8. Orthogonality

Gram Schmidt Proces (GSP):

Given a basis, GSP provides a method for obtaining an orthogonal basis for any non-zero subspace W:

Assume a basis $\{\bar{x}_1 \dots \bar{x}_p\}$, let

$$\bar{v}_1 = \bar{x}_1$$

$$\bar{v}_2 = \bar{x}_2 - \frac{x_2 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} \bar{v}_1$$

$$\bar{v}_3 = \bar{x}_3 - \frac{x_3 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} \bar{v}_1 - \frac{x_3 \cdot \bar{v}_2}{\bar{v}_2 \cdot v_2} \bar{v}_2$$

$$\vdots$$

$$\bar{v}_p = \bar{x}_p - \frac{\bar{x}_p \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} \bar{v}_1 - \frac{\bar{x}_p \cdot \bar{v}_2}{\bar{v}_2 \cdot \bar{v}_2} \bar{v}_2 - \dots - \frac{\bar{x}_p \cdot \bar{v}_{p-1}}{\bar{v}_{p-1} \cdot \bar{v}_{p-1}} \bar{v}_{p-1}$$

Then $\{\bar{v}_1 \dots \bar{v}_p\}$ will be an orthogonal set.

$$\bar{v}_p = \bar{x}_p - \text{proj}_{\bar{v}_1} \bar{x}_p - \text{proj}_{\bar{v}_2} \bar{x}_p - \dots \text{proj}_{\bar{v}_{p-1}} \bar{x}_p$$

$$= \bar{x}_p - \left(\sum_{i=1}^{p-1} \text{proj}_{\bar{v}_i} \bar{x}_p \right) = \bar{x}_p - \left(\sum_{i=1}^{p-1} \frac{x_p \cdot \bar{v}_i}{\bar{v}_i \cdot \bar{v}_i} \cdot \bar{v}_i \right)$$

Ex:

$$\bar{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \quad \bar{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\bar{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$\bar{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}}{\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

EX:

$$\bar{X}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{X}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{X}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Using Python.

```
x1 = Matrix([[1],[1],[1],[1]])
x2 = Matrix([[0],[1],[1],[1]])
x3 = Matrix([[0],[0],[1],[1]])
```
✓ 0.0s

```
v1 = x1
v2 = x2 - x2.project(v1)
v3 = x3 - x3.project(v1) - x3.project(v2)
v1, v2, v3
```
✓ 0.0s

$$\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \right)$$

EX

Assignment 4

Let

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}, u_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, and\ v = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix},$$

a) Show that $\{u_1, u_2, u_3, u_4\}$ is an orthogonal basis for $\mathbb{R}^4$
b) Write *v* as the sum of two vectors, one in $span\{u_1, u_2\}$ and the other in $span\{u_3, u_4\}$.
c) Determine the (shortest) distance between *v* and the subspace spanned by $\{u_1, u_2, u_3\}$

Using Pyton

```
u1 = Matrix([[1],[2],[1],[1]])
u2 = Matrix([[-2],[1],[-1],[1]])
u3 = Matrix([[1],[1],[-2],[-1]])
u4 = Matrix([[-1],[1],[1],[-1]])

v = Matrix([[4],[2],[-1],[0]])

U = Matrix.hstack(u1, u2, u3, u4)
U.T * U , U.rref()
```
✓ 0.0s

$$\left(\begin{bmatrix} 7 & 0 & 0 & 1 \\ 0 & 7 & 0 & 1 \\ 0 & 0 & 7 & -1 \\ 1 & 1 & -1 & 4 \end{bmatrix}, \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, (0, 1, 2, 3) \right) \right)$$

```
# indpendent, not orthogonal
u1, u2, u3, u4 = GramSchmidt((u1, u2, u3, u4))
```
✓ 0.0s

```
z1 = v.project(u1) + v.project(u2)
z2 = v.project(u3) + v.project(u4)

z1, z2
```
✓ 0.0s

[illegible]

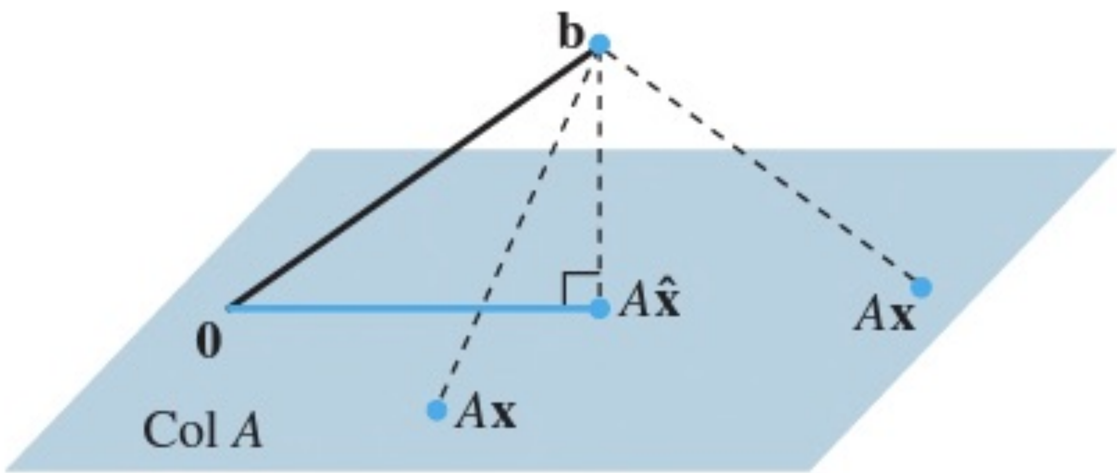
```
((v - (v.project(u1) + v.project(u2) + v.project(u3))).norm(),
float((v - (v.project(u1) + v.project(u2) + v.project(u3))).norm()))
```
✓ 0.0s

$$\left(\frac{3\sqrt{7}}{7}, 1.13389341902768 \right)$$

Least Squares Problem: Best Approximation

Problem: Not all $A\bar{x}=\bar{b}$ have solutions

Solution: Find the best approximation $\hat{b} \in \text{Col}\,A$: $\exists\, \hat{x}$ s.t. $A\hat{x}=\hat{b}$



$\hat{x}$ is called the least squares solution to $A\bar{x}=\bar{b}$.

$$\|\bar{b}-A\hat{x}\| \leq \|\bar{b}-A\bar{x}\|$$

$$Ax=b$$

$$A=[\bar{a}_1 \;\; \bar{a}_2 \;\; \cdots \;\; \bar{a}_n]$$

$$a_i\cdot(\bar{b}-\hat{b})=0$$

$$A^T=\begin{bmatrix} \text{---} & \bar{a}_1 & \text{---} \\ \text{---} & \bar{a}_2 & \text{---} \\ & \vdots & \\ \text{---} & a_n & \text{---} \end{bmatrix}$$

$$A^T\cdot(\bar{b}-\hat{b})=0$$

$$A^T\cdot(\bar{b}-A\hat{x})=0$$

$$A^T\cdot\bar{b}-A^TA\hat{x}=0$$

$$A^TA\hat{x}=A^Tb$$

$$[A^T A \;\; A^T b] \overset{RREF}{\sim} [I \;\; \hat{x}]$$

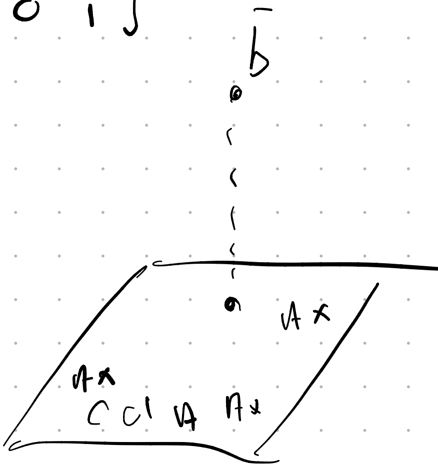
Method: Finding least squares

1) Find A^TA
2) Find A^Tb
3) Solve $[A^TA \;\; A^Tb]$
4) state solution, $\hat{x}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix} \overset{RREF}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ex

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}, \quad [A \;\; \bar{b}] \overset{RREF}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



1) $A^TA = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 6 \end{bmatrix}$

2) $A^T\bar{b} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 11 \end{bmatrix}$

3) $\begin{bmatrix} 5 & 4 & 9 \\ 4 & 6 & 11 \end{bmatrix} \overset{RREF}{\sim} \begin{bmatrix} 1 & 0 & 5/7 \\ 0 & 1 & 19/14 \end{bmatrix}$

4) $\hat{x} = \begin{bmatrix} 5/7 \\ 19/14 \end{bmatrix}$

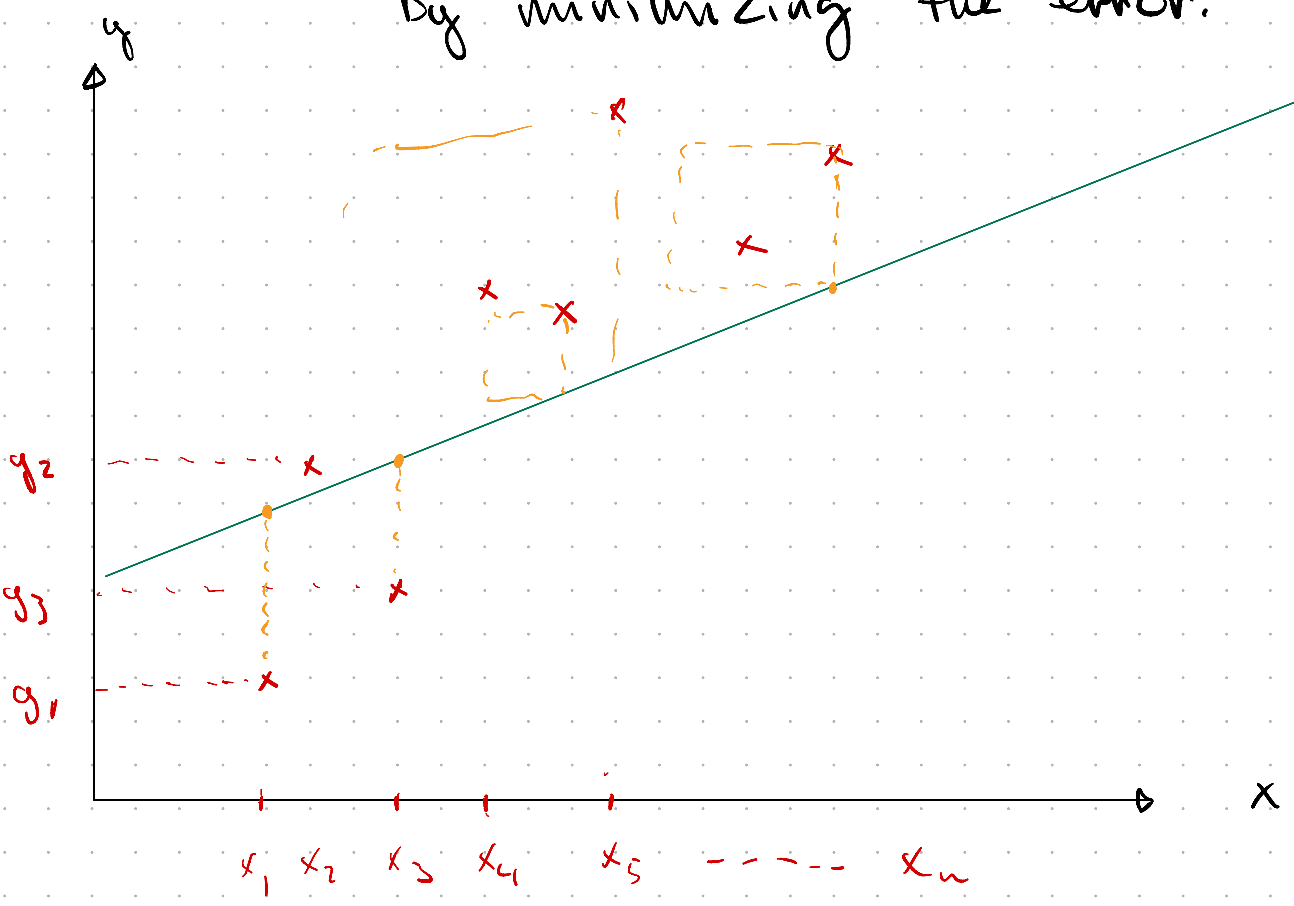
We now know that $\underset{A}{\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}} \cdot \underset{\hat{x}}{\begin{bmatrix} 5/7 \\ 19/14 \end{bmatrix}} = \underset{\hat{b}}{\begin{bmatrix} 24/7 \\ 39/14 \\ 19/14 \end{bmatrix}}$ is

the closest point in Col A to $\bar{b}$.

Linear Models:

Problem: Real life problems rarely follow mathematical functions perfectly.

Solution: Use least square to find the function/model that yield the best approximation of the observation by minimizing the error.



Predicted y-value, $\hat{y}$	Observed y-value, y
$\hat{B}_0 + \hat{B}_1 x_1 \approx$	y_1
$\hat{B}_0 + \hat{B}_1 x_2$	y_2
$\vdots$	$\vdots$
$\hat{B}_0 + \hat{B}_1 x_n$	y_n

$$y = bx + ax^2 + dx^3$$

Design matrix | Parameter vector | Observation vector.

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad \hat{\beta} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix}, \quad \bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

To create a simple linear regression model means to find estimates of β_0 and β_1 that minimizes the error, i.e. find $\hat{\beta}_0$ and $\hat{\beta}_1$ s.t.

$$\|\bar{y} - X\hat{\beta}\| \leq \|\bar{y} - X\beta\|$$

Using least squares:

1) $X^t X$
2) $X^t y$
3) solve $[X^t X \quad X^t y] \overset{RREF}{\sim} \begin{bmatrix} 1 & 0 & \hat{\beta}_0 \\ 0 & 1 & \hat{\beta}_1 \end{bmatrix}$
4) state $\hat{\beta}_0$ and $\hat{\beta}_1$

Ex:

Fit a linear model of the form $\beta_0 + \beta_1 \cdot \ln w = p$

w	44	61	81	113	131
p	91	98	103	110	112

$$X = \begin{bmatrix} 1 & \ln 44 \\ 1 & \ln 61 \\ 1 & \ln 81 \\ 1 & \ln 113 \\ 1 & \ln 131 \end{bmatrix} = \begin{bmatrix} 1 & 3{,}78 \\ 1 & 4{,}11 \\ 1 & 4{,}39 \\ 1 & 4{,}73 \\ 1 & 4{,}88 \end{bmatrix}, \quad \bar{p} = \begin{bmatrix} 91 \\ 98 \\ 103 \\ 110 \\ 112 \end{bmatrix}$$

1) $$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3.78 & - & - & - & 4.88 \end{bmatrix} \begin{bmatrix} 1 & 3.78 \\ 1 & \vdots \\ 1 & \vdots \\ 1 & 4.88 \end{bmatrix}$$

2) $$X^T p = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3.78 & - & - & - & 4.88 \end{bmatrix} \begin{bmatrix} 91 \\ \vdots \\ \vdots \\ 112 \end{bmatrix}$$

3) $$\begin{bmatrix} X^T X & X^T p \end{bmatrix} \xrightarrow{\text{RRef}} \begin{bmatrix} 1 & 0 & 17.92 \\ 0 & 1 & 19.39 \end{bmatrix}$$

So $$\boxed{p = 17.92 + 19.39 \cdot \ln w}$$