

7. Orthogonality I

Def: Dot Product (Inner Product):

The dot product of two vectors

$$\bar{U} = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix}, \bar{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$$

is the sum of the entry-by-entry products:

$$\bar{U} \cdot \bar{V} = \bar{U}^T \cdot \bar{V} = [U_1 \ U_2 \ \dots \ U_n] \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = U_1 \cdot V_1 + U_2 \cdot V_2 + \dots + U_n \cdot V_n$$

The dot product is a **scalar**

Def: length of vector (norm)

$$\|\bar{V}\| = \sqrt{\bar{V} \cdot \bar{V}} = \sqrt{V_1^2 + V_2^2 + \dots + V_n^2}$$

A vector whose length is 1 is denoted a **unit vector**:

$$\bar{U} = \frac{1}{\|\bar{V}\|} \cdot \bar{V}$$

This is also called normalising a vector

Ex

$\bar{V} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$. Find a unit vector at \bar{V} :

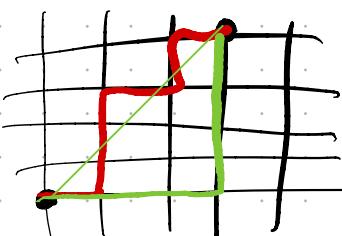
$$\bar{U} = \frac{1}{\|\bar{V}\|} \bar{V} = \frac{1}{\sqrt{1^2 + (-2)^2 + 2^2 + 0^2}} \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

Notice

$$\bar{U} = \begin{bmatrix} \frac{V_1}{\|\bar{V}\|} \\ \frac{V_2}{\|\bar{V}\|} \\ \vdots \\ \frac{V_n}{\|\bar{V}\|} \end{bmatrix}$$

Distance in \mathbb{R}^n :

$$\text{dist}(\bar{v}, v) = \|\bar{v} - v\| = \sqrt{(\bar{v}_1 - v_1)^2 + (\bar{v}_2 - v_2)^2 + \dots + (\bar{v}_n - v_n)^2}$$



↳ Also called Euclidean Distance or L_2 -norm

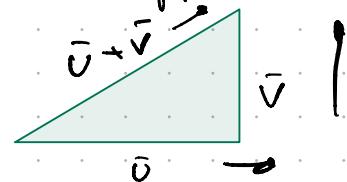
Orthogonal Vectors:

We say that two vectors in \mathbb{R}^n are orthogonal to each other if $\bar{v} \cdot v = 0$

We can also reformulate this in terms of Pythagorean Theorem:

Two vectors are orthogonal iff.:

$$\|\bar{v} + v\|^2 = \|\bar{v}\|^2 + \|v\|^2$$



Note the geometry of the sum of two vectors.

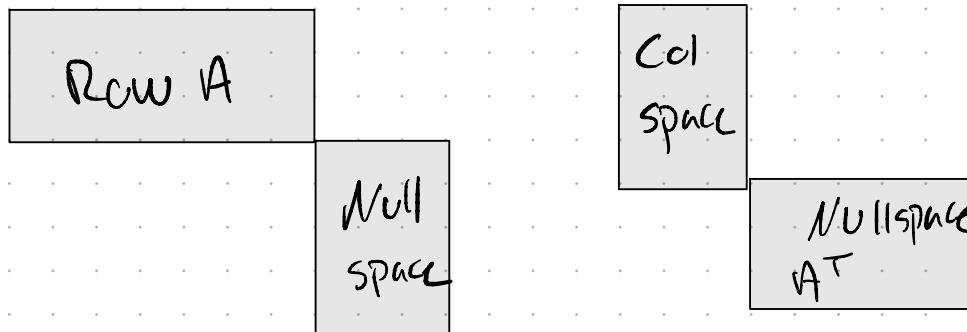
Orthogonal complement:

If W is a subspace in \mathbb{R}^n , and \bar{z} is orthogonal to all vectors in W , we say that \bar{z} is orthogonal to W . The set of all orthogonal vectors is called the Orthogonal Complement of W and is denoted W^\perp . Note, W^\perp is also a subspace of \mathbb{R}^n .

Null A, Col A, Row A:

Let A be an $m \times n$ matrix:

$$(\text{Row } A)^\perp = \text{Null } A \quad \text{and} \quad (\text{Col } A)^\perp = \text{Null } A^T$$



Orthogonal sets:

If $S = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p\}$ is an orthogonal set of nonzero vectors in \mathbb{R}^n , then S is also linearly independent which means S is a basis for \mathbb{R}^p . If $p=n$ then S is a basis for \mathbb{R}^n .

Note: Each distinct pair of vectors are orthogonal.
Also note independence does NOT entail orthogonality.

$$\bar{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \bar{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \bar{v} \cdot \bar{v} = (-1) \cdot (-1) + 2(-1) = -1$$

$$[\bar{v} \bar{v}] = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Orthogonality is stricter than independence

$$\bar{y} = c_1 \bar{v}_1 + c_2 \bar{v}_2 + \dots + c_n \bar{v}_n$$

The c_j 's are called weights

$$c_j = \frac{\bar{y} \cdot \bar{v}_j}{\bar{v}_j \cdot \bar{v}_j}$$

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EX

$$S = \left\{ \begin{bmatrix} \bar{v}_1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} \bar{v}_2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} \bar{v}_3 \\ -1 \\ -4 \\ 7 \end{bmatrix} \right\}, \quad \bar{g} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$$

a) Check whether S is an orthogonal set

b) Express \bar{g} as a lin. comp. of S :

a) $\bar{v}_1 \cdot \bar{v}_2 = 0$ What if $\bar{v}_1, \dots, \bar{v}_{100}$

$$\bar{v}_1 \cdot \bar{v}_3 = 0 \quad U^T U = \text{diag}(d_{11}, d_{22}, \dots, d_{nn})$$

$$\bar{v}_2 \cdot \bar{v}_3 = 0$$

b) $c_1 = \frac{\bar{g} \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} = \frac{11}{11} = 1, c_2 = \frac{-12}{6} = -2, c_3 = \frac{-66}{66} = -1$

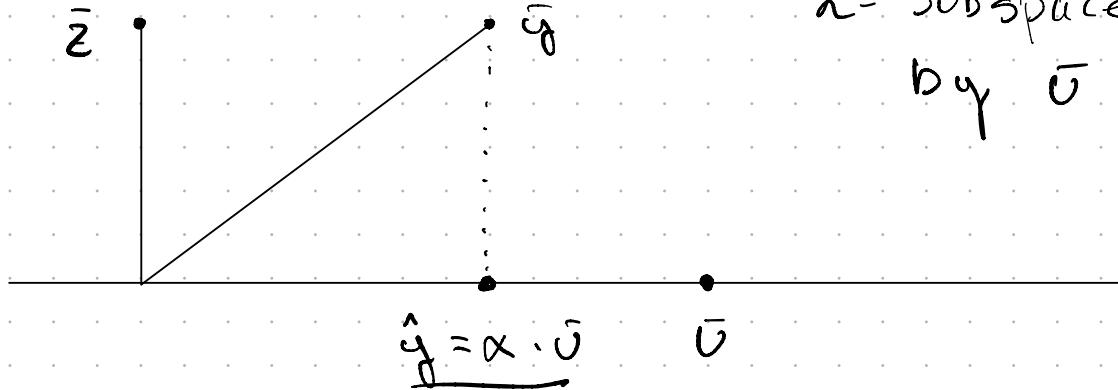
$$\bar{g} = \bar{v}_1 - 2\bar{v}_2 - \bar{v}_3$$

Exercise:

Let S be like above. Express $\bar{g} = \begin{bmatrix} 3 \\ 1/2 \\ -4 \end{bmatrix}$ as a linear comb. of S .

$$\bar{g} = \frac{1}{2}\bar{v}_1 - \bar{v}_2 - \frac{1}{2}\bar{v}_3$$

Orthogonal Projection:



$L = \text{Subspace spanned by } \bar{v}$

$$\bar{y} = \bar{y} + \bar{z}$$

$$\bar{y} = \bar{y} - \bar{z}$$

$$\bar{z} = \bar{y} - \bar{y} = \bar{y} - \alpha \cdot \bar{v}$$

$$z \perp \bar{v} \quad (\bar{y} - \alpha \bar{v}) \cdot \bar{v} = 0 \Leftrightarrow \bar{y} \bar{v} - \alpha \bar{v} \cdot \bar{v} = 0 \Rightarrow$$

$$\alpha \cdot \bar{v} \cdot \bar{v} = \bar{y} \cdot \bar{v} \Rightarrow \alpha = \frac{\bar{y} \cdot \bar{v}}{\bar{v} \cdot \bar{v}}$$

$$\bar{y} = \alpha \cdot \bar{v} = \frac{\bar{y} \cdot \bar{v}}{\bar{v} \cdot \bar{v}} \bar{v}$$

$$\bar{y} = \text{Proj}_L \bar{y} = \frac{\bar{y} \cdot \bar{v}}{\bar{v} \cdot \bar{v}} \bar{v}$$

The Orthogonal Decomposition Theorem:

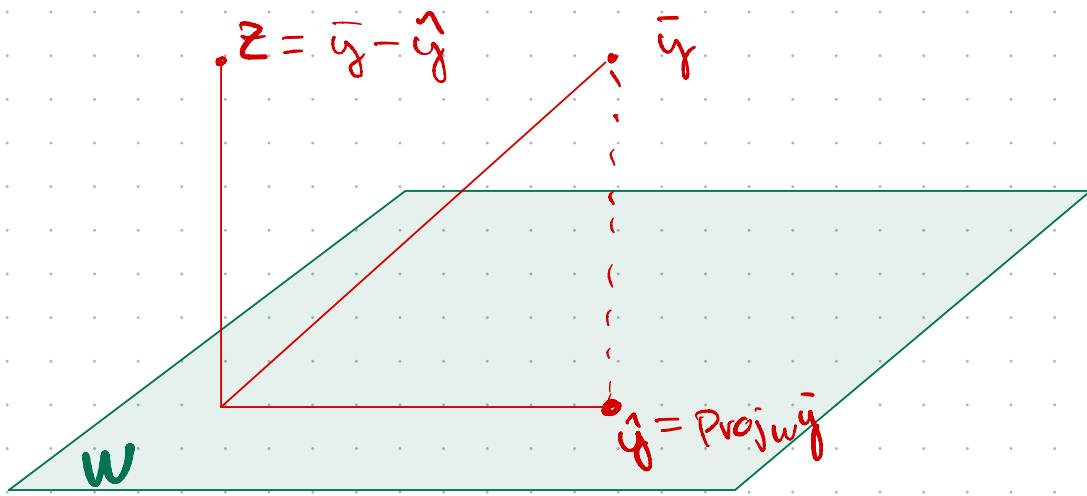
Let W be a subspace of \mathbb{R}^n . Then each \bar{y} in \mathbb{R}^n can be written uniquely as

$$\bar{y} = \bar{y} + \bar{z}$$

Where \bar{y} is in W and \bar{z} is in W^\perp . Even further if $\{\bar{v}_1, \dots, \bar{v}_p\}$ is an orthogonal basis:

$$\bar{y} = \frac{\bar{y} \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} \bar{v}_1 + \frac{\bar{y} \cdot \bar{v}_2}{\bar{v}_2 \cdot \bar{v}_2} \bar{v}_2 + \dots + \frac{\bar{y} \cdot \bar{v}_p}{\bar{v}_p \cdot \bar{v}_p} \bar{v}_p$$

and $\bar{z} = \bar{y} - \bar{y}$



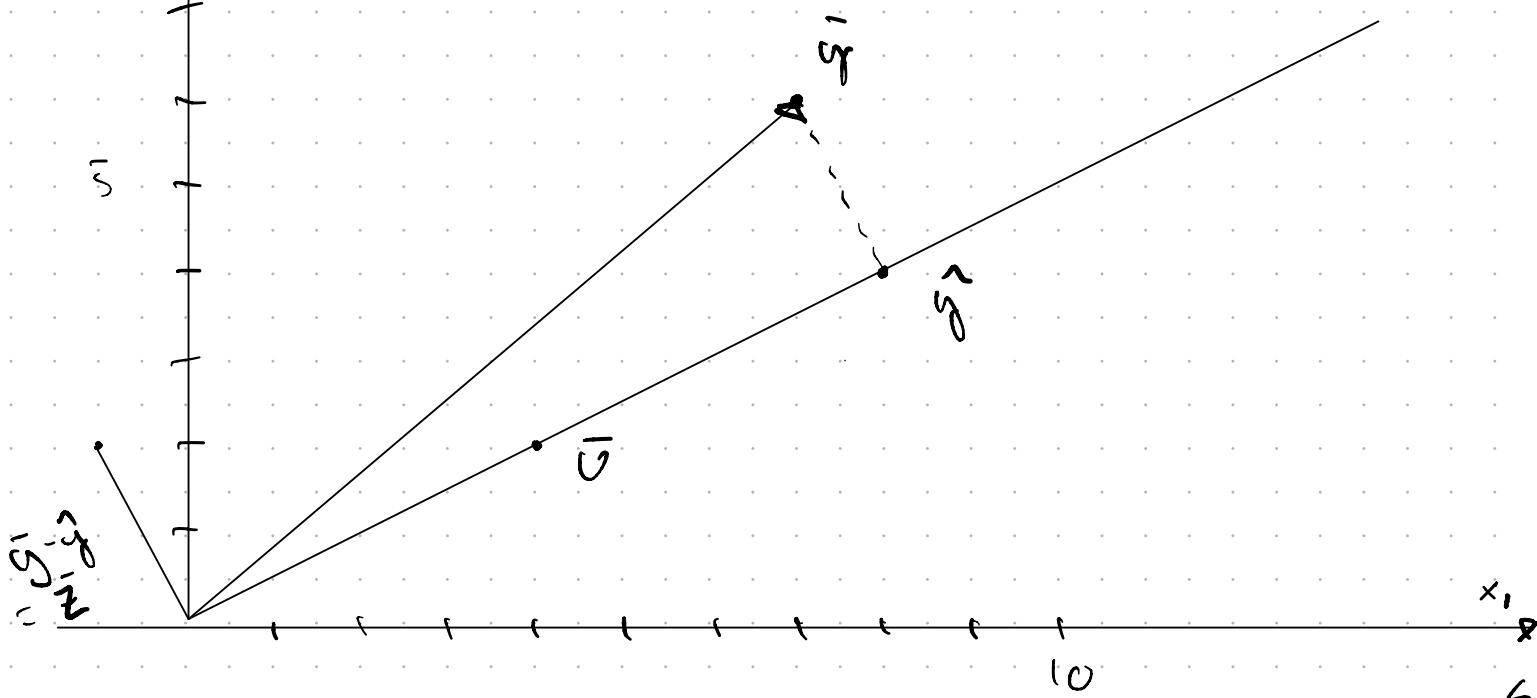
Ex:

$$\bar{g} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}, \bar{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

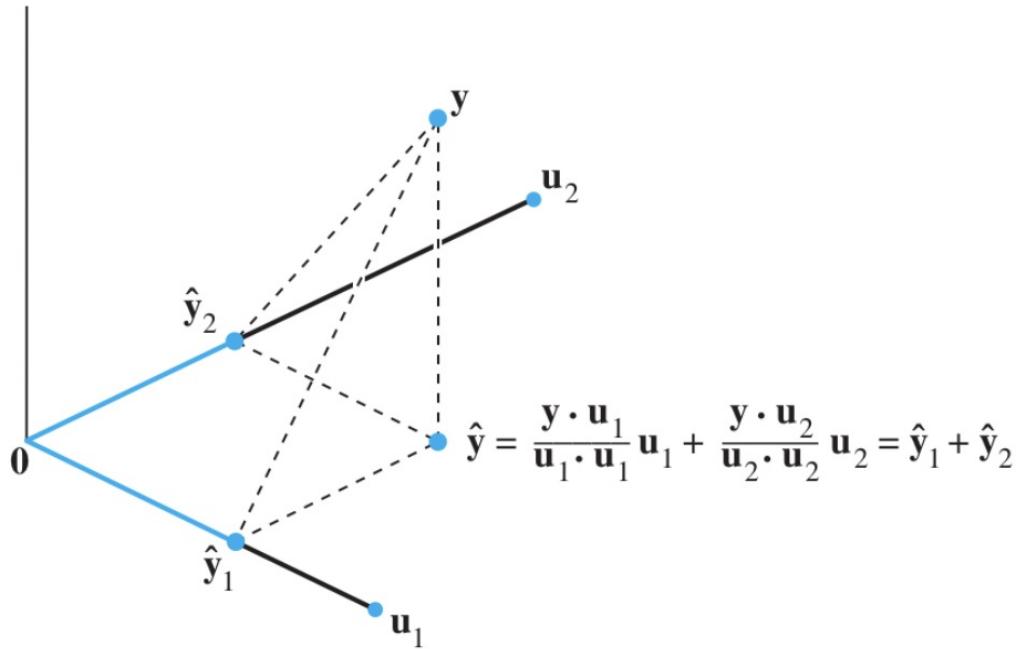
- a) Find $\text{proj}_{\bar{v}} \bar{g}$
- b) Write \bar{g} as the sum of two orthogonal vectors: one in $\text{span } \bar{v}$ and one orthogonal to \bar{v}

$$a) \hat{g} = \frac{40}{20} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$b) \bar{g} - \hat{g} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow \bar{g} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



A geometric view



Orthonormal sets:

- * An orthogonal set of unit vectors!
- * If W is spanned by this set, then the set is an Orthonormal basis for W .

An $m \times n$ matrix U has orthonormal columns iff

- $U^T U = I$
- $\|U \bar{x}\| = \|\bar{x}\|$
- $(U \bar{x}) \cdot (\bar{U} \bar{y}) = \bar{x} \cdot \bar{y}$
- $(U \bar{x}) \cdot (\bar{U} \bar{y}) = 0 \text{ iff. } \bar{x} \cdot \bar{y} = 0 \rightarrow \bar{x} \perp \bar{y}$.

$$Ex: \quad \bar{v}_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$$

Show that $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is an orthonormal basis for \mathbb{R}^3

Step 1: Show orthogonal:

- a) $\bar{v}_i \cdot \bar{v}_j \{ \forall i, j \in \mathbb{N}_n \mid i \neq j \}$
- b) $U^T U = I$

Step 2: Show length = 1

- a) $\|\bar{v}_i\| = 1$
- b) $U^T U = I$

Ex:

$$\bar{v}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Let $W = \{\bar{v}_1, \bar{v}_2\}$. Write \bar{y} as the sum of two orthonormal vectors, one in W and the other in W^\perp .

$$\bar{y} = \frac{\bar{y} \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} \bar{v}_1 + \frac{\bar{y} \cdot \bar{v}_2}{\bar{v}_2 \cdot \bar{v}_2} \bar{v}_2 = \frac{9}{30} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/5 \\ 3/2 \\ -3/10 \end{bmatrix} + \begin{bmatrix} -1 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} \quad \text{---} \quad \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\bar{z} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}$$

$$\bar{g} = \| \hat{g} \| \begin{bmatrix} -7/5/\| \hat{g} \| \\ \vdots \\ \vdots \end{bmatrix}$$

$$\bar{g} = \| \hat{g} \| \begin{bmatrix} \hat{g}_1/\| \hat{g} \| \\ \hat{g}_2/\| \hat{g} \| \\ \hat{g}_3/\| \hat{g} \| \end{bmatrix} + \| \bar{z} \| \begin{bmatrix} z_1/\| \bar{z} \| \\ z_2/\| \bar{z} \| \\ z_3/\| \bar{z} \| \end{bmatrix}$$