In [ ]:
```
import sympy as sp
from scipy import *
from sympy import *
init_printing()
from IPython.display import display, Latex, HTML, Math
import numpy as np
import pandas as pd
from sympy import Rational as R
```

Assignment 1

In [ ]:
```
# a)
# i. True
# ii. False
# iii.True. In fact these are equivalent conditions.
# iv. False. This would mean the eigenvectors form a line, but it could be a pla
# v. False. A^3 = PD^3P^-1
# vi. False. It might be diagonalizable and it might not be.
# vii. True
# viii. True
# ix. False
# x. False. It would be true with the condition that v != 0.
```

You will need very few and simple calculation in order to solve the problems of this assignment.

a. Mark each statement as either true or false

i. Two vectors are linearly independent if one is not a scalar multiple of the other.
ii. Every 2×2 matrix is diagonalizable.
iii. If A is diagonalizable, then there is a basis of eigenvectors of A
iv. If λ is an eigenvalue for A, then the eigenvectors with eigenvalue λ are scalar multiples of each other.
v. If $A = PDP^{-1}$, then $A^3 = P^3D^3P^{-3}$
vi. If A does not have n distinct eigenvalues, then A is not diagonalizable.
vii. If $S = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^n$ and $\text{span}(S) = \mathbb{R}^n$ then S is a basis for $\mathbb{R}^n$
viii. A is invertible if and only if $\det(A) \neq 0$
ix. If A is diagonalizable, then A is invertible.
x. If $A\overline{v} = c\overline{v}$ for some $c \in \mathbb{R}$, then $\overline{v}$ is an eigenvector for A.

a. Mark each statement as true or false.

i.	◉ True ✓	○ False
ii.	○ True	◉ False ✓
iii.	◉ True ✓	○ False
iv.	○ True	◉ False ✓
v.	○ True	◉ False ✓
vi.	○ True	◉ False ✓
vii.	◉ True ✓	○ False
viii.	◉ True ✓	○ False
ix.	○ True	◉ False ✓
x.	○ True	◉ False ✓

In [ ]:
```
# b
b = symbols('b')
solve(3*b*4*3 - 72, b)
```

Out[ ]: [2]

b. It is known that the determinant of A is 72. Based on this information, determine the value of b:

$$\begin{bmatrix} 3 & 7 & 1 & 2 \\ 0 & b & 6 & 8 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

State your answer as a positive integer.

$b = \boxed{2}$ ✓

In [ ]:
```
# c)
# Symmetric matrix so l1 + l2 + l3 = sum of diagonal (called trace) = 6
l1, l2, l3 = symbols('λ₁ λ₂ λ₃')
l2 = l1+sqrt(l1)
l3 = l1 - sqrt(l1)

x1 = solve(l1 + l2+ l3  - 6, l1)[0]
x2 = l2.subs(l1, x1)
x3 = l3.subs(l1, x1)

x1, x2, x3
```

Out[ ]: $\left(2,\ \sqrt{2}+2,\ 2-\sqrt{2}\right)$

c. Consider the following matrix

$$A = \begin{bmatrix} 2 & a & b \\ a & 2 & c \\ b & c & 2 \end{bmatrix}$$

It is known that $\lambda_2 = \lambda_1 + \sqrt{\lambda_1}$ and $\lambda_3 = \lambda_1 - \sqrt{\lambda_1}$. What are the eigenvalues of A?

State your answer as positive integers.

$\lambda_1 = \boxed{2},\quad \lambda_2 = \boxed{2} + \sqrt{\boxed{2}},\quad \lambda_3 = \boxed{2} - \sqrt{\boxed{2}}$

Assignment 2

Let A be the following matrix:

$$\begin{bmatrix} 1 & 3 \\ -2 & -8 \end{bmatrix}$$

a. Find numbers p and q, such that $A^2 = pA + qI$, where I is the 2×2 identity matrix.

b. Let $B = A - tI$, where t is a scalar. For which values of t is B not invertible?

c. What are the values (t) found in part (b) called relative to A?

d. Find all the orthonormal eigenvectors of A.

In [ ]:
```
# a)
p, q = symbols('p q')
A = Matrix([[1,3],[-2,-8]])
A2 = A**2
res = A2 - p* A-q * eye(2)
p = solve(res[0,-1], p)[0]
q = solve(-p-q-5, q)[0]
display(Math(r'p = ' + latex(p) + r'\text{ and } q = ' + latex(q)))
```

$p = -7 \text{ and } q = 2$

In [ ]:
```
# b)
t = symbols('t')
```

```
B = A-t*eye(2)
B.echelon_form()
```

Out[ ]:

$$\begin{bmatrix} -2 & -t-8 \\ 0 & -t^2-7t+2 \end{bmatrix}$$

In [ ]:

```
solve(B.echelon_form()[-1,-1], t)
```

Out[ ]:

$$\left[-\frac{7}{2} + \frac{\sqrt{57}}{2}, \ -\frac{\sqrt{57}}{2} - \frac{7}{2} \right]$$

In [ ]:

```
# c) Eigenvalues
```

c. Choose the correct answer below. Only one answer is correct.

- A The singular values of A
- B The determinants of A
- C The explicit values of A
- D The discriminants of A
- E The eigenvalues of A
- F The inverse values of A
- G The ranks of A

In [ ]:

```
# d)

A.eigenvects()[0][2][0].normalized().evalf(4), A.eigenvects()[1][2][0].normalize
```

Out[ ]:

$$\left(\begin{bmatrix} -0.972 \\ 0.2349 \end{bmatrix}, \ \begin{bmatrix} -0.3408 \\ 0.9401 \end{bmatrix} \right)$$

d. Choose **one** or **multiple** options below. The values are rounded to four decimal precision.

A	$\begin{bmatrix} -0.7201 \\ 0.1207 \end{bmatrix}$
B	$\begin{bmatrix} -0.647 \\ 0.6603 \end{bmatrix}$
C	$\begin{bmatrix} -0.3408 \\ 0.9401 \end{bmatrix}$
D	$\begin{bmatrix} -0.972 \\ 0.2349 \end{bmatrix}$
E	$\begin{bmatrix} 0.5608 \\ 0.0921 \end{bmatrix}$
F	$\begin{bmatrix} -0.8408 \\ 0.0401 \end{bmatrix}$

Assignment 3

In [ ]:
```
h = symbols('h')
solve(Matrix([[-1, 4, 10], [-4, 5, 7], [3, 0, h+3]]).echelon_form()[-1,-1], h)[0
```

Out[ ]: 3

Consider the following vectors in $\mathbb{R}^3$

$$\vec{u} = \begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 10 \\ 7 \\ h+3 \end{bmatrix}$$

For which values of h do these three vectors form a basis for $\mathbb{R}^3$?

State your answer as a positive integer.

$h \neq \boxed{3}$

Assignment 4

In [ ]:
```
# a)

v1 = Matrix([[2],[2],[-2],[-2]])
v2 = Matrix([[1],[-1],[1],[-1]])
```

```
v3 = Matrix([[0],[3],[3],[0]])
x = Matrix([[0],[1],[2],[3]])

V = Matrix.hstack(v1, v2, v3)
G = V.T * V
G
```

Out[ ]:

$$\begin{bmatrix} 16 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

The following three vectors are given in $\mathbb{R}^4$

$$\bar{v}_1 = \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix}$$

a. Construct the Gram matrix, G, for $\bar{v}_1, \bar{v}_2$, and $\bar{v}_3$ and confirm that this set of vectors form an orthogonal set V.

b. Calculate the projection, $\text{proj}_V\, \bar{x}$, of

$$\bar{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

onto the subspace spanned by V.

c. Use $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{x}$ and $\text{proj}_V\, \bar{x}$ to construct an orthonormal basis B for $\mathbb{R}^4$

a. State your answer as positive integers. Indexing is done according to the vector index.

$$G = \begin{bmatrix} 16 & \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{4} & 0 \\ 0 & \boxed{0} & \boxed{18} \end{bmatrix}$$

In [ ]:

```
# b)
proj = x.project(v1) + x.project(v2) + x.project(v3)
proj
```

Out[ ]:

$$\begin{bmatrix} -\frac{3}{2} \\ 1 \\ 2 \\ \frac{3}{2} \end{bmatrix}$$

In [ ]:

```
# c)
v4 = x - proj
v1.normalized(), v2.normalized(), v3.normalized(), v4.normalized()
```

Out[ ]:

$$\left(\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} \right)$$

c. State your answer as positive integers.

$$\left\{ \begin{bmatrix} \frac{1}{\boxed{2}} \\ \frac{1}{2} \\ -\frac{1}{\boxed{2}} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{\boxed{1}}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \boxed{0} \\ \frac{1}{\sqrt{\boxed{2}}} \\ \frac{1}{\sqrt{\boxed{2}}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{\boxed{2}}} \\ 0 \\ \boxed{0} \\ \frac{1}{\sqrt{\boxed{2}}} \end{bmatrix} \right\}$$

Assignment 5

In [ ]:
```
# a)
k = symbols('k')
A = Matrix([[1,3,0],[1,1,1],[4,1,1]])
B = Matrix([[2,5,3],[1,1,2],[4,0,1]])
C = Matrix([[1,1,3],[3,2,1],[0,1,2]])
D = Matrix([[3,1,k],[3*k,k,2*k],[9,3,3*k]])

X2 = (A+B)**-1*C
detX = sqrt(det(X2))
detX
```

Out[ ]: $\frac{\sqrt{6}}{11}$

Consider the following matrices:

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 4 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 & 3 \\ 1 & 1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 1 & k \\ 3k & k & 2k \\ 9 & 3 & 3k \end{bmatrix}$$

a. Find the determinant of matrix X in the following matrix equation $AX^2 + BX^2 = C$.

b. Let $Y = D^T D$. What is the sum of the eigenvalues of Y?

c. For which value(s) of k, if any, is dim Nul $D = 2$?

a. State your answer as a positive integer.

$$\det X = \frac{\sqrt{\boxed{6}}}{11}$$

In [ ]:
```
# b)
Y = D.T * D
Y.trace()
```

Out[ ]: $24k^2 + 100$

In [ ]:
```
# c)
D.echelon_form()
```

Out[ ]:

$$\begin{bmatrix} 3 & 1 & k \\ 0 & 0 & -3k^2 + 6k \\ 0 & 0 & 0 \end{bmatrix}$$

In [ ]:
```
solve(D.echelon_form()[1,-1], k)
```

Out[ ]: $[0, 2]$

c. Choose **one or multiple** options below.

A $k = 10$

B $k \in \{\varnothing\}$, i.e. no such value exists

C $k \in \mathbb{R}$

D $k \in \mathbb{R} \setminus \{0\}$

E $k = -2\sqrt{10}$

F $k = 2$

G $k = 0$

H $k = -4$

Assignment 6

In [ ]:
```
# a)

x = np.array([0.66, 1.32, 1.98, 2.64, 3.3, 3.96, 4.62, 3.28, 5.94, 6.6])
y = np.array([7.32, 12.22, 16.34, 23.66, 28.06, 33.39, 34.12, 39.21, 44.21, 47.4

X = Matrix.hstack(Matrix(x**3), Matrix(x**2), Matrix(x), ones(len(x), 1))
X.evalf(5)
```

Out[ ]:

$$\begin{bmatrix} 0.2875 & 0.4356 & 0.66 & 1.0 \\ 2.3 & 1.7424 & 1.32 & 1.0 \\ 7.7624 & 3.9204 & 1.98 & 1.0 \\ 18.4 & 6.9696 & 2.64 & 1.0 \\ 35.937 & 10.89 & 3.3 & 1.0 \\ 62.099 & 15.682 & 3.96 & 1.0 \\ 98.611 & 21.344 & 4.62 & 1.0 \\ 35.288 & 10.758 & 3.28 & 1.0 \\ 209.58 & 35.284 & 5.94 & 1.0 \\ 287.5 & 43.56 & 6.6 & 1.0 \end{bmatrix}$$

Consider the following data points

t	0.66	1.32	1.98	2.64	3.3	3.96	4.62	3.28	5.94	6.6
h	7.32	12.22	16.34	23.66	28.06	33.39	34.12	39.21	44.21	47.48

We would like to model the data according to the following polynomial: pt^3+qt^2+rt+s. This corresponds to solving the system of linear equations given by $Ax = b$.

a. Determine matrix A in the above system.

b. Find the least squares solution to $A\bar{x} = \bar{b}$, and state the estimates of p, q, r, s. Remember to store the exact values and not the rounded values.

c. Using the exact value estimates found above, find the least squares error of the system

d. Using the model found in (b) with the exact values for p, q, r, s, determine the value of h when $t = 8$

a. State only the **first** and last **row** of matrix A. State your answer as positive integers.

$$\begin{bmatrix} \boxed{0}.29 & \boxed{0}.44 & \boxed{0}.66 & \boxed{1} \\ \vdots & \vdots & \vdots & \vdots \\ \boxed{287}.50 & \boxed{43}.56 & \boxed{6}.6 & \boxed{1} \end{bmatrix}$$

In [ ]:

```
# b)
B = Matrix.hstack(X.T * X, X.T * Matrix(y)).rref()[0][:,-1]
B
```

Out[ ]:

$$\begin{bmatrix} 0.0737485703615477 \\ -1.43455749869459 \\ 13.7797191051537 \\ -2.64941752283648 \end{bmatrix}$$

In [ ]:

```
round((Matrix(y)-X*B).norm(), 2)
```

Out[ ]: 11.06

In [ ]:

```
# d)
def f(t): return B[0]*t**3+B[1]*t**2 + B[2]*t + B[3]
round(f(8), 2)
```

Out[ ]: 53.54

Assignment 7

In [ ]:
```
a = 5
b = 3
c = a+b
d = a

A = 200
B = 100

Y = Matrix([[-R(c,A), R(b,B)], [R(c,A), -R((b+d),B)]])
Y.eigenvects()
```

Out[ ]: $\left[\left(-\frac{1}{10},\ 1,\ \left[\begin{bmatrix}-\frac{1}{2}\\1\end{bmatrix}\right]\right),\ \left(-\frac{1}{50},\ 1,\ \left[\begin{bmatrix}\frac{3}{2}\\1\end{bmatrix}\right]\right)\right]$

In [ ]:
```
l1 = Y.eigenvects()[0][0]
l2 = Y.eigenvects()[1][0]
v1 = Y.eigenvects()[0][2][0]
v2 = Y.eigenvects()[1][2][0]

y0 = Matrix([100, 10])

Matrix.hstack(v1, v2, y0).rref()
```

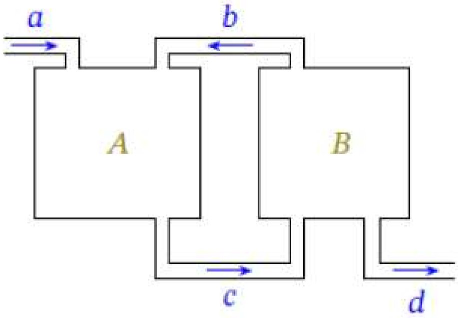
Out[ ]: $\left(\begin{bmatrix}1 & 0 & -\frac{85}{2}\\0 & 1 & \frac{105}{2}\end{bmatrix},\ (0,\ 1)\right)$

In [ ]:
```
c1 = Matrix.hstack(v1, v2, y0).rref()[0][0,-1]
c2 = Matrix.hstack(v1, v2, y0).rref()[0][1,-1]
```

In [ ]:
```
def y(t): return c1*v1*exp(t*l1) + c2*v2*exp(t*l2)
y(11).evalf(2)
```

Out[ ]: $\begin{bmatrix}70.0\\28.0\end{bmatrix}$

Two containers are connected by pipes, as in the figure below. Container A holds 200 l and container B 100 l. Initially, there is 100 g of salt in container A and 10 g of salt in container B. Through pipe a 5 l/min of pure water is added to A and through pipe b 3 l/min of a salt-water mixture is added to container A.



State the above case in terms of a system of differential equations and find the unique solution. How much salt (in grams) will there be in A and B, respectively, at time $t = 11$?. Please round to the nearest integer.

State your answer as positive integers. Remember to round to the nearest integer.

A: 100 g B: 10 g