3.1. + 3.2. Determinants

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We know if $\det A = ad - bc = 0$, then a is not invertible. But why?

Assume $a \neq 0$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{r_2 \to ar} \begin{bmatrix} a & b \\ ac & ad \end{bmatrix} \xrightarrow{r_2 \to r_2 - cr_1} \begin{bmatrix} a & b \\ 0 & \boxed{ad-bc} \end{bmatrix}$$

If $ad - bc \neq 0$ we have pivots in all colums

↳ Invertible

Det < -0.0003

Det > 0.0003

Submatrix:

Let A_{ij} denote the submatrix obtained by deleting the i'th row and j'th column.

EX

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad A_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

Determinants:

$\{ a_{11} \; a_{12} \; a_{13}$
a_{21}

For $n \geq 2$, the determinant of $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{ij} \cdot \det A_{ij}$ with the signs alternating:

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det A_{ij}$$

OR

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det A_{ij}$$

Cofactor:

Given $A = [a_{ij}]$, we will call $(-1)^{i+j} \cdot \det A_{ij}$ the (i,j)-cofactor of A:

$$C_{ij} = (-1)^{i+j} \cdot \det A_{ij}$$

The determinant can thus be stated as:

$$\det A = \sum_{i=1}^{n} a_{ij} \cdot C_{ij}$$

OR

$$\det A = \sum_{j=1}^{n} a_{ij} \cdot C_{ij}$$

Ex: Find det A. Use $i=1$:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (-1)^{1+1} \cdot a_{11} \cdot \det A_{11} + (-1)^{1+2} a_{12} \cdot \det A_{12} + (-1)^{1+3} \cdot a_{13} \cdot \det A_{13}$$

$$= 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1 \cdot (-3) - 2(-6) + 3 \cdot (-3) = 0$$

A is not invertible

Ex: Find det A. Use $j=3$

$$\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 0 \cdot \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} + 1 \begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} + 0 \cdot \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = -2$$

Cofactor Expansion:

The method above is called cofactor expansion and can be done on any row or any column. We restate the results from above:

Cofactor Expansion along any row:

$$\det A = (-1)^{i+1} \cdot a_{i1} \cdot A_{i1} + (-1)^{i+2} \cdot a_{i2} \cdot A_{i2} + \dots + (-1)^{i+n} \cdot a_{in} \cdot A_{in}$$
$$= a_{i1} \cdot C_{i1} + a_{i2} \cdot C_{i2} + \dots + a_{in} \cdot C_{in}$$
$$= \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot A_{ij} \quad = \sum_{j=1}^{n} a_{ij} \cdot C_{ij}$$

Cofactor Expansion along any column:

$$\det A = (-1)^{1+j} \cdot a_{1j} \cdot A_{1j} + (-1)^{2+j} \cdot a_{2j} \cdot A_{2j} + \dots + (-1)^{2+j} \cdot a_{nj} \cdot A_{nj}$$
$$= a_{1j} \cdot C_{1j} + a_{2j} \cdot C_{2j} + \dots + a_{nj} \cdot C_{nj}$$
$$= \sum_{=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot A_{ij} \quad = \sum_{=1}^{n} a_{ij} \cdot C_{ij}$$

Ex: Expand along row 3:

$$\begin{vmatrix} 2 & 1 & 0 \\ 5 & 3 & 1 \\ 0 & 2 & 0 \end{vmatrix} = -2 \begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} = -4$$

Note: Always expand along row/column with most zeros. Use the algorithm recursively.

Ex

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix} = 2 \begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix} = 2 \cdot 5 \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} = 2 \cdot 5(-18+20)$$

$$= 20$$

Exercise

ⓐ

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix} = 3 \cdot 2 \cdot 1 \begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} = -12$$

ⓑ

$$A = \begin{bmatrix} 1 & -4 & 2 & -1 \\ 0 & 4 & 0 & 2 \\ 3 & -2 & 0 & 5 \\ 0 & 1 & 0 & 7 \end{bmatrix} = 2(-3) \begin{vmatrix} 4 & 2 \\ 1 & 7 \end{vmatrix} = -156$$

Empty space reserved for people who fail to choose easiest solution !! :)

Consider the following:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix} = 1 \cdot 6 \cdot 1 \cdot (4 \cdot 6 - 5 \cdot 0)$$

$$= 1 \cdot 6 \cdot 1 \cdot 4 \cdot 6$$

$$= 144$$

Theorem:

If A is a triangular matrix, $\det A$ is the product of the entries on the main diagonal.

Note: Echelon form is a triangular matrix. If Echelon Form has a free variable, the determinant $= 0 \Rightarrow$ Not invertible, etc., etc. ..

Theorem:

Let A be a square matrix. If B is obtained from A

i) Replacement : $\det B = \det A$

ii) Swap : $\det B = -\det A$

iii) Scaling : $\det B = K \cdot \det A$

Ex

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 2 \\ -3 & -5 & 2 \end{bmatrix} \overset{r_2 \to r_2 + r_1}{\sim} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 4 \\ -3 & -5 & 2 \end{bmatrix} \overset{r_3 \to r_3 + 3r_1}{\sim} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 4 \\ 0 & 1 & 8 \end{bmatrix}$$

$$\overset{r_2 \leftrightarrow r_3}{\sim} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 8 \\ 0 & 0 & 4 \end{bmatrix}, \quad \det A = -4$$

Exercise: Find det A using E.F.

$$A = \begin{bmatrix} 3 & 6 & 9 & 12 \\ 0 & 3 & 1 & 7 \\ 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \overset{r_3 \to r_3 - \frac{1}{3} r_1}{\sim} \begin{bmatrix} 3 & 6 & 9 & 12 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 2 & 1 \end{bmatrix} \overset{r_4 \to r_4 + 2r_3}{\sim}$$

$$\begin{bmatrix} 3 & 6 & 9 & 12 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & -5 \end{bmatrix}, \quad \det A = 45$$

It is always possible to obtain Echelon form using only replacement and swap:

$$\det A = (-1)^r \cdot \det U \quad , \; U = \text{Echelon Form}, \; r = \text{no. swap.}$$

Ex

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 0 \end{bmatrix} \overset{r_2 \to r_2 + r_3}{\sim} \begin{bmatrix} & \text{———} & & \\ 0 & -9 & 6 & 8 \\ & \text{———} & & \\ & \text{———} & & \end{bmatrix} \overset{r_1 \to r_1 - 2r_4}{\sim} \begin{bmatrix} 0 & 0 & 6 & 8 \\ 0 & -9 & 6 & 8 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 0 \end{bmatrix}$$

$$\overset{r_3 \to r_3 + 3r_4}{\sim} \begin{bmatrix} & \text{———} & & \\ & \text{———} & & \\ 0 & -12 & 1 & -2 \\ & & & \end{bmatrix} \overset{r_1 \leftrightarrow r_4}{\sim} \begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & -9 & 6 & 8 \\ 0 & -12 & 1 & -2 \\ 0 & 0 & 6 & 8 \end{bmatrix}$$

$$r_3 \to r_3 - \frac{4}{3} r_2 \begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & -9 & 6 & 8 \\ 0 & 0 & -7 & -\frac{38}{3} \\ 0 & 0 & 6 & 8 \end{bmatrix} \overset{r_4 \to r_4 + \frac{6}{7} r_3}{\sim} \begin{bmatrix} 1 & & & \\ & -9 & & \\ & & -7 & -\frac{38}{3} \\ & & 0 & -\frac{20}{7} \end{bmatrix}$$

$\det A = 180$

Theorem:

i) A square matrix is invertible iff $\det A \neq 0$

ii) $\det A^T = \det A$

iii) $\det AB = \det A \cdot \det B$

Ex

$$A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}, \quad \det(AB) = 25 \cdot 13 - 20 \cdot 14 = 45$$

$\det A = 9$

$\det B = 5$ $\to \det AB = 9 \cdot 5$