In [ ]:
```
import sympy as sp
from scipy import *
from sympy import *
init_printing()
from IPython.display import display, Latex, HTML, Math
import numpy as np
import pandas as pd
from sympy import Rational as R
```

Assignment 1

In [ ]:
```
# a)
# 0, 1, and 2 that correspond to that they do not intersect, they intersect in a
```

In [ ]:
```
# b) For any two distinct...is the right answer since this is just nonsense.
# Why would distinct vectors be linearly independent?
```

In [ ]:
```
# c) Only 2 is the right answer. A^2 contain negatives.
A = Matrix([[0,-1],[1,0]])
A, A**2
```

Out[ ]: $\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)$

In [ ]:
```
# Here's an example where 3 is false. In general it will not be true when we hav
# with different signs (1, -1; 2, -2, etc)
A = Matrix([[1,1],[0,-1]])
A.eigenvals(), (A**2).eigenvals()
```

Out[ ]: $(\{-1 : 1,\ 1 : 1\},\ \{1 : 2\})$

$$\det\left(A^{-1}\right) = (\det(A))^{-1}$$
$$\det(k \cdot A) = k^n \det(A), \text{ so}$$
$$\det\left(-2A^{-1}\right) = (-2)^n = 8^{-1} = 2$$

$$\det\left(A^\top A\right) = \det\left(A^\top\right) \cdot \det(A) = 8 \times 8 = 64$$

$$\det\left(A^{-1} \cdot A^\top \cdot A\right) = (\det(A))^{-1} \cdot \det(A) \cdot \det\left(A^\top\right) = 8$$

$$\det(A) = 2^4 \det\,(A) = 2^4 \cdot 8 = 2^4 \cdot 2^3 = 2^7 = 128$$

Assignment 2

In [ ]:
```
# a)
c, d = symbols('c d')
A = Matrix([[1, R(1,3)], [c, d]])
X = A**2 + eye(2)
solve((X[0,0]+X[0,1], X[1,0]+X[1,1]), (c,d))
```

Out[ ]: $[(-6, \ -1)]$

a. Let $A = \begin{bmatrix} 1 & 1/3 \\ c & d \end{bmatrix}$. Find the the values of c and d such that $A^2 = -I_2$. State your answer as positive integers. Please note that negative signs have been pre-printed.

$c = -\boxed{6}$

$d = -\boxed{1}$

In [ ]:
```python
# Check
A = Matrix([[1, R(1,3)], [-6, -1]])
A**2 == - eye(2)
```

Out[ ]: True

In [ ]:
```python
# b)
A = Matrix([[1,0,0],[4,1,0],[0,4,1]])
b = Matrix([[1,0],[0,1],[1,1]])

A**-1 * b
```

Out[ ]:

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \\ 17 & -3 \end{bmatrix}$$

b. Find a matrix X such that $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

State all inputs as positive integers.

$$X = \begin{bmatrix} \boxed{1} & \boxed{0} \\ -\boxed{4} & \boxed{1} \\ \boxed{17} & -\boxed{3} \end{bmatrix}$$

Assignment 3

In [ ]:
```python
A = Matrix([[-1, -2, 0, 1, -2, 2], [1, 2, 3, -5, 15, -11], [2, 4, 1, -1, 6, 7]])
A[2,:] = A[2,:] + 2*A[0,:]
A
```

Out[ ]:

$$\begin{bmatrix} -1 & -2 & 0 & 1 & -2 & 2 \\ 1 & 2 & 3 & -5 & 15 & -11 \\ 0 & 0 & 1 & 1 & 2 & 11 \end{bmatrix}$$

In [ ]:
```python
A[1,:] = A[1,:] + A[0,:]
A
```

Out[ ]:

$$\begin{bmatrix} -1 & -2 & 0 & 1 & -2 & 2 \\ 0 & 0 & 3 & -4 & 13 & -9 \\ 0 & 0 & 1 & 1 & 2 & 11 \end{bmatrix}$$

In [ ]:
```
A[1,:] = A[1,:] - 3*A[2,:]
A
```

Out[ ]:

$$\begin{bmatrix} -1 & -2 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & -7 & 7 & -42 \\ 0 & 0 & 1 & 1 & 2 & 11 \end{bmatrix}$$

In [ ]:
```
A[1,:] = -R(1,7) * A[1,:]
A
```

Out[ ]:

$$\begin{bmatrix} -1 & -2 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & 1 & 2 & 11 \end{bmatrix}$$

In [ ]:
```
R2 = A[1,:]
R3 = A[2,:]
```

In [ ]:
```
A[1,:] = R3
A[2,:] = R2
A
```

Out[ ]:

$$\begin{bmatrix} -1 & -2 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 & 2 & 11 \\ 0 & 0 & 0 & 1 & -1 & 6 \end{bmatrix}$$

In [ ]:
```
A[1,:] = A[1,:] - A[2,:]
A
```

Out[ ]:

$$\begin{bmatrix} -1 & -2 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & -1 & 6 \end{bmatrix}$$

In [ ]:
```
A[0,:] = -1*A[0,:]
A
```

Out[ ]:

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 2 & -2 \\ 0 & 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & -1 & 6 \end{bmatrix}$$

In [ ]:
```
A[0,:] = A[0,:] + A[2,:]
A
```

Out[ ]:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & -1 & 6 \end{bmatrix}$$

In [ ]:
```
Matrix([[-1, -2, 0, 1, -2, 2], [1, 2, 3, -5, 15, -11], [2, 4, 1, -1, 6, 7]]).rre
```

Out[ ]:

$$\left(\begin{bmatrix} 1 & 2 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & -1 & 6 \end{bmatrix}, \ (0, 2, 3) \right)$$

So we get

$$\left.\begin{aligned} a &= -2b - e + 4 \\ b &= b \\ c &= -3e + 5 \\ d &= e + 6 \\ e &= e \end{aligned}\right\} = \begin{bmatrix} 4 - 2b - e \\ b \\ 5 - 3e \\ 6 + e \\ e \end{bmatrix}$$

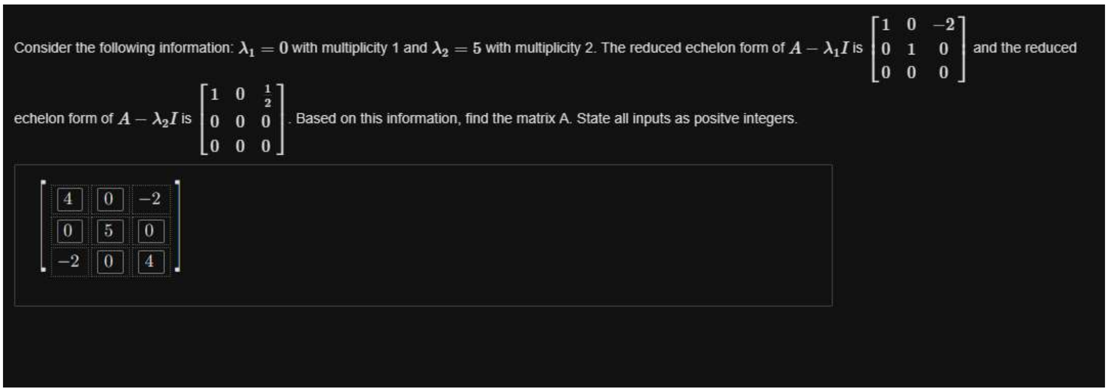
Assignment 4

In [ ]:

```
A0 = Matrix([[1,0,-2],[0,1,0],[0,0,0]])
A5 = Matrix([[1,0,(1)/(2)],[0,0,0],[0,0,0]])
v1 = A0.nullspace()[0]
v2 = A5.nullspace()[0]
v3 = A5.nullspace()[1]
P = Matrix.hstack(v1, v2, v3)
D = diag(0,5,5)
Pinv = P**-1
A = P*D*Pinv
A
```

Out[ ]:

$$\begin{bmatrix} 1.0 & 0 & -2.0 \\ 0 & 5 & 0 \\ -2.0 & 0 & 4.0 \end{bmatrix}$$



Consider the following information: $\lambda_1 = 0$ with multiplicity 1 and $\lambda_2 = 5$ with multiplicity 2. The reduced echelon form of $A - \lambda_1 I$ is $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and the reduced echelon form of $A - \lambda_2 I$ is $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Based on this information, find the matrix A. State all inputs as positve integers.

$$\begin{bmatrix} 4 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

Assignment 5

In [ ]:

```
# a)
A = Matrix([1, -2, 3]).T
v1 = A.nullspace()[0]
v2 = A.nullspace()[1]

# Make them orthonormal
u1 = GramSchmidt([v1, v2], True)[0]
u2 = GramSchmidt([v1, v2], True)[1]

u1, u2
# Remember to factor out so that the values match
```

Out[ ]:

$$\left(\begin{bmatrix} \frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3\sqrt{70}}{70} \\ \frac{3\sqrt{70}}{35} \\ \frac{\sqrt{70}}{14} \end{bmatrix} \right)$$

Let

$$W = \left\{ \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x - 2y + 3z = 0 \right\}$$

a. Find an ortonormal basis of W.

$$\left(\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{\sqrt{70}} \\ \frac{3\sqrt{70}}{35} \\ \frac{\sqrt{70}}{14} \end{bmatrix} \right)$$

In [ ]:

```
# b)
Matrix.vstack(u1.T, u2.T).nullspace()[0].normalized()
```

Out[ ]:

$$\begin{bmatrix} \frac{\sqrt{14}}{14} \\ -\frac{\sqrt{14}}{7} \\ \frac{3\sqrt{14}}{14} \end{bmatrix}$$

Assignment 6

In [ ]:

```
# a)
u1 = Matrix([[1],[2],[1],[1]])
u2 = Matrix([[-2],[1],[-1],[1]])
u3 = Matrix([[1],[1],[-2],[-1]])
u4 = Matrix([[-1],[1],[1],[-2]])
v = Matrix([[4],[2],[-1],[0]])
U = Matrix.hstack(u1, u2, u3, u4)
U.T * U
```

Out[ ]:

$$\begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

In [ ]:

```
# b)
vhat = v.project(u1) + v.project(u2) + v.project(u3)
(v-vhat).norm()
```

Out[ ]:

$$\frac{3\sqrt{7}}{7}$$

Assignment 7

In [ ]:
```
AAt = Matrix([[5,-2,-4],[-2,8,-2],[-4,-2,5]])
AtA = Matrix([[5,-2,-4],[-2,8,-2],[-4,-2,5]])
AAt, AtA
```

Out[ ]:

$$\left(\begin{bmatrix}5 & -2 & -4\\-2 & 8 & -2\\-4 & -2 & 5\end{bmatrix}, \begin{bmatrix}5 & -2 & -4\\-2 & 8 & -2\\-4 & -2 & 5\end{bmatrix}\right)$$

In [ ]:
```
# a)
AtA.eigenvects()
```

Out[ ]:

$$\left[\left(0,\ 1,\ \left[\begin{bmatrix}1\\\frac{1}{2}\\1\end{bmatrix}\right]\right),\ \left(9,\ 2,\ \left[\begin{bmatrix}-\frac{1}{2}\\1\\0\end{bmatrix},\ \begin{bmatrix}-1\\0\\1\end{bmatrix}\right]\right)\right]$$

In [ ]:
```
# a)
vecs = AtA.eigenvects()
vecs
vecs[0][2][0]*2
```

Out[ ]:

$$\begin{bmatrix}2\\1\\2\end{bmatrix}$$

In [ ]:
```
# b) First find values for SVD. The given matrix is U so we need to find V
s1 = sqrt(vecs[1][0])
s2 = sqrt(vecs[1][0])
s3 = sqrt(vecs[0][0])
v1 = GramSchmidt([vecs[1][2][0], vecs[1][2][1]], True)[0]
v2 = GramSchmidt([vecs[1][2][0], vecs[1][2][1]], True)[1]
v3 = vecs[0][2][0].normalized()

# So V is
V = Matrix.hstack(v1, v2, v3)
Vt = V.T

# We get U from the assignment
U = Matrix([[(sqrt(5))/(3),0,(2)/(3)],
            [-(2*sqrt(5))/(15),-(2*sqrt(5))/(5),(1)/(3)],
            [-(4*sqrt(5))/(15),(sqrt(5))/(5),(2)/(3)]])

# We need Sigma
S = diag(s1, s2, s3)

# We find A
display(Latex("$$A = {}{}{}$$".format(latex(U), latex(S), latex(Vt))))
A = U*S*Vt
A
```

$$A = \begin{bmatrix} \frac{\sqrt{5}}{3} & 0 & 0.666666666666667 \\ -\frac{2\sqrt{5}}{15} & -\frac{2\sqrt{5}}{5} & 0.333333333333333 \\ -\frac{4\sqrt{5}}{15} & \frac{\sqrt{5}}{5} & 0.666666666666667 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} & 0 \\ -\frac{4\sqrt{5}}{15} & -\frac{2\sqrt{5}}{15} & \frac{\sqrt{5}}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Out[ ]:

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

b. It is known that the columns of the matrix below are the left singular vectors of A corresponding to the eigenvectors and eigenvalues found in (a).

$$\begin{bmatrix} \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \\ -\frac{2\sqrt{5}}{15} & -\frac{2\sqrt{5}}{5} & \frac{1}{3} \\ -\frac{4\sqrt{5}}{15} & \frac{\sqrt{5}}{5} & \frac{2}{3} \end{bmatrix}$$

Find matrix A. State all inputs as positive integers.

$$A = \begin{bmatrix} -1 & \boxed{2} & \boxed{0} \\ \boxed{2} & \boxed{0} & -2 \\ \boxed{0} & -2 & \boxed{1} \end{bmatrix}$$

In [ ]:

```
# C)

# we get n and m from size of U and V so n = 3, m = 3
# Col A = Rank. 2 nonzero singular values, so Dim Col A = 2
# Dim Nul A = n - Dim Col A = 1
# Dim Row = Rank A = 2
# Dim Nul A.T = n - Rank A = 1
```

Assignment 8

It is suggested that the data

t	p
-7,0	-11,23
-5,0	-4,47
-3,0	-1,12
-1,0	-0,05
1,0	0,06
3,0	1,14
5,0	4,53
7,0	11,56

can be modelled by an equation of the form $p = \beta_0 t + \beta_1 \cdot t^3$. Find the model of this type that produces the least-squares fit of the data and state the length of the error vector. State your answer as an integer between 10 and 99 so that the result is displayed with two decimal precision.

In [ ]:

```
x = pd.DataFrame([-7, -5, -3, -1, 1, 3, 5, 7])
y = pd.DataFrame([-11.23, -4.47, -1.12, -0.05, 0.06, 1.14, 4.53, 11.56])
```

In [ ]:

```
X1 = Matrix(x).row_join(Matrix(x**3))

display(Math(r'X_1 = ' + latex(X1)))
```

$$X_1 = \begin{bmatrix} -7 & -343 \\ -5 & -125 \\ -3 & -27 \\ -1 & -1 \\ 1 & 1 \\ 3 & 27 \\ 5 & 125 \\ 7 & 343 \end{bmatrix}$$

In [ ]:
```
X1tX1 = X1.T*X1
X1ty = X1.T*Matrix(y)
Mat, _ = X1tX1.row_join(X1ty).rref()
B1 = Mat[:,-1]
B1
```

Out[ ]:
$$\begin{bmatrix} 0.109443542568544 \\ 0.0310542929292929 \end{bmatrix}$$

In [ ]:
```
display(Math(r'X_1^TX_1 = ' + latex(X1tX1)))
display(Math(r'X_1^Ty = ' + latex(X1ty)))
```

$$X_1^T X_1 = \begin{bmatrix} 168 & 6216 \\ 6216 & 268008 \end{bmatrix}$$

$$X_1^T y = \begin{bmatrix} 211.42 \\ 9003.1 \end{bmatrix}$$

In [ ]:
```
display(Latex("$$y_1(t) = {}+{}t^3$$".format(round(B1[0],2), round(B1[1], 5))))
```

$$y_1(t) = 0.11 + 0.03105t^3$$

In [ ]:
```
display(Latex("$$e_1 = {}$$".format(round((Matrix(y)-X1*B1).norm(), 4))))
```

$$e_1 = 0.2914$$

Assignment 9

In [ ]:
```
# We translate the problem to a matrix problem

A = Matrix([[4, 0, 4],[0, 4, 4], [4, 4, 8]])
display(Math(r'A = ' + latex(A)))
A.charpoly()
```

$$A = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix}$$

Out[ ]: $\mathrm{PurePoly}\left(\lambda^3 - 16\lambda^2 + 48\lambda, \lambda, domain = \mathbb{Z}\right)$

In [ ]:
```
# We find the eigenvalues and the corresponding eigenspaces.
l = symbols('l')
l1, l2, l3 = solve(det(A-l*eye(np.shape(A)[0])))
```

```
display(Math(r'\lambda_1 =' +latex(l1) + r'\approx' + latex(round(l1, 2))))
display(Math(r'\lambda_2 =' +latex(l2) + r'\approx' + latex(round(l2, 2))))
display(Math(r'\lambda_3 =' +latex(l3) + r'\approx' + latex(round(l3, 2))))

v1 = (A-l1*eye(np.shape(A)[0])).nullspace()[0]
v2 = (A-l2*eye(np.shape(A)[0])).nullspace()[0]
v3 = (A-l3*eye(np.shape(A)[0])).nullspace()[0]
display(Math(r'v_1 = '+ latex(v1) + r'=' + latex(v1.evalf(4))))
display(Math(r'v_2 = '+ latex(v2) + r'=' + latex(v2.evalf(4))))
display(Math(r'v_3 = '+ latex(2*v3) + r'=' + latex(2*v3.evalf(4))))
```

$\lambda_1 = 0 \approx 0$

$\lambda_2 = 4 \approx 4$

$\lambda_3 = 12 \approx 12$

$$v_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.0 \\ -1.0 \\ 1.0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1.0 \\ 1.0 \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 1.0 \\ 2.0 \end{bmatrix}$$

In [ ]:

```
y0 = Matrix([6, 12, 12])

# To solve the system we form the following augmented matrix and solve. The c's
C = v1.row_join(v2).row_join(2*v3).row_join(y0).rref()[0][:, -1]
C
```

Out[ ]:

$$\begin{bmatrix} -2 \\ 3 \\ 7 \end{bmatrix}$$

So the system in vector form is:

$$\bar{y}(t) = -2\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{0 \cdot t} + 3\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot e^{4t} + 7\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot e^{12t}$$

We can transform this into three equations by multiplying into the vectors and then stating each entry as an equation:

$$\bar{y}(t) = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix} \cdot e^{4t} + \begin{bmatrix} 7 \\ 7 \\ 14 \end{bmatrix} \cdot e^{12t}$$

We get

$$y_1 = 2 - 3 \cdot e^{4t} + 7e^{12t}$$
$$y_2 = 2 + 3 \cdot e^{4t} + 7e^{12t}$$
$$y_3 = -2 + 14 \cdot e^{12t}$$