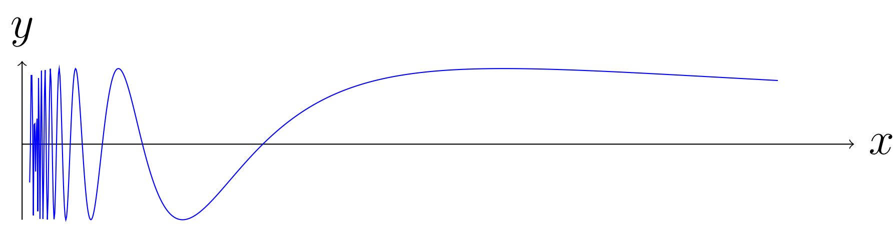
Glossary for Topology



Xiang Gao

July 28, 2022

Introduction

Oh Boy! Topology! The one class that absolutely kicked my ass. If you happened to come across this “glossary” and only know the word “Topology” from the “coffee cup is a doughnuts” joke. Well, I have bad news and good news for ya:

B). You won’t see and “coffee cups” or “doughnuts” for quite some time;

G). But you are going to enjoy all the materials nonetheless, and you’ll love “doughnut” as an object for the rest of your life, spoiler alert: it’s called a 2-D Torus.

Truth is, you need to have some Analysis background with proofs experience, and remember, you best friend and worst enemy $\frac{1}{n}$.

My thought of topologists only caring about “coffee cups” or “doughnuts” caused me to name the “Table of Contents” as “Clay Mathematics” as you can see on the top of each and every single page, which is a quick link to go back, as well as the bottom right. Note: I was not aware of the Clay Mathematics Institution at all when I began to make this “glossary”, it was a happy coincidence.

So what do topologists actually do? Well...

Main goal of topology: Classify topological spaces up to homeomorphism.

But that is way too hard! So...

More realistic goal: Given topological spaces X and Y, decide whether or not they are homeomorphic.

Also note that: topologists are sometimes savages. So $\subset$ means “subset” or “inclusion” where as $\subsetneq$ means “proper subset” or “proper inclusion”. They also denote collection of set using script letter, like $\mathcal{P}$(The power set), $\mathcal{B}$(The set of basis), $\ldots$

Anyway! So what’s this “glossary” actually about? Well, when I was busying getting my life totally destroyed by Topology, I found this way to be a really neat way for me to learn harder mathematical concept, and since I will go to graduate school for mathematics one day, it only makes sense for me keep on adding more and more to this document as well as the following:

1. On-Going Analysis
2. Never-Ending Algebra

The creation of this document involves but not limited to

1. *Topology* by James Munkers;
2. *Counterexamples in Topology* by Lynn Arthur Steen.

And here’s the most sincere thanks for all the support and encouragement I got when I was so beaten down by Topology and this Glossary wouldn’t be made without you all: Dr. Matt Young, Jan-Luca, Jasper Swensen, Jessica Jorgensen, Hyrum Hansen, Dr. Micheal Shutlz, Dr. Justin Heavilin, and Dr. David Euler Brown.

Copyright © 2022 Xiang Gao

Contents

1. Reminders on Continuity

1. Continuity

Definition 1.1 (Continuous Function form $\mathbb{R} \longrightarrow \mathbb{R}$). *Click here if you want to know another definition*

1). A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at a point $x \in \mathbb{R}$, if, for all $\epsilon > 0$, there exists a δ such that

$$|f(x) - f(y)| < \epsilon \quad \textit{whenever} \quad |x - y| < \delta$$

2). A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous if it is continuous at every point $x \in \mathbb{R}$.

Definition 1.2 (Euclidean Distance). *Recall that the (Euclidean) distance between two points*

$$x = (x_1, \dots, x_n), \quad y = (y_1, \dots, y_n)$$

in $\mathbb{R}^n$ is

$$||x - y|| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

Definition 1.3 (Continuous Function form $\mathbb{R}^n \longrightarrow \mathbb{R}^m$). *(So far still good? Well, check this out!)*

1). A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous at a point $x \in \mathbb{R}^n$, if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$||f(x) - f(y)|| < \epsilon \quad \textit{whenever} \quad ||x - y|| < \delta$$

2). A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous if it is continuous at every point $x \in \mathbb{R}^n$.

Definition 1.4 (Open Ball). *Let $x \in \mathbb{R}^n$ and $\delta > 0$. The open ball at radius δ centered at x is the set*

$$B_\delta(x) = \left\{ y \in \mathbb{R}^n \middle| ||x - y|| < \delta \right\}$$

Definition 1.5 (Continuity (Reformulated)). *A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous at a point $x \in \mathbb{R}^n$, if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$f(y) \in B_\epsilon(f(x)) \textit{ whenever } y \in B_\delta(x)$$

That is the same as

$$f(B_\delta(x)) \subset B_\epsilon(f(x))$$

In words, f is continuous at $x \in \mathbb{R}^n$ if for any $\epsilon > 0$, there exists a $\delta > 0$ such that f sends to open ball of radius δ at x into the open ball at radius ϵ at f(x). This sentence gives a precise definition of "f preserves closeness".
In my own words, the image of the open ball of x has to be a subset of the open ball centered at the image of x.

Definition 1.6 (Open Sets). *(Oooohhhhhh! Boy! Here we go!!!!)*
A subset $U \subset \mathbb{R}^n$ is called open if for all $u \in U$, there exist an $r > 0$ such that

$$B_r(u) \subset U$$

Open sets, in their generalized form, are fundamental objects in topology.

Definition 1.7 (Pre-image). *If $f : X \longrightarrow Y$ is a function between sets and $U \subset Y$ is a subset, its pre-image is*

$$f^{-1}(U) = \left\{ x \in X \middle| f(x) \in U \right\}$$

Theorem 1.1. *A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous if and only if, for any open set $U \subset \mathbb{R}^m$, the set $f^{-1}(u) \subset \mathbb{R}^n$ is open.*

Proof. First assume that f is continuous and let $U \subset \mathbb{R}^m$ be open. We need to show that $f^{-1}(u) \subset \mathbb{R}^n$ is open. So, let $x \in f^{-1}(u)$, that is

$$f(x) \in U$$

Since U is open, there exists an $\epsilon > 0$ such that

$$B_\epsilon(f(x)) \subset U$$

Since f is continuous, there exists a $\delta > 0$ such that

$$f(B_\delta(x)) \subset B_\epsilon(f(x)) \subset U$$

This implies that

$$B_\delta(x) \subset f^{-1}(U),$$

proving that $f^{-1}(u)$ is open.
To prove the converse, suppose that if $U \subset \mathbb{R}^m$ is open, then so too if $f^{-1}(u)$. Let $x \in \mathbb{R}^n$ and $\epsilon > 0$ be given. Then $B_\epsilon(f(x)) \subset \mathbb{R}^m$ is open, so that

$$f^{-1}\left(B_\epsilon(f(x))\right) \subset \mathbb{R}^n$$

is open. Note that $x \in f^{-1}\left(B_\epsilon(f(x))\right)$. So there exists a $\delta > 0$ such that

$$B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$$

that is

$$f(B_\delta(x)) \subset B_\epsilon(f(x))$$

This proves that f is continuous at $x \in \mathbb{R}^n$. Since x was arbitrary, f is continuous. ■

Definition 1.8 (Closed Sets)**.** *A subset $A \subset \mathbb{R}^n$ is called closed if its complement*

$$A^c = \mathbb{R}^n \setminus A = \{x \in \mathbb{R}^n \mid x \notin A\}$$

is an open subset.
A is closed $\iff$ for each $u \notin A$, there exists an open ball $B_r(u)$ which does not intersect A.

2. Topological Spaces

1. Topology

Definition 2.1 (Topology). *(Here's where everything starts to go south...)*

1). *Let X be a set. A topology on X is a set τ of subsets of X, called open sets, such that*

   i). *$\varnothing$, X are open sets*

   ii). *If $\{U_i\}_{i\in I}$ is an arbitrary collection of open sets, then $\bigcup_{i\in I} U_i$ is open*

   iii). *If $\{U_i\}_{i\in I}$ is a finite collection of open sets, then $\bigcap_{i\in I} U_i$ is open*

2). *A topological space is a set X with a chosen topology τ.*
For example, the Euclidean topology on $\mathbb{R}^n$, $\tau_{Euclidean} = \{U \subset \mathbb{R}^n \mid U \text{ open in the sense of Definition 1.6}\}$

Definition 2.2 (Discrete Topology). *Let X be a set. The discrete topology on X is*

$$\tau_{dis} = \{U \subset X\}$$

That is, every subset of X is open. It is immediate that τ_{dis} is indeed a topology.
Book: If X is any set, the collection of all subsets of X is a topology on X, it is called discrete topology.

Definition 2.3 (Indiscrete(trivial) Topology). *Let X be a set. The indiscrete(trivial) topology on X is*

$$\tau_{ind} = \{\varnothing, X\}$$

That is, every subset of X is open. It is immediate that τ_{dis} is indeed a topology.
The indiscrete topology is very "small" in the sense that all points of X are clumped together! On the other hand, the discrete topology is very "large" (Spoiler Alert: is it Hausdorff?).
Book: The collection of consisting of X and $\varnothing$ only is a topology on X, we shall call it the indiscrete topology.

Definition 2.4 (The Finite Complement Topology). *Let X be a set. The finite Complement topology on X is*

$$\tau_{fin} = \{U \subset X \mid X \setminus U \text{ is a finite set or is } X\}$$

That is, every subset of X is open. It is immediate that τ_{dis} is indeed a topology.
Book: Let X be a set; let τ_f be the collection of all subsets U of X such that $X - U$ either is finite or is all of X. Then τ_f is a topology on X, called the finite complement topology.

Proof. Check that 2.4 is a topology

1). $\varnothing \in \tau_f$ since $X \setminus X$ is finite. $X \in \tau_f$ since $X \setminus \varnothing$ is all of X.

2). Say $\{U_i\}_{i\in I}$ are open, i.e.

$$X \setminus U_i \text{ is finite on } X$$

then

$$X \setminus \bigcup_{i\in I} U_i = \bigcap_{i\in I} (X \setminus U_i)$$

is the intersection of finite sets, so is itself finite. Hence, $\bigcup_{i\in I} U_i$ is open.

3). Say $\{U_i\}_{i\in I}$ is a finite collection of open sets. Then

$$X \setminus \bigcap_{i\in I} U_i = \bigcup_{i\in I} (X \setminus U_i)$$

is a finite union of finite sets and so is itself finite. Hence, $\bigcap_{i\in I} U_i$ is open.

Therefore, τ_f is a topology. ■

Definition 2.5 (Coarser and Finer). *Let X be a set with topologies τ and τ'. We call τ coarser than $\tau \subset \tau'$. In this situation, we call τ' finer than τ. If either $\tau \subset \tau'$ or $\tau' \subset \tau$, then we call τ and τ' comparable.*

2. Basis for Topology

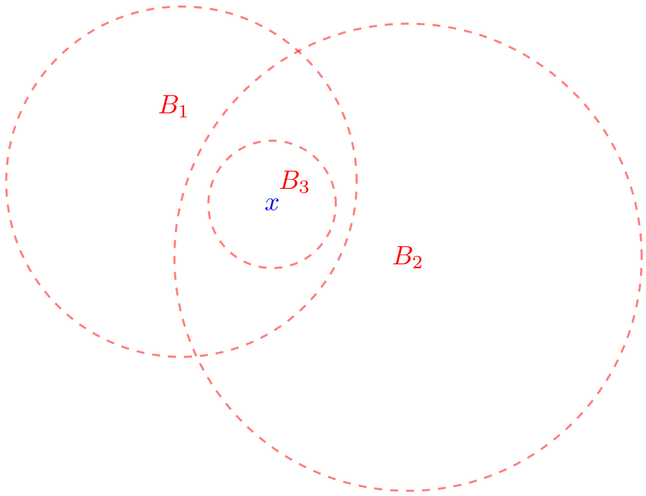
Definition 2.6 (Basis for a Topology)**.** *Let X be a set. A basis for a topology on X is a collection $\mathcal{B}$ of subsets of X such that*

1). Every $x \in X$ is contained in some $B \in \mathcal{B}$

2). If $x \in X$ with $B_1, B_2 \in \mathcal{B}$ containing x then there exists $B_3 \in \mathcal{B}$ such that

$$x \in B_3 \subset B_1 \cap B_2$$

In pictures:



Theorem 2.1. *Let $\mathcal{B}$ be a basis for a topology on X. Then*

$$\tau = \{U \subset X \mid \text{ for all } u \in U, \text{ there exists a } B \in \mathcal{B} \text{ such that } u \in B \subset U\}$$

is a topology on X.

Definition 2.6. *The topology τ of Theorem 2.1 is called the topology generated by $\mathcal{B}$.*
Book: A subset U of X is said to be open in X (that is, to be an element of τ) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of τ.

Lemma 2.2. *Let $\mathcal{B}$ be a basis for a topology τ on a set X. Then $U \subset X$ is open (i.e. $U \in \tau$) if and only if U is a union of elements of $\mathcal{B}$.*

Proof. Given a collection of elements of $\mathcal{B}$, they are also elements in τ. Because τ is a topology, their union is in τ. Conversely, given $U \in \tau$, choose for each $x \in U$ an element B_x of $\mathcal{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathcal{B}$. ∎

Lemma 2.3. *Let (X, τ) be a topological space. Let $\mathcal{C}$ be a collection of open sets such that for each open set U and $x \in U$, there exists a $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for τ.*

Proof. We need to show two things:

1). $\mathcal{C}$ is a basis for a topology on X.
The first condition for a basis is easy: Given $x \in X$, since X is itself an open set, there is by hypothesis an element C of $\mathcal{C}$ such that $x \in C \subset X$. To check the second condition, let x belong $C_1 \cap C_2$, where C_1 and C_2 are elements of $\mathcal{C}$. Since C_1 and C_2 are open, so is $C_1 \cap C_2$. Therefore, there exists by hypothesis an element C_3 in $\mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.

2). The topology $\tau_{\mathcal{C}}$ generated by $\mathcal{C}$ is τ.
Let $U \subset X$ be open. For each $u \in U$, there exists a $C_u \in \mathcal{C}$ with $u \in C_u \subset U$. Then

$$U = \bigcup_{u \in U} C_u$$

By Lemma 2.2, the right hand side is in $\tau_{\mathcal{C}}$, that is $\tau \subset \tau_{\mathcal{C}}$. Since $\mathcal{C} \subset \tau$, again by Lemma 2.2, we have $\tau_{\mathcal{C}} \subset \tau$. So, $\tau = \tau_{\mathcal{C}}$.

∎

Definition 2.7. *A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union is equal to X:*

$$\bigcup_{S\in\mathcal{S}} S = X$$

Lemma 2.4. *Let $\mathcal{S}$ be a subbasis for a topology on X. Then*

$$\tau_{\mathcal{S}} = \{U \subset X \mid U \text{ is a union of finite intersections of elements of } \mathcal{S}\}$$

is a topology on X.

3. Product Topology

Definition 2.8 (Product Topology)**.** *Let X, Y be topological spaces. The product topology on*

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

is the topology generated by the basis

$$\mathcal{B} = \{u \times v \mid u \subset X \text{ is open}, v \subset Y \text{ is open}\}$$

Proof. Check that $\mathcal{B}$ is indeed a basis for a topology on $X \times Y$.

1). $\mathcal{B}$ covers $X \times Y$: Let $(x, y) \in X \times Y$. Since $X \subset X$ and $Y \subset Y$ are open, $X \times Y \in \mathcal{B}$ and $\mathcal{B}$ covers. Or, it's already trivial, since $X \times Y$ is itself a basis element.

2). Say $u \times v,\ u' \times v' \in \mathcal{B}$ and

$$(x, y) \in (u \times v) \cap (u' \times v') = (u \cap u') \times (v \cap v')$$

Since $u \cap u',\ v \cap v'$ are open, we have

$$(u \times v) \cap (u' \times v') \in \mathcal{B}$$

■

Theorem 2.5. *Let X and Y be topological spaces with bases $\mathcal{B}$ and $\mathcal{C}$, respectively, then*

$$\mathcal{D} = \{u \times v \mid u \in \mathcal{B}, v \in \mathcal{C}\}$$

is a basis for the product topology on $X \times Y$.

Proof. We will use Lemma 2.3. So, let $W \subset X \times Y$ be open and $(x, y) \in W$. By the definition of the product topology,

$$W = \bigcup_i u_i \times v_i$$

for some $u_i \in \mathcal{B}$ and $v_i \in \mathcal{C}$. Then

$$(x, y) \in \underbrace{u_j \times v_j}_{in\,\mathcal{D}} \subset W$$

for some j. By Lemma 2.3, we are done. ■

Definition 2.9 (Projections)**.** *Let $\pi_x : X \times Y \longrightarrow X$ be defined by the equation:*

$$\pi_x(x, y) = x$$

Let $\pi_y : X \times Y \longrightarrow Y$ be defined by the equation:

$$\pi_y(x, y) = y$$

The maps π_x and π_y are called the projections of $X \times Y$ onto (surjective) its its first and second factors, respectively. Given any subset $U \subset X$ is open, we have

$$\pi_x^{-1}(U) = \{(x, y) \in X \times Y \mid x \in U\}$$

In particular, if X and Y are topological spaces and $U \subset X$ is open, then so too is $\pi_x^{-1}(U)$.

Theorem 2.6. *Let X, Y be topological spaces. Then*

$$\mathcal{S} = \left\{\pi_x^{-1}(U) \mid U \subset X \text{ is open}\right\} \cup \left\{\pi_y^{-1}(U) \mid V \subset Y \text{ is open}\right\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let τ be the product topology on $X \times Y$ and $\tau_{\mathcal{S}}$ the topology generated by $\mathcal{S}$. It is immediate that $\tau_{\mathcal{S}} \subset \tau$. To see that $\tau \subset \tau_{\mathcal{S}}$, note that if $U \subset X$ and $V \subset Y$ are open, then

$$\pi_x^{-1}(U) \cap \pi_y^{-1}(V) = U \times V \in \tau_{\mathcal{S}}$$

Since $\{U \times V \mid U \subset X \text{ open}, V \subset Y \text{ open}\}$ is a basis for τ, we get $\tau \subset \tau_{\mathcal{S}}$. ■

4. Subspace Topology

Definition 2.10 (Subspace Topology)**.** *Let (X, τ) be a topological space and $Y \subset X$ a subset. Then*

$$\tau_Y = \{Y \cap U \mid U \subset X \textit{ is open}\}$$

is the subspace topology on Y. With this topology, Y is called a subspace of X; its open sets consist of all intersections of opens sets of X and Y.

Lemma 2.7. *The subspace topology τ_Y is a topology.*

Proof. We have

$$\varnothing = Y \cap \varnothing \in \tau_Y$$
$$Y = Y \cap X \in \tau_Y$$

Let $\{U_i\}_{i \in I}$ be open in Y (so $U_i \in \tau_Y$). Then

$$U_i = Y \cap V_i$$

for some $V_i \subset X$ open and

$$\bigcup_{i \in I} U_i = Y \cap \underbrace{\left(\bigcup_{i \in I} V_i\right)}_{\text{open in } X} \in \tau_Y$$

$$\bigcap_{i \in I} U_i = Y \cap \overbrace{\left(\bigcap_{i \in I} V_i\right)} \in \tau_Y$$

if $|I| < \infty$. ■

Lemma 2.8. *Let $Y \subset X$ be a subspace. If $U \subset Y$ is open and $Y \subset U$ is open, then $U \subset X$ is open.*

Proof. Since U is open in Y, $U = Y \cap V$ for some set V open in X. Since Y and V are both open in X, so is $Y \cap V$. ■

Lemma 2.9. *Let X be a topological space with basis $\mathcal{B}$ and $Y \subset X$ a subset. Then*

$$\mathcal{B}_Y = \{Y \cap B \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology of Y. In words, a basis for X induces a basis for the subspace $Y \subset X$.

Proof. Given U is open in X and given $y \in U \cap Y$, we can choose an element B of $\mathcal{B}$ such that $y \in B \subset U$. Then $y \in B \cap Y \subset U \cap Y$. It follows from Lemma 2.3 that $\mathcal{B}_Y$ is a basis for the subspace topology on Y. ■

Theorem 2.10. *Let X, Y be topological spaces with subsets $A \subset X$ and $B \subset Y$. Then the subspace topology on $A \times B \subset X \times Y$ is the same as the product topology on $A \times B$.*

Proof. A basis element of the product topology on $X \times Y$ is $U \times V$, where $U \subset X$ and $V \subset Y$ are open. Then

$$(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V)$$

is a basis element for the subspace topology on $A \times B$. But the RHS is a basis element for the product topology on $A \times B$. ■

Definition 2.11. *Let X be a topological space. A subset $A \subset X$ is called closed if $A^c = X \setminus A \subset X$ is open.*

Theorem 2.11. *Let X be a topological space*

1). *$\varnothing$ and X are closed.*

2). *Arbitrary intersections of closed sets are closed:*
If $\{A_i\}_{i \in I}$ is a collection of closed sets, then $\bigcap_{i \in I} A_i$ is closed.

3). *Finite unions of closed sets are closed:*
If $\{A_i\}_{i \in I}$ is a finite collection of closed sets, then $\bigcup_{i \in I} A_i$ is closed.

Proof. 1). $\varnothing$ and X are closed because they are the complements of the open sets X and $\varnothing$, respectively.

2). Given a collection of closed sets $\{A_i\}_{i\in I}$, we apply DeMorgan's Law,

$$X - \bigcap_{i\in I} A_i = \bigcup_{i\in I}(X - A_i)$$

Since the sets $X - A_i$ are open by definition, the right side of this equation represents an arbitrary union of open sets, and is thus open. Therefore, $\bigcap\limits_{i\in I} A_i$ is closed.

3). Similarly, if A_i is closed for $i = 1, 2, \ldots, n \in I$, consider the equation

$$X - \bigcup_{i\in I} A_i = \bigcap_{i\in I}(X - A_i)$$

The set on the right hand side is a finite intersection of open sets and is therefore open. Hence $\bigcup\limits_{i\in I} A_i$ is closed. ■

Lemma 2.12. *Let $Y \subset X$ be a subspace and $A \subset Y$ a subset. Then $A \subset Y$ is closed if and only if $A = Y \cap C$ for a closed set $C \subset X$.*

Proof. If and only if...

1). Assume $A \subset Y$ is closed. Then $Y \setminus A$ is open so that $Y \setminus A = Y \cap U$ for an open $U \subset X$. Then

$$\begin{aligned} A &= Y \setminus (Y \setminus A) \\ &= Y \setminus (Y \cap U) \\ &= Y \cap (X \setminus U) \end{aligned}$$

The set $X \setminus U$ is closed in X, so we can take $C = X \setminus U$.

2). Conversely, assume that $A = Y \cap C$ for $C \subset X$ closed. Then, by the above, we see that

$$Y \setminus A = Y \cap (X \setminus C)$$

The set $X \setminus C$ is open in X, so that $Y \setminus A$ is open in Y. Therefore, $A \subset Y$ is closed.

■

5. Interior and Closure

Definition 2.12 (Interior and Closure)**.** *Let X be a topological space and $Y \subset X$ a subset.*

1). The interior of Y (in X) is the union of all open sets contained in Y, and is denoted by $Int(Y)$.

2). The closure of Y (in X) is the intersection of all closed sets of X which contain Y, and is denoted by $\overline{Y}$.

$$Int(Y) \subset Y \subset \overline{Y}$$

If Y is open, $Y = Int(Y)$; while if Y is closed, $Y = \overline{Y}$

Lemma 2.13. *Let X be a topological space and $Y \subset X$ a subset. Then $Int(Y) \subset X$ is open and $\overline{Y} \subset X$ is closed.*

Lemma 2.14. *Let X be a topological space and $Y \subset X$ a subset.*

1). Y is open if and only if $Int(Y) = Y$

2). Y is closed if and only if $Y = \overline{Y}$

Theorem 2.15. *Let X be a topological space with a subset Y.*

1). Then $x \in \overline{Y}$ if and only if any open set $x \in U \subset X$ intersects Y.

2). Let $\mathcal{B}$ be a basis for the topology on X. Then $x \in \overline{Y}$ if and only if every $B \in \mathcal{B}$ with $x \in B$ intersects Y.

Definition 2.13 (Limit Points)**.** *Let X be a topological space and $Y \subset X$ a subset. A point $l \in X$ is called a limit point of Y (or cluster/accumulation point) if every open set $U \subset X$ which contains l intersects Y at a point different from l. Said differently, l is a limit point of A if it belongs to the closure of $A - \{x\}$. The point l may lie in A or not; for this definition it does not matter.*

Theorem 2.16. *Let X be a topological space with a subset $Y \subset X$. Then*

$$\overline{Y} = Y \cup \{\text{limit points of } Y\} = Y \cup Y'. \quad (Y' := \{\text{limit points of } Y\})$$

Proof. Let l be a limit point of Y. Then every open set containing l intersects Y. By Theorem 2.15, $x \in \overline{Y}$. Hence, $\overline{Y} \supset Y \cup Y'$. For the reverse inclusion, let $x \in \overline{Y}$. Say $x \notin Y$; we prove that $x \in Y'$. Again by Theorem 2.15, any open set $x \in U \subset X$ intersects Y. Since $x \notin Y$, we see that any open set containing x intersects Y at a point different from l. So, $x \in Y'$. ∎

Corollary 2.17. *A subset $Y \subset X$ is closed if and only if it contains all of its limit points.*

Definition 2.13 (Boundary)**.** *Let X be a topological space. The* boundary *of a subset $A \subset X$ is defined to be*

$$\partial A = \overline{A} \cap \overline{X \setminus A},$$

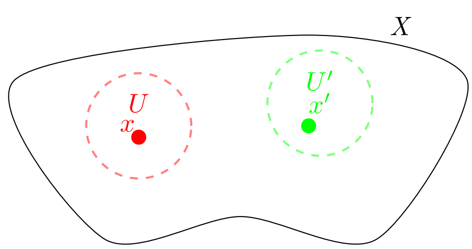
where $\overline{(-)}$ denotes closure.

6. Hausdorff Spaces

Definition 2.14 (Hausdorff Spaces)**.** *A topological space X is called Hausdorff if for every pair of distinct points $x, x' \in X$, there exists open sets*

$$x \in U \subset X, \quad x' \in U' \subset X$$

such that $U \cap U' = \varnothing$
In pictures:



Theorem 2.18. *Let X be a Hausdorff (topological) space. Then any finite subset $A \subset X$ is closed. In particular, points are closed in X.*

Definition 2.15. *Let X be a topological space. A sequences of points $\{x_n\}_{n \geqslant 1}$ in X converges to $x \in X$ if for any open set $x \in U \subset X$, there exists an $N > 0$ such that $x_n \in U$ whenever $n > N$.*

Theorem 2.19. *Let X be a Hausdorff space. A sequence in $\{x_n\}_{n \geqslant 1}$ in X converges to at most one point of X; if it exists, this point is called the limit of $\{x_n\}$ and denoted by $\lim\limits_{n \to \infty} x_n$.*

Proof. Suppose that $\{x_n\}_{n \geqslant 1}$ converges to distinct points $a, b \in X$. Since X is Hausdorff, there exists open set

$$a \in U \subset X, \quad b \in V \subset X$$

such that $U \cap V = \varnothing$. Since $\{x_n\}_{n \geqslant 1}$ converges to a, there exists $N > 0$ such that $x_n \in U$ if $n > N$. Similarly, there exists an $M > 0$ such that $x_n \in V$ if $n > M$. Let $L = max\{M, N\}$. Then $x_n \in U \cap V$ if $n > L$. But $U \cap V = \varnothing$, a contradiction. ∎

Theorem 2.20 (Fantastic Top-spaces and Where to Find Them)**.** *(Hausdorff)*

1). Let X, Y be Hausdorff spaces. Then $X \times Y$ is Hausdorff.

2). Let X be a Hausdorff space and $Y \subset X$ a subset. Then Y, with the subspace topology, is Hausdorff.

Proof. 1). Let $(x,y),(x',y') \in X \times Y$ be distinct. Suppose that $x \neq x'$. Since X is Hausdorff, there exist open sets

$$x \in U \subset X, \quad x' \in U' \subset X$$

such that $U \cap U' = \varnothing$. Then

$$(x,y) \in U \times Y, \quad (x',y') \in U' \times Y$$

are disjoint open sets. If $x = x'$, then $y = y'$ and we can modify the previous argument.

2). Let $y, y' \in Y$ be distinct. Since X is Hausdorff, there exist open sets $y \in U \subset X,\ y' \in U' \subset X$ such that $U \cap U' = \varnothing$. Then $y \in U \cap Y \subset Y,\ y' \in U' \cap Y \subset Y$ are open sets in Y such that

$$(U \cap Y) \cap (U' \cap Y) = (U \cap U') \cap Y = \varnothing$$

So, Y is Hausdorff.

■

3. Continuous Functions

1. Continuous Functions

Definition 3.1 (Continuous Functions)**.** *Click here to be more sane.*
Let X, Y be topological space. A continuous function $f : X \longrightarrow Y$ is a function at underlying sets such that, for each open set $U \subset Y$, the preimage

$$f^{-1}(U) = \{x \in X \mid f(x) \in U\}$$

is open (in X).
To know more about the notation please go to Appendix 1.

Lemma 3.1. *Let $\mathcal{B}$ be a basis for the topology on Y. A function $f : X \longrightarrow Y$ is continuous if and only if $f^{-1}(\mathcal{B})$ is open for all $B \in \mathcal{B}$.*

Theorem 3.2. *Let $f : X \longrightarrow Y$ be a function between topological spaces. The following statements are equivalent:*

1). *f is continuous.*

2). *For any subset $A \subset X$, we have $f(\overline{A}) \subset \overline{f(A)}$.*

3). *For any closed set $C \subset Y$, the preimage $f^{-1}(C) \subset X$ is closed.*

4). *For each $x \in X$ and open set $f(x) \in V \subset Y$, there exists an open set $x \in U \subset X$ such that $f(u) \subset V$.*

If the condition in (4) holds for the point x of X, we say that f is continuous at the point x.

Proof. We show that (1) $\implies$ (2) $\implies$ (3) $\implies$ (1) and (1) $\implies$ (4) $\implies$ (1)

1) $\implies$ 2)**.** Assume that f is continuous. Let A be a subset of X. We show that if $x \in \overline{A}$, then $f(x) \in \overline{f(A)}$. Let V be a neighborhood of $f(x)$ (V is an open set containing x). Then $f^{-1}(V)$ is an open set of X containing x; it must intersect A in some point y. Then V intersects $f(A)$ in the point $f(y)$, so that $f(x) \in \overline{f(A)}$, as desired.

2) $\implies$ 3)**.** Let B be closed in Y and let $A = f^{-1}(B)$. We wish to prove that A is closed in X; we show that $\overline{A} = A$. By elementary set theory, we have $f(A) = f(f^{-1}(B)) \subset B$. Therefore, if $x \in \overline{A}$,

$$f(x) \in f(\overline{A}) \subset \overline{f(A)} \subset \overline{B} = B,$$

so that $x \in f^{-1}(B) = A$. Thus $\overline{A} \subset A$, so that $\overline{A} = A$, as desired.

3) $\implies$ 1)**.** Let V be an open set of Y. Set $B = Y - V$. Then

$$f^{-1}(B) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V)$$

Now B is a closed set of Y. Then $f^{-1}(B)$ is closed in X by hypothesis, so that $f^{-1}(V)$ is open in X, as desired.

1) $\implies$ 4)**.** Let $x \in X$ and let V be a neighborhood of $f(x)$. Then the set $U = f^{-1}(V)$ is a neighborhood of x such that $f(U) \subset V$.

4) $\implies$ 1)**.** Let V be an open set of Y; let x be a point of $f^{-1}(V)$. Then $f(x) \in V$, so that by hypothesis there is a neighborhood U_x of x such that $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$. It follows that $f^{-1}(V)$ can be written as the union of the open sets U_x, so that it is open.

■

Theorem 3.3 (Constructing Continuous Functions)**.** *Let X, Y, Z be topological spaces.*

a). *(Constant Function) If $f : X \longrightarrow Y$ maps to all of X into the single point y_0 of Y, then f is continuous.*

b). *(Inclusion) If A is subspace of X, the inclusion function $j : A \longrightarrow X$ is continuous.*

c). *(Composites) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous, then the map $g \circ f : X \longrightarrow Z$ is continuous.*

d). *(Restricting the domain) If $f : X \longrightarrow Y$ is continuous, and if A is a subspace of X, then the restricted function $f_{|A} : A \longrightarrow Y$ is continuous (where $A \subset Y$ is given the subspace topology).*

e). *(Restricting or expanding the range) Let $f : X \longrightarrow Y$ is continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \longrightarrow Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \longrightarrow Z$ obtained by expanding the range of f is continuous.*

f). *(Local formulation of continuity) The map $f : X \longrightarrow Y$ is continuous if X can be written as the union of open sets $X = \bigcup\limits_{i \in I} U_\alpha$ such that $f_{|U_\alpha} : U_\alpha \longrightarrow Y$ is continuous for each α.*

Proof. Let's proof every single one of them:

a). Let $f(x) = y_0$ for every $x \in X$. Let V be open in Y. The set $f^{-1}(V)$ equals X or $\varnothing$, depending on whether V contains y_0 or not. In either case, it is open.

b). If U is open in X, then $j^{-1}(U) = A \cap U$, which is open in A by definition of the subspace topology.

c). If U is open in Z, then $g^{-1}(U)$ is open in Y and $f^{-1}(g^{-1}(U))$ is open in X. But

$$f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U),$$

by elementary set theory. (The book really need to drop this "elementary" notation, makes me feels even dumber.)

d). The function $f_{|A}$ equals the composite of the inclusion map $j : A \longrightarrow X$ and the map $f : X \longrightarrow Y$, both of which are continuous.

e). Let $f : X \longrightarrow Y$ be continuous. If $f(X) \subset Z \subset Y$, we show that the function $g : X \longrightarrow Z$ obtained from f is continuous. Let B open in Z. Then $B = Z \cap U$ for some open set U of Y. Because Z contains the entire image set $f(X)$,

$$f^{-1}(U) = f^{-1}(B),$$

by the elementary set theory (Shut the f**k up!). Since $f^{-1}(U)$ is open, so is $g^{-1}(B)$.
To show $h : X \longrightarrow Z$ is continuous if Z has Y as a subspace, note that h is the composite of the map $f : X \longrightarrow Z$ and the inclusion map $j : Y \longrightarrow Z$.

f). By hypothesis, we can write X as a union of open sets U_α, such that $f_{|U_\alpha}$ is continuous for each α. Let V be an open set in Y. Then

$$f^{-1}(V) \cap U_\alpha = \left(f_{|U_\alpha}\right)^{-1}(V),$$

because both expressions represent the set of those points x lying in U_α for which $f(x) \in V$. Since $f_{|U_\alpha}$ is continuous, this set is open in U_α, and hence open in X. But

$$f^{-1}(V) = \bigcup_\alpha \left(f^{-1}(V) \cap U_\alpha\right),$$

so that $f^{-1}(V)$ is also open in X.

■

Theorem 3.4 (Another Version (Slightly Weaker) of Gluing (f in the previous theorem) For Closed Sets)**.** *(Ready?) Let X be a topological space with closed subsets $A, B \subset X$ such that $X = A \cup B$. Let*

$$g : A \longrightarrow Z, \quad h : B \longrightarrow Z$$

be continuous functions such that

$$g_{|A\cap B} = h_{|A\cap B}$$

Then g and h glue to a continuous function

$$\begin{aligned} f : X &\longrightarrow Z \\ x &\longrightarrow \begin{cases} g(x), & \text{if } x \in A \\ h(x), & \text{if } x \in B \end{cases} \end{aligned}$$

Proof. Recall (Theorem 3.2(3)) that f is continuous if and only if $f^{-1}(C) \subset X$ is closed for all $C \subset Z$ closed. Basic set theory gives:

$$f^{-1}(C) = g^{-1}(C) \cap h^{-1}(C).$$

Since g, h are continuous, $g^{-1}(C) \subset A$ and $h^{-1}(C) \subset B$ are closed. Since A and B are closed in X, so too are $g^{-1}(C)$ and $h^{-1}(C)$. Since finite intersections of closed sets are closed, we can conclude that $f^{-1}(C) \subset X$ is closed. ■

2. Homeomorphisms

Definition 3.2 (Homeomorphisms). *("Sameness")*

1). A continuous function $f : X \longrightarrow Y$ with a continuous inverse $f^{-1} : Y \longrightarrow X$ is called a homeomorphism.

2). Topological spaces X and Y are called homeomorphic if there exists a homeomorphism $X \longrightarrow Y$.

Book: Let X and Y be topological spaces; let $f : X \longrightarrow Y$ be a bijection. If both the function f and the inverse function

$$f^{-1} : Y \longrightarrow X$$

are continuous, then f is called a homeomorphism.
Comments on homeomorphism:

1). For $f : X \longrightarrow Y$ to be a homeomorphism, we need

i). f is continuous

ii). f is a bijection (so that $f^{-1} : Y \longrightarrow X$ exists)

iii). f^{-1} is continuous

2). The identity map $id : X \longrightarrow X$ is a homeomorphism.
If $f : X \longrightarrow Y$ is a homeomorphism, then so too is $f^{-1} : Y \longrightarrow X$; its inverse if f.
If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are homeomorphisms, then so too is $g \circ f : X \longrightarrow Z$.
It follows that homeomorphism is an equivalence relation.

3). Homeomorphisms preserves all topological properties of topological spaces. Just like in modern algebra, an isomorphism *is a bijective correspondence that preserves the algebraic structure involved.*

Definition 3.3. *A function $f : X \longrightarrow Y$ is called open if $f(U) \subset Y$ is open for all $U \subset X$ open.*

Theorem 3.5. *A continuous bijection $f : X \longrightarrow Y$ is a homeomorphism if and only if f is open.*

Proof. Since f is a continuous bijection, it is a homeomorphism if and only if $\left(f^{-1}\right)^{-1}(U) \subset Y$ is open for all $U \subset X$ open. The theorem now follows form the (set theory) identity

$$\left(f^{-1}\right)^{-1}(A) = A$$

for all $A \subset X$. ■

4. More Topological Spaces

1. Metric Topological Spaces

Definition 4.1 (Metric)**.** *A metric on a set X is a function $d : X \times X \longrightarrow \mathbb{R}$ such that*

1). $d(x, y) \geqslant 0$ with equality if and only if $x = y$

2). $d(x, y) = d(y, x)$

3). (Triangle equality) $d(x, y) + d(y, z) \geqslant d(x, z)$

for all $x, y, z, \in X$.

Theorem 4.1. *Let X be a set with metric d. Then the collection*

$$\mathcal{B}_d = \{B_\epsilon(x; d) \mid x \in X, \epsilon > 0\}$$

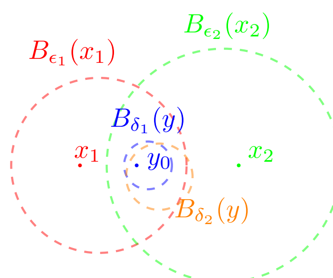
is a basis for a topology on X, called the metric topology induced by d.

Proof. Since $d(x, x) = 0$, we see that $x \in B_\epsilon(x; d)$ for every $\epsilon > 0$. So, $\mathcal{B}_d$ covers X.
Let $B_{\epsilon_1}(x_1), B_{\epsilon_3}(x_2) \in \mathcal{B}_d$ with

$$y \in B_{\epsilon_1}(x_1) \cap B_{\epsilon_3}(x_2)$$

Let $\delta_1 = \epsilon_1 - d(x_1, y)$. If $z \in B_{\delta_1}(y)$, then

$$\begin{aligned} d(x, z) &\leqslant d(x, y) + d(y, z) \\ &< d(x, y) + \epsilon_1 - d(x, y) \\ &< \epsilon_1 \end{aligned}$$



so that $B_{\delta_1}(y) \subset B_{\epsilon_1}(x_1)$. Similarly, setting $\delta_2 = \epsilon_2 - d(x_2, y)$ gives $B_{\delta_2}(y) \subset B_{\epsilon_2}(x_2)$. Let

$$\delta = \min\{\delta_1, \delta_2\}$$

Then

$$B_{delta}(y) \subset B_{\epsilon_1}(x_1) \cap B_{\epsilon_2}(x_2)$$

■

Definition 4.2 (Metrizable)**.** *A topological space X is called metrizable if there exists a metric d on X whose associated metric topology agrees with that of X.*
Metrizability of a topological space is a subtle problem, but is ultimately known via Urysohn's Metrization Theorem.

Definition 4.3. *Let (X, d) be a metric space.*

1). A subset $A \subset X$ is called bounded if there exists an $N > 0$ such that $d(a_1, a_2) \leqslant N$ for all $a_i \in A$.

2). The diameter of a non-empty bounded subset $A \subset X$ is

$$diam A = sup\{d(a_1, a_2) \mid a_i \in A\}.$$

Theorem 4.2. *Let (X, d) be a metric space. Then define $\overline{d} : X \times X \longrightarrow \mathbb{R}$ by*

$$\overline{d} = min\{d(x, y), 1\}$$

in other words

$$\begin{aligned} \overline{d} : X \times X &\longrightarrow \mathbb{R} \\ (x, y) &\longrightarrow min\{d(x, y), 1\} \end{aligned}$$

Then $\overline{d}$ is a metric that includes the same topology as d.

Theorem 4.3. *The following statements hold.*

1). Subspaces of metric space are metrizable.

2). Metrizable spaces are Hausdorff.

In particular, Lemma 2.19 *and the* 2) *imply that, if a sequence* x_n *in a metric space converges, then if has a unique limit.*

Proof. 1). Let (X, d) be a metric space and $A \subset X$ a subspace. We claim that

$$d_{|A} : A \times A \longrightarrow \mathbb{R}$$

is a metric on A and that the metric topology on A agrees with the subspace topology. That $d_{|A}$ is a metric follows immediately from the metric axioms hold. For the second, we wish to apply Lemma 2.3. Let

$$\mathcal{A} = \left\{ \underbrace{B_\epsilon(y)}_{\text{with respect to } d_{|A}} \subset Y \mid y \in Y, \, \epsilon > 0 \right\}$$

(This is the set $\mathcal{C}$ in Lemma 2.3). Note that

$$B_\epsilon(y) = \underbrace{B_\epsilon}_{\text{open in } X} (y; d) \cap Y$$

So that $B_\epsilon(y) \subset Y$ is open. Let $U \subset Y$ be open and $u \in U$. Then

$$U = \bigcup_i (B_{\epsilon_i}(x_i, d) \cap Y)$$

for some i. Then

$$u \in B = B_{\epsilon_i - d(x_i, u)}(u) \subset U.$$

Lemma 2.3 therefore applies, proving the second statement.

2). Let (X, d) be a metric space and $x, y \in X$ distinct points. Let $\epsilon = d(x, y) > 0$(by the non-negativity axiom). Then $x \in B_{\epsilon/2}(x)$ and $y \in B_{\epsilon/2}(y)$ and

$$B_{\epsilon/2}(x) \cap B_{\epsilon/2}(y) = \varnothing$$

(by the Triangle Inequality). So, (X, d) is Hausdorff.

■

Theorem 4.4. *Finite products of metrizable topological spaces are metrizable.*

Theorem 4.5. *Let* (X, d) *and* (Y, e) *be metric spaces. A function* $f : X \longrightarrow Y$ *is continuous if and only if for each* $x \in X$ *and* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that*

$$e(f(x), f(y)) < \epsilon$$

whenever $d(x, y) < \delta$.
In other words,

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$$

Continuous functions between metric spaces admit an $\epsilon - \delta$ *characterization, just like in analysis.*

Lemma 4.6. *Let* $A \subset X$ *be a subset of a topological space.*

1). If $\{x_n\}_{n \geqslant 0}$ *is a sequence in* A *converging to a point* x, *then* $x \in \overline{A}$.

2). If X *is metrizable and* $x \in \overline{A}$, *then there exists a sequence* $\{x_n\}_{n \geqslant 0}$ *in* A *converging to* x.

Theorem 4.7. *Let* $f : X \longrightarrow Y$ *be a function between topological spaces.*

1). If f *is continuous and* $\{x_n\}_{n \geqslant 1}$ *converges to* x, *then* $\{f(x_n)\}_{n \geqslant 1}$ *converges to* $f(x)$.

2). If X *is metrizable* $\{f(x_n)\}_{n \geqslant 1}$ *converges to* $f(x)$ *for all sequences* x_{nn} *converging to* x, *then* f *is continuous.*

Proof. 1). Let $\{x_n\}_{n\geqslant 1}$ converges to $x \in X$ and $U \subset Y$ be an open set containing $f(x)$. Since f is continuous, $f^{-1}(U) \subset X$ is open and, by construction, contains x. Hence, there exists an $N > 0$ such that $x_n \in f^{-1}(U)$ whenever $n > N$. But then $f(x_n) \in U$ whenever $n > N$. So, $\{f(x_n)\}_{n\geqslant 1}$ converges to $f(x)$.

2). Let (X, d) be a metric space and $\{x_n\}_{n\geqslant 1}$ a convergent sequence, say $x = \lim_{n\to\infty} x_n$. (Note: Since X is metrizable, $\{x_n\}_{n\geqslant 1}$ converges to a unique point, so that we can safely write $\lim_{n\to\infty} x_n$).
Recall that f is continuous if and only if

$$f\left(\overline{A}\right) \subset \overline{f(A)}$$

for all $A \subset X$; see Theorem 3.2. Let $x \in \overline{A}$. By Lemma 4.6, there exists $\{x_n\}_{n\geqslant 1}$ with $x = \lim_{n\to\infty} x_n$. By hypothesis, $\{f(x_n)\}_{n\geqslant 1}$ converges to $f(x)$, that is, $f(x) \in \overline{f(A)}$, again by Lemma 4.6.

■

Definition 4.4. *Let (X, d) be a metric space.*

1). *A sequence $\{x_n\}_{n\geqslant 1}$ is called Cauchy if for all $\epsilon > 0$ there exists an $N > 0$ such that*

$$d(d_n, d_{n+1}) < \epsilon$$

whenever $m, n > N$

2). *(X, d) is called complete if every Cauchy sequence converges.*

Example 4.1. *The following are examples about complete metric space.*

1). *$(\mathbb{R}, d_E)$ is complete;*

2). *$(0, 1)$ is not complete, for example*

$$\left\{\frac{1}{n}\right\}_{n\geqslant 1} \text{ is Cauchy but does not converge in } (0,1).$$

2. The Quotient Topology

Definition 4.5 (Equivalence Relationship)**.** *Let X be a set. An equivalence relation on X is $x \sim y$, such that*

1). *(Reflexivity) $x \sim x$,*

2). *(Symmetry) $x \sim y$ if and only if $y \sim x$,*

3). *(Transitivity) If $x \sim y$ and $y \sim z$, then $x \sim z$.*

The equivalence class of x is

$$[x] = \{y \in X \mid x \sim y\} \subset X$$

So

$$X = \bigsqcup_{x\in X/\sim} [x]$$

where $X/\sim = \{$set of all equivalence classes$\}$.

Definition 4.6 (Quotient Topology)**.** *Let X be a topological space, Y a set and*

$$q : X \longrightarrow Y$$

a surjective function. The quotient topology on Y (induced by the function q) is

$$\tau_q = \left\{U \subset Y \mid q^{-1}(U) \subset X \text{ is open }\right\}$$

We call (Y, τ_q) the quotient topological space.

Proof. We should check this is indeed a topology:

1.) $\varnothing \in \tau_q$ because $q^{-1}(\varnothing) = \varnothing \in \tau$ (topology on X). And $Y \in \tau_q$ because $q^{-1}(Y) = X \in \tau$ (q is surjective.)

2.) Let $\{U_i\}_{i \in I}$ be an arbitrary collection in τ_q. Then

$$q^{-1}\left(\bigcup_{i \in I} U_i\right) = \underbrace{\bigcup_{i \in I} \underbrace{q^{-1}(U_i)}_{\text{open in } X}}_{\text{open in} X} \in \tau_q$$

2.) Let $\{U_i\}_{i \in I}$ be a finite collection in τ_q. Then

$$q^{-1}\left(\bigcap_{i \in I} U_i\right) = \underbrace{\bigcap_{i \in I} \underbrace{q^{-1}(U_i)}_{\text{open in } X}}_{\text{open in} X} \in \tau_q$$

■

Example 4.2. *Let $X = \mathbb{R}$. Define an equivalence relation on X by*

$$x \sim y \iff x - y \in \mathbb{Q}$$

Consider the quotient $X/\sim$. The points $[0], [\pi] \in X/\sim$ are distinct. Let

$$[0] \in U, \ [\pi] \in V$$

be open sets in (in $X/\sim$). Then $q^{-1}(U) \subset \mathbb{R}$ is an open set which contains $\mathbb{Q}$ and $q^{-1}(V) \subset \mathbb{R}$ is an open set which contains

$$\pi + \mathbb{Q} = \{\pi + r \mid r \in \mathbb{Q}\}$$

But then $q^{-1}(U) \cap q^{-1}(V) \neq \varnothing$. So, $U \cap V \neq \varnothing$, and $X/\sim$ is not Hausdorff. This example shows that quotient spaces do not, in general inherit the Hausdorff property.

Definition 4.7 (Quotient Map)**.** *Let X and Y be topological spaces; let $f : X \longrightarrow Y$ be a surjective map. The map f is said to be a quotient map provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.*

Definition 4.8 (Saturated)**.** *Let $f : X \longrightarrow Y$ be a surjective map of topological spaces. A subset $S \subset X$ is called saturated (with respect to f) is any $y \in Y$ which satisfies*

$$S \cap f^{-1}(\{y\}) \neq \varnothing$$

also satisfies

$$f^{-1}(\{y\}) \subset S$$

Book: *We say that a subset C of X is saturated (with respect to the surjective map $p : X \longrightarrow Y$) if C contains every set $p^{-1}(\{y\})$ that it intersects. Thus C is saturated if it equals the complete inverse image of a subset of Y. To say that p is a quotient map is equivalent to saying that p is continuous and p maps* saturated *open sets of X to open sets of Y (or saturated closed sets of X to closed sets of Y).*
More: *A set X is saturated if*

$$S = \bigcup_{y \in q(S)} f^{-1}(\{y\})$$

Lemma 4.8. *Let $q : X \longrightarrow Y$ be a continuous surjection. Then q is a quotient map if and only if whenever $V \subset X$ is open and saturated, $q(V) \subset Y$ is open.*

Proof. It's another if-and-only-if, you know the drill...

$\Longrightarrow$: Assume that q is a quotient map. Then q is continuous. Let $V \subset X$ be open and saturated. We claim that

$$V = q^{-1}(q(v))$$

Clearly, $V \subset q^{-1}(q(v))$. Let $x \in q^{-1}(q(v))$, that is, $q(x) \in q(v)$. Then $q^{-1}(q(x)) \cap V \neq \varnothing$. Since V is saturated, $q^{-1}(q(v)) \subset V$. Or

$$V = \bigcup_{y \in q(V)} q^{-1}(\{y\})$$

So that

$$V = q^{-1}(q(v))$$

Since q is a quotient map and $q^{-1}(q(v))$ is open, we know $q(V) \subset Y$ is open.

$\Longleftarrow$: Assume that q is continuous and, if $V \subset X$ is open and saturated, then $q(V)$ is open . Let $U \subset Y$ be a subset such that $q^{-1}(U)$ is open; we need to show that $U \subset Y$ is open. Since $q\left(q^{-1}(U)\right) = U$, it suffices to show that $q^{-1}(U)$ is saturated. Say $y \in Y$ satisfies $q^{-1}(\{y\}) \cap q^{-1}(U) \neq \varnothing$. Then $y \in U$ and $q^{-1}(\{y\}) \subset q^{-1}(U)$

$\blacksquare$

Theorem 4.9. *Let $g : X \longrightarrow Y$ be a quotient map and Z a topological space. If $f : X \longrightarrow Z$ is continuous and takes a single value in each $g^{-1}(\{y\})$, $y \in Y$, then f determines a (unique) continuous function*

$$\widetilde{f} : Y \longrightarrow Z$$

such that

$$f : \widetilde{f} \circ g$$

In pictures:

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[dr, "f"] & \\ Y \arrow[rr, "\widetilde{f}"'] & & Z \end{tikzcd}$$

Proof. (Tells you the quotient definition is the right one.)
Let $f : X \longrightarrow Z$ be continuous and constant in each $q^{-1}(\{y\})$. So, f takes a single value, $z_y \in Z$, in $q^{-1}(\{y\})$. Define

$$\begin{aligned} \widetilde{f} : & Y \longrightarrow Z \\ & y \longmapsto z_y \end{aligned}$$

Then $f = \widetilde{f} \circ g$. Let $U \subset Z$ be open; want $\widetilde{f}^{-1}(U) \subset Y$ is open. Because $U \subset Z$ open, and f is continuous

$$f^{-1}(U) \subset X$$

is open. Then

$$f^{-1}(U) = q^{-1}\left(\widetilde{f}^{-1}(U)\right)$$

Because q is a quotient map and $f^{-1}(U)$ is open, $\widetilde{f}^{-1}(U)$ is open. $\blacksquare$

Theorem 4.10. *Let $p : X \longrightarrow Y$ be a quotient map; let A be a subspace of X that is saturated with respect to p; let $q : A \longrightarrow p(A)$ be the map obtained by restricting p.*

1). If A is either open or closed in X, then q is a quotient map.

2). If p is either an open map or a closed map, then q is a quotient map.

5. Connectedness and Compactness

1. Connectedness

Definition 5.1 (Connected). *A topological space X is called connected if it* **cannot** *be written as a union of disjoint non-empty open sets, $U_1, U_2 \subset X$.*
Book: *Let X be a topological space. A* **separation** *of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be* **connected** *if there does not exist a separation of X.*
Another Way: *A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.*
Extra Comment: *So, to prove that X is connected, assume that*

$$X = U_1 \cup U_2$$

with $U_1, U_2 \subset X$ disjoint and open, and then prove that $U_1 = \varnothing$ or $U_2 = \varnothing$.

Theorem 5.1. *The closed interval $[0, 1]$ is connected (in Euclidean topology). More generally, the intervals (a, b), $[a, b]$, $(a, b]$, $[a, b)$, with $-\infty \leqslant a < b \leqslant \infty$, are connected. In fact, those are the only connected subsets of $\mathbb{R}$.*

Theorem 5.2. *Let $U \subset \mathbb{R}$ (with its subspace topology) be connected. Then U is one of the intervals from theorem* 5.1.

Proof. It suffices to prove that if $a, b \in U$, then $[a, b] \subset U$. Suppose that $c \in [a, b]$ is not in U. The sets

$$V_1 = (-\infty, c) \cap U, \quad V_2 = (c, \infty) \cap U$$

are then open, disjoint and satisfy $V_1 \cup V_2 = U$. Since U is connected, $V_1 = \varnothing$ or $V_2 = \varnothing$, a contradiction, as $a \in V_1$ and $b \in V_2$. ■

Theorem 5.3. *A topological space X is connected if and only if the only subsets of X which are open and closed are $\varnothing, X$.*

Proof. Assume that X is connected and $U \subset X$ is open and closed. Then $X = U \cup (X \setminus U)$. Note that $U \cap (X \setminus U) = \varnothing$ and $X \setminus U$ is open. Since X is connected, $U = \varnothing$ or $U = X$.
Conversely, assume that X is such that the only open sets are $\varnothing$ or X. Say $X = U_1 \cup U_2$ for disjoint open sets $U_i \subset X$. Then $X \setminus U_1 = U_2$ is closed. Hence, $U_2 = \varnothing$ or $U_2 = X$; in the latter case $U_1 = \varnothing$. ■

Theorem 5.4. *Let $f : X \longrightarrow Y$ be continuous function. If X is connected, the image of f, $\{f(x) \mid x \in X\} \subset Y$, is connected.*

Corollary 5.5 (Intermediate Value Theorem). *Let $f : X \longrightarrow \mathbb{R}$ be a continuous function. Assume that X is connected. If $a, b \in f(x)$, then*

$$[a, b] \subset f(x).$$

Book: *Let $f : X \longrightarrow Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.*
The intermediate value theorem of calculus is the special case of this theorem that occurs when we take X to be a closed interval in $\mathbb{R}$ and Y to be $\mathbb{R}$. Click here for examples.

Corollary 5.6. *The quotient of connected topological space is connected. Click here for examples.*

Definition 5.2 (Connected Components). *Let X be a topological space. Define an equivalence relation $\sim$ on X by*

$$x \sim y \text{ if there exists } A \subset X \text{ connected such that } x, y \in A$$

The equivalence class of $\sim$ are called connected components of X.
Book: *Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called the* **components** *(or the "connected components") of X.*
Comment: *Note that a space X is connected is and only if it has a single connected component. So,* $[0, 1]$ *has one connected component, while* $[0, 1] \cup [2, 3]$ *has two.*

Proof. Symmetry and reflexivity of the relation are obvious. Transitivity follows by noting that if A is a connected subspace containing x and y, and if B is a connected subspace containing y and z, then $A \cup B$ is a subspace containing x and z that is connected because A and B have the point y in common.
Or to be cooler: Let $U, V \subset A \cup B$ be open and disjoint such that

$$U \cup V = A \cup B.$$

Assume without loss of generality that $y \in U$. Then

$$U \cap A \neq \varnothing, \;\; U \cap B \neq \varnothing$$

Then

$$A = (U \cap A) \cup (V \cap A)\,, \;\; B = (U \cap B) \cup (V \cap B)$$

Since A is connected, $V \cap A = \varnothing$, that is, $V \subset B$. But then $U \cap B = \varnothing$, as B is connected. This is a contradiction. ■

Theorem 5.7. *Let X, Y be connected topological spaces. Then $X \times Y$ is connected. Click here for examples.*

Theorem 5.8. *The connected components of a topological space X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.*
Comment: *A* **stronger version** *is the connected components of a topological space X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X* **lies** *in exactly one connected component.*

Proof. The construction of connected components as equivalence classes implies that they are pairwise disjoint and cover X. Let $U \subset X$ be connected. Say U intersects two connected components C_1 and C_2. Then $C_1 \cap C_2 \neq \varnothing$ so that $C_1 = C_2$.
We need to check that each connected component C is connected. Fix $x \in C$. if $y \in C$, then there exists a connected subset $A_y \subset X$, such that

$$x, y \in A_u \;\; (\implies \; x \sim y)$$

Then

$$C \subset \bigcup_{y \in C} \underbrace{A_y}_{\text{connected}}$$

And by the result just proved, $A_y \subset C$. Therefore,

$$C = \bigcup_{y \in C} A_y$$

■

Definition 5.3 (Path Connectedness)**.** *Let X be a topological space. Click here for examples.*

1). *A* **path** *in X is a continuous function $f : [a, b] \longrightarrow X$ for some $-\infty < a < b < \infty$.*

2). *Call X* **path connected** *if, for each $x, y \in X$, there exists a path from x to y, that is, a path $f : [a, b] \longrightarrow X$ such that $f(a) = x$, $f(b) = y$.*

3). *The* **path components** *of X are the equivalence classes of X under the equivalence relation*

$$x \sim y \text{ if there exists a path from } x \text{ to } y.$$

2. Compactness

Definition 5.4 (Open Cover)**.** *Let X be a topological space.*

1). *An* **open cover** *of X is an arbitrary collection $\{U_i\}_{i \in I}$ of open subsets of X such that*

$$\bigcup_{i \in I} U_i = X.$$

2). *A* **subcover** *of an open cover $\{U_i\}_{i \in I}$ is a subset $J \subset I$ such that $\{U_j\}_{j \in J}$ is an open cover of X.*

3). *A* **finite subcover** *of an open cover is a subcover as in* 2) *with J a finite set. (A* **finite subcover** *subcover is a subcover with $|J| < \infty$.)*

Definition 5.5 (Compactness)**.** *Click here for examples.*
A topological space X is called **compact** *if every open cover of X has a finite subcover.*
Book: *A space X is said to be* **compact** *if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.*
Comment: *Really difficult to construct examples using this definition. Since it may be easy to show a space is non-compact using it.*

Theorem 5.9. *If X is compact and $C \subset X$ is a closed subset, then C is compact.*
Recall: *If Y is a subspace of X, a collection $\mathcal{A}$ of subsets of X is said to* **cover** *Y if the union of its elements contains Y.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of C. Then $U_i = \widetilde{U}_i \cap C$ for some open subset $\widetilde{U}_i \subset X$. Then

$$\{X \setminus C\} \cup \left\{\widetilde{U}_i\right\}_{i\in I}$$

is an open cover of X. (Since C is closed, $X \setminus C$ is open.)
Since X is compact, there is a finite subcover, say

$$\{X \setminus C\} \cup \left\{\widetilde{U}_j\right\}_{j\in J},$$

with $J \subset I$ finite. Then $\{U_j\}_{j\in J}$ is a finite subcover of $\{U_i\}_{i\in I}$, and C is compact. ■

Theorem 5.10. *Let X be Hausdorff. If $C \subset X$ is a compact subspace, then C is closed.*
Comment: *This the converse of the previous theorem is true only if we made X Hausdorff.*

Proof. We prove that $X \setminus C \subset X$ is open. Let $x \in X \setminus C$, want an open set $W \subset X$ such that $x \in W \subset X \setminus C$. For each $c \in C$, there exists open sets

$$c \in U_c, \;\; x \in V_c$$

and

$$U_c \cap V_c = \varnothing$$

Then $\{U_c \cap C\}_{c\in C}$ is an open cover of C. By compactness, there exists $C_1, \ldots, C_n \in C$ such that

$$\{U_{C_i} \cap C\}_{i=1,\ldots,n}$$

covers C (finite subcover). Then

$$W = V_{C_1} \cap \cdots \cap V_{C_n}$$

is an open subset of X which contains x and is disjoint from C. Note that $x \in V$ and W is open. So, $x \in W \subset X \setminus C$, proving that $X \setminus C$ is open. ■

Theorem 5.11. *Let $f : X \longrightarrow Y$ be continuous function with X is compact. Then $f(X) \subset Y$ is compact.*

Proof. Let $\{U_i\}_{i\in I}$ and open cover of $f(X)$. Write

$$U_i = \widetilde{U}_i \cap f(X)$$

for some $\widetilde{U}_i \subset Y$ is open. Then

$$\left\{f^{-1}\left(\widetilde{U}_i\right)\right\}_{i\in I}$$

is an open cover of X. By compactness of X, there is a finite subcover

$$\left\{f^{-1}\left(\widetilde{U}_j\right)\right\}_{j\in J}$$

Then

$$\{U_j\}_{j\in J}$$

is a finite subcover of $f(X)$. ■

Theorem 5.12. *For any $a < b$, the closed interval $[a, b] \subset \mathbb{R}$ is compact. (Proof is pretty insane, will come back when having more time.)*

Theorem 5.13 (Heine-Borel)**.** *A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.*

Theorem 5.14 (The Theorem of Life Appreciation)**.** *Click here for examples.*
Let $f : X \longrightarrow Y$ be a continuous function which is bijective. If X is compact and Y is Hausdorff, than f^{-1} is continuous. So f is a homeomorphism.

Proof. (Would be a shame to not give it a proof)
We need to show that the images of closed sets of X under f are closed in Y, this will prove the continuity of the map f^{-1}. Let $C \subset X$ be closed. Since X is compact, so is C (Theorem, 5.9). Then $f(C) \subset Y$ is compact (Theorem 5.11). Since Y is Hausdorff, $f(C)$ is closed (Theorem 5.10). ■

Theorem 5.15. *Let X, Y be compact topological spaces. Then $X \times Y$ is compact. (Proof is again pretty insane, will come back when having more time.)*

Corollary 5.16. *The following topological spaces are compact:*

1). $[0,1]^n$ *for each* $n \geqslant 1$.

2). The n*-sphere* $S^n = [0,1]^n/_{\partial[0,1]^n}$ *for each* $n \geqslant 1$.

3). The n*-torus* $\Pi^n = \left(S^1\right)^n$ *for each* $n \geqslant 2$.

4). The real projective spaces $\mathbb{RP}^n$ *for each* $n \geqslant 1$.

Proof. The proofs for the previous corollary:

1). This follows from Theorem 5.12 and 5.15

2) $\sim$ 3). These follow from 1) and the fact that the quotient of a compact space is compact (by Theorem 5.11).

4). Consider the n-sphere

$$S^n = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=1}^{n+1} (x_i)^2 = 1 \right\}$$

Define an equivalence relation on S^n by

$$p \sim -p, \;\; p \in S^n$$

(Note: Equivalence classes in S^n are the antipodal points.) The quotient space $S^n/_\sim$ is $\mathbb{RP}^n$. It is compact since S^n is. (To use that S^n is compact, one can prove that $[0,1]^n/_{\partial[0,1]^n} \simeq S^n$) using Theorem 5.14 or use that $S^n \subset \mathbb{R}^{n+1}$ is closed and bounded and apply Heine-Borel.)

■

Theorem 5.17 (Extreme Value Theorem)**.** *Let $f : X \longrightarrow \mathbb{R}$ be a continuous function. If X is compact, then there exist points $x_m, x_M \in X$ such that*

$$f(x_m) \leqslant f(x) \leqslant f(x_M), \quad x \in X.$$

Proof. The image $f(X) \subset \mathbb{R}$ is compact, by Theorem 5.11. We show that $f(X)$ has a maximal element M and then choose $x_M \in X$ such that $f(x_M) = M$. Let's proceed by contradiction. In this case, if $f(X)$ has no largest element, the collection

$$\{(-\infty, y) \cap f(X) \mid y \in f(X)\}$$

is an open cover of $f(X)$. By compactness, there exists $y_1 < \cdots < y_n$ in $f(X)$ such that

$$\{(-\infty, y_i) \cap f(X) \mid i = 1, \ldots, n\}$$

covers $f(X)$. But then $y_n \notin f(X)$, a contradiction. ■

Definition 5.6 (Distance)**.** *Let (X, d) be a metric space and $A \subset X$ a non-empty subset. The* **distance** *from $x \in X$ to A is*

$$d(x, A) = inf\{d(x, a) \mid a \in A\}$$

Comment: *To show that for fixed A, the function $d(x, A)$ is a continuous function of x: Given $x, y \in X$, one has the inequalities*

$$d(x, A) \leqslant d(x, a) \leqslant d(x, y) + d(y, a),$$

for each $a \in A$. It follows that

$$d(x, A) - d(x, y) \leqslant inf d(y, a) = d(y, A),$$

so that

$$d(x, A) - d(y, A) \leqslant d(x, y).$$

The same inequality holds with x and y interchanged; continuity of the function $d(x, A)$ follows. Recall the **diameter** *of a bounded subset A of a metric space (X, d) is the number*

$$sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}.$$

Theorem 5.18 (The Lebesgue Number Lemma)**.** *Let $\{U_i\}_{i\in I}$ be an open cover of a compact metric space (X, d). Then there exists a $\delta > 0$ (called a* **Lebesgue Number***) such that each (bounded) subset of X of diameter less than δ is contained in some U_i. (Can't even understand the theorem yet, leave the proof for later)*

Definition 5.7 (Uniform Continuity)**.** *A function f from the metric space (X, d_X) to the metric space (Y, d_Y) is said to be* **uniformly continuous** *if given $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points x_0, x_1 of X.*

$$d_X(x_0, x_1) < \delta \implies d_Y(f(x_0), f(x_1)) < \epsilon.$$

Definition 5.8 (Limit Point Compact)**.** *A space X is said to be* **limit point compact** *if every infinite subset of X has a limit point.*

Theorem 5.19. *Compactness implies limit point compactness, but not conversely.*

Definition 5.9 (Sequentially Compact)**.** *Let X be a topological space. If (x_n) is a sequence of points of X, and if*

$$n_1 < n_2 < \cdots < n_i < \ldots$$

is an increasing sequence of positive integers, then the sequence (y_i) defined by setting $y_i = x_{n_i}$ is called a **subsequence** *of the sequence (x_n). The space X is said to be* **sequentially compact** *if every sequence of points of X has a convergent subsequence.*

Theorem 5.20. *Let X be a metrizable space. Then the following are equivalent:*

1). X is compact.

2). X is limit point compact.

3). X is sequentially compact.

6. Surfaces

1. Manifolds

Definition 6.1 (Second Countable)**.** *Click here for examples.*
A topological space X is called **second countable** *if it has a countable (remember "countable", huh? Probably the hard way, eh?) basis for its topology.*
Comment: *Like compactness, second countability should be seen as a "smallness" condition on a topological space.*

Definition 6.2 (n-Dimensional Manifold)**.** *Click here for examples.*

1) *Let $n \geqslant 1$. An $n-$dimensional (topological) manifold is a topological space M such that*
   i) *M is Hausdorff*
   ii) *M is second countable*
   iii) *M is locally Euclidean: for any $m \in M$, there exists an open set $x \in U \subset M$, an open set $\widetilde{U} \subset \mathbb{R}^n$ and a homeomorphism $\varphi : U \longrightarrow \widetilde{U}$.*
2) *A* **surface** *is a 2-manifold.*

Definition 6.3. *Let X, Y be topological spaces. The disjoint union $X \bigsqcup Y$ is the topological space with set $X \bigsqcup Y$ with $U \subset X \bigsqcup Y$ open if*

$$U \cap X \subset X, \text{ and } U \cap Y \subset Y$$

are open.

Definition 6.4 (Connected Sum)**.** *Let M, N be n-manifolds, e.g(exempli gratia), surfaces. The* **connected sum** *$M\#N$ is the n-manifold defined as follows:*

i) *Pick $a \in M$, an open set $a \in U_a \subset M$ and a homeomorphism $\varphi_a : U_a \xrightarrow{\sim} B_1(0) \subset \mathbb{R}^n$, and similarly for $b \in N$, giving (b, U_b, φ_b). (Note: The sets U_a must be sufficiently nice - "regular Euclidean")*

ii) *Consider $M \setminus U_a$ and $N \setminus U_b$. The composition*

$$U_a \xrightarrow{\varphi_a} B_1(0) \xrightarrow{\varphi_b^{-1}} U_b$$

extends to a homeomorphism

$$\Phi : \overline{U}_a \longrightarrow \overline{U}_b$$

sending the boundary S^{n-1} to the boundary S^{n-1}.

iii) *Define*

$$M\#N = (M \setminus U_a) \bigsqcup (N \setminus U_b) /_{\sim}$$

where

$$\partial\overline{U}_a \ni x \sim \Phi(x) \in \partial\overline{U}_b.$$

Remark: *If M and N are connected, then connected sum is connected. That is $M\#N$ is independent of all choices.*

Theorem 6.1. *Every compact connected surface is homeomorphic to exactly one of $\{\Sigma_g\}_{g \geqslant 0}$ or $\{N_g\}_{g \geqslant 0}$.*

1. $\Sigma_g = S^2\#\underbrace{\Pi^2\#\Pi^2\#\ldots\#\Pi^2}_{g\ times}$ *is called closed orientable surface of genus g.*
2. $N_g = \mathbb{RP}^2\#\underbrace{\mathbb{RP}^2\#\mathbb{RP}^2\#\ldots\#\mathbb{RP}^2}_{g\ times}$ *is called closed non-orientable surface of genus g.*

Definition 6.5 (Simplex)**.** *Let k be a positive integer. We say that the vectors $v_0, \ldots, v_k$ are in general position, i.e. they do not all lie in an affine $(k-1)$-plane in $\mathbb{R}^n$. For low $k =$*

1*: The vectors are not equal.*

2*: The vectors are not colinear.*

Define the k-dimensional simplex, or k-simplex spanned by them is

$$\triangle^n = \left\{ a_0 v_0 + a_1 v_1 + \cdots + a_k v_k \mid 0 \leqslant a_i \leqslant 1, \sum_{i=0}^{k} a_i = 1 \right\}$$

In particular, the standard k-simplex is

$$\left\{ (a_0, \ldots, a_k) \in \mathbb{R}^{k+1} \mid 0 \leqslant a_i \leqslant 1, \sum_{i=0}^{k} a_i = 1 \right\}$$

Remark: *A 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and so on.*

1. Fundamental Concepts

1 Sets

1. *DeMorgan's Laws*

$$A - (B \cup C) = (A - B) \cap (A - C)$$
$$A - (B \cap C) = (A - B) \cup (A - C)$$

2. Verbalized version of the *DeMorgan's Laws*

The complement of the union equals the intersection of the complements.
The complement of the intersection equals the union of the complements.

3. *Distributive Law*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

4. *Arbitrary Unions and Intersections*

   i) Given a collection $\mathcal{A}$ of sets, the union of the elements of $\mathcal{A}$ is defined by the equation

$$\bigcup_{A \in \mathcal{A}} A = \{x \mid x \in A \text{ for at least one } A \in \mathcal{A}\}.$$

   ii) The intersection of the elements of $\mathcal{A}$ is defined by the equation

$$\bigcap_{A \in \mathcal{A}} A = \{x \mid x \in A \text{ for every } A \in \mathcal{A}\}.$$

5. *Cartesian Products*
Given sets A and B, we define their cartesian product $A \times B$ to be the set of all ordered pairs (a, b) for which a is an elements of A and b is an element of B. Formally,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

6. *This guy*

$$(a, b) = \bigcup_{i=1}^{\infty} \left[a - \frac{b-a}{i}, b\right)$$

2. Functions

1. *Rule of Assignment*
A **rule of assignment** is a subset r of the cartesian product $C \times D$ of two sets, having the property that each element of C appears as the first coordinate of *at most one* ordered pair belonging to r. Thus, a subset r of $C \times D$ is a rule of assignment if

$$[(c, d) \in r \text{ and } (c, d') \in r] \implies [d = d']$$

2. *Domain and the Image Set*
Given a rule of assignment r, the **domain** of r is defined to be the subset of C consisting of all first coordinates of elements of r, and the **image set** of r is defined as the subset of D consisting of all second coordinates of elements of r. Formally,

$$\text{domain } r = \{c \mid \text{there exists } d \in D \text{ such that } (c, d) \in r\}.$$
$$\text{image } r = \{d \mid \text{there exits } c \in C \text{ such that } (c, d) \in r\}$$

3. *Function*
A **function** f is a rule of assignment r, together with a set B that contains the image set of r, The domain A of the rule r is also called the **domain** of the function f; the image set of r is also called the **image set** of f; and the set B is called the **range** of f.

$$f : A \longrightarrow B$$

4. *Restriction*
   If $f : A \longrightarrow B$ and if A_0 is a subset of A, we define the **restriction** of f to A_0 to be the function mapping A_0 into B whose rule is
   $$\{(a, f(a)) \mid a \in A_0\}$$
   It is denoted by $f_{|A_0}$, which is read "f restricted to A_0."

5. *Composite Function*
   Given functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$, we define the **composite** $g \circ f$ of f and g as the function $g \circ f : A \longrightarrow C$ defined by the equation $(g \circ f)(a) = g(f(a))$. Formally, $g \circ f : A \longrightarrow C$ is the function whose rule is
   $$\{(a, c) \mid \text{ For some } c \in B, f(a) = b \text{ and } g(b) = c\}.$$

6. *Injective, Surjective, Bijective, and Inverse*

   i) A function is $f : A \longrightarrow B$ is said to be **injective** (or **one-to-one**) if for each pair of distinct points of A, their images under f are distinct. Formally, f is injective if
   $$[f(a) = f(a')] \implies [a = a']$$

   ii) A function is $f : A \longrightarrow B$ is said to be **surjective** (or f is said to map A **onto** B) if for every element of B is the image of some element of A under the function f. Formally, f is injective if
   $$[b \in B] \implies [b = f(a) \text{ for at least one } a \in A]$$

   iii) If $f : A \longrightarrow B$ is both injective and surjective, it is said to be **bijective** (or is called a **one-to-one correspondence**).

   iv) If f is bijective, there exists a function from B to A called the **inverse** of f. It is denoted by f^{-1} and is defined by letting $f^{-1}(b)$ be that unique element a of A for which $f(a) = b$.

   Comment: This comment is added solely because the importance it plays in the homework and stuff:

   a). The composite of two injective functions is injective, and the composite of two surjective functions is surjective; it follows that the composite of two bijective functions is bijective.

   b). Given $b \in B$, the fact that f is surjective implies that there exists such an element $a \in A$; the fact that f is injective implies that there is only one such element a. It is easy to see that if f is bijective f^{-1} is also bijective.

   c). The inverse of a composite function $(g \circ f)(a)$ is given by $(g \circ f)^{-1}(a) = f^{-1}\left(g^{-1}(r)\right)$.

   d). Given that $f : A \longrightarrow B$ and $g : B \longrightarrow C$:
   - If $g \circ f$ is injective then f is injective.
   - If $g \circ f$ is surjective then g is surjective.

   so if $g \circ f$ is bijective then f is injective and g is onto, but the converse is not true.

   Proof. The two bullet points and the comment:
   - $f(x) = f(x') \implies g(f(x)) = g(f(x')) \implies (g \circ f)(x) = (g \circ f)(x') \implies x = x'$
   - If $c \in C$ then $\exists\, a$ such that
   $$(a \in A \wedge (g \circ f)(a) = c) \implies (a' \in A \wedge g(f(a')) = c)$$
   Consider $b = f(a) \in B$ then $g(b) = c$.

   Now take $A = \{1, 2\}$, $B = \{3, 4, 5\}$ $C = \{6, 7\}$ and
   $$f(1) = 3, f(2) = 4; g(3) = g(4) = 6, g(5) = 7$$
   f is injective and g is onto but
   $$(g \circ f)(1) = (g \circ f)(2) = 6$$
   which means that $g \circ f$ is not a bijective map. ∎

7. *Image and Preimage*
   Let $f : A \longrightarrow B$. If A_0 is a subset of A, we denote $f(A_0)$ the set of all images of points of A_0 under the function f; this set is called the **image** of A_0 under f. Formally,

$$f(A_0) = \{b \mid b = f(a) \text{ for at least one } a \in A_0\}.$$

   On the other hand, if B_0 is a subset of B, we denote by $f^{-1}(B_0)$ the set of all elements of A whose images under f lie in B_0; it is called the **preimage** of B_0 under f (or the "counterimage," or the "inverse image," of B_0). Formally,

$$f^{-1}(B_0) = \{a \mid f(a) \in B_0\}.$$

   Comment: Note that if $f : A \longrightarrow B$ is bijective and $B_0 \subset B$, we have two meanings for the notation $f^{-1}(B_0)$. It can taken to denote the **preimage** of B_0 under the function f or to denote the **image** of B_0 under the function $f^{-1} : B \longrightarrow A$. These two meanings give precisely the same subset of A, however, so there is, in fact, no ambiguity.

8. $f^{-1}(f(A_0)) \stackrel{?}{=} A_0$ and $f\left(f^{-1}(B_0)\right) \stackrel{?}{=} B_0$
   If $f : A \longrightarrow B$ and if $A_0 \subset A$ and $B_0 \subset B$, then

$$A_0 \subset f^{-1}(f(A_0)) \quad \text{and} \quad f\left(f^{-1}(B_0)\right) \subset B_0.$$

   The first inclusion is an equality if f is injective, and the second inclusion is an equality if f is surjective.

3. Relations

1. *Relation*
   A **relation** on a set A is a subset C of the cartesian product $A \times A$. A relation R can have the following properties:

   i) (Reflexive) xRx for all $x \in A$;
   ii) (Symmetric) $xRy \implies yRx$ for all $x, y \in A$;
   iii) (Transitive) $xRy \wedge yRz \implies xRz$ for all $x, y, z \in A$;
   iv) (Antisymmetric) $xRy \wedge yRx \implies x = y$ for all $x, y, z \in A$;
   v) (Asymmetric) $xRy \implies y\not{R}x$ for all $x, y \in A$;
   vi) (Irreflexive) $x\not{R}x$ for all $x \in A$.

   with $i) \sim iii)$ defines an **equivalence relation** (usually write R as $\sim$) on set A.

2. *Equivalence Class*
   Given an equivalence relation $\sim$ on a set A and an element x of A, we define a certain subset E of A, called the **equivalence class** determined by x, by the equation

$$E = \{y \mid y \sim x.\}$$

   Note that the equivalence class E determined by x contains x, since $x \sim x$.

3. *Partition*
   A **partition** of a set A is a collection of disjoint nonempty subsets of A whose union is all of A.

4. *Order Relation*
   Not mentioned in class and not really needed for the exams and homework, will come back later to add the information.

5. Connectedness and Compactness

Example c.5.5. *Here is some an example for Corollary* 5.5 *(Intermediate Value Theorem).*
Let $f : S^1 \longrightarrow \mathbb{R}$ *be continuous.*
Claim: such a function can never be injective. (Borsuk–Ulam theorem)
Let

$$g : S^1 \longrightarrow \mathbb{R}, x \mapsto f(x) - f(-x)$$

Note that $g(x) = -g(-x)$*. So,* g *is either the zero function or not.*

1) *If* $g = 0$*, then* $f(x) = f(-x)$ *for all* $x \in S^1$*, then* f *is not injective.*
2) *If* $g \neq 0$*, then there exists a* $x_* \in S^1$ *such that*

$$\begin{aligned} &g(x_*) > 0 \text{ or } g(x_*) < 0 \\ \Longrightarrow\ &g(-x_*) < 0 \text{ or } g(-x_*) > 0 \end{aligned}$$

By the IVT, there exists a $x_0 \in S^1$ *such that* $g(x_0) = 0$*, which means* $f(x_0) = f(-x_0)$*, that is,* f *is not injective.*

Example c.5.6. *Here are some examples for Corollary* 5.6*.*

1). S^1 *is connected: it is the quotient of* $[0, 1]$ *by an equivalence relation* $0 \sim 1$*.*
2). *The nasty space* $\mathbb{R}/\sim$ *is connected even not Hausdorff.*

Example t.5.7. *Here are some examples for Theorem* 5.7*.*

1). *The* 2*-sphere* S^2 *is connected:*
$$[0, 1] \times [0, 1]/_{\partial([0,1]\times[0,1])}$$
2). *The two dimensional torus is connected:*
$$[0, 1] \times [0, 1]/_\sim$$
3). *The Klein bottle is connected:*
$$[0, 1] \times [0, 1]/_\sim$$

Click here for the amazing drawings of the three (might update later): Classification of Surfaces

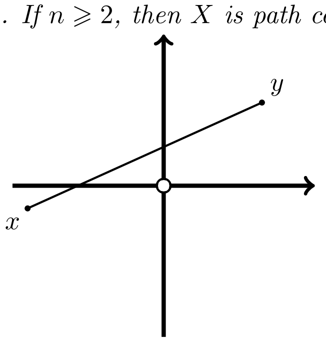
Example d.5.3. *Here are some examples for Definition* 5.3 *(Path Connectedness).*

1). $\mathbb{R}^n$ *is connected. Let* $X = \mathbb{R}^n$*, then* X *is path connected: given* $x, y \in R^n$*, the function*

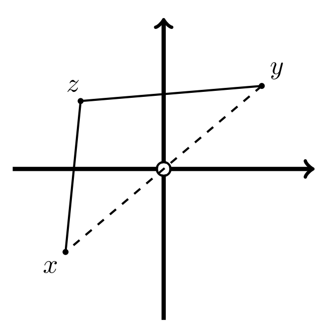
$$f : [0, 1] \longrightarrow \mathbb{R}^n, \quad t \mapsto xt + (1 - t)y$$

is a path from y *to* x*.*

2). *Let* $X = \mathbb{R}^n \setminus \{0\}$*. If* $n \geqslant 2$*, then* X *is path connected. In pictures:*



Case 1



Case 2

Explicitly in case 2*,* $z \in X$ *is a point not on the line though* x *and* y *and* f *is the path*

$$f = \begin{cases} x(1-t) + zt & \text{if } t \in [0, 1] \\ z(2-t) + (t-1)y & \text{if } t \in [1, 2] \end{cases}$$

Moral: if $n = 1$*, then* X *is not path connected. Indeed,* X *has two path components, namely* $(-\infty, 0)$ *and* $(0, \infty)$*, so that any path in* X *has image contained in* $(-\infty, 0)$ *or* $(0, \infty)$*. In particular, there is no path from* 1 *to* -1 *in* X*.*

3). *For each $n \geqslant 1$, the n-sphere*

$$S^n = \{x \in \mathbb{R}^n \mid ||x||^2 = 1\}$$

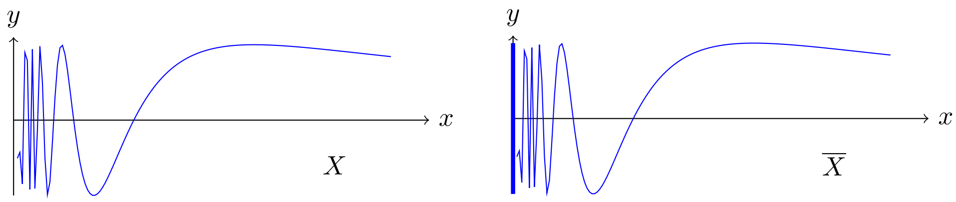
is (path) connected. Indeed, let $x, y \in S^n$. The great arc from x to y is a path from x to y.

4). *Let*

$$X = \left\{ \left(x, \sin\frac{1}{x}\right) \Big| 0 < x \leqslant 1 \right\} \subset \mathbb{R}^2$$

Then X is (path) connected, being the image of the (path) connected space $(0,1]$ under the continuous map $\sin\frac{1}{x}$. The closure $\overline{X}$ is called the **Topologist's Sine Curv e**.

$$\overline{X} = X \cup \{(0, y) \mid -1 \leqslant y \leqslant 1\}.$$



(Note: Change the sample size in the LaTeX source to make the graph smoother.)

The closure is connected. However, $\overline{X}$ is **not** *path connected. Say $f : [0,1] \longrightarrow \overline{X}$ is a continuous function with*

$$f(0) = (0,0), \quad f(1) \in X$$

Since $A = \{0\} \times [-1,1] \subset \overline{X}$ is closed, so too is $f^{-1}(A) \subset [0,1]$. So, $f^{-1}(A)$ has a maximal element, say, a. Then $f(a) \in A$ while $f(t) \in X$ if $t > a$.
Write f in components: $f(t) = (x(t), y(t))$. Then $x(a) = 0$ and $x(t) > 0$ and $y(t) = \sin\frac{1}{x(t)}$ if $t > a$. By the condition $x(t) > 0$ if $t > a$, for each $n \geqslant 1$, there exists a

$$0 < u < x\left(a + \frac{1}{n}\right)$$

such that $\sin\frac{1}{u} = (-1)^n$. By the Intermediate Value Theorem, there exists a $0 < t_n < \frac{1}{n}$ such that $x(t_n) = u$. Then $\{t_n\}_n$ converges to 0 but $y(t) = (-1)^n$ does not converge, a contradiction.

Example d.5.5. *Here are some examples for Definition 5.5.*

1). *Let X be a finite topological space. Then X is compact.*
Let $\{U_i\}_{i \in I}$ be an arbitrary open cover. So,

$$X = \bigcup_{i \in I} U_i$$

For each $x \in X$, choose $i(x) \in I$ such that

$$x \in U_{i(x)}$$

Then

$$X = \bigcup_{x \in X} U_{i(x)}$$

So, $\{U_{i(x)}\}_{x \in X}$ is a finite subcover.

2). *Let $X = \left\{\frac{1}{n} \mid n \geqslant 1, n \in \mathbb{Z}\right\} \subset \mathbb{R}$. To prove it's not compact, we need to provide an open cover with no finite cover.*
For each $n \geqslant 1$, let $\widetilde{U}_n \subset \mathbb{R}$ be the open interval

$$\widetilde{U}_n = \left(\frac{1}{n} - \widetilde{\epsilon}_n, \frac{1}{n} + \epsilon_n\right)$$

3). Let $X = \overline{\left\{\frac{1}{n} \mid n \geqslant 1\right\}} = \{0\} \cup \left\{\frac{1}{n} \mid n \geqslant 1\right\}$. *Let* $\{U_i\}_{i \in I}$ *be an open cover of* X. *Let* $i_0 \in I$ *such that* $0 \in U_{i_0}$. *Since* U_{i_0} *is open, there exists an* $N > 0$ *such that* $\frac{1}{n} \in U_{i_0}$ *whenever* $n \geqslant N$. *For* $1, \ldots, N-1$, *choose open sets* $U_{i_1}, \ldots, U_{i_{N-1}}$ *such that* $\frac{1}{n} \in U_{i_n}$ *for* $n = 1, \ldots, N-1$. *Then*

$$\left\{U_{i_0}, U_{i_1}, \ldots, U_{i_{N-1}}\right\}$$

is a finite subcover. So, X *is compact.*

4). The open interval $(0,1)$ *is not compact. Indeed, consider the open cover* $\left\{\left(\frac{1}{n}, \frac{1}{n+2}\right) \mid n \geqslant 1\right\}$.

Let $I \subset \mathbb{Z}_{>0}$ *be a finite subset. Let* $N > 0$ *such that* $N > i$ *for all* $i \in I$. *Then* $\frac{1}{N+4} \notin \bigcup_{i \in I} U_i$. *So, this open cover has no finite subcover.*

Example t.5.14. *Here is an example for Theorem* 5.14.
Recall the diagram:

$$\begin{tikzcd}
& {[0,1]} \arrow[dl, "q"'] \arrow[dr, "f(x) = (\cos 2\pi x, \sin 2\pi x)"] & \\
{[0,1]/\sim} \arrow[rr, "\widetilde{f}"'] & & S^1
\end{tikzcd}$$

The space $[0,1]$ *is compact (Theorem* 5.12*), and hence so too is* $[0,1]/\sim$ *(Theorem* 5.11*). The space* S^1 *is Hausdorff, being a subspace of the Hausdorff space* $\mathbb{R}^2$ *(Theorem 2.20). The function* $\widetilde{f}$ *is a continuous bijection and so is a homeomorphism by Theorem* 5.14.

6. Surfaces

Example d.6.1. *Here are some examples for Definition* 6.1.

1). $\mathbb{R}$ *is second countable:*

$$\mathcal{B}' = \{B_\epsilon(x) \mid x \in \mathbb{Q}, \epsilon \in \mathbb{Q} \cap (0, \infty)\}$$

2). If X, Y *are second countable, then* $X \times Y$ *is second countable.*

3). If X *is second countable,* $A \subset Y$ *is second countable.*

4). $\mathbb{R} \times \mathbb{R}_{dis}$ *is not second countable. (Note: the discrete topology by itself is not second countable.)*

Comment: $1) \sim 3)$ *implies that any subspace of* $\mathbb{R}^n$ *is second countable.*

Example d.6.2. *Here are some examples for Definition* 6.2.

1). The sphere S^n *is an* n*-manifold:*

- *i). It is a subspace of* $\mathbb{R}^{n+1}$*, hence Hausdorff*
- *ii). It is a subspace of* $\mathbb{R}^{n+1}$*, hence second countable*
- *iii). The locally Euclidean condition can be seen by intersection with open hemispheres:*

1. Sets and Continuity

Question 1.1 (Finite and Infinite Sets). *The following are checking basic understandings of set theory and the foundations of the axioms of topology.*

(a). *A finite subset of $\mathbb{R}^n$ is closed.*

Proof. This is true. Consider $A = \{x_1, \ldots, x_n\} \subset \mathbb{R}^n$ a finite subset. Then we can write

$$A = \left(\bigcup_{i=1}^{n} (\mathbb{R}^n \setminus \{x_i\}) \right)^c$$

As the sets $\mathbb{R}^n \setminus \{x_i\}$ are open, so is their intersection. Thus A is closed, being the complement of an open set. ∎

(b). *If τ and σ are topologies on a set X, then so too are*

$$\tau \cap \sigma := \{U \mid U \in \tau \text{ and } Y \in \sigma\} \quad \text{and} \quad \tau \cup \sigma := \{U \mid U \in \tau \text{ or } Y \in \sigma\}$$

Proof. This is false. The intersection of two topologies is again a topology using the definition of a topology. However the union of two topologies is not necessarily a topology again. Consider $A = \{1, 2, 3\}$. Then $\tau = \{\varnothing, \{1\}, \{1, 2\}, A\}$ is a topology on A and $\sigma = \{\varnothing, \{3\}, \{2, 3\}, A\}$ is as well, but in their union the finite intersection $\{1, 2\} \cap \{2, 3\} = \{2\}$ is not contained. ∎

(c). *Let $\{A_i\}_{i \in I}$ be an arbitrary collection of closed subsets of $\mathbb{R}^n$. Then $\bigcup_{i \in I} A_i \subset \mathbb{R}^n$ is closed.*

Proof. This is false. It suffices to produce a counterexample for $n = 1$. For each positive integer i, define a closed subset of $\mathbb{R}$ by

$$A_i = \left[0, 1 - \frac{1}{i}\right]$$

The union $\bigcup_{i=1}^{\infty} A_i = [0, 1)$ is not closed: every open interval centered at 1 intersects A. ∎

Question 1.2 (Proving Topologies). *The following are checking the understanding the axioms of topology.*

(a). *Let*

$$\tau = \{U \subset \mathbb{R} \mid \text{ for all } u \in U, \text{ there exists } a < b \text{ such that } u \in [a, b) \subset U\}.$$

Prove that τ is a topology on $\mathbb{R}$.

Proof. We need to verify the three axioms for a topology:

i. Any real number lies in an interval of the form $[a, b)$, thus it is clear that $\varnothing$, $\mathbb{R}$ lie in τ.

ii. Let $\{U_i\}_{i \in I}$ be an arbitrary collection of elements of τ and set $U = \bigcup_{i \in I} U_i$. Let $u \in U$. Then $u \in U_i$ for some $i \in I$. Since U_i is open, there exists an $a \in \mathbb{R}$ such that $u \in [a, b) \subset U_i$. But then $u \in [a, b) \subset U$, so that $U \in \tau$.

iii. Let $\{U_i\}_{i \in I}$ be an finite collection of elements of τ and set $U = \bigcap_{i \in I} U_i$. Let $u \in U$. Then $u \in U_i$ for each $i \in I$. Since U_i is open, there exists $a_i, b_i \in \mathbb{R}$ such that $u \in [a_i, b_i)$. Let $a = \max\{a_i \mid i \in I\}$ and $b = \min\{b_i \mid i \in I\}$. Since I is finite, a and b are well-defined real numbers, and they still fulfill $a < b$. Then $u \in [a, b)$ and, since $[a, b) \subset [a_i, b_i) \subset U_i$ for all $i \in I$, we have $u \in [a, b) \subset U$. Hence, $U \in \tau$.

∎

(b). *Let*

2. Topological Spaces

Question 2.1 (Basis). *The following are checking basic understandings of basis of a topology.*

(a). What is a basis for a topology on $\mathbb{R}$?

Proof. ■

Question 2.2 (Subspace Topology). *The following are checking basic understandings of subspace construction of topological spaces.*

(a). Proof. ■

Question 2.3 (Product Topology). *The following are checking basic understandings of product construction of topological spaces.*

(a). Proof. ■

Question 2.4 (Hausdorff). *Prove that a topological space X is Hausdorff if and only if the diagonal*

$$\Delta = \{(x,x) \mid x \in X\}$$

is closed in $X \times X$

Proof. ■

Question 2.? (Interiors and Closures). *The following are checking the understandings of interiors and closures*

(a). $Int(\mathbb{Q}) = \varnothing$, $\overline{\mathbb{Q}} = \mathbb{R}$

Proof. For the interior, since it's the union of all open sets contained in $\mathbb{Q}$, then $\forall q \in \mathbb{Q}$, $\forall \epsilon > 0$, $B_\epsilon q = \{x \in \mathbb{R} \mid |x - q| < \epsilon\}$ contains irrational numbers that are not in $\mathbb{Q}$. So the only open interval we can have is the $\varnothing$, hence, $\text{Int}(\mathbb{Q}) = \varnothing$. Note, if the whole set is $\mathbb{Q}$, then $\text{Int}(\mathbb{Q}) = \mathbb{Q}$.
For the closure, since it's the intersection of all closed sets of $\mathbb{R}$ which contain $\mathbb{Q}$, we can assume $\mathbb{Q} \subset C \subset \mathbb{R}$, for some closed set C. Then, $\mathbb{R} \setminus C$ must be open since the complement of a closed set is open. However, every single open interval of $\mathbb{R}$ has to include some rational numbers, and $\mathbb{Q}$ is out of the picture because $\mathbb{Q} \subset C$. Therefore, $\mathbb{R} \setminus C$ can only be $\varnothing$, then $C = \mathbb{R}$. ■

(b). Find the closure, interior and boundary of each of the following subsets of $\mathbb{R}^2$

i)

3. Continuous Functions

4. More Topological Spaces

5. Connectedness and Compactness

6. Surfaces