Computer Vision, Assignment 1
Elements of Projective Geometry

1   Instructions

In this assignment you will study the basics of projective geometry. You will study the representations of points lines and planes, as well as transformations and camera matrices. The data for the assignments is available from the course page `https://canvas.education.lu.se/courses/21938`

The assignment is due at the end (Sunday 29/1 at midnight) of study week 2. The deadline is firm and reports are to be handed in on time through the canvas page. Not everything has to be correct the first time you hand in however you have to present solutions/attempts for each mandatory exercise. It is not ok to hand in blank solutions. (If you have problems solving something you should come to the Q & A-sessions or contact the lecturer by email in good time before the deadline.) In exceptional cases extensions can be offered (due to unforeseen circumstances). In such cases contact the lecturer by email before the deadline (or as soon as possible) for instructions on what to do.

Make sure you answer all questions and provide complete solutions to the exercises. Your solutions should be be submitted as a single pdf-file. Note that it is fine to submit handwritten solutions (by including scans in the pdf-file) as long as they are well structured and readable. After each exercise there is a gray box with instructions on what should be included in the report. In addition, all the code should be submitted as m- and mat-files in a zip-archive. Make sure that your matlab scripts are well commented and can be executed directly, that is, without loading any data, setting parameters etc. Such things should be done in the script.

If you run into problems with any of the exercises you can go to the Q & A-sessions to get help (see course schedule) or send an email to the lecturer (`viktor.larsson@math.lth.se`).

The report should be written individually, however you are encouraged to work together. Keep in mind that everyone is responsible for their own report and should be able to explain all the solutions.

Some exercises are marked as OPTIONAL. You do not have to do these to pass the assignment. However if you submit good solutions to these you will be awarded at most 0.3 bonus points for the home-exam.

2 Points in Homogeneous Coordinates.

Exercise 1. What are the 2D Cartesian coordinates of the points with homogeneous coordinates

$$\mathbf{x}_1 = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} \text{ and } \mathbf{x}_3 = \begin{pmatrix} 4\lambda \\ -2\lambda \\ 2\lambda \end{pmatrix}, \lambda \neq 0? \tag{1}$$

What is the interpretation of the point with homogeneous coordinates

$$\mathbf{x}_4 = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}? \tag{2}$$

For the report: Answers are enough.

Computer Exercise 1. Write a matlab function `pflat` that divides the homogeneous coordinates with their last entry for points of any dimensionality. (You may assume that none of the points have last homogeneous coordinate zero.) Note that the function should output a matrix of the same size as the input. For example, if `x2D` is a matrix with 3 rows, then the output of `pflat(x2D)` should also have three rows. Apply the function to the points in `x2D` and `x3D` in the file `compEx1.mat`, and plot the result.

```
Useful matlab commands:

x(end,:) %Extracts the last row of x

repmat(a,[m n]) %Creates a block matrix with mn copies of a.

a./b %Elementwise division.
%Divides the elements of a by the corresponding element of b.

plot(a(1,:),a(2,:),'.')   %Plots a point at (a(1,i),a(2,i)) for each i.

plot3(a(1,:),a(2,:),a(3,:),'.') %Same as above but 3D.

axis equal %Makes sure that all axes have the same scale.
```

For the report: Submit the m-file, and the plots.

3 Lines

Exercise 2. Compute the homogeneous coordinates of the intersection (in $\mathbb{P}^2$) of the lines

$$l_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } l_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}. \tag{3}$$

What is the corresponding point in $\mathbb{R}^2$?

Compute the intersection (in $\mathbb{P}^2$) of the lines

$$l_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } l_4 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \tag{4}$$

What is the geometric interpretation in $\mathbb{R}^2$?

Compute the line that goes through the points with Cartesian coordinates

$$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } x_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}. \tag{5}$$

Hint: Re-use the calculations from the line intersections above.

> For the report: Submit a complete solution.

Exercise 3. The nullspace of an $m \times n$ matrix A is the set

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n; Ax = 0\}, \tag{6}$$

that is, all the x for which the multiplication Ax gives the zero vector. Explain why the intersection point (in homogeneous coordinates) of l_1 and l_2 (from Exercise 2) is in the null space of the matrix

$$M = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{7}$$

The equation $Mx = 0$ has infinitely many solutions, that is, there are many vectors in the nullspace of M. If we interpret the (non-zero) vectors as points in $\mathbb{P}^2$ is there any other point besides the intersection point?

> For the report: Answer all the questions.

Computer Exercise 2. Load and plot the image in `compEx2.jpg` (see Figure 1).

Figure 1: compEx2.jpg

In the file `compEx2.mat` there are three pairs of image points. Plot the image points in the same figure as the image.

For each pair of points compute the line going through the points. Use the function `rital` to plot the lines in the same image. Do these lines appear to be parallel (in 3D)?

Compute the intersection of the second and third line (the lines obtained from the pairs `p2` and `p3`). Plot this point in the same image.

The distance between a 2D-point $x = (x_1, x_2)$ in Cartesian coordinates and a line $l = (a, b, c)$ can be computed using the distance formula

$$d = \frac{|ax_1 + bx_2 + c|}{\sqrt{a^2 + b^2}}, \tag{8}$$

see your linear algebra book. Compute the distance between the first line and the the intersection point. Is it close to zero? Why/why not?

```
Useful matlab commands:

imread('compEx2.JPG') %Loads the image compEx2.JPG

imagesc(im) %Displays the image

colormap gray %changes the colormap of the current image to gray scale

hold on %Prevents the plot command from clearing the figure before plotting

hold off %Makes the plot command clear the figure before plotting

null(A) %computes the nullspace of A
```

For the report: Submit the m-file, the plot, and the computed distance between the intersection point and the first line.

4   Projective Transformations

Exercise 4. Let H be the projective transformation

$$H = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{9}$$

Compute the transformations $\mathbf{y}_1 \sim H\mathbf{x}_1$ and $\mathbf{y}_2 \sim H\mathbf{x}_2$ if

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \tag{10}$$

Compute the lines l_1, l_2 containing $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{y}_1, \mathbf{y}_2$ respectively.

Compute $(H^{-1})^T l_1$ and compare to l_2.

Show that projective transformations preserve lines. That is, for each line l_1 there is a corresponding line l_2 such that if $\mathbf{x}$ belongs to l_1 then the transformation $\mathbf{y} \sim H\mathbf{x}$ belongs to l_2. (Hint: If $l_1^T \mathbf{x} = 0$ then $l_1^T H^{-1} H \mathbf{x} = 0$.)

For the report: Submit the answers and the proof.

Computer Exercise 3. The file `compEx3.mat` contains the start and end points of a set of lines. Plotting the lines gives the grid in Figure 2.

For each of the projective mappings given by the matrices

$$H_1 = \begin{pmatrix} \sqrt{3} & -1 & 1 \\ 1 & \sqrt{3} & 1 \\ 0 & 0 & 2 \end{pmatrix}, \; H_2 = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{11}$$

$$H_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } H_4 = \begin{pmatrix} \sqrt{3} & -1 & 1 \\ 1 & \sqrt{3} & 1 \\ 1/4 & 1/2 & 2 \end{pmatrix}, \tag{12}$$

compute the transformations of the given start and endpoints and plot the lines between them. (Note that you do not need to loop over the points. One matrix multiplication for the start and end points is enough. To compute Cartesian coordinates you can use your `pflat` function. Don't forget to use the `axis equal` command, otherwise the figures might look distorted.)
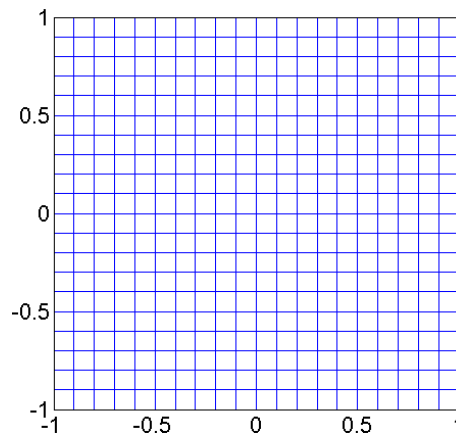
Figure 2: Lines between start and end points.

Which of the transformations preserve lengths between points? Which preserve angles between lines? Which map parallel lines to parallel lines? Classify the transformations into euclidean, similarity, affine and projective transformations.

```
Useful matlab commands:

plot([startpoints(1,:); endpoints(1,:)], ...
        [startpoints(2,:); endpoints(2,:)],'b-');
%Plots a blue line between each startpoint and endpoint
```

For the report: Submit the plots, and the answers to all the questions.

5 The Pinhole Camera

Exercise 5. Compute the projections of the 3D points with homogeneous coordinates

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \ \mathbf{X}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{X}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \tag{13}$$

in the camera with camera matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{14}$$

What is the geometric interpretation of the projection of $\mathbf{X}_3$?

Compute the camera center (position) of the camera and the principal axis (viewing direction).

For the report: Answers are enough.

Computer Exercise 4. Load and plot the images `compEx4im1.jpg` and `compEx4im2.jpg` (see Figure 3). The file `compEx4.mat` contains the matrices `K, R1, t1, R2, t2` and a point model `U` of the statue. The camera matrices for `compEx4im1.jpg` and `compEx4im2.jpg` are $P_1 = K \begin{bmatrix} R_1 & t_1 \end{bmatrix}$ and $P_2 = K \begin{bmatrix} R_2 & t_2 \end{bmatrix}$ respectively.

Compute the camera centers and principal axes of the cameras.

Figure 3: compEx4im1.jpg and compEx4im2.jpg

Plot the 3D-points in `U` and the camera centers in the same 3D plot (make sure that the 4th coordinate of `U` is one before you plot by using `pflat`). In addition plot a vector in the direction of the principal axes (viewing direction) from the camera center.

Project the points contained in `U` into the cameras P_1 and P_2 and plot the result in the same plots as the images. Does the result look reasonable?

```
Useful matlab commands:

null(P) %computes the nullspace of P

P(3,1:3) %extracts elements P31, P32 and P33.

quiver3(a(1),a(2),a(3),v(1),v(2),v(3),s) %Plots a vector v starting from
                                 %the point a and rescales the sise by s

plot(x1(1,:),x1(2,:),'.','Markersize',2);%Same as plot but with smaller points
```

For the report: Submit the m-file, the plots, the camera centers in Cartesian coordinates, and the principal axes normalized to length one.

Exercise 6. (OPTIONAL.) Consider the calibrated camera pair $P_1 = [I\ 0]$ and $P_2 = [R\ t]$. If $\mathbf{x}$ is the 2D projection of the 3D point $\mathbf{U}$ in P_1, that is $\mathbf{x} \sim P_1\mathbf{U}$ verify that

$$\mathbf{U} \sim \begin{pmatrix} \mathbf{x} \\ s \end{pmatrix}, \tag{15}$$

where $s \in \mathbb{R}$. That is, for any s the point of the form $\mathbf{U}(s) = (\mathbf{x}^T, s)^T$ projects to $\mathbf{x}$. What kind of object is this collection of points? Is it possible to determine s using only information from P_1?

Assume that $\mathbf{U}$ belongs to the plane

$$\Pi = \begin{pmatrix} \pi \\ 1 \end{pmatrix}, \tag{16}$$

where $\pi \in \mathbb{R}^3$. Compute the s that makes $\mathbf{U}(s)$ belong to the plane, that is, find s such that $\Pi^T\mathbf{U}(s) = 0$.

Verify that if $\mathbf{x} \sim P_1\mathbf{U}$, $\mathbf{y} \sim P_2\mathbf{U}$ and $\Pi^T\mathbf{U} = 0$ then the homography

$$H = (R - t\pi^T), \tag{17}$$

where $P_2 = [R\ t]$, maps $\mathbf{x}$ to $\mathbf{y}$. (Hint: What is $P_2\mathbf{U}(s)$ for the s from above?)

For the report: The exercise is optional, but if you want bonus points for the exam submit a complete solution.

Computer Exercise 5. (OPTIONAL.) Figure 4 shows an image (`compEx5.jpg`) of a poster located somewhere in the Centre for Mathematical Sciences. (Here the axes units are pixels.)
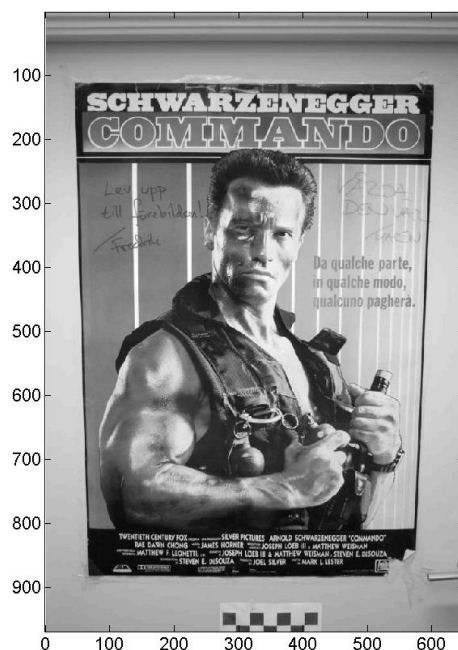
Figure 4: Arnold looking mean.

The file `compEx5.mat` contains the inner parameters `K`, the corner points of the poster and the 3D plane `v` that contains the poster. The camera matrix for the camera that generated this image is

$$P_1 = K[I\ 0]. \tag{18}$$

The goal of this exercise is to create an image of the poster taken by a camera 2m to the right of the poster.

Start by plotting the corner points and the image in the same 2D-figure. (Use `axis equal` to get the correct aspect ratio.) Note the scale on the axis. Where is the origin of the image coordinate system located?

To be able to use the formulas derived in Exercise 6 we must first ensure that we have calibrated cameras. To do this normalize the corner points by multiplying with K^{-1} and plot them in a new 2D-figure. (Use `axis ij` to make the y-axis point downwards (as for the previous image) and `axis equal`. Note the difference in scale compared to the previous figure. Where is the origin of the image coordinate system located?

Since the inner parameters K have been removed our calibrated camera is $[I\ 0]$. Using the results from Exercise 6 compute the 3D points in the plane `v` that project onto the corner points. Compute the camera center and principal axis, and plot together with the 3D-points. Does it look reasonable?

Next compute a new camera with camera center in $(2, 0, 0)$ (2 units to the right of $P1$) and orientation

$$R = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \tag{19}$$

(minus 30 degrees rotation around the y-axis. Note that the y-axis points down in the first image.) Compute the new camera and plot in the same figure. (Don't forget to use `axis equal`.)

Compute the homography in Exercise 6 and transform the normalized corner points to the new (virtual) image. Plot the transformed points (don't forget to divide by the third coordinate) in a new 2D-figure. Does the result look like you would expect it to when moving the camera like this? Also project the 3D points into the same image using the camera matrix. Does this give the same result?

We now have a homography that takes normalized points in the first camera and transforms them to normalized points in the second camera. To remove the need for normalization we simply include the normalization in the homography. If $\tilde{\mathbf{x}} = K^{-1}\mathbf{x}$ and $\tilde{\mathbf{y}} = K^{-1}\mathbf{y}$ then for the normalized points $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ we have

$$\tilde{\mathbf{y}} \sim H\tilde{\mathbf{x}} \Leftrightarrow K^{-1}\mathbf{y} \sim HK^{-1}\mathbf{x} \Leftrightarrow \mathbf{y} \sim KHK^{-1}\mathbf{x} \tag{20}$$

Therefore the total transformation is $H_{tot} = KHK^{-1}$. Transform the original image and the corner points using the homography H_{tot} and plot both in a new 2D-figure.

```matlab
Useful matlab commands:

plot(corners(1,[1:end 1]), corners(2,[1:end 1]),'*-');
%Plots the cornerpoints and connects them with lines.

axis ij %Makes the y-axis point down (as in an image)

tform = maketform('projective',Htot');
%Creates a projective transformation that can be used in imtransform
%NOTE: Matlab uses the transposed version of the homography.

[new_im,xdata,ydata] = imtransform(im,tform,'size',size(im));
%Creastes a transformed image (using tform)
%of the same size as the original one.

imagesc(xdata,ydata,new_im);
%plots the new image with xdata and ydata on the axes
```

For the report: The exercise is optional, but if you want bonus points for the exam submit the m-file, the plots and the new camera matrix.