

5. Eigen vectors and Eigen values

Eigenvector:

A nonzero vector \bar{x} s.t. $A\bar{x} = \lambda\bar{x}$, $\lambda \in \mathbb{R}$

Eigenvalue:

A scalar λ is an eigenvalue of A if there is a non-trivial solution \bar{x} of $A\bar{x} = \lambda\bar{x}$

↳ In such cases we say that \bar{x} is an eigenvector corresponding to λ .

$$T: \bar{x} \rightarrow A\bar{x}$$

Ex

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}, \bar{v} = \begin{bmatrix} -12 \\ 10 \end{bmatrix}, \bar{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Are \bar{v} and \bar{v} eigenvectors of A ?

$$A\bar{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} -12 \\ 10 \end{bmatrix} = \begin{bmatrix} -12 + 60 \\ -60 + 20 \end{bmatrix} = \begin{bmatrix} 48 \\ -40 \end{bmatrix} = -4 \begin{bmatrix} -12 \\ 10 \end{bmatrix}$$

so \bar{v} is an eigenvector of A with corresponding eigenvalue $\lambda = -4$.

$$A\bar{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 - 12 \\ 15 - 4 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}, \bar{v} \text{ is not an eig. vec.}$$

Ex: Determine if $\lambda = 7$ is eig. val. of A : $\boxed{\quad} = 0$

$$\text{Note } A\bar{x} = 7\bar{x} \Rightarrow A\bar{x} - 7I\bar{x} = 0 \Rightarrow (A - 7I)\bar{x} = 0$$

so we need to check if $A - 7I$ has a non-trivial solution:

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 1-7 & 6 \\ 5 & 2-7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

so

$$\sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow \text{free variable} = \text{non-trivial}$$

so $\lambda = 7$ is an eigenvalue.

What about the corresponding eigenvect. of 7.

↳ The vectors in PVF of $(A-7I)$:

$\bar{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, all multiples of this is also an eigenvector of A corresponding to $\lambda = 7$.
except $x_2 = 0$

Theorem

λ is an eigenvalue of A iff. $A - \lambda I = 0$ has a non trivial solution

Eigen space:

Is the nullspace of $A - \lambda_i I_n$, denoted $E_A(\lambda)$

Ex: Find a basis for the eigenspace of $\lambda = 2$

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}, \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} = \begin{bmatrix} 4-2 & -1 & 6 \\ 2 & 1-2 & 6 \\ 2 & -1 & 8-2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Note: If $\dim \text{Nul}(A - \lambda I) > 1$, we are not guaranteed that the vectors are independent.

Theorem:

The eigenvalues of a triangular matrix are the entries on the main diagonal

Ex:

$$A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$$

$$\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 3 \quad \lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 4$$

0 is an eigenvalue iff. A is not invertible!

$$\lambda = 0:$$

$A\bar{x} = \lambda\bar{x} = 0\bar{x} = \bar{0}$ so only if $A\bar{x} = 0$ has non-trivial

Theorem:

If $\bar{v}_1, \dots, \bar{v}_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix, then $\{\bar{v}_1, \dots, \bar{v}_r\}$ is linearly independent.

Note: The eigenspace at $\lambda=0$ is the same as the nullspace of A: $A\bar{x} = 0\bar{x} = \bar{0}$

The Characteristic Equation:

Ex: Find eigenvalues of

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$$

$$\text{We have } (A - \lambda I)\bar{x} = \bar{0}$$

We need non-trivial solution

↳ free variables

↳ non-invertible

$$\rightarrow \det(A - \lambda I) = 0$$

so $A - \lambda I = \bar{0}$ will have a non-trivial solution when $\det(A - \lambda I) = 0$:

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

$$\begin{aligned} &= \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = (2-\lambda)(-6-\lambda) - 9 \\ &= -12 - 2\lambda + 6\lambda + \lambda^2 - 9 \\ &= \lambda^2 + 4\lambda - 21 \\ &= (\lambda + 7)(\lambda - 3) = 0 \Rightarrow \\ &\lambda = \begin{cases} -7 \\ 3 \end{cases} \end{aligned}$$

Theorem:

A scalar λ is an eigenvalue of an $n \times n$ matrix iff. λ satisfies

$$\det(A - \lambda I) = 0$$

This equation is called the characteristic equation (or polynomial).

Ex: Find all the eigenvalues!

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 1 \\ 0 & 3 & 1 & 7 & -1 \\ 0 & 0 & 2 & 6 & 5 \\ 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(A - \lambda I_5)$$

$$\begin{bmatrix} 2-\lambda & 1 & 0 & 0 & 2 \\ 0 & 3-\lambda & 1 & 7 & -1 \\ 0 & 0 & 2-\lambda & 6 & 5 \\ 0 & 0 & 0 & 4-\lambda & 3 \\ 0 & 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

$$= (2-\lambda)(3-\lambda)(2-\lambda)(4-\lambda)(1-\lambda) = 0$$

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 2, \lambda_4 = 3, \lambda_5 = 4$$

Algebraic and Geometric Multiplicities:

Ex

$$\text{let } A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 2 = \lambda^2 - \lambda - 2 = 0$$

$$\lambda = -1$$

$$A + I = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \text{so } \bar{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ for } \lambda = -1 \quad = (\lambda - 2)(\lambda + 1)$$

The geometric multiplicity of an eigenvalue λ of A is the dimension of the eigenspace.

$$\hookrightarrow \dim \text{Nul}(A - \lambda I)$$

The algebraic multiplicity $m(\lambda)$ of an eigenvalue λ at A is the number of times λ appears as a root of the characteristic equation.

$$\hookrightarrow \text{In the diagonal matrix above } m(2) = 2$$

algebraic multiplicity \geq geometric multiplicity.

Ex

$$\text{let } A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) = \underline{(1-\lambda)^2} = 0$$

$$\hookrightarrow \text{eig. vec : } \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} ? \quad \begin{bmatrix} 0 & 2 \\ c & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_2 = 0 \quad \text{so : } \begin{bmatrix} x_1 \\ 0 \end{bmatrix} - x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\dim \text{Nul}(A - \lambda I) = \underline{1}$$

Diagonalisation:

An $n \times n$ matrix A is said to be diagonalisable if A is similar to a diagonal matrix D :

$$A = PDP^{-1},$$

Where the diagonals of D are the eigenvalues of A and the columns of P are the corresponding Eigenvectors

When is A diagonalisable?

A is diagonalisable iff. A has n linearly independent eigenvectors.

↳ Guaranteed if for every eigenvalue of A, the geometric multiplicity equals the algebraic multiplicity

↳ n distinct eigenvalues is a special case at this scenario.

Method: Diagonalisation

1. Find eigenvalues of A:

a) If triangular \rightarrow diagonal

b) roots of $\det(A - \lambda I_n) = 0$

2. For each λ , find a basis for the eigenspace:

$$(A - \lambda I_n) \vec{x} = \vec{0} \text{ in P.V.F.}$$

3. Construct D by placing the eigenvalues on the diagonal of an $n \times n$ zero matrix. Note it is good practice to place them in descending order: $d_{11} \geq d_{22} \geq d_{33} \dots \geq d_{nn}$

4. Construct P from the vectors found in step 2. Note the eigenvector must be placed in the same column as its corresponding eigenvalue

5. Find P^{-1}

6. Test $A = PDP^{-1}$

7. Enjoy a job well-done!!

$$EX \quad A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Step 1: Find λ

$$\begin{aligned} & \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} - 3 \begin{vmatrix} -3 & -3 \\ 3 & 1-\lambda \end{vmatrix} + 3 \begin{vmatrix} -3 & -5-\lambda \\ 3 & 3 \end{vmatrix} \\ &= (1-\lambda)((-5-\lambda)(1-\lambda)+9) - 3(-3(1-\lambda)+9) + 3(-9 - ((-5-\lambda) \cdot 3)) \\ &= (1-\lambda)(-5+5\lambda-\lambda+\lambda^2+9) - 3(-3+3\lambda+9) + 3(-9+15+3\lambda) \\ &= (1-\lambda)(\lambda^2+4\lambda+4) - 3(\cancel{3\lambda+6}) + 3(\cancel{3\lambda+6}) \\ &= (1-\lambda)(\lambda+2)^2 \quad \text{so } \lambda_1=1, \lambda_2=-2, \lambda_3=-2 \end{aligned}$$

Step 2: Find eigenvectors

$$\underline{\lambda=1}: \quad \begin{bmatrix} 1-1 & 3 & 3 \\ -3 & -5-1 & -3 \\ 3 & 3 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda=-2$

$$\begin{bmatrix} 1+2 & 3 & 3 \\ -3 & -5+2 & -3 \\ 3 & 3 & 1+2 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Step 3: Setup D

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Step 4: Setup P

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 5: Find P^{-1}

$$\left[\begin{array}{cccccc} 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array} \right]$$

Step 6: Test $A = P \cdot D \cdot P^{-1}$

$$A = P D P^{-1}$$

Step 7: Enjoy



A question of Power:

$$A^2 = P D P^{-1} P D P^{-1} = P D D D P^{-1} = P D^2 P^{-1}$$

$$A^3 = P D P^{-1} P D^2 P^{-1} = P D D D^2 P^{-1} = P D^3 P^{-1}$$

=

In general:

$$A^K = P D^K P^{-1}$$

Ex

$$A^{1000} = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \times \dots \times \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

v.s.

$$A^{1000} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (-2) & 0 \\ 0 & 0 & (-2) \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$