In [ ]:
```
import sympy as sp
from scipy import *
from sympy import *
init_printing()
from IPython.display import display, Latex, HTML, Math
import numpy as np
import pandas as pd
from sympy import Rational as R
import math
```

Assignment 1

a)

i: True. eig = 0 means free variable hence rank < n

ii.: True. Since nullspace is nonempty, so we get free variables, the system is consistent

iii: False. If $P(\lambda) = \det(A - \lambda I)$ does not have n real roots, counting multiplicities (in other words, if it has some complex roots), then A is not diagonalizable.

iv : False. It might be if the arithmetic multiplicity matches the geometric multiplicity, i.e. dim Nul $A - \lambda I = n$

v: True. There will always be a set of vectors that are orthogonal to each other.

a. Select the appropriate truth value of each statement:

i. If 0 is an eigenvalue of an $n \times n$ matrix A, then rank$(A) < n$.

ii. If A is a 3×5 matrix such that Nul A is 2 dimensional, then the equation $Ax = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$ has infinitely many solutions.

iii. Let A be a real 2×2 matrix, whose characteristic polynomial does not have real roots. Then A is diagonalizable.

iv. If A is an $n \times n$ matrix with fewer than n distinct eigenvalues, then A is not diagonalizable.

v. Every subspace of $\mathbb{R}^n$ has at least one orthogonal basis.

i.	◉ True ✔	○ False
ii.	◉ True ✔	○ False
iii.	○ True	◉ False ✔
iv.	○ True	◉ False ✔
v.	◉ True ✔	○ False

b)

Sum of diagonal of symmetric matrix = sum of eigenvalues, so $\lambda_1 + \lambda_2 + \lambda_3 = 56$. We substitute the information supplied: $\lambda_1 + \lambda_1^2 + 9\lambda_1 = 56$. Now we just solve:

In [ ]:
```
l = symbols('lambda')
solve(l**2 + 10*l - 56, l)
```

Out[ ]: $[-14,\ 4]$

We know λ_1 must be positive and use it to find the other values:

In [ ]:
```
l1 = 4
l2 = l1**2 # Pow
l3 = 9*l1
display(l1, l2, l3)
```

4

16

36

b. Consider the following matrix

$$A = \begin{bmatrix} 20 & a & b \\ a & 20 & c \\ b & c & 16 \end{bmatrix}$$

It is known that $\lambda_2 = \lambda_1^2$ and $\lambda_3 = 9\lambda_1$ and that all eigenvalues are positive integers. What are the eigenvalues of A?

State your answer as positive integers such that $\lambda_1 < \lambda_2 < \lambda_3$

$\lambda_1 =$ 4 ✔

$\lambda_2 =$ 16 ✔

$\lambda_3 =$ 36 ✔

c)

So since we get distinct eigenvalues we know A is diagonalizable. We are given the eigenvalues and eigenvectors and can simply set up the factorization:

In [ ]:
```
P = Matrix([[3, 0, 1], [-3, 0, 1],[7, 1, 1]])
Pinv = P**-1
D = diag(3, 6, 7)
display(Math(r'A = ' + latex(P*D*Pinv)))
```

$$A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{bmatrix}$$

c. The eigenvalues and corresponding eigenvectors of a 3 x 3 matrix A are as follows:

$$\lambda_1 = 3 \text{ and } \bar{v}_1 = \begin{bmatrix} 3 \\ -3 \\ 7 \end{bmatrix}, \quad \lambda_2 = 6 \text{ and } \bar{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \lambda_3 = 7 \text{ and } \bar{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find matrix A. State your answers as positive integers.

$$A = \begin{bmatrix} 5 & \boxed{2} & \boxed{0} \\ \boxed{2} & 5 & \boxed{0} \\ -3 & \boxed{4} & 6 \end{bmatrix}$$

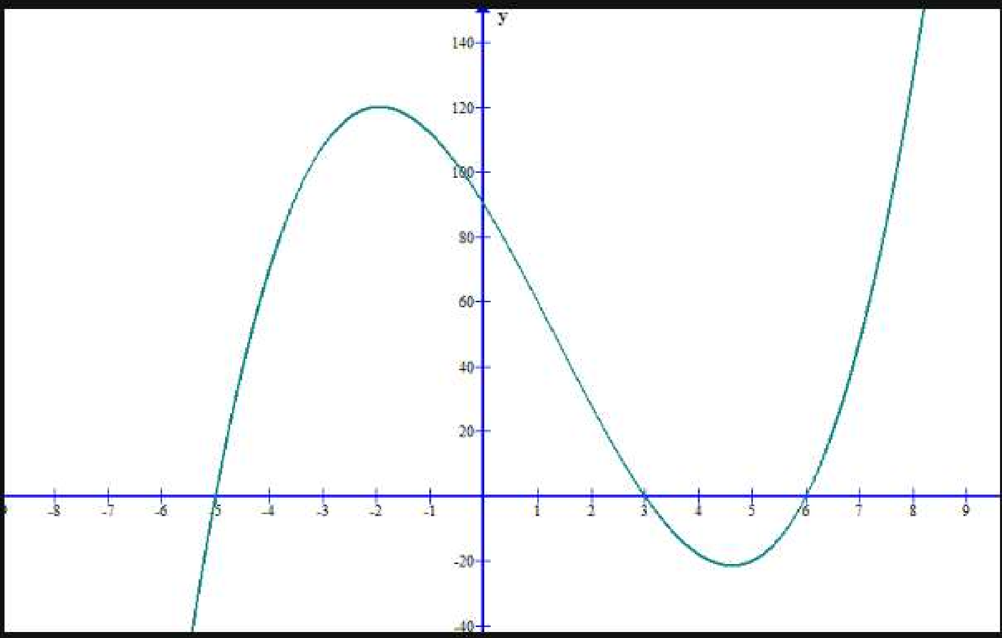
Assignment 2

a)

So the eigenvalues are the roots of the characteristic polynomial which means that we check where the graph intersects the x-axis and get:

$$\lambda_1 = -5$$
$$\lambda_2 = 3$$
$$\lambda_3 = 6$$

Below you see a plot of the characteristic polynomial of a 3 x 3 matrix A.



a. What are the eigenvalues of A. State your answer as positive integers such that $\lambda_1 < \lambda_2 < \lambda_3$

$\lambda_1 = -$ 5 ✓

$\lambda_2 =$ 3 ✓

$\lambda_3 =$ 6 ✓

b) The determinant is the product of the eigenvalues:

In [ ]:
```
display(Math(r'det(A) = ' + latex(-5*3*6)))
```

$det(A) = -90$

c)

Since $det(A) \neq 0$ A has pivots in all columns which means no free variables which means an empty nullspace so dim Nul $A = 0$.

Assignment 3

In [ ]:
```
k = symbols('k')
A = Matrix([[1,1,0],[0,1,1],[0,0,1]])
B = Matrix([[2,0,0],[1,1,2],[2,0,1]])
C = Matrix([[1,1,0],[0,1,0],[0,1,2]])
D = Matrix([[k+1,1,0],[1,k+8,2],[4,1,k+2]])
```

a) So here we must "isolate" matrix X, remembering all the rules of matrix algebra:

$$XB = XA - X + C$$
$$XB - XA + X = C$$
$$X(B - A + I_3) = C$$
$$X = C(B - A + I_3)^{-1}$$

We get:

In [ ]:
```
X = C*(B-A+eye(3))**-1
display(Math(r'det(X)= ' + latex(det(X))))
```

$det(X) = 2$

Consider the following matrices:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} k+1 & 1 & 0 \\ 1 & k+8 & 2 \\ 4 & 1 & k+2 \end{bmatrix}$$

a. Find the determinant of matrix X in the following matrix equation $XB = XA - X + C$.

b. Let $Y = A^T A$. What is the sum of the eigenvalues of Y?

c. Let $Z = A - D$. For which value(s) of k, if any, is dim Nul $Z = 0$?

a. State your answer as a positive integer.

det $X =$ $\boxed{2}$

b)

Remember the sum of the eigenvalues of a symmetric matrix is always the sum of its diagonal (sum of diagonal is also called trace):

In [ ]:
```
Y = A.T * A
Y
```

Out[ ]:
$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

In [ ]:
```
trace(Y)
```

Out[ ]: 5

c)

A lot of students had mistakes in this exercise. One student even asked me if it contained errors. You just need to think a little bit. Dim Nul Z = 0 means that the nullspace is empty. This only happens when there is a pivot in each column. A pivot in each column occurs when there is a non-zero value in all pivot positions:

In [ ]:
```
Z = A-D
Z.echelon_form().expand()
```

Out[ ]:
$$\begin{bmatrix} -1 & -k-7 & -1 \\ 0 & -k^2-7k & -k \\ 0 & 0 & -k^3-8k^2-6k \end{bmatrix}$$

We really only need to look at the last row two rows. We now know that dim Nul Z = 0 when $-k^3-8k^2-6k \neq 0 \text{ and } -k^2-7k \neq 0$

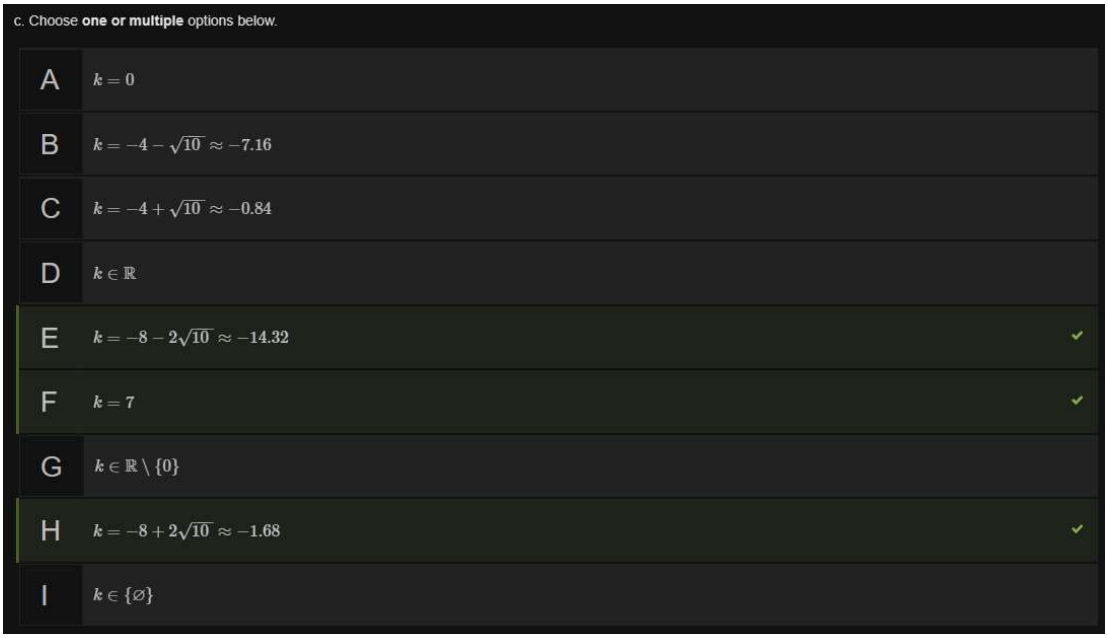
In [ ]:
```
solve(-k**3-8*k**2-6*k, k), solve(-k**2-7*k, k)
```

Out[ ]: $\left(\left[0,\ -4-\sqrt{10},\ -4+\sqrt{10}\right],\ \left[-7,\ 0\right]\right)$

So we just need to make sure that k is **not** one of these values. That means that only three of the choices will make dim Nul Z = 0:

$$k = -8 - 2\sqrt{10} \approx -14.32$$
$$k = -8 + 2\sqrt{10} \approx -1.68$$
$$k = 7$$

(So kind of the "opposite" of what many answered).



Assignment 4

a)

In [ ]:
```
A = Matrix([[-1, -2, 0, 1, -2, 2],
            [1, 2, 3, -5, 15, -11],
            [2, 4, 1, -1, 6, 7]])
A
```

Out[ ]:
$$\begin{bmatrix} -1 & -2 & 0 & 1 & -2 & 2 \\ 1 & 2 & 3 & -5 & 15 & -11 \\ 2 & 4 & 1 & -1 & 6 & 7 \end{bmatrix}$$

b)

In [ ]:
```
A.rref()[0]
```

Out[ ]:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & -1 & 6 \end{bmatrix}$$

c) So this is simply derived from row reduced echelon form (b):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 5 \\ 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 1 \end{bmatrix}$$

d) I prefer just rref'ing the coeeficient matrix and look at the output:

In [ ]:
```
A[:,0:-1].rref()
```

Out[ ]:

$$\left(\begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \ (0,\ 2,\ 3) \right)$$

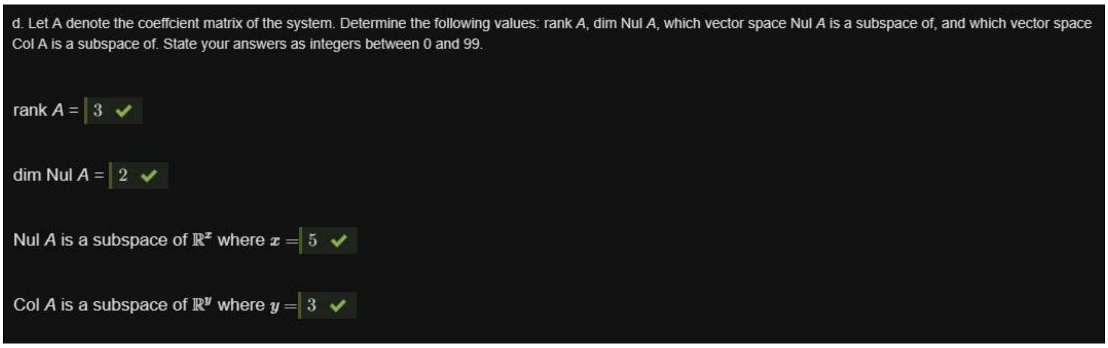
So we get the following info: 3 pivots, 2 free variables. There are 5 columns and 3 rows which leads us to

rank A is 3 since it is given by number of pivots

dim Nul A is 2 since it is given by number of free variables

Nul A is a subspace of $\mathbb{R}^5$ since there are five entries in the vectors of the nullspace (also see answer to (c)). In a $n \times m$ matrix the nullspace is a subspace of $\mathbb{R}^m$

Col A is a subspace of $\mathbb{R}^3$ since there are three entries in the vectors that make up the columns of A. In a $n \times m$ matrix the column space is a subspace of $\mathbb{R}^n$


d. Let A denote the coeffcient matrix of the system. Determine the following values: rank *A*, dim Nul *A*, which vector space Nul *A* is a subspace of, and which vector space Col A is a subspace of. State your answers as integers between 0 and 99.

rank *A* = 3 ✔

dim Nul *A* = 2 ✔

Nul *A* is a subspace of $\mathbb{R}^x$ where $x =$ 5 ✔

Col *A* is a subspace of $\mathbb{R}^y$ where $y =$ 3 ✔

e)

We look at the output of the rref. Notice that the free variables are a_2 and a_5. That means that a_2 kan only be a linear combination of a_1 while a_5 can be a linear combination of a_1, a_3 and a_4. We can see what the combination for a_2 is by looking at the values in column 2 and similarly for column 5 and a_5. We get:

$a_2 = 2a_1$

$a_5 = a_1 + 3a_3 - a_4$

e. Let a_i denote the column vectors of A such that $i \in \{1,2,3,4,5\}$. Based on the reduced row echelon form of A, write the non-pivot columns of A as linear combinations of the pivot columns.

State your answer as positive integers. **Note**: You also have to insert the indices of the column vectors.

$a_{\boxed{2}} = \boxed{2} \cdot a_{\boxed{1}}$

$a_{\boxed{5}} = a_{\boxed{1}} + \boxed{3} \cdot a_{\boxed{3}} - a_{\boxed{4}}$

Assignment 5

Very few had this one right. The method is simply set the two expressions equal to each other. Place the "intersections" on one side and the Matrices with their paramaters on the other side. We get:

$$\begin{bmatrix}2\\0\\0\\0\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & -1 & 1\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}r\\s\\t\end{bmatrix} = \begin{bmatrix}0\\1\\2\\3\end{bmatrix} + \begin{bmatrix}-1\\2\\1\\0\end{bmatrix}u$$

$$\begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & -1 & 1\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}r\\s\\t\end{bmatrix} - \begin{bmatrix}-1\\2\\1\\0\end{bmatrix}u = \begin{bmatrix}0\\1\\2\\3\end{bmatrix} - \begin{bmatrix}2\\0\\0\\0\end{bmatrix}$$

We can now combine the left-hand sides in to one matrix since the two parts are merely linear combinations. Similarly we subtract the right-hand sides.

$$\begin{bmatrix}1 & 0 & 0 & 1\\-1 & 1 & 0 & -2\\0 & -1 & 1 & -1\\0 & 0 & -1 & 0\end{bmatrix}\begin{bmatrix}r\\s\\t\\u\end{bmatrix} = \begin{bmatrix}-2\\1\\2\\3\end{bmatrix}$$

Now we just have a good-old system which we can solve by rref. This will give us the values for r, s, t, u:

In [ ]:
```
Matrix([[1,0,0,1, -2],
        [-1,1,0,-2, 1],
        [0,-1,1,-1, 2],
        [0,0,-1,0, 3]]).rref()[0]
```

Out[ ]:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

So now we just use the value of u and plug it in to W and get the values of the intersection:

In [ ]:

```
Matrix([[0],[1],[2],[3]]) -2*Matrix([[-1],[2],[1],[0]])
```

Out[ ]:

$$\begin{bmatrix} 2 \\ -3 \\ 0 \\ 3 \end{bmatrix}$$

In [ ]:

```
# Same as above just in one go:
V = Matrix([[1,0,0],[-1,1,0],[0,-1,1],[0,0,-1]])
v = Matrix([[2],[0],[0],[0]])
W = Matrix([[-1],[2],[1],[0]])
w = Matrix([[0],[1],[2],[3]])

u = Matrix.hstack(V, -W, w-v).rref()[0][-1,-1]
w+W*u
```

Out[ ]:

$$\begin{bmatrix} 2 \\ -3 \\ 0 \\ 3 \end{bmatrix}$$

5 of 9

Consider the following two subspaces of $\mathbb{R}^4$:

$$V: \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

$$W: \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} u$$

Find the intersection of V and W.

State your answer as positive integers.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \boxed{2} \\ \boxed{-3} \\ \boxed{0} \\ \boxed{3} \end{bmatrix}$$

Assignment 6

a)

Gram Matrix is the $A^T A$ matrix

In [ ]:
```
v1 = Matrix([[1],[-1],[0],[1],[1]])
v2 = Matrix([[-1],[1],[2],[1],[1]])
v3 = Matrix([[3],[3],[0],[-3],[3]])
V = Matrix.hstack(v1, v2, v3)
V.T*V
```

Out[ ]:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 36 \end{bmatrix}$$

The following three vectors are given in $\mathbb{R}^5$

$$\bar{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix}$$

a. Construct the Gram matrix, G, for $\bar{v}_1, \bar{v}_2$, and $\bar{v}_3$ and confirm that this set of vectors form an orthogonal set V.

b. Calculate the projection, $\text{proj}_V\, \bar{w}$, of

$$\bar{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

onto the subspace spanned by V.

c. Use $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}$ and $\text{proj}_V\, \bar{w}$ to construct an orthonormal basis B for $\mathbb{R}^4$

d. Find an orthonormal basis for $\mathbb{R}^5$ that includes the vectors of B found in (c) and an additional vector. You only need to state the new vector.

a. State your answer as positive integers.

$$G = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 36 \end{bmatrix}$$

b)

We just use the built in formula:

In [ ]:
```
w = Matrix([1,1,1,1,1])

proj = w.project(v1) + w.project(v2) + w.project(v3)
proj
```

Out[ ]:

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

b. State your answer as positive integers.

$$proj_V w = \begin{bmatrix} \frac{1}{\boxed{2}} \\ \frac{1}{\boxed{2}} \\ \boxed{1} \\ \frac{1}{\boxed{2}} \\ \frac{\boxed{3}}{2} \end{bmatrix}$$

c)

So the orthonormal basis will simply be the normalised versions of $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, and $\bar{w} - \text{proj}_V\, \bar{w}$

In [ ]:
```python
v4 = w-proj
display(Math(latex(v1.normalized()) + ' , '
             + latex(v2.normalized()) + ' , '
             + latex(v3.normalized()) + ' , '
             + latex(v4.normalized())))
```

$$\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

c. State your answer as positive integers.

$$\left\{ \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \boxed{0} \\ \frac{1}{2} \\ \frac{1}{\boxed{2}} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{\boxed{2}} \\ \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{\boxed{2}} \\ \frac{1}{2} \\ \boxed{0} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \boxed{0} \\ \frac{1}{2} \\ -\frac{\boxed{1}}{2} \end{bmatrix} \right\}$$

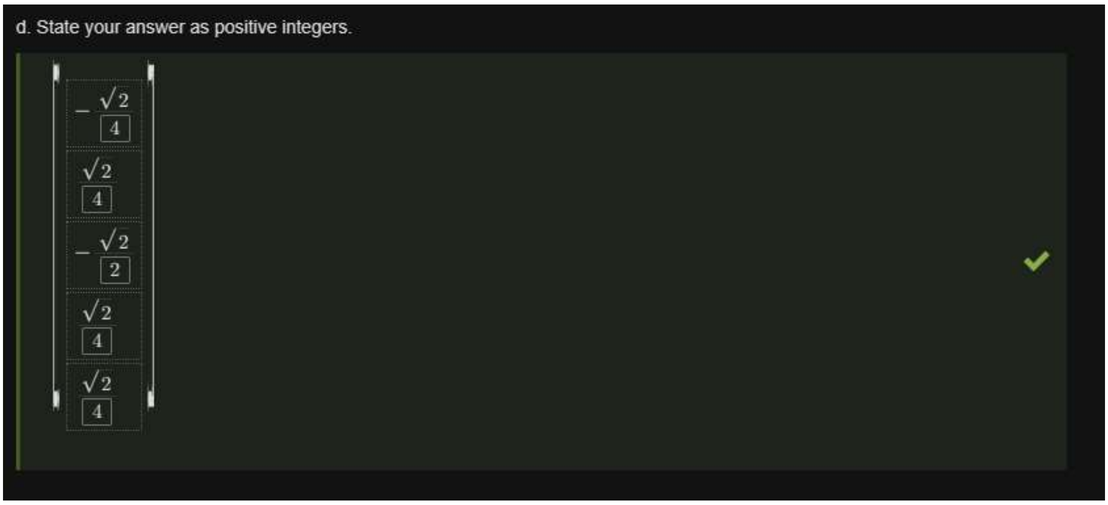
d)

So the last vector will be the orthogonal complement of the basis found in (c). We can determine this by using "the trick", i.e. find the nullspace of the transpose of the above basis:

In [ ]:
```
Matrix.vstack(v1.T, v2.T, v3.T, v4.T).nullspace()[0].normalized()
```

Out[ ]:

$$\begin{bmatrix} -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} \end{bmatrix}$$



Assignment 7

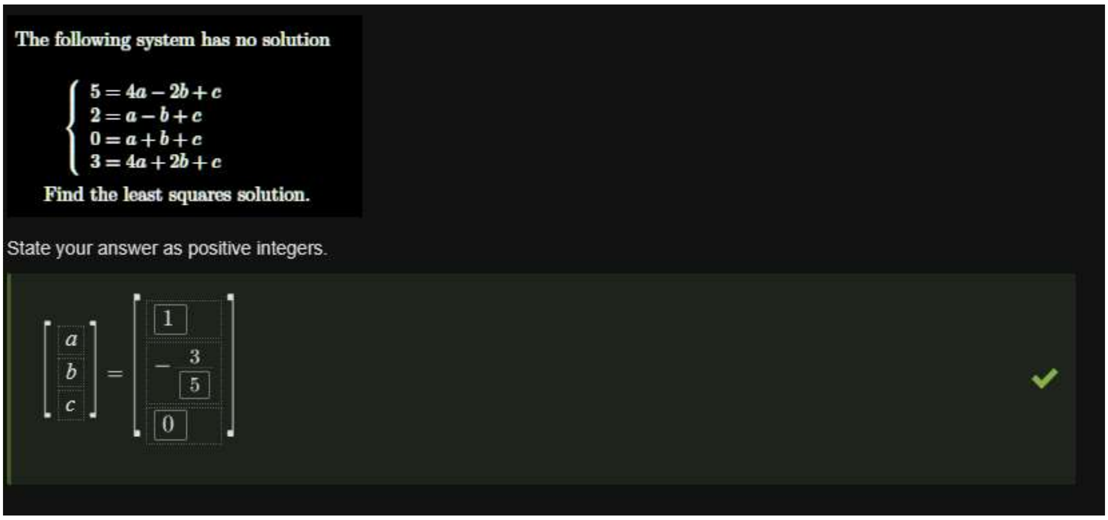
So we just use the formula which I went through in class.

In [ ]:
```
A = Matrix([[4, -2, 1],
            [1, -1, 1],
            [1, 1, 1],
            [4, 2, 1]])
y = Matrix([5, 2, 0, 3])
AtA = A.T * A
Aty = A.T * y

Matrix.hstack(AtA, Aty).rref()[0][:,-1]
```

Out[ ]:

$$\begin{bmatrix} 1 \\ -\frac{3}{5} \\ 0 \end{bmatrix}$$



The following system has no solution

$$\begin{cases} 5 = 4a - 2b + c \\ 2 = a - b + c \\ 0 = a + b + c \\ 3 = 4a + 2b + c \end{cases}$$

Find the least squares solution.

State your answer as positive integers.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{5} \\ 0 \end{bmatrix}$$

Assignment 8

So we just need to use the formula which I derived in class to find the vectors of V. As we can see, we will be missing one vector. We can find this vector by using "the trick".

In [ ]:

```
A = Matrix([[4,11,14],[8,7,-2]])
U = Matrix([[(3*sqrt(10))/(10),-(sqrt(10))/(10)],
            [(sqrt(10))/(10),(3*sqrt(10))/(10)]])

u1 = U[:,0]
u2 = U[:,1]

s1 = 6*sqrt(10)
s2 = 3*sqrt(10)

vt1 = s1**-1 * u1.T * A
vt2 = s2**-1 * u2.T * A

vt3 = Matrix.vstack(vt1, vt2).nullspace()[0].T.normalized()

Vt = Matrix.vstack(vt1, vt2, vt3)
V = Vt.T
V
```

Out[ ]:

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Below you see matrices U and A of a singular value decomposition $A = U\Sigma V^T$:

$$U = \begin{bmatrix} \frac{3\sqrt{10}}{10} & -\frac{\sqrt{10}}{10} \\ \frac{\sqrt{10}}{10} & \frac{3\sqrt{10}}{10} \end{bmatrix}, \qquad A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

It is known that the nonzero singular values of A are $3\sqrt{10}$ and $6\sqrt{10}$. Derive the columns of matrix V from the columns of U and the singular values and state matrix V. **Note:** You are asked to find V and not V transposed.

State your answer as positive integers.

$$V = \begin{bmatrix} \frac{1}{3} & \frac{\boxed{2}}{3} & \frac{2}{3} \\ \frac{\boxed{2}}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{\boxed{1}}{3} \end{bmatrix}$$

Assignment 9

In total, 5 + 3 litres enter A, which means 8 must leave A from pipe c. If 8 enters B, 8 must leave, 3 from b and 5 from d. We get:

In [ ]:
```python
# We translate the problem to a matrix problem

A = Matrix([[-R(8,200), R(3,100)],[R(8,200), -R(8,100)]])
display(Math(r'A = ' + latex(A)))
```

$$A = \begin{bmatrix} -\frac{1}{25} & \frac{3}{100} \\ \frac{1}{25} & -\frac{2}{25} \end{bmatrix}$$

In [ ]:
```python
# We find the eigenvalues and the corresponding eigenspaces.
l = symbols('l')
l1, l2 = solve(det(A-l*eye(np.shape(A)[0])))
display(Math(r'\lambda_0 =' +latex(l1) + r'\approx' + latex(round(l1, 2))))
display(Math(r'\lambda_1 =' +latex(l2) + r'\approx' + latex(round(l2, 2))))

v1 = (A-l1*eye(np.shape(A)[0])).nullspace()[0]
v2 = (A-l2*eye(np.shape(A)[0])).nullspace()[0]
display(Math(r'v_0 = '+ latex(v1) + r'=' + latex(v1.evalf(4))))
display(Math(r'v_1 = '+ latex(v2) + r'=' + latex(v2.evalf(4))))
```

$$\lambda_0 = -\frac{1}{10} \approx -0.1$$

$$\lambda_1 = -\frac{1}{50} \approx -0.02$$

$$v_0 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 1.0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix}$$

We get the following general solution:

$$y(0) = c_0 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} e^{-\frac{t}{10}} + c_1 \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} e^{-\frac{t}{50}}$$

To get the particular solution (sometimes called the unique solution) we use the initial value and solve the above general solution with regard to c_0 and c_1.

In [ ]:
```
y0 = Matrix([100, 50])
# To solve the system we form the following augmented matrix and solve. The c's
rref = v1.row_join(v2).row_join(y0).rref()[0]
#rref.applyfunc(lambda x: round(x, 2)) # this I can use when the values are horr
rref
```

Out[ ]:

$$\begin{bmatrix} 1 & 0 & -\frac{25}{2} \\ 0 & 1 & \frac{125}{2} \end{bmatrix}$$

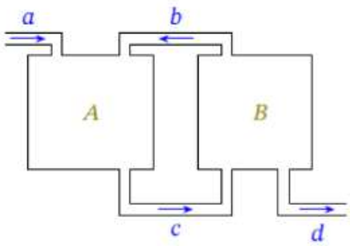
In [ ]:
```
C = rref[:,-1]
t = symbols('t')
y1 = C[0]*v1[0]*math.e**(l1*t) + C[1]*v2[0]*math.e**(l2*t)
y2 = C[1]*v1[1]*math.e**(l1*t) + C[1]*v2[1]*math.e**(l2*t)
z=limit(y1/y2,t,0)
float(z)
```

Out[ ]: 0.8

So we get:

$$y(0) = -75 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} e^{-\frac{t}{10}} + 75 \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} e^{-\frac{t}{50}}$$

Two containers are connected by pipes, as in the figure below. Container A holds 200 l and container B 100 l. Initially, there is 150 g of salt in container A and only clean water in container B. Through pipe a 5 l/min of pure water is added to A and through pipe b 3 l/min of a salt-water mixture is added to container A.



Let $y_0(t)$ and $y_1(t)$ be the amount of salt in A and B, respectively. State the above case in terms of a system of differential equations, find the unique solution and set it up in the following vector form:

$$y(t) = \begin{bmatrix} y_0(t) \\ y_1(t) \end{bmatrix} = c_0\bar{v}_0 e^{\lambda_0 t} + c_1\bar{v}_1 e^{\lambda_1 t}$$

State your answer as a positive integer.

$$y(t) = \begin{bmatrix} y_0(t) \\ y_1(t) \end{bmatrix} = -\boxed{75} \begin{bmatrix} -\frac{1}{\boxed{2}} \\ \boxed{1} \end{bmatrix} e^{-\frac{t}{\boxed{10}}} + \boxed{75} \begin{bmatrix} \frac{\boxed{3}}{2} \\ \boxed{1} \end{bmatrix} e^{-\frac{t}{\boxed{50}}}$$