past few decades.

In DC plans, contributions are fixed, and associated risks fall on the employees themselves. Pension benefits are related to the investment return earned on the plan surplus. In recent decades, extensive research has been devoted to finding optimal investment strategies on stochastic optimization for DC pension schemes. For example, [5] proposes that the employee can choose from a range of investment strategies and discusses investment decision-making to improve the design of DC pension plans. [6] consider optimal investment strategies from a DC plan's entry time to retirement time. [16] assume that the retiree can defer annuitization and study the risk management of a DC pension plan. [31] investigate the optimal investment policy of an insurance company, which trades in a money market account, a stock, and an inflation-linked bond while financing its investment by the pension plan's members contributions. [11] and [14] derive the optimal asset allocation under the CEV and Heston stochastic volatility model in DC pension plans, respectively. Further investigations on the modeling of DC plans in the presence of optimal problems include [30], [3], [23], [21] and [26].

While the trustee in DC plans fixes contributions, individuals might have better options. As pointed out by [24], individuals need more financial skills for retirement planning in the complex financial market. Individuals can easily make poor decisions on the investment with DC Plans. This inadequacy leads to the development of collective defined contribution (CDC) pension plans.

In CDC pension plans, the fund's assets are concentrated and managed by the financial institution, and benefits depend on the fund's financial position. According to [12], who considers that future generations are obliged to participate in pension plans, the collective pension scheme with compulsory participation could transfer risk among different generations. The corresponding risks are shared by current and future generations, which leads to improving welfare. Due to the success of CDC pension plans in actuarial studies, this type of pension design has drawn significant attention from researchers. [4] discuss the costs and benefits of collective pension schemes compared to traditional plans. [8] show a welfare comparison between the collective plans and the optimal individual pension scheme without risk sharing. [28] study an optimal investment strategy and adjusted benefit payment policy for a target benefit plan to minimize the combination of benefit risk and intergenerational transfers. [18] show the cost of unstable contribution and discontinuity risks in a DB-PAYGO pension system. [29] consider the optimal investment and benefit payment problem for a target benefit plan (TBP) with default risk and model uncertainty. The literature assumes that the labor income process is deterministic or follows a geometric Brownian motion. However, labor income could be related to the financial market. The relationship between labor income and stock markets should be considered. [25] first empirically investigates the connection between aggregate labor income and asset prices. [1] then point out that labor income is co-integrated with dividends. They also propose that in the absence of co-integration, labor income and capital of GDP may go to zero or one in the long run. Further, [2] introduce the co-integration between labor and stock markets in portfolio selection problems. They present that the high correlation between labor income and dividend flows is compelling by analyzing data, and the young agent's human capital is stock-like due to the co-integration. The optimal investment and consumption strategies are also provided for an agent with constant relative risk aversion. [13] assume a positive long-run correlation between average labor earnings and the stock market and presents the importance of investigating this correlation.

In this paper, we investigate a stochastic optimal problem for a CDC pension fund scheme in continuous time, where contribution rate is assumed to be fixed, and benefit payments depend on the final salary rate. The pension fund is allowed to be invested in the financial market consisting of one risky asset and one risk-free asset. Inspired by [2], we let the labor income process be described as a geometric random walk with a drift that depends on the ratio of current dividend to current labor income, where the dividend process follows a geometric Brownian motion. Labor income is assumed to be a product of aggregate labor income and member's idiosyncratic shocks whose constant growth rate is unknown. It can be modeled by a continuous-time two-state hidden Markov chain. We also assume that pension members are divided into active and retired groups. The active members are working members who contribute to the pension fund, while the retired members obtain benefits from the pension fund. Each participant enters the pension scheme at a_0 and retires at a_1. These additional complexities pose mathematical challenges. When addressing optimal investment problems with a constant relative risk aversion (CRRA) utility function, the traditional guess-verify approach is less feasible for solving the associated HJB equations. We look for a mathematically tractable framework under model uncertainty. Thus, we study this optimal problem under a constant absolute risk aversion (CARA) utility function and assume that the objective for the fund manager is to seek the optimal proportional investment and benefit replacement policies to maximize social welfare and the terminal surplus wealth. The explicit solutions for optimal asset allocation and benefit payment policies are obtained after using Hamilton-Jacobi-Bellman (HJB) equations.

Compared with the existing optimal designs for CDC fund scheme problems, our work considers co-integration between labor and stock markets. Labor income is tied up in the financial market in reality. The stochastic labor income and return on the financial market are sources of uncertainty the fund manager faces. Thus, we build a genuine relationship between the two in our model by letting the difference between logs of labor income and dividends follow a mean-reverting process. Trends of labor income and price of risky assets are co-integrated, implying that they will have the same trend in the long run. When co-integration is strong, the retirement income is more volatile than the fund portfolio return. Our model covers more realistic financial markets by considering the relationship between the labor and stock markets. Besides, we consider the fact that there exists uncertainty of idiosyncratic labor income shocks. As results in [7] and [9], labor income consists of aggregate labor income and idiosyncratic labor income shocks. The growth rate of labor income shocks may either be high or low, leading the fund manager to face estimation uncertainty of the income shocks. Thus, in addition to letting the logarithm of labor income shocks follow an arithmetic Brownian motion, such as [2], our model presents that the growth rate of labor shocks may change randomly over time. We model this uncertainty by a continuous time two-state hidden Markov chain, which adds mathematical difficulties to solving HJB equations. Furthermore, this paper is the first to consider the uncertainty of idiosyncratic labor income shocks. It investigates the impact of co-integration between the labor and stock markets on the benefit replacement ratio in a CDC pension system. These assumptions make our model more practical in life insurance.

The rest of paper is organized as follows. The formulation of asset and labor income, and assumptions are presented in Section 2. Section 3 derives explicit expressions for the optimal investment strategy and benefit replacement rate. An application is also showed in Section 3. Section 4 uses numerical examples to illustrate our results. Concluding remarks are provided in Section 5.

2. Formulation

Let $T > 0$ be a finite time horizon and three standard Brownian motions $Z_D = \{Z_D(t), t \geq 0\}$, $Z_1 = \{Z_1(t), t \geq 0\}$ and $Z_L = \{Z_L(t), t \geq 0\}$ are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ to be an augmented filtration generated by Brownian motions. We assume that the standard Brownian motions $Z_D(t)$, $Z_1(t)$ and $Z_L(t)$ are independent. Throughout the paper, in a filtered complete probability, for $p \geq 1$, we define:

$$
\begin{aligned}
L^p_{\mathcal{F}_t}(\Omega;\mathbb{R}) &= \Big\{ X : \Omega \to \mathbb{R} \Big| X(t) \text{ is } \mathcal{F}_t\text{-measurable}, \mathbb{E}[|X^p|] < \infty \Big\},\\
L^2_{\mathbb{F}}(s,t;\mathbb{R}) &= \Big\{ X : [s,t]\times\Omega \to \mathbb{R} \Big| X(t) \text{ is } \mathbb{F}\text{-adapted},\\
&\qquad \mathbb{E}\Big[\int_s^t |X(v)|^2 dv \Big] < \infty \Big\},\\
L^p_{\mathbb{F}}\left(\Omega; L^2(s,t;\mathbb{R})\right) &= \Big\{ X : [s,t]\times\Omega \to \mathbb{R} \Big| X(t) \text{ is } \mathbb{F}\text{-adapted},\\
&\qquad \mathbb{E}\Big[\Big(\int_s^t |X(v)|^2 dv \Big)^p \Big] < \infty \Big\},\\
L^p_{\mathbb{F}}(\Omega; C([s,t];\mathbb{R})) &= \Big\{ X : [s,t]\times\Omega \to \mathbb{R} \Big| X(t) \text{ is bounded } \mathbb{F}\text{-adapted},\\
&\qquad \text{has continuous paths and } \mathbb{E}\Big[\sup_{v\in[s,t]} |X(v)|^p \Big] < \infty \Big\}.
\end{aligned}
$$

2.1. Financial market

In our model, we assume that a fund manager can make an investment in the financial market consisting of one risky asset and one risk-free asset within the time horizon $[0,T]$. The risky asset pays a continuous flow of dividend to this investor. Let $D(t)$ represent the dividend process of the risky asset at time t. The dynamics of dividends process is given by

$$
\begin{cases}
\frac{dD(t)}{D(t)} = g_D dt + \sigma dZ_D(t), \quad t \in [0,T],\\
D(0) = d_0 > 0,
\end{cases}
$$

where g_D, σ are the growth and volatility rate of dividends, respectively. Let the dynamics of pricing kernel $M(t)$ be described as

$$
\frac{dM(t)}{M(t)} = -r dt - \lambda_m dZ_D(t),
$$

where $r > 0$ denotes a constant risk-free interest rate and λ_m is a constant price of risk.

Denote $X(t)$ to be the process of the risky asset. The price can be described as the expected discounted sum of dividends and generated by

$$
X(t) = \int_t^\infty \mathbb{E}_t \left[\frac{M(s)}{M(t)} D(s) \right] ds.
$$

Hence, we can derive that $D(t)$ is proportional to $X(t)$, that is

$$
X(t) = \frac{D(t)}{r + \lambda_m \sigma - g_D}.
$$

The ratio of dividend and stock price is constant, thus the dynamics of the risky asset is given by the following geometric Brownian motion:

$$
\begin{cases}
\frac{dX(t)}{X(t)} = g_D dt + \sigma dZ_D(t), \quad t \in [0,T],\\
X(0) = x_0 > 0.
\end{cases}
$$

Let $S(t)$ to be the surplus process over time t, described as

$$\begin{cases} \frac{dS(t)}{S(t)} = \frac{dX(t)+D(t)dt}{X(t)} = \mu dt + \sigma dZ_D(t), \quad t \in [0,T], \\ S(0) = s_0 > 0, \end{cases} \tag{2.1}$$

where the expected return follows from the definition of the pricing kernel that is $\mu = r + \lambda_m \sigma$. The evolution of the risk-free asset $S_0(t)$ is given by

$$\begin{cases} dS_0(t) = rS_0(t)dt, \\ S_0(0) = s_{00} > 0. \end{cases} \tag{2.2}$$

2.2. Labor income

Following the work of [2], we suppose that labor income $L(t)$ is a product of aggregate labor income $L_1(t)$ and the member's idiosyncratic shocks $L_2(t)$ at time t. Taking the logarithm of labor income, we have

$$l(t) = l_1(t) + l_2(t), \tag{2.3}$$

where $l(t) = \log L(t)$, $l_1(t) = \log L_1(t)$ and $l_2(t) = \log L_2(t)$. In a similar spirit to [2], we model that aggregate labor income and the stock market are co-integrated by letting the difference between logs of aggregate labor income and aggregate dividends follow a mean-reverting process. Let the difference $y(t)$ satisfy

$$y(t) = \log L_1(t) - \log D(t) - \lambda, \tag{2.4}$$

where the positive constant λ is the long run log-ratio of aggregate labor income to dividends. The dynamics of the difference $y(t)$ is governed

$$\begin{cases} dy(t) = -ky(t)dt + \upsilon_L dZ_L(t) - \upsilon_D dZ_D(t), \quad t \in [0,T], \\ y(0) = y_0, \end{cases}$$

where k determines the speed of the variable $y(t)$ towards long-run mean and captures the co-integration of aggregate labor income and dividends. That is, when $k = 0$, there exists no co-integration between two variables. υ_L and υ_D are conditional volatilities, and $Z_L(t)$ is a standard Brownian motion concerning the uncertainty in aggregate labor income.

The process for the log-idiosyncratic shocks is given by

$$\begin{cases} dl_2(t) = \left(\alpha(t) - \frac{\upsilon_1^2}{2}\right) dt + \upsilon_1 dZ_1(t), \quad t \in [0,T], \\ l_2(0) = l_{20}, \end{cases}$$

where $\alpha(t)$ is the growth rate and υ_1 is the corresponding volatility rate. $Z_1(t)$ is another standard Brownian motion, which is independent on $Z_L(t)$ and $Z_D(t)$.

2.3. Pension system

We now consider a pension scheme in which the plan members can be divided into two groups: active members and retired members. The active members are the working members who make contributions to the pension fund, while the retired members obtain benefits from the pension fund. All members are assumed to join the pension plan at age a_0, keep active up to a retirement age a_1 and death at age a_2. We also assume that the survival function is $s(x)$ with $s(a_0) = 1$.

We denote $n(t)$ to describe the density of new members entering in a pension plan at age a_0 at time t. The function $n(t)$ is a non-negative function, reflecting that the density of new members entering the pension plan cannot be negative. Then the number of those who are x-year-old at time t is

$$n(t - (x - a_0))s(x), \quad t > 0.$$

$t - (x - a_0)$ could be negative, meaning that an individual aged x joined the plan $x - a_0$ years ago, with $t > 0$. $n(t - (x - a_0))$ equals zero, when the individuals of age x have not yet entered the pension plan.

The total number of members in the labor market and the retired members is given by $M_1(t)$ and $M_2(t)$, respectively.

$$M_1(t) = \int_{a_0}^{a_1} n(t - (x - a_0))s(x)dx,$$

$$M_2(t) = \int_{a_1}^{a_2} n(t - (x - a_0))s(x)dx.$$

The number of active members determines the aggregate contribution rate. We assume that C_0 is the instantaneous rate at time 0 and η_1 is the contribution's exponential growth rate. Thus, the aggregate contribution rate of the pension fund at time t is given by

$$C(t) = \int_{a_0}^{a_1} n(t - x + a_0)s(x)C_0 e^{\eta_1 t} dx.$$

The number of retired members determines the total benefit, the salary structure, and the initial annual pension payment rate. The initial pension payment rate is assumed to be a proportion of the final salary. For a member who retires at t, the initial pension payment is $b(t)L(t)$, where $b(t)$ is the replacement rate and can be regarded as a control variable. As mentioned above, labor income is described as (3.3). At time t, for members who age at x, the final salary is obtained $x - a_1$ years ago. Similar to [28]. For determining the annual pension payment rate applicable to a retired member aged x at time t. We introduce a new quantity $FL(x,t)$, representing the member's assumed salary at retirement $x - a_1$ years ago. This quantity is defined as

$$FL(x,t) = L(t)e^{-\eta_0(x-a_1)}, \quad t \geq 0, \quad x \geq a_1,$$

where $L(t)$ denotes the salary of members retiring at time t and the projection is made backwards deterministically using the exponential growth rate η_0. This approach differs from the member's actual salary at retirement when $\eta_0 > 0$ and $x > a_1$, Notably, the expected difference between the actual salary and the assumed salary at retirement increases with age. We assume that there is one adjustment factors applied to the assumed salary rate. Thus, the pension payment rate at time t for those who aged x is

$$B(x,t) = b(t)L(t)e^{-\eta_0(x-a_1)}.$$

The actual aggregate retirement benefit for all the retirees for all ages between a_1 and a_2 at time t is easy to obtain, which is shown as

$$B(t) = \int_{a_1}^{a_2} n(t - x + a_0)s(x)B(x,t)dx = F(t)b(t)L(t),$$

where $F(t)$ is a positive function, given by

$$F(t) = \int_{a_1}^{a_2} n(t - x + a_0)s(x)e^{-\eta_0(x-a_1)}dx.$$

To increase benefits of the pension fund, the fund manager invests the surplus between the contribution and benefit in the financial market dynamically. In our model, the financial market consists of one risky asset and one risk-free asset. Let $\pi(t)$[1] be the proportional investment invested in the risky asset at time t. The dynamics of wealth process of pension fund becomes

$$\begin{cases} dW(t) = \pi(t)W(t)\frac{dX(t)+D(t)dt}{X(t)} + (1-\pi(t))W(t)\frac{dS_0(t)}{S_0(t)} + C(t)dt - B(t)dt, \\ W(0) \ = w_0 > 0. \end{cases} \tag{2.5}$$

In the following, we show the definition of admissible strategies corresponding to (3.4) and propose the main problem that we study in this paper.

Definition 2.1. For any fixed $t \in [0,T]$, the strategy pair $(\pi(t), b(t))$ is said to be admissible if it satisfies the following conditions:

- the investment strategy and initial benefit payment policy $(\pi(t), b(t))$ is $\mathcal{F}_t$-adapted such that the SDE (3.4) admits a unique solution $W^{\pi,b}(t)$.
- $\pi(t) \in L^2_{\mathbb{F}}\left(0,T;\mathbb{R}^+\right)$ and $b(t) \in L^2_{\mathbb{F}}\left(0,T;\mathbb{R}^+\right)$, for all $t > 0$.

Problem 2.1. For and initial state (t, W_t), the objective function for the pension fund manager is to maximize

$$J(t,w,l,y) = \mathbb{E}_{\pi,b}\Big[\int_t^T e^{-rs}U(b(s)F(s)L(s))ds + \lambda_1 e^{-rT}U(W(T))\Big], \tag{2.6}$$

where λ_1 is a non-negative constant, interpreted as the weight put on the utility derived from the terminal wealth. $\mathbb{E}_{\pi,b}$ is the conditional expectation under probability measure $\mathbb{P}$ given $W(t) = w$, $L(t) = l$ and $y(t) = y$. Then, the value function of this problem is given by

$$V(t,w,l,y) = \sup_{(\pi,b)\in\mathcal{U}} J(t,w,l,y),$$

[1] In this paper, we consider an optimal investment strategy without borrowing constraints to derive a closed-form solution. In reality, companies may excessively rely on debt financing without borrowing constraints, leading to increased financial leverage ratios. Companies can effectively manage their exposure to market risks by setting limits on the investment proportion. We will consider this problem in the future to enhance the model's realism.

where $\mathcal{U}$ is a set of control pair.

3. Optimal strategies for the pension fund plan

In this section, we investigate this stochastic optimal control problem by using a standard dynamic programming approach. We obtain the explicit expressions for optimal controls when the labor and stock market are co-integrated. We consider two situations for the labor income process: identifying idiosyncratic shocks with an unknown growth rate and considering without idiosyncratic shocks.

3.1. Optimal strategies with idiosyncratic shocks

In this subsection, we assume that the employees face uncertainty about the labor income by considering idiosyncratic shocks with an unknown growth rate. We capture the inherent uncertainty and dynamics of labor income shocks by employing a continuous-time, two-state, hidden Markov chain. To be more specific, $\alpha(t)$ is unknown growth rate, which is modeled by a continuous-time, two-state, hidden Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$ and varies between α_1 and α_2, where $\alpha_1 > \alpha_2$. The growth rate $\alpha(t)$ may get a high value α_1 or get a low value α_2. For a small time interval Δt, $1 - p_1\Delta t$ is the probability that the growth rate $\alpha(t)$ takes a high value α_1 at time t and $1 - p_2\Delta t$ is the probability that the growth rate $\alpha(t)$ remains a low value α_2 at time t. p_1 and p_2 are the transition intensities of the two-state hidden Markov chain. The growth rate moves from high value to α_2 with probability $p_1\Delta t$. Similarly, the probability that the growth rate $\alpha(t)$ increases to α_1 from low value is $p_2\Delta t$. For any t, let $P(t)$ be the conditional probability,

$$P(t) = \mathbb{P}\big(\alpha(t) = \alpha_1 | \mathcal{F}_t\big).$$

That is, given the observed information $\mathcal{F}_t$, $\alpha(t)$ takes a high value α_1. The expected growth rate of log-idiosyncratic shocks $\mu_0(t)$ is the weighted average of two possible growth rates, given by

$$\mu_0(t) = P(t)\alpha_1 + (1 - P(t))\alpha_2 = \alpha_2 + \beta P(t), \tag{3.1}$$

where $\beta = \alpha_1 - \alpha_2$. Given a small time period Δt, the total change of log-labor shock is $l_2(t + \Delta t) - l_2(t)$ and the expected change is $\mu_0(t)\Delta t - \frac{v_1^2}{2}$. Following the idea incorporated in [27], we normalize the corresponding volatility and obtain

$$dZ_1(t) = \frac{1}{v_1\Delta t}\left(l_2(t + \Delta t) - l_2(t) - \mu_0(t)\Delta t - \frac{v_1^2}{2}\right). \tag{3.2}$$

Plugging (3.2) into (3.1), we have

$$dl_2(t) = (\alpha_2 + \beta P(t) - \frac{v_1^2}{2})dt + v_1 dZ_1(t).$$

Applying the results in [22], we find

$$dP(t) = (p_2 - (p_1 + p_2)P(t))dt + v_1^{-1}\beta P(t)(1 - P(t))dZ_1(t).$$

According to (2.3) and (2.4), the log-labor income can be rewritten as

$$l(t) = y(t) + d(t) + \lambda + l_2(t),$$

where $d(t) = \log D(t)$ and the dynamics of log-dividend process is generated by

$$\begin{cases} dd(t) = (g_D - \frac{\sigma^2}{2})dt + \sigma dZ_D(t), & t \in [0, T], \\ d(0) = d_0. \end{cases}$$

Applying Itô's formula, the dynamics of log-labor income is derived as

$$\begin{aligned} dl(t) = {} & [-ky(t) + g_D - \frac{1}{2}\sigma^2 + \lambda + \beta P(t) + \alpha_2 - \frac{1}{2}v_1^2]dt \\ & + (\sigma - v_D)dZ_D(t) + v_L dZ_L(t) + v_1 dZ_1(t). \end{aligned}$$

Then, using Itô's formula again, we have

$$\begin{cases} \frac{dL(t)}{L(t)} &= [-ky(t) + g_D + \lambda - \frac{1}{2}\sigma^2 + \beta P(t) + \alpha_2 + \frac{1}{2}v_L^2 + \frac{1}{2}(\sigma - v_D)^2]dt \\ & \quad +(\sigma - v_D)dZ_D(t) + v_L dZ_L(t) + v_1 dZ_1(t), \quad t \in [0, T], \\ L(0) &= l_0. \end{cases} \tag{3.3}$$

Combining (2.2) and (2.1), we can easily rewrite

$$\begin{cases} \frac{dW(t)}{W(t)} = [(\mu - r)\pi + r + \frac{C(t)}{W(t)} - \frac{F(t)L(t)b(t)}{W(t)}]dt + \pi\sigma dZ_D, \\ W(0) \ = w_0 > 0, \end{cases} \tag{3.4}$$

with

$$\begin{aligned} \frac{dL(t)}{L(t)} &= [-ky(t) + g_D + \lambda - \frac{1}{2}\sigma^2 + \beta P(t) + \alpha_2 + \frac{1}{2}v_L^2 + \frac{1}{2}(\sigma - v_D)^2]dt \\ &\quad + (\sigma - v_D)dZ_D(t) + v_L dZ_L(t) + v_1 dZ_1(t), \\ dP(t) &= (p_2 - (p_1 + p_2)P(t))dt + v_1^{-1}\beta P(t)(1 - P(t))dZ_1(t), \\ dy(t) &= -ky(t)dt + v_L dZ_L(t) - v_D dZ_D(t). \end{aligned} \tag{3.5}$$

Thereafter, we use P to denote $P(t)$ with a slight abuse of notation. For and initial state (t, W_t), the objective function for the pension fund manager is to maximize

$$J(t, w, l, y, P) = \mathbb{E}_{\pi,b}\Big[\int_t^T e^{-rs}U(b(s)F(s)L(s))ds + \lambda_1 e^{-rT}U(W(T))\Big]. \tag{3.6}$$

The value function of this problem is given by

$$V(t, w, l, y, P) = \sup_{(\pi,b)\in\mathcal{U}} J(t, w, l, y, P), \tag{3.7}$$

where $\mathcal{U}$ is a set of control pair. For a mathematically tractable framework, we assume that the fund manager's utility function is the following expression:

$$U(w) = -\frac{1}{\gamma}e^{-\gamma w},$$

where $\gamma > 0$ is a constant absolute risk aversion coefficient of the fund manager. The objective of fund manager is to maximize the social welfare together with the terminal surplus wealth. For convenience, we denote

$$\begin{aligned} &C^{1,2,2,2}([0,T] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+) \\ &\quad = \{V(t, w, l, y, P) | V(t, \cdot, \cdot, \cdot, \cdot)\} \text{ is once continuously differentiable on}[0,T], \\ &\qquad V(\cdot, w, l, y, P) \text{ is twice continuously differentiable for } w \text{ on } \mathbb{R}^+, l \text{ on } \mathbb{R}^+, y \text{ on } \mathbb{R}^+, P \text{ on } \mathbb{R}^+\}. \end{aligned}$$

Using standard methods, we get the following HJB equation satisfied by the value function $V(t, w, l, y, P) \in C^{1,2,2,2}$

$$\begin{aligned} 0 = \sup_{\pi,b}\Big\{ & V_t + wV_w[r + (\mu - r)\pi + \frac{C(t)}{w} - \frac{F(t)lb}{w}] + \frac{1}{2}w^2V_{ww}\sigma^2\pi^2 \\ & + lv_L[-ky + g_D + \lambda - \frac{1}{2}\sigma^2 + \beta P + \alpha_2 + \frac{1}{2}v_L^2 + \frac{1}{2}(\sigma - v_D)^2] \\ & + \frac{1}{2}l^2V_{ll}\big[(\sigma - v_D)^2 + v_1^2 + v_L^2\big] - kyV_y + \frac{1}{2}V_{yy}(v_D^2 + v_L^2) + wlV_{wl}\pi\sigma(\sigma - v_D) \\ & - wV_{wy}\pi\sigma v_D + lV_{ly}[v_L^2 - v_D(\sigma - v_D)) + lPV_{lP}\beta(1 - P) + V_P[p_2 - (p_1 + p_2)P] \\ & + \frac{1}{2v_1^2}V_{PP}P^2(1 - P)^2\beta^2 - e^{-rt}\frac{1}{\gamma}e^{-\gamma F(t)lb}\Big\}, \end{aligned} \tag{3.8}$$

with the boundary condition

$$V(T, w, l, y, P) = -e^{-rT}\frac{\lambda_1}{\gamma}e^{-\gamma w}, \tag{3.9}$$

where $V_t, V_w, V_{ww}, V_l, V_{ll}, V_P, V_{PP}, V_y, V_{yy}, V_{wl}, V_{wy}, V_{ly}$ and V_{lP} are denoted as partial derivatives of $V(t, w, l, y, P)$ for simplification.

In the following theorem, we present our investigations on this stochastic control problem.

Theorem 3.1. *For any $t \in [0,T]$, the optimal investment strategy and benefit adjustment policy are given, respectively, by*

$$\pi^*(t, w, l, y, P) = \frac{(\mu - r)}{\gamma f_1(t)\sigma^2 w}, \tag{3.10}$$

$$b^*(t, w, l, y, P) = \frac{\ln\lambda_1 + \ln f_1(t) - \gamma f_1(t)w - \gamma f_2(t) - \gamma f_5(P)}{-\gamma F(t)l}, \tag{3.11}$$

and the corresponding value function is

$$V(t, w, l, y, P) = -\frac{\lambda_1}{\gamma}e^{-\gamma[f_1(t)w + f_2(t) + f_5(P)] - rt},$$

where

$$f_1(t) = \left[e^{-\int_t^T r ds} + \int_t^T e^{-\int_t^s r du} ds \right]^{-1},$$

$$f_2(t) = \int_t^T e^{-\int_t^s f_1(u)du} \times \left[f_1(s) \left(C(s) - \frac{1 - \ln f_1(s) - \ln \lambda_1}{\gamma} \right) + \frac{1}{2} \frac{(\mu - r)^2}{\gamma \sigma^2} + \frac{r}{\gamma} \right] ds - f_5(P(T)).$$

$f_5(P)$ satisfies the following ODE, for $0 < P < 1$,

$$f_1(t) f_5(P) = f_5'(P) \left[p_2 - (p_1 + p_2) P \right] - \frac{1}{2v^2} \left[\gamma f_5'(P) - f_5''(P) \right] P^2 (1 - P)^2 \beta^2,$$

with boundary conditions

$$\begin{cases} f_1(t) f_5(0) = p_2 f_5'(0), \\ f_1(t) f_5(1) = -p_1 f_5'(1), \end{cases} \tag{3.12}$$

where $f_5'(P) = \frac{\partial f_5(P)}{\partial P}$ and $f_5''(P) = \frac{\partial^2 f_5(P)}{\partial P^2}$.

Proof. From the HJB equation (3.8), we easily derive that the first-order conditions for two controls are

$$0 = w V_w (\mu - r) + w^2 V_{ww} \sigma^2 \pi + w l V_{wl} \sigma (\sigma - v_D) - w V_{wy} \sigma v_D,$$

$$0 = -V_w F(t) l + F(t) l e^{-rt} e^{-\gamma F(t) l b}.$$

Thus, the optimal investment and benefit strategies are given by

$$\begin{aligned} \pi^* &= \frac{V_{wy} \sigma v_D - V_w (\mu - r) - l V_{wl} \sigma (\sigma - v_D)}{w V_{ww} \sigma^2}, \\ b^* &= \frac{\ln V_w + rt}{-\gamma F(t) l}. \end{aligned} \tag{3.13}$$

The value function of the fund manager is assumed to be the following form:

$$V(t, w, l, y, P) = -\frac{\lambda_1}{\gamma} \exp\{-\gamma [f_1(t) w + f_2(t) + f_3(t, l) + f_4(t) y + f_5(P)] - rt\},$$

where $f_1(t), f_2(t), f_3(t)$ and $f_4(t)$ are functions of t to be determined. The terminal conditions are $f_1(T) = 1$, $f_2(T) = -f_5(P(T))$ and $f_3(T) = f_4(T) = 0$.

From the expression of the value function, we obtain

$$\begin{aligned} & V_t = -\gamma V [f_{1t} x + f_{2t} + f_{3t}(t, l) + f_{4t} y + \frac{r}{\gamma}], \\ & V_w = -\gamma f_1 V, \qquad V_{ww} = \gamma^2 f_1^2 V, \\ & V_L = -\gamma f_3' V, \qquad V_{ll} = \gamma^2 f_3' f_3' V - \gamma f_3'' V, \\ & V_y = -\gamma f_4 V, \qquad V_{yy} = 0, \\ & V_{wl} = \gamma^2 f_1 f_3' V, \quad V_{wy} = \gamma^2 f_1 f_4 V, \\ & V_{ly} = \gamma^2 f_3' f_4 V, \quad V_P = -\gamma f_5' V, \\ & V_{PP} = \gamma^2 f_5' V - \gamma f_5'' V, \quad V_{lP} = \gamma^2 f_3' f_5' V, \end{aligned} \tag{3.14}$$

where $f_3' = \frac{\partial f_3}{\partial l}$ and $f_3'' = \frac{\partial^2 f_3}{\partial l^2}$.

Plugging (3.13) and (3.14) into HJB equation (3.8), we get

$$
\begin{aligned}
0 =& \big[-\ln \lambda_1 - \ln f_1(t) + \gamma(f_1(t)w + f_2(t) + f_3(t,l) + f_4(t)y + f_5(P))\big] f_1(t) \\
&+ f_1(t) - \gamma \left[f_{1t} w + f_{2t} + f_{3t}(t,l) + f_{4t} y + \frac{r}{\gamma} \right] - \gamma f_1(t) w r \\
&- \frac{(\mu - r)\big[\gamma f_4(t)\sigma v_D + \mu - r - l\gamma^2 f_3' \sigma(\sigma - v_D)\big]}{\sigma^2} - \gamma f_1(t) C(t) \\
&+ \frac{\big[\gamma f_4(t)\sigma v_D + \mu - r - l\gamma^2 f_3' \sigma(\sigma - v_D)\big]^2}{2\sigma^2} \\
&- l\gamma f_3' \left[-ky + g_D - \frac{1}{2}\sigma^2 + \beta P + \alpha_2 + \frac{1}{2} v_L^2 + \frac{1}{2}(\sigma - v_D)^2 \right] \\
&+ \frac{1}{2} l^2 \big[\gamma^2 f_3' f_3' - \gamma f_3''\big] \big[(\sigma - v_D)^2 + v_1^2 + v_L^2\big] \\
&- \gamma f_4(t) y k + \frac{\big[l\gamma f_3'(\sigma - v_D)\big]\big[\gamma f_4(t)\sigma v_D + \mu - r - l\gamma^2 f_3' \sigma(\sigma - v_D)\big]}{\sigma} \\
&+ \frac{\gamma f_4(t) v_D \big[\gamma f_4(t)\sigma v_D + \mu - r - l\gamma^2 f_3' \sigma(\sigma - v_D)\big]}{\sigma} \\
&+ l\gamma^2 f_3' f_4(t)\big[v_L^2 - v_D(\sigma - v_D)\big] + lP\gamma^2 f_3' f_5'(P)\beta(1-P) \\
&- \gamma f_5'(P)\big[p_2 - (p_1 + p_2)P\big] + \frac{1}{2v_1^2}\big[\gamma^2 f_5'(y) - \gamma f_5''(y)\big] P^2 (1-P)^2 \beta^2.
\end{aligned} \tag{3.15}
$$

We postulate that $f_3(t,l)$ is of the form $f_3(t)\ln l$. Comparing the coefficients, we obtain

$$
\begin{cases}
f_{1t} - f_1(t)^2 + f_1(t) r = 0, \\
f_1(T) = 1.
\end{cases} \tag{3.16}
$$

$$
\begin{cases}
f_1(t) f_5(P) = f_5'(P)\left[p_2 - (p_1+p_2)P\right] - \frac{1}{2v_1^2}\left[\gamma f_5'(P) - f_5''(P)\right] P^2(1-P)^2\beta^2 + f_3(t)\beta P \\
\qquad\qquad + P f_3(t) f_5'(P)\beta(1-P), \\
f_1(t) f_5(0) = p_2 f_5'(0), \\
f_1(t) f_5(1) = -p_1 f_5'(1).
\end{cases} \tag{3.17}
$$

$$
\begin{cases}
f_{2t} + \frac{1}{\gamma}\left[\ln \lambda_1 + \ln f_1(t) - \frac{1}{\gamma}\right] f_1(t) - f_2(t) f_1(t) - \frac{(\mu-r)^2}{\gamma\sigma^2} + f_1(t) C(t) + \frac{r}{\gamma} \\
+ \frac{(\mu-r)\left[f_4(t)\sigma v_D - \gamma f_3(t)\sigma(\sigma - v_D)\right]}{\sigma^2} - \frac{\left[\gamma f_4(t)\sigma v_D + \mu - r - \gamma^2 f_3(t)\sigma(\sigma - v_D)\right]^2}{2\sigma^2\gamma} \\
+ f_3(t)\left[g_D - \frac{1}{2}\sigma^2 + \alpha_2 + \frac{1}{2}(\sigma - v_D)^2\right] \\
- \frac{1}{2}\left[\gamma f_3^2(t) + f_3(t)\right]\left[(\sigma - v_D)^2 + v_1^2 + v_L^2\right] \\
- \frac{\left[f_3(t)(\sigma - v_D)\right]\left[\gamma f_4(t)\sigma v_D + \mu - r - \gamma^2 f_3(t)\sigma(\sigma - v_D)\right]}{\sigma} \\
+ \frac{f_4(t) v_D\left[\gamma f_4(t)\sigma v_D + \mu - r - \gamma f_3(t)\sigma(\sigma - v_D)\right]}{\sigma} \\
- \gamma f_3(t) f_4(t)\left[v_L^2 - v_D(\sigma - v_D)\right] = 0, \\
f_2(T) = -f_5(P(T)).
\end{cases} \tag{3.18}
$$

$$
\begin{cases}
f_{3t} - f_1(t) f_3(t) = 0, \\
f_3(T) = 0.
\end{cases} \tag{3.19}
$$

$$
\begin{cases}
f_{4t} - f_1(t) f_4(t) + f_4(t) k + k f_3(t) = 0, \\
f_4(T) = 0.
\end{cases} \tag{3.20}
$$

According to the expression in (3.19), we can easily get $f_3(t) = 0$. Solving (3.20) gives $f_4(t) = 0$. Hence, the form functions $f_1(t)$ and $f_2(t)$ follows from (3.16) and (3.18). Then, following the standard methods of [10], we demonstrate the sufficiency of both the optimal investment strategy and the optimal benefit rate.

Let $\hat{V} \in C^{1,2,2,2}$ be a classical solution to (3.8) that satisfies (3.9), then for any fixed $t \in [0,T]$, for every admissible control pair $(\pi^*(t), b^*(t))$

$$
\hat{V}(t,w,l,y) \geq J(t,w,l,y).
$$

If there exists an admissible control pair $(\pi^*(t), b^*(t))$, such that

$$
\begin{aligned}
0 = \sup_{\pi^*,b^*} \Big\{ & V_t + wV_w[r + (\mu - r)\pi^* + \frac{C(t)}{w} - \frac{F(t)lb^*}{w}] + \frac{1}{2}w^2V_{ww}\sigma^2\pi^{*2} \\
& + lv_L[-ky + g_D + \lambda - \frac{1}{2}\sigma^2 + \beta P + \alpha_2 + \frac{1}{2}v_L^2 + \frac{1}{2}(\sigma - v_D)^2] \\
& + \frac{1}{2}l^2V_{ll}\left[(\sigma - v_D)^2 + v_1^2 + v_L^2\right] - kyV_y + \frac{1}{2}V_{yy}(v_D^2 + v_L^2) + wlV_{wl}\pi^*\sigma(\sigma - v_D) \\
& - wV_{wy}\pi^*\sigma v_D + lV_{ly}[v_L^2 - v_D(\sigma - v_D) + lPV_{lP}\beta(1 - P) + V_P[p_2 - (p_1 + p_2)P] \\
& + \frac{1}{2v_1^2}V_{PP}P^2(1 - P)^2\beta^2 - e^{-rt}\frac{1}{\gamma}e^{-\gamma F(t)lb^*} \Big\},
\end{aligned}
$$

holds for all $(t,w,l,y,P) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$. Then,

$$\widehat{V}(t,w,l,y,P) = J(t,w,l,y,P). \tag{3.21}$$

If we let $\widehat{V}(t,w,l,y,P) = V(t,w,l,y,P)$, then it provides the assumptions hold. The corresponding control pair $(\pi^*(t), b^*(t))$ is admissible, the optimal investment strategy $\pi^*(t)$ and the optimal benefit rate $b^*(t)$ can be shown. □

Remark 3.2. It is worthy of noting that the optimal investment strategy $\pi^*(t)$ is independent of labor income $L(t)$. This means that the investment decisions are not influenced by the salary model for retired members, indicating that the investment strategy remains consistent regardless of variations in labor income. The benefit adjustment policy $b^*(t)$ is contingent upon labor income, impacting the value of retirement benefits. That is because benefit payments depend on the final salary rate.

Remark 3.3. Note that the optimal investment proportion given in (3.10) depends on the wealth process w at time t. As we know, the return rate from a risky asset is usually higher than risk-free rate, that is $\mu > r$. Equation (3.10) shows that the coefficient of optimal allocation invested in the risky asset is positive, which implies as the expected return rate increases, the fund manager will make a larger investment. This result coincides with previous results in [15] and [32].

3.2. Optimal strategies without idiosyncratic shocks

In this subsection, we consider a special case that labor income is co-integrated with the stock in the absence of idiosyncratic shocks. We model the long-run relation between labor and stock market by letting the log labor-dividend ratio follows a mean-reverting process. Let $\bar{L}(t)$ be the labor income process, then $\bar{l}(t) = \log \bar{L}(t)$. The difference between log-labor and log-dividend is $\bar{y}(t)$. Let $\bar{\pi}(t)$ be the proportion of investment invested in the risky asset and $\bar{b}(t)$ is the benefit strategy in this case, the wealth process becomes

$$
\begin{cases}
\frac{d\bar{W}(t)}{\bar{W}(t)} = [(\mu - r)\bar{\pi} + r + \frac{C(t)}{\bar{W}(t)} - \frac{F(t)\bar{L}(t)\bar{b}(t)}{\bar{W}(t)}]dt + \bar{\pi}\sigma dZ_D, \\
\bar{W}(0) \ = \bar{w}_0 > 0,
\end{cases} \tag{3.22}
$$

with

$$
\begin{aligned}
\frac{d\bar{L}(t)}{\bar{L}(t)} &= [-k\bar{y}(t) + g_D + \bar{\lambda} - \frac{1}{2}\sigma^2 + \frac{1}{2}(\sigma - v_D)^2]dt \\
&\quad + (\sigma - v_D)dZ_D(t), \\
d\bar{y}(t) &= -k\bar{y}(t)dt - v_D dZ_D(t).
\end{aligned}
$$

We have the following definition of admissible strategies corresponding to stochastic differential equation (SDE) (3.22).

Definition 3.4. For any $t \in [0,T]$, the strategy pair $(\bar{\pi}(t), \bar{b}(t))$ is said to be admissible, if the strategy pair is $\mathcal{F}_t$-adapted, $\mathbb{E}[\int_t^T[\bar{b}^2(s)]ds] < +\infty$, $\mathbb{E}[\int_t^T[\bar{\pi}^2(s)]ds] < +\infty$ and SDE (3.22) has unique solution.

The fund manager aims to maximize

$$\bar{J}(t,\bar{w},\bar{l},\bar{y}) = \mathbb{E}_{\bar{\pi},\bar{b}}\Big[\int_t^T e^{-rs}U(\bar{b}(s)F(s)\bar{L}(s))ds + \lambda_1 e^{-rT}U(\bar{W}(T))\Big], \tag{3.23}$$

where λ_1 is a non-negative constant, interpreted as the weight put on the utility derived from the terminal wealth. Then the value function of this problem is given by

$$\bar{V}(t,\bar{w},\bar{l},\bar{y}) = \sup_{\bar{\pi},\bar{b}} \bar{J}(t,\bar{w},\bar{l},\bar{y}), \tag{3.24}$$

$$U(\bar{w}) = -\frac{1}{\gamma_1}e^{-\gamma_1\bar{w}}, \tag{3.25}$$

where $\gamma_1 > 0$ is the constant absolute risk aversion coefficient of the fund manager. Further, the HJB equation can be derived as

$$
\begin{aligned}
0 = \sup_{\bar{\pi},\bar{b}} \Big\{ & \bar{V}_t + \bar{w}\bar{V}_{\bar{w}}[r + (\mu - r)\bar{\pi} + \frac{C(t)}{\bar{w}} - \frac{F(t)\bar{l}\bar{b}}{\bar{w}}] + \frac{1}{2}\bar{w}^2 V_{\bar{w}\bar{w}}\sigma^2\bar{\pi}^2 \\
& + \bar{l}\bar{V}_{\bar{l}}[-k\bar{y} + g_D + \bar{\lambda} - \frac{1}{2}\sigma^2 + \frac{1}{2}(\sigma - v_D)^2] + \frac{1}{2}\bar{l}^2\bar{V}_{\bar{l}\bar{l}}(\sigma - v_D)^2 \\
& - k\bar{y}\bar{V}_{\bar{y}} + \frac{1}{2}\bar{V}_{\bar{y}\bar{y}}v_D^2 + \bar{w}\bar{l}\bar{V}_{\bar{w}\bar{l}}\bar{\pi}\sigma(\sigma - v_D) - \bar{w}\bar{V}_{\bar{w}\bar{y}}\bar{\pi}\sigma v_D - \bar{l}\bar{V}_{\bar{l}\bar{y}}v_D(\sigma - v_D) \\
& - e^{-rt}\frac{1}{\gamma_1}e^{-\gamma_1 F(t)\bar{l}\bar{b}} \Big\},
\end{aligned}
\tag{3.26}
$$

with the boundary condition

$$
\bar{V}(T, \bar{w}, \bar{l}, \bar{y}) = -e^{-rT}\frac{1}{\gamma_1}e^{-\gamma_1\bar{w}},
\tag{3.27}
$$

where $\bar{V}_t$, $\bar{V}_{\bar{w}}$, $\bar{V}_{\bar{w}\bar{w}}$, $\bar{V}_{\bar{l}}$, $\bar{V}_{\bar{l}\bar{l}}$, $\bar{V}_{\bar{y}}$, $\bar{V}_{\bar{y}\bar{y}}$, $\bar{V}_{\bar{w}\bar{l}}$, $\bar{V}_{\bar{w}\bar{y}}$ and $\bar{V}_{\bar{l}\bar{y}}$ are partial derivatives of $\bar{V}(t, \bar{w}, \bar{l}, \bar{y})$.

The optimal investment strategy and benefit adjustment policy can be derived in the following theorem.

Theorem 3.5. *For any $t \in [0, T]$, the optimal investment strategy and benefit adjustment policy are given, respectively, by*

$$
\begin{aligned}
\bar{\pi}^*(t, \bar{w}, \bar{l}, \bar{y}) &= \frac{(\mu - r)}{\gamma_1\bar{g}_1(t)\sigma^2\bar{w}}, \\
\bar{b}^*(t, \bar{w}, \bar{l}, \bar{y}) &= \frac{\ln\lambda_1 + \ln\bar{g}_1(t) - \gamma_1\bar{g}_1(t)\bar{w} - \gamma_1\bar{g}_2(t)}{-\gamma_1 F(t)\bar{l}},
\end{aligned}
$$

and the corresponding value function is

$$
\bar{V}(t, \bar{w}, \bar{l}, \bar{y}) = -\frac{\lambda_1}{\gamma_1}e^{-\gamma_1[\bar{g}_1(t)\bar{w} + \bar{g}_2(t)] - rt},
$$

where

$$
\begin{aligned}
\bar{g}_1(t) &= \left[e^{-\int_t^T r\,ds} + \int_t^T e^{-\int_t^s r\,du}\,ds \right]^{-1}, \\
\bar{g}_2(t) &= \int_t^T e^{-\int_t^s \bar{g}_1(u)du} \times \left[\bar{g}_1(s)\left(C(s) - \frac{1 - \ln\bar{g}_1(s) - \ln\lambda_1}{\gamma_1} \right) + \frac{1}{2}\frac{(\mu - r)^2}{\gamma_1\sigma^2} + \frac{r}{\gamma_1} \right] ds.
\end{aligned}
$$

The proof of Theorem 3.5 is provided in Appendix A.

Remark 3.6. We compare the optimal investment and benefit replacement policies obtained for two cases. We find that the optimal allocation strategies remain the same without model ambiguity. This is because the optimal investment strategy is independent of labor income. The labor income process does not affect the investment strategy. However, the uncertainty of the growth rate influences the benefit adjustment policy. The benefit payment strategy is different in two situations.

4. Numerical analysis

In this section, we explore that the impact of parameters of the financial market on the optimal investment allocation and benefit adjustment policy. Inspired by [28], we have the following assumption:

$$
\mu_1(x) = A + Bc^x,
$$

where $\mu_1(x)$ is the force of mortality of the individuals at age x. Thus, the survival function is given by

$$
\begin{aligned}
s(x) &= e^{-\int_0^{x-a_0}\mu(a_0+s)ds} \\
&= e^{-A(x-a_0) = \frac{B}{\ln c}(c^x - c_0^a)}, \quad a_0 \le x \le a_1.
\end{aligned}
$$

As used in [28], the basic parameters are given by: $A = 0.00022$, $B = 2.7 \times 10^{-6}$, $c = 1.124$, $a_0 = 30$, $a_1 = 65$, $a_2 = 100$, $n = 10$, According to the function of $s(x)$ and $F(t)$

$$
F(t) = \int_{a_1}^{a_2} n(t - x + a_0)s(x)e^{-\eta_0(x-a_1)}dx,
$$

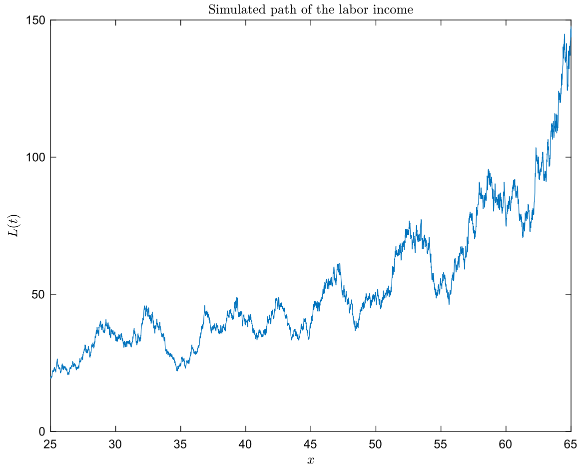


Fig. 4.1. Simulated path of the labor income.

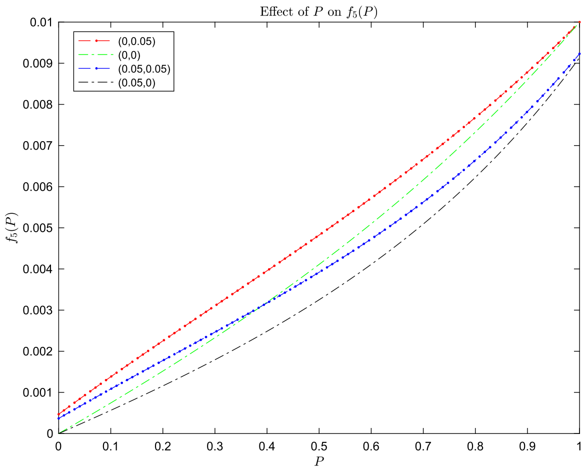


Fig. 4.2. Effect of P on the certainty wealth $f_5(P)$.

producing an aggregate benefit factor $F = 188.8688$. $\eta_0 = 0.01$, $\eta_1 = 0.02$, $\lambda = 0.3$, $g_D = 0.01$, $w_0 = 150$, $y_0 = 0$, $\lambda_m = 1$, $\lambda_1 = 0.3$, $\gamma_1 = 0.3$, $C_0 = 0.1$. The real risk-free rate $r = 0.01$, the key co-integration coefficient parameter $k = 0.15$, $\upsilon_D = 0.16$ and $\sigma = 0.16$ are taken from [2]. Firstly, we illustrate a simulated path of the labor income, which is derived by simulating the belief process $P(t)$, the difference process $y(t)$ and the labor income process $L(t)$ from equations in (3.5). We assume that a member aged at 25 makes salary $20 \times 1K$ per annum. The simulated path can be seen from Fig. 4.1. Throughout the numerical analysis, we consider the time horizon for choosing optimal strategies is 20 years, that is, $T = 20$.

Under conditions of an uncertain growth rate of idiosyncratic shocks, the growth rate may decrease, subject to a transition probability denoted as p_1. As shown in Fig. 4.2, the certain equivalent wealth $f_5(P)$ is an increasing and convex function of P. Comparing the setting of $(p_1, p_2) = (0.05, 0.05)$ and $(p_1, p_2) = (0, 0.05)$, it is obvious that the $f_5(P)$ is larger when $p_2 = 0.05$. A larger p_2 suggests a higher probability of transitioning from low to high growth, which leads to a higher average growth rate and results in a larger $f_5(P)$. Fig. 4.3 illustrates that the impact of belief P on the benefit rate $b^*(t)$. We can see that the value of $b^*(t)$ increases, when the belief P increases. A larger in P indicates a higher probability of a high growth rate. The greater the fund manager's confidence in the future growth rate of members' labor income, the more likely members will be assigned a higher benefit rate. Values in the figure are relatively small, which may due to the specific parameterization of our model.

Follow the work of [2], we also use the analytic solutions for $\mathbb{E}_t[\bar{y}_s]$ and $\mathbb{E}_t[\bar{L}_s]$, $s \in [t, T]$. To simulate the value of $\bar{y}(t)$ and $\bar{L}(t)$. Labor income $\bar{L}(t)$ is assumed to be normally distributed, implying that

$$\mathbb{E}_t[\bar{L}(s)] = e^{\mathbb{E}_t[\bar{l}(s)] + 0.5\mathbb{V}ar_t[\bar{l}(s)]},$$

where

$$\mathbb{E}_t[\bar{l}_s] = \bar{l}_0 - \bar{y}_0(1 - e^{-ks}) + (g_D - \bar{\lambda} - \frac{\sigma^2}{2})s,$$

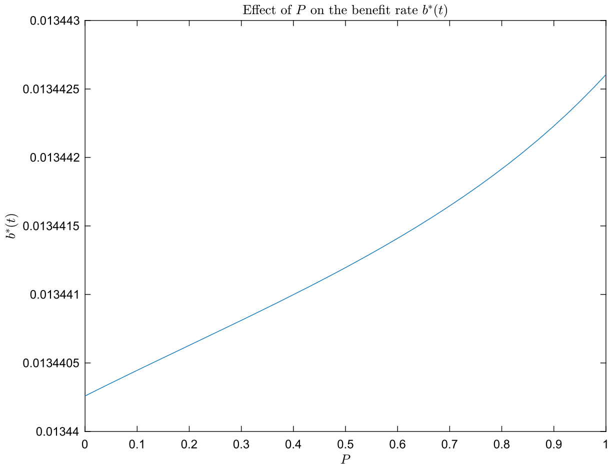


Fig. 4.3. Effect of P on the benefit rate.

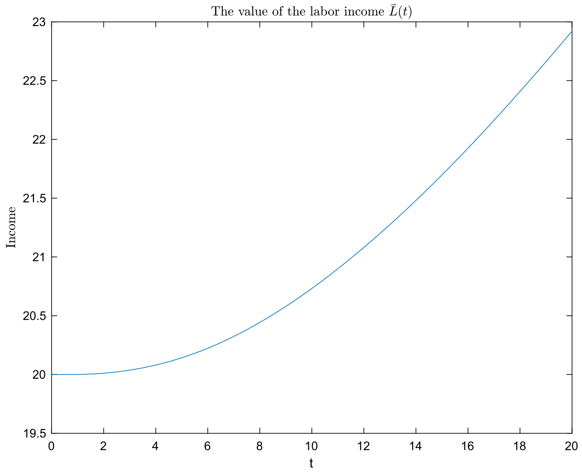


Fig. 4.4. The values of the labor income with time t.

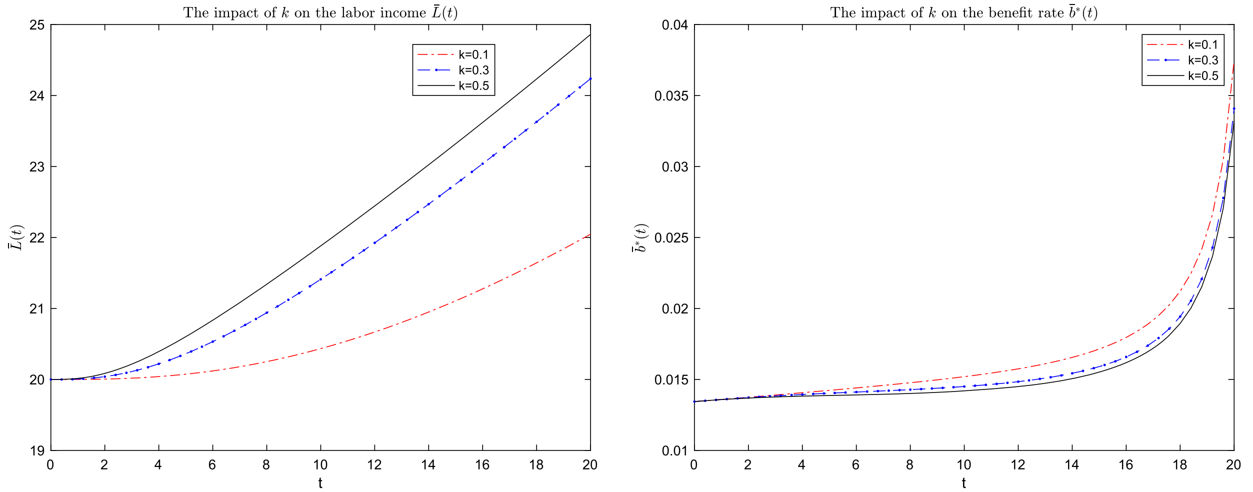
$$\mathbb{V}ar_t[\bar{l}_s] = \upsilon_D^2[s - \frac{1}{2k}(3 - e^{-ks})(1 - e^{-kt})] + (\sigma - \upsilon_D)^2 s + 2[\upsilon_D(\sigma - \upsilon_D)(s - \frac{1}{k}(1 - e^{-ks}))].$$

Fig. 4.4 depicts the deterministic labor income pattern $\bar{L}(t)$ with time t. When optimizing social welfare, the impact of labor income has to be considered by the pension fund manager. It is obvious that the trend of the labor income is on the rise over time t, which means that the salary received by the labor is higher and higher as time goes by. The trend of the labor income is similar to the simulation path with time t.

Fig. 4.5 shows the effect of the mean reversion coefficient on the salary and benefit payment rate for different co-integration controlled by the parameter k. The parameter measures the speed or slowness with which labor income returns to its average, offering insights into the stability or volatility of income over time. From Fig. 4.5.1, we see that for any fixed time, labor income increases as k increases. That is because when k increases, labor income becomes more 'stock like'. The parameter k determines the speed of the variable $\bar{y}(t)$ towards the long-run mean and captures the co-integrated time scale, which is tied up in labor income and dividends. In addition, co-integration means long-run dependence thus, labor income does not change in the short time with different levels of the co-integration. Fig. 4.5.2 shows that the benefit adjustment payment rate and the growth of k are reduced. A lower benefit payout rate $\bar{b}^*(t)$ means more money is left in the fund. The increasing labor income leads to a downward benefit replacement payment ratio, which captures the rate of a pension system to retirement income.

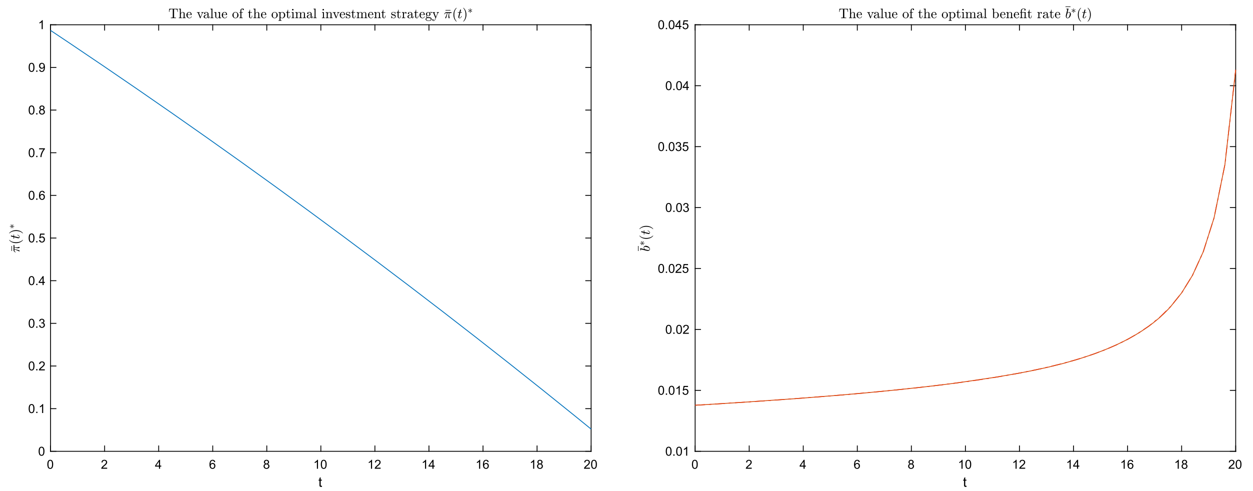
Fig. 4.6 illustrates the change of the optimal values $\bar{\pi}^*(t)$ and $\bar{b}^*(t)$ at each time point, respectively. We observe from Fig. 4.6.1 that the optimal investment allocation $\bar{\pi}^*(t)$ decreases, implying that the proportion invested in the risky asset is less over the given period. It seems to make sense that a younger decision maker invests heavily in a risky asset, which coincides the investment wisdom. It can also be seen that Fig. 4.6.2 shows that the benefit adjustment payout rate $\bar{b}^*(t)$ has an upward trend, which demonstrates that there exists an increasing payment ratio given by the fund manager as time moves.

Now, we study optimal investment and benefit strategies concerning the risk-aversion parameter γ_1. Fig. 4.7 shows how the coefficient of risk-aversion parameter γ_1 impacts the optimal strategies. In Fig. 4.7.1, we observe that $\bar{\pi}^*$ decreases with an increasing



4.5.1 4.5.2

Fig. 4.5. The impacts of k on the labor income and benefit rate.



4.6.1 4.6.2

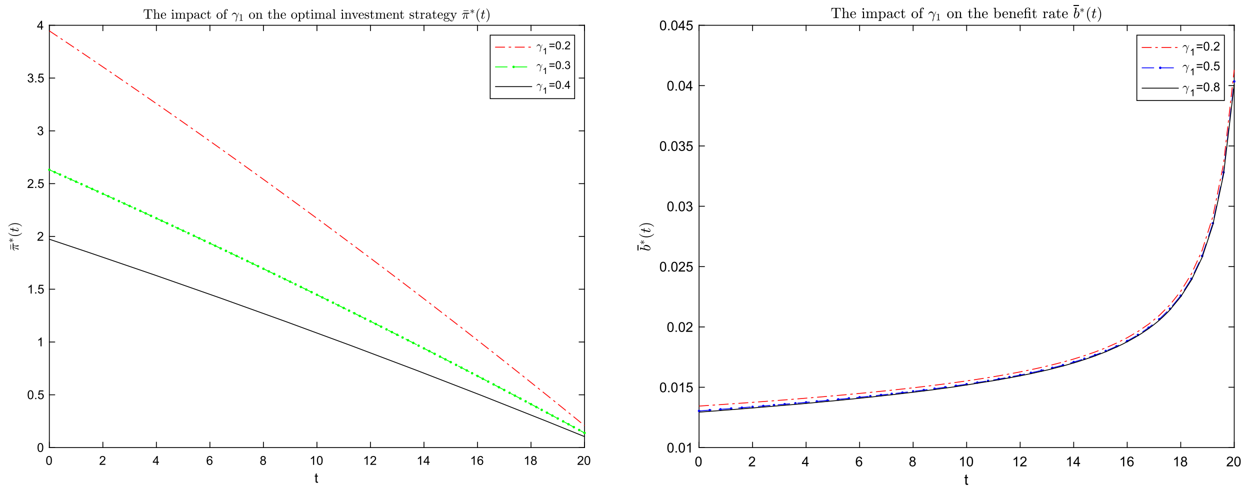
Fig. 4.6. The values of optimal strategies.

coefficient of the risk-aversion parameter, which means that a lower and lower proportion is being invested in the risky asset. A larger γ_1 implies that the fund manager is more risk averse and he will be cautious about actively investing in the risky asset when making investment decisions. In Fig. 4.7.2, we see that the value of the optimal benefit adjustment rate almost does not change with different risk-aversion parameters γ_1, demonstrating that the profit received by pension members can not be changed with different levels of the risk preference investor.

5. Conclusion

This paper focuses on a collective defined contribution (CDC) pension fund scheme in continuous time where the stock is co-integrated with labor income. We assume that member's idiosyncratic shocks whose constant growth rate is unknown, which is modeled by a continuous-time two-state hidden Markov chain. Using the standard HJB equation, we obtain the closed-form expressions for optimal investment strategy and benefit adjustment policy. Moreover, we provide numerical examples to show that how parameters impact on optimal strategies. To be more specific, we find that the level of the belief P impact on the benefit payment rate. The greater the fund manager's confidence in the future growth rate of members' labor income, the more likely members will be assigned a higher benefit rate.

It would be interesting to extend our model by considering human capital. When introducing human capital, the optimal investment strategy of the fund manager may change from different levels of co-integration. Labor income is "stock-like" in the long run when co-integration is firm, which reduces a fund manager's appetite to invest in the risky asset. In this way, we need a more



4.7.1 4.7.2

Fig. 4.7. The impacts of γ_1 on optimal strategies.

complicated technique to obtain explicit expressions for the equilibrium strategies. In addition, it would be meaningful to study this optimal problem under constant relative risk aversion (CRRA) utility function. CRRA models account for the idea that an individual's willingness to take risks varies with relative changes in wealth rather than absolute amounts, which adds significant complexity and reflects a more realistic scenario in economic decisions.

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Appendix A. Proof of Theorem 3.5

Proof. From the HJB equation (3.26), we easily get the first-order conditions for the two controls are

$$\begin{aligned}
&0=\bar{w}\bar{V}_{\bar{w}}(\mu-r)+\bar{w}^2\bar{V}_{\bar{w}\bar{w}}\sigma^2\bar{\pi}+\bar{w}\bar{l}\bar{V}_{\bar{w}\bar{l}}\sigma(\sigma-v_D)-\bar{w}\bar{V}_{\bar{w}\bar{y}}\sigma v_D,\\
&0=-\bar{V}_{\bar{w}}F(t)\bar{l}+F(t)\bar{l}e^{-rt}e^{-\gamma_1F(t)\bar{l}\bar{b}}.
\end{aligned}$$

Thus, the optimal investment and benefit strategies are given by

$$\begin{aligned}
&\bar{\pi}^*=\frac{\bar{V}_{\bar{w}\bar{y}}\sigma v_D-\bar{V}_{\bar{w}}(\mu-r)-\bar{l}\bar{V}_{\bar{w}\bar{l}}\sigma(\sigma-v_D)}{\bar{w}\bar{V}_{\bar{w}\bar{w}}\sigma^2},\\
&\bar{b}^*=\frac{\ln\bar{V}_{\bar{w}}+rt}{-\gamma_1F(t)\bar{l}}.
\end{aligned}\tag{A.1}$$

We assume that the value function of the fund manager could be written in the following form:

$$\bar{V}(t,\bar{w},\bar{l},\bar{y})=-\frac{\lambda_1}{\gamma_1}\exp\{-\gamma_1[\bar{g}_1(t)\bar{w}+\bar{g}_2(t)+\bar{g}_3(t,\bar{l})+\bar{g}_4(t)\bar{y}]-rt\},$$

where $\bar{g}_1(t),\bar{g}_2(t),\bar{g}_3(t)$ and $\bar{g}_4(t)$ are functions of t to be determined. The terminal conditions are $\bar{g}_1(T)=1$, $\bar{g}_2(T)=\bar{g}_3(T)=\bar{g}_4(T)=0$.

From the expression of the value function, we obtain

$$\begin{aligned}
&\bar{V}_t=-\gamma_1\bar{V}[\bar{g}_{1t}\bar{w}+\bar{g}_{2t}+\bar{g}_{3t}(t,\bar{l})+\bar{g}_{4t}\bar{y}+\frac{r}{\gamma_1}],\\
&\bar{V}_{\bar{w}}=-\gamma_1\bar{g}_1\bar{V},\qquad \bar{V}_{\bar{w}\bar{w}}=\gamma_1^2\bar{g}_1^2\bar{V},\\
&\bar{V}_{\bar{l}}=-\gamma_1\bar{g}_3'\bar{V},\qquad \bar{V}_{\bar{l}\bar{l}}=\gamma_1^2\bar{g}_3'\bar{g}_3'\bar{V}-\gamma_1\bar{g}_3''\bar{V},\\
&\bar{V}_y=-\gamma_1\bar{g}_4\bar{V},\qquad \bar{V}_{yy}=0,\\
&\bar{V}_{wl}=\gamma_1^2\bar{g}_1\bar{g}_3'\bar{V},\quad \bar{V}_{wy}=\gamma_1^2\bar{g}_1\bar{g}_4\bar{V},\\
&\bar{V}_{ly}=\gamma_1^2\bar{g}_3'\bar{g}_4\bar{V},
\end{aligned}\tag{A.2}$$

where $\bar{g}_3'=\frac{\partial\bar{g}_3}{\partial\bar{l}}$ and $\bar{g}_3''=\frac{\partial^2\bar{g}_3}{\partial\bar{l}^2}$. Plugging (A.2) and (A.1) into the HJB equation (3.26), we get

$$
\begin{aligned}
0 = & \left[-\ln \lambda_1 - \ln \bar{g}_1(t) + \gamma_1(\bar{g}_1(t)\bar{w} + \bar{g}_2(t) + \bar{g}_3(t,\bar{l}) + \bar{g}_4(t)\bar{y}) \right] \bar{g}_1(t) \\
& + \bar{g}_1(t) - \gamma_1 \left[\bar{g}_{1t}\bar{w} + \bar{g}_{2t} + \bar{g}_{3t}(t,\bar{l}) + \bar{g}_{4t}\bar{y} + \frac{r}{\gamma_1} \right] - \gamma_1 \bar{g}_1(t)\bar{w}r \\
& - \frac{(\mu - r)\left[\gamma_1 \bar{g}_4(t)\sigma v_D + \mu - r - \bar{l}\gamma_1^2 \bar{g}_3' \sigma(\sigma - v_D)\right]}{\sigma^2} - \gamma_1 \bar{g}_1(t)C(t) \\
& + \frac{\left[\gamma_1 \bar{g}_4(t)\sigma v_D + \mu - r - \gamma_1^2 \bar{g}_3(t)\sigma(\sigma - v_D)\right]^2}{2\sigma^2} \\
& - \bar{l}\gamma_1 \bar{g}_3' \left[-k\bar{y} + g_D + \bar{\lambda} - \frac{1}{2}\sigma^2 + \frac{1}{2}(\sigma - v_D)^2 \right] + \frac{1}{2}\bar{l}^2 \left[\gamma_1^2 \bar{g}_3' \bar{g}_3' - \gamma_1 \bar{g}_3''\right](\sigma - v_D)^2 \\
& - \gamma_1 \bar{g}_4(t)\bar{y}k + \frac{\left[\gamma_1 \bar{g}_3(\sigma - v_D)\right]\left[\gamma_1 \bar{g}_4(t)\sigma v_D + \mu - r - \bar{l}\gamma_1^2 \bar{g}_3' \sigma(\sigma - v_D)\right]}{\sigma} \\
& - \frac{\gamma_1 \bar{g}_4(t) v_D \left[\gamma_1 \bar{g}_4(t)\sigma v_D + \mu - r - \bar{l}\gamma_1^2 \bar{g}_3' \sigma(\sigma - v_D)\right]}{\sigma} \\
& - \bar{l}\gamma_1^2 \bar{g}_3' \bar{g}_4(t) v_D(\sigma - v_D).
\end{aligned}
\tag{A.3}
$$

To solve the SDE, we assume that $\bar{g}_3(t,l) = \bar{g}_3(t)\ln \bar{l}$. Note that (A.3) can be taken as a sum of four differential equations, it follows that

$$
\begin{cases}
\bar{g}_{1t} - \bar{g}_1(t)^2 + \bar{g}_1(t)r = 0, \\
\bar{g}(T) = 1.
\end{cases}
\tag{A.4}
$$

$$
\begin{cases}
\bar{g}_{3t} - \bar{g}_1(t)\bar{g}_3(t) = 0, \\
\bar{g}_3(T) = 0.
\end{cases}
\tag{A.5}
$$

$$
\begin{cases}
\bar{g}_{4t} - \bar{g}_1(t)\bar{g}_4(t) + \bar{g}_4(t)k + k\bar{g}_3(t) = 0, \\
\bar{g}_4(T) = 0.
\end{cases}
\tag{A.6}
$$

$$
\begin{cases}
\bar{g}_{2t} + \frac{1}{\gamma_1}\left[\ln \lambda_1 + \ln \bar{g}_1(t) - \frac{1}{\gamma_1}\right]\bar{g}_1(t) - \bar{g}_2(t)\bar{g}_1(t) - \frac{(\mu - r)^2}{\gamma_1 \sigma^2} + \bar{g}_1(t)C(t) + \frac{r}{\gamma_1} \\
+ \frac{(\mu - r)\left[\bar{g}_4(t)\sigma v_D - \gamma_1 \bar{g}_3(t)\sigma(\sigma - v_D)\right]}{\sigma^2} - \frac{\left[\gamma_1 \bar{g}_4(t)\sigma v_D + \mu - r - \gamma_1^2 \bar{g}_3(t)\sigma(\sigma - v_D)\right]^2}{2\sigma^2 \gamma_1} \\
+ \bar{g}_3(t)\left[g_D + \bar{\lambda} - \frac{1}{2}\sigma^2 + \frac{1}{2}(\sigma - v_D)^2\right] - \frac{1}{2}\left[\gamma_1 \bar{g}_3^2(t) + \bar{g}_3(t)\right](\sigma - v_D)^2 \\
- \frac{\left[\bar{g}_3(\sigma - v_D)\right]\left[\gamma_1 \bar{g}_4(t)\sigma v_D + \mu - r - \gamma_1^2 \bar{g}_3(t)\sigma(\sigma - v_D)\right]}{\sigma} - \frac{\bar{g}_4(t) v_D \left[\gamma_1 \bar{g}_4(t)\sigma v_D + \mu - r - \gamma_1 \bar{g}_3(t)\sigma(\sigma - v_D)\right]}{\sigma} \\
+ \gamma_1 \bar{g}_3(t)\bar{g}_4(t) v_D(\sigma - v_D) = 0, \\
\bar{g}_2(T) = 0.
\end{cases}
\tag{A.7}
$$

According to the expression in (A.5), we can easily get $\bar{g}_3(t) = 0$. Solving A.6 gives $\bar{g}_4(t) = 0$. Hence, the form functions $\bar{g}_1(t)$ and $\bar{g}_2(t)$ follow from (A.4) and (A.7). Then, using the standard methods of [10], we show the sufficiency of the optimal investment strategy and optimal benefit rate.

Let $\widehat{W} \in C^{1,2,2}$ be a classical solution to (3.26) that satisfies (3.27), then the value function $\bar{V}$ given by (3.24) coincides with $\widehat{W}$. That is

$$
\widehat{W}(t,\bar{w},\bar{l},\bar{y}) = \bar{V}(t,\bar{w},\bar{l},\bar{y}).
\tag{A.8}
$$

Furthermore, set the pair $(\bar{\pi}^*, \bar{b}^*)$ such that

$$
\begin{aligned}
0 = \sup_{\bar{\pi}^*, \bar{b}^*} \Big\{ & \bar{V}_t + \bar{w}\bar{V}_{\bar{w}}[r + (\mu - r)\bar{\pi}^* + \frac{C(t)}{\bar{w}} - \frac{F(t)\bar{l}\bar{b}^*}{\bar{w}}] + \frac{1}{2}\bar{w}^2 V_{\bar{w}\bar{w}}\sigma^2 \bar{\pi}^{*2} \\
& + \bar{l}\bar{V}_{\bar{l}}[-k\bar{y} + g_D + \bar{\lambda} - \frac{1}{2}\sigma^2 + \frac{1}{2}(\sigma - v_D)^2] + \frac{1}{2}\bar{l}^2 \bar{V}_{\bar{l}\bar{l}}(\sigma - v_D)^2 \\
& - k\bar{y}\bar{V}_{\bar{y}} + \frac{1}{2}\bar{V}_{\bar{y}\bar{y}} v_D^2 + \bar{w}\bar{l}\bar{V}_{\bar{w}\bar{l}}\bar{\pi}^*\sigma(\sigma - v_D) - \bar{w}\bar{V}_{\bar{w}\bar{y}}\bar{\pi}^*\sigma v_D - \bar{l}\bar{V}_{\bar{l}\bar{y}} v_D(\sigma - v_D) \\
& - e^{-rt}\frac{1}{\gamma_1} e^{-\gamma_1 F(t)\bar{l}\bar{b}^*} \Big\},
\end{aligned}
$$

holds for all $(t,\bar{w},\bar{l},\bar{y}) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$. Then $(\bar{\pi}^*(t), \bar{b}^*(t))$ is the optimal strategy pair.

□

Data availability

No data was used for the research described in the article.

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