6. Differential Equation

Disclaimer: This topic requires pre-existing Calculus skills, more specifically knowledge about differential equations. I recap a bit here.

Definition:

An equation that relates one or more functions and their derivatives is called a **Differential Equation**. The "unknowns" are thus functions:

$$y'(t) = a \cdot y(t) \qquad , \; a \text{ is a constant.}$$

This means that solving the equation boils down to finding $y(t)$:

① $y(t) = e^{at}$, so $y'(t) = (e^{at})' = (at)' \cdot e^{at} = a e^{at} = a \cdot y(t)$

② $y(t) = c \cdot e^{at}$, so $y'(t) = (c e^{at})' = c \cdot (at)' e^{at} = c \cdot a \cdot e^{at} = a y(t)$

In summary:

The general solution to

$$y'(t) = a \cdot y(t)$$

is given by

$$y(t) = c \cdot e^{at} ,$$

Solutions are uniquely determined by an initial value $y(t_0) = r_0$:

$$y(t_0) = r_0$$
$$c \cdot e^{at_0} = r_0$$
$$c = r_0 \cdot e^{-at_0}$$

Ex Given $y'(t) = 2y(t)$ and $y(0) = -3$ find $y(t)$

$$c = -3e^{-2\cdot 0} = -3$$

$$y(t) = -3e^{2t}$$

Systems of first order differential equations:

$$y_1'(t) = a_{11}y_1(t) + a_{12}y_2(t) + \dots + a_{1n}y_n(t)$$

$$y_2'(t) = a_{21}y_1(t) + a_{22}y_2(t) + \dots + a_{2n}y_n(t)$$

$$\vdots$$

$$y_n'(t) = a_{n1}y_1(t) + a_{n2}y_2(t) + \dots + a_{nn}y_n(t)$$

In vector notation:

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

Or $y'(t) = A\,y(t)$, $A = [a_{ij}] \in \mathbb{R}^{n\times n}$

Decoupling:

When A is diagonal, $A = \mathrm{diag}(d_1, d_2, d_3 \dots d_n)$

$$y_1'(t) = d_1 \cdot y_1(t)$$

$$y_2'(t) = d_2 \cdot y_2(t)$$

$$\vdots$$

$$y_n'(t) = d_n\, y_n(t)$$

Ex

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$y_1'(t) = 3\, y_1(t)$$
$$y_2'(t) = -5\, y_2(t)$$

which we can solve

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \cdot e^{3t} \\ c_2 \cdot e^{-5t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot e^{3t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot e^{-5t}$$

Ex

Find a solution to $y'(t) = A\, y(t)$ for $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ and $y(0) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -3\, e^{2t} \\ 2\, e^{-3t} \end{bmatrix}$$

So we may just hope to always get diagonal matrices? Well No!

$$A \bar{v} = \lambda \bar{v} \quad , \quad \bar{v} \neq \bar{0}$$

This will give us:

$$y(t) = c \cdot \bar{v} \cdot e^{\lambda t}$$

as a solution to

$$y'(t) = A \cdot y(t)$$

Proof:

$$y'(t) = \frac{d(c \cdot \bar{v} \cdot e^{\lambda t})}{dt} = \frac{c \cdot \bar{v}\, d(e^{\lambda t})}{dt} = c \cdot \underbrace{\bar{v} \cdot \lambda}_{A\bar{v}} e^{\lambda t}$$

$$= c \cdot A \bar{v} \cdot e^{\lambda t} = A \cdot \underbrace{c \bar{v} \cdot e^{\lambda t}}_{y(t)} = A \cdot y(t)$$

Theorem:

If A is diagonalisable with $\bar{v}_1, \bar{v}_2 \dots \bar{v}_n$ of eigenvectors and eigenvalues $\lambda_1, \lambda_2, \dots \lambda_n$, the solution to $y'(t) = A y(t)$ is

$$y(t) = c_1 \cdot \bar{v}_1 \cdot e^{\lambda_1 t} + c_2 \cdot \bar{v}_2 \cdot e^{\lambda_2 t} + \dots c_n \cdot \bar{v}_n \cdot e^{\lambda_n t}$$

Given an initial vector $y(t_0) = \bar{r} \in \mathbb{R}^n$, a unique solution can be found using

$$c_1 \bar{v}_1 \cdot e^{\lambda_1 t_0} + c_2 \bar{v}_2 \cdot e^{\lambda_2 t_0} + \dots + c_n \bar{v}_n \cdot e^{\lambda_n t_0} = \bar{r}$$

And it will have a unique solution for $c_1 \dots c_n$ since $\{\bar{v}_1 \dots \bar{v}_n\}$ is a basis.

Ex:

$$\left.\begin{aligned} y_1'(t) &= y_1(t) + y_2(t) \\ y_2'(t) &= 4 \cdot y_1(t) + y_2(t) \end{aligned}\right\} \to y'(t) = A y(t) \to \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

so we first need to find λ_1 and λ_2

$$\det(A - \lambda I) = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) \to \lambda = \begin{cases} 3 \\ -1 \end{cases}$$

<u>For $\lambda = 3$</u>

$$\underbrace{\begin{bmatrix} 1-3 & 1 \\ 4 & 1-3 \end{bmatrix}}_{A - \lambda I} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

<u>For $\lambda = -1$</u>

$$\begin{bmatrix} 1+1 & 1 \\ 4 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} -1 \\ 2 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

General solution:

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot e^{-t} \\ c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot e^{3t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot e^{3t}$$

$$y_1(t) = c_1 \cdot e^{-t} + c_2 \cdot e^{3t}$$

$$y_2(t) = -2c_1 \cdot e^{-t} + 2c_2 \cdot e^{3t}$$

Assume $g(1) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, find unique solution

$$3 = C_1 \cdot e^{-1} + C_2 \cdot e^{3\cdot 1} = e^{-1} C_1 + e^{3} C_2 = 3$$

$$2 = -2 C_1 \cdot e^{-1} + 2 C_2 \cdot e^{3\cdot 1} = -2 e^{-1} C_1 + 2 e^{3} C_2 = 2$$

$$\begin{bmatrix} e^{-1} & e^{3} & 3 \\ -2e^{-1} & 2e^{3} & 2 \end{bmatrix} \overset{r_2 \to r_2 + 2r_1}{\sim} \begin{bmatrix} e^{-1} & e^{3} & 3 \\ 0 & 4e^{3} & 8 \end{bmatrix} \overset{r_2 \to \frac{1}{4} r_2}{\sim}$$

$$\begin{bmatrix} e^{-1} & e^{3} & 3 \\ 0 & e^{3} & 2 \end{bmatrix} \overset{r_1 \to r_1 - r_2}{\sim} \begin{bmatrix} e^{-1} & 0 & 1 \\ 0 & e^{3} & 2 \end{bmatrix} \overset{\substack{r_1 \to e^{1} \cdot r_1 \\ r_2 \to e^{-3} \cdot r_2}}{\sim} \begin{bmatrix} 1 & 0 & e^{1} \\ 0 & 1 & \frac{2}{e^{3}} \end{bmatrix}$$

So

$$g_1(t) = e^{1} \cdot e^{-t} + 2 e^{-3} \cdot e^{3t} = e^{1-t} + 2 e^{3(t-1)}$$

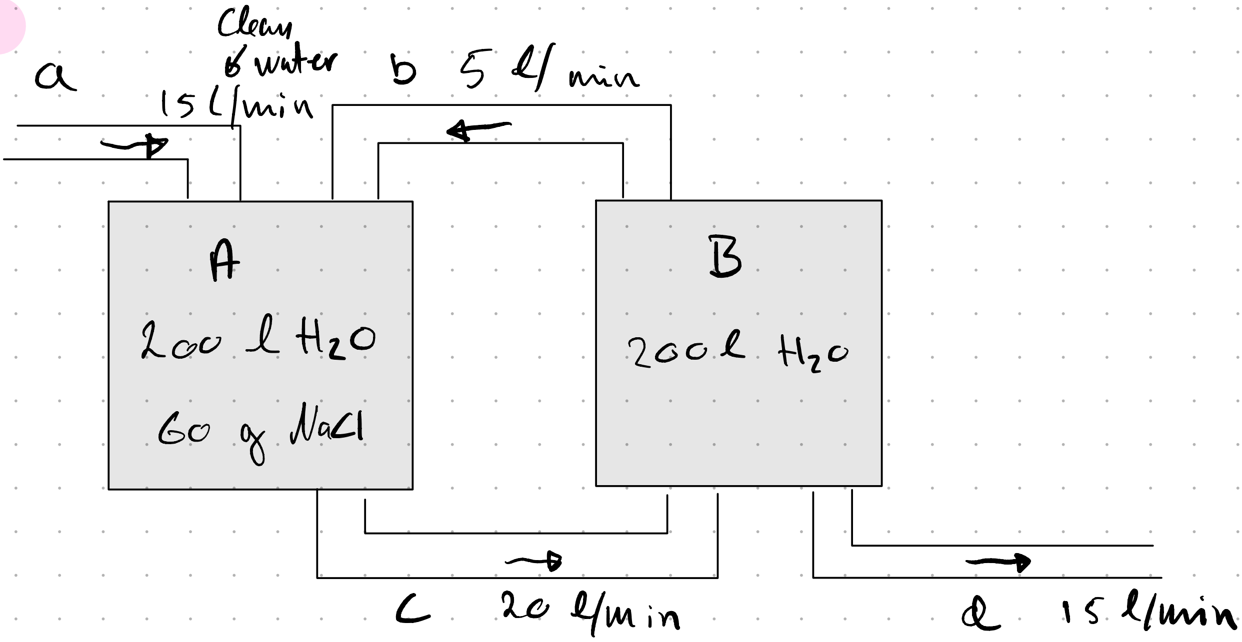
$$g_2(t) = -2 \cdot e^{1} \cdot e^{-t} + 2 \cdot 2 \cdot e^{-3} e^{3t} = -2 e^{1-t} + 4 e^{3(t-1)}$$

Usually the solution is stated as (for 2d-problem):

$$\begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} = C_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cdot e^{\lambda_1 t} + C_2 \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{\lambda_2 t}$$

$$\begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} = \underbrace{e}_{} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot e^{-t} + 2 \cdot e^{-3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$$

EX



Let $y_1(t)$ be amount of salt in A at time t

Let $y_2(t)$ be amount of salt in B at time t

$$y_1'(t) = -\frac{20}{200} y_1(t) + \frac{5}{200} y_2(t) = -\frac{1}{10} y_1(t) + \frac{1}{40} y_2(t)$$

$$y_2'(t) = \frac{20}{200} y_1(t) - \frac{20}{200} y_2(t) = \frac{1}{10} y_1(t) - \frac{1}{10} y_2(t)$$

$$y'(t) = A\, y(t) = \begin{bmatrix} -1/10 & 1/40 \\ 1/10 & -1/10 \end{bmatrix}, \quad y(0) = \begin{bmatrix} 60 \\ 0 \end{bmatrix}$$

Charpoly A:

$$\lambda^2 - \left(-\frac{1}{10} - \frac{1}{10}\right)\lambda + \frac{1}{100} - \frac{1}{400} = \lambda^2 + \frac{1}{5}\lambda + \frac{3}{400}$$

$$\lambda = \frac{-\frac{1}{5} \pm \sqrt{\left(\frac{1}{5}\right)^2 - 4 \cdot 1 \cdot \frac{3}{400}}}{2 \cdot 1} = \begin{cases} -\frac{1}{20} \\ -\frac{3}{20} \end{cases}$$

For $\lambda = -\frac{1}{20}$:

$$\begin{bmatrix} -\frac{1}{10} + \frac{1}{20} & 1/40 \\ 1/10 & -1/10 + 1/20 \end{bmatrix} \sim \begin{bmatrix} -\frac{1}{20} & 1/40 \\ 1/10 & -1/20 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For $\lambda = -\frac{3}{20}$

$$\begin{bmatrix} -\frac{1}{10} + \frac{3}{20} & 1/40 \\ \text{———} & \end{bmatrix} \sim \begin{bmatrix} 1/20 & 1/40 \\ & \end{bmatrix} \to \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

General Solution:

$$\bar{y}(t) = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot e^{-t/20} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-3t/20}$$

for $t=0$: $y(0) = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 60 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 60 \\ 2 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 30 \\ 0 & 1 & 30 \end{bmatrix}$$

Unique solution:

$$\bar{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = 30\begin{bmatrix}1\\2\end{bmatrix} e^{-\frac{t}{20}} + 30\begin{bmatrix}1\\-2\end{bmatrix} \cdot e^{-\frac{3}{20}t}$$

You will often see the notation $\bar{x}(t)$ instead of $\bar{y}(t)$ to emphasise that $\bar{x}(t)$ is a vector function.

e.g. $\bar{x}(t) = 3\begin{bmatrix}1\\2\end{bmatrix} e^{-t/2} - 2\begin{bmatrix}-1\\1\end{bmatrix} e^{-2t}$

Trajectory

$$\mathbf{x}(t) = 3\begin{bmatrix}1\\2\end{bmatrix}e^{-.5t} - 2\begin{bmatrix}-1\\1\end{bmatrix}e^{-2t}$$

or

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3e^{-.5t} + 2e^{-2t} \\ 6e^{-.5t} - 2e^{-2t} \end{bmatrix}$$

Figure 2 shows the graph, or *trajectory*, of $\mathbf{x}(t)$, for $t \geq 0$, along with trajectories for some other initial points. The trajectories of the two eigenfunctions $\mathbf{x}_1$ and $\mathbf{x}_2$ lie in the eigenspaces of A.

The functions $\mathbf{x}_1$ and $\mathbf{x}_2$ both decay to zero as $t \to \infty$, but the values of $\mathbf{x}_2$ decay faster because its exponent is more negative. The entries in the corresponding eigenvector $\mathbf{v}_2$ show that the voltages across the capacitors will decay to zero as rapidly as possible if the initial voltages are equal in magnitude but opposite in sign. ■

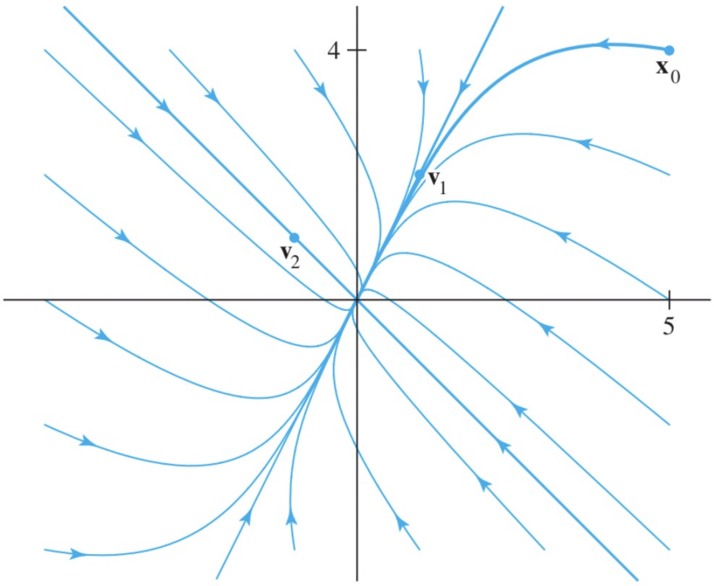


FIGURE 2 The origin as an attractor.

Negative eig. value
↳ attractor
or
sink

Greatest attraction is along function with greatest eig. value (numerically)

Here x_2 since $\lambda_2 = -2 > -1/2 = \lambda_1$

If the eig. values at positive, the origin would be a repeller (or source)
↳ greatest repulsion a long line of greatest eig. value.

$$\mathbf{x}(t) = \frac{-3}{70}\begin{bmatrix} -5 \\ 2 \end{bmatrix} e^{6t} + \frac{188}{70}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

Trajectories of $\mathbf{x}$ and other solutions are shown in Fig. 3. ■

In Fig. 3, the origin is called a **saddle point** of the dynamical system because some trajectories approach the origin at first and then change direction and move away from the origin. A saddle point arises whenever the matrix A has both positive and negative eigenvalues. The direction of greatest repulsion is the line through $\mathbf{v}_1$ and $\mathbf{0}$, corresponding to the positive eigenvalue. The direction of greatest attraction is the line through $\mathbf{v}_2$ and $\mathbf{0}$, corresponding to the negative eigenvalue.

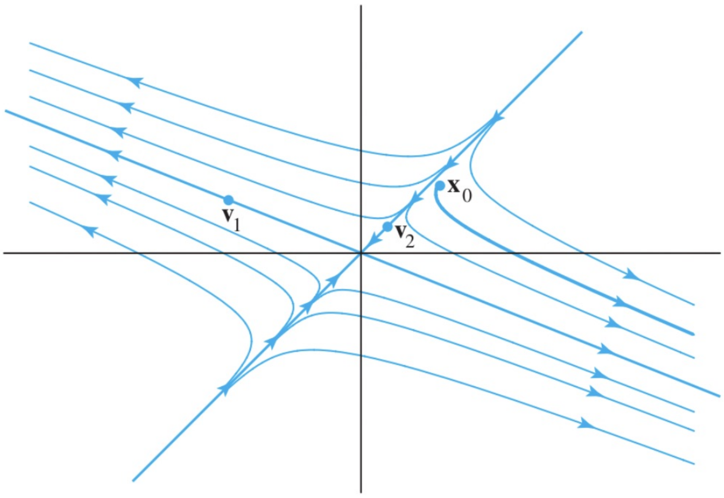


FIGURE 3 The origin as a saddle point.