Applied Linear Algebra Final Exam

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Please state all answers in the 'ALI-exam.ipynb'. If you have handwritten answer which is scanned, please state 'In paper' or similar so the examiner knows where to find the answer. Also, please include all ipynb files when you hand in.

Assignment 1 (12 %)

Determine whether statements a-d are true or false. State the reasons for your answers. **The points for this problem are given entirely for your reasons.**

a) Two matrices are row equivalent if they have the same number of rows.

b) If a system of linear equations has no free variables, then it has a unique solution.

c) There exists a 5×6 matrix A and a 5×1 vector $\vec{b}$ for which $\mathrm{A}\vec{\mathrm{x}} = \vec{b}$ has only one solution.

d) A is diagonalizable where A is a 9×9 matrix with three distinct eigenvalues of which two of the eigenvalues have multiplicity 3; and one eigenvalue has multiplicity 2.

Assignment 2 (8 %)

Let A be the matrix $A = \begin{bmatrix} 4 & 8 & -2 \\ -6 & 2 & 10 \\ -2 & 6 & 6 \end{bmatrix}$, and let $\vec{b}$ be the vector $\vec{b} = \begin{bmatrix} 2 \\ 18 \\ 15 \end{bmatrix}$

a) Determine whether $\vec{b}$ is in the span of the columns of A.

b) Let $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ denote the columns of the matrix A. Is the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly independent or linearly dependent? If it is linearly dependent, find a linear dependency relation.

Assignment 3 (25 %)

a) Evaluate the following determinant and write your answer as a polynomial in x.

$$\begin{vmatrix} a-x & b & c \\ 1 & -x & 0 \\ 0 & 1 & -x \end{vmatrix}$$

b) Find all values of λ for which the following determinant will equal 0.

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix}$$

c) Evaluate the following determinant by using cofactor expansion.

$$\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix}$$

d) Let A $= \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & a \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 3 \\ b \end{bmatrix}$. Find the values of a and b for which the linear system whose augmented matrix is of the form $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ has one solution, infinitely many solutions, and no solutions, and then find the value of a for which the determinant of A is 1.

Assignment 4 (10%)

Let $W \subseteq \mathbb{R}^4$ be the subspace of vectors (x_1, x_2, x_3, x_4) satisfying

$$2x_1 - x_3 + x_4 = 0$$

Find an orthonormal basis for W.

Assignment 5 (15 %)

Let A be the matrix A $= \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$

a) Show that A is diagonalizable.

b) Find an invertible matrix X and a diagonal matrix D such that $X^{-1}AX = D$.

c) Next, find an expression for A^k, where k is an arbitrary positive integer.

Assignment 6 (20 %)

For this assignment, you will need to load the file "Smoking_and_Cancer.xlsx". The data was collected in 1960 from the National Cancer Institute and provides death rates (per 100.000) for lung cancer as a function of the per capita numbers of cigarettes sold. The National Cancer Institute believe that the relationship can be modelled by either a linear function or a quadratic function:

$$f_1(x) = \beta_0 + \beta_1 x \text{ or } f_2(x) = \delta_1 x + \delta_2 x^2$$

a) Fit the two proposed models for lung cancer.
b) Determine which of the two models provides the best fit for the measured data.
c) Use the best fitted model to predict death rate of that specific cancer when the number of cigarettes sold per capita reaches 50.

Assignment 7 (10 %)

Do a full singular value decomposition of matrix A below. This includes the following steps:

1) Finding the eigenvalues of either $A^T A$ or AA^T
2) Determining the corresponding eigenvectors
3) Finding the columns of U or V depending on which method you used in step 1, where U or V are matrices made up of the eigenvectors of AA^T and $A^T A$, respectively
4) Setting up $A = U\Sigma V^T$
5) Testing that $A = U\Sigma V^T$

$$\text{A} = \begin{bmatrix} 2 & 5 & 4 \\ 6 & 3 & 0 \\ 6 & 3 & 0 \\ 2 & 5 & 4 \end{bmatrix}$$