4. Vector spaces

A "bunch" of vectors where you are able to take linear combinations (add and scale):

- A set of vectors
- Is non-empty
- Addition and scalar multiplication are defined
- Always contains the zero vector
- 10 Axioms apply, see p. 190

Ex

$\mathbb{R}^2$ = All 2-d real vectors: $\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\pi\\e\end{bmatrix}$

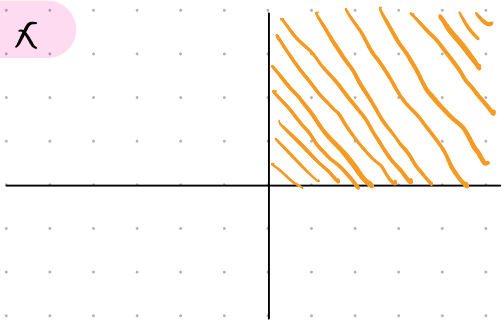
$\mathbb{R}^3$ = All 3-d real vectors: $\bar{v} = \begin{bmatrix}3\\2\\0\end{bmatrix} \in \mathbb{R}^3$

$\mathbb{R}^n$ = All n-dim real vectors: $\bar{w} = \begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix} \in \mathbb{R}^n$

Closure:

We say a vector space must be closed under multiplication and addition

Ex



← all vectors with non-negative components.

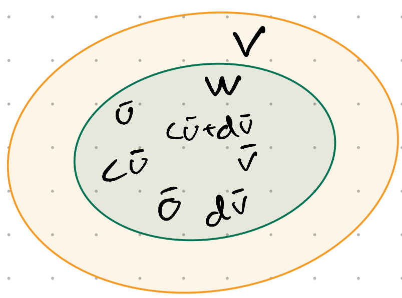
- Can I add and stay?
- Can I scale and stay?
  - No e.g. $-5\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}-10\\-5\end{bmatrix}$

Subspace:

A "smaller" space that satisfies closure. Let W be a subspace of V:

↓

> A subspace is a vector space in its own right.



Ex

Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$? **No!** not even subset.

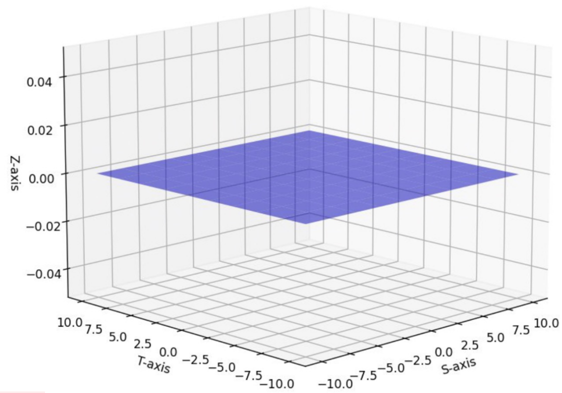
All vectors in $\mathbb{R}^2$ have 2-entries $\Big\}$ logically
All vectors in $\mathbb{R}^3$ have 3-entries $\Big\}$ distinct

Ex

Is $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} \right\}$, $s,t \in \mathbb{R}$ a subspace?

① contains $\bar{0}$ when $s,t = 0$

② Is closed since last entry is 0.



Theorem:

If $\bar{v}_1 \dots \bar{v}_p$ are in a vector space V,
then $\text{span}\{\bar{v}_1 \dots \bar{v}_p\}$ is a subspace of V

Ex

a and b are scalars and H is the set of all vectors of the form $(a+2b, b, a-b, a)$.
Is H a subspace of $\mathbb{R}^4$?

$$\begin{bmatrix} a+2b \\ b \\ a-b \\ a \end{bmatrix} = a \underset{\bar{v}_1}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}} + b \underset{\bar{v}_2}{\begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}}$$

Thus $H = \text{span}\{\bar{v}_1, \bar{v}_2\}$ is a subspace of $\mathbb{R}^4$

Ex

$$\bar{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \bar{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \bar{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

For what values h will y be in the subspace spanned by $\bar{v}_1$, $\bar{v}_2$ and $\bar{v}_3$?

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix} \underset{\sim}{E.F.} \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}, \quad h=5$$

The Nullspace

The set of all solutions of the homogeneous equation $A\bar{x}=\bar{0}$ and is denoted $Nul\, A$

The null space contains x's.

Consider:

$$\begin{aligned} x_1 - 3x_2 - 2x_3 &= 0 \\ -5x_1 + 9x_2 + x_3 &= 0 \end{aligned} \rightarrow \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \underset{\sim}{E.F} \begin{bmatrix} 1 & 0 & 5/2 \\ 0 & 1 & 3/2 \end{bmatrix}$$

The set of all $\bar{x}$ that satisfy this is the solution set. We call the set that satisfies $A\bar{x}=0$ the null space.

EX

Does $\bar{v} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ belong to nul A?

$$\begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5-9+4 \\ -25+27-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \bar{0}$$

Parametric vector Form (PVF):

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -5/2 \\ -3/2 \\ 1 \end{bmatrix}$$. or a scale of this $\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$

So I could write PVF as:

$$\bar{x} = x_3 \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

Exercise

Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 1 & 0 \end{bmatrix}$ and $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\bar{v}_2 = \begin{bmatrix} 1 \\ -4 \\ 6 \end{bmatrix}$.

Are $\bar{v}_1$ and $\bar{v}_2$ in Nul A?

$A\bar{v}_1 = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \rightarrow$ No!

$A\bar{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow$ yes!

Theorem

The Null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$

Finding Null space

The vectors in PVF make up the null space!

Ex

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \overset{RREF}{\sim} \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}_1} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_2} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\bar{v}_3}$$

$\text{Nul } A = \text{span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$

Every linear combination of $\bar{v}_1, \bar{v}_2$ and $\bar{v}_3$ is an element in Nul A.

In Sympy: A.nullspace():

```python
A = Matrix([[-3,6,-1,1,-7],[1,-2,2,3,-1],[2,-4,5,8,-4]])
A.rref()
```

✓ 0.0s

$$\left(\begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, (0, 2)\right)$$

```python
A = Matrix([[-3,6,-1,1,-7],[1,-2,2,3,-1],[2,-4,5,8,-4]])
A.nullspace()
```

✓ 0.0s

$$\left[\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}\right]$$

The column space

The set of all linear combinations of the columns of A where A is a $m \times n$ matrix

$$\text{Col } A = \text{span}\{\bar{a}_1, \bar{a}_2 \ldots \bar{a}_n\}$$

The column space is a subspace of $\mathbb{R}^m$ and is the set of all vectors $\bar{b}$ in $\mathbb{R}^m$ s.t. $A\bar{x} = \bar{b}$ for some $\bar{x}$ in $\mathbb{R}^n$

Ex

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

a) For what value of K is Col A a subspace of $\mathbb{R}^K$? $\mathbb{R}^3$

b) For what value of k is Nul A a subspace of $\mathbb{R}^k$? $\mathbb{R}^4$

c) Find a non-zero vector in Col A

→ Any entry, Always possible unless $A = \bar{0}$

d) Find non-zero vector in Nul A.

→ only possible if A is singular

since $m < n$ we have non-trivial

$$\begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \rightarrow \text{Nul } A = \left\{ \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$$

e) Are $\bar{u}$ and/or $\bar{v}$ in Nul/Col A?

$$\bar{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \bar{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\, A \quad \mathbf{0} \,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\, A \quad \mathbf{v} \,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

Bases

A linearly independent set that span a subspace. So basically the minimum set of vectors needed to span a sub-space.

Bases for Null space:

The vectors in PVF (or scales here of)

Bases for Col space:

The Pivot columns

Ex:

$$A = \begin{array}{c} \begin{matrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \bar{a}_4 & \bar{a}_5 \end{matrix} \\ \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \end{array} \overset{RREF}{\sim} \begin{array}{c} \begin{matrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 & \bar{u}_5 \end{matrix} \\ \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

so a basis for Col A would be

$\{\bar{a}_1, \bar{a}_3, \bar{a}_5\}$ but also $\{\bar{u}_1, \bar{u}_3, \bar{u}_5\}$

Dimension of a vector space V

Is the number of vectors in a basis for V.

Ex

$$H = \left\{ \begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix} \right\} : a \overset{\bar{v}_1}{\begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}} + b \overset{\bar{v}_2}{\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}} + c \overset{\bar{v}_3}{\begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}} + d \overset{\bar{v}_4}{\begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}}$$

$H = \text{Span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$

But $\bar{v}_3 = -2\bar{v}_2$, so

$H = \text{span}\{\bar{v}_1, \bar{v}_2, \bar{v}_4\} \leftarrow$ Basis.

$\dim H = 3$

In general:

Nullity

$\dim \text{Nul } A = \#$ of free variables

$\dim \text{Col } A = \#$ of pivots in A

Row space

- Is the linear comb. of the row vectors:

$$\text{Row } A = \text{Col } A^T$$

- If A and B are equivalent, they share the same row space and same basis for row space.
- Basis for Row A: all nonzero rows in Echelon F.

EX

Find Basis for Row A, Col A and Nul A:

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Row A $= \{ \bar{r}_1, \bar{r}_2, \bar{r}_3 \}$

Basis for Col A $= \{ \bar{a}_1, \bar{a}_2, \bar{a}_4 \}$

Basis for Nul A $= \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}$

Rank

The dimension of the col space

$$\text{rank } A = \dim \text{Col } A = \dim \text{Row } A$$

$$\text{rank } A + \dim \text{Nul } A = n \quad (A_{m \times n})$$

EX

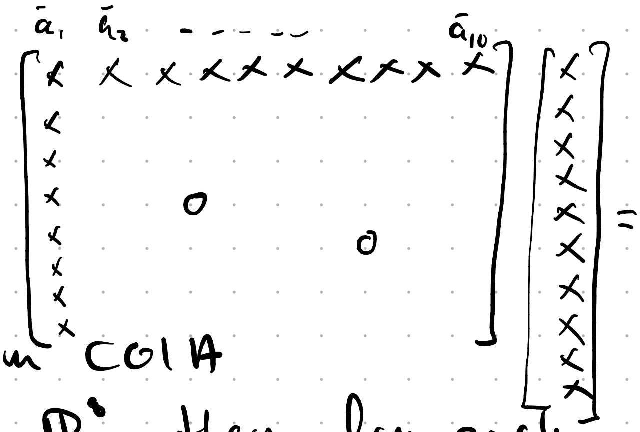
suppose we have two <u>independent</u> solutions to a 8×10 system: $\mathbb{R}^n \to \mathbb{R}^m$ $\quad n \times m$

$A\bar{x} = 0$

What can we deduce?

$\dim \operatorname{Nul} A = 2$

$\operatorname{rank} A = 8 = \dim$ Col A



since dim Col A spans $\mathbb{R}^8$ then for each $\bar{b}$ in $\mathbb{R}^8$ $A\bar{x} = \bar{b}$ has a solution!

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. $\operatorname{Col} A = \mathbb{R}^n$

o. $\dim \operatorname{Col} A = n$

p. $\operatorname{rank} A = n$

q. $\operatorname{Nul} A = \{\mathbf{0}\}$

r. $\dim \operatorname{Nul} A = 0$

$T: \mathbb{R}^{10} \to \mathbb{R}^8$