In [ ]:
```
import sympy as sp
from scipy import *
from sympy import *
init_printing()
from IPython.display import display, Latex, HTML, Math
import numpy as np
import pandas as pd
```

Assignment 1

In [ ]:
```
# a) Trace (i.e. sum of main diagonal) of any (real) matrix is equal to sum of e

# b) determinant is product of diagonal, which is -1, and since one entry is -1

# c) i) A is in echelon form, ii) nullity A + rank A = 4, iii) The matrix has ei

# d) det of orthogonal matrix always either -1 or 1. Since det B > 0, det B = 1,

# e) pivots = 3, free var = 3, dim nul A = 6, dim col A = 3
```

a. It is known that the sum of the eigenvalues of the below matrix is 7. Based on this information, fill in the missing value. State your answer as an integer between 0 and 99.

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \boxed{2} & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Check answer

b. It is known that the determinant of the following matrix is -1. It is also know that one of its eigenvalues has algebraic multiplicity of 2. Based on this information, fill in the missing values. State your answers as integers between 0 and 99.

$$\begin{bmatrix} \boxed{1} & 3 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

c. Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & -1 & -2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Mark all correct statements below. You need to mark all and only the correct statements in order to receive points.

- [ ] A is singular
- [x] A is in row-echelon form
- [ ] nullity A = 1
- [ ] rank A = 3
- [x] nullity A + rank A = 4
- [ ] The number 0 is an eigenvalue of A
- [ ] A is in reduce row-echelon form
- [ ] There is a vector $b \in \mathbb{R}^4$, such that $A\mathbf{x} = \mathbf{b}$ is not consistent.
- [ ] det A = 4
- [x] The matrix has an eigenvalue with a multiplicity greater than 1.

e. The reduced row-echelon form of the matrix

$$A = \begin{bmatrix} -2 & 2 & 3 & 1 & -3 & -2 \\ 0 & 2 & 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & -3 & -2 & 2 \end{bmatrix}$$

is given by

$$R = \begin{bmatrix} 1 & 0 & 0 & 10 & 2 & -7 \\ 0 & 1 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 1 & 7 & 0 & -5 \end{bmatrix}$$

Specify the following values. State your answers as integers between 0 and 99.

The number of pivots of A is 3 ✔

The number of free variables in the system of equations $A\mathbf{x} = \mathbf{0}$ is 3 ✔

The null space of A is a subspace of $\mathbb{R}^x$ where $x =$ 6 ✔

The column space of A is a subspace of $\mathbb{R}^y$ where $y =$ 3 ✔

In [ ]:
```
Matrix([[1,3,1],[0,-1,0],[0,0,1]]).eigenvects()
```

Out[ ]:

$$\left[\left(-1,\ 1,\ \left[\begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \right] \right),\ \left(1,\ 2,\ \left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right] \right) \right]$$

Assignment 2

In [ ]:
```
A = Matrix([[1,1,0],[0,1,1],[0,0,1]])
B = Matrix([[2,0,0],[1,1,2],[2,0,1]])
C = Matrix([[1,1,0],[0,1,0],[0,1,2]])
```

In [ ]:
```
# a)
display(Math(r'\text{Det}(X) =' + latex(((B+C)*A**-1).det())))
```

$$\text{Det}(X) = 13$$

Let three matrices be given by:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

a. Find the determinant of matrix X in the following matrix equation $XA = B + C$. State your answer as an integer between 0 and 99.

det $X =$ 13 ✓

In [ ]:
```
# b)
C = Matrix([[9,-7,7],[20,3,-10]])
D = Matrix([[9,-2,20],[25,-3,-8]])
B = Rational(1,3)*(D-2*C)
A = Rational(1,3)*(C+B)
display(Math(r'A =' + latex(A)))
display(Math(r'B =' + latex(B)))
```

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 0 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & 4 & 2 \\ -5 & -3 & 4 \end{bmatrix}$$

b. Determine the matrices A and B from the following system of matrix equations

$$\begin{cases} 3A - B = \begin{bmatrix} 9 & -7 & 7 \\ 20 & 3 & -10 \end{bmatrix} \\ 6A + B = \begin{bmatrix} 9 & -2 & 20 \\ 25 & -3 & -8 \end{bmatrix} \end{cases}$$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 4 & 2 \\ -5 & -3 & 4 \end{bmatrix}$$ ✓

Assignment 3

In [ ]:
```
# a)
A = Matrix([[0,2,3,4],[1,-3,4,5],[-3,10,-6,-7]])
display(Math(r'A =' + latex(A)))
```

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 1 & -3 & 4 & 5 \\ -3 & 10 & -6 & -7 \end{bmatrix}$$

In [ ]:
```
# b)
R = Matrix([[0,2,3,4,1],[1,-3,4,5,2],[-3,10,-6,-7,-4]])
R.row_swap(0,1)
R1 = R
display(Math(r'R1 =' + latex(R1)))
R1[2,:] = R1[2,:]+3*R1[0,:]
R2 = R1
display(Math(r'R2 =' + latex(R2)))
R2.row_swap(1,2)
R3 = R2
```

```
display(Math(r'R3 =' + latex(R3)))
R3[2,:] = R[2,:]-2*R[1,:]
R4 = R3
display(Math(r'R4 =' + latex(R4)))
R4[2,:] = -Rational(1,9) * R4[2,:]
R5 = R4
display(Math(r'R5 =' + latex(R5)))
```

$$R1 = \begin{bmatrix} 1 & -3 & 4 & 5 & 2 \\ 0 & 2 & 3 & 4 & 1 \\ -3 & 10 & -6 & -7 & -4 \end{bmatrix}$$

$$R2 = \begin{bmatrix} 1 & -3 & 4 & 5 & 2 \\ 0 & 2 & 3 & 4 & 1 \\ 0 & 1 & 6 & 8 & 2 \end{bmatrix}$$

$$R3 = \begin{bmatrix} 1 & -3 & 4 & 5 & 2 \\ 0 & 1 & 6 & 8 & 2 \\ 0 & 2 & 3 & 4 & 1 \end{bmatrix}$$

$$R4 = \begin{bmatrix} 1 & -3 & 4 & 5 & 2 \\ 0 & 1 & 6 & 8 & 2 \\ 0 & 0 & -9 & -12 & -3 \end{bmatrix}$$

$$R5 = \begin{bmatrix} 1 & -3 & 4 & 5 & 2 \\ 0 & 1 & 6 & 8 & 2 \\ 0 & 0 & 1 & \frac{4}{3} & \frac{1}{3} \end{bmatrix}$$

In [ ]:

```
# c)
R.rref()[0]
```

Out[ ]:

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} & \frac{1}{3} \end{bmatrix}$$

In [ ]:

```
# d)
v1 = Matrix([0, 2, 3, 4])
v2 = Matrix([1, 3, 4, 5])
v3 = Matrix([-3, 10, -6, -7])
V = v1.row_join(v2).row_join(v3)
display((V.T).nullspace()[0])
```

$$\begin{bmatrix} \frac{1}{3} \\ 0 \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} + \frac{1}{3}x_4 \\ 0 \\ \frac{1}{3} - \frac{4}{3}x_4 \\ x_4 \end{bmatrix}$$

In [ ]:
```
# e)
```

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{1}{3} \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \frac{1}{3} \\ 0 \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

In [ ]:
```
# f)
display((V.T).nullspace()[0])
```

$$\begin{bmatrix} \frac{1}{3} \\ 0 \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

In [ ]:
```
# g) rank A = 3, dim Nul A = 1, x = 4
```

In [ ]:
```
# h)
R = Matrix([[0,2,3,4,1],[1,-3,4,5,2],[-3,10,-6,-7,-4]])
A*Matrix([1,1,1,1]) == R[:,-1]
```

Out[ ]: False

Assignment 4

In [ ]:
```
# a)
x1 = Matrix([[1],[-1],[0],[2]])
x2 = Matrix([[1],[1],[1],[3]])
x3 = Matrix([[3],[1],[1],[5]])
v1 = GramSchmidt([x1, x2, x3])[0]
v2 = GramSchmidt([x1, x2, x3])[1]
v3 = GramSchmidt([x1, x2, x3])[2]
display(Latex("$${} , {} , {}$$".format(latex(v1), latex(v2), latex(v3))))
```

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 5 \end{bmatrix} \right\}$$

is a basis for the subspace $W \subseteq \mathbb{R}^4$.

a. Using the Gram-Schmidt procedure, find an orthogonal basis for W. State all answers as integers between 0 and 99.

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ \boxed{2} \end{bmatrix}, \begin{bmatrix} 0 \\ \boxed{2} \\ \boxed{1} \\ 1 \end{bmatrix}, \begin{bmatrix} \boxed{1} \\ \frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} \right\}$$

```
In [ ]: # b)
u1 = GramSchmidt([v1, v2, v3], True)[0]
u2 = GramSchmidt([v1, v2, v3], True)[1]
u3 = GramSchmidt([v1, v2, v3], True)[2]
display(Latex("$${} , {} , {}$$".format(latex(u1), latex(u2), latex(u3))))
```

$$\begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ 0 \\ \frac{\sqrt{6}}{3} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} \end{bmatrix}$$

b. Determine an orthonormal basis for W. State all answers as integers between 0 and 99

$$\left\{ \begin{bmatrix} \frac{\sqrt{6}}{\boxed{6}} \\ -\frac{\sqrt{6}}{6} \\ 0 \\ \frac{\sqrt{6}}{\boxed{3}} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{\sqrt{6}}{\boxed{3}} \\ \frac{\sqrt{6}}{\boxed{6}} \\ \frac{\sqrt{6}}{6} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{3}}{\boxed{2}} \\ \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{\boxed{6}} \\ -\frac{\sqrt{3}}{6} \end{bmatrix} \right\}$$

```
In [ ]: # c)
u4 = Matrix([u1.T, u2.T, u3.T]).nullspace()[0].normalized()
display(Math(r'u =' + latex(u4)))
```

$$u = \begin{bmatrix} -\frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{6} \end{bmatrix}$$

c. Create a unit vector **u** that is orthogonal to the orthogonal basis for W, i.e. **u** will be the orthogonal complement to the bases found in (a) and (b) and as a set all four vectors will make up an orthogonal basis ***B*** for $\mathbb{R}^4$.

$$u = \begin{bmatrix} -\frac{\sqrt{\boxed{3}}}{\boxed{6}} \\ \frac{\sqrt{\boxed{3}}}{6} \\ -\frac{\sqrt{3}}{\boxed{2}} \\ \frac{\sqrt{3}}{6} \end{bmatrix}$$

```python
# d)
display(Math(r'\text{det}(M) =' + latex(Matrix([u1.T, u2.T, u3.T, u4.T]).det())))
```

$\det(M) = -1$

d. Find the determinant of the orthonormal matrix *M* whose columns are made up of the vectors from the basis ***B*** from part (c) and vector u from part (d). Think hard and long about what this may indicate about the determinants of orthonormal matrices! State your answer as an integer between 0 and 99.

$\det M = -\boxed{1}$

Assignment 5

```python
# a)
```

Let *A* denote the matrix

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

a. Find the eigenvalues using the characteristic equation and cofactor expansion on the second row. State your answers as integers between 0 and 99.

$$\det\begin{bmatrix} \boxed{1}-\lambda & 0 & -2 \\ 0 & \boxed{5}-\lambda & 0 \\ -2 & \boxed{0} & \boxed{4}-\lambda \end{bmatrix} = (\boxed{5}-\lambda)\det\begin{bmatrix} \boxed{1}-\lambda & -2 \\ -2 & \boxed{4}-\lambda \end{bmatrix}$$

$$\det\begin{bmatrix} 1-\lambda & 0 & -2 \\ 0 & 5-\lambda & 0 \\ -2 & 0 & 4-\lambda \end{bmatrix} = (5-\lambda)\det\begin{bmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{bmatrix}$$

$$= (5-\lambda)\big((1-\lambda)(4-\lambda)-4\big) = -\lambda(\lambda-5)^2$$

```python
A = Matrix([[1,0,-2],[0,5,0],[-2,0,4]])
lamda = symbols('lamda')
p = A.charpoly(lamda)
factor(p.as_expr())
```

Out[ ]: $\lambda(\lambda-5)^2$

```python
# b)
```

```
A.eigenvects()
```

Out[ ]:

$$\left[\left(0,\ 1,\ \left[\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right] \right),\ \left(5,\ 2,\ \left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right] \right) \right]$$

So the eigenvalues are ($\lambda_1 < \lambda_2$)

$\lambda_1 =$ 0 ✔ with algebraic multiplicity 1 ✔

$\lambda_2 =$ 5 ✔ with algebraic multiplicity 2 ✔

In [ ]:

```
# c)
u1 = A.eigenvects()[0][2][0].normalized()
u2 = A.eigenvects()[1][2][1].normalized()
u3 = A.eigenvects()[1][2][0].normalized()
GramSchmidt([u1,u2,u3])
```

Out[ ]:

$$\left[\begin{bmatrix} \frac{2\sqrt{5}}{5} \\ 0 \\ \frac{\sqrt{5}}{5} \end{bmatrix},\ \begin{bmatrix} -\frac{\sqrt{5}}{5} \\ 0 \\ \frac{2\sqrt{5}}{5} \end{bmatrix},\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right]$$

b. Find vectors that form an orthonormal basis $\boldsymbol{B}$ of $\mathbb{R}^3$. Note $\vec{v}_1$ corresponds to λ_1, and so on. State your answers as integers between 0 and 99.

$$\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},\quad \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix},\quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ ✔

In [ ]:

```
S = GramSchmidt([u1,u2,u3])[0].row_join(GramSchmidt([u1,u2,u3])[1]).row_join(Gra
Sinv = S**-1
D = diag(0,5,5)
display(Latex("$${}{}{}$$".format(latex(S), latex(D), latex(Sinv))))
```

$$\begin{bmatrix} \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \\ -\frac{\sqrt{5}}{5} & 0 & \frac{2\sqrt{5}}{5} \\ 0 & 1 & 0 \end{bmatrix}$$

The values from the matrices can be factorised so that you obtain

$$\begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}$$

c. Use basis **B** to describe *A* in terms of a similar matrix *D* such that $A = SDS^{-1}$ where the columns of *S* are the vectors from the orthonormal basis ***B***, and *D* is made up of the eigenvalues of *A*. Note, since *S* is orthogonal, $S^{-1} = S^T$. State your answers as integers between 0 and 99.

$$\begin{bmatrix} \frac{\boxed{2}}{\sqrt{5}} & -\frac{\boxed{1}}{\sqrt{5}} & 0 \\ \boxed{0} & \boxed{0} & \boxed{1} \\ \frac{\boxed{1}}{\sqrt{5}} & \frac{\boxed{2}}{\sqrt{5}} & \boxed{0} \end{bmatrix} \begin{bmatrix} 0 & \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{5} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{5} \end{bmatrix} \begin{bmatrix} \frac{\boxed{2}}{\sqrt{5}} & \boxed{0} & \frac{\boxed{1}}{\sqrt{5}} \\ -\frac{\boxed{1}}{\sqrt{5}} & \boxed{0} & \frac{\boxed{2}}{\sqrt{5}} \\ 0 & \boxed{1} & \boxed{0} \end{bmatrix}$$

In [ ]:
```python
k = symbols('k')
S*(D**k)*Sinv
```

Out[ ]:

$$\begin{bmatrix} \frac{4 \cdot 0^k}{5} + \frac{5^k}{5} & 0 & \frac{2 \cdot 0^k}{5} - \frac{2 \cdot 5^k}{5} \\ 0 & 5^k & 0 \\ \frac{2 \cdot 0^k}{5} - \frac{2 \cdot 5^k}{5} & 0 & \frac{0^k}{5} + \frac{4 \cdot 5^k}{5} \end{bmatrix}$$

The values from the matrices can be factorised so that you obtain

$$A^k = \begin{bmatrix} 5^{k-1} & 0 & -(2)5^{k-1} \\ 0 & 5^k & 0 \\ -(2)5^{k-1} & 0 & (4)5^{k-1} \end{bmatrix}$$

d. Using the fact that $A = SDS^{-1}$, for any integer *k*, write an explicit formula for A^k. "Explicit" here means "in one matrix". State your answers as integers between 0 and 99.

$$A^k = \begin{bmatrix} \boxed{5}^{k-1} & \boxed{0} & -(2)5^{k-1} \\ \boxed{0} & 5^k & \boxed{0} \\ -(2)5^{k-1} & \boxed{0} & (\boxed{4})5^{k-1} \end{bmatrix}$$

Assignment 6

In [ ]:
```python
x = pd.DataFrame([1, 11, 21, 30, 40, 50, 60, 70, 81, 90, 99, 109, 118])
y = pd.DataFrame([2450, 2757, 3262, 3551, 3844, 4281, 4585, 4938,5124, 5135, 531
x.head()
```

Out[ ]:

	0
0	1
1	11
2	21
3	30
4	40

In [ ]:
```
x1 = ([1, 11, 21, 30, 40, 50, 60, 70, 81, 90, 99, 109, 118])
x1
```

Out[ ]: [1, 11, 21, 30, 40, 50, 60, 70, 81, 90, 99, 109, 118]

In [ ]:
```
X1 = Matrix([ones(len(x), 1)]).row_join(Matrix(x**3))
X2 = Matrix([ones(len(x), 1)]).row_join(Matrix(x)).row_join(Matrix(x**3))
X3 = Matrix([ones(len(x), 1)]).row_join(Matrix(x))

display(Math(r'X_1 = ' + latex(X1) + r'\quad X_2 = ' + latex(X2) + r'\quad X_3 =
            + latex(Matrix(y))))
```

$$X_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1331 \\ 1 & 9261 \\ 1 & 27000 \\ 1 & 64000 \\ 1 & 125000 \\ 1 & 216000 \\ 1 & 343000 \\ 1 & 531441 \\ 1 & 729000 \\ 1 & 970299 \\ 1 & 1295029 \\ 1 & 1643032 \end{bmatrix} \quad X_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 11 & 1331 \\ 1 & 21 & 9261 \\ 1 & 30 & 27000 \\ 1 & 40 & 64000 \\ 1 & 50 & 125000 \\ 1 & 60 & 216000 \\ 1 & 70 & 343000 \\ 1 & 81 & 531441 \\ 1 & 90 & 729000 \\ 1 & 99 & 970299 \\ 1 & 109 & 1295029 \\ 1 & 118 & 1643032 \end{bmatrix} \quad X_3 = \begin{bmatrix} 1 & 1 \\ 1 & 11 \\ 1 & 21 \\ 1 & 30 \\ 1 & 40 \\ 1 & 50 \\ 1 & 60 \\ 1 & 70 \\ 1 & 81 \\ 1 & 90 \\ 1 & 99 \\ 1 & 109 \\ 1 & 118 \end{bmatrix} \quad y = \begin{bmatrix} 2450 \\ 2757 \\ 3262 \\ 3551 \\ 3844 \\ 4281 \\ 4585 \\ 4938 \\ 5124 \\ 5135 \\ 5314 \\ 5511 \\ 5806 \end{bmatrix}$$

In [ ]:
```
X1tX1 = X1.T*X1
X1ty = X1.T*Matrix(y)
Mat, _ = X1tX1.row_join(X1ty).rref()
B1 = Mat[:,-1]


X2tX2 = X2.T*X2
X2ty = X2.T*Matrix(y)
Mat, _ = X2tX2.row_join(X2ty).rref()
B2 = Mat[:,-1]


X3tX3 = X3.T*X3
X3ty = X3.T*Matrix(y)
Mat, _ = X3tX3.row_join(X3ty).rref()
B3 = Mat[:,-1]
```

In [ ]:
```
# a)
display(Math(r'X_1^TX_1 = ' + latex(X1tX1)))
```

```
display(Math(r'X_1^Ty = ' + latex(X1ty)))
display(Math(r'X_2^TX_2 = ' + latex(X2tX2)))
display(Math(r'X_2^Ty = ' + latex(X2ty)))
display(Math(r'X_3^TX_3 = ' + latex(X3tX3)))
display(Math(r'X_3^Ty = ' + latex(X3ty)))
```

$$X_1^TX_1 = \begin{bmatrix} 13 & 5954394 \\ 5954394 & 6316847487430 \end{bmatrix}$$

$$X_1^Ty = \begin{bmatrix} 56558 \\ 31894029580 \end{bmatrix}$$

$$X_2^TX_2 = \begin{bmatrix} 13 & 780 & 5954394 \\ 780 & 64330 & 586551382 \\ 5954394 & 586551382 & 6316847487430 \end{bmatrix}$$

$$X_2^Ty = \begin{bmatrix} 56558 \\ 3885466 \\ 31894029580 \end{bmatrix}$$

$$X_3^TX_3 = \begin{bmatrix} 13 & 780 \\ 780 & 64330 \end{bmatrix}$$

$$X_3^Ty = \begin{bmatrix} 56558 \\ 3885466 \end{bmatrix}$$

In [ ]:

```
# b)
display(Latex("$$y_1(t) = {}+{}t^3$$".format(round(B1[0],2), round(B1[1], 4))))
display(Latex("$$y_2(t) = {}+{}t+{}t^3$$".format(round(B2[0],2), round(B2[1], 4)
display(Latex("$$y_3(t) = {}+{}t$$".format(round(B3[0],2), round(B3[1], 4))))
```

$$y_1(t) = 3586.45 + 0.0017t^3$$

$$y_2(t) = 2419.75 + 37.9504t + -0.0008t^3$$

$$y_3(t) = 2666.69 + 28.0654t$$

In [ ]:

```
# c)
display(Latex("$$e_1 = {}$$".format(round((Matrix(y)-X1*B1).norm(), 2))))
display(Latex("$$e_2 = {}$$".format(round((Matrix(y)-X2*B2).norm(), 2))))
display(Latex("$$e_3 = {}$$".format(round((Matrix(y)-X3*B3).norm(), 2))))
```

$$e_1 = 2062.15$$

$$e_2 = 314.56$$

$$e_3 = 660.48$$

In [ ]:

```
# Model 2 is the best
```

c. State, with substantiation, which of the three models is the best fit for the measured data. State your answers as integers between 0 and 9999.

The error of $y_1 =$ 2062 ✔ .15

The error of $y_2 =$ 314 ✔ .56

The error of $y_3 =$ 660 ✔ .48

Since model number 2 ✔ has the smallest error, we conclude that it is the best fitted model.

In [ ]:

```
# d)
display(Latex("$$f_2(150) ={}$$".format(round(float(B2[0]+B2[1]*150 + B2[2]*150*
```

$$f_2(150) = 5562$$

Assignment 7

In [ ]:

```
A = Matrix([[3,-3,2],[1,1,2],[-3,3,2],[-1,-1,2]])
AtA = A.T*A
vecs1 = AtA.eigenvects()
vecs1
```

Out[ ]:

$$\left[\left(4,\ 1,\ \left[\left[\begin{matrix}1\\1\\0\end{matrix}\right]\right]\right),\ \left(16,\ 1,\ \left[\left[\begin{matrix}0\\0\\1\end{matrix}\right]\right]\right),\ \left(36,\ 1,\ \left[\left[\begin{matrix}-1\\1\\0\end{matrix}\right]\right]\right)\right]$$

In [ ]:

```
# a)
s1 = sqrt(vecs1[2][0])
s2 = sqrt(vecs1[1][0])
s3 = sqrt(vecs1[0][0])

v1 = vecs1[2][2][0].normalized()
v2 = vecs1[1][2][0].normalized()
v3 = vecs1[0][2][0].normalized()

u1 = ((s1**-1)*A*v1)
u2 = ((s2**-1)*A*v2)
u3 = ((s3**-1)*A*v3)

# We need one more eigenvectors for U

u4 = Matrix([u1.T, u2.T, u3.T]).nullspace()[0].normalized()

U1 = u1.row_join(u2).row_join(u3).row_join(u4)
V = v1.row_join(v2).row_join(v3)
Vt1 = V.T
S1 = diag(s1,s2,s3).col_join(zeros(1,3))

display(Math('U \Sigma V^T = {}{}{}'.format(latex(U1), latex(S1), latex(Vt1))))
display(Latex("Test:"))
display(U1*S1*Vt1)
display(A)
display(U1*S1*Vt1==A)
```

$$U\Sigma V^T = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

Test:

$$\begin{bmatrix} 3 & -3 & 2 \\ 1 & 1 & 2 \\ -3 & 3 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -3 & 2 \\ 1 & 1 & 2 \\ -3 & 3 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

True

In [ ]:

```
# b)
A = Matrix([[3,-3,2],[1,1,2],[-3,3,2],[-1,-1,2]])
AAt = A*A.T
vecs = AAt.eigenvects()
vecs
```

Out[ ]:

$$\left[\left(0, \ 1, \ \left[\left[\begin{matrix}-1\\1\\-1\\1\end{matrix}\right]\right]\right), \ \left(4, \ 1, \ \left[\left[\begin{matrix}0\\-1\\0\\1\end{matrix}\right]\right]\right), \ \left(16, \ 1, \ \left[\left[\begin{matrix}1\\1\\1\\1\end{matrix}\right]\right]\right), \ \left(36, \ 1, \ \left[\left[\begin{matrix}-1\\0\\1\\0\end{matrix}\right]\right]\right)\right]$$

In [ ]:

```
s1 = sqrt(vecs[3][0])
s2 = sqrt(vecs[2][0])
s3 = sqrt(vecs[1][0])

u1 = vecs[3][2][0].normalized()
u2 = vecs[2][2][0].normalized()
u3 = vecs[1][2][0].normalized()
u4 = vecs[0][2][0].normalized()

U = u1.row_join(u2).row_join(u3).row_join(u4)
S = diag(s1,s2,s3).col_join(zeros(1,3))

v1 = Rational(1,6) * u1.T*A
v2 = Rational(1,4) * u2.T*A
v3 = Rational(1,2) * u3.T*A

Vt = v1.col_join(v2).col_join(v3)

display(Math('U \Sigma V^T = {}{}{}'.format(latex(U), latex(S), latex(Vt))))
display(Latex("Test:"))
display(U*S*Vt)
display(A)


display(U*S*Vt==A)
display(U1 ==U)
```

```
display(Vt1 ==Vt)
display(S==S1)
```

$$U\Sigma V^T = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

Test:

$$\begin{bmatrix} 3 & -3 & 2 \\ 1 & 1 & 2 \\ -3 & 3 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -3 & 2 \\ 1 & 1 & 2 \\ -3 & 3 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

```
True
False
False
True
```

b. Now, try to redo the SVD but this time depart from AA^T. The question is whether any of the matrices are identical to the matrices found in question (a). Please note that each incorrect answer is awarded with a negative score and no answer is awarded a score of 0.

The U matrix of AA^T is not identical ✓ to the U matrix of A^TA.

The Σ matrix of AA^T is identical ✓ to the Σ matrix of A^TA.

The V^T matrix of AA^T is not identical ✓ to the V^T matrix of A^TA.

Check answer