Computer Vision, Assignment 3
Epipolar Geometry

1   Instructions

In this assignment you study epipolar geometry. You will use the Fundamental matrix and the Essential matrix for simultaneously reconstructing the structure and the camera motion from two images. The data for the assignments is available from the course page https://canvas.education.lu.se/courses/21938

The assignment is due at the end (Sunday 26/2 at midnight) of study week 6. The deadline is firm and reports are to be handed in on time through the canvas page. Not everything has to be correct the first time you hand in however you have to present solutions/attempts for each mandatory exercise. It is not ok to hand in blank solutions. (If you have problems solving something you should come to the Q & A-sessions or contact the lecturer by email in good time before the deadline.) In exceptional cases extensions can be offered (due to unforeseen circumstances). In such cases contact the lecturer by email before the deadline (or as soon as possible) for instructions on what to do.

Make sure you answer all questions and provide complete solutions to the exercises. Your solutions should be be submitted as a single pdf-file. Note that it is fine to submit handwritten solutions (by including scans in the pdf-file) as long as they are well structured and readable. After each exercise there is a gray box with instructions on what should be included in the report. In addition, all the code should be submitted as m- and mat-files in a zip-archive. Make sure that your matlab scripts are well commented and can be executed directly, that is, without loading any data, setting parameters etc. Such things should be done in the script.

If you run into problems with any of the exercises you can go to the Q & A-sessions to get help (see course schedule) or send an email to the lecturer (viktor.larsson@math.lth.se).

The report should be written individually, however you are encouraged to work together. Keep in mind that everyone is responsible for their own report and should be able to explain all the solutions.

Some exercises are marked as OPTIONAL. You do not have to do these to pass the assignment. However if you submit good solutions to these you will be awarded at most 0.2 bonus points for the home-exam.

2   The Fundamental Matrix

Exercise 1. If $P_1 = [I \ 0]$ and

$$
P_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{1}
$$

Compute the fundamental matrix.

Suppose that the point $x = (1, 1)$ is the projection of a 3D-point $\mathbf{X}$ into P_1. Compute the epipolar line in the second image generated from x.

Which of the points $(2, 0)$,$(2, 1)$ and $(4, 2)$ could be a projection of the same point $\mathbf{X}$ into P_2?

> For the report: Answers are enough.

Exercise 2. If $P_1 = [I \ 0]$ and

$$
P_2 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2}
$$

Compute the epipoles, by projecting the camera centers.

Compute the fundamental matrix, its determinant and verify that $e_2^T F = 0$ and $Fe_1 = 0$.

The rest of this exercise is OPTIONAL.
For a general camera pair $P_1 = [I \ 0]$ and $P_2 = [A \ t]$. Compute the epipoles, by projecting the camera centers. (You may assume that A is invertible.)

Verify that for the fundamental matrix $F = [t]_\times A$ the epipoles will always fulfill $e_2^T F = 0$ and $Fe_1 = 0$.

Given the above result explain why the fundamental matrix has to have determinant 0.

> For the report: Complete solution, except for the optional parts. If you want bonus points you also have to submit solutions the optional parts.

Exercise 3. When computing the fundamental matrix F using the 8-point algorithm it is recommended to use normalization. Suppose the image points have been normalized using

$$
\tilde{\mathbf{x}}_1 \sim N_1 \mathbf{x}_1 \text{ and } \tilde{\mathbf{x}}_2 \sim N_2 \mathbf{x}_2. \tag{3}
$$

If $\tilde{F}$ fulfills $\tilde{\mathbf{x}}_2^T \tilde{F} \tilde{\mathbf{x}}_1 = 0$ what is the fundamental matrix F that fulfills $\mathbf{x}_2^T F \mathbf{x}_1 = 0$ for the original (un-normalized) points?

> For the report: Answer is enough.

Computer Exercise 1. In this exercise you will compute the fundamental matrix for the two images in Figure 1 of a part of the fort Kronan in Gothenburg.

Figure 1: kronan1.jpg and kronan2.jpg.

The file `compEx1data.mat` contains a cell `x` with matched points for the two images.

First compute normalization matrices N_1 and N_2. These matrices should subtract the mean and re-scale using the standard deviation, as in Assignment 2. Normalize the image points of the two images with N_1 and N_2 respectively.

Set up the matrix M in the eight point algorithm (use all the points), and solve the homogeneous least squares system using SVD. Check that the minimum singular value and $||Mv||$ are both small.

Construct the normalized fundamental matrix from the solution v. Don't forget to make sure that $\det(\tilde{F}) = 0$ for your solution. Check that the epipolar constraints $\tilde{\mathbf{x}}_2^T \tilde{F} \tilde{\mathbf{x}}_1 = 0$ are roughly fulfilled.

Compute the un-normalized fundamental matrix F (using the formula from exercise 3) and the epipolar lines $l = F\mathbf{x}_1$. Pick 20 points in the second image at random and plot these in the same figure as the image. Also plot the corresponding epipolar lines in the same image using the function `rital.m`. Are they close to each other?

Compute the distance between all the points and their corresponding epipolar lines and plot these in a histogram with 100 bins. What is the mean distance?

Try what happens if you do everything without normalization (that is, set $N_1 = N_2 = I$ and run the code again). What is the mean distance in this case?

```
Useful matlab commands:

xx = x2n(:,i)*x1n(:,i)'; %Computes a 3x3 matrix containing all multiplications
%of coordinates from x1n(:,i) and x2n(:,i).
M(i,:) = xx(:)'; %Reshapes the matrix above and adds to the M matrix

Fn = reshape(v,[3 3]);
%Forms an F-matrix from the solution v of the least squares problem

plot(diag(x2n'*Fn*x1n));
%Computes and plots all the epipolar constraints (should be roughly 0)

l = F*x{1};   %Computes the epipolar lines
l = l./sqrt(repmat(l(1,:).^2 + l(2,:).^2,[3 1]));
%Makes sure that the line has a unit normal
%(makes the distance formula easier)
hist(abs(sum(l.*x{2})),100);
%Computes all the the distances between the points
%and there corresponding lines, and plots in a histogram
```

> For the report: Submit the m-file, the fundamental matrix for the original (un-normalized) points (where $F(3,3) = 1$, use $F = F./F(3,3)$), the histogram and the plot of the epipolar lines. Also submit the the mean epipolar distances obtained when F is computed both with and without normalization.

Exercise 4. Consider the fundamental matrix

$$F = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Verify that the projection of the scene points $(1, 2, 3)$ and $(3, 2, 1)$ in the cameras $P_1 = [I \ 0]$ and $P_2 = [[e_2]_\times F \ e_2]$, fulfill the epipolar constraint $(x_2^T F x_1 = 0)$. What is the camera center of P_2?

> For the report: Complete solution.

Computer Exercise 2. Use the fundamental matrix F that you obtained in Computer Exercise 1 to compute the camera matrices in Exercise 4. Also use triangulation (with DLT) to compute the 3D-points. Plot both the image, the image points, and the projected 3D points in the same figure.

Don't forget to normalize when triangulating. Note that since the point sets are the same as in Computer Exercise 1 the normalization matrices will also be the same. (Alternatively one could

compute cameras and 3D points from the fundamental matrix $\tilde{F}$ obtained with the normalized points and transform the cameras afterwards. This also gives a valid solution, but it is a different one.)

Plot the 3D-points in a 3D plot. Does it look like you expected? (Recall Computer Exercise 1 in Assignment 2...)

Is the solution you have obtained Quasi Affine with respect to the true point cloud? If not switch columns 3 and 4 of the camera matrices and rows 3 and 4 of the the 3D point coorindates and plot again.

```
Useful matlab commands:

e2 = null(F'); %Computes the epipole
e2x = [0 -e2(3) e2(2); e2(3) 0 -e2(1); -e2(2) e2(1) 0];
%Constructs the cross product matrix
```

For the report: Submit the m-file, the 2D plot and the camera matrices.

3   The Essential Matrix

Exercise 5. (OPTIONAL.) The goal of this exercise is to show that an essential matrix has two nonzero identical singular values.

Suppose the 3×3 skew symmetric matrix $[t]_\times$ has a singular value decomposition

$$[t]_\times = USV^T, \tag{4}$$

where U, V are orthogonal and S diagonal with non-negative elements. Show that the eigenvalues of $[t]_\times^T[t]_\times$ are the squared singular values. (Hint: Show that $S^TS = S^2$ is the diagonalization of $[t]_\times^T[t]_\times$, see your linear algebra book.)

Verify that the eigenvectors of $[t]_\times^T[t]_\times$ fulfill

$$-t \times (t \times w) = \lambda w. \tag{5}$$

In "Linjär Algebra" by Sparr (on page 96) we find the formula

$$u \times (v \times w) = (u \cdot w)v - (u \cdot v)w. \tag{6}$$

Show that $w = t$ is an eigenvector to $[t]_\times^T[t]_\times$ with eigenvalue 0 and that any w that is perpendicular to t is an eigenvector with eigenvalue $||t||^2$. Are these all of the eigenvectors?

Show that the singular values of $[t]_\times$ are $0, ||t||$ and $||t||$.

If $E = [t]_\times R$ and $[t]_\times$ has the svd in (4), state an svd of E. What are the singular values of E?

For the report: Complete solution if you want bonus, otherwise nothing.

Computer Exercise 3. The file `compEx3data.mat` contains the calibration matrix K for the two images in Computer Exercise 1.

Normalize the image points using the inverse of K.

Set up the matrix M in the eight point algorithm, and solve the homogeneous least squares system using SVD. Check that the minimum singular value and Mv are both small.

Construct the Essential matrix from the solution v. Don't forget to make sure that E has two equal singular values and the third one zero. Check that the epipolar constraints $\tilde{x}_2^T E \tilde{x}_1 = 0$ are roughly fulfilled.

Compute the fundamental matrix for the un-normalized coordinate system from the essential matrix and compute the epipolar lines $l = F\mathbf{x}_1$. Pick 20 of the detected points in the second image at random

and plot these in the same figure as the image. Also plot the corresponding epipolar lines in the same figure using the function `rital.m`.

Compute the distance between the points and their corresponding epipolar lines and plot these in a histogram with 100 bins. How does this result compare to the corresponding result Computer Exercise 1?

```
Useful matlab commands:

[U,S,V] = svd(Eapprox);
if det(U*V')>0
    E = U*diag([1 1 0])*V';
else
    V = -V;
    E = U*diag([1 1 0])*V';
end
% Creates a valid essential matrix from an approximate solution.
% Note: Computing svd on E may still give U and V that does not fulfill
% det(U*V') = 1 since the svd is not unique.
% So don't recompute the svd after this step.
```

For the report: Submit the m-file, the Essential matrix (where $E(3,3) = 1$, use $E = E./E(3,3)$), the histogram and the plot of the epipolar lines.

Exercise 6. An essential matrix has the singular value decomposition

$$E = U\mathrm{diag}([1\ 1\ 0])V^T \tag{7}$$

where

$$U = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{8}$$

Verify that $\det(UV^T) = 1$.

Compute the essential matrix and verify that $x_1 = (0,0)$ (in camera 1) and $x_2 = (1,1)$ (in camera 2) is a plausible correspondence.

If x_1 is the projection of $\mathbf{X}$ in $P_1 = [I\ 0]$ show that $\mathbf{X}$ must be one of the points

$$\mathbf{X}(s) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ s \end{pmatrix}. \tag{9}$$

For each of the solutions

$$P_2 = [UWV^T\ u_3] \text{ or } [UWV^T\ -u_3] \text{ or } [UW^TV^T\ u_3] \text{ or } [UW^TV^T\ -u_3], \tag{10}$$

where

$$W = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

and u_3 is the third column of U, compute s such that $\mathbf{X}(s)$ projects to x_2.

For which of the camera pairs is the 3D point $\mathbf{X}(s)$ in front of both cameras?

For the report: Complete solution.

Computer Exercise 4. For the essential matrix obtained in Computer Exercise 3 compute four camera solutions in (10).

Triangulate the points using DLT for each of the four camera solutions, and determine for which of the solutions the points are in front fo the cameras. Since there is noise involved it might not be possible to find a solution with all points in front of the cameras. In that case select the one with the highest number of points in front of the cameras. (Note that the SVD of E is not unique. Different solutions may result in permutations of the order of the solutions for P_2. Therefore the code has to determine the number of points in front of the cameras to find the right solution.)

Compute the corresponding camera matrices for the original (un-normalized) coordinate system and plot the image the points and the projected 3D-points in the same figure. Do the errors look small?

Plot the 3D points and camera centers and principal axes in a 3D plot. Does it look like you expected it to? (Recall Exercise 2 in Assignment 2...)

For the report: Submit the m-file and the plots.