Matrix

Echelon form

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

Reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reduce Echelon form with inf solutions

b. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Echelon form

9. a. In order for a matrix B to be the inverse of A, the equations $AB = I$ and $BA = I$ must both be true.

b. If A and B are $n \times n$ and invertible, then $A^{-1}B^{-1}$ is the inverse of AB.

c. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ab - cd \neq 0$, then A is invertible.

d. If A is an invertible $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for *each* $\mathbf{b}$ in $\mathbb{R}^n$.

e. Each elementary matrix is invertible.

9. a. True, by definition of *invertible*.

b. False. See Theorem 6(b).

c. False. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then $ab - cd = 1 - 0 \neq 0$, but Theorem 4 shows that this matrix is not invertible, because $ad - bc = 0$.

d. True. This follows from Theorem 5, which also says that the solution of $A\mathbf{x} = \mathbf{b}$ is unique, for each $\mathbf{b}$.

e. True, by the box just before Example 6.

11. a. If the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix.

b. If the columns of A span $\mathbb{R}^n$, then the columns are linearly independent.

c. If A is an $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

d. If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then A has fewer than n pivot positions.

e. If A^T is not invertible, then A is not invertible.

12. a. If there is an $n \times n$ matrix D such that $AD = I$, then $DA = I$.

b. If the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$, then the row reduced echelon form of A is I.

c. If the columns of A are linearly independent, then the columns of A span $\mathbb{R}^n$.

d. If the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not one-to-one.

e. If there is a $\mathbf{b}$ in $\mathbb{R}^n$ such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then the solution is unique.

13. An $m \times n$ **upper triangular matrix** is one whose entries *below* the main diagonal are 0's (as in Exercise 8). When is a square upper triangular matrix invertible? Justify your answer.

14. An $m \times n$ **lower triangular matrix** is one whose entries *above* the main diagonal are 0's (as in Exercise 3). When is a square lower triangular matrix invertible? Justify your answer.

15. Is it possible for a 4×4 matrix to be invertible when its columns do not span $\mathbb{R}^4$? Why or why not?

16. If an $n \times n$ matrix A is invertible, then the columns of A^T are linearly independent. Explain why.

17. Can a square matrix with two identical columns be invertible? Why or why not?

18. Can a square matrix with two identical rows be invertible? Why or why not?

19. If the columns of a 7×7 matrix D are linearly independent, what can be said about the solutions of $D\mathbf{x} = \mathbf{b}$? Why?

20. If A is a 5×5 matrix and the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^5$, is it possible that for some $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has more than one solution? Why or why not?

21. If the equation $C\mathbf{u} = \mathbf{v}$ has more than one solution for some $\mathbf{v}$ in $\mathbb{R}^n$, can the columns of the $n \times n$ matrix C span $\mathbb{R}^n$? Why or why not?

22. If $n \times n$ matrices E and F have the property that $EF = I$, then E and F commute. Explain why.

23. Assume that F is an $n \times n$ matrix. If the equation $F\mathbf{x} = \mathbf{y}$ is inconsistent for some $\mathbf{y}$ in $\mathbb{R}^n$, what can you say about the equation $F\mathbf{x} = \mathbf{0}$? Why?

11. **a**. True, by the IMT. If statement (d) of the IMT is true, then so is statement (b).

b. True. If statement (h) of the IMT is true, then so is statement (e).

c. False. Statement (g) of the IMT is true only for invertible matrices.

d. True, by the IMT. If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then statement (d) of the IMT is false. In this case, all the lettered statements in the IMT are false, including statement (c), which means that A must have fewer than n pivot positions.

e. True, by the IMT. If A^T is not invertible, then statement (1) of the IMT is false, and hence statement (a) must also be false.

12. **a**. True. If statement (k) of the IMT is true, then so is statement (j). Use the first box after the IMT.

b. False. Notice that (i) if the IMT uses the work *onto* rather than the word *into.*

c. True. If statement (e) of the IMT is true, then so is statement (h).

d. False. Since (g) if the IMT is true, so is (f).

e. False, by the IMT. The fact that there is a $\mathbf{b}$ in $\mathbf{R}^n$ such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent, does not imply that statement (g) of the IMT is true, and hence there could be more than one solution.

13. If a square upper triangular $n \times n$ matrix has nonzero diagonal entries, then because it is already in echelon form, the matrix is row equivalent to I_n and hence is invertible, by the IMT. Conversely, if the matrix is invertible, it has n pivots on the diagonal and hence the diagonal entries are nonzero.

14. If A is lower triangular with nonzero entries on the diagonal, then these n diagonal entries can be used as pivots to produce zeros below the diagonal. Thus A has n pivots and so is invertible, by the IMT. If one of the diagonal entries in A is zero, A will have fewer than n pivots and hence be singular.

15. Part (h) of the IMT shows that a 4×4 matrix cannot be invertible when its columns do not span $\mathbf{R}^4$.

16. If A is invertible, so is A^T, by (l) of the IMT. By (e) of the IMT applied to A^T, the columns of A^T are linearly independent.

17. If A has two identical columns then its columns are linearly dependent. Part (e) of the IMT shows that A cannot be invertible.

18. If A contains two identical rows, then it cannot be row reduced to the identity because subtracting one row from the other creates a row of zeros. By (b) of the IMT, such a matrix cannot be invertible.

19. By (e) of the IMT, D is invertible. Thus the equation $D\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^7$, by (g) of the IMT. Even better, the equation $D\mathbf{x} = \mathbf{b}$ has a *unique* solution for each $\mathbf{b}$ in $\mathbf{R}^7$, by Theorem 5 in Section 2.2. (See the paragraph following the proof of the IMT.)

20. By (g) of the IMT, A is invertible. Hence, each equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, by Theorem 5 in Section 2.2. This fact was pointed out in the paragraph following the proof of the IMT.

21. The matrix C cannot be invertible, by Theorem 5 in Section 2.2 or by the box following the IMT. So (h) of the IMT is false and the columns of C do not span $\mathbf{R}^n$.

22. By the box following the IMT, E and F are invertible and are inverses. So $FE = I = EF$, and so E and F commute.

23. Statement (g) of the IMT is false for F, so statement (d) is false, too. That is, the equation $F\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

Vector

28. A steam plant burns two types of coal: anthracite (A) and bituminous (B). For each ton of A burned, the plant produces 27.6 million Btu of heat, 3100 grams (g) of sulfur dioxide, and 250 g of particulate matter (solid-particle pollutants). For each ton of B burned, the plant produces 30.2 million Btu, 6400 g of sulfur dioxide, and 360 g of particulate matter.

a. How much heat does the steam plant produce when it burns x_1 tons of A and x_2 tons of B?

b. Suppose the output of the steam plant is described by a vector that lists the amounts of heat, sulfur dioxide, and particulate matter. Express this output as a linear combination of two vectors, assuming that the plant burns x_1 tons of A and x_2 tons of B.

c. [**M**] Over a certain time period, the steam plant produced 162 million Btu of heat, 23,610 g of sulfur dioxide, and 1623 g of particulate matter. Determine how many tons of each type of coal the steam plant must have burned. Include a vector equation as part of your solution.

a. The amount of heat produced when the steam plant burns x_1 tons of anthracite and x_2 tons of bituminous coal is $27.6x_1 + 30.2x_2$ million Btu.

b. The total output produced by x_1 tons of anthracite and x_2 tons of bituminous coal is given by the vector $x_1\begin{bmatrix}27.6\\3100\\250\end{bmatrix}+x_2\begin{bmatrix}30.2\\6400\\360\end{bmatrix}$.

c. **[M]** The appropriate values for x_1 and x_2 satisfy $x_1\begin{bmatrix}27.6\\3100\\250\end{bmatrix}+x_2\begin{bmatrix}30.2\\6400\\360\end{bmatrix}=\begin{bmatrix}162\\23{,}610\\1{,}623\end{bmatrix}$.

To solve, row reduce the augmented matrix:

$$\begin{bmatrix}27.6 & 30.2 & 162\\3100 & 6400 & 23610\\250 & 360 & 1623\end{bmatrix}\sim\begin{bmatrix}1.000 & 0 & 3.900\\0 & 1.000 & 1.800\\0 & 0 & 0\end{bmatrix}$$

The steam plant burned 3.9 tons of anthracite coal and 1.8 tons of bituminous coal.

Linearly independent

8. august 2023 17:22

Determine by inspection whether the vectors in Exercises 15–20 are linearly *independent*. Justify each answer.

15. $\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix}$ **16.** $\begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -12 \end{bmatrix}$

17. $\begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix}$ **18.** $\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

19. $\begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$ **20.** $\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

15. The set is linearly dependent, by Theorem 8, because there are four vectors in the set but only two entries in each vector.

16. The set is linearly dependent because the second vector is –3/2 times the first vector.

17. The set is linearly dependent, by Theorem 9, because the list of vectors contains a zero vector.

18. The set is linearly dependent, by Theorem 8, because there are four vectors in the set but only two entries in each vector.

19. The set is linearly independent because neither vector is a multiple of the other vector. [Two of the entries in the first vector are – 4 times the corresponding entry in the second vector. But this multiple does not work for the third entries.]

20. The set is linearly dependent, by Theorem 9, because the list of vectors contains a zero vector.

Vector Spaces

10. august 2023 18:04

For the matrices in Exercises 17–20, (a) find k such that Nul A is a subspace of $\mathbb{R}^k$, and (b) find k such that Col A is a subspace of $\mathbb{R}^k$.

17. $A = \begin{bmatrix} 6 & -4 \\ -3 & 2 \\ -9 & 6 \\ 9 & -6 \end{bmatrix}$

18. $A = \begin{bmatrix} 5 & -2 & 3 \\ -1 & 0 & -1 \\ 0 & -2 & -2 \\ -5 & 7 & 2 \end{bmatrix}$

19. $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$

20. $A = \begin{bmatrix} 1 & -3 & 2 & 0 & -5 \end{bmatrix}$

17. The matrix A is a 4×2 matrix. Thus
(a) Nul A is a subspace of $\mathbf{R}^2$, and
(b) Col A is a subspace of $\mathbf{R}^4$.

18. The matrix A is a 4×3 matrix. Thus
(a) Nul A is a subspace of $\mathbf{R}^3$, and
(b) Col A is a subspace of $\mathbf{R}^4$.

19. The matrix A is a 2×5 matrix. Thus
(a) Nul A is a subspace of $\mathbf{R}^5$, and
(b) Col A is a subspace of $\mathbf{R}^2$.

20. The matrix A is a 1×5 matrix. Thus
(a) Nul A is a subspace of $\mathbf{R}^5$, and
(b) Col A is a subspace of $\mathbf{R}^1 = \mathbf{R}$.