2.1. Matrix Operations

Adding and Subtracting

* Matrices must be same size
* Entry by Entry

Ex

$$\begin{bmatrix} 0 & 1 & -1 \\ 3 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 4 & 1 & 5 \end{bmatrix}$$

$$A + B = B + A$$
$$(A+B) + C = A + (B+C)$$
$$A + \bar{O} = A$$
$$A + (-A) = O$$
$$A - B = A + (-B)$$
$$A + B = A + C \iff B = C$$

Scaling:

* Multiply each entry by scalar

Ex

$$3\begin{bmatrix} 0 & 1 & -1 \\ 3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 3 & -3 \\ 9 & 0 & 6 \end{bmatrix}$$

$$s(A+B) = sA + sB \quad , s \in \mathbb{R}$$
$$(A+B)s = sA + sB \quad , s \in \mathbb{R}$$
$$(r+s)A = rA + sA \quad , r, s \in \mathbb{R}$$
$$r(sA) = (rs)A$$

Multiplying Matrices

A: $m \times n$ $\quad 3\underbrace{\begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix}}_{5}$ $\quad \overset{3}{m} \times \overset{5}{n}$ $\quad$ 3 rows, 5 columns

B: $k \times l$ $\quad 5\underbrace{\begin{bmatrix} x & x \\ x & x \\ x & x \\ x & x \\ x & x \end{bmatrix}}_{2}$ $\quad \overset{5}{k} \times \overset{2}{l}$ $\quad$ 5 rows, 2 columns

AB is only defined if $n = k$: Column of first matrix must match row of second

$$\begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix} \begin{bmatrix} x & x \\ x & x \\ x & x \\ x & x \\ x & x \end{bmatrix}$$

$3 \times \underbrace{5 \quad\quad 5}_{\text{Match}} \times 2$

size of AB: 3×2

Note: BA is not defined

$5 \times \underbrace{2 \quad\quad 3}_{\text{no match}} \times 5$

$$AB = A\begin{bmatrix} \bar{b}_1 & \bar{b}_2 & \cdots & \bar{b}_l \end{bmatrix} = \begin{bmatrix} A\bar{b}_1 & A\bar{b}_2 & \cdots & A\bar{b}_l \end{bmatrix}$$

$$= \left[A\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{k1} \end{bmatrix} \quad A\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{k2} \end{bmatrix} \quad \cdots \quad A\begin{bmatrix} b_{1l} \\ b_{2l} \\ \vdots \\ b_{kl} \end{bmatrix} \right]$$

EX

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \quad AB = \left[A\begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad A\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right] = \left[\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 2+6 & -1+2 \\ 4+9 & -2+3 \\ 6+12 & -3+4 \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 13 & 1 \\ 18 & 1 \end{bmatrix}$$

Row-Column rule:

If AB is defined, then the ij entry of AB is the dot product of the i'th row of A with the j'th column of B. $A: m \times n$ and $B: n \times l$

$$AB_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \cdots + a_{in} \cdot b_{nj}$$

Remember that we are indexing from 1.

EX

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 3 & 2 \\ 9 & 7 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & -1 \\ 3 & 4 & 5 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 0 = 14$$

$$AB_{23} = 4 \cdot 1 + 3 \cdot 2 + 2 \cdot 5 = 4 + 6 + 10 = 20 \rightarrow \begin{array}{cc} & 4 \\ 3 & \begin{bmatrix} x & x & x & x \\ x & x & 20 & x \\ x & x & x & x \end{bmatrix} \end{array}$$

Given B and C are defined appropriately, where A is $m \times n$:

$$A(BC) = (AB)C$$
$$A(B+C) = AB + AC$$
$$(B+C)A = BA + CA$$
$$rAB = (rA)B = A(rB)$$
$$I_m A = A = A I_n$$

, I is the identity matrix

If $AB = BA$ we say that **A and B commute** with one another.

Power

If A is square, i.e an $n \times n$ matrix:

$$A^k = \underbrace{A \cdot A \cdot A \cdots A}_{k \text{ times}}$$

Transpose

If A is an $m \times n$ matrix, then the transpose of A, A^T, is the $n \times m$ matrix whose columns are formed from the rows of A:

Ex

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \\ 7 & 8 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 4 & 3 & 7 \\ 1 & 2 & 8 \end{bmatrix}$$

$$(A^T)^T = A$$

$$(A+B)^T = A^T + B^T$$

$$(rA)^T = rA^T \quad , \; r \in \mathbb{R}$$

$$(AB)^T = B^T A^T$$

Ex

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \quad , \; \bar{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$(A\bar{x})^T = \left(\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right)^T = \begin{pmatrix} 5-9 \\ -10+12 \end{pmatrix}^T = \begin{bmatrix} -4 \\ 2 \end{bmatrix}^T = \begin{bmatrix} -4 & 2 \end{bmatrix}$$

$$x^T \cdot A^T = \begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 5-9 & -10+12 \end{bmatrix} = \begin{bmatrix} -4 & 2 \end{bmatrix}$$

Note: $(A\bar{x})^T = \bar{x}^T \cdot A^T$

$$\bar{x} \cdot x^T = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \begin{bmatrix} 5 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 15 \\ 15 & 9 \end{bmatrix}$$

$$\bar{x}^T \cdot x = \begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 34 \end{bmatrix}$$

$$A^T \cdot x^T = [2 \times \underbrace{2] \; [1}_{\neq} \times 2$$

$$\begin{array}{cccc} A & A^T & & \\ 2 \times 3 & 3 \times 2 & \rightarrow & 2 \times 2 \end{array}$$

$$\begin{array}{cccc} A^T & A & & \\ 3 \times 2 & 2 \times 3 & \rightarrow & 3 \times 3 \end{array}$$

Inverse

We say that a square matrix is invertible iff

$$A^{-1} \cdot A = I \quad \text{and} \quad A \cdot A^{-1} = I$$

Where A^{-1} is the inverse of A.

2 x 2 Matrices

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, A is invertible:

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $ad-bc$ is called the determinant of A:

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Ex

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \det A = 1 \cdot 4 - 2 \cdot 3 = -2$$

$$A^{-1} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

$$A \cdot A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} -2+3 & 1-1 \\ -6+\frac{12}{2} & 3-\frac{4}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$n \times n$ Matrices

Note: If A is invertible, $A\bar{x} = \bar{b}$ has a unique solution:

$$A\bar{x} = \bar{b}$$
$$A^{-1} \cdot A\bar{x} = A^{-1} \cdot b$$
$$\bar{x} = A^{-1} b$$

$$ax = 7$$
$$a^{-1} a x = a^{-1} \cdot 7$$
$$x = a^{-1} \cdot 7$$

Ex Use the inverse to solve

$$x_1 + 2x_2 = 5$$
$$3x_1 + 4x_2 = 6$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -10 + 6 \\ 15/2 - 6/2 \end{bmatrix} = \begin{bmatrix} -4 \\ 9/2 \end{bmatrix}$$

Let A and B be invertible $n \times n$ matrices

$$(A^{-1})^{-1} = A$$
$$(AB)^{-1} = B^{-1} A^{-1}$$
$$(A^T)^{-1} = (A^{-1})^T$$

Algorithm for finding A^{-1}:

An $n \times n$ matrix is invertible iff. it is row equivalent with I, i.e. we get the identity matrix when we reduce to reduced echelon form.

1. Setup $[A \; I]$, i.e A concatenated with I
2. Row reduce until A reaches reduced echelon form. The matrix I will now have changed into A^{-1}:

$$[A \; I] \overset{RREF}{\sim} [I \; A^{-1}]$$

EX $A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

$$
\left[\begin{array}{cccccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 4 & 5 & 3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]
\overset{r_2 \to r_2 - 4r_1}{\sim}
\left[\begin{array}{cccccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & -4 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]
$$

$$
\overset{r_3 \to \frac{1}{2}\cdot r_3}{\sim}
\left[\begin{array}{cccccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]
$$

$$
\overset{\substack{r_1 \to r_1 - r_3 \\ r_2 \to r_2 + r_3}}{\sim}
\left[\begin{array}{cccccc} 1 & 2 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & -3 & 0 & -4 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]
$$

$$
\overset{r_2 \to -\frac{1}{3}\cdot r_2}{\sim}
\left[\begin{array}{cccccc} 1 & 2 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]
$$

$$
\overset{r_1 \to r_1 - 2r_2}{\sim}
\left[\begin{array}{cccccc} 1 & 0 & 0 & -\frac{5}{3} & \frac{2}{3} & -\frac{1}{6} \\ 0 & 1 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]
$$

x^2

Theorem (The Invertible Matrix Theorem)

Let A be an $n \times n$ matrix. Then the following statements are equivalent.

(a) A is invertible.
(b) A is row equivalent to I_n.
(c) A has n pivot positions (i.e. one for each row and column).
(d) The equation $A\bar{x} = \bar{0}$ has *only* the trivial solution.
(e) The columns of A are linearly independent.
(f) The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\bar{x}) = A\bar{x}$ is one-to-one.
(g) The equation $A\bar{x} = \bar{b}$ has at least one solution for each $\bar{b}$ in $\mathbb{R}^n$.
(h) The columns of A span $\mathbb{R}^n$.
(i) The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\bar{x}) = A\bar{x}$ is onto.
(j) There is an $n \times n$ matrix C such that $CA = I$.
(k) There is an $n \times n$ matrix D such that $AD = I$.
(l) A^T is invertible.