PART 1

Item 1

It is possible to obtain 10 points in this assignment, two points for each assignment. In each assignment, it is specified how you should answer. No documentation is required for this assignment.

a. Consider a m\times n matrix *A* with the following properties:

- A has 6 pivot columns
- There exists a \vec{b}\ \in \mathbb{R}^m, such that A\vec{x}=\vec{b} is inconsistent

Specify the **minimal** value for *m* using an integer between 0 and 99:

1 6 ✕

Specify the **minimal** value for *n* using an integer between 0 and 99:

2 6 ✓

Correct answers:

1 7

b. State whether each statement is true or false.

	True	False
There is given four vectors $\vec{x}, \vec{y}, \vec{u}, \vec{v} \in \mathbb{R}^4$ such that both $\{\vec{x}, \vec{y}\}$ and $\{\vec{u}, \vec{v}\}$ are linearly independent. $\{\vec{x}, \vec{y}, \vec{u}, \vec{v}\}$ is always linearly independent.	◉ ✗	○ ✓
All invertible matrices can be diagonalized	◉ ✗	○ ✓
Let W be a subspace of $\mathbb{R}^6$ with dimension 4. Then it is always true that 4 linearly independent vectors in W constitute a basis.	◉ ✓	○

c. Find the determinant of A:

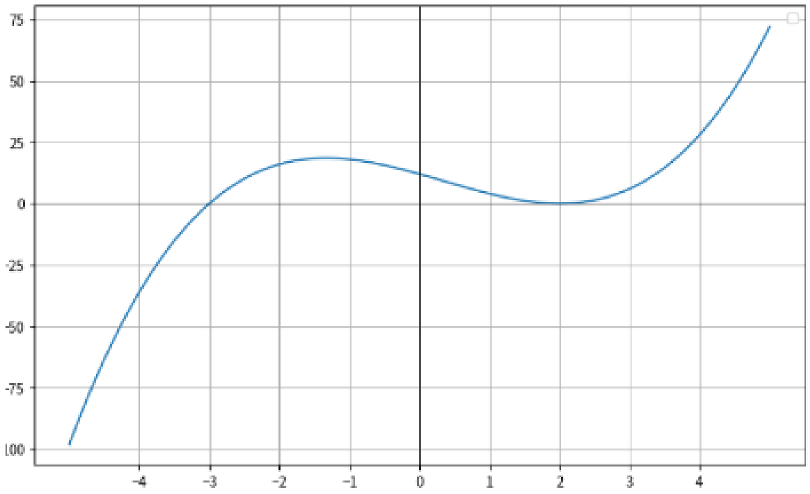
$$A = \begin{bmatrix} 2 & 8 & 3 & 6 & 7 \\ 1 & 2 & 4 & 5 & 5 \\ 4 & 16 & 6 & 12 & 14 \\ 5 & 7 & 8 & 9 & 0 \\ 3 & 4 & 5 & 7 & 1 \end{bmatrix}$$

$\det(A) = 48$ ✗

Correct answers:

$\det(A) = 0$

d. Below you see a plot of the characteristic polynomial of a 3 x 3 matrix A.



State the eigenvalues and their multiplicities by choosing the correct value from the drop-down menu.

✕

The smallest eigenvalue is 1 -3 ✓ and it has multiplicity 2 1 ✓ . The largest eigenvalue is 3 2 ✓ and it has multiplicity 4 1 ✕ . Note: Negative eigenvalues are smaller than positive eigenvalues.

Correct answers:

4 2

e. Assume the results of a singular value decomposition of a $m \times n$ matrix *A* shows that there are three left singular vectors, five right singular vectors, and one non-zero singular value. Determine the values below. State all answers as integers between 0 and 99.

$dim\ Col\ A =$ 1 3 ✕

$dim\ Nul\ A =$ 2 2 ✕

$dim\ Nul\ A^T =$ 3 2 ✓

$dim\ Col\ A^T =$ 4 3 ✕

Correct answers:

1 1 2 4 4 1

Item 2

It is possible to obtain 5 points in this assignment. You must document how you obtained the result.

Find a vector function of the form

$$\bar{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} = \sum_{i=1}^{n} C_i \bar{v}_i e^{\lambda_i t}$$

such that such that $y_1(0) = 15$ and $y_2(0) = -10$ and such that they are related by the linear system of differential equations,

$$y_1' = y_1 + y_2$$
$$y_2' = 4y_1 + y_2$$

State your answers as integers between 0 and 99 (note negative signs have been pre-printed).

$$\bar{y}(t) = -10\begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t} + 15\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$$ ✔

Item 3

It is possible to obtain 5 points in this assignment, one point for each problem. You must document how you obtained the result.
Let A and B both be 3 x 3 matrices with det (A) = 4 and det (B) = 8. Calculate the following, stating all inputs as integers between 0 and 99. Please note, in the last problem the determinant is between 12 and 9912 given that your input must be between 0 and 99 and since 12 is pre-printed.

$\det\left((AB)^{-1}\right) =$ 1 32 ✔ $^{-1}$

$\det\left(\frac{1}{8}A^3B^2\right) =$ 2 512 ✘

$\det\left(AB^T\right) =$ 3 32 ✔

$\det\left(4A^{-2}\right) =$ 4 16 ✘

$\det\left((3A)(2B)\right) =$ 5 192 ✘ 12

Correct answers:

2 8 4 4 5 69

Item 4

It is possible to obtain 10 points in this assignment, two points for each assignment. In each assignment, it is specified how you should answer. No documentation is required for this assignment.
Consider the following system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + x_4 &= 14 \\ -x_1 - 2x_2 - 3x_4 &= -4 \\ 2x_1 + 4x_2 + 8x_3 - 2x_4 &= 48. \end{aligned}$$

Let A denote the coefficient matrix of this system such that the system takes the form $A\vec{x} = \vec{b}$.

a. Translate this system into an augmented matrix. State your answers as integers between 0 and 99. Note all and any negative sign have been pre-printed.

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 14 \\ -1 & -2 & 0 & -3 & -4 \\ 2 & 4 & 8 & -2 & 48 \end{bmatrix}$$

✔

b. In the rest of the assignment you can use the fact that the row reduced echelon form of the system above is

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer the questions below. State all your answers as integers between 0 and 99.

The pivot columns are (in ascending order, indexing from 1): Column 1 1 ✔ and Column

2 3 ✔

The free variables are located in columns (in ascending order, indexing from 1): Column

3 2 ✔ and Column 4 4 ✔

The rank of the matrix is 5 2 ✔

c. Assuming that the free variables are set to 0, fill in the values of x_i that gives a non-trivial solution to the system. That means you have to put a 0 into the fields of the free variables below. State all answers as integers between 0 and 99.

$x_1 =$ 1

$x_2 =$ 2 0 ✓

$x_3 =$ 3

$x_4 =$ 4 0 ✓

Correct answers:

1 4 3 5

d. Determine the solution to $A\vec{x} = \vec{0}$ and state the solution in parametric vector form. State all answers as integers between 0 and 99. Please remember to also fill out the subscripts of the free variables.

$$\vec{x} = \begin{bmatrix} 4 \\ 0 \\ 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

✗

Correct answers:

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

e. Determine whether the statements below are true or false about the system.

The system will always have a solution.	○ True ✓	◉ False ✗
The coefficient matrix is invertible.	◉ True ✗	○ False ✓
The columns of A span $\mathbb{R}^2$.	◉ True ✗	○ False ✓
The dimension of the nullspace of A is 4.	◉ True ✗	○ False ✓
0 will be a singular value of A.	○ True ✓	◉ False ✗

Item 5

It is possible to obtain 5 points in this assignment. You must document how you obtained the result.
Find an expression for the determinant of matrix A, expressed in terms of (x, y, z). State your inputs as as integers between 0 and 99.

$$A = \begin{bmatrix} 1 & 2 & x \\ 2 & 1 & y \\ 3 & 5 & z \end{bmatrix}$$

$\det(A) = 2x + y - 3z$ ✗

Correct answers:

$$\det(A) = 7x + y - 3z$$

Item 6

It is possible to obtain 10 points in this assignment, two points each for the first two problems and six for the last. In each assignment, it is specified how you should answer. You must document how you obtained the results, except in (b).

a. Express $\vec{v}_3$ as a linear combination of the two other vectors. State all inputs as integers between 0 and 99.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 5 \\ 11 \end{bmatrix}$$

$\vec{v}_3 = 3\vec{v}_1 + 5\vec{v}_2$ ✓

b. Determine which sets are dependent and independent and categorise them below.

$$L_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 9 \\ 1 \end{bmatrix} \right\},$$

$$L_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix} \right\}, \quad L_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\},$$

$$L_4 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

✗

	Dependent	Independent
L_1	◉ ✓	○
L_2	○	◉ ✓
L_3	◉ ✓	○
L_4	○ ✓	◉ ✗

c. Determine which, if any, of the following vectors are eigenvectors for the matrix A and use this information to state all the eigenvalues of A such that $\lambda_1 \leq \lambda_2 \leq \lambda_3$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

	Is an eigenvector	Is not an eigenvector
$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$	◉ ✗	○ ✓
$\begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$	○ ✓	◉ ✗
$\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$	○	◉ ✓
$\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$	◉ ✓	○

✗

$\lambda_1 = -$ 1 3 ✗

$\lambda_2 =$ 2 1 ✓

$\lambda_3 =$ 3 2 ✗

Correct answers:

1 1 3 6

Item 7

It is possible to obtain 5 points in this assignment. You must document how you obtained the result.

Let the set Y be a basis for a subspace in $\mathbb{R}^4$:

$$Y = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 2 \end{bmatrix} \right\} \subset \mathbb{R}^4$$

Find a basis for $Y^\perp$. You must use the following method:

1. Form the matrix A whose rows are the vectors from Y.
2. Find the null space (or kernel) of A. The basis of this null space will be the basis for $Y^\perp$.

State all inputs as integers between 0 and 99.

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

✕

Correct answers:

$$\left\{ \begin{bmatrix} 4 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

PART 2

Item 1

It is possible to obtain 15 points in this assignment, five points for each problem. You must document how you obtained the result.

Let *U* be the subspace in $\mathbb{R}^3$ given by

$$U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| 2x - y - 2z = 0 \right\}$$

Find the orthogonal projection of $\vec{p} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ onto the subspace spanned by *U*. State your answer as integers between 0 and 99.

$$\frac{1}{3}\begin{bmatrix} 5 \\ \square \\ \square \end{bmatrix}$$

Correct answers:

$$\frac{1}{3}\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$$

b. Now find the distance from *U* to $\vec{q} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$. State you answer as integers between 0 and 99 such that they make up an irreducible fraction.

$$dist(U, \vec{q}\,) = \frac{\square}{\square}$$

Correct answers:

$$\boldsymbol{dist}\,(U,\ \vec{q}) = \frac{5}{3}$$

c. Find $U^{\perp}$ and state it as a normalised vector. State your answer as integers between 0 and 99.

$$\frac{1}{3}\begin{bmatrix} -2 \\ \square \\ \square \end{bmatrix}$$

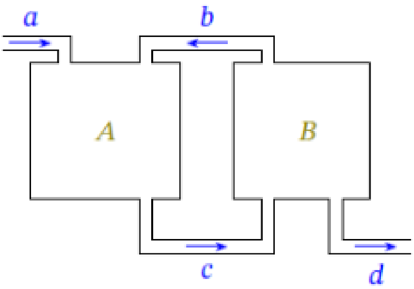
Correct answers:

$$\frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

Item 2

It is possible to obtain 10 points in this assignment, equally divided between the four problems. You must document how you obtained the result.

Two containers are connected by pipes, as in the figure below. Container A holds 200 l and container B 100 l. Initially, there is 100 g of salt in container A and no salt in container B. Through pipe a 5 l/min of pure water is added to A and through pipe c 8 l/min of a salt-water mixture is added to container B. Assume that the water level is constant.



Let $y_1(t)$ and $y_2(t)$ be the amount of salt in A and B, respectively.

a. How many grams of salt will there be in tank B after 25 minutes? Round down to the nearest integer.

Salt in tank B after 25 minutes: 1

Correct answers:

1 26

b. At what time will the amount of salt in A be a quarter of what it was initially? State your answer as a positive integer. Always **round up** to the nearest integer in this type of problem. Note, the answer is less than t = 100.

$t =$ 1 56

c. In the long run, what is the ratio of salt in tank B relative to tank A $\left(\frac{y2}{y1}\right)$. State you answers as positive integers so that the final answer is an irreducible fraction.

Correct answers:

$\frac{2}{3}$

d. At what time will the difference between between the two tanks be less than 1 gram of salt. Round up to the nearest integer. The answer is somewhere between t = 100 and t = 200.

Time at which difference between the two tanks is less than 1 gram: 1

Correct answers:

1 161

Item 3

It is possible to obtain 5 points in this assignment. You must document how you obtained the result.
Let

$$U = \text{span}\{\vec{u}_1, \vec{u}_2\} = \text{span}\left\{\begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}\right\}, \quad V = \text{span}\{\vec{v}_1, \vec{v}_2\} = \text{span}\left\{\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\1\\-1\end{bmatrix}\right\}$$

Find a basis for $U + V$ and state the dimension of the basis as an integer between 0 and 99.

$\dim U + V =$ 1

Correct answers:

1 3

Item 4

It is possible to obtain 5 points in this assignment, divided equally between the two problems. You must document how you obtained the result.

Let A and B be the following matrices:

$$A = \begin{bmatrix} 1 & 2 & -4 \\ 3 & 1 & 0 \\ 2 & 3 & -3 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 3 & -1 \end{bmatrix}$$

a. Solve the equation $AX = B$

$$\begin{bmatrix} 0 & 1 \\ 1 & -3 \\ 0 & -2 \end{bmatrix}$$

✔

b. Now consider this small modification to A

$$A = \begin{bmatrix} 1 & 2 & -4 \\ 3 & k & 0 \\ 2 & 3 & -3 \end{bmatrix}$$

Find the value of k such that the equation $AX = B$ has no solution. State your inputs as two integers between 0 and 99 such that the answer is an irreducible fraction.

$$\frac{0}{0}$$

✘

Correct answers:

$$\frac{18}{5}$$

Item 5

It is possible to obtain 5 points in this assignment, three for the first problem and two for the second. You must document how you obtained the result.
Consider the following data:
x = [0.2, 0.3, 1.1, 1.3, 2.2, 2.4, 2.5]
y = [1.6, -1.2, -0.3, 2.1, 2.3, -0.2, -1.3]
It is assumed that the data can be approximated by a model of the form $y_1(t) = \beta_0 + \beta_2 t^3$ or of the form $y_2(t) = \beta_1 t^2 + \beta_2 t^3$ or of the form $y_3 = \beta_0 + \beta_1 t$. Note: in all parts, you are expected to use the exact values in your calculations and not the rounded off values that you input.

a. State the first two rows and the last two rows for the Design Matrix of y_2. State your inputs as decimal values, correctly rounded to two-decimal precision (e.g. 3.45).

$$\begin{bmatrix} \square & \square \\ \square & \square \\ \vdots & \vdots \\ \square & \square \\ \square & \square \end{bmatrix}$$

Correct answers:

$$\begin{bmatrix} 0.04 & 0.01 \\ 0.09 & 0.03 \\ \vdots & \vdots \\ 5.76 & 13.82 \\ 6.25 & 15.63 \end{bmatrix}$$

and now state $X_3^T X_3$ where X_3 is the Design Matrix for y_3. Round all values to the nearest integer and state these as your inputs.

$$\begin{bmatrix} 7 & 10 \\ 10 & 20 \end{bmatrix}$$

✔

b. Find the best fitted model and state its error. Remember to also input the subscript which indicates which model is the best. State the subscript as an integer between 1 and 3, and state the error as a decimal value with two-decimal precision, correctly rounded off.

$e_2 = 3.38$ ✘

Correct answers:

$e_2 = 3.15$

Item 6

Assume the following is known with regard to a matrix A:

$$AA^T = \begin{bmatrix} 2 & 4 & 4 \\ 4 & 8 & 8 \\ 4 & 8 & 8 \end{bmatrix}$$

a. Find the eigenvalues of AA^T, such that $\lambda_1 \geq \lambda_2 \geq \lambda_3$, their multiplicity and the corresponding eigenvectors of AA^T, denoted x_1, x_2, x_3, respectively, below.

$\lambda_1 =$ 1 18 with multiplicity 2 1

And the corresponding eigenvector for λ_1 is

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Correct answers:

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$\lambda_2 =$ 1 0 with multiplicity 2 2

And the corresponding eigenvectors for λ_2 are

$$x_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

b. It is known that the columns of the matrix below are the right singular vectors of A corresponding to the eigenvectors and eigenvalues found in (a).

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Find matrix A and state the sum of all its entries as an integer between 0 and 99.

$A = 10$