

```
In [ ]: from main import *
```

Assignment 1

```
In [ ]: # a
# m = 7 and n = 6. Here is an example:
```

```
A = Matrix([[1,0,0,0,0,0],[0,1,0,0,0,0],[0,0,1,0,0,0],[0,0,0,1,0,0],[0,0,0,0,1,0],[0,0,0,0,0,1],[0,0,0,0,0,0]])
b = Matrix([[0],[0],[0],[0],[0],[0],[1]])
A, b
```

Out[]:

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

```
In [ ]: # b
```

The statement is false. While $\{x, y\}$ and $\{u, v\}$ are both linearly independent sets, it doesn't guarantee that the combined set $\{x, y, u, v\}$ is linearly independent.

In \mathbb{R}^4 , the maximum number of linearly independent vectors you can have is 4. So, it's possible for $\{x, y, u, v\}$ to be linearly independent, but it's not a certainty. For a counterexample, consider:

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Both $\{x, y\}$ and $\{u, v\}$ are linearly independent sets in \mathbb{R}^4 . However, the combined set $\{x, y, u, v\}$ is not linearly independent because v can be expressed as a linear combination of x, y , and u :

$$v = x + y + u$$

So, even in \mathbb{R}^4 , the statement " $\{x, y, u, v\}$ is always linearly independent" is false.

No, not all invertible matrices are diagonalizable. A matrix is diagonalizable if and only if it has enough linearly independent eigenvectors to form a basis for the space it acts on. This means that for an $n \times n$ matrix to be diagonalizable, it must have n linearly independent eigenvectors. Being invertible (having a non-zero determinant) does not

guarantee that a matrix will have n linearly independent eigenvectors. For example, consider the matrix:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

This matrix is invertible because its determinant is non-zero. However, it only has one distinct eigenvalue, $\lambda = 2$, with algebraic multiplicity 2. The geometric multiplicity of this eigenvalue (i.e., the number of linearly independent eigenvectors associated with this eigenvalue) is 1, which is less than its algebraic multiplicity. Thus, the matrix A is not diagonalizable. So, while many invertible matrices are diagonalizable, it's not a guarantee.

The statement is true. By definition, a basis for a subspace W is a set of linearly independent vectors that spans W . If W is a subspace of \mathbf{R}^6 with dimension 4, then any set of 4 linearly independent vectors in W will span W and thus constitute a basis for W . So, if you have 4 linearly independent vectors in W , they will indeed form a basis for W .

```
In [ ]: # c: Determinant is zero since row 3 is a multiple of row 1.
A = Matrix([[2,8,3,6,7],[1,2,4,5,5],[4,16,6,12,14],[5,7,8,9,0],[3,4,5,7,1]])
A.det()
```

Out[]: 0

```
In [ ]: # d Roots are -3, and 2. since 2 is a global minimum, it has multiplicity 2 (sin
```

```
In [ ]: A = Matrix([[2,8,3,6,7],[1,2,4,5,5],[4,16,6,12,14],[5,7,8,9,0],[3,4,5,7,1]])
display(A.rref())
A.columnspace()
```

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{387}{92} \\ 0 & 1 & 0 & 0 & \frac{25}{23} \\ 0 & 0 & 1 & 0 & \frac{22}{23} \\ 0 & 0 & 0 & 1 & \frac{59}{92} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, (0, 1, 2, 3) \right)$$

```
Out[ ]: [[2], [8], [3], [6], [1], [2], [4], [5], [5], [16], [6], [12], [4], [7], [8], [9], [0], [3], [4], [5], [7], [1]]]
```

```
In [ ]: At = A.T
display(At.rref())
display(At.nullspace())
```

$$\left(\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, (0, 1, 3, 4) \right)$$

$$\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

```
In [ ]: # e
# dim col A = number of non-zero singular values, 1 (same as rank)
# dim nul A = n - rank, n is size of right singular vector so 5-1 = 4
# dim Nul A^T is the number of free variables in A^T - this will be 3 - 1 = 2
# dim Col A^T is the row space, same dimension as Col A = 1
```

Assignment 2

```
In [ ]: A = Matrix([[1, 1], [4, 1]])
(Matrix.hstack(A.eigenvecs()[0][2][0], A.eigenvecs()[1][2][0], Matrix([15, -10, 5])), A.eigenvals())
Out[ ]: (([[1 0 -20], [0 1 10]], (0, 1)), (([-1, 1, [[-1/2]]], (3, 1, [[1/2]])))
```

Assignment 3

```
In [ ]: A = 4
B = 8
n = 3

R((A*B)**-1), R(1/8)**n * A**3 * B**2, A*B, 4**n * R(A**-2), 3**3*2**3*4*8
Out[ ]: (1/32, 8, 32, 4, 6912)
```

Assignment 4

```
In [ ]: # A
A_aug = Matrix([[1,2,2,1,14],[-1,-2,0,-3,-4],[2,4,8,-2,48]])
A_aug
Out[ ]: [[1, 2, 2, 1, 14], [-1, -2, 0, -3, -4], [2, 4, 8, -2, 48]]
```

```
In [ ]: # b)
pivot1 = A_aug.rref()[1][0] + 1
pivot2 = A_aug.rref()[1][1] + 1

pivot1, pivot2, 2, 4, A_aug.rank()
```

```
Out[ ]: (1, 3, 2, 4, 2)
```

```
In [ ]: # c From RREF we can see that x1 will be 4 and x3 will be 4 when x2 and x4 are 0
```

```
In [ ]: # d
# Remember we are looking for the null space, so the first vector will be all zero
A = A_aug[:, 0:-1]
A.nullspace()
```

Out[]: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

Assignment 5

```
In [ ]: # So you would get the result by cofactor expanding on any row/column
x, y, z = symbols('x y z')
Matrix([[1,2,x],[2,1,y],[3,5,z]]).det()
```

Out[]: $7x + y - 3z$

Assignment 6

```
In [ ]: # a: Notice that v3 is just 2*v1 + v2 so v3 = 2v1 + 1v2
```

```
In [ ]: # L1 is dependent since it has 5 vectors but spans R^4 so one will be dependent
# L2 is independent
# L3 is dependent since row 2 and 4 are the same and will produce a free variable
# L4 includes L3 so is also dependent
```

```
In [ ]: # c So you check by multiplying each vector with A
A = Matrix([[1,1,0],[1,4,3],[0,3,1]])
x = Matrix([1, -1, 3])
y = Matrix([1, 5, 3])
z = Matrix([5, 0, 1])
w = Matrix([-3, [0], [1]])
display(A*x, A*y, A*z, A*w)
```

$$\begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 \\ 30 \\ 18 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 8 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

```
In [ ]: # We see that y and w are eigenvectors with eigenvalues 6 and 1, respectively.
```

```
In [ ]: # d
# Since the Matrix is symmetric, the trace is the sum of the eigen values. The t
# on the main diagonal. The is 1 + 4 + 1 = 6. We have 1 + 6, so we can deduce th
```

Assignment 7

```
In [ ]: # Find the null space of A^T
v1 = Matrix([[1],[3],[-1],[1]])
v2 = Matrix([[1],[4],[0],[2]])
A = Matrix.hstack(v1, v2)
B = A.T

display(A.T)

B.nullspace()
```

$$\begin{bmatrix} 1 & 3 & -1 & 1 \\ 1 & 4 & 0 & 2 \end{bmatrix}$$

```
Out[ ]: \left[\begin{bmatrix} 4 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right]
```

```
In [ ]: B[1,:] = B[1,:] - B[0,:]
B
```

$$\begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

```
In [ ]: B[0,:] = B[0,:] - 3*B[1,:]
B
```

$$\begin{bmatrix} 1 & 0 & -4 & -2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Part 2

Assignment 1

```
In [ ]: # a
A = Matrix([2, -1, -2])
v1 = (A.T).nullspace()[0]
v2 = (A.T).nullspace()[1]
p = Matrix([1, 1, 2])
proj = p.project(v1) + p.project(v2)
proj = proj.normalized()
v1, v2, p
```

$$\left(\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right)$$

```
In [ ]: u1, u2 = GramSchmidt((2*v1, v2))
u1, u2
```

$$\text{Out[]: } \left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{4}{5} \\ -\frac{2}{5} \\ 1 \end{bmatrix} \right)$$

```
In [ ]: p.project(u1)+p.project(u2)
```

$$\text{Out[]: } \begin{bmatrix} \frac{5}{3} \\ \frac{2}{3} \\ \frac{4}{3} \end{bmatrix}$$

```
In [ ]: # You could also solve it using least squares
X = Matrix.hstack(v1, v2)
X
```

$$\text{Out[]: } \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
In [ ]: Xtx = X.T * X
Xtp = X.T * p

b = Matrix.hstack(Xtx, Xtp).rref()[0][:,-1]
X*b
```

$$\text{Out[]: } \begin{bmatrix} \frac{5}{3} \\ \frac{2}{3} \\ \frac{4}{3} \end{bmatrix}$$

```
In [ ]: # b
q = Matrix([3,1,0])
Xtx = X.T * X
Xtq = X.T * q

b = Matrix.hstack(Xtx, Xtq).rref()[0][:,-1]
(q-X*b).norm()
```

$$\text{Out[]: } \frac{5}{3}$$

```
In [ ]: # same using projection
(q-(q.project(u1)+q.project(u2))).norm()
```

$$\text{Out[]: } \frac{5}{3}$$

```
In [ ]: # c
Matrix.vstack(u1.T, u2.T).nullspace()[0].normalized() * 3
```

Out[]:
$$\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$



Assignment 2

In []: *# We translate the problem to a matrix problem*

```
A = Matrix([[-R(8,200), R(3,100)],[R(8,200), -R(8,100)]])
display(Math(r'A = ' + latex(A)))
```

$$A = \begin{bmatrix} -\frac{1}{25} & \frac{3}{100} \\ \frac{1}{25} & -\frac{2}{25} \end{bmatrix}$$

In []: *# We find the eigenvalues and the corresponding eigenspaces.*

```
l = symbols('l')
l1, l2 = solve(det(A-l*eye(np.shape(A)[0])))
display(Math(r'\lambda_1 = ' + latex(l1) + r'\approx' + latex(round(l1, 2))))
display(Math(r'\lambda_2 = ' + latex(l2) + r'\approx' + latex(round(l2, 2)))

v1 = (A-l1*eye(np.shape(A)[0])).nullspace()[0]
v2 = (A-l2*eye(np.shape(A)[0])).nullspace()[0]
display(Math(r'v_1 = ' + latex(v1) + r'= ' + latex(v1.evalf(4))))
display(Math(r'v_2 = ' + latex(v2) + r'= ' + latex(v2.evalf(4))))
```

$$\lambda_1 = -\frac{1}{10} \approx -0.1$$

$$\lambda_2 = -\frac{1}{50} \approx -0.02$$

$$v_1 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 1.0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix}$$

In []: *y0 = Matrix([100, 0])*
c1 = Matrix.hstack(v1, v2, y0).rref()[0][0, -1]
c2 = Matrix.hstack(v1, v2, y0).rref()[0][1, -1]
c1, c2

Out[]: $(-50, 50)$

In []: *# a*
*def y(t): return c1*v1*exp(t*l1) + c2*v2*exp(t*l2)*
y(25).evalf(10)

Out[]:
$$\begin{bmatrix} 47.54192444 \\ 26.22228305 \end{bmatrix}$$

In []: *import math*
t = symbols('t')
*y1 = c1*v1[0]*math.e**(l1*t) + c2*v2[0]*math.e**(l2*t)*
*y2 = c1*v1[1]*math.e**(l1*t) + c2*v2[1]*math.e**(l2*t)*

```
In [ ]: # b
#z=limit(y2/y1,t,oo)
#z
```

```
In [ ]: for x in range (100,201):
    f1 = c1*v1[0]*math.e**(l1*x) + c2*v2[0]*math.e**(l2*x)
    f2 = c1*v1[1]*math.e**(l1*x) + c2*v2[1]*math.e**(l2*x)
    if f1-f2 <= 1:
        print(x)
        break
```

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Assignment 3

```
In [ ]: u1 = Matrix([0,1,-1,-2])
u2 = Matrix([1,0,1,0])
v1 = Matrix([0,-1,0,1])
v2 = Matrix([1,0,0,-1])
Matrix.hstack(v1, v2, u1, u2).rref()
```

Out[]:

$$\left(\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, (0, 1, 2) \right)$$

```
In [ ]: # so dim U+V = 3
```

Assignment 4

```
In [ ]: A = Matrix([[1,2,-4],[3,1,0],[2,3,-3]])
B = Matrix([[2,3],[1,0],[3,-1]])
X = A**-1 * B
X, A*X, A
```

Out[]:

$$\left(\begin{bmatrix} 0 & 1 \\ 1 & -3 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -4 \\ 3 & 1 & 0 \\ 2 & 3 & -3 \end{bmatrix} \right)$$

```
In [ ]: k = symbols('k')
A = Matrix([[1,2,-4],[3,k,0],[2,3,-3]])
B = Matrix([[2,3],[1,0],[3,-1]])
L, U, _ = A.LUdecomposition()
s = solve(U[2,2], k)
s
```

Out[]:

$$\left[\frac{18}{5} \right]$$

S

Assignment 5

```
In [ ]: import pandas as pd
x = pd.DataFrame([0.2, 0.3, 1.1, 1.3, 2.2, 2.4, 2.5])
y = pd.DataFrame([1.6, -1.2, -0.3, 2.1, 2.3, -0.2, -1.3])
```

```
In [ ]: # a)
X1 = Matrix([ones(len(x), 1)]).row_join(Matrix(x**3))
X2 = Matrix((Matrix(x**2)).row_join(Matrix(x**3)))
X3 = Matrix([ones(len(x), 1)]).row_join(Matrix(x))

display(Math(r'X_1 = ' + latex(X1) + r'\quad X_2 = ' + latex(X2) + r'\quad X_3 = '
+ latex(Matrix(y))))
```

$$X_1 = \begin{bmatrix} 1 & 0.008 \\ 1 & 0.027 \\ 1 & 1.331 \\ 1 & 2.197 \\ 1 & 10.648 \\ 1 & 13.824 \\ 1 & 15.625 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0.04 & 0.008 \\ 0.09 & 0.027 \\ 1.21 & 1.331 \\ 1.69 & 2.197 \\ 4.84 & 10.648 \\ 5.76 & 13.824 \\ 6.25 & 15.625 \end{bmatrix} \quad X_3 = \begin{bmatrix} 1 & 0.2 \\ 1 & 0.3 \\ 1 & 1.1 \\ 1 & 1.3 \\ 1 & 2.2 \\ 1 & 2.4 \\ 1 & 2.5 \end{bmatrix} \quad y = \begin{bmatrix} 1.6 \\ -1.2 \\ -0.3 \\ 2.1 \\ 2.3 \\ -0.2 \\ -1.3 \end{bmatrix}$$

```
In [ ]: X1tX1 = X1.T*X1
X1ty = X1.T*Matrix(y)
Mat, _ = X1tX1.row_join(X1ty).rref()
B1 = Mat[:, -1]

X2tX2 = X2.T*X2
X2ty = X2.T*Matrix(y)
Mat, _ = X2tX2.row_join(X2ty).rref()
B2 = Mat[:, -1]

X3tX3 = X3.T*X3
X3ty = X3.T*Matrix(y)
Mat, _ = X3tX3.row_join(X3ty).rref()
B3 = Mat[:, -1]
```

```
In [ ]: # b)
display(Math(r'X_1^T X_1 = ' + latex(X1tX1)))
display(Math(r'X_1^T y = ' + latex(X1ty)))
display(Math(r'X_2^T X_2 = ' + latex(X2tX2)))
display(Math(r'X_2^T y = ' + latex(X2ty)))
display(Math(r'X_3^T X_3 = ' + latex(X3tX3.evalf(5))))
display(Math(r'X_3^T y = ' + latex(X3ty)))
```

$$X_1^T X_1 = \begin{bmatrix} 7 & 43.66 \\ 43.66 & 555.222668 \end{bmatrix}$$

$$X_1^T y = \begin{bmatrix} 3.0 \\ 5.6079 \end{bmatrix}$$

$$X_2^T X_2 = \begin{bmatrix} 99.9956 & 234.145 \\ 234.145 & 555.222668 \end{bmatrix}$$

$$X_2^T y = \begin{bmatrix} 4.997 \\ 5.6079 \end{bmatrix}$$

$$X_3^T X_3 = \begin{bmatrix} 7.0 & 10.0 \\ 10.0 & 19.88 \end{bmatrix}$$

$$X_3^T y = \begin{bmatrix} 3.0 \\ 3.69 \end{bmatrix}$$

```
In [ ]: display(Latex("$$y_1(t) = {}+{}t^3$".format(round(B1[0],2), round(B1[0],4))))
display(Latex("$$y_2(t) = {}t^2+{}t^3$".format(round(B2[0],2), round(B2[1], 4)))
display(Latex("$$y_3(t) = {}+{}t$".format(round(B3[0],2), round(B3[1], 4))))
```

$$y_1(t) = 0.72 + 0.7175t^3$$

$$y_2(t) = 2.10t^2 + -0.8754t^3$$

$$y_3(t) = 0.58 + -0.1065t$$

```
In [ ]: # c)
display(Latex("$$e_1 = {}$".format(round((Matrix(y)-X1*B1).norm(), 2))))
display(Latex("$$e_2 = {}$".format(round((Matrix(y)-X2*B2).norm(), 2))))
display(Latex("$$e_3 = {}$".format(round((Matrix(y)-X3*B3).norm(), 2))))
```

$$e_1 = 3.69$$

$$e_2 = 3.15$$

$$e_3 = 3.76$$

Assignment 6

```
In [ ]: AAt = Matrix([[2,4,4],[4,8,8],[4,8,8]])
AAt.eigenvecs()
u1 = 2*AAt.eigenvecs()[1][2][0]
u2 = AAt.eigenvecs()[0][2][0]
u3 = AAt.eigenvecs()[0][2][1]

u1, u2, u3 = GramSchmidt((u1, u2, u3), True)
U = Matrix.hstack(u1, u2, u3)

U
```

$$\begin{bmatrix} \frac{1}{3} & -\frac{2\sqrt{5}}{5} & -\frac{2\sqrt{5}}{15} \\ \frac{2}{3} & \frac{\sqrt{5}}{5} & -\frac{4\sqrt{5}}{15} \\ \frac{2}{3} & 0 & \frac{\sqrt{5}}{3} \end{bmatrix}$$

```
In [ ]: # a)
AAt.eigenvecs() # multiply the first by 2
```

Out[]: $\left(\left(0, 2, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right), \left(18, 1, \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix} \right) \right)$

```
In [ ]: # b)
s1 = sqrt(AAt.eigenvecs()[1][0])
V = Matrix([(sqrt(2))/2, -(sqrt(2))/2, [(sqrt(2))/2, (sqrt(2))/2]])
Vt = V.T
S = diag(s1, 0).col_join(zeros(1,2))
S

display(S)
display(U * S * Vt)
display(AAt)
display(Vt)
```

$$\begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 4 \\ 4 & 8 & 8 \\ 4 & 8 & 8 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

```
In [ ]: V = Matrix([(sqrt(2))/2, -(sqrt(2))/2, [(sqrt(2))/2, (sqrt(2))/2]])
U * S * V.T
```

Out[]: $\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 2 & 2 \end{bmatrix}$

```
In [ ]: display(1+1+2+2+2)
display(Math('Sum of A matrix is= {}'.format(latex(np.sum(U * S * V.T)))))
```

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Sum of A matrix is = 10

In []: